概率统计 (A) 课程作业: 离散变量的数字特征

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(1)

$$\begin{cases}1=\sum_{n\geqslant 0}\mathbb{P}\left(X=n\right)=\sum_{n\geqslant 0}A\cdot\frac{B^n}{n!}=A\mathrm{e}^B\\ a=\mathbb{E}\left[X\right]=\sum_{n\geqslant 1}n\mathbb{P}\left(X=n\right)=\sum_{n\geqslant 1}A\cdot\frac{B^n}{(n-1)!}=AB\sum_{n\geqslant 0}\frac{B^n}{n!}=AB\mathrm{e}^B\end{cases}\Rightarrow\begin{cases}A=\mathrm{e}^{-a}\\ B=a\end{cases}$$

(2)

令 $\frac{a}{1+a}=p$.

$$\begin{aligned}\mathbb{E}\left[X\right]&=\sum_{n\geqslant 0}n\mathbb{P}\left(X=n\right)=\frac{1}{1+a}\sum_{n\geqslant 0}np^n=\frac{1}{1+a}\cdot\frac{p}{(1-p)^2}=a\\ \mathbb{E}\left[X^2\right]&=\sum_{n\geqslant 0}n^2\mathbb{P}\left(X=n\right)=\frac{1}{1+a}\sum_{n\geqslant 0}n^2p^n=\frac{1}{1+a}\cdot\frac{p(1+p)}{(1-p)^3}=a(1+2a)\\ \mathrm{Var}\left[X\right]&=\mathbb{E}\left[X^2\right]-\mathbb{E}\left[X\right]^2=a+a^2\end{aligned}$$

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$$\begin{aligned}\mathbb{E}\left[\min\{X,Y\}\right]&=\sum_{n\geqslant 0}n\mathbb{P}\left(\min\{X,Y\}=n\right)\\ &\leqslant\sum_{n\geqslant 0}n(\mathbb{P}\left(X=n\right)\mathbb{P}\left(Y\geqslant n\right)+\mathbb{P}\left(Y=n\right)\mathbb{P}\left(X\geqslant n\right))\\ &\leqslant\sum_{n\geqslant 0}n(\mathbb{P}\left(X=n\right)+\mathbb{P}\left(Y=n\right))\\ &=\mathbb{E}\left[X\right]+\mathbb{E}\left[Y\right]\\ &=\mathbb{E}\left[X+Y\right]\end{aligned}$$

由于 $X,Y>0$, 故后者收敛能推出前者也收敛.

$$\begin{aligned}\mathbb{E}\left[\min\{X,Y\}\right]&=\sum_{n\geqslant 0}n\mathbb{P}\left(\min\{X,Y\}=n\right)\\ &=\sum_{n\geqslant 1}\mathbb{P}\left(\min\{X,Y\}\geqslant n\right)\\ &=\sum_{n\geqslant 1}\mathbb{P}\left(X\geqslant n\right)\mathbb{P}\left(Y\geqslant n\right)\end{aligned}$$

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(1)

$$\int_{-\infty}^{+\infty}|x|f_X(x)\mathrm{d}x=2\int_0^{+\infty}\frac{x}{\pi(x^2+1)}\mathrm{d}x=+\infty$$

广义积分 $\int_{-\infty}^{+\infty}xf_X(x)\mathrm{d}x$ 不绝对收敛, 因此 $\mathbb{E}\left[X\right]$ 不存在.

(2)

$$\begin{aligned}\int_{-\infty}^{+\infty}|\arctan(x)|f_X(x)\mathrm{d}x&=2\int_0^{+\infty}\frac{\arctan(x)}{\pi(x^2+1)}\mathrm{d}x\\&=\frac{2}{\pi}\int_0^{+\infty}\arctan(x)\mathrm{d}\arctan(x)\\&=\left.\frac{\arctan^2(x)}{\pi}\right|_0^{+\infty}\\&=\frac{\pi}{4}\end{aligned}$$

广义积分 $\int_{-\infty}^{+\infty}\arctan(x)f_X(x)\mathrm{d}x$ 绝对收敛, 因此 $\mathbb{E}\left[\arctan X\right]$ 存在. 容易观察到积分函数是奇函数, 因此有 $\mathbb{E}\left[\arctan X\right]=0$.

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(1)

$$\mathbb{E}\left[\mathrm{e}^{aX}\right]=\int_{-\infty}^{+\infty}\mathrm{e}^{ax}\cdot\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x=\mathrm{e}^{\frac{a^2}{2}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(x-a)^2}{2}}\mathrm{d}x=\mathrm{e}^{\frac{a^2}{2}}$$

(2)

$$\mathbb{E}\left[|X|\right]=2\int_0^{+\infty}x\cdot\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x=\frac{\sqrt{2}}{\sqrt{\pi}}\int_0^{+\infty}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}\left(\frac{x^2}{2}\right)=-\left.\frac{\sqrt{2}}{\sqrt{\pi}}\mathrm{e}^{-u}\right|_0^{+\infty}=\frac{\sqrt{2}}{\sqrt{\pi}}$$

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(1)

$$\mathbb{E}\left[X_N\right]=\sum_{k=1}^{N}\mathbb{P}\left(X_N\geqslant k\right)=\sum_{k=1}^{N}\left[1-\left(\frac{k-1}{N}\right)^n\right]=N-\sum_{k=0}^{N-1}\left(\frac{k}{N}\right)^n$$

(2)

$$\lim_{N\to\infty}\mathbb{E}\left[X_N/N\right]=1-\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\left(\frac{k}{N}\right)^n=1-\int_0^1x^n\mathrm{d}x=\frac{n}{n+1}$$

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记 X 表示需要支付的金额, x 表示乙猜测的号码.

(1)

$$\mathbb{E}[X]=\frac{1}{15}\left[(x-1)^2+2(x-2)^2+3(x-3)^2+4(x-4)^2+5(x-5)^2\right]=x^2-\frac{22}{3}x+15$$

当 x 取 $\frac{11}{3}$ 时上式取到最小值 $\frac{14}{9}$.

(2)

$$\mathbb{E}[X]=\frac{1}{15}\left[|x-1|+2|x-2|+3|x-3|+4|x-4|+5|x-5|\right]$$

当 x 取 4 时上式取到最小值 1.

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(0)

主要需要验证非负性与归一性.

当 $\max\{|x|,|y|\}\geqslant\pi$ 时, $g(x)g(y)=0$, 故 $f_{X,Y}(x,y)=\phi(x)\phi(y)>0$. 当 $|x|,|y|<\pi$ 时, $\phi(x)\phi(y)>\frac{\mathrm{e}^{-\pi^2}}{2\pi}>\frac{\mathrm{e}^{-\pi^2}}{2\pi}g(x)g(y)$, 从而也有 $f_{X,Y}(x,y)=\phi(x)\phi(y)>0$.

易知 $\iint f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y=1$, 而 $\int_x g(x)\mathrm{d}x=0$, 因此 $\iint g(x)g(y)\mathrm{d}x\mathrm{d}y=0$, 故归一性满足.

(1)

$$f_X(x)=\int_{-\infty}^{+\infty}f_{X,Y}(x,y)\mathrm{d}y=\phi(x)\int_{-\infty}^{+\infty}\phi(y)\mathrm{d}y+\frac{\mathrm{e}^{-\pi^2}}{2\pi}g(x)\int_{-\infty}^{+\infty}g(y)\mathrm{d}y=\phi(x)$$

故 X 的边缘分布是正态分布. Y 显然是对称的, 故也是正态分布.

(2)

$$\begin{aligned}\rho_{X,Y}&=\frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}[X]\operatorname{Var}[Y]}}=\mathbb{E}\left[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])\right]=\mathbb{E}[XY]\\&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xy\left(\phi(x)\phi(y)+\frac{\mathrm{e}^{-\pi^2}}{2}g(x)g(y)\right)\mathrm{d}x\mathrm{d}y\\&=\left[\int_{-\infty}^{+\infty}x\phi(x)\mathrm{d}x\right]\left[\int_{-\infty}^{+\infty}y\phi(y)\mathrm{d}y\right]+\frac{\mathrm{e}^{-\pi^2}}{2}\left[\int_{-\infty}^{+\infty}xg(x)\mathrm{d}x\right]\left[\int_{-\infty}^{+\infty}yg(y)\mathrm{d}y\right]\\&=0\end{aligned}$$

但 $f_{X,Y}(x,y)=f_X(x)f_Y(y)$ 不恒成立, 因此 X,Y 不线性相关但不独立.

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(1)

$$\rho_{X_1,X_1-X_2} = \frac{\mathrm{Cov}\left(X_1, X_1 - X_2\right)}{\sqrt{\mathrm{Var}\left[X_1\right]\mathrm{Var}\left[X_1 - X_2\right]}} = \frac{\mathrm{Var}\left[X_1\right] - \mathrm{Cov}\left(X_1, X_2\right)}{\sqrt{\mathrm{Var}\left[X_1\right]\left(\mathrm{Var}\left[X_1\right] + \mathrm{Var}\left[X_2\right] - 2\mathrm{Cov}\left(X_1, X_2\right)\right)}} = \sqrt{\frac{1-\rho}{2}}$$

(2)

记 $Z = X_1 - X_2$. 考虑 Z 的密度函数.

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{+\infty} f_{X_1,X_2}(x, x-z)\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho x(x-z) + (x-z)^2\right)\right]\mathrm{d}x \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2(1-\rho^2)}\left(2(1-\rho)\left(x-\frac{z}{2}\right)^2 + \frac{z^2(1+p)}{2}\right)\right]\mathrm{d}x \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{z^2}{4(1-\rho)}\right) \int_{-\infty}^{+\infty} \exp\left(-\frac{\left(x-\frac{z}{2}\right)^2}{1+\rho}\right)\mathrm{d}x \\
&= \frac{1}{2\sqrt{\pi(1-\rho)}} \exp\left(-\frac{z^2}{4(1-\rho)}\right)
\end{aligned}$$

于是

$$\begin{aligned}
\mathbb{E}\left[|Z|\right] &= 2\int_0^{+\infty} z f_Z(z)\mathrm{d}z \\
&= \frac{1}{\sqrt{\pi(1-\rho)}} \int_0^{+\infty} z\exp\left(-\frac{z^2}{4(1-\rho)}\right)\mathrm{d}z \\
&= \frac{2\sqrt{1-\rho}}{\sqrt{\pi}} \int_0^{+\infty} \exp\left(-\frac{z^2}{4(1-\rho)}\right)\mathrm{d}\frac{z^2}{4(1-\rho)} \\
&= \frac{2\sqrt{1-\rho}}{\sqrt{\pi}}
\end{aligned}$$

(3)

考虑 $\mathbb{E}\left[\max\{X_1, X_2\}\right] = \mathbb{E}\left[X_2\right] + \mathbb{E}\left[\max\{0, X_1 - X_2\}\right]$, 显然 $\mathbb{E}\left[X_2\right] = 0$, 而

$$\mathbb{E}\left[\max\{0, X_1 - X_2\}\right] = \int_0^{+\infty} z f_Z(z)\mathrm{d}z = \frac{\sqrt{1-\rho}}{\sqrt{\pi}}$$

因此 $\mathbb{E}\left[\max\{X_1, X_2\}\right] = \frac{\sqrt{1-\rho}}{\sqrt{\pi}}$.

9

用 X_i, Y_i $(1 \leqslant i \leqslant n)$ 分别表示第 i 次摸球时是否摸到了白球/黑球, 则显然有 $X = \sum_{i=1}^n X_i, Y = \sum_{i=1}^n Y_i$.
由于 $\forall i, \mathbb{E}\left[X_i\right] = p, \mathbb{E}\left[Y_i\right] = q$, 故 $\mathbb{E}\left[X\right] = np, \mathbb{E}\left[Y\right] = nq$.
简单分析可知 $\mathbb{E}\left[X_iY_j\right] = \begin{cases} 0, & i = j \\ pq, & i \neq j \end{cases}, \mathbb{E}\left[X_iX_j\right] = \begin{cases} p, & i = j \\ p^2, & i \neq j \end{cases}, \mathbb{E}\left[Y_iY_j\right] = \begin{cases} q, & i = j \\ q^2, & i \neq j \end{cases}$.

(1)

$$\begin{aligned}\operatorname{Cov}(X,Y) &= \mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}[X_iY_j]-\mathbb{E}[X]\mathbb{E}[Y]\\ &= n(n-1)pq-n^2pq\\ &= -npq\end{aligned}$$

(2)

$$\begin{aligned}\operatorname{Var}[X] &= \mathbb{E}\left[X^2\right]-\mathbb{E}[X]^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}[X_iX_j]-\mathbb{E}[X]^2\\ &= np+(n^2-n)p^2-(np)^2=np(1-p)\\ \operatorname{Var}[Y] &= nq(1-q)\\ \rho_{X,Y} &= \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}[X]\operatorname{Var}[Y]}}=\frac{-npq}{\sqrt{n^2p(1-p)q(1q)}}=-\sqrt{\frac{pq}{(1-p)(1-q)}}\end{aligned}$$

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(1)

$\mathbb{E}[X]=\mathbb{E}[X_1+X_2+\cdots+X_n]$ 即为从 N 个球中抽出总计 n 个, 其数码和的期望. 因此有

$$\begin{aligned}\mathbb{E}[X] &= \sum_k k\binom{a}{k}\binom{b}{n-k}\Big/\binom{N}{n}\\ &= \sum_k a\binom{a-1}{k-1}\binom{b}{n-k}\Big/\binom{N}{n}\\ &= a\binom{N-1}{n-1}\Big/\binom{N}{n}\\ &= \frac{an}{N}\end{aligned}$$

(2)(3)

不难验证 $\mathbb{E}[X_i]=\frac{a}{N}$ (因为 **(1)** 的结论对所有 n 均成立) 以及 $\mathbb{E}[X_iX_j]=\begin{cases}\frac{a}{N}, & i=j\\ \frac{a(a-1)}{N(N-1)}, & i\neq j\end{cases}$.

$$\begin{aligned}\operatorname{Var}[X] &= \mathbb{E}\left[X^2\right]-\mathbb{E}[X]^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}[X_iX_j]-\mathbb{E}[X]^2\\ &= n\frac{a}{N}+(n^2-n)\frac{a(a-1)}{N(N-1)}-\frac{n^2a^2}{N^2}\\ &= \frac{na(N-n)(N-a)}{N^2(N-1)}\\ \operatorname{Cov}(X_i,X_j) &= \mathbb{E}[X_iX_j]-\mathbb{E}[X_i]\mathbb{E}[X_j]=\begin{cases}\frac{a(N-a)}{N^2}, & i=j\\ \frac{-a(N-a)}{N^2(N-1)}, & i\neq j\end{cases}\end{aligned}$$

特别的, 当 $n=N$ 时, 有 $\operatorname{Var}[X]=0$.