Machine Learning Homework: Week 1

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Statement

Random variable $X \sim \mathcal{N}(0,1)$, let $f(x) = \mathbb{P}\left[X \geqslant x\right]$, find an elementary function $g(x)$ satisfying $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$. Furthermore, find such elementary functions $g_u(x)$ and $g_l(x)$ satisfying $g_l(x) \leqslant f(x) \leqslant g_u(x)$ for $\forall t \in \mathbb{R}^+$.

Solution

$$g_l(x) = \left(\frac{1}{x} - \frac{1}{x^3}\right)\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}, \qquad g_u(x) = \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}x}.$$

We first prove that $g_l(x)$ and $g_u(x)$ are the lowerbound and upperbound of $f(x)$ respectively. Notice that

$$f(x) = \int_x^{+\infty} \frac{\mathrm{e}^{-t^2/2}}{\sqrt{2\pi}}\mathrm{d}t$$

by the technique of **integration by parts**, we have

$$\sqrt{2\pi}f(x) = \int_x^{+\infty} \frac{1}{t}\mathrm{d}\left(-\mathrm{e}^{-t^2/2}\right) = -\frac{\mathrm{e}^{-t^2/2}}{t}\bigg|_x^{+\infty} - \int_x^{+\infty}\frac{\mathrm{e}^{-t^2/2}}{t^2}\mathrm{d}t \leqslant \frac{\mathrm{e}^{-x^2/2}}{x} = \sqrt{2\pi}g_u(x) \tag{1}$$

Furthermore, applying such technique to the remaining part above gives

$$\int_x^{+\infty}\frac{\mathrm{e}^{-t^2/2}}{t^2}\mathrm{d}t = \int_x^{+\infty}\frac{1}{t^3}\mathrm{d}\left(-\mathrm{e}^{-t^2/2}\right) = -\frac{\mathrm{e}^{-t^2/2}}{t^3}\bigg|_x^{+\infty} - \int_x^{+\infty}\frac{3\mathrm{e}^{-t^2/2}}{t^4}\mathrm{d}t \leqslant \frac{\mathrm{e}^{-x^2/2}}{x^3} \tag{2}$$

and thus

$$\sqrt{2\pi}f(x) \geqslant \left(\frac{1}{x} - \frac{1}{x^3}\right)\mathrm{e}^{-x^2/2} = \sqrt{2\pi}g_l(x) \tag{3}$$

$\lim\limits_{x\to+\infty}\frac{f(x)}{g_u(x)} = 1$ and $\lim\limits_{x\to+\infty}\frac{f(x)}{g_l(x)} = 1$ can be easily proved using **L'Hôpital's rule**:

$$\lim_{x\to+\infty}\frac{f(x)}{g_u(x)} = \lim_{x\to+\infty}\frac{f'(x)}{g_u'(x)} = \lim_{x\to+\infty}\frac{-\mathrm{e}^{-x^2/2}}{-\left(1+\frac{1}{x^2}\right)\mathrm{e}^{-x^2/2}} = 1 \tag{4}$$

$$\lim_{x\to+\infty}\frac{f(x)}{g_l(x)} = \lim_{x\to+\infty}\frac{f'(x)}{g_l'(x)} = \lim_{x\to+\infty}\frac{-\mathrm{e}^{-x^2/2}}{-\left(1+\frac{3}{x^4}\right)\mathrm{e}^{-x^2/2}} = 1 \tag{5}$$