Machine Learning Homework: Week 7 & 8

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April 24, 2022

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Statement

Prove the following lemma.

Lemma 1. For any $\delta > 0$,

$$\mathbb{P}_{S\sim D^n}\left(\mathbb{E}_{h\sim\mathcal{P}}[\mathrm{e}^{n(err_D(h)-err_S(h))^2}] \geqslant 3/\delta\right) \leqslant \delta$$

Solution

First we prove that for some fixed $h \sim \mathcal{P}$,

$$\mathbb{E}_{S\sim D^n}[\mathrm{e}^{n(err_D(h)-err_S(h))^2}] \leqslant 3$$

Denote $|err_D(h) - err_S(h)|$ by $\Delta(h)$, by applying Chernoff bound,

$$\mathbb{P}_{S\sim D^n}(\Delta(h) \geqslant \varepsilon) \leqslant 2\exp(-2n\varepsilon^2)$$

Thus

$$\begin{aligned}
\mathbb{E}_{S\sim D^n}\left[\mathrm{e}^{n\Delta(h)^2}\right] &= \int_0^{+\infty} \mathbb{P}_{S\sim D^n}\left(\mathrm{e}^{n\Delta(h)^2} \geqslant t\right)\mathrm{d}t \\
&= \int_1^{+\infty} \mathbb{P}_{S\sim D^n}\left(\Delta(h) \geqslant \sqrt{\frac{\ln t}{n}}\right)\mathrm{d}t + 1 \\
&\leqslant \int_1^{+\infty} 2\mathrm{e}^{-2\ln t}\mathrm{d}t + 1 \\
&= 3
\end{aligned}$$

Then by applying Markov Inequality we get

$$\mathbb{P}_{S\sim D^n}\left(\mathbb{E}_{h\sim\mathcal{P}}[\mathrm{e}^{n\Delta(h)^2}] \geqslant 3/\delta\right) \leqslant \frac{\mathbb{E}_{S\sim D^n}\left(\mathbb{E}_{h\sim\mathcal{P}}[\mathrm{e}^{n\Delta(h)^2}]\right)}{3/\delta} = \frac{\mathbb{E}_{h\sim\mathcal{P}}\left(\mathbb{E}_{S\sim D^n}[\mathrm{e}^{n\Delta(h)^2}]\right)}{3/\delta} \leqslant \frac{\mathbb{E}_{h\sim\mathcal{P}}(3)}{3/\delta} = \delta$$

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Statement

Determine $D_{KL}(\mathcal{Q}\|\mathcal{P})$ for $\mathcal{P} = \mathcal{N}(\mathbf{0}, I_d)$ and $\mathcal{Q} = \mathcal{N}(\mu\mathbf{w}, I_d)$, where $\mathbf{w} \in \mathbb{R}^d$ satisfies $\|\mathbf{w}\|^2 = 1$, μ stands for the scale factor.

Solution

$$
\begin{aligned}
D_{KL}(\mathcal{Q}\|\mathcal{P}) &= \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \exp\left[-\frac{1}{2}\|\mathbf{x}-\mu\mathbf{w}\|^2\right] \frac{1}{2}\left(\|\mathbf{x}\|^2 - \|\mathbf{x}-\mu\mathbf{w}\|^2\right) \mathrm{d}\mathbf{x} \\
&= \int_\lambda \int_{\mathbf{y}\in\mathbb{R}^{d-1}, \mathbf{y}\perp\mathbf{w}} \frac{1}{(2\pi)^{d/2}} \exp\left[-\frac{1}{2}\|\lambda\mathbf{w}+\mathbf{y}-\mu\mathbf{w}\|^2\right] \frac{1}{2}\left(\|\lambda\mathbf{w}+\mathbf{y}\|^2 - \|\lambda\mathbf{w}+\mathbf{y}-\mu\mathbf{w}\|^2\right) \mathrm{d}\lambda\mathrm{d}\mathbf{y} \\
&= \int_\lambda \int_{\mathbf{y}\in\mathbb{R}^{d-1}, \mathbf{y}\perp\mathbf{w}} \frac{1}{(2\pi)^{d/2}} \exp\left[-\frac{1}{2}(\lambda-\mu)^2 - \frac{1}{2}\|\mathbf{y}\|^2\right] \frac{1}{2}\left(\lambda^2 + \|\mathbf{y}\|^2 - (\lambda-\mu)^2 - \|\mathbf{y}\|^2\right) \mathrm{d}\lambda\mathrm{d}\mathbf{y} \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(\lambda-\mu)^2\right] \frac{1}{2}(2\lambda\mu-\mu^2)\mathrm{d}\lambda \left[\int_{\mathbf{y}\in\mathbb{R}^{d-1}, \mathbf{y}\perp\mathbf{w}} \frac{1}{(2\pi)^{(d-1)/2}} \exp\left(-\frac{1}{2}\|\mathbf{y}\|^2\right) \mathrm{d}\mathbf{y}\right] \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(\lambda-\mu)^2\right] (\lambda\mu-\mu^2)\mathrm{d}\lambda + \frac{\mu^2}{2} \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(\lambda-\mu)^2\right] \mu\mathrm{d}\frac{(\lambda-\mu)^2}{2} + \frac{\mu^2}{2} \\
&= \frac{\mu^2}{2}
\end{aligned}
$$