Sheaves in a nutshell

Topology as a Dietary choice

What is Topology?

When talking about *topology*, people draw cups with handles turning into donuts. When I think of *topology*, I see nutritious food. In mathematics, *topology* is defined as a family of *subsets* of some *space* X. We call these *subsets open*. *Open* sets are like meaty, skinless fruits.

Figure 1: Skinless fruits, are open set

For instance, in standard *topology*, the inside of a ball in *3-d* is considered meaty. Contrast this with an empty *sphere*, a *curve*, or a point-these are skinny when embedded in *3-d* - they have no nutritional value.

In one dimension (on a line), the inside of a *segment* is meaty, but a *segment* with endpoints is not *open*, because it has a rind (the endpoints).

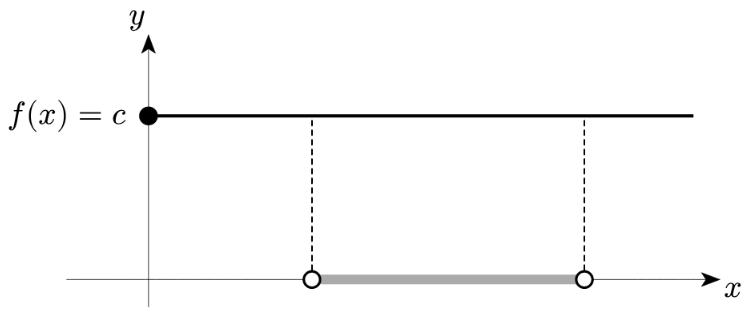
These four conditions define a topology.

1. The intersection of any two open sets is again an open set. This is what I mean by skinlessness. If you included skins, the intersection could end up skiny.
2. A union of open sets is again open. It's even more juicy, and no skin can be produced by a union. There is subtlety there: You can take a union of an arbitrary number of open sets and it's still open. But you have to be carful wiith intersections–only finite intersections are allowed. That's because by intersecting an infinite number of open sets you could end up with something very skinny-like a single point.
3. The whole space X is open. In a sense, it defines what it means to be juicy and it doesn't have a skin because it has no contact with outisde-it is its own Universe.
4. As usual, an empty set is an add item. Even though it's emppty, it's considered open. You may think of it as a union of zero open sets.

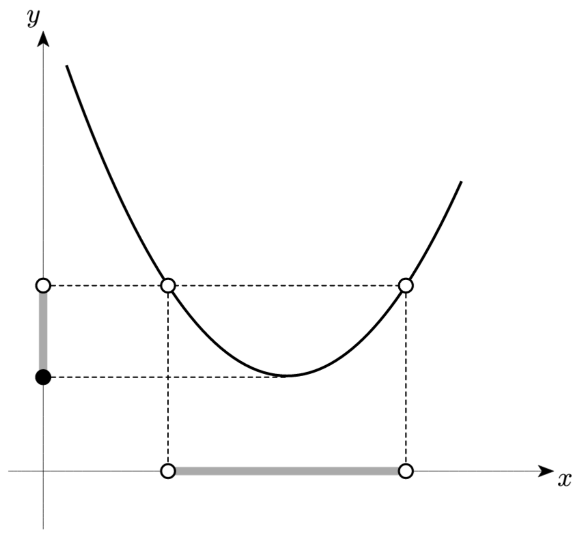
There are some extreme topologies, like the discrete topology in which all subsets are open (even individual points)a and a trivial (indiscrete) topology where only the whole space and the empty set are open. But most topologies are reasonable and adhere to our intuitions. So let's not worry about pathologies.

Continuity

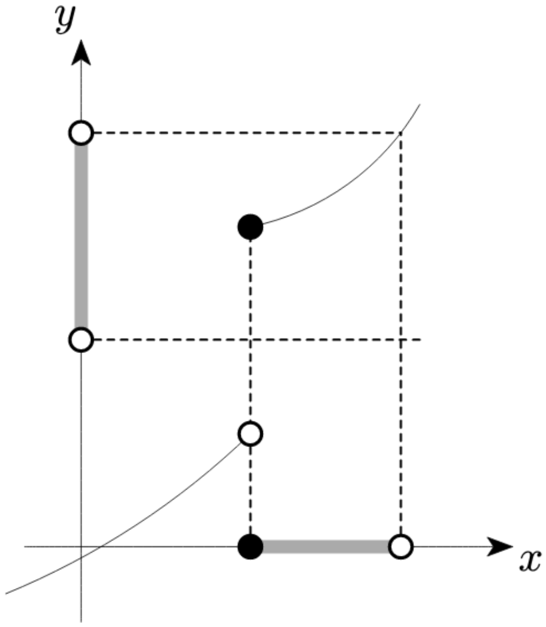
Consider a function from one topological space X to another topological space Y. Intuitively, a function is continuous if it doesn't make sudden jumps. So naively you might think that a continuous function maps any open set to a point which, in most topologies, is not open.



In fact any time a function stalls, or makes a turnaround (like the function $y = x^2$ at $x = 0$) you get a skinny point in its image.



The correct definition foes in the opposite direction: a function is continous if and only if the pre-image of every open set is open. First of all, a function cannot stall or turn around in the x direction, since that would mean mapping one point to many. Secondly, if a function makes a jump at some point x, it's possible to surround $f(x)$ with a samll open set whose counter-image contains x as its boundary.



It's also possible to define a *continous function* as a pair of *functions*. One *function* f is the usual mapping of *points* from X to Y. The other *function* g maps *open sets* in Y to *open sets* in X. The pair (f, g) defines a *continuous function* if for all *points* $x \in X$ and *open sets* O in Y we have the following equivalence:

$$f(x) \in O \Leftrightarrow x \in g(O)$$

The left-hand side tells us that x is the pre-image of O under f. The right-hand side tells us that g maps O to thus *preimage*. This formula looks a bit like an *adjunction* between f and g. It's an example of a more general notion of *Chu constructions*.

Finally, the cups and donuts magic trick uses invertible continous functions called *homeomorphisms* to deform shapes without tearing or gluing them.

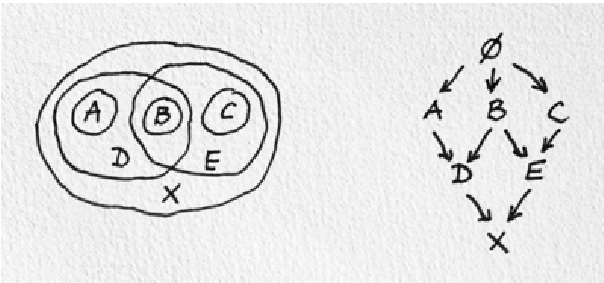
Presheaves and Topology.

Category theory lets us change the focus from individual objects to relationships between them. Since *topology* is defined using *open sets*, we'd start by concentrating on relations between *sets*.

One such obvious relation is *inclusion*. It imposes a categorical structure on the subsets of a given set X. We draw *arrows* between two sets whenever one is a *subset* of the other.

These arrows satisfy the axioms of a category:
- There is an identity arrow for every object (every set is its own subset)
- Arrows compose (inclusion is transitive).

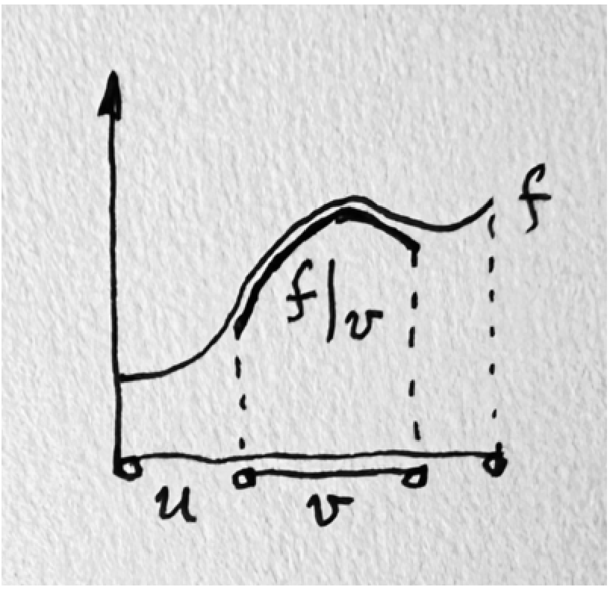
Not every pair of object is connected by an arrow some sets are disjoint, others overlap only partially. We may include the whole space as the *terminal object* (with arrows coming from every subset) and the *empty set* 0 as the *initial object* (with arrows going to every set). As categories go, this is a *thin category*, because there is at most one arrow between any two objects.



Every topological space gives thus rise to a thin category that abstracts the structure of ots open sets. But the real reason for defining a topology is to be able to talk about continous functions. These are fnctions between topological spaces such that thhe inverse image of every open set if open. Here, again, category theory tells us not to think about the details of how these functions are defined, but rather what we can do with them. And not just one function at a time, but the whole bucn at once.

So let's talk about sets of functions. We have our topological sapce X, and to each open subset u we will assign a set of continous function on it. These could be functions to real or complex numbers, or whatever-we don't care. All we care about is that they form a set.

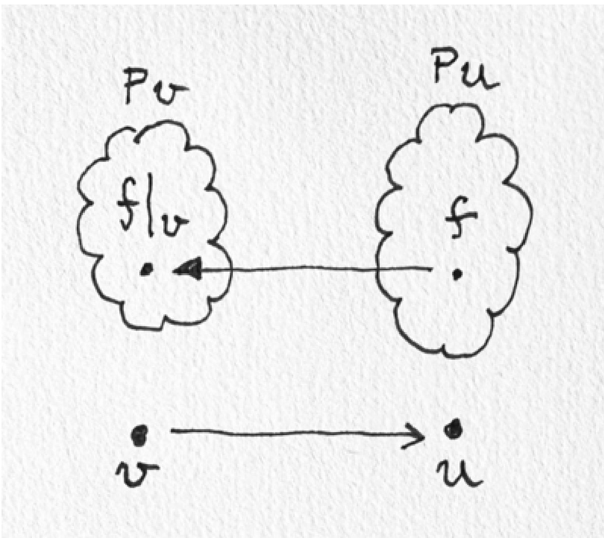
Since open sets in X form a (thin) category, we are talking about assigning to each object (open set) u its own set (of continous functions) P_u. Notice however that these sets of functions are not independent of each other. If one open set is a subset of anothher, it inherits all the funcions defined over the larger set. These are the same functions, the inly difference being that their arguments are restricted to a smaller subset. For instance, given two sets $vinu$ and a function $fu \to R$, there us a function such that on v.



Let's restate these statements in thhe categorical language. We already have a category X of open sets with inclusion. The sets of functions on these open sets are objects in the category *Set*. We have defined a mapping P between these two categories that assigns sets of functions to open sets.

Notice that we are dealing with two different categories whose objects are sets. One has inclusions as arrows, the other has functions as arrows. (To confuse matters even more, the objects in the second category represent sets of functions.)

To define a functor between categories, we also need a mapping of arrows to accompany the mapping of objects. An arrow $v \to u$ means that $vinu$. Corresponding to it, we have a function $P_u \to P_v$ that assigns to each $finP_u$ its restriction $f_v \in P_v$



Together, these mappings define a functor P: X^{op} -> S. The *op* notation means thhat the directions of arrows are reversed: the function is *contravariant.*

A functor must preserve the structure of a category, that is identity and composition. In our case this follows from the fact that an identity u in u maps to a trivial do-nothing restriction, and thhat restriction compose:

There is a special name for contravariant functors from any category C to *Set*. They are called *presheaves*, exactly because they were first introduced in the context of *topology* as precursors of *sheaves*. Consequently, the simpler functor C -> *Set* had to be confusingly called *co-presheaves*.

Presheaves on C form their own category, often denoted by C^{^}, with natural transformations as arrows.

Sheaves and Topology.

In all branches of science we sooner or later encounter the global vs. local duality. Topology is no different.

In topology we have the global definition of continuity: counter-images of all open sets are open. But we perceive a discontinuity as a local jump. How are the two pictures related, and can we express this topologically, thhat is without talking about sizes and distances?

All we have at our disposal are open sets, so exactly what properties of open sets are te most relevant? They do form a (thin) category with inclusions as arrows, but so does any set of subsets. As it turns out open sets can be stiched togther to create coverings. Such coverings let us zoom in on finer and finer details, thus creating the bridge between the global and the local picture.

Open sets are plump-they can easily fill the bulk of space. They are also skinless, so they can't touch each other without some overlap. That makes them perfect for constructing covers.

An open cover of a set u is a family of open sets $\{u_i\}$ such that u is their union:

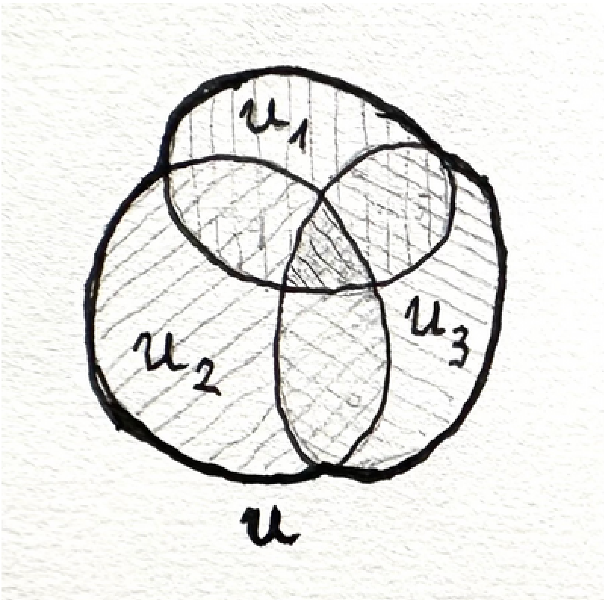


Figure 8: Skinless fruits, are open set

$$u = \cup_{\{i \in I\}} u_i$$

Here I is a set used for indexing the family.

If we have a continous function f defined over u, thhen all its restrictions $F|\{u_i\}$ are also continous (this follows from the condition that an intersection of open sets is open). Thus going from global to local is easy.

The converse is more interesting. Suppose that we have a family of functions f_i, one per each open set u_i, and we want to reconstruct the global function f defined over the set u cover by u_i's. This is only possible if the individual functions agree on overlaps.

Take two functions: f_i defined over u_i, and f_j defined over u_j. If the two sets overlap, each of the functions can be restricted to the overlap …

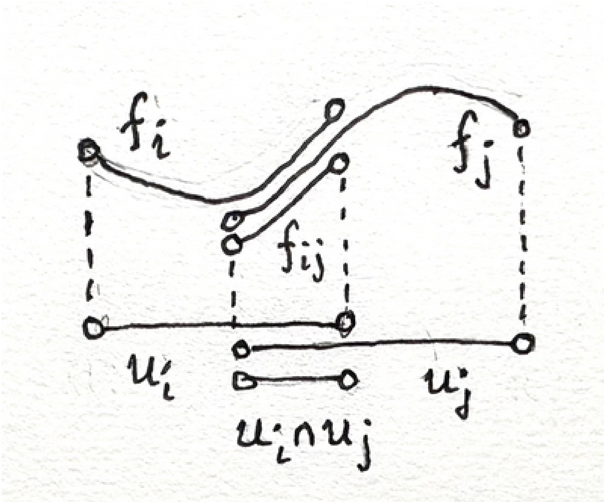


Figure 9: Skinless fruits, are open set

Take two functions: f_i defined over u_i, and f_j defined over u_j. If the two sets overlap, each of the functions can be restricted to the overlap ...

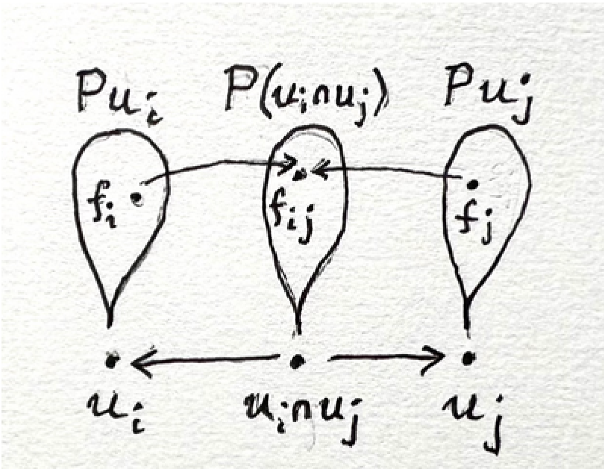


Figure 10: Skinless fruits, are open set