概率统计 (A) 课程作业: 多元随机变量及其分布

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1.1

$$\begin{cases} \mathbb{P}(XY \neq 0) = a + c + 0.2 = 0.4 \\ \mathbb{P}(X \leqslant 0 | Y \leqslant 0) = \frac{a+b+0.1}{a+b+0.3} = \frac{2}{3} \\ a + b + c + 0.6 = 1 \end{cases} \Rightarrow \begin{cases} a = 0.1 \\ b = 0.2 \\ c = 0.1 \end{cases}$$

1.2

X/Y	-1	0	1	$\mathbb{P}(X = x_i)$
-1	0.1	0	0.2	0.3
0	0.1	0.2	0.1	0.4
1	0	0.2	0.1	0.3
$\mathbb{P}(Y = y_j)$	0.2	0.4	0.4	

1.3

z_k	-2	-1	0	1	2
$\mathbb{P}(X + Y = z_k)$	0.1	0.1	0.4	0.3	0.1

2

2.1

$$1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x, y) \mathrm{d}y \mathrm{d}x = \int_{-1}^{1} \int_{-1}^{1} C(1 + xy) \mathrm{d}y \mathrm{d}x = \int_{-1}^{1} 2C \mathrm{d}x = 4C \tag{1}$$

故 $C = \frac{1}{4}$.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} f_{X,Y}(x, y) \mathrm{d}y = \begin{cases} \frac{1}{2}, & |x| < 1 \\ 0, & |x| \geqslant 1 \end{cases} \\ f_Y(y) &= \int_{-\infty}^{+\infty} f_{X,Y}(x, y) \mathrm{d}x = \begin{cases} \frac{1}{2}, & |y| < 1 \\ 0, & |y| \geqslant 1 \end{cases} \end{aligned} \tag{2}$$

2.2

X, Y 不独立, 因为 $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$ 对几乎所有 (x, y) 都不成立.

2.3

$$\begin{aligned}
F_{X^2,Y^2}(x,y) &= \mathbb{P}\left(-\sqrt{x} \leqslant X \leqslant \sqrt{x}, -\sqrt{y} \leqslant Y \leqslant \sqrt{y}\right) = \int_{-\sqrt{x}}^{\sqrt{x}}\int_{-\sqrt{y}}^{\sqrt{y}} f_{X,Y}(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2 = \min\{\sqrt{x},1\}\cdot\min\{\sqrt{y},1\} \\
F_{X^2}(x) &= \mathbb{P}\left(-\sqrt{x} \leqslant X \leqslant \sqrt{x}\right) = \int_{-\sqrt{x}}^{\sqrt{x}} f_X(t)\mathrm{d}t = \min\{\sqrt{x},1\} \\
F_{Y^2}(y) &= \mathbb{P}\left(-\sqrt{y} \leqslant Y \leqslant \sqrt{y}\right) = \int_{-\sqrt{y}}^{\sqrt{y}} f_Y(t)\mathrm{d}t = \min\{\sqrt{y},1\}
\end{aligned} \tag{3}$$

因此 X^2, Y^2 相互独立.

3

3.1

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[\frac{-1}{2(1-\rho^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right)\right] \tag{4}$$

其中 $\mu_1 = 4, \mu_2 = 3, \sigma_1^2 = 1, \sigma_2^2 = 2$.

3.2

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_1}\exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-4)^2}{2}\right) \tag{5}$$

3.3

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{\left(y-(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1))\right)^2}{2(1-\rho^2)\sigma_2^2}\right] = \frac{1}{\sqrt{\pi}}\exp\left[-\frac{(y+x-7)^2}{2}\right] \tag{6}$$

4

令 $Z = X+Y$. $X \sim NB(1,p) \Rightarrow \mathbb{P}(X=r) = p(1-p)^{r-1}, Y \sim NB(2,p) \Rightarrow \mathbb{P}(Y=r) = (r-1)p^2(1-p)^{r-2}$.

$$\begin{aligned}
\mathbb{P}(Z=r) &= \sum_{k=0}^{r-1}\mathbb{P}(X=k)\mathbb{P}(Y=r-k) \\
&= \sum_{k=0}^{r-1} p(1-p)^{k-1}\cdot(r-k-1)p^2(1-p)^{r-k-2} \\
&= p^3(1-p)^{r-3}\sum_{k=0}^{r-1}(r-k-1) \\
&= \binom{r-1}{2}p^3(1-p)^{r-3}
\end{aligned} \tag{7}$$

从而 $Z \sim NB(3,p)$.

5

$$
\begin{aligned}
\mathbb{P}\left(Y=k\right) &= \sum_{x\geqslant k}\mathbb{P}\left(X=x\right)\binom{x}{k}\left(\frac{1}{3}\right)^{k}\left(\frac{2}{3}\right)^{x-k} \\
&= \sum_{x\geqslant k}\mathrm{e}^{-6}\frac{6^x}{x!}\binom{x}{k}\left(\frac{1}{3}\right)^{k}\left(\frac{2}{3}\right)^{x-k} \\
&= \mathrm{e}^{-2}\frac{2^k}{k!}\sum_{x-k\geqslant 0}\mathrm{e}^{-4}\frac{4^{x-k}}{(x-k)!} \\
&= \mathrm{e}^{-2}\frac{2^k}{k!}
\end{aligned} \tag{8}
$$

从而 $Y\sim\pi(2)$, 说明 $\arg\max\limits_{k}\mathbb{P}\left(Y=k\right)=2$.

6

6.1

$$
\mathbb{P}\left(X=x,Z=z\right)=\begin{cases}\mathbb{P}\left(X=x\right)\mathbb{P}\left(y\leqslant x\right), & x=z\\ \mathbb{P}\left(X=x\right)\mathbb{P}\left(y=z\right), & x<z\\ 0, & x>z\end{cases}=\begin{cases}p(1-p)^z(1-(1-p)^{z+1}), & x=z\\ p^2(1-p)^{x+z}, & x<z\\ 0, & x>z\end{cases} \tag{9}
$$

6.2

注意到

$$
\begin{aligned}
\mathbb{P}\left(Z=z\right) &= \mathbb{P}\left(Z\leqslant z\right)-\mathbb{P}\left(Z\leqslant z-1\right) \\
&= \left[\mathbb{P}\left(X\leqslant z\right)\right]^2-\left[\mathbb{P}\left(X\leqslant z-1\right)\right]^2 \\
&= [1-(1-p)^{z+1}]^2-[1-(1-p)^z]^2 \\
&= p(1-p)^z(2-(1-p)^z-(1-p)^{z+1})
\end{aligned} \tag{10}
$$

因此

$$
\mathbb{P}\left(X=x|Z=z\right)=\frac{\mathbb{P}\left(X=x,Z=z\right)}{\mathbb{P}\left(Z=z\right)}=\begin{cases}\frac{1-(1-p)^{z+1}}{2-(1-p)^z-(1-p)^{z+1}}, & x=z\\ \frac{(1-p)^x}{2-(1-p)^z-(1-p)^{z+1}}, & x<z\\ 0, & x>z\end{cases} \tag{11}
$$

7

考虑对 n 归纳. 当 $n=1$ 时, $f_{Y_1}(y)=\begin{cases}\lambda\mathrm{e}^{-\lambda y}, & y>0\\ 0, & y\leqslant 0\end{cases}$ 恰好是指数分布, 故成立.

假设 $f_{Y_k}(y)=\begin{cases}y^{k-1}\lambda^k\mathrm{e}^{-\lambda y}/(k-1)!, & y>0\\ 0, & y\leqslant 0\end{cases}$ 成立, 考虑计算 $Y_{k+1}=Y_k+X_{k+1}$ 的密度分布, 当 $y>0$ 时, 有

$$\begin{aligned} f_{Y_{k+1}}(y) &= \int_{-\infty}^{+\infty} f_{Y_k}(t) f_X(y-t)\mathrm{d}t = \int_0^y \frac{t^{n-1}\lambda^n \mathrm{e}^{-\lambda t}}{(n-1)!}\cdot \lambda \mathrm{e}^{-\lambda(y-t)}\mathrm{d}t \\ &= \frac{\lambda^{n+1}\mathrm{e}^{-\lambda x}}{(n-1)!}\frac{t^n}{n}\bigg|_0^y = \frac{y^n\lambda^{n+1}\mathrm{e}^{-\lambda x}}{n!} \end{aligned} \tag{12}$$

而显然 $f_{Y_{k+1}}(y) = 0\ (y \leqslant 0)$, 因此结论对于 $k+1$ 也成立, 从而对任意 n 均成立.

8

8.1

$$\begin{aligned} F_{Z,W}(z,w) &= \mathbb{P}\left(X^2 \leqslant z, X+Y \leqslant w\right) \\ &= \int_0^w \int_0^{\min(w-y,\sqrt{z})} f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^w \int_0^{\min(w-y,\sqrt{z})} \lambda^2 \mathrm{e}^{-\lambda(x+y)}\mathrm{d}x\mathrm{d}y \\ &= \begin{cases} \int_0^w \int_0^{w-y} \lambda^2\mathrm{e}^{-\lambda(x+y)}\mathrm{d}x\mathrm{d}y, & \sqrt{z} > w \\ \mathbb{P}\left(X \leqslant \sqrt{z}\right)\mathbb{P}\left(Y \leqslant w-\sqrt{z}\right) + \int_{w-\sqrt{z}}^w \int_0^{w-y} \lambda^2\mathrm{e}^{-\lambda(x+y)}\mathrm{d}x\mathrm{d}y, & \sqrt{z} \leqslant w \end{cases} \\ &= \begin{cases} 1 - w\lambda\mathrm{e}^{-\lambda w} - \mathrm{e}^{-\lambda w}, & \sqrt{z} > w \\ 1 - \sqrt{z}\lambda\mathrm{e}^{-\lambda w} - \mathrm{e}^{-\lambda\sqrt{z}}, & \sqrt{z} \leqslant w \end{cases} \\ f_{Z,W}(z,w) &= \frac{\partial^2 F_{Z,W}(z,w)}{\partial z\partial w} = \begin{cases} 0, & \sqrt{z} > w \\ \frac{\lambda^2\mathrm{e}^{-\lambda w}}{2\sqrt{z}}, & \sqrt{z} \leqslant w \end{cases} \end{aligned} \tag{13}$$

8.2

$$f_{Z|W}(z|w) = \frac{f_{Z,w}(z,w)}{f_W(w)} = \frac{f_{Z,w}(z,w)}{w\lambda^2\mathrm{e}^{-\lambda w}} = \begin{cases} 0, & \sqrt{z} > w \\ \frac{1}{2\sqrt{z}w}, & \sqrt{z} \leqslant w \end{cases} \tag{14}$$

9

9.1

$$\begin{aligned} F_n(x) &= \mathbb{P}\left(Y_n \leqslant x\right) = 1 - \mathbb{P}\left(Y_n > x\right) = 1 - \prod_{i=1}^n \mathbb{P}\left(X_i > x\right) = \begin{cases} 0, & x < 0 \\ 1-(1-x)^n, & 0 \leqslant x < 1 \\ 1, & x \geqslant 1 \end{cases} \\ \lim_{n\to\infty} F_n(x) &= \begin{cases} 0, & x \leqslant 0 \\ 1, & x > 0 \end{cases} \end{aligned} \tag{15}$$

9.2

$$\begin{aligned}\mathbb{P}(Z>3)&=\mathbb{P}(X_1\leqslant 1\wedge X_1+X_2\leqslant 1\wedge X_1+X_2+X_3\leqslant 1)\\&=\int_0^1\int_0^{1-x_1}\int_0^{1-x_1-x_2}\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1\\&=\int_0^1\int_{y_1}^1\int_{y_2}^1\mathrm{d}y_3\mathrm{d}y_2\mathrm{d}y_1\end{aligned}\tag{16}$$

引理 1. 对于 $n\in\mathbb{N}^+$ 和 $L>0$, 有

$$\int_0^L\int_{y_1}^L\cdots\int_{y_{n-1}}^L\mathrm{d}y_n\cdots\mathrm{d}y_2\mathrm{d}y_1=\frac{L^n}{n!}\tag{17}$$

证明. 考虑对 n 作归纳. $n=1$ 时显然成立. 假设结论对于 k 成立, 则

$$\begin{aligned}\int_0^L\left[\int_{y_1}^L\cdots\int_{y_k}^L\mathrm{d}y_{k+1}\cdots\mathrm{d}y_2\right]\mathrm{d}y_1&=\int_0^L\left[\int_0^{L-y_1}\cdots\int_{z_k}^{L-y_1}\mathrm{d}z_{k+1}\cdots\mathrm{d}z_2\right]\mathrm{d}y_1\\&=\int_0^L\frac{(L-y_1)^k}{k!}\mathrm{d}y_1\\&=-\frac{(L-y_1)^{k+1}}{(k+1)!}\bigg|_0^L\\&=\frac{L^{k+1}}{(k+1)!}\end{aligned}\tag{18}$$

结论对 $k+1$ 亦成立, 从而对所有正整数 n 均成立. □

结合上述引理, 可知 $\mathbb{P}(Z>3)=\frac{1}{6}$.

10

10.1

因为 X_1,X_2,X_3 是独立随机变量, 故 $f_{X_1,X_2,X_3}(x_1,x_2,x_3)=f_{X_1}(x_1)f_{X_2}(x_2)f_{X_3}(x_3)=(2\pi)^{-\frac{3}{2}}\exp\left(-\frac{x_1^2+x_2^2+x_3^2}{2}\right)$.

10.2

$$\begin{aligned}f_{Z,X_2,X_3}(z,x_2,x_3)&=f_{X_1,X_2,X_3}(x_1,x_2,x_3)\left|\frac{\partial(z,x_2,x_3)}{\partial(x_1,x_2,x_3)}\right|^{-1}\\&=f_{X_1,X_2,X_3}(x_1,x_2,x_3)\begin{vmatrix}\frac{1}{\sqrt{1+x_3^2}} & \frac{x_3}{\sqrt{1+x_3^2}} & \frac{x_3}{\sqrt{x_3^2+1}}-\frac{x_3(x_1+x_2x_3)}{(x_3^2+1)^{\frac{3}{2}}}\\0&1&0\\0&0&1\end{vmatrix}^{-1}\\&=(2\pi)^{-\frac{3}{2}}\sqrt{1+x_3^2}\exp\left[-\frac{1}{2}\left(\left(z\sqrt{1+x^3}-x_2x_3\right)^2+x_2^2+x_3^2\right)\right]\end{aligned}\tag{19}$$

10.3

$$
\begin{aligned}
f_{Z,X_3}(z,x_3) &= \int_{-\infty}^{+\infty} f_{Z,X_2,X_3}(z,x_2,x_3)\mathrm{d}x_2 \\
&= \int_{-\infty}^{+\infty} (2\pi)^{-\frac{3}{2}}\sqrt{1+x_3^2}\exp\left[-\frac{1}{2}\left(\left(z\sqrt{1+x^3}-x_2x_3\right)^2+x_2^2+x_3^2\right)\right]\mathrm{d}x_2 \\
&= \int_{-\infty}^{+\infty} (2\pi)^{-\frac{3}{2}}\sqrt{1+x_3^2}\exp\left[-\frac{1}{2}\left(\left(x_2\sqrt{1+x^3}-zx_3\right)^2+z^2+x_3^2\right)\right]\mathrm{d}x_2 \\
&= \frac{1}{2\pi}\exp\left[-\frac{1}{2}(z^2+x_3^2)\right]\int_{-\infty}^{+\infty}\frac{\sqrt{1+x_3^2}}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\sqrt{1+x_3^2}x_2-zx_3\right)^2\right]\mathrm{d}x_2 \\
&= \frac{1}{2\pi}\exp\left[-\frac{1}{2}(z^2+x_3^2)\right]
\end{aligned}
\tag{20}
$$

可以发现 $f_Z(z)=\int_{-\infty}^{+\infty} f_{Z,X_3}(z,x_3)\mathrm{d}x_3=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)$ 故 $Z\sim\mathcal{N}(0,1)$, 而 $f_{Z,X_3}(z,x_3)=f_Z(z)f_{X_3}(x_3)$ 说明 Z,X_3 是独立随机变量, 因此 $Z,X_3\sim$ i.i.d. $\mathcal{N}(0,1)$.