Appendix B

Some Function Spaces and Norms

1. Convexity. A set V in a vector space W is said to be a *convex set* if the line segment joining any two points in V is contained in V. A real-valued function $f(x)$ defined on a convex set V is said to be a *convex function* if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$, $0 < \lambda < 1$ for all $x, y \in V$. The function is said to be concave if the inequality is reversed. If the equality is excluded, then we have a *strictly convex (concave) function*.

A Hilbert space X is *strictly convex* in the sense that if $x, y \in X$ and $\|x\| = \|y\| = 1, x \neq y$, then $\|x + y\| < 2$.

2. Hölder continuity and Lipschitz continuity. A function $f(x)$ defined on a bounded closed set Ω of R^n is said to be *Hölder continuous* in Ω with exponent α, $0 < \alpha < 1$, if there exists a constant δ such that $|f(x) - f(y)| \leq \delta\, |x - y|^\alpha$ for all $x, y \in \Omega$. The smallest value δ_0 of δ for which the inequality holds is called a *Hölder coefficient*.

If $\alpha = 1$, then f is called a *Lipschitz continuous* function.

3. Equicontinuity. Let F be a set of real- or complex-valued functions such that $f \in F$ $(f(x) = f(x_1, x_2, \ldots, x_n))$ is defined on a compact subset B of R^n. The functions in F are *uniformly bounded* on B if there exists a constant M with the property that $|f(x)| \leq M$ holds for all $x \in B$ and all $f \in F$.

The collection of functions F is *equicontinuous* on B if for every $\varepsilon > 0$ there exists a $\delta > 0$ which depends only on ε, such that for $x', x'' \in B$, $\left\|x' - x''\right\| < \delta$ implies

$$\left|f(x') - f(x'')\right| < \varepsilon \text{ for all } f \in F.$$

Note that equicontinuity of F implies uniform continuity of each member of F, but not vice versa (δ may depend on f). If instead of F, a sequence $\{x_n\}$ of functions is considered, then $\{x_n\}$ is said to be equicontinuous on B if for every $\varepsilon > 0$ there exists a $\delta > 0$, depending only on ε such that for all x_n and $y_1, y_2 \in B$ satisfying $\|y_1 - y_2\| < \delta$, we have

$$|x_n(y_1) - x_n(y_2)| < \varepsilon.$$

4. Lower semicontinuity. Let W be a normed space and $f : W \to R$ and let $N(x_0)$ be the family of neighbourhoods of a point $x_0 \in W$. f is said to be *lower semicontinuous* (l.s.c.) at $x_0 \in W$ if for all $\varepsilon > 0$, there exists a $V_\varepsilon \in N(x_0)$ such that for all $y \in V_\varepsilon, f(y) \geq f(x_0) - \varepsilon$. An *upper semicontinuous function* can be defined in an analogous manner.

5. The space $C^m(\overline{\Omega})$. The space $C^m(\Omega)$, where m is a nonnegative integer and $\Omega \subset R^n$, is a vector space of all functions $f(x)$, $x \in \Omega$ which together with all their partial derivatives $D^\beta f$ of orders $0 \leq |\beta| \leq m$, are continuous on Ω. Here, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ and $|\beta|$ is defined as follows. $C^0(\Omega) = C(\Omega)$ and $C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$. The subspaces $C_0(\Omega)$ and $C_0^\infty(\Omega)$ consist of all those functions in $C(\Omega)$ and $C^\infty(\Omega)$, respectively, which have compact support in Ω.

If $f \in C(\Omega)$ is bounded and uniformly continuous on Ω (Ω is bounded), then it possesses a unique, bounded continuous extension to $\overline{\Omega}$. The vector space $C^m(\overline{\Omega})$ consists of all those functions $f \in C^m(\Omega)$ for which $D^\beta f$, $0 \le |\beta| \le m$, are bounded and uniformly continuous on Ω. $C^m(\overline{\Omega})$ is a Banach space if the norm of $f \in C^m(\overline{\Omega})$ is defined as

$$\|f\|_m = \sum_{|\beta| \le m} \sup_{x \in \Omega} \left| D^\beta f(x) \right|,$$

$D^\beta = D_1^{\beta_1} D_2^{\beta_2} \ldots D_n^{\beta_n}$, $|\beta| = \sum_{J=1}^{n} \beta_J$ and $D_J = \partial/\partial x_J$, $J = 1, 2 \ldots n$. Here, all β_j's are nonnegative integers.

6. The space $H_\alpha(\overline{\Omega})$ or $C^{0,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$. The set of all Hölder continuous functions on $\overline{\Omega}$ with exponent α, $0 < \alpha < 1$ is denoted by $H_\alpha(\overline{\Omega})$ or $C^\alpha(\overline{\Omega})$ or $C^{0,\alpha}(\overline{\Omega})$. If Ω is be a bounded open set in R^n, then $f(x)$ is *locally Hölder continuous* on Ω if $f(x)$ is Hölder continuous in every bounded closed set B of Ω. The constant δ may depend on B. If the constant δ (δ as in **2**. of this Appendix) is independent of the set B, then f is said to be *uniformly Hölder continuous* with exponent α. If $f \in H_\alpha(\overline{\Omega})$, then we define its norm as

$$\|f\|_{H_\alpha} = \|f\|_0 + \sup_{\substack{x,y \in \Omega \\ x \neq y}} |f(x) - f(y)| \,/\, |x - y|^\alpha .$$

Here, $\|f\|_0$ is the *uniform norm* of f defined as

$$\|f\|_0 = \sup_{x \in \Omega} |f(x)| .$$

7. The space $H_{m+\alpha}(\overline{\Omega})$ or $C^{m,\alpha}(\overline{\Omega})$, $m \ge 0$, $0 < \alpha < 1$. For $0 < \alpha < 1$, the space $H_{m+\alpha}(\overline{\Omega})$ or $C^{m,\alpha}(\overline{\Omega})$ or $C^{m+\alpha}(\overline{\Omega})$ is a subspace of $C^m(\overline{\Omega})$, $m \ge 0$ and consists of those functions f for which $D^\beta f$, $0 \le |\beta| \le m$, satisfy in Ω a Hölder condition of exponent α, i.e. there exists a constant $\delta > 0$ such that

$$\left| D^\beta f(x) - D^\beta f(y) \right| \le \delta \, |x - y|^\alpha , \quad x, y \in \Omega.$$

Here, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, $D^\beta = D_1^{\beta_1} D_2^{\beta_2} \ldots, D_n^{\beta_n}$, $|\beta| = \sum_{J=1}^{n} \beta_J$ and $D_J = \partial/\partial x_J$, $J = 1, 2, \ldots, n$.

If $f \in C^{m,\alpha}(\overline{\Omega})$ and $\|f\|$ is defined as

$$\|f\|_{m+\alpha} = \|f\|_m + \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{\left| D^{|\beta|} f(x) - D^{|\beta|} f(y) \right|}{|x - y|^\alpha} = \|f\|_{m-1} + \left\| D^m f \right\|_{H_\alpha} , \quad |\beta| = m,$$

is finite, then $C^{m,\alpha}(\overline{\Omega})$ is a Banach space in this norm.

For norms in *anisotropic Hölder continuous spaces* the reader is referred to [295].

In the one-dimensional parabolic problems, we consider $\Omega_{t_*} = D \times (0, t_*)$, $D \subset R$, $0 < t < t_*$. Let $P = (x, t)$, $P_1 = (x_1, t_1)$ and $P_2 = (x_2, t_2)$, where P_0, P_1, $P_2 \in \Omega_{t_*}$. Define the distance P_1P_2 as

$$P_1P_2 = \left\{(x_1 - x_2)^2 + |t_1 - t_2|\right\}^{1/2}.$$

Let $f(x, t)$ be a continuous function in $\overline{\Omega}_{t_*}$. We say that $f \in C_\alpha(\overline{\Omega}_{t_*})$, $0 < \alpha < 1$, if the norm of f defined below is finite.

$$\|f\|_{C_\alpha(\overline{\Omega}_{t_*})} = \|f\|_0 + \sup_{P_1, P_2 \in \Omega_{t_*}} |f(P_1) - f(P_2)| / (P_1P_2)^\alpha.$$

Here, $\|f\|_0 = \sup_{P \in \Omega_{t_*}} |f(P)|$. The spaces $C_{1+\alpha}$ $(\overline{\Omega}_{t_*})$ and $C_{2+\alpha}$ $(\overline{\Omega}_{t_*})$ are Banach spaces of functions f provided the norms defined below are finite.

$$\|f\|_{C_{1+\alpha}} (\overline{\Omega}_{t_*}) = \|f\|_{C_\alpha(\overline{\Omega}_{t_*})} + \|f_x\|_{C_\alpha(\overline{\Omega}_{t_*})}, \text{ (norm in } C_{1+\alpha}(\overline{\Omega}_{t_*})).$$
$$\|f\|_{C_{2+\alpha}} (\overline{\Omega}_{t_*}) = \|f\|_{C_{1+\alpha}(\overline{\Omega}_{t_*})} + \|f_{xx}\|_{C_\alpha(\overline{\Omega}_{t_*})} + \|f_t\|_{C_\alpha(\overline{\Omega}_{t_*})}, \text{ (norm in } C_{2+\alpha}(\overline{\Omega}_{t_*})).$$

8. Imbedding. A normed space X is said to be *embedded* in the normed space Y and written as $X \hookrightarrow Y$, provided

(i) X is a subspace of Y,

(ii) the *identity operator* defined on X into Y by $Ix = x$ for all $x \in X$ is continuous.

9. The space $L^P(\Omega)$. Let Ω be a domain in R^n and let p be a positive real number. We denote by $L^p(\Omega)$ the class of all measurable functions f defined on Ω such that

$$\int_\Omega |f(x)|^p \, dx < \infty.$$

Here, the integration is taken in the Lebesgue sense.

Two functions in $L^p(\Omega)$ are equal if they are equal almost everywhere (a.e.) on Ω, i.e. they are equal except on a set of measure zero. If $1 \le p \le \infty$, then the norm of a function $f \in L^p(\Omega)$ can be defined as

$$\|f\|_{L^p(\Omega)} = \left\{\int_\Omega |f(x)|^p \, dx\right\}^{1/p}.$$

The space $L^p(\Omega)$, $1 \le p \le \infty$ is a Banach space in the above norm. $L^2(\Omega)$ is a Hilbert space with respect to the inner product defined as

$$(f, g) = \int_\Omega f(x)\overline{g(x)}dx, \; f, \; g \in L^2(\Omega), \text{ bar stands for the complex conjugate.}$$

10. Essentially bounded function. A function f, measurable on Ω is said to be *essentially bounded* on Ω if there exists a constant δ such that $|f(x)| \leq \delta$ a.e. on Ω. The set of all essentially bounded functions on Ω denoted by $L^{\infty}(\Omega)$ is a vector space. The greatest lower bound of all such constants δ is called the *essential supremum* of $|f|$ on Ω and is denoted by $\operatorname{ess}\sup_{x\in\Omega}|f(x)|$. If $f \in L^{\infty}(\Omega)$ and $\|f\|_{\infty} = \operatorname{ess}\sup_{x\in\Omega}|f(x)| < \infty$, then $L^{\infty}(\Omega)$ is a Banach space in the norm $\|f\|_{\infty}$.

11. A locally integrable function. A function f defined a.e. on Ω is said to be *locally integrable* on Ω written as $f \in L^1_{loc}(\Omega)$ provided $f \in L^1(A)$ for every measurable compact subset A of Ω.

12. Locally compact space. A normed space is said to be *locally compact* if each point of the space has a compact neighbourhood.

13. Graph of an operator. The *graph of a linear operator* $P : H_1 \to H_2$, where H_1 and H_2 are normed spaces is the set of points G_A such that

$$G_A = \{(x, y) : x \in \text{Domain}(A),\ y = Ax\}.$$

14. Maximal monotone graph. Let A be a *multivalued operator*, i.e. $A : H \to 2^H$ from H to itself. A will be viewed as a subset of $H \times H$ and A will not be distinguished from its graph. A subset $A \subset H \times H$ is called *monotone* if

$$\forall u, v \in H,\ \forall \xi \in A(v),\ \eta \in A(v),\ (\xi - \eta,\ u - v) \geq 0.$$

A monotone subset of $H \times H$ is called *maximal monotone* if it is not properly contained in any other monotone subset of $H \times H$.

15. The $C^{m+\alpha}$ boundary $\partial\Omega$ of Ω. If each point x of $\partial\Omega$ has a neighbourhood B such that the graph of the intersection of B with $\partial\Omega$ belongs to $C^{m+\alpha}$, then $\partial\Omega \in C^{m+\alpha}$.