Appendix C

Fixed Point Theorems and Maximum Principles

Fixed point theorems play a key role in proving the existence and uniqueness theorems in the analysis. They also provide rigorous justification for the convergence of iterations in the numerical computations. In many situations, the behaviour of the temperature and/or its derivatives can be analysed by simple application of maximum principles.

1. Banach fixed point theorem. A *fixed point of a mapping* $P : X \to X$ of a set X into itself is an element $x \in X$ which is mapped onto itself, i.e. $Px = x$. A mapping $P : X \to X$, where X is a normed space is called a *contraction* on X if there exists a positive real number $\alpha < 1$ such that $\|Px - Py\| \leq \alpha \|x - y\|$ for all $x, y \in X$. This definition implies that the mapping P is uniformly continuous on X.

Let X be a Banach space and $P : X \to X$, a contraction. *Banach fixed point theorem* which is also called *Banach contraction mapping theorem* states that there exists a unique element $x^* \in X$ such that $P(x^*) = x^*$. x^* is called the fixed point of P.

The following results hold for contraction mappings:

(i) If X is a Banach space and $P : X \to X$ is such that $P^r = PP \ldots r$ times is a contraction for some integer $r > 1$, then P has a unique fixed point.

(ii) Let Z be a closed subset of a Banach space X and $P : Z \to Z$ be such that $\|Px - Py\| \leq \alpha \|x - y\|$, $0 < \alpha < 1$, for all $x, y \in Z$. Then there exists a unique vector $x^* \in Z$ such that $P(x^*) = x^*$ and x^* may be obtained as the limit of a sequence $\{x_n\}$ where $x_n = P(x_{n-1})$, $x_0 \in Z$.

2. Schauder's fixed point theorem. Let P be a continuous operator on a Banach space X which maps a closed convex set Z of X into itself. Assuming that the image set PZ is relatively compact, P has at least one fixed point in Z.

Several versions of fixed point theorem are available in the literature and for this the reader is referred to the functional analysis books mentioned in the bibliography.

3. Ascoli–Arzela theorem. Let Z be a compact metric space, and $W(Z)$ a Banach space of real- or complex-valued continuous functions $f(x)$ normed by $\|f\| = \sup_{x \in Z} |f(x)|$. Then a sequence $\{f_n(x)\} \subseteq W(Z)$ is relatively compact (also called *precompact* in $W(Z)$) if the following two conditions are satisfied:

(i) $f_n(x)$ is *equibounded* (in n), i.e. $\sup_{n\geq 1}\sup_{x\in Z}|f_n(x)|<\infty$,

(ii) $f_n(x)$ is *equicontinuous* (in n), i.e.

$$\lim_{\delta\to 0}\sup_{n\geq 1,\ \mathrm{dis}(\mathrm{x}',\mathrm{x}'')\leq\delta}\left|f_n(x')-f_n(x'')\right|=0.$$

Another way of stating *Ascoli–Arzela* theorem is as follows: Let Ω be a bounded domain in R^n. A subset F of $C(\overline{\Omega})$ is relatively compact in $C(\overline{\Omega})$ provided the following two conditions hold:

(i) There exists a constant α such that for every $f\in F$ and $x\in\Omega$, $|f(x)|\leq\alpha$.

(ii) For every $\varepsilon>0$, there exists a $\delta>0$ such that if $f\in F$, $x,y\in\Omega$, and $|x-y|<\delta$, then $|f(x)-f(y)|<\varepsilon$.

4. The strong maximum principle. The *strong maximum principle* is associated with Nirenberg [320] and is concerned with the parabolic operators. Consider the parabolic operator L defined as

$$LT=\sum_{i,J=1}^{n}a_{iJ}(x,t)\frac{\partial^2 T}{\partial x_i\partial x_J}+\sum_{i=1}^{n}b_i(x,t)\frac{\partial T}{\partial x_i}+d(x,t)T-\frac{\partial T}{\partial t},$$

in an $(n+1)$-dimensional parabolic domain $D_{t_*}=\Omega\times(0,t_*)$, where Ω (open and bounded) $\subset R^n$ and $t_*<\infty$. D_{t_*} is bounded by the planes $t=0$ and $t=t_*<\infty$. Let $D_0=\Omega\times\{t=0\}$. The lateral surface of this parabolic cylinder is given by $D_l=\partial\Omega\times(0,t_*)$. $D_l\cup\overline{D}_0$ is called the *parabolic boundary* of D_{t_*}

We make the following assumptions:

(a) L is parabolic in D_{t_*} (see Eq. 7.3.26 for parabolicity).

(b) The coefficients of L are continuous functions in D_{t_*}.

(c) $T(x,t)\in C^{2,1}(D_{t_*})$ and the coefficient $d(x,t)\leq 0$ in D_{t_*}.

Let $P^0=(x^0,t^0)$ be any point in D_{t_*} and let $Z(P^0)$ be the set of all points in D_{t_*} which can be connected to P^0 by a 'simple' continuous curve in D_{t_*} along which the $t-$ coordinate is nondecreasing as we move from any point in $Z(P^0)$ to P^0.

The *strong maximum principle* asserts the following:

Let the assumptions (a)–(c) given above hold. If $LT\geq 0$ in D_{t_*} or $LT\leq 0$ in D_{t_*} and the temperature T has a positive maximum (negative minimum) in D_{t_*} which is attained at a point $P^0(x^0,t^0)$, then $T(P)=T(P^0)$ for all $P\in Z(P^0)$.

The strong maximum principle holds even if $P^0\in\Omega\times\{t_*\}$ provided $T(x,t)$ is continuous in $D_{t_*}\cup\Omega\times\{t_*\}$.

Let $L=a^2\partial^2/\partial x^2-\partial/\partial t$, $a>0$, then we say T is *subparabolic* if $LT\geq 0$ and *superparabolic* if $LT\leq 0$.

5. The weak maximum principle. Let the assumptions (a)–(c) hold and $T(x,t)$ be a continuous function in $\overline{D}_{t_*}$. Let $LT\geq 0$ ($LT\leq 0$) in D_{t_*}. Then the *weak maximum principle* asserts that the maximum (minimum) of $T(x,t)$ is attained on the parabolic boundary of D_{t_*}. Note that the same maximum (minimum) of $T(x,t)$ can be attained in D_{t_*} also.

For further extensions of these principles see [9].