

# 概率统计 (A) 课程作业: 多元随机变量及其分布

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## 1

### 1.1

$$\begin{cases} \mathbb{P}(XY \neq 0) = a + c + 0.2 = 0.4 \\ \mathbb{P}(X \leq 0 | Y \leq 0) = \frac{a+b+0.1}{a+b+0.3} = \frac{2}{3} \\ a + b + c + 0.6 = 1 \end{cases} \Rightarrow \begin{cases} a = 0.1 \\ b = 0.2 \\ c = 0.1 \end{cases}$$

### 1.2

$X/Y$	-1	0	1	$\mathbb{P}(X = x_i)$
-1	0.1	0	0.2	0.3
0	0.1	0.2	0.1	0.4
1	0	0.2	0.1	0.3
$\mathbb{P}(Y = y_j)$	0.2	0.4	0.4	

### 1.3

$z_k$	-2	-1	0	1	2
$\mathbb{P}(X + Y = z_k)$	0.1	0.1	0.4	0.3	0.1

## 2

### 2.1

$$1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dy dx = \int_{-1}^1 \int_{-1}^1 C(1+xy) dy dx = \int_{-1}^1 2C dx = 4C \quad (1)$$

故  $C = \frac{1}{4}$ .

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dy = \begin{cases} \frac{1}{2}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \\ f_Y(y) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dx = \begin{cases} \frac{1}{2}, & |y| < 1 \\ 0, & |y| \geq 1 \end{cases} \end{aligned} \quad (2)$$

### 2.2

$X, Y$  不独立, 因为  $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$  对几乎所有  $(x,y)$  都不成立.

## 2.3

$$\begin{aligned}
 F_{X^2, Y^2}(x, y) &= \mathbb{P}(-\sqrt{x} \leq X \leq \sqrt{x}, -\sqrt{y} \leq Y \leq \sqrt{y}) = \int_{-\sqrt{x}}^{\sqrt{x}} \int_{-\sqrt{y}}^{\sqrt{y}} f_{X, Y}(t_1, t_2) dt_1 dt_2 = \min\{\sqrt{x}, 1\} \cdot \min\{\sqrt{y}, 1\} \\
 F_{X^2}(x) &= \mathbb{P}(-\sqrt{x} \leq X \leq \sqrt{x}) = \int_{-\sqrt{x}}^{\sqrt{x}} f_X(t) dt = \min\{\sqrt{x}, 1\} \\
 F_{Y^2}(y) &= \mathbb{P}(-\sqrt{y} \leq Y \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f_Y(t) dt = \min\{\sqrt{y}, 1\}
 \end{aligned} \tag{3}$$

因此  $X^2, Y^2$  相互独立.

## 3

### 3.1

$$f_{X, Y}(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right)\right] \tag{4}$$

其中  $\mu_1 = 4, \mu_2 = 3, \sigma_1^2 = 1, \sigma_2^2 = 2$ .

### 3.2

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-4)^2}{2}\right) \tag{5}$$

### 3.3

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{\left(y - (\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1))\right)^2}{2(1-\rho^2)\sigma_2^2}\right] = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(y+x-7)^2}{2}\right] \tag{6}$$

## 4

令  $Z = X + Y$ .  $X \sim NB(1, p) \Rightarrow \mathbb{P}(X = r) = p(1-p)^{r-1}, Y \sim NB(2, p) \Rightarrow \mathbb{P}(Y = r) = (r-1)p^2(1-p)^{r-2}$ .

$$\begin{aligned}
 \mathbb{P}(Z = r) &= \sum_{k=0}^{r-1} \mathbb{P}(X = k) \mathbb{P}(Y = r - k) \\
 &= \sum_{k=0}^{r-1} p(1-p)^{k-1} \cdot (r-k-1)p^2(1-p)^{r-k-2} \\
 &= p^3(1-p)^{r-3} \sum_{k=0}^{r-1} (r-k-1) \\
 &= \binom{r-1}{2} p^3(1-p)^{r-3}
 \end{aligned} \tag{7}$$

从而  $Z \sim NB(3, p)$ .

## 5

$$\begin{aligned}
 \mathbb{P}(Y = k) &= \sum_{x \geq k} \mathbb{P}(X = x) \binom{x}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{x-k} \\
 &= \sum_{x \geq k} e^{-6} \frac{6^x}{x!} \binom{x}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{x-k} \\
 &= e^{-2} \frac{2^k}{k!} \sum_{x-k \geq 0} e^{-4} \frac{4^{x-k}}{(x-k)!} \\
 &= e^{-2} \frac{2^k}{k!}
 \end{aligned} \tag{8}$$

从而  $Y \sim \pi(2)$ .  $\arg \max_k \mathbb{P}(Y = k) = \{1, 2\}$ .

## 6

### 6.1

$$\mathbb{P}(X = x, Z = z) = \begin{cases} \mathbb{P}(X = x) \mathbb{P}(y = z), & x < z \\ \mathbb{P}(X = x) \mathbb{P}(y \leq x), & x = z \\ 0, & x > z \end{cases} = \begin{cases} p^2(1-p)^{x+z}, & x < z \\ p(1-p)^z(1 - (1-p)^{z+1}), & x = z \\ 0, & x > z \end{cases} \tag{9}$$

### 6.2

注意到

$$\begin{aligned}
 \mathbb{P}(Z = z) &= \mathbb{P}(Z \leq z) - \mathbb{P}(Z \leq z-1) \\
 &= [\mathbb{P}(X \leq z)]^2 - [\mathbb{P}(X \leq z-1)]^2 \\
 &= [1 - (1-p)^{z+1}]^2 - [1 - (1-p)^z]^2 \\
 &= p(1-p)^z(2 - (2-p)(1-p)^z)
 \end{aligned} \tag{10}$$

因此

$$\mathbb{P}(X = x|Z = z) = \frac{\mathbb{P}(X = x, Z = z)}{\mathbb{P}(Z = z)} = \begin{cases} \frac{p(1-p)^x}{2 - (2-p)(1-p)^z}, & x < z \\ \frac{1 - (1-p)^{z+1}}{2 - (2-p)(1-p)^z}, & x = z \\ 0, & x > z \end{cases} \tag{11}$$

## 7

考虑对  $n$  归纳. 当  $n = 1$  时,  $f_{Y_1}(y) = \begin{cases} \lambda e^{-\lambda y}, & y > 0 \\ 0, & y \leq 0 \end{cases}$  恰好是指数分布, 故成立.

假设  $f_{Y_k}(y) = \begin{cases} y^{k-1} \lambda^k e^{-\lambda y} / (k-1)!, & y > 0 \\ 0, & y \leq 0 \end{cases}$  成立, 考虑计算  $Y_{k+1} = Y_k + X_{k+1}$  的密度分布, 当  $y > 0$  时,

有

$$\begin{aligned} f_{Y_{k+1}}(y) &= \int_{-\infty}^{+\infty} f_{Y_k}(t) f_X(y-t) dt = \int_0^y \frac{t^{n-1} \lambda^n e^{-\lambda t}}{(n-1)!} \cdot \lambda e^{-\lambda(y-t)} dt \\ &= \frac{\lambda^{n+1} e^{-\lambda y}}{(n-1)!} \frac{t^n}{n} \Big|_0^y = \frac{y^n \lambda^{n+1} e^{-\lambda y}}{n!} \end{aligned} \quad (12)$$

而显然  $f_{Y_{k+1}}(y) = 0$  ( $y \leq 0$ ), 因此结论对于  $k+1$  也成立, 从而对任意  $n$  均成立.

## 8

### 8.1

$$\begin{aligned} f_{Z,W}(z, w) &= f_{X,Y}(x, y) \left| \frac{\partial(x, y)}{\partial(z, w)} \right| \\ &= f_{X,Y}(x, y) \begin{vmatrix} \frac{1}{2\sqrt{z}} & 1 \\ 0 & 1 \end{vmatrix} \\ &= \begin{cases} \frac{\lambda^2 e^{-\lambda(x+y)}}{2\sqrt{z}}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{\lambda^2 e^{-\lambda w}}{2\sqrt{z}}, & 0 \leq \sqrt{z} \leq w \\ 0, & \text{otherwise} \end{cases} \end{aligned} \quad (13)$$

### 8.2

根据 section 7 有  $f_W(w) = w\lambda^2 e^{-\lambda w}$ , 故

$$f_{Z|W}(z|w) = \frac{f_{Z,W}(z, w)}{f_W(w)} = \frac{f_{Z,W}(z, w)}{w\lambda^2 e^{-\lambda w}} = \begin{cases} \frac{1}{2\sqrt{zw}}, & 0 \leq \sqrt{z} \leq w \\ 0, & \text{otherwise} \end{cases} \quad (14)$$

## 9

### 9.1

$$F_n(x) = \mathbb{P}(Y_n \leq x) = 1 - \mathbb{P}(Y_n > x) = 1 - \prod_{i=1}^n \mathbb{P}(X_i > x) = \begin{cases} 0, & x < 0 \\ 1 - (1-x)^n, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases} \quad (15)$$

$$\lim_{n \rightarrow \infty} F_n(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

## 9.2

$$\begin{aligned}\mathbb{P}(Z > 3) &= \mathbb{P}(X_1 \leq 1 \wedge X_1 + X_2 \leq 1 \wedge X_1 + X_2 + X_3 \leq 1) \\ &= \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} dx_3 dx_2 dx_1 \\ &= \int_0^1 \int_{y_1}^1 \int_{y_2}^1 dy_3 dy_2 dy_1\end{aligned}\quad (16)$$

**引理 1.** 对于  $n \in \mathbb{N}^+$  和  $L > 0$ , 有

$$\int_0^L \int_{y_1}^L \cdots \int_{y_{n-1}}^L dy_n \cdots dy_2 dy_1 = \frac{L^n}{n!} \quad (17)$$

证明. 考虑对  $n$  作归纳.  $n = 1$  时显然成立. 假设结论对于  $k$  成立, 则

$$\begin{aligned}\int_0^L \left[ \int_{y_1}^L \cdots \int_{y_k}^L dy_{k+1} \cdots dy_2 \right] dy_1 &= \int_0^L \left[ \int_0^{L-y_1} \cdots \int_{z_k}^{L-y_1} dz_{k+1} \cdots dz_2 \right] dy_1 \\ &= \int_0^L \frac{(L-y_1)^k}{k!} dy_1 \\ &= -\frac{(L-y_1)^{k+1}}{(k+1)!} \Big|_0^L \\ &= \frac{L^{k+1}}{(k+1)!}\end{aligned}\quad (18)$$

结论对  $k+1$  亦成立, 从而对所有正整数  $n$  均成立. □

结合上述引理, 可知  $\mathbb{P}(Z > 3) = \frac{1}{6}$ .

## 10

### 10.1

因为  $X_1, X_2, X_3$  是独立随机变量, 故  $f_{X_1, X_2, X_3}(x_1, x_2, x_3) = f_{X_1}(x_1)f_{X_2}(x_2)f_{X_3}(x_3) = (2\pi)^{-\frac{3}{2}} \exp\left(-\frac{x_1^2 + x_2^2 + x_3^2}{2}\right)$ .

### 10.2

$$\begin{aligned}f_{Z, X_2, X_3}(z, x_2, x_3) &= f_{X_1, X_2, X_3}(x_1, x_2, x_3) \left| \frac{\partial(z, x_2, x_3)}{\partial(x_1, x_2, x_3)} \right|^{-1} \\ &= f_{X_1, X_2, X_3}(x_1, x_2, x_3) \left| \begin{array}{ccc} \frac{1}{\sqrt{1+x_3^2}} & \frac{x_3}{\sqrt{1+x_3^2}} & \frac{x_3}{\sqrt{x_3^2+1}} - \frac{x_3(x_1+x_2x_3)}{(x_3^2+1)^{\frac{3}{2}}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right|^{-1} \\ &= (2\pi)^{-\frac{3}{2}} \sqrt{1+x_3^2} \exp \left[ -\frac{1}{2} \left( \left( z\sqrt{1+x_3^2} - x_2x_3 \right)^2 + x_2^2 + x_3^2 \right) \right]\end{aligned}\quad (19)$$

### 10.3

$$\begin{aligned}
 f_{Z, X_3}(z, x_3) &= \int_{-\infty}^{+\infty} f_{Z, X_2, X_3}(z, x_2, x_3) dx_2 \\
 &= \int_{-\infty}^{+\infty} (2\pi)^{-\frac{3}{2}} \sqrt{1+x_3^2} \exp \left[ -\frac{1}{2} \left( (z\sqrt{1+x_3^2} - x_2 x_3)^2 + x_2^2 + x_3^2 \right) \right] dx_2 \\
 &= \int_{-\infty}^{+\infty} (2\pi)^{-\frac{3}{2}} \sqrt{1+x_3^2} \exp \left[ -\frac{1}{2} \left( (x_2\sqrt{1+x_3^2} - z x_3)^2 + z^2 + x_3^2 \right) \right] dx_2 \quad (20) \\
 &= \frac{1}{2\pi} \exp \left[ -\frac{1}{2}(z^2 + x_3^2) \right] \int_{-\infty}^{+\infty} \frac{\sqrt{1+x_3^2}}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \sqrt{1+x_3^2} x_2 - z x_3 \right)^2 \right] dx_2 \\
 &= \frac{1}{2\pi} \exp \left[ -\frac{1}{2}(z^2 + x_3^2) \right]
 \end{aligned}$$

可以发现  $f_Z(z) = \int_{-\infty}^{+\infty} f_{Z, X_3}(z, x_3) dx_3 = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{z^2}{2} \right)$  故  $Z \sim \mathcal{N}(0, 1)$ , 而  $f_{Z, X_3}(z, x_3) = f_Z(z)f_{X_3}(x_3)$  说明  $Z, X_3$  是独立随机变量, 因此  $Z, X_3 \sim \text{i.i.d. } \mathcal{N}(0, 1)$ .