概率统计 (A) 课程作业: 随机变量及其分布

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1. $\mathbb{P}(X=3) = F(3) - \lim\limits_{x\to 3^-} F(x) = \frac{1}{12}$.
2. $\mathbb{P}\left(\frac{1}{2} < X \leqslant \frac{3}{2}\right) = F(\frac{3}{2}) - F(\frac{1}{2}) = \frac{5}{8} - \frac{1}{8} = \frac{1}{2}$.
3. $\mathbb{P}(X<2) = \lim\limits_{x\to 2^-} F(x) = \frac{3}{4}$.

2

$$\mathbb{P}(S_n = 2k-n) = \begin{cases} \binom{n}{k} p^k q^{n-k}, & k \in [0,n] \cap \mathbb{Z} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

3

设 L 表示左口袋火柴盒先空, R 表示右口袋火柴盒先空. 设随机变量 X 表示剩余火柴数量.

$$\begin{aligned} \mathbb{P}(X=r) &= \mathbb{P}(L \wedge X=r) + \mathbb{P}(R \wedge X=r) \\ &= 2\mathbb{P}(L \wedge X=r) \\ &= 2\binom{2N-r}{N}\left(\frac{1}{2}\right)^{2N-r+1} \\ &= \binom{2N-r}{N}\left(\frac{1}{2}\right)^{2N-r} \end{aligned} \tag{2}$$

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设成虫数量为 X, 产卵数量为 Y.

$$\begin{aligned} \mathbb{P}(X=x) &= \sum_{y=x}^{\infty} \mathbb{P}(X=x|Y=y)\,\mathbb{P}(Y=y) \\ &= \sum_{y=x}^{\infty} \binom{y}{x} p^x (1-p)^{y-x} \mathrm{e}^{-\lambda} \frac{\lambda^y}{y!} \\ &= \mathrm{e}^{-\lambda} \frac{(\lambda p)^x}{x!} \sum_{y=x}^{\infty} \frac{(\lambda(1-p))^{y-x}}{(y-x)!} \\ &= \mathrm{e}^{-\lambda} \frac{(\lambda p)^x}{x!} \mathrm{e}^{\lambda(1-p)} \\ &= \mathrm{e}^{-\lambda p} \frac{(\lambda p)^x}{x!} \end{aligned} \tag{3}$$

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废品数量 $X \sim B(1000, 0.005)$, 可以用 Poisson 分布 $\pi(\lambda = 5)$ 来近似.

1. $\mathbb{P}(X \geqslant 2) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) \approx 1 - \mathrm{e}^{-5} - 5\mathrm{e}^{-5} \approx 0.95957$.
2. $\mathbb{P}(X \leqslant 5) \approx \sum_{k=0}^{5} \mathrm{e}^{-5} \frac{5^k}{k!} \approx 0.61596$.
3. $\mathbb{P}(X \leqslant 7) \approx 0.86662, \mathbb{P}(X \leqslant 8) \approx 0.93190$. 能以 90% 的概率希望废品件数不超过 8 件.

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注意到 $S(x) = 1 - F(x), S'(x) = -f(x)$, 于是有 $\lambda(x) = -\frac{S'(x)}{S(x)} = -\left[\ln(S(x))\right]'$, 因此

$$S(x) = 1 - F(x) = 1 - \int_0^x f(t)\mathrm{d}t = \exp\left(-\int_0^x \lambda(t)\mathrm{d}t\right)$$

当 $\xi \sim \mathrm{Exp}(\lambda)$ 时, $f(x) = \lambda \mathrm{e}^{-\lambda x}, F(x) = 1 - \mathrm{e}^{-\lambda x}, S(x) = \mathrm{e}^{-\lambda x}, \lambda(x) \equiv \lambda$. 易验证上述关系成立.

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已知 $f_\theta(x) = \begin{cases} 1/\pi, & x \in [-\pi/2, \pi/2] \\ 0, & \text{otherwise} \end{cases}$, ψ 的取值范围是 $(-\infty, +\infty)$.

$$\begin{aligned} F_\psi(x) &= \mathbb{P}(\psi \leqslant x) \\ &= \mathbb{P}(\tan\theta \leqslant x) \\ &= \mathbb{P}(\theta \leqslant \arctan x) \\ &= F_\theta(\arctan x) \end{aligned} \tag{4}$$

因此 $f_\psi(x) = f_\theta(\arctan x)(\arctan x)' = \frac{1}{\pi(x^2+1)}$.

8

$X \sim \mathcal{N}(0,1)$ 的概率密度函数为 $f_X(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}$.

1. 记 $Y = X^2$, 显然 Y 的取值范围是 $[0, +\infty)$.
   $F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}\left(X^2 \leqslant y\right) = 2\mathbb{P}\left(X \leqslant \sqrt{y}\right) - 1$, 故 $f_Y(y) = 2f_X(\sqrt{y})(\sqrt{y})' = \frac{1}{\sqrt{2\pi}} \frac{\mathrm{e}^{-y/2}}{\sqrt{y}}$.
2. 记 $Z = \mathrm{e}^X$, 显然 Z 取值范围是 $(0, +\infty)$.
   $F_Z(z) = \mathbb{P}(Z \leqslant z) = \mathbb{P}\left(\mathrm{e}^X \leqslant z\right) = \mathbb{P}(X \leqslant \ln z) = f_X(\ln z)$, 故 $f_Z(z) = f_X(\ln z)(\ln z)' = \frac{1}{\sqrt{2\pi}} \frac{\mathrm{e}^{-\frac{\ln^2 z}{2}}}{z}$.

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1. 令 $\alpha = X^2$, $f_\alpha(x) = \frac{1}{2\sqrt{x}} f_X(\sqrt{x}) = \frac{1}{2N\sqrt{x}}$.
2. 令 $\beta = [\alpha]$, 此时 β 是离散型随机变量, $\mathbb{P}(\beta = k) = \int_k^{k+1} f_\alpha(x)\mathrm{d}x = \begin{cases} \frac{\sqrt{k+1} - \sqrt{k}}{N}, & k \in [0, N^2 - 1] \cap \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$.

3. 令 $\gamma = \alpha - \beta$, 易知 $\gamma \in [0,1)$, 而对于 $x \in [0,1)$ 有 $\mathbb{P}(\gamma \leqslant x) = \sum\limits_{k=0}^{N^2-1} \mathbb{P}(k \leqslant \alpha \leqslant k+x) = \sum\limits_{k=0}^{N^2-1} \frac{\sqrt{k+x}-\sqrt{k}}{N}$, 因此 γ 的分布函数为

$$F_\gamma(x) = \begin{cases} 0, & x \leqslant 0 \\ \sum\limits_{k=0}^{N^2-1} \frac{\sqrt{k+x}-\sqrt{k}}{N}, & 0 \leqslant x < 1 \\ 1, & x \geqslant 1 \end{cases} \tag{5}$$

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1. 在 Pascal 分布 $\binom{k-1}{r-1}p^r q^{k-r}$ 中, 代入 $k = \frac{t}{\Delta t}, p = \lambda\Delta t$, 同时令 $\Delta t \to 0$, 可以得到

$$\begin{aligned} &\lim_{\Delta t\to 0} \binom{\frac{t}{\Delta t}-1}{r-1} (\lambda\Delta t)^r (1-\lambda\Delta t)^{\frac{t}{\Delta t}-r+1}/\Delta t \\ =&\lim_{\Delta t\to 0} \frac{(\frac{t}{\Delta t}-1)\cdots(\frac{t}{\Delta t}-r+1)}{(r-1)!} \left(\frac{\lambda\Delta t}{1-\lambda\Delta t}\right)^r (1-\lambda\Delta t)^{\frac{t}{\Delta t}}/\Delta t \\ =&\lim_{\Delta t\to 0} \left(\prod_{i=1}^{r-1}(1-\frac{i\Delta t}{t})\right) \frac{\lambda^r t^{r-1}}{(r-1)!}\mathrm{e}^{-\lambda t} \\ =&\frac{\lambda^r t^{r-1}}{(r-1)!}\mathrm{e}^{-\lambda t} \\ =&f_{\lambda,r}(t) \end{aligned} \tag{6}$$

即 “Pascal 分布的极限是 Erlang 分布” .

2.

引理 1 (几何分布的极限是指数分布). 随机变量 X 服从几何分布 $G(\lambda\Delta t)$, 对于任意 $x > 0$, 有

$$\lim_{\Delta t\to 0} \mathbb{P}\left(X = \left\lfloor \frac{x}{\Delta t} \right\rfloor\right)/\Delta t = \lambda \mathrm{e}^{-\lambda x} \tag{7}$$

证明.

$$\begin{aligned} \lim_{\Delta t\to 0} \mathbb{P}\left(X = \left\lfloor \frac{x}{\Delta t} \right\rfloor\right)/\Delta t &= \lim_{\Delta t\to 0} \lambda\Delta t(1-\lambda\Delta t)^{\lfloor \frac{x}{\Delta t}\rfloor}/\Delta t \\ &= \lambda \lim_{\Delta t\to 0} (1-\lambda\Delta t)^{\lfloor \frac{x}{\Delta t}\rfloor} \\ &= \lambda \mathrm{e}^{-\lambda x} \end{aligned} \tag{8}$$

□