Machine Learning Homework: Week 3 & 4

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Statement

Prove that

$$\begin{aligned}\frac{1}{2}\mathbb{P}\left(\sup_{\phi\in\Phi}\left|\mathbb{E}_z\phi(z)-\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)\right|\geqslant 2\varepsilon\right)&\leqslant\mathbb{P}\left(\sup_{\phi\in\Phi}\left|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)\right|\geqslant\varepsilon\right)\\&\leqslant 2\mathbb{P}\left(\sup_{\phi\in\Phi}\left|\mathbb{E}_z\phi(z)-\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)\right|\geqslant\frac{\varepsilon}{2}\right)\end{aligned}\tag{1}$$

Solution

$$\begin{aligned}&\mathbb{P}\left(\sup_{\phi\in\Phi}\left|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)\right|\geqslant\varepsilon\right)\\=&\mathbb{P}\left(\forall\delta>0,\exists\phi\in\Phi\text{ s.t. }\left|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)\right|>\varepsilon-\delta\right)\\\leqslant&\mathbb{P}\left(\forall\delta>0,\exists\phi\in\Phi\text{ s.t. }\left|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\mathbb{E}_z\phi(z)\right|>\frac{\varepsilon-\delta}{2}\text{ or }\left|\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)-\mathbb{E}_z\phi(z)\right|>\frac{\varepsilon-\delta}{2}\right)\\\leqslant&\mathbb{P}\left(\forall\delta>0,\exists\phi\in\Phi\text{ s.t. }\left|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\mathbb{E}_z\phi(z)\right|>\frac{\varepsilon-\delta}{2}\right)+\mathbb{P}\left(\forall\delta>0,\exists\phi\in\Phi\text{ s.t. }\left|\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)-\mathbb{E}_z\phi(z)\right|>\frac{\varepsilon-\delta}{2}\right)\\=&\mathbb{P}\left(\sup_{\phi\in\Phi}\left|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\mathbb{E}_z\phi(z)\right|\geqslant\frac{\varepsilon}{2}\right)+\mathbb{P}\left(\sup_{\phi\in\Phi}\left|\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)-\mathbb{E}_z\phi(z)\right|\geqslant\frac{\varepsilon}{2}\right)\\=&2\mathbb{P}\left(\sup_{\phi\in\Phi}\left|\mathbb{E}_z\phi(z)-\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)\right|\geqslant\frac{\varepsilon}{2}\right)\end{aligned}\tag{2}$$

which provides the proof for the inequality on the right side.

$$\begin{aligned}&\mathbb{P}\left(\sup_{\phi\in\Phi}\left|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)\right|\geqslant\varepsilon\right)\\=&\mathbb{P}\left(\forall\delta>0,\exists\phi\in\Phi\text{ s.t. }\left|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)\right|>\varepsilon-\delta\right)\\\geqslant&\mathbb{P}\left(\forall\delta>0,\exists\phi\in\Phi\text{ s.t. }\left|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\mathbb{E}_z\phi(z)\right|>2\varepsilon-\delta\text{ and }\left|\frac{1}{n}\sum_{i=n+1}^{2n}\phi(z_i)-\mathbb{E}_z\phi(z)\right|<\varepsilon\right)\\=&\mathbb{P}\left(\forall\delta>0,\exists\phi_0\in\Phi\text{ s.t. }\left|\frac{1}{n}\sum_{i=1}^{n}\phi_0(z_i)-\mathbb{E}_z\phi_0(z)\right|>2\varepsilon-\delta\right)\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=n+1}^{2n}\phi_0(z_i)-\mathbb{E}_z\phi_0(z)\right|<\varepsilon\right)\end{aligned}\tag{3}$$

By Chernoff bound we obtain that $\mathbb{P}\left(\left|\frac{1}{n}\sum\limits_{i=n+1}^{2n}\phi(z_i)-\mathbb{E}_z\phi(z)\right|<\varepsilon\right)\geqslant\frac{1}{2}$ when $n\geqslant\frac{\ln 2}{\varepsilon^2}$, which gives the proof for the inequality on the left side

$$\begin{aligned}\cdots&\geqslant\frac{1}{2}\mathbb{P}\left(\forall\delta>0,\exists\phi\in\Phi\text{ s.t. }\left|\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)-\mathbb{E}_z\phi(z)\right|>2\varepsilon-\delta\right)\\&=\frac{1}{2}\mathbb{P}\left(\sup_{\phi\in\Phi}\left|\mathbb{E}_z\phi(z)-\frac{1}{n}\sum_{i=1}^{n}\phi(z_i)\right|\geqslant2\varepsilon\right)\end{aligned}\tag{4}$$

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Statement

Prove that

$$\sum_{k=0}^{d}\binom{n}{k}<\left(\frac{\mathrm{e}n}{d}\right)^d$$

for any $n\in\mathbb{N}^+$ and $1\leqslant d\leqslant n$.

Solution

By Taylor expansion we have

$$\mathrm{e}^d=\sum_{k=0}^{\infty}\frac{d^k}{k!}>\sum_{k=0}^{d}\frac{d^k}{k!}$$

where immediately follows the proof

$$\begin{aligned}\left(\frac{\mathrm{e}n}{d}\right)^d&>n^d\sum_{k=0}^{d}\frac{d^{k-d}}{k!}\\&\geqslant n^d\sum_{k=0}^{d}\frac{n^{k-d}}{k!}\\&=\sum_{k=0}^{d}\frac{n^k}{k!}\\&\geqslant\sum_{k=0}^{d}\frac{n^{\underline{k}}}{k!}\\&=\sum_{k=0}^{d}\binom{n}{k}\end{aligned}\tag{5}$$

3

Statement

Prove that the following two programming problems are equivalent. For simplicity, we assume that the first problem has non-trivial solution, i.e. there exists $t>0$ satisfying $y_i(\mathbf{w}^\mathrm{T}\mathbf{x}_i+b)\geqslant t\ (\forall i\in[n])$.

$$\begin{aligned}\max_{\mathbf{w},b,t}\quad&t\\\text{s.t.}\quad&y_i(\mathbf{w}^\mathrm{T}\mathbf{x}_i+b)\geqslant t\ (\forall i\in[n])\\&\|\mathbf{w}\|=1\end{aligned}\qquad\qquad\begin{aligned}\min_{\mathbf{w},b,t}\quad&\tfrac{1}{2}\|\mathbf{w}\|\\\text{s.t.}\quad&y_i(\mathbf{w}^\mathrm{T}\mathbf{x}_i+b)\geqslant1\ (\forall i\in[n])\end{aligned}$$

Solution

Let the solution to the two problems above be t_0 and $\mathbf{w}_0$ respectively. By the assumption, we have $t_0 > 0$.

By dividing the inequality in the first problem by t_0 we obtain $y_i\left(\frac{\mathbf{w}^{\mathrm{T}}}{t_0}\mathbf{x}_i + \frac{b}{t_0}\right) \geqslant 1$, so $\mathbf{w}' = \frac{\mathbf{w}}{t_0}$ is a candidate for the second problem, and thus $\|\mathbf{w}_0\| \leqslant \|\mathbf{w}'\| = \frac{1}{t_0}$.

By dividing the inequality in the second problem by $\|w_0\|$ we obtain $y_i\left(\frac{\mathbf{w}_0}{\|\mathbf{w}_0\|}\mathbf{x}_i + \frac{b}{\|\mathbf{w}_0\|}\right) \geqslant \frac{1}{\|\mathbf{w}_0\|}$, thus $t \geqslant \frac{1}{\|\mathbf{w}_0\|}$ since $\left\|\frac{\mathbf{w}_0}{\|\mathbf{w}_0\|}\right\| = 1$.

So we know that $\|\mathbf{w}_0\|t_0 = 1$, which shows the equivalence relationship between these two programming problems.