Chapter 7

Elliptic and Parabolic Variational Inequalities

7.1 INTRODUCTION

Rigorous definitions of elliptic and parabolic variational inequalities will be given a little later. Before that we ask a question, what are the essential features of a variational inequality formulation? Let us consider a simple problem of finding the point $x_0 \in (a,b)$ at which the unique minimum of a real valued function $f(x) \in C^1[a,b]$ exists. If $f(x)$ is a convex function, then x_0 can be obtained by solving the equation $f'(x_0) = 0$. In general for $x_0 \in [a,b]$, three cases arise:

1. If minimum is attained at $x_0 = a$, then $f'(x_0) \geq 0$.
2. If minimum is attained at $x_0 = b$, then $f'(x_0) \leq 0$.
3. If x_0 is an interior point, then $f'(x_0) = 0$.

These three conditions can be expressed in terms of a single inequality

$$f'(x_0)(x - x_0) \geq 0, \quad \forall x \in [a,b]. \tag{7.1.1}$$

Eq. (7.1.1) is an example of a *variational inequality* whose characteristic features are that it is an inequality and it is satisfied for all x varying over the interval $[a,b]$.

We shall now generalize this notion and consider a space of functions, say, $H^1(\Omega)$, $\Omega \subset R^n$, $n \geq 1$ is an open bounded set and let f be a functional on $H^1(\Omega)$. Find a function $u = u_0 \in H^1(\Omega)$ or belonging to a subset of $H^1(\Omega)$ such that $f(u_0)$ is minimum as u varies over $H^1(\Omega)$ or over a subset of $H^1(\Omega)$. Immediately several questions arise such as: What sort of function spaces should be considered so that a solution can be found? What should be the form of the functional defined on these spaces? Are there equivalent formulations? Answers to these require sophisticated functional analysis tools whose detailed description is beyond the scope of this book. In the next few sections an attempt will be made to answer some of the above questions in simple mathematical terms. After discussing the theoretical background of elliptic and parabolic variational inequalities, the formulations of some classical Stefan problems as variational inequalities have been given. It may be noted that for any given classical Stefan problem it may not be possible to formulate it as a variational inequality problem. Weak formulations of Stefan problems, which are continuum models, are easily

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amenable to variational inequality formulations as variational inequalities are themselves continuum models but this is not the case with classical formulations. To make this volume self-contained, some relevant definitions and theorems are given in Appendices A–D.

The elliptic variational inequalities will be discussed first. Whenever it is possible to formulate transient Stefan problems as variational inequalities, they are formulated as parabolic variational inequalities. Elliptic variational inequalities, which are concerned with elliptic or steady-state free boundary problems, serve as a good starting point for discussing parabolic variational inequalities. This is because many of the ideas involved and approaches used in proving results in the analysis of elliptic variational inequalities can be extended, with appropriate changes, to the analysis of parabolic variational inequalities. This does not mean that every elliptic problem can be extended to a time-dependent parabolic problem.

7.2 THE ELLIPTIC VARIATIONAL INEQUALITY

7.2.1 Definition and the Basic Function Spaces

Let Ω be a bounded open subset of R^n with smooth boundary $\partial\Omega$. An inequality of the form

$$a(u, v-u) \geq (q, v-u), \quad \forall v \in \mathcal{M}; \ \ u \in \mathcal{M}, \tag{7.2.1}$$

where $a(u, v)$ is a *quadratic bilinear form* (cf. Appendix A), $a(u, v) : H^1(\Omega) \times H^1(\Omega) \to R$, and

$$(q, v) = \int_{\Omega} qv dx, \quad q \in L^2(\Omega), \tag{7.2.2}$$

$$\mathcal{M} = \left\{ v \in H^1(\Omega); v - d \in H_0^1(\Omega); v \geq \psi, \psi \in C(\bar{\Omega}) \right\} \tag{7.2.3}$$

is an example of an *elliptic variational inequality*. q, d and ψ, are known functions. $v = d$ on $\partial\Omega$ in the trace sense. ψ is called an *obstacle* and the problem is called an *obstacle problem* or an *elliptic variational inequality with obstacle*. If there exists a function $u_0 \in \mathcal{M}$ which satisfies Eq. (7.2.1) for all $v \in \mathcal{M}$, then u_0 is called a solution of the elliptic variational inequality. It can be proved that $\mathcal{M}$ is a nonempty closed convex set (cf. Section 7.2.4 for the proof). Problems with more than one obstacle can also be studied (cf. [178]) but such problems will not be discussed here. It is well known in the calculus of variations that the problem formulated in a proposed class of functions may not possess a solution in that class. This difficulty can be overcome by broadening the class of functions in which the problem is formulated and therefore the admissible functions are considered in ‘Sobolev spaces’ or the ‘space of distributions’. A ‘Hilbert space’ $H^1(\Omega)$ has been considered in Eq. (7.2.1) as we would like that at least the first-order *weak derivatives* of functions belonging to $\mathcal{M}$ exist. The choice of an appropriate Sobolev space depends on the physical problem under consideration. For example, if in a given physical problem, a solution of the form $y = |x|$, $x \in R$, makes sense then $H^1(R)$ is an appropriate space for the admissible functions. Several other questions concerning variational formulations arise which will be explained after an adequate mathematical exposition of the concepts and notions of variational inequalities.

Let $A : H^1(\Omega) \to H^{-1}(\Omega)$ (H^{-1} is the *dual space* of the Hilbert space H^1, i.e. the space of continuous linear real valued functions on $H^1(\Omega)$) be an operator such that for a fixed $u \in H^1(\Omega)$, $A_u(v) = a(u, v)$, $v \in H^1(\Omega)$. A_u defines the mapping $v \to a(u, v)$. It can be easily proved that A_u is linear and if $a(u, v)$ is continuous, i.e. $|a(u, v)| \leq M\|u\|\|v\|$ for some $M \in R$ then

$$\|A_u\|_{H^{-1}(\Omega)} = \sup_{\|v\|_H=1} |A_u(v)| \leq M \|u\|_{H^1(\Omega)} \tag{7.2.4}$$

and hence

$$\|A\|_{\mathcal{L}(H^1,H^{-1})} \leq M_1 \quad \text{(constant)}. \tag{7.2.5}$$

Therefore A is continuous and belongs to $H^{-1}(\Omega)$ as A is a linear continuous real valued function on $H^1(\Omega)$. By *Riesz representation theorem*, we have

$$_{H^{-1}(\Omega)}\langle A_u, v\rangle_{H^1(\Omega)} = (u^*, v), \quad \forall v \in H^1(\Omega), \tag{7.2.6}$$

where u^* is some fixed element of $H^1(\Omega)$ and depends on A_u.

Conversely, if A is linear and satisfies Eq. (7.2.4), then $a(u, v)$ is a continuous bilinear form. It is clear that

$$a(u, v) = {}_{H^{-1}(\Omega)}\langle A_u, v\rangle_{H^1(\Omega)}. \tag{7.2.7}$$

With each $q \in L^2(\Omega)$, we can associate a continuous linear functional $B_q \in H^{-1}(\Omega)$ which is defined as

$$B_q : v \to (q, v) = \int_\Omega qv dx, \quad v \in H^1(\Omega). \tag{7.2.8}$$

For a rigorous proof of this statement, see [22]. A sketchy proof can be given using the following arguments. With each $q \in L^2(\Omega)$, an element of the dual space of $L^2(\Omega)$ can be associated. If the dual space of $L^2(\Omega)$ is denoted by $L^{2*}(\Omega)$ then $L^{2*}(\Omega) \hookrightarrow H^{-1}(\Omega)$.

In view of these arguments, we can write

$$_{L^2(\Omega)}\langle q, v\rangle_{H^1(\Omega)} = {}_{H^{-1}(\Omega)}\langle B_q, v\rangle_{H^1(\Omega)} = \int_\Omega qv dx = (q, v), \quad v \in H^1(\Omega). \tag{7.2.9}$$

The pairing $\langle q, v\rangle$ is the value of the functional B_q at v in the sense of Eq. (7.2.9). We shall not justify at other places the use of (q, v) in the place of $\langle q, v\rangle$. The above discussion suggests that a true variational inequality (or equality) in the context of Eq. (7.2.1) should be of the form

$$a(u, v-u) = {}_{H^{-1}(\Omega)}\langle A_u, v-u\rangle_{H^1(\Omega)} \geq {}_{H^{-1}(\Omega)}\langle q, v-u\rangle_{H^1(\Omega)}, \forall v \in H^1(\Omega), \quad u \in H^1(\Omega). \tag{7.2.10}$$

The space H^k, $k \geq 1$ can also be considered and weaker or stronger conditions can also be imposed on the functions involved in the variational inequality formulation.

The form of $a(u, v)$ depends on the elliptic operator considered in a given physical problem. For example, if the elliptic operator $A : H^1(\Omega) \to H^{-1}(\Omega)$ is defined as

$$Au = \nabla^2 u, \quad u \in H^1(\Omega), \tag{7.2.11}$$

then

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx. \tag{7.2.12}$$

If the elliptic operator A is defined as

$$Au = -\sum_{i,J}^{n} a_{iJ}(x)\frac{\partial^2 u}{\partial x_i \partial x_J} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + e(x)u, \quad x \in \Omega, \tag{7.2.13}$$

then the bilinear form in Eq. (7.2.1) is defined by the relation

$$a(u,v) = \int_\Omega \left\{ \sum_{i,J}^{n} a_{iJ}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_J} + \sum_{i}^{n}\left(b_i + \sum_J \frac{\partial a_{iJ}}{\partial x_J}\right)\frac{\partial u}{\partial x_i}v + euv \right\} dx. \tag{7.2.14}$$

The relationship between an elliptic operator and the bilinear form which occurs in the inequality will be discussed later (cf. Eqs 7.2.24, 7.2.37). The coefficients a_{iJ}, b_i and e in the elliptic operator in Eq. (7.2.13) should belong to appropriate function spaces so that the variational inequality formulation makes sense. We shall see later that for the existence and uniqueness of solutions, it will be required that $a(u,v)$ satisfies some conditions such as continuity, coercivity and these conditions also put some restrictions on the coefficients in the elliptic operator. The nonhomogeneous Dirichlet problem corresponding to the elliptic operator A in Eq. (7.2.13) can be stated as the problem of finding a function $u(x)$, $x \in \Omega$ such that

$$Au = q, \quad \text{in } \Omega; \quad u = d, \quad \text{on } \partial\Omega. \tag{7.2.15}$$

If continuous derivatives of u exist in Ω and d is continuous, then Eq. (7.2.15) is satisfied in the classical sense or pointwise sense. If q is square integrable in Ω, then $Au = q$ is satisfied in the sense of distributions. If only the weak derivatives of u exist, then $Au = q$ is satisfied in the distributional sense and $u = d$ is satisfied on $\partial\Omega$ in the trace sense.

7.2.2 Minimization of a Functional

A minimization problem and its equivalent elliptic variational inequality

The variational inequality (7.2.1) can be expressed in some other forms also. We shall first show that the variational inequality problem

$$(z, v-z) + a(z, v-z) \geq (q, v-z), \quad \forall v \in \mathcal{M}^*, \ \ z \in \mathcal{M}^*, \ \ q \in L^2(\Omega), \tag{7.2.16}$$

where $\mathcal{M}^*$ is a convex set of an inner product space and Ω is as in Eq. (7.2.1), is equivalent to the problem of minimization of a functional $P(v)$, where

$$P(v) = (v,v) + a(v,v) - 2(q,v), \quad v \in \mathcal{M}^*, \ \ q \in L^2(\Omega). \tag{7.2.17}$$

Let $a(u,v)$ be symmetric and z be a solution of Eq. (7.2.16). By definition $a(u,v)$ is linear in both the arguments but it need not be symmetric. We shall show that $P(z) \leq P(v)$, $\forall v \in \mathcal{M}^*$ so that z is the minimum of $P(v)$.

$$P(v) - P(z) = (v, v) - (z, z) + a(v, v) - a(z, z) - 2(q, v) + 2(q, z)$$
$$= (z - v, z - v) + 2(z, v - z) + a(v, v) - a(z, z) - 2(q, v - z). \quad (7.2.18)$$

Also

$$a(u, u) = a(u - w, u - w) - a(w, w) + 2a(u, w). \quad (7.2.19)$$

Take $u = z$ and $w = z - v$ in Eq. (7.2.19). It can be seen that

$$a(v, v) - a(z, z) \geq 2a(z, v - z). \quad (7.2.20)$$

On using Eq. (7.2.20) in Eq. (7.2.18), we get

$$P(v) - P(z) \geq (z - v, z - v) + 2(z, v - z) + 2a(z, v - z) - 2(q, v - z). \quad (7.2.21)$$

If Eq. (7.2.16) holds, then from Eq. (7.2.21), $P(v) \geq P(z)$, $\forall v \in \mathcal{M}^*$.

To prove the converse, suppose that $P(z) \leq P(v)$, $\forall v \in \mathcal{M}^*$. Since $\mathcal{M}^*$ is a convex set, if $v \in \mathcal{M}^*$ and $z \in \mathcal{M}^*$ then $(1 - \alpha)z + \alpha v \in \mathcal{M}^*$, $0 < \alpha < 1$. Therefore

$$P(z + \alpha(v - z)) - P(z) \geq 0$$

or

$$\alpha(v - z, v - z) + \alpha a(v - z, v - z) + 2\{(z, v - z) + a(z, v - z) - (q, v - z)\} \geq 0. \quad (7.2.22)$$

For Eq. (7.2.22) to hold for an arbitrarily small α, Eq. (7.2.16) should hold.

7.2.3 The Complementarity Problem

By considering the minimization of the functional (a real valued function)

$$f(v) = a(v, v) - (q, v), \quad (7.2.23)$$

we shall now obtain other forms of elliptic variational inequalities. Although $a(u, v)$ given in Eq. (7.2.14) can also be considered, a simple bilinear form given by

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx, \quad (7.2.24)$$

will be considered for illustration. It will be assumed that a minimizing function $u(x)$ exists such that

$$f(u) \leq f(v), \quad \forall v \in \mathcal{M}, \quad u \in \mathcal{M}, \quad (7.2.25)$$

where the set $\mathcal{M}$ is defined by Eq. (7.2.3). Since $\mathcal{M}$ is a convex set, for $0 \leq \alpha \leq 1$, we have

$$f(u + \alpha(v - u)) - f(u) \geq 0, \quad \forall v \in \mathcal{M} \quad (7.2.26)$$

or

$$\int_{\Omega} \{\nabla(u+\alpha(v-u)) \cdot \nabla(u+\alpha(v-u)) - \nabla u \cdot \nabla u\} dx - 2p(u+\alpha(v-u)) + 2p(u) \geq 0. \tag{7.2.27}$$

Here p is a continuous linear functional on $H^1(\Omega)$ and is defined by

$$p(v) = \int_{\Omega} qv dx. \tag{7.2.28}$$

On using the linearity of both p and the gradient operator from Eq. (7.2.27), we obtain

$$\alpha^2 \int_{\Omega} \nabla(v-u) \cdot \nabla(v-u) dx + \alpha \int_{\Omega} \{\nabla u \cdot \nabla(v-u) - q(v-u)\} \, dx \geq 0. \tag{7.2.29}$$

For Eq. (7.2.29) to hold for an arbitrarily small α, we should have

$$\int_{\Omega} \{\nabla u \cdot \nabla(v-u) - q(v-u)\} \, dx \geq 0. \tag{7.2.30}$$

If it is assumed that $u \in H^2(\Omega) \cap C(\Omega)$, the first term in Eq. (7.2.30) can be integrated by parts (application of Green's formula). Since v satisfies the same boundary conditions as u, we have $v - u = 0$ on $\partial\Omega$. On using this condition on $\partial\Omega$ and doing the integration by parts, we obtain

$$\int_{\Omega} (\nabla^2 u + q)(v-u) dx \leq 0, \quad \forall v \in \mathcal{M}, \ \ u \in \mathcal{M}. \tag{7.2.31}$$

Let $v = u + \zeta$, $\zeta \geq 0$, $\zeta \in C_0^{\infty}(\Omega)$. If $u \geq \psi$, then $v \geq \psi$. Substituting $\zeta = (v-u)$ in Eq. (7.2.31) and remembering that $\zeta \geq 0$ and $\zeta \in C_0^{\infty}(\Omega)$, we conclude that

$$\nabla^2 u + q \leq 0, \quad \text{a.e. in } \Omega, \ \ u \geq \psi. \tag{7.2.32}$$

Next, consider a subset Ω_1 of Ω, where

$$\Omega_1 = \{x \in \Omega; u(x) > \psi(x)\}, \quad \Omega_1 \subset \Omega. \tag{7.2.33}$$

If $u(x)$ and $\psi(x)$ both are continuous in Ω, then for every $x \in \Omega_1$ there exists a neighbourhood of x in which $u(x) - \psi(x) > 0$. Therefore Ω_1 cannot be a closed set. For any $\zeta \in C_0^{\infty}(\Omega_1)$, $v = u - \varepsilon\zeta$ is in $\mathcal{M}$ provided $|\varepsilon|$ is sufficiently small. On substituting $v = u - \varepsilon\zeta$ in Eq. (7.2.31), we get

$$-\varepsilon \int_{\Omega} (\nabla^2 u + q)\zeta dx \leq 0. \tag{7.2.34}$$

If Eq. (7.2.34) holds for ε positive or negative and ζ is arbitrary, then it can be easily concluded that

$$\nabla^2 u + q = 0, \quad \text{a.e. in } \Omega_1, \ \ u > \psi. \tag{7.2.35}$$

The set Ω_1 in which $u(x) > \psi(x)$ is called a *noncoincidence set* and the set $\Omega_2 = \{x \in \Omega : u(x) = \psi(x)\}$ is called a *coincidence set*. The boundary of the noncoincidence set is called a free boundary. If Γ is the free boundary, then

$$\Gamma = \partial\Omega_1 \cap \Omega. \tag{7.2.36}$$

On combining the results obtained in Eqs (7.2.32), (7.2.35), we get the following variational inequality problem.

Find $u \in H^2(\Omega) \cap C(\Omega)$ such that

$$\left.\begin{aligned} \nabla^2 u + q &\le 0, \\ u &\ge \psi, \\ (\nabla^2 u + q)(u - \psi) &= 0, \end{aligned}\right\} \quad \text{a.e. in } \Omega, \tag{7.2.37}$$

$$u - d \in H_0^1(\Omega).$$

The boundary condition $u = d$ is satisfied on $\partial\Omega$ in the trace sense. The free boundary problem to be studied in Ω_1 is to find a pair (u, Γ) such that

$$\nabla^2 u + q = 0, \quad \text{in } \Omega_1, \tag{7.2.38}$$

$$\left. u = \psi; \quad \frac{\partial u}{\partial x_i} = \frac{\partial \psi}{\partial x_i}, \quad 1 \le i \le n \right\} \quad \text{on } \Gamma, \tag{7.2.39}$$

$$u = d, \quad \text{on } \partial\Omega_1 \cap \partial\Omega. \tag{7.2.40}$$

The second condition on Γ in Eq. (7.2.39) arises due to the fact that $u - \psi$ takes its minimum on Γ (assuming u and ψ to be smooth). The problem stated in Eq. (7.2.37) is called a *complementarity problem* and it is a standard problem in 'quadratic programming'. Its 'finite-difference discretization' will have the following form:

$$\left.\begin{aligned} BU + D &\le 0, \\ U &\ge \Psi, \\ (BU + D)(U - \Psi)^T &= 0, \end{aligned}\right\} \tag{7.2.41}$$

where B, U, D and Ψ are appropriate matrices obtained after discretization of Eq. (7.2.37). For example, B could be a $n \times n$ matrix and U, D and Ψ could be $n \times 1$ matrices (n stands for the number of nodal points). On introducing the following substitutions:

$$U - \Psi = E \quad \text{and} \quad Y = -(BU + D), \tag{7.2.42}$$

in Eq. (7.2.41), we have a problem of finding E such that

$$BE = -Y - (B\Psi + D), \quad E^T Y = 0; \;\; E \ge 0, \;\; Y \ge 0. \tag{7.2.43}$$

Matrices B, D and Ψ are known. The problem (7.2.43) is equivalent to the following programming problem provided B is symmetric and positive definite [179]. Minimize

$$(B\Psi + D)^T E + \frac{1}{2} E^T B E, \quad \text{for } E \ge 0. \tag{7.2.44}$$

If a more general form of $a(u, v)$ is considered such as the one considered in Eq. (7.2.14), then by following the procedure indicated in Eqs (7.2.26)–(7.2.35), one can easily obtain the complementarity problem of the form (7.2.37) in which in the place of $\nabla^2 u$ we shall have Au given in Eq. (7.2.13).

The variational inequality (7.2.37) can be transformed into an inequality of the form (7.2.1). Let $\zeta \in C_0^\infty(\Omega)$, $\zeta \geq 0$ and u be the solution of Eq. (7.2.37). If $v = u + \zeta$, then $v \geq \psi$ and v belongs to $\mathcal{M}$. Multiplying the first equation in Eq. (7.2.37) by ζ and integrating over Ω, we get

$$\int_\Omega (\nabla^2 u + q)\zeta \leq 0. \tag{7.2.45}$$

Integrating Eq. (7.2.45) by parts and remembering that $\zeta = v - u = 0$ on $\partial\Omega$, we get an inequality of the form (7.2.1). The arguments used in obtaining Eq. (7.2.45) hold good even if $\nabla^2 u$ is replaced by Au given in Eq. (7.2.13). The inequality (7.2.1) also implies the problem in Eq. (7.2.37). To show this, we first consider the case $u \geq \psi$. Take $\zeta \in C_0^\infty(\Omega)$, $\zeta \geq 0$. If $v = u + \zeta$, then we have $v \geq \psi$ and therefore v belongs to $\mathcal{M}$. Substituting ζ in place of $(v - u)$ in Eq. (7.2.1) and integrating by parts it can be easily seen that $\nabla^2 u + q \leq 0$, a.e., in Ω provided $u \in H^2(\Omega) \cap C(\Omega)$. Next consider the set $\Omega_1 = \{x \in \Omega : u(x) > \psi(x)\} \subset \mathcal{M}$. Follow the procedure which was used to arrive at Eq. (7.2.35). It is then easy to show that $(\nabla^2 u + q)(u - \psi) = 0$ for $u \geq \psi$.

7.2.4 Some Existence and Uniqueness Results Concerning Elliptic Inequalities

The minimization problem associated with the functional in Eq. (7.2.23) and the complementarity problem in Eq. (7.2.37) are also called variational inequality problems. We shall see later that variational inequalities can be expressed in some other forms as well. The equivalence of some different formulations of variational inequalities will be discussed in Section 7.2.5 in the context of Problem 1.1.12. Questions pertaining to the existence, uniqueness and stability of the solutions of variational inequalities arise at this point. As mentioned earlier, sophisticated functional analytic tools are required to answer them (cf. [178, 180, 181]). To understand the basic concepts we shall first discuss some results on the existence and uniqueness of the solutions of elliptic variational inequalities. These results will be helpful in parabolic variational inequalities also. It may be noted that for the existence of the minimum of a function (functional) it is not necessary for the function to be continuous. For example, if $f(x) = |x|$, $x \neq 0$ and $f(0) = -2$ then f has a minimum value -2. In this case $f(x)$ is not continuous but is *lower semi continuous* (l.s.c.). The conditions under which the unique minimum of some of the functionals exist are discussed below in the form of theorems which provide answers to some problems.

Problem 7.2.1. Given a real vector space X, a function $f : X \to R$ and a set $Y \subset X$. Find the minimum of f in Y, i.e. find $y_0 \in Y$ such that $f(y_0) = \inf_{y \in Y} f(y)$.

The answer to this problem is contained in the following theorem.

Theorem 7.2.1. *If X is a reflexive Banach space, $f : X \to R$ is a convex and l.s.c. function, $Y \neq \emptyset$ is a closed convex subset of X, and either Y is bounded or f is coercive,* *then Problem 7.2.1 has a solution. This solution is unique if f is strictly convex. This theorem is called 'theorem of minimization of convex functionals'.*

Problem 7.2.2. Let W be a Hilbert space and $\mathcal{P} \subset W$ be a nonempty closed convex set, $g \in W'$ (W' is the dual space of W and the elements of W' are linear, continuous real valued functions) and $f : W \to R$ is defined by

$$f(v) = \frac{1}{2}\|v\|_W^2 - g(v). \tag{7.2.46}$$

Find $u_0 \in \mathcal{P}$ such that $f(u_0) = \inf_{v\in\mathcal{P}} f(v)$.

Theorem 7.2.2. *The solution to Problem 7.2.2 exists and is unique.*

Proof. We shall show that f is both strictly convex and coercive and so Theorem 7.2.1 can be applied. Since W is a Hilbert space, it is reflexive. Let λ_1 and λ_2 be any two scalars such that $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$ and v_1 and v_2 ($v_1 \neq v_2$) be any two vectors belonging to W. We have

$$\lambda_1 \|v_1\|^2 + (1-\lambda_1)\|v_2\|^2 - \left\{\lambda_1^2 \|v_1\|^2 + 2\lambda_1(1-\lambda_1)(v_1, v_2) + (1-\lambda_1)^2 \|v_2\|^2\right\}$$

$$= \lambda_1(1-\lambda_1)\left\{\|v_1\|^2 - 2(v_1, v_2) + \|v_2\|^2\right\} \tag{7.2.47}$$

$$\geq \lambda_1(1-\lambda_1)\left\{\|v_1\|^2 - 2\|v_1\|\|v_2\| + \|v_2\|^2\right\} \tag{7.2.48}$$

$$\geq \lambda_1(1-\lambda_1)\{\|v_1\| - \|v_2\|\}^2 > 0, \quad \text{if } v_1 \neq v_2 \quad \text{and} \quad \lambda_1 \neq 0. \tag{7.2.49}$$

Also

$$\begin{aligned}
f(\lambda_1 v_1 + \lambda_2 v_2) &= \frac{1}{2}\|\lambda_1 v_1 + \lambda_2 v_2\|_W^2 - g(\lambda_1 v_1 + \lambda_2 v_2) \\
&= \frac{1}{2}\left\{\lambda_1^2 \|v_1\|^2 + \lambda_2^2 \|v_2\|^2 + 2\lambda_1\lambda_2(v_1, v_2)\right\} - \lambda_1 g(v_1) - \lambda_2 g(v_2) \\
&< \frac{1}{2}\left\{\lambda_1 \|v_1\|^2 + \lambda_2 \|v_2\|^2\right\} - \lambda_1 g(v_1) - \lambda_2 g(v_2) \\
&< \lambda_1 f(v_1) + \lambda_2 f(v_2).
\end{aligned}$$

Therefore f is strictly convex. From the Riesz representation theorem, we have

$$g(v) \leq \|g\|_{W'}\|v\|_W. \tag{7.2.50}$$

On using Eq. (7.2.50) in Eq. (7.2.46), we get

$$\begin{aligned}
f(v) &\geq \frac{1}{2}\|v\|_W^2 - \|g\|_{W'}\|v\|_W \\
&\geq \gamma\|v\|_W^2, \quad \text{for some positive constant } \gamma \text{ as } \|v\| \to \infty, \quad \forall v
\end{aligned} \tag{7.2.51}$$

and so f is coercive. □

Problem 7.2.3. Given a Hilbert space W, $g \in W'$ and $\mathcal{P} \neq \emptyset$, a closed convex set of W, find $u_0 \in \mathcal{P}$ such that

$$(u_0, u_0 - v)_W \leq g(u_0 - v), \quad \forall v \in \mathcal{P}. \tag{7.2.52}$$

Theorem 7.2.3. *u_0 is the solution of Problem 7.2.2 if and only if u_0 is a solution of Problem 7.2.3 or in other words Problems 7.2.2 and 7.2.3 are equivalent.*

Proof. First, we prove that Problem 7.2.2 implies Problem 7.2.3. If $u_0 \in \mathcal{P}$ and $v \in \mathcal{P}$, then $u_0 + \alpha(v - u_0) \in \mathcal{P}$ for $0 \le \alpha \le 1$ as $\mathcal{P}$ is a convex set. If the minimum is attained at u_0, then

$$F(\alpha) = f(u_0 + \alpha(v - u_0)) \ge f(u_0) \quad \text{and} \quad F'(0) \ge 0. \tag{7.2.53}$$

Let $G(\alpha) = u_0 + \alpha(v - u_0)$, then

$$\frac{d}{d\alpha}F(G(\alpha))|_{\alpha=0} = \frac{1}{2}\left(\frac{d}{d\alpha}\|G(\alpha)\|_W^2\right)_{\alpha=0} - \left(\frac{d}{d\alpha}g(G(\alpha))\right)_{\alpha=0} \ge 0. \tag{7.2.54}$$

The first term on the r.h.s. of Eq. (7.2.54) can be easily calculated from the first principles and we obtain

$$\frac{1}{2}\left(\frac{d}{d\alpha}\|G(\alpha)\|_W^2\right)_{\alpha=0} = (u_0, v - u_0)_W. \tag{7.2.55}$$

On using the linearity of g

$$\left.\frac{d}{d\alpha}g(G(\alpha))\right|_{\alpha=0} = \left.\frac{d}{d\alpha}(g(u_0) + \alpha g(v - u_0))\right|_{\alpha=0} = g(v - u_0). \tag{7.2.56}$$

On combining the results in Eqs (7.2.55), (7.2.56), we obtain Eq. (7.2.52). To prove that Problem 7.2.3 implies Problem 7.2.2, it will be assumed that the solution of Problem 7.2.3 exists. For the existence proof, see [22]. It can be easily shown that if the solution exists then it is unique. Let u_1 and u_2 be two solutions of Problem 7.2.3. We have

$$(u_1, u_1 - u_2)_W \le g(u_1 - u_2), \quad u_2 \in \mathcal{P} \tag{7.2.57}$$

and

$$(u_2, u_2 - u_1)_W \le g(u_2 - u_1), \quad u_1 \in \mathcal{P}. \tag{7.2.58}$$

Adding Eqs (7.2.57), (7.2.58) and using the definition of 'scalar product' and the linearity of g, we get

$$(u_1 - u_2, u_1 - u_2) \le 0, \tag{7.2.59}$$

which implies $u_1 = u_2$. In view of the fact that Problem 7.2.2 implies Problem 7.2.3 the unique solution of Problem 7.2.3 is also the unique solution of Problem 7.2.2. The inequality (7.2.52) is also called a variational inequality and here the inequality is defined with the help of a scalar product, whereas, in Eq. (7.2.1) it has been defined with the help of a bilinear form. □

The functional (7.2.46) is a special case of the functional

$$f(v) = \frac{1}{2}a(v, v) - g(v), \tag{7.2.60}$$

where $a : W \times W \to R$ is a bilinear form. If with each pair $(u, v) \in W \times W$, $a(u, v)$ associates the scalar product $(u, v)_W$ or in other words $a(u, v) = (u, v)_W$ then $|a(u, v)| = |(u, v)_W| \le$

$\gamma \|u\|_W \|v\|_W (\gamma = 1)$ and a is continuous. $a(u,v) = (u,v)_W = (v,u)_W = a(v,u)$ and a is symmetric. a has also the coercivity property on W, i.e. $a(u,u) \geq \alpha \|u\|_W^2$ for $\alpha = 1$ as $(u,v) = \|u\|^2 \geq \|u\|^2$. On the other hand if a is coercive, symmetric and continuous then a scalar product $((u,v))$ can be defined with the help of a bilinear form as

$$((u,v)) = a(u,v), \quad \forall u, v \in W. \tag{7.2.61}$$

Let $|||\cdot|||_W$ be the norm associated with the inner product defined in Eq. (7.2.61). We shall show that $\|\cdot\|_W$ and $|||\cdot|||_W$ are equivalent and therefore the continuous linear functionals defined on W for the two norms are the same. $\alpha \|u\|_W^2 \leq a(u,u) = ((u,u))_W = |||u|||_W^2 = ((u,u))_W = a(u,u) \leq \gamma \|u\|_W^2$. Therefore the two norms are equivalent and $a(u,v)$ is coercive and continuous with respect to the norm $|||\cdot|||_W$. Equivalence of norms implies that if $a(u,v)$ in Eq. (7.2.60) is symmetric, coercive and continuous with the respect to $\|\cdot\|_W$, then the functional f in Eq. (7.2.46) is no more general then the functional in Eq. (7.2.60) as the functional in Eq. (7.2.60) can be written as

$$f(v) = \frac{1}{2} |||\, v \,|||_W^2 - g(v), \tag{7.2.62}$$

where $|||v|||_W = ((v,v)) = a(v,v),\ v \in W$. Next we ask whether the minimum of the functional (7.2.60) can be obtained as the solution of a variational inequality.

Problem 7.2.4. Let W be a Hilbert space, $\mathcal{P} \subset W$, a nonempty closed convex set and $f : W \to R$, a functional defined by Eq. (7.2.60) in which the bilinear form $a(u,v)$ is continuous, find $u_0 \in \mathcal{P}$ such that

$$f(u_0) \leq f(v), \quad \forall v \in \mathcal{P}. \tag{7.2.63}$$

Problem 7.2.5. Let W, $\mathcal{P}, f$ and $a(u,v)$ be the same as in Problem 7.2.4. Find $u_0 \in \mathcal{P}$ such that

$$a(u_0, u_0 - v) \leq g(u_0 - v), \quad \forall v \in \mathcal{P}. \tag{7.2.64}$$

If u_0 is the minimum of $f(v)$ in Eq. (7.2.60), then

$$\left. \frac{d}{d\alpha} f(u_0 + \alpha(v - u_0)) \right|_{\alpha=0} \geq 0, \quad \forall v \in \mathcal{P}. \tag{7.2.65}$$

The derivative in Eq. (7.2.65) can be easily calculated by using the bilinearity of a and the linearity of g and it can be shown that Eq. (7.2.63) implies Eq. (7.2.64) only if $a(u,v)$ is symmetric. If a solution of Problem 7.2.5 exists, then it can be proved that it is unique [22]. If it can be proved that a unique solution of Problem 7.2.4 exists, then the equivalence of Problems 7.2.5 and 7.2.4 can be established. If a is coercive on $\mathcal{P} - \mathcal{P}$, i.e.

$$\exists \alpha, \ \ \alpha > 0, \quad \text{e.g. } a(u - v, u - v) \geq \alpha \|u - v\|_W^2, \quad \forall u, v \in \mathcal{P}, \tag{7.2.66}$$

then the solution of Problem 7.2.5 exists [22]. $\mathcal{P} - \mathcal{P} = \{x - y : x \in \mathcal{P}, y \in \mathcal{P}\}$. It can be proved that if a is symmetric and, continuous on $\mathcal{P}$ and coercive on $\mathcal{P} - \mathcal{P}$ then it is coercive on $\mathcal{P}$.

$$\begin{aligned} a(u,u) &= a(u-\zeta, u-\zeta) - a(\zeta,\zeta) + 2a(u,\zeta), \quad u \in \mathcal{P}, \quad \zeta \text{ fixed in } \mathcal{P} \\ &\geq \alpha\|u-\zeta\|_W^2 - \gamma\|\zeta\|_W^2 - 2\gamma\|u\|\|\zeta\|_W, \quad \alpha > 0, \quad \gamma > 0 \\ &\geq \alpha\|u-\zeta\|_W^2 - \gamma\|\zeta\|_W(3\|\zeta\|_W + 2\|u-\zeta\|_W). \end{aligned} \tag{7.2.67}$$

It is now easy to prove that $a(u,u) \to +\infty$ as $\|u\| \to \infty$.

Lions–Stampacchia Theorem

Theorem 7.2.4. *Let W be a Hilbert space. $\mathcal{P} \subset W$ be a nonempty closed convex set, $g \in W'$ and $a : W \times W \to R$, a continuous bilinear form on W and coercive on $\mathcal{P} - \mathcal{P}$. There is one and only one $u_0 \in \mathcal{P}$ such that*

$$a(u_0, v - u_0) \geq g(v - u_0), \quad \forall v \in \mathcal{P}, \tag{7.2.68}$$

and further the application which associates u_0 to every g is continuous, i.e. Problem 7.2.5 is well-posed.

Let $\mathcal{P} = \mathcal{M}$ (for $\mathcal{M}$ see Eq. 7.2.3) in Theorem 7.2.4, so that we can discuss stability of the solution of the variational inequality (7.2.1). A comparison of Eqs (7.2.68), (7.2.1) suggests that the linear functional g is defined as

$$g(v) = (q, v) = \int_\Omega qv dx, \quad \forall v \in \mathcal{M}. \tag{7.2.69}$$

Note that a test function v satisfies the boundary condition. Therefore to prove continuous dependence of the solution on the data, it is to be proved that if u_1 and u_2 are two different solutions of Eq. (7.2.68) corresponding to the data q_1 and q_2, then

$$\alpha\|u_1 - u_2\|_W \leq \|q_1 - q_2\|_{W'(\Omega)}, \quad \alpha > 0. \tag{7.2.70}$$

Let g_1 and g_2 be associated with q_1 and q_2 through Eq. (7.2.69). Eq. (7.2.68), we have

$$a(u_1, u_2 - u_1) \geq g_1(u_2 - u_1) \quad (\text{take } v = u_2) \tag{7.2.71}$$

and

$$a(u_2, u_1 - u_2) \geq g_2(u_1 - u_2) \quad (\text{take } v = u_1). \tag{7.2.72}$$

Adding Eqs (7.2.71), (7.2.72) and changing the sign, we obtain

$$\begin{aligned} a(u_1 - u_2, u_1 - u_2) &\leq g_1(u_1 - u_2) - g_2(u_1 - u_2) \\ &\leq \|g_1 - g_2\|_{W'}\|u_1 - u_2\|_W. \end{aligned} \tag{7.2.73}$$

Using the coercivity of $a(u,v)$ on $\mathcal{P}$, Eq. (7.2.73) can be written as

$$\alpha\|u_1 - u_2\|_{W(\Omega)} \leq \|q_1 - q_2\|_{W'(\Omega)}, \quad \alpha > 0. \tag{7.2.74}$$

This proves continuous dependence of the solution on the data. Problem 7.2.5 is a generalization of Problem 7.2.3 as it imposes weaker conditions on the form of $a(u,v)$. There is an interesting geometrical interpretation of the minimization of the functional (7.2.46). Using Riesz representation theorem, $f(v)$ in Eq. (7.2.46) can be written as

$$f(v) = \frac{1}{2}\|v\|_W^2 - (u^*, v), \quad v \in \mathcal{P}, \; u^* \in W. \tag{7.2.75}$$

Define a functional $F(v)$ as

$$\begin{aligned} F(v) &= f(v) + \frac{1}{2}\|u^*\|_W^2 \\ &= \frac{1}{2}\|v\|_W^2 - (u^*, v)_W + \frac{1}{2}\|u^*\|_W^2 \\ &= \frac{1}{2}\|v - u^*\|_W^2. \end{aligned} \tag{7.2.76}$$

If there is an element of $\mathcal{P}$ which minimizes F, then it also minimizes f and minimizing F amounts to finding a $v \in \mathcal{P}$ whose distance from a fixed $u^* \in W$ is minimum, i.e. $\|u_0 - u^*\|$ is minimum. If $u_0 \in \mathcal{P}$ is such that $\|u_0 - u^*\|$ is minimum then

$$\|u_0 - u^*\| \leq \|v - u^*\|, \quad \forall v \in \mathcal{P}. \tag{7.2.77}$$

It can be proved that the inequality (7.2.77) is equivalent to the following inequality:

$$(u_0 - u^*, v - u_0) \geq 0, \quad \forall v \in \mathcal{P}. \tag{7.2.78}$$

Hint: Let u_0, u^* and v be the vertices of a triangle. Use the proposition that the sum of the two sides of the triangle is greater than the third side. Problems 7.2.2 and 7.2.3 are both equivalent to the problem of finding $u_0 \in \mathcal{P}$ for a given u^* (determined by g) satisfying Eq. (7.2.78) for $\forall v \in \mathcal{P}$. u_0 is called the projection of u^* on $\mathcal{P}$. The inequality (7.2.78) is yet another way of writing the variational inequality associated with functional (7.2.46).

Some of the results discussed above can be proved under weaker conditions by considering a reflexive Banach space in the place of a Hilbert space but in a Hilbert space more interesting results can be obtained which in some cases have interesting interpretations.

In Eq. (7.2.64) (we consider this inequality as it is more general than Eq. 7.2.46), g is linear. Therefore Eq. (7.2.64) can be written as

$$a(u_0, u_0 - v) - g(u_0) \leq -g(v), \quad \forall v \in \mathcal{P}. \tag{7.2.79}$$

This suggests that we can consider variational inequalities of the type

$$a(u, v - u) + p(u) \leq p(v), \quad \forall v \in \mathcal{P}, \tag{7.2.80}$$

where $p : W \to R$ is not necessarily linear. In view of Eq. (7.2.80), we want to study now the minimization problem associated with the functional

$$f(v) = m(v) + p(v); \quad f : W \to R \cup \{\infty\}, \tag{7.2.81}$$

where p is not linear and $m(v)$ could be a functional whose *Gâteaux derivative* denoted by $\nabla m(v)$ exists. The motivation for considering Gâteaux derivative comes from the fact that in seeking the minimum of a function defined from $R \to R$ we look for those points at which the classical derivative of the function is zero. On using the linearity of $a(u, v)$ it was possible to calculate the derivative in Eq. (7.2.65) and therefore the Gâteaux derivative of the bilinear form $a(u, v)$ exists. If the continuity of $a(u, v)$ is assumed, then it can be proved that the mapping $v \to a(u, v)$ is continuous (see Eq. 7.2.4). What type of functional $m(v)$ should be? Note that $p(v)$ can be easily handled as it is enough if it is l.s.c. and a *proper convex functional*. A question similar to the one raised about the minimum of the functional in Problem 7.2.4 can be asked for the functional (7.2.81).

Problem 7.2.6. Given a Hilbert space W, $\mathcal{P} \subset W$, a nonempty closed convex set and $f : W \to R \cup \{+\infty\}$, a functional of the form $f(u) = m(u) + p(u)$, find $u_0 \in \mathcal{P}$ such that

$$f(u_0) \leq f(v), \quad \forall v \in \mathcal{P}. \tag{7.2.82}$$

The conditions under which u_0 is the solution of Eq. (7.2.82) are given in the following theorem.

Theorem 7.2.5. *If $m(u)$ is finite, convex and G-differentiable (Gâteaux differentiable) on $\mathcal{P}$ and p is convex and a proper functional on $\mathcal{P}$, then $u_0 \in \mathcal{P}$ is the minimum of $f(u)$ in Problem 7.2.6 if and only if u_0 satisfies the inequality*

$$_{W'}\langle \nabla m(u_0), u_0 - v\rangle_W + p(u_0) \leq p(v), \quad \forall v \in \mathcal{P}. \tag{7.2.83}$$

The inequality (7.2.83) presents yet another form of variational inequality. We shall not pursue further the generalizations of the functionals whose minimum could be obtained.

Variational Equation

If $\mathcal{P} = W$ in Problem 7.2.4 and we seek the minimum of f for $\forall v \in W$, then in Eq. (7.2.65), α could be any real number and not restricted to $0 \leq \alpha \leq 1$. In this situation, in the place of Eq. (7.2.65), we shall have an equation $f'(0) = 0$ and we get a *variational equation* to determine the solution u_0 and we have

$$a(u_0, z) = g(z), \quad z \in W. \tag{7.2.84}$$

However, this is not the case always. For example, if f is of the form (7.2.81) in which $m(v)$ is G-differentiable but not $p(v)$, then even if $\mathcal{P} = W$, we have an inequality of the form (7.2.83) (cf. [22]).

The bilinear form $a(u, v)$ in Eq. (7.2.14) is not symmetric and therefore the scalar product which by definition is symmetric cannot be defined in terms of $a(u, v)$. The bilinear form (7.2.24) is symmetric and let us examine what happens if a scalar product is formally defined in terms of $a(u, v)$ by the equation

$$(u, v) = a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx, \quad u, v \in H^1(\Omega). \tag{7.2.85}$$

All other properties of the scalar product are satisfied by (u, v) if it is defined by Eq. (7.2.85), except the property $(u, u) = 0 \Rightarrow u = 0$. We have already seen that if a bilinear form can be equated with a scalar product then the bilinear form is continuous and coercive and both these properties are very useful in analysing the solutions of the variational inequalities.

It is possible to define a scalar product with the help of Eq. (7.2.85) after some modifications in the formulation of the variational inequality. Let $\Omega \subset R^n$ be an open bounded set with $\partial\Omega$ of class C^0, i.e. every point of $\partial\Omega$ has a neighbourhood U such that $\partial\Omega \cap U$ is the graph of a continuous function (cf. [182]), $u \in H^k(\Omega)$, $k \in N$ and $D^\alpha u$ (αth order weak derivative, see Appendix D) $\in L^2(\Omega)$, $|\alpha| \le k$. H^k can be provided with the scalar product

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha|\le k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)} \tag{7.2.86}$$

and

$$(u, u)_{H^k(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^2 d\Omega. \tag{7.2.87}$$

The norm $\|u\|_{H^k(\Omega)} = (u, u)^{1/2}$. The scalar product in $H_0^k(\Omega)$ ($H_0^k(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$ in $W^{k,2}(\Omega)$, see Appendix D) can be defined as

$$(u, v)_{H_0^k(\Omega)} = \sum_{|\alpha|=k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}. \tag{7.2.88}$$

In the space $H_0^k(\Omega)$, the equivalence of the norms generated by the scalar products defined in Eqs (7.2.86), (7.2.88) can be asserted with the help of the following theorem.

Theorem 7.2.6. *Let $\Omega \subset R^n$ be an open bounded set and $k \in N$ (N is the set of positive integers). There exist two constants β_1 and β_2 such that for every $u \in H_0^k(\Omega)$ the following inequality holds:*

$$\beta_1 \left(\sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^2 dx \right)^{1/2} \le \left(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^2 dx \right)^{1/2} \le \beta_2 \left(\sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^2 dx \right)^{1/2}. \tag{7.2.89}$$

The inequality on the l.h.s. of Eq. (7.2.89) can be easily proved and we have $\beta_1 = 1$. To prove the inequality on the r.h.s., the Poincare's inequality [22] can be invoked which states that if $\psi \in C_0^\infty(\Omega)$ then

$$\int_\Omega |\psi|^2 dx \le \beta \sum_{i=1}^n \int_\Omega \left| \frac{\partial \psi}{\partial x_i} \right|^2 dx, \tag{7.2.90}$$

for some constant $\beta > 0$. Since $H_0^k(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$, Eq. (7.2.90) holds even if $\psi \in H_0^k(\Omega)$. If the r.h.s. in Eq. (7.2.90) tends to zero, then the l.h.s. also tends to zero which implies $\psi \equiv 0$. The above discussion suggests that the scalar product can be defined with the help of Eq. (7.2.88) provided the space $H_0^1(\Omega)$ is considered in the place of $H^1(\Omega)$.

Next we show that the existence and uniqueness results are not affected if the space $H_0^1(\Omega)$ is considered in the place of $H^1(\Omega)$. Let u be the solution to the variational inequality

$$\int_\Omega \nabla u \cdot \nabla(v-u)dx \geq \int_\Omega q(v-u)dx, \quad \forall v \in \mathcal{M}, \;\; u \in \mathcal{M}, \tag{7.2.91}$$

where $\mathcal{M}$ is defined by Eq. (7.2.3). If $w = u - d$, then $w = 0$ on $\partial\Omega$ and $w \in H_0^1(\Omega)$. u is the solution to Eq. (7.2.91) if and only if $w = u - d$ is the solution to

$$\int_\Omega \nabla w \cdot \nabla(v-u)dx \geq \langle q, v-u \rangle - \int_\Omega \nabla d \cdot \nabla(v-u)dx. \tag{7.2.92}$$

We have $v - u = v - d - w = \zeta - w$, and both ζ and w belong to $H_0^1(\Omega) \subset H^1(\Omega)$. The inequality (7.2.92) can be written as

$$\int_\Omega \nabla w \cdot \nabla(\zeta - w)dx \geq \langle F, \zeta - w \rangle, \quad \forall \zeta \in M_1, \tag{7.2.93}$$

where

$$\langle F, \zeta \rangle = \langle q, \zeta \rangle - \int_\Omega \nabla d \cdot \nabla \zeta dx, \quad dx_i \in L^2(\Omega), \;\; 1 \leq i \leq n \tag{7.2.94}$$

and

$$\mathcal{M}_1 = \{\zeta \in H_0^1(\Omega) : \zeta \geq \psi - d\}. \tag{7.2.95}$$

It can be seen that if $d \in H^1(\Omega)$ and $q \in H^{-1}(\Omega)$, then F is linear and continuous on $H^1(\Omega)$, i.e. $F \in H^{-1}(\Omega)$.

It will now be shown that $\mathcal{M}_1$ is a nonempty closed convex set, $a(w, \zeta)$ is continuous and coercive on $\mathcal{M}_1$, so that Theorem 7.2.4 can be applied. If $\zeta_1, \zeta_2 \in \mathcal{M}_1$, then $\alpha\zeta_1 + (1-\alpha)\zeta_2 \geq \alpha(\psi - d) + (1-\alpha)(\psi - d) \geq \psi - d$, for $0 < \alpha < 1$. Therefore $\mathcal{M}_1$ is a convex set. The functions of $H_0^1(\Omega)$ are absolutely continuous functions if Ω is an open subset of R^1. Let ψ be continuous on $\Omega = (0, b)$. Consider the function $[\psi - d]^+ = [\psi - d + |\psi - d|]/2$. $[\psi - d]^+$ is the positive part of $\psi - d$, therefore $[\psi - d]^+ \geq \psi - d$. As $\zeta = v - d = 0$ on $\partial\Omega$, $\psi - d \leq 0$ on $\partial\Omega$ and so $[\psi - d]^+ = 0$ on $\partial\Omega$. $\mathcal{M}_1$ is nonempty as it contains $[\psi - d]^+$. We shall now show that the space $\mathcal{M}_1$ is closed (complete in the norm defined through Eq. 7.2.88). Let $\{v_n\}$ be a convergent sequence in $\mathcal{M}_1$ whose limit is $\hat{v}$. It is to be proved that $\hat{v} \in \mathcal{M}_1$. The sequence $\{v_n\}$ can be thought of as a subsequence of a sequence $\left\{\hat{v}_n\right\}$ in $H_0^1(\Omega)$ and $\{v_n\}$ converges to $\hat{v} \in H_0^1(\Omega)$ in the a.e. sense. Since $v_n \in \mathcal{M}_1$, $v_n \geq \psi$ for $\forall n$. $\{v_n\}$ is a subsequence of $\left\{\hat{v}_n\right\}$ therefore $\hat{v} \geq \psi$ and so $\hat{v} \in \mathcal{M}_1$. In the space $H_0^1(\Omega)$, scalar product can be defined as

$$(w, \zeta) = a(w, \zeta) = \int \nabla w \cdot \nabla \zeta dx. \tag{7.2.96}$$

We have $a(w, w) = (w, w) \geq \|w\|^2$ and therefore $a(w, \zeta)$ is coercive and continuous on $H_0^1(\Omega)$ or coercive on $\mathcal{M}_1 - \mathcal{M}_1$. The conditions of Theorem 7.2.4 are satisfied and a unique solution w_0 of Eq. (7.2.92) exists in $\mathcal{M}_1$. The unique solution u_0 of Eq. (7.2.91) is then given by $u_0 = w_0 + d$.

Till now only Dirichlet problem has been considered for the elliptic operator of the form ∇^2. We now consider a problem of the form

$$-\nabla^2 u + \lambda u = q, \quad \text{a.e. in } \Omega, \quad u \in H^2(\Omega) \cap C(\Omega), \quad q \in L^2(\Omega), \tag{7.2.97}$$

$$u = d, \quad \text{a.e. on } \partial\Omega, \quad d \in L^2(\Omega). \tag{7.2.98}$$

The differential equation (7.2.97) can be obtained as an *Euler equation* of an appropriate minimization problem, for example, in Eq. (7.2.60) take $a(u, v)$ given by Eq. (7.2.100), g defined by Eq. (7.2.69) and $v \in H^2(\Omega) \cap C(\Omega)$. If an obstacle is introduced such as $v \geq \psi$ then the obstacle problem for Eqs (7.2.97)–(7.2.98) can be formulated as a variational inequality of the form (see the derivation of the complementarity problem in Eq. 7.2.37 obtained from the minimization problem) given below in Eq. (7.2.99)

$$a(w, \zeta - w) \geq \langle q, \zeta - w \rangle, \quad \forall \zeta \in H_0^1(\Omega), \quad w \in H_0^1(\Omega), \tag{7.2.99}$$

where

$$a(w, \zeta) = \int_\Omega \nabla w \cdot \nabla \zeta \, dx + \lambda \int_\Omega w \zeta \, dx, \quad \forall w, \quad \zeta \in H_0^1(\Omega). \tag{7.2.100}$$

It can be proved that if $\lambda > -1/\beta$, where β is the same as in the Poincare's inequality (7.2.90) then $a(w, \zeta)$ is coercive on $H_0^1(\Omega)$ (cf. [22]).

If the boundary condition is of the Neumann type such as

$$\frac{\partial u}{\partial n} = r, \quad \text{a.e. on } \partial\Omega, \quad r \in L^2(\Omega), \tag{7.2.101}$$

where $\vec{n}$ is the unit outward normal to $\partial\Omega$, then the obstacle problem results in a variational inequality of the form

$$a(u, v - u) \geq \int_\Omega q(v - u) dx + \int_{\partial\Omega} r(v - u) dx, \quad \forall v \in H^1(\Omega), \quad u \in H^1(\Omega). \tag{7.2.102}$$

In this case $a(u, v)$ has the same form as in Eq. (7.2.100) but $u, v \in H^1(\Omega)$ (boundary condition is different). By choosing $\lambda > 0$ and $\nu = \min(1, \lambda)$, it can be shown that (cf. [22])

$$a(v, v) \geq \nu \, \|v\|^2_{H^1(\Omega)}, \tag{7.2.103}$$

where the norm in $H^1(\Omega)$ is defined by Eq. (7.2.86). Note that the Neumann boundary condition (7.2.101) has already been incorporated in Eq. (7.2.102). The test function need not satisfy the Neumann boundary condition as it occurs naturally in the inequality. When $-\nabla^2 u + \lambda u \leq q$ is multiplied by $(v - u)$ and integrated over Ω and integration by parts is done we get Eq. (7.2.102). The integration by parts will give an integral over $\partial\Omega$, and the Neumann boundary condition is incorporated. If $\lambda = 0$, the form $a(u, v)$ is not coercive, and in this case the problem with Neumann condition does not always have a solution; and when it does, it is not unique.

The boundary conditions could be of mixed type, for example, let $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$ and

$$-\nabla^2 u + \lambda = q, \quad \text{a.e. in } \Omega, \tag{7.2.104}$$

$$u = d, \quad \text{a.e. on } \partial\Omega_1, \tag{7.2.105}$$

$$\frac{\partial u}{\partial n} = r, \quad \text{a.e. on } \partial\Omega_2. \tag{7.2.106}$$

Then the variational inequality formulation is still given by Eq. (7.2.102) with $\partial\Omega$ in the integral on the r.h.s. of Eq. (7.2.101) replaced by $\partial\Omega_2$. Since $a(u, v)$ remains unchanged, in this case also, Eq. (7.2.103) holds good and $a(u, v)$ is coercive. The test functions have to satisfy the boundary condition (7.2.105).

Operators such as the one considered in Eq. (7.2.13) or even more general operators can be considered and the existence and uniqueness of solutions of the variational inequalities so obtained can be discussed under suitable assumptions (cf. [22]). It is not always possible to formulate a given physical problem as a variational inequality problem or obtain a variational equation for the problem.

Regularity of the solution is an important aspect of the study of variational inequalities. Some results on the regularity of solutions will be given in the context of parabolic variational inequalities. With the help of the obstacle problem of the string discussed earlier in Problem 1.1.12 (see Section 1.1), some other aspects of variational inequalities such as the choice of appropriate spaces in which the solutions are sought, and the restrictions on the obstacle so that the set $\mathcal{M}$ in Eq. (7.2.3) is nonempty, will be discussed in the next section. In general the space of unknown functions should be large enough so that the existence of solutions can be discussed, but small enough so that a unique solution can be obtained. In physical problems, smoothness of solutions cannot be ignored. In principle, the data spaces should be general so as to accommodate various types of data but the continuous dependence of the solution on the data is required for well-posedness.

7.2.5 Equivalence of Different Inequality Formulations of an Obstacle Problem of the String

We restate briefly an obstacle problem of the string as a problem of finding a continuous function $v(x)$ which minimizes the energy functional $f(v)$, where

$$f(v) = \frac{1}{2}\int_0^b (v')^2 dx, \quad v \in \mathcal{M}_2, \tag{7.2.107}$$

$$\begin{aligned}\mathcal{M}_2 &= \{v \in H^1(0,b) : v(0) = v(b) = 0, v(x) \geq \psi(x), \forall x \in (0,b)\}\\ &= \left\{v \in H_0^1(0,b) : v(x) \geq \psi(x), \forall x \in (0,b)\right\}.\end{aligned} \tag{7.2.108}$$

Since we are dealing with a physical problem, v and ψ should be continuous functions. It is clear from Fig. 7.2.1 that the first-order derivatives of $v(x)$ and $\psi(x)$ need not be continuous but they should belong to $L^2(\Omega)$ which is also suggested by the integral in Eq. (7.2.107).

In Fig. 7.2.2, although $v(x) \geq \psi(x)$, v does not satisfy the boundary conditions and so the set $\mathcal{M}_2$ is empty. For $\mathcal{M}_2$ to be nonempty, ψ must satisfy one of the following two conditions:

$$(\psi 1): \quad \psi \in C^0(\Omega), \quad \psi(0+) < 0 \quad \text{and} \quad \psi(b-) < 0. \tag{7.2.109}$$

$$(\psi 2): \quad \psi \in H^1(\Omega), \quad \psi(0) \leq 0 \quad \text{and} \quad \psi(b) \leq 0. \tag{7.2.110}$$

Note that by increasing the smoothness of ψ, the smoothness of the solution is not increased. For example, if ψ is taken as a parabola then $\psi \in C^\infty(\Omega)$, $\Omega = (0, b)$, but still the second derivative of the solution $u'' \notin C^2(0, b)$ if the parabola does not pass through the points 0 and b. In showing the equivalence of different forms of variational inequality formulations given

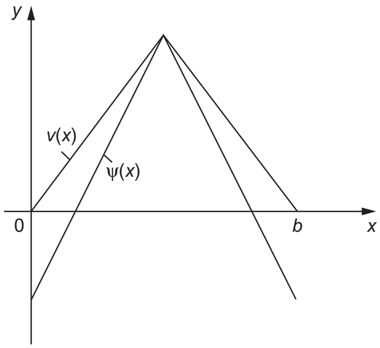

Fig. 7.2.1 Differentiability of $v(x)$ is not required.

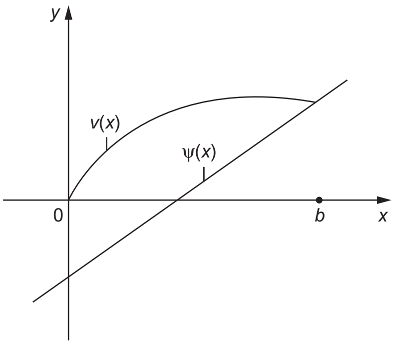

Fig. 7.2.2 The set $\mathcal{M}_2$ is empty.

below for the obstacle problem, the treatment is not strictly rigorous. A rigorous treatment would require that the condition $u'' \leq 0$ is considered in the sense of distributions in the formulation (I) given below. If $u \in H^1(0, b)$, then it is not necessary that u'' belongs to $L^1(0, b)$. It will be assumed for the present discussion that $u, v \in H^2(\Omega) \cap C(\Omega)$ and u'' belongs to $L^1(\Omega)$. The obstacle ψ satisfies one of the two conditions ($\psi 1$) or ($\psi 2$). For clarity, the three different formulations are restated below.

(I) Find $u \in \mathcal{M}_2$ ($\mathcal{M}_2$ as in Eq. 7.2.108) such that $u(0) = u(b) = 0$, $u'' \leq 0$, $u \geq \psi$ and $(u - \psi)u'' = 0$.

(II) Find u such that

$$\int_0^b u'(x)(v'(x) - u'(x))dx \geq 0, \quad \forall v \in \mathcal{M}_2, \;\; u \in \mathcal{M}_2. \tag{7.2.111}$$

(III) Minimize

$$\int_\Omega (\partial v/\partial x)^2 dx, \quad \forall v \in \mathcal{M}_2.$$

If $u', v' \in L^2(\Omega)$, then u'^2 and $u'v' \in L^1(\Omega)$. Further let $u'' \in L^1(\Omega)$. It will be assumed in the following that all the equations hold good in the a.e. sense.

Equivalence of Formulations (I), (II) and (III)

(III) $\Rightarrow$ (II). Assume that there exists a $u \in \mathcal{M}_2$ such that

$$\int_0^b u'^2 dx \leq \int_0^b w'^2 dx, \quad \forall w \in \mathcal{M}_2. \tag{7.2.112}$$

Since $\mathcal{M}_2$ is a convex set, if $u, v \in \mathcal{M}_2$ then $w = u + \lambda(v - u) \in \mathcal{M}_2$, $0 < \lambda < 1$. From Eq. (7.2.112), we get

$$\int_0^b u'^2 dx \le \int_0^b \{u'^2 + \lambda^2(v' - u')^2 + 2\lambda u'(v' - u')\}dx \tag{7.2.113}$$

or

$$0 \le 2\lambda \int_0^b u'(v' - u')dx + O(\lambda^2). \tag{7.2.114}$$

For Eq. (7.2.114) to be true for $\forall\lambda > 0$, we should have

$$\int_0^b u'(v' - u')dx \ge 0.$$

(II) $\Rightarrow$ (III). If $a(u, v)$ is defined by Eq. (7.2.24), then $a(u - v, u - v) \ge 0$ implies

$$\int_0^b 2u'(u' - v')dx - \int_0^b u'^2 dx + \int_0^b v'^2 dx \ge 0 \tag{7.2.115}$$

or

$$\int_0^b u'^2 dx \le \int_0^b v'^2 dx - 2\int_0^b u'(v' - u')dx \le \int_0^b v'^2 dx. \tag{7.2.116}$$

As Eq. (7.2.111) holds, the relation $\int_0^b u'(v' - u')dx \ge 0$ can be used in Eq. (7.2.116). The inequality (7.2.116) implies that u gives the minimum in (III).

(I) $\Rightarrow$ (II).

$$\begin{aligned}\int_0^b u'(v' - u')dx &= u'(v - u)_0^b - \int_0^b u''(v - u)dx \\ &= -\int_0^b u''(v - u)dx.\end{aligned} \tag{7.2.117}$$

Let $\Omega = (0, b) = I_1 \cup I_2$, where

$$I_1 = \{x \in \Omega : u(x) = \psi(x)\}; \quad u'' \le 0 \ \text{ in } I_1 \text{ if formulation (I) holds,} \tag{7.2.118}$$

$$I_2 = \{x \in \Omega : u(x) > \psi(x)\}; \quad u'' = 0 \ \text{ in } I_2 \text{ if formulation (I) holds.} \tag{7.2.119}$$

Using Eqs (7.2.118), (7.2.119) in Eq. (7.2.117), we get

$$\int_0^b u'(v' - u')dx = -\int_{I_1} u''(v - \psi(x))dx \ge 0. \tag{7.2.120}$$

The integration by parts in Eq. (7.2.117) requires that u'' should belong to $L^1(\Omega)$. Therefore (I) $\Rightarrow$ (II) provided the solution u of (I) is such that $u'' \in L^1(\Omega)$. From Fig. 7.2.1 it is clear that the solution may exist even if $u'' \notin L^1(\Omega)$. It has been shown in [183] that if ψ be such that ψ'' is a *Radon measure* and its positive part can be represented by a function belonging to $L^1(0, b)$ then the condition $u'' \notin L^1(\Omega)$ is not required.

(II) ⇒ (I). Consider Eq. (7.2.117) and the case $u \geq \psi$. Let $v = u + \zeta$, $\zeta \geq 0$, $\zeta \in C_0^\infty(\Omega)$ and $v \in \mathcal{M}_2$. If (II) holds, then

$$-\int_0^b u''\zeta\,dx \geq 0. \tag{7.2.121}$$

The relation (7.2.21) implies $u'' \leq 0$ in Ω.

Next consider the case $u > \psi$. Let $v = u + \varepsilon\zeta$, $\zeta \in C_0^\infty(I_2)$. Extend ζ to Ω trivially, i.e. $\zeta \equiv 0$ in I_1. If ε is sufficiently small, v will be greater than ψ and we have

$$\int_0^b u'(v' - u')dx = -\int_0^b u''(v-u)dx = -\varepsilon\int_{I_2} u''\zeta\,dx \geq 0. \tag{7.2.122}$$

Since Eq. (7.2.122) holds for all sufficiently small ε which could be positive or negative, it can be concluded that

$$\int_{I_2} u''\zeta\,dx = 0. \tag{7.2.123}$$

As ζ is arbitrary, $u'' = 0$ on I_2.

The equivalence of different variational inequality formulations of problems involving more general elliptic operators can also be discussed. For example, it can be proved that the variational inequality (7.2.1) with $a(u, v)$ defined by Eq. (7.2.14) is equivalent to the variational inequality (7.2.37) in which the operator A is given by Eq. (7.2.13) provided u, the coefficients in the elliptic operator and the obstacle ψ satisfy some conditions [22].

7.3 THE PARABOLIC VARIATIONAL INEQUALITY

7.3.1 Formulation in Appropriate Spaces

Let Ω be an open bounded domain in R^n and $\Omega_{t_*} = \Omega \times \{0 < t < t_*\}$. One of the problems of mathematical physics is to solve the initial-boundary value problem stated below in Eqs (7.3.1)–(7.3.3)

$$T_t + AT = f, \quad \text{in } \Omega_{t_*} \tag{7.3.1}$$

$$T = g, \quad \text{on } \partial\Omega_{t_*}, \tag{7.3.2}$$

$$T = T_0, \quad \text{in } \Omega \times \{0\}, \tag{7.3.3}$$

where

$$A = -\sum_{i,J=1}^{n} a_{iJ}(x,t)\frac{\partial^2 T(x,t)}{\partial x_i \partial x_J} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial T(x,t)}{\partial x_i} + d(x,t)T(x,t), \tag{7.3.4}$$

$$\partial\Omega_{t_*} = \partial\Omega \times (0, t_*). \tag{7.3.5}$$

$\partial\Omega_{t_*} \cup \Omega \times \{0\}$ is called a *parabolic boundary*. The weak form of problem (7.3.1)–(7.3.4) can be stated in terms of finding a function $T(x,t) \in H^1(\Omega_{t_*})$ satisfying the equation

$$(T_t, v - T) + a(t; T, v - T) = (f, v - T), \tag{7.3.6}$$

for a.a. $t \in (0, t_*)$ and all $v \in H^1(\Omega_{t_*})$. v satisfies the given initial and boundary conditions,

$$a(t; T, v) = \int_\Omega \left(\sum_{i,J} a_{iJ} \frac{\partial T}{\partial x_i} \frac{\partial v}{\partial x_J} + \sum_i \hat{b}_i \frac{\partial T}{\partial x_i} v + dTv \right) dx, \tag{7.3.7}$$

$$\hat{b}_i = b_i + \sum \frac{\partial a_{iJ}}{\partial x_J}, \tag{7.3.8}$$

$$(v, w) = \int_\Omega vwdx. \tag{7.3.9}$$

It has been assumed in Eq. (7.3.4) that the data, the temperature derivatives with respect to x and t, and coefficients a_{iJ} are sufficiently smooth functions (cf. [1] and Proposition 7.3.1). There does not exist a minimization principle for parabolic problems. Often Biot's variational statement [184] is taken as a variational principle. Every elliptic problem cannot be generalized to a transient problem, for example, the obstacle problem for the string (Problem 1.1.12) does not have a time-dependent generalization. If an obstacle is introduced in formulations (7.3.1)–(7.3.4), then we may obtain an inequality. Parabolic variational inequality for the problem in Eqs (7.3.1)–(7.3.4) can be defined in any of the following two equivalent forms.

(I) For a.a. $t \in (0, t_*)$, let

$$E(t) = \{v \in H^1(\Omega); v \geq \psi(x, t) \text{ a.e. in } \Omega\}. \tag{7.3.10}$$

$E(t)$ is a closed convex subset of $H^1(\Omega)$. To make $E(t)$ a nonempty set, we shall take $\psi(x, t) \leq g(x, t)$ a.e. on $\partial\Omega$ for $t \in (0, t_*)$. A function $T(x, t) \in L^2(0, t_*; H^1(\Omega)), T(t) \in E(t)$, for a.a. $t \in (0, t_*)$, i.e. for each fixed t, $T(x, t) = T(x) \in H^1(\Omega)$ is called a solution of the parabolic variational inequality (7.3.11) if

$$(T_t, v - T) + a(t; T, v - T) \geq (f, v - T), \quad \text{for a.a. } t \in (0, t_*), \tag{7.3.11}$$

is satisfied for $\forall v \in L^2(0, t_*; H^1(\Omega))$ with $v(t) \in E(t)$ for a.a. $t \in (0, t_*)$, $v(x, 0) = T_0$ and v satisfying Eq. (7.3.2). It is required that $\partial T/\partial t \in L^2(0, t_*; L^2(\Omega))$. $\psi(x, t)$ is called an obstacle. $f(x, t)$, $\psi(x, t)$ and $\partial\Omega$ should satisfy some smoothness conditions. Some of these conditions will be discussed for some specific problems to be discussed later.

(II) The second variational inequality formulation is the integrated form of Eq. (7.3.11), i.e. in the place of Eq. (7.3.11), we have

$$\int_0^{t_*} \left\{ \left(\frac{\partial T}{\partial t}, v - T \right) + a(t; T, v - T) - (f, v - T) \right\} dt \geq 0. \tag{7.3.12}$$

Let

$$E = \{v \in L^2(0, t_*; H^1(\Omega)); v \geq \psi(x, t) \text{ a.e. in } \Omega_{t_*}, v(x, 0) = T_0\}. \tag{7.3.13}$$

$v(x, t)$ satisfies the boundary condition (7.3.2). If there exists a function $T(x, t) \in E$ with $\partial T/\partial t \in L^2(0, t_*; L^2(\Omega))$ such that Eq. (7.3.12) is satisfied for all $v \in E$, then $T(x, t)$ is called a solution of the variational inequality (7.3.12).

Any solution of Eq. (7.3.11) will satisfy Eq. (7.3.12) and so the formulation (I) implies formulation (II). We assume that a solution $T(x,t)$ of Eq. (7.3.12) exists. For some $\varepsilon > 0$ such that $(t_0 - \varepsilon, t_0 + \varepsilon) \in (0, t_*)$, consider the following function:

$$\left.\begin{aligned} v &= T(x,t), \quad t \notin (t_0 - \varepsilon, t_0 + \varepsilon), \\ &= \hat{v}, \qquad\quad\; t \in (t_0 - \varepsilon, t_0 + \varepsilon). \end{aligned}\right\} \tag{7.3.14}$$

If v defined in Eq. (7.3.14) is used in Eq. (7.3.12), then

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \left\{ \left(\frac{\partial T}{\partial t}, \hat{v} - T \right) + a(t; T, \hat{v} - T) - (f, \hat{v} - T) \right\} dt \geq 0. \tag{7.3.15}$$

Since the interval $(t_0 - \varepsilon, t_0 + \varepsilon) \in (0, t_*)$ is arbitrary, we obtain Eq. (7.3.11). The *complementarity problem* corresponding to the parabolic variational inequality (7.3.11) in which the operator A is given by Eq. (7.3.4) consists of finding $T(x,t)$ satisfying the following system of equations:

$$\left.\begin{aligned} T_t + AT &\geq f, \\ T &\geq \psi(x,t), \\ (T_t + AT - f)(T - \psi(x,t)) &= 0, \end{aligned}\right\} \quad \text{in } \Omega_{t_*}. \tag{7.3.16}$$

The relations in Eq. (7.3.16) are satisfied in the a.e. sense and $T \in L^2(0, t_*; H^2(\Omega))$. The equivalence of the variational inequality (7.3.12) and the complementarity problem (7.3.16) can be established. The proof is similar to the one discussed in the elliptic case (see Section 7.2.3). Consider first the case $T \geq \psi$, where T is the solution of Eq. (7.3.12). If $\phi \in C_0^\infty(\Omega_{t_*})$, $\phi \geq 0$ and $\phi(x,0) = 0$ then $v = T + \phi$ belongs to the set E

$$\int_{\Omega_{t_*}} (T_t + AT - f)\phi dx dt = \int_0^{t_*} \left\{ \left(\frac{\partial T}{\partial t}, v - T \right) + a(t; T, v - T) - (f, v - T) \right\} dt \geq 0. \tag{7.3.17}$$

Since ϕ is arbitrary, we conclude that $\partial T/\partial t + AT - f \geq 0$ in Ω_{t_*}. Next, we consider the case $T > \psi$ and use the procedure followed in the elliptic case (Eqs 7.2.32–7.2.36) with a difference that now $v = T - \varepsilon\phi$, $|\varepsilon| > 0$, ε very small, $\phi \in C_0^\infty(\Omega_{t_*})$ and $\phi(x,0) = 0$. It can be seen that $\partial T/\partial t + AT - f = 0$. Combining the two cases $T \geq \psi$ and $T > \psi$, Eq. (7.3.16) is obtained. To prove that Eq. (7.3.16) implies Eq. (7.3.12), consider the first integral on the l.h.s. of Eq. (7.3.17). On replacing ϕ by $(v - T)$, writing $v - T = v - \psi - (T - \psi)$ and using the third relation is Eq. (7.3.16) and the fact $v \geq \psi$, we obtain

$$\int_{\Omega_{t_*}} (T_t + AT - f)(v - T) dx dt \geq 0, \quad \forall v \in E. \tag{7.3.18}$$

Since Eq. (7.3.18) holds for all $v \in E$, T is the solution of Eq. (7.3.12).

In the elliptic case, the coercivity of the bilinear form played an important role in proving well-posedness and uniqueness of the solution. If the transformation

$$\hat{T} = e^{-\alpha t} T, \quad \alpha > 0 \tag{7.3.19}$$

is used then $T_t + AT = f$ is transformed into

$$\hat{T}_t + (A+\alpha)\hat{T} = e^{-\alpha t} f \tag{7.3.20}$$

and $a(t; T, v)$ is transformed into

$$\hat{a}(t; \hat{T}, \hat{v}) = a(t; \hat{T}, \hat{v}) + \alpha(\hat{T}, \hat{v}). \tag{7.3.21}$$

If α is sufficiently large, then irrespective of the fact that $a(t; T, v)$ is coercive or not $\hat{a}(t; \hat{T}, \hat{v})$ is coercive. Without any loss of generality it can be assumed from the start that $a(t; T, v)$ is coercive and if for a.a. t, $T \in H^1(\Omega)$ then

$$a(t; T, T) \geq \lambda \|T\|_{H^1_\Omega} \geq \lambda \int_\Omega (|\nabla T|^2 + T^2)dx, \quad \lambda > 0. \tag{7.3.22}$$

Uniqueness and stability of the solution of Eq. (7.3.12) can be proved if coercivity condition is imposed on $a(t; T, v)$ in Eq. (7.3.12). Let T_1 and T_2 be two different solutions of Eq. (7.3.12) satisfying the same initial and boundary conditions. Taking $v = T_2$ in Eq. (7.3.12) and $T = T_1$, we obtain

$$\int_0^{t_*} \left\{ \left(\frac{\partial T_1}{\partial t}, T_2 - T_1 \right) + a(t; T_1, T_2 - T_1) - (f, T_2 - T_1) \right\} dt \geq 0. \tag{7.3.23}$$

Similarly by taking $T = T_2$ and $v = T_1$, a second relation is obtained. Adding the two relations so obtained, it is easy to obtain the relation

$$\int_0^{t_*} \left\{ \frac{1}{2} \frac{d}{dt}(T_1 - T_2, T_1 - T_2) + a(t; T_1 - T_2, T_1 - T_2) \right\} dt \leq 0. \tag{7.3.24}$$

On integrating the first term under the integral in Eq. (7.3.24), remembering that $T_1 - T_2 = 0$ at $t = 0$ and using the coercivity condition (7.3.22), we obtain

$$\frac{1}{2} \|T_1 - T_2\|^2_{t=t_*} + \lambda \int_0^{t_*} \|T_1 - T_2\|^2_{H^1(\Omega)} dt \leq 0. \tag{7.3.25}$$

Since both the terms in Eq. (7.3.25) are positive, each should be zero and this implies $T_1 = T_2$. The stability of the solution can be proved by following the proof given earlier in the elliptic case (cf. Section 7.2.3 and Eq. 7.2.74). For proving the existence of the solution *penalty method* is generally used. The conditions under which a unique solution of the variational inequality (7.3.11) exists for the parabolic operator given in Eq. (7.3.4) have been discussed in [1] and are briefly given in Proposition 7.3.1.

Proposition 7.3.1. *Assume that the following conditions are satisfied.*

(1) The operator $T_t + AT$, where A is given by Eq. (7.3.4) with coefficients defined in Ω_{t_*} is of parabolic type at (x, t), i.e.

$$\sum_{i,J=1}^{n} a_{iJ}(x,t)\xi_i\xi_J \geq \lambda\,|\xi|^2, \quad \forall \xi \in R^n, \;\; \lambda > 0. \tag{7.3.26}$$

It is said to be uniformly parabolic in Ω_{t_*} *if Eq. (7.3.26) holds for all* (x, t) *in* Ω_{t_*} *and* λ *is independent of* (x, t).

(2) $\sum \|a_{iJ}\|_\alpha + \sum \|b_i\|_\alpha + \|d\|_\alpha \leq M$ *(Constant),* $\quad 0 < \alpha < 1,$ (7.3.27)

where

$$\|T(x,t)\|_\alpha = \|T(x,t)\|_0 + H_\alpha(T(x,t)), \tag{7.3.28}$$

$$\|T\|_0 = \sup_{(x,t)\in\Omega_{t_*}} |T(x,t)|, \tag{7.3.29}$$

$$H_\alpha(u) = \sup_{(x,t),(x',t')} \frac{|T(x,t) - T(x',t')|}{(p_d((x,t),(x',t')))^\alpha}, \tag{7.3.30}$$

$$p_d((x,t),(x',t')) = |x - x'| + |t - t'|^{1/2}. \tag{7.3.31}$$

$p_d((x,t),(x',t'))$ *is called parabolic distance.*

(3) $\partial\Omega$ *is in* $C^{2+\alpha}$ *(cf. Appendix D).*

(4) $d(x,t) \geq 0$ *and* f, g, $D_x g$, $D_x^2 g$, $D_t g$ *belong to* $C^\alpha(\bar{\Omega}_{t_*})$. $C^\alpha(\bar{\Omega}_{t_*})$ *is the space of functions* $u(x,t)$ *which are Hölder continuous with exponent* α, $H_\alpha(u) < \infty$.

(5) $|D_x a_{iJ}| \leq M_1$ *(constant) so that* $a(t; u, v)$ *can be defined.* (7.3.32)

If the above conditions are satisfied, then there exists a unique solution of the obstacle problem (7.3.11) (Eq. 7.3.12) and

$$D_x T,\ D_x^2 T,\ D_t T \text{ belongs to } L^p(\Omega_{t_*}), \quad \forall p, \quad 1 < p < \infty. \tag{7.3.33}$$

7.4 SOME VARIATIONAL INEQUALITY FORMULATIONS OF CLASSICAL STEFAN PROBLEMS

7.4.1 One-Phase Stefan Problems

The variational inequality formulation is a fixed domain formulation and therefore weak enthalpy formulations of the Stefan problems are more suitable for variational inequality formulations than the classical formulations of Stefan problems. Even in the classical Stefan problems, variational inequality formulations of one-phase Stefan problems can be handled easily than the two-phase Stefan problems. This is because a one-phase Stefan problem can be formulated in terms of the *freezing index* which together with its gradient is continuous throughout the region. Also the constraint that the temperature is greater than or equal to (less than or equal to) the melting temperature holds throughout the phase-change region. We describe below some variational inequality formulations of one-phase Stefan problems. Some results concerning analysis of solutions of these problems are also given.

Consider an open bounded domain Ω in R^3. At time $t = 0$, Ω is filled with ice cold water at temperature $T = 0$ where 0 is the dimensionless freezing temperature of water. Let $\partial\Omega = \partial\Omega_f(t) \cup \partial\Omega_e(t) \cup \partial\Omega_r(t)$ and on these three disjoint portions of the boundary, three different types of boundary conditions are prescribed. On the portion $\partial\Omega_f$ cooling is done so that ice formation takes place. For $t > 0$, $\Omega = \Omega_0(t) \cup \Omega_1(t) \cup \Gamma(t)$. $\Omega_0(t)$ is the ice region at time t, $0 \leq t \leq t_*$, $\Omega_1(t)$ contains water at temperature $T = 0$ and $\Gamma(t)$ is the ice–water phase-change boundary whose equation is taken as $t = \phi(x)$, $x \in \Omega$. $\Omega_0(t)$ and $\Omega_1(t)$ are disjoint regions. The mathematical formulation of this problem is as follows:

$$\frac{\partial T}{\partial t} = \nabla \cdot (k\nabla T), \quad \text{in } \Omega_0(t), \; 0 < t < t_*, \tag{7.4.1}$$

$$T = 0, \quad \text{in } \Omega_1(t), \; 0 < t < t_*, \tag{7.4.2}$$

$$T(x,t) = g(t) < 0, \quad \text{on } \partial\Omega_f, \; 0 < t < t_*, \tag{7.4.3}$$

$$T(x,t) = 0, \quad \text{on } \partial\Omega_r, \; 0 < t < t_*, \tag{7.4.4}$$

$$\alpha T = k\frac{\partial T}{\partial n}, \quad \text{on } \partial\Omega_e, \; 0 < t < t_*, \tag{7.4.5}$$

$$k\nabla T \cdot \nabla\phi(x) = l_c \quad \text{and} \quad T(x,\phi(x)) = 0, \tag{7.4.6}$$

$$T(x,0) = 0, \quad \text{in } \Omega. \tag{7.4.7}$$

Here $\vec{n}$ is the unit outward normal on $\partial\Omega_e$, $l_c = l/C > 0$, k is the thermal diffusivity and α is the constant for heat radiation. It has been assumed that $\Gamma(t)$ is monotone and ice is progressing on water.

Although the constraint $T \leq 0$ holds throughout Ω, the variational inequality formulation of this problem presents two difficulties. The above formulation is not a continuum model (because of the free boundary conditions) and secondly ∇T is discontinuous across $t = \phi(x)$ and so integration by parts which is required to obtain the bilinear form cannot be done in Ω unless we write the integral as the union of two integrals (see Eq. 7.3.7).

Duvait's Transformation

Using *Duvait's transformation* [185], temperature $T(x,t)$ is transformed into another dependent variable $u(x,t)$, called *freezing index* [186], such that u and ∇u are continuous throughout Ω. $u(x,t)$ is defined as

$$u(x,t) = \int_{\phi(x)}^{t} T(x,\tau)d\tau, \quad \text{in } \Omega_0(t); \quad u(x,t) = 0, \quad \text{in } \Omega_1. \tag{7.4.8}$$

Note that $u(x,t)$ is continuous in $\Omega \times (0,t_*)$.

From Eq. (7.4.8), we can easily obtain the equation

$$\nabla u(x,t) = \int_{\phi(x)}^{t} \nabla T(x,\tau)d\tau - T(x,\phi(x)) \cdot \nabla\phi(x), \quad \text{in } \Omega_0$$

or

$$\nabla u(x,t) = \int_{\phi(x)}^{t} \nabla T(x,\tau)d\tau, \quad \text{in } \Omega_0(t), \tag{7.4.9}$$

$$\nabla u(x,t) = 0, \quad \text{in } \Omega_1(t). \tag{7.4.10}$$

Also

$$\nabla \cdot (k\nabla u) = \int_{\phi(x)}^{t} \nabla \cdot (k\nabla T)d\tau - k\nabla(T(x,\phi(x))).\nabla\phi(x)$$
$$= \int_{\phi(x)}^{t} \frac{\partial T}{\partial \tau} d\tau - l_c$$
$$= \frac{\partial}{\partial t}\int_{\phi(x)}^{t} T d\tau - l_c$$
$$= \frac{\partial u}{\partial t} - l_c, \quad \text{in } \Omega_0(t) \tag{7.4.11}$$

or

$$\frac{\partial u}{\partial t} - \nabla \cdot (k\nabla u) - l_c = 0, \quad \text{in } \Omega_0(t). \tag{7.4.12}$$

If $u = 0$, then $\partial u/\partial t - \nabla \cdot (k\nabla u) - l_c = -l_c \le 0$. Therefore

$$\frac{\partial u}{\partial t} - \nabla \cdot (k\nabla u) - l_c \le 0, \quad \text{in } \Omega, \ \ 0 < t < t_*. \tag{7.4.13}$$

The initial and boundary conditions for $u(x,t)$ can be obtained from those given for $T(x,t)$ and the problem for the dependent variable $u(x,t)$ can be stated as follows:

$$\left.\begin{aligned} \partial u/\partial t - \nabla \cdot (k\nabla u) - l_c \le 0, &\quad \text{in } \Omega_{t_*} = \Omega \times (0,t_*),\\ u \le 0, &\quad \text{in } \Omega_{t_*},\\ u(\partial u/\partial t - \nabla \cdot (k\nabla u) - l_c) = 0, &\quad \text{in } \Omega_{t_*}. \end{aligned}\right\} \tag{7.4.14}$$

$$u(x,t) = \int_0^t g(\tau)d\tau = \hat{g}(t) < 0 \quad \text{and} \quad \hat{g}(0) = 0; \quad \text{on } \partial\Omega_f(t), \ \ 0 < t < t_*, \tag{7.4.15}$$

$$u(x,t) = 0, \quad \text{on } \partial\Omega_r(t), \ \ 0 < t < t_*, \tag{7.4.16}$$

$$\alpha u(x,t) = k\frac{\partial u}{\partial n}, \quad \text{on } \partial\Omega_e(t), \ \ 0 < t < t_*, \tag{7.4.17}$$

$$u(x,0) = 0, \quad \text{in } \Omega. \tag{7.4.18}$$

The above equations are to be understood in the a.e. sense.

A variational inequality of the form (7.3.12) will now be obtained for Eqs (7.4.14)–(7.4.18). Define a set D as

$$D = \{v \in L^2(0,t_*;H^1(\Omega)), \dot{v} \in L^2(0,t_*;L^2(\Omega))\},$$

where

$$\begin{aligned} &v \le 0, \quad \text{a.e. in } \Omega_{t_*}, \quad v(x,0) = 0, \quad \text{in } \Omega,\\ &v = 0, \quad \text{on } \partial\Omega_r, \ \ 0 < t < t_*,\\ &v = \hat{g}(t), \quad \text{on } \partial\Omega_f, \ \ 0 < t < t_*. \end{aligned} \tag{7.4.19}$$

The test function $v(x,t)$ need not satisfy Eq. (7.4.17) as it will appear as a natural boundary condition in the formulation.

It can be seen that

$$\begin{aligned}&\{(u_t, v-u) + (k\nabla u, \nabla(v-u)) - (l_c, v-u)\}_\Omega \\ &\quad = (u_t - \nabla\cdot(k\nabla u) - l_c, v-u)_\Omega + (k\nabla u\cdot\vec{n}, v-u)_{\partial\Omega_e}, \quad 0 \le t \le t_*. \end{aligned} \tag{7.4.20}$$

The last term on the r.h.s. of Eq. (7.4.20) arises on doing the integration by parts of the term $(k\nabla u, \nabla(v-u))$. On $\partial\Omega_r$ and $\partial\Omega$, $v-u=0$. In view of the third condition in Eq. (7.4.14), we have

$$(u_t - \nabla\cdot(k\nabla u) - l_c, v-u)_\Omega = (u_t - \nabla\cdot(k\nabla u) - l_c, v)_\Omega \ge 0. \tag{7.4.21}$$

It may be noted that in Ω_{t_*}, $u_t - \nabla\cdot(k\nabla u) - l_c \le 0$ and $v \le 0$. On using Eqs (7.4.21), (7.4.17) in Eq. (7.4.20), it can be written as

$$\{(u_t, v-u) + (k\nabla u, \nabla(v-u)) - (l_c, v-u)\}_\Omega - (\alpha u, v-u)_{\partial\Omega_e} \ge 0, \quad \text{for a.a. } t\in(0,t_*). \tag{7.4.22}$$

In Eq. (7.4.22), the boundary conditions (7.4.16), (7.4.17) have been incorporated. *Neumann boundary conditions* such as in Eq. (7.4.17) are called *natural boundary conditions* because they are automatically taken care in the variational formulation and the test functions are not required to satisfy them. Dirichlet boundary conditions are to be satisfied by the test functions. On integrating Eq. (7.4.22) with respect to time, we get

$$\int_0^{t_*}\left[\{(u_t, v-u) + (k\nabla u, \nabla(v-u)) - (l_c, v-u)\}_\Omega - (\alpha u, v-u)_{\partial\Omega_e}\right] dt \ge 0. \tag{7.4.23}$$

The inequality (7.4.23) is the parabolic variational inequality associated with the one-phase Stefan problem (7.4.1)–(7.4.7). If there exists a function $u \in D$ (D as above) such that Eq. (7.4.23) is satisfied for all $v \in D$, then u is called a solution of the variational inequality (7.4.23). If $u \in L^2(0,t_*;H^1(\Omega))$ and the norm of u is defined as

$$\int_0^{t_*} \|u\|^2_{H^1(\Omega)}\, dt, \tag{7.4.24}$$

then $L^2(0,t_*;H^1(\Omega))$ is a Banach space.

The formulation given in Eqs (7.4.14)–(7.4.18) is also a variational inequality formulation and in this case we are looking for $u \in L^2(0,t_*;H^2(\Omega))$. It has been proved in [187, 188] that if the meas $(\partial\Omega_f) > 0$, then there exists a unique solution $u(x,t)$ of the variational inequality (7.4.23) subject to the initial and boundary conditions (7.4.15)–(7.4.18) such that

$$\dot{u} \in L^2(0,t_*;H^1(\Omega)) \cap L^\infty(0,t_*;L^2(\Omega)) \quad \text{and} \quad \dot{u} - \nabla\cdot(k\nabla u) \in L^2(\Omega_{t_*}). \tag{7.4.25}$$

The numerical solution of the variational inequality

$$(\dot{u}, v-u)_\Omega + (k\nabla u, \nabla(v-u))_\Omega + (\alpha u, v-u)_{\partial\Omega_e} \ge (l_c, v-u)_\Omega, \quad \text{for a.a. } t\in(0,t_*), \tag{7.4.26}$$

has been discussed in [188] by 'Galerkin approximation' in space and discretization by finite difference in time. A variational formulation of a two-phase Stefan problem has been given in [189] and its numerical solution has been obtained by finite-difference methods.

The initial condition (7.4.18) implies that at $t = 0$, Ω is occupied by ice cold water. We shall now consider the problem in which at $t = 0$ ice occupies a region G_0 with temperature $\tilde{g}(x) \leq 0$, $G_0 \subset \Omega$. $\Omega - G_0$ is filled with water at temperature $T = 0$. Let $\Omega_0(t)$ be the region occupied by ice at any time t and $\Omega_1(t)$ be the water region so that $\Omega = \Omega_0(t) \cup \Omega_1(t) \cup \Gamma(t)$, where, Ω_0 and Ω_1 are disjoint regions and $\Gamma(t)$ is the ice–water interface whose equation is given by $t = \phi(x)$. If $x \in G_0$, then $\phi(x) = 0$. The initial and boundary conditions are given by Eqs (7.4.16)–(7.4.18). We assume that ice is progressing on water. The freezing index $u(x,t)$ in this case is defined as

$$u(x,t) = \int_{\phi(x)}^{t} T(x,\tau)d\tau, \quad \text{in } \Omega_0(t) - G_0, \ \ 0 < t < t_*, \tag{7.4.27}$$

$$u(x,t) = \int_{0}^{t} T(x,\tau)d\tau, \quad \text{in } G_0, \ \ 0 < t < t_*, \tag{7.4.28}$$

$$u(x,t) = 0, \quad \text{in } \Omega_1(t), \ \ 0 < t < t_* \quad \text{and} \quad u(x,0) = 0, \quad \text{in } \Omega_1(0). \tag{7.4.29}$$

The above transformation was suggested in [190]. It is easy to check (see Eq. 7.4.11) that

$$\nabla \cdot (k\nabla u) = u_t - l_c, \quad \text{in } \Omega_0(t) - G_0, \ \ 0 < t < t_*, \ \ l_c = l/C, \tag{7.4.30}$$

$$\nabla \cdot (k\nabla u) = u_t - \tilde{g}(x), \quad \text{in } G_0; \quad \text{and} \quad u = 0, \quad \text{in } \Omega_1(t), \ \ 0 < t < t_*. \tag{7.4.31}$$

Define a function $f(x)$ such that

$$\left.\begin{aligned} f(x) = \tilde{g}(x), \quad \tilde{g}(x) \leq 0, \ \ x \in G_0(x), \\ f(x) = l_c > 0, \quad x \in \Omega - G_0. \end{aligned}\right\} \tag{7.4.32}$$

In view of Eqs (7.4.30)–(7.4.32), we have

$$u_t - \nabla \cdot (k\nabla u) - f = 0, \ \ u < 0, \quad \text{in } \Omega_0(t), \ \ 0 < t < t_* \tag{7.4.33}$$

and when $u = 0$, we have

$$u = 0; \quad u_t - \nabla \cdot (k\nabla u) - f = -f = -l_c, \quad x \in \Omega - \Omega_0(t), \ \ 0 < t < t_*. \tag{7.4.34}$$

Combining Eqs (7.4.33), (7.4.34), we get

$$u(u_t - \nabla \cdot (k\nabla u) - f) = 0, \quad \text{in } \Omega, \ \ 0 < t < t_*. \tag{7.4.35}$$

For $(x,t) \in \Omega_{t_*}$,

$$\begin{aligned} (u_t - \nabla \cdot (k\nabla u) - f)(v - u) &= -(u_t - \nabla \cdot (k\nabla u) - f)u + (u_t - \nabla \cdot (k\nabla u) - f)v, \\ &\geq 0, \quad \text{a.e. in } \Omega_{t_*}, \ \ v \leq 0, \quad \text{in } \Omega_{t_*}. \end{aligned} \tag{7.4.36}$$

Therefore the variational inequality problem corresponding to Eqs (7.4.30)–(7.4.32) can be stated as the problem of finding $u \in \Omega$ such that

$$(u_t, v-u)_\Omega + (k\nabla u, \nabla(v-u))_\Omega - (f, v-u)_\Omega - (\alpha u, v-u)_{\partial\Omega_e} \geq 0,$$
$$\text{for a.a. } t \in (0, t_*) \quad \text{and} \quad \forall v \in D. \tag{7.4.37}$$

Here D is the same as defined in the context of problem (7.4.14)–(7.4.18).

By taking $\Omega = \{x : 0 \leq x \leq R_0 < \infty\}$, and $\Omega = \Omega_0(t) \cup \Omega_1(t) \cup \Gamma(t)$, where $\Omega_0(t)$ and $\Omega_1(t)$ are disjoint regions and $\Gamma(t)$ is the phase-change boundary, a one-dimensional one-phase problem has been considered in [178]. $\Omega_0(0) = \{x : 0 \leq x < S_0, 0 < S_0 < R_0\}$ is the region initially occupied by hot water at the temperature $b(x) > 0$. $\Omega_1(0) = \{x : S_0 < x < R_0\}$ is occupied by ice at the melting temperature zero. A variational inequality with $u = \hat{g}(t) > 0$ (sufficiently smooth) prescribed at $x = 0$ has been studied in [178] when water is progressing on ice, i.e. $u \geq 0$ in $\Omega(t_*) = \Omega(0) \times (0, t_*)$. Using the 'penalty method' (cf. [1]), it has been shown that under suitable assumptions there exists a unique solution u to the variational inequality such that

$$u,\ u_x,\ u_t,\ u_{xx} \in L^\infty(\Omega(t_*)) \tag{7.4.38}$$

and $u \geq 0$, $u_t \geq 0$ in $\Omega_0(t_*)$. The region $\Omega_0(t) = \{x : u(x,t) > 0\}$, $0 \leq t \leq t_*$ is expanding continuously with time. Further, the water–ice boundary Γ admits the representation

$$\Gamma : x = \sigma(t), \quad 0 \leq t \leq t_*,$$

where σ is a continuously increasing function of t with $S_0 = \sigma(0) < \sigma(t)$ for $t > 0$. For each $(x_0, t_0) \in \Gamma$, there exists a neighbourhood B_r of (x_0, t_0) such that

$$u_{xx},\ u_{xt} \in C(\bar{\Omega}_0 \cap B_r).$$

The one-phase Stefan problem has been studied by many authors with the help of the variational inequality formulation and references of several such studies can be found in [191]. In the water region, convection can be included in the formulation (cf. Eq. 7.4.43). It is possible to obtain the complementarity problem in a constant-velocity case by applying Baiocchi transformation [192]

$$u(x,t) = \int_0^t T(x,\tau)d\tau, \quad x \in \bar{\Omega} \times (0, t_*) \tag{7.4.39}$$

to Eq. (7.4.43). Even in this simple situation, it is not possible to convert the complementarity problem to a variational inequality formulation due to difficulties arising in the transformation of boundary conditions and the geometry of the domain (cf. [191]). Several problems connected with the one-phase Stefan problem, such as the exterior problem, the continuous casting model and the degenerate case pertaining to a quasi-steady state model have been studied in [191]. In the interior problem, the melting of ice takes place due to a prescribed nonnegative temperature on one part of the boundary of $\Omega \subset R^n$ and the Neumann boundary condition prescribed on the remaining part of the boundary of Ω. Some results of the regularity of the freezing index and its continuous dependence on the data in a strong sense can be found in [191]. The geometry of the exterior problem is similar to that of the expanding core model in the Hele-Shaw problem (cf. Section 6.2.1). The variational inequality formulation of the exterior problem can be obtained in terms of the freezing index by using transformation (7.4.39). As discussed earlier, for large times, cusp-like singularities may develop on the free boundary. A strong geometric assumption about the data in the exterior problem leads to a star-shaped configuration without singular points [190].

A fundamental question which arises concerning the solution of a variational inequality is its relationship with the classical solution of the Stefan problem in which the temperature of the phase-change boundary is the melting temperature and the energy balance condition is satisfied pointwise on it. This question was essentially solved in [193, 194]. For the smoothness of the free boundary and the 'Caffarelli's criterion' on the local existence of the classical solutions, see Refs. [193, 195]. Under some assumptions it has been proved that for each $t > 0$, the n-dimensional Lebesgue measure of the free boundary $t = \phi(x)$ is equal to zero; $\phi(x)$ is a locally Lipschitz function and there exists a neighbourhood G_0 of the point (x, t) on the free boundary where the free boundary can be represented in suitable coordinates by the graph of a C^1 function in the form

$$x_n = S(x_1, x_2, \ldots, x_{n-1}, t), \quad S \in C^1 \tag{7.4.40}$$

and all the second derivatives of u are continuous up to the free boundary.

One-Phase Continuous Casting Model and Its Variational Inequality Formulation

The formulation of the problem and the analysis presented below is not classical. This problem is being discussed here to introduce a continuous casting model and because of some novelty in the expressions of bilinear form and the inner product used in the formulation.

The thermal energy conservation equation in a heat conducting body $\Omega \subset R^3$ is given by

$$\rho \frac{de}{dt} + \nabla \cdot \vec{q} = 0, \quad \text{in } \Omega_{t_*} = \Omega \times (0, t_*), \tag{7.4.41}$$

where e is the specific energy and $\vec{q}$ is the heat flux vector. If the phase change is taking place in Ω at temperature T_m and the latent heat l is released, then e can be expressed as

$$e = CT + lH_v(T - T_m), \tag{7.4.42}$$

where $H_v(T)$ is the *Heaviside function* ($H_v(x) = 0$, if $x \leq 0$ and $H_v(x) = 1$, for $x < 1$). If the body is moving with a velocity $\vec{v}$, then the total derivative with respect to time is to be taken and Eq. (7.4.41) can be written as

$$\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T - \nabla \cdot (\nabla T) = -l \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) H_v(T), \tag{7.4.43}$$

For simplicity, it has been assumed in Eq. (7.4.43) that $C = 1.0$, $\rho = 1.0$, $k = 1.0$, $T_m = 0$ and l is a constant. Eq. (7.4.43) holds in Ω_{t_*} in the distributional sense and for its derivation see [196]. We consider ice–water system in which the temperature of water is zero and ice is progressing on water. If θ represents the difference between the solidification temperature and the actual temperature of ice, then Eq. (7.4.43) can be written as (cf. [196])

$$\frac{\partial \theta}{\partial t} + \vec{v} \cdot \nabla \theta - \nabla^2 \theta = -l \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \chi_{(\theta > 0)}, \quad \text{a.e. in } \Omega_{t_*}, \tag{7.4.44}$$

where χ is the *characteristic function* of the set $\{\theta > 0\}$ ($\chi = 1$ for $\{\theta > 0\}$) which is now ice region and water region is the set $\{\theta \leq 0\}$($\chi = 0$ for $\{\theta \leq 0\}$). If the ice–water sharp boundary is written as $t = \phi(x)$, $x \in R^3$, $x = (x_1, x_2, x_3)$, then the classical Stefan condition and the isotherm condition can be written as

$$\{\nabla\theta \cdot \nabla\phi + l(1 + \vec{v} \cdot \nabla\phi)\}(x, \phi(x)) = 0 \quad \text{and} \quad \theta = 0. \tag{7.4.45}$$

To complete the formulation, the initial and boundary conditions should be added to Eqs (7.4.44), (7.4.45).

We shall now discuss variational inequality formulation of a problem arising in connection with the formation of the continuous ingot by the process of continuous casting. A simple diagram depicting solidification of the melt in continuous casting is shown in Fig. 7.4.1. As soon as a sufficiently stable core is formed, the platform B begins to drop down with the velocity $\vec{v}$ in the direction x_1 and thus it draws out a continuous cast of cooling liquid. For appropriate assumptions under which a simple mathematical model of continuous cast is formulated, see [21]. The portion of the ingot taken into account in the present formulation occupies a cylindrical open bounded domain $\Omega = \Omega_1 \times \Omega_2 \subset R^n$, $n = 2, 3$ where $\Omega_1 = (0, b)$, b is the height of the lateral mould and $\Omega_2 = (0, a)$ if $n = 2$ because of the symmetry, or if $n = 3$, $\Omega_2 \subset R^2$ is a domain with Lipschitz boundary $\partial\Omega_2$ representing half the section of the ingot. Let $\Gamma_0 = \{0\} \times \Omega_2$, $\Gamma_1 = (0, b) \times \partial\Omega_2$ and $\Gamma_2 = \{b\} \times \Omega_2$. A point $x \in \bar{\Omega}$ has the coordinates $x = (x_1, x')$ where $x' = x_2$ if $n = 2$ and $x' = (x_2, x_3)$ if $n = 3$. In the notation $\nabla = (\partial_1, \nabla')$, $\nabla' = \partial_2$ if $n = 2$ and $\nabla' = (\partial_2, \partial_3)$ if $n = 3$. Here ∂_1, ∂_2 and ∂_3 denote the partial derivatives with respect to x_1, x_2 and x_3, respectively. The initial temperature of the melt is given by

$$\theta(x_1, x', 0) = \theta_0(x_1, x') \leq 0, \quad (x_1, x') \in \Omega. \tag{7.4.46}$$

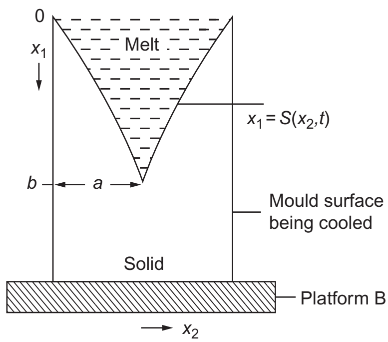

Fig. 7.4.1 Formation of continuous ingot.

Boundary conditions which are prescribed on the lateral, bottom and upper surfaces of the cylinder are:

$$\theta(0, x', t) = 0, \quad \text{on } \Gamma_0, \ t > 0, \tag{7.4.47}$$

$$-\nabla\theta \cdot \vec{n} = \alpha(\theta - \theta_1), \quad \text{on } \Gamma_1, \ \theta_1 \geq 0, \ \alpha > 0, \tag{7.4.48}$$

$$\theta(b, x', t) = \theta_2(x', t), \quad \text{on } \Gamma_2, \ t > 0. \tag{7.4.49}$$

The geometry considered in [191] is different from the geometry shown in Fig. 7.4.1 as only one branch of the free boundary has been considered in [191]. By symmetry consideration, one branch of the free boundary can be considered in Fig. 7.4.1 also, but the boundary condition at $x_2 = 0$ will be different from that at $x_2 = a$ and not as in Eq. (7.4.48).

The boundary Γ_1 is being cooled. $\vec{n}$ is the unit outward normal on Γ_1. The energy equation in $\Omega_{t_*} = \Omega \times (0, t_*)$ is given by

$$\frac{\partial \theta}{\partial t} + v_0 \frac{\partial \theta}{\partial x_1} - \nabla^2 \theta = -l \left(\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x_1} \right) \chi_{(\theta > 0)}, \quad \text{a.e. in } \Omega_{t_*}. \tag{7.4.50}$$

Here $\bar{v} = (v_0, 0, 0)$ and v_0 is the constant casting velocity in the direction of x_1. Eq. (7.4.50) holds in the sense of distributions. Let the free boundary be given by

$$x_1 = S(x', t), \quad x' \in \Omega_2, \quad t > 0, \tag{7.4.51}$$

where $S(x', t)$ is a smooth function, $S : \Omega_2 \times (0, t_*) \to [0, b)$. The conditions at the free boundary can be written as

$$\theta = 0 \quad \text{and} \quad \frac{\partial \theta}{\partial x_1} - \nabla' \theta \cdot \nabla' S = l \left(v_0 - \frac{\partial S}{\partial t} \right). \tag{7.4.52}$$

The second condition in Eq. (7.4.52) can be obtained by using relations of the type (1.4.19), (1.4.21) and remembering that $\partial(x_1 - S)/\partial t = v_0 - \partial S/\partial t$. The region $x_1 > S(x', t)$ is the solid region (see Fig. 7.4.1) and the region $x_1 < S(x', t)$ is the melt region.

Using the maximum principle, it can be proved that under the boundary conditions considered, for $x_1 > S(x', t)$, $\theta > 0$. Only $\partial S/\partial t \le v_0$ is physically admissible. In order to obtain variational inequality formulation of this casting problem, the following transformation has been used in [196] which regularizes Eq. (7.4.50)

$$u(x, t) = v_0 \int_0^t \theta(x_1 + v_0(\tau - t), x', \tau) d\tau, \quad x = (x_1, x') \in \bar{\Omega}, \quad t > 0, \tag{7.4.53}$$

where θ is a nonnegative function and is extended to $-\infty \le x_1 \le S(x', t)$ by taking it to be zero in this extension. If $\theta \in L^1_{loc}(\Omega_{t_*})$, then θ can be obtained in terms of u by the relation (cf. [196])

$$\frac{\partial u}{\partial t} + v_0 \frac{\partial u}{\partial x_1} = v_0 \theta, \quad \text{a.e. in } \Omega_{t_*}. \tag{7.4.54}$$

It can be shown that $u(x, t)$ formally satisfies the following equations:

$$\frac{\partial u}{\partial t} + v_0 \frac{\partial u}{\partial x_1} - \nabla^2 u = f \chi_{(u > 0)}, \quad \text{a.e. in } \Omega_{t_*}, \ u \ge 0, \tag{7.4.55}$$

$$u = 0, \ \text{ on } \Gamma_0 \quad \text{and} \quad u = 0, \ \text{ at } t = 0, \tag{7.4.56}$$

$$\frac{\partial u}{\partial n} = \alpha(g - u), \quad \text{on } \Gamma_1, \tag{7.4.57}$$

$$\frac{\partial u}{\partial t} + v_0 \frac{\partial u}{\partial x_1} = v_0 \theta_2(x', t), \quad \text{on } \Gamma_2, \tag{7.4.58}$$

where f and g in Eqs (7.4.55), (7.4.57) are given by (cf. [196])

$$f(x,t) = v_0\theta_0(x_1 - bt, x')\chi_I - l v_0[1 - \chi_I], \quad x \in \Omega, \ \ t \ge 0, \tag{7.4.59}$$

$$g(x_1, x', t) = \int_{[x_1 - bt]^+}^{x_1} \theta_1\left(\xi, t + \frac{\xi - x_1}{v_0}, x'\right) d\xi, \quad 0 \le x_1 \le b, \ \ t \ge 0. \tag{7.4.60}$$

Here $[x_1 - bt]^+$ is the positive part of $x_1 - bt$ and χ_I is the characteristic function of the set I related to the initial position of the free boundary $\Phi(0)$, and I is given by

$$I = \{(x_1, x', t) \in \Omega_{t_*}; x_1 > S(x', 0) + bt\}, \tag{7.4.61}$$

$$I \subset \Omega^+ = \{(x_1, x', t) \in \Omega_{t_*}; x_1 > S(x', t)\} = \{\theta > 0\},$$

$$\Phi(0) : x_1 = S(x', 0). \tag{7.4.62}$$

Eqs (7.4.54), (7.4.55) hold in the sense of distributions and the reader is referred to [196] for their derivation which is not straightforward.

For $t \ge b/v_0$, $\chi_I = 0$ and $\chi_{(u>0)} \ge \chi_I$ for $t > 0$ because $\partial S/\partial t \le v_0$. To write the variational inequality formulation, we introduce the following notations:

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v + v_0 \int_\Omega v \frac{\partial u}{\partial x} + \alpha \int_{\Gamma_1} uv, \quad u(t), \ \ v(t) \in M_1(t), \tag{7.4.63}$$

$$(u,v) = \int_\Omega uv + \frac{1}{v_0}\int_{\Gamma_2} uv, \quad u(t), \ \ v(t) \in M_1(t), \tag{7.4.64}$$

$$\langle G(t), v\rangle = \int_\Omega f(t)v + \alpha \int_{\Gamma_1} g(t)v + \int_{\Gamma_2} \theta_2(t)v, \quad v(t) \in M_1(t). \tag{7.4.65}$$

For a.a. $t \in (0.t_*)$

$$M(t) = \{v \in H^1(\Omega); v = 0, \ \text{on } \Gamma_0\}, \quad M_1(t) = \{v \in M(t); v \ge 0, \ \text{on } \Omega\}. \tag{7.4.66}$$

Problem (7.4.55)–(7.4.58) has the following variational inequality formulation.

$$\begin{aligned}(\partial u/\partial t, v - u) + a(u, v-u) \ge \langle G(t), v - u\rangle, \quad \text{a.a. } t \in (0, t_*),\\ \forall v(t) \in M_1, \ \ u(t) \in M_1.\end{aligned} \tag{7.4.67}$$

If there exists a $u(t) \in M_1(t)$ such that Eq. (7.4.67) holds for all $v(t) \in M_1(t)$, then $u(t)$ is called a solution of the variational inequality (7.4.67). It may be noted that the boundary conditions (7.4.57), (7.4.58) occur as natural boundary conditions in the formulation and so only the boundary condition on Γ_0 has been imposed on v.

Under suitable assumptions, existence of the unique solution of the variational inequality (7.4.67) has been discussed in [191, 196]. It has been shown that the free boundary has in fact the representation (7.4.51) such that $\partial S/\partial t \le v_0$ and (θ, S) is the classical solution of the continuous casting problem.

Oxygen-Diffusion Problem

In the background of the previous formulations, variational inequality formulation of oxygen-diffusion problem (ODP) discussed earlier in Section 3.3.2 does not present much difficulty. We consider Eqs (3.3.34)–(3.3.38) and take $\Omega = [0,1]$, $\Omega_{t_*} = \Omega \times (0, t_*)$, $\Omega_0(t) = \{x; 0 < x < S(t)\}$, $0 < t < t_*$. In the region $\Omega_0(t)$, the concentration $c > 0$ and in $\Omega \backslash \Omega_0(t)$, $c = 0$. Therefore in Ω, $c \geq 0$. Note that in ODP, the region under consideration is only $\Omega_0(t)$ and no constraint of the form $c \geq 0$ is imposed. The variational inequality problem is studied in the fixed domain Ω and the constraint $c \geq 0$ has been added to study an obstacle problem. It can be seen that

$$\int_0^{t_*}\int_\Omega \left(\frac{\partial c}{\partial t} - \frac{\partial^2 c}{\partial x^2} + 1\right)(v-c)dxdt = \int_0^{t_*}\int_0^1 \frac{\partial c}{\partial t}(v-c)dxdt$$
$$- \int_0^{t_*}\left[(v-c)\frac{\partial c}{\partial x}\right]_0^1 dx + \int_0^{t_*}\int_0^1 \frac{\partial c}{\partial x}\frac{\partial}{\partial x}(v-c)dxdt + \int_0^{t_*}\int_0^1 (v-c)dxdt. \tag{7.4.68}$$

The test function $v(x,t)$ satisfies the same initial and boundary conditions as the concentration $c(x,t)$ and so the second term on the r.h.s. of Eq. (7.4.68) is zero. On rearranging Eq. (7.4.68), we obtain

$$\int_0^{t_*}\int_0^1 \frac{\partial c}{\partial t}(v-c)dxdt + \int_0^{t_*}\int_0^1 \frac{\partial c}{\partial x}\frac{\partial}{\partial x}(v-c)dxdt$$
$$+ \int_0^{t_*}\int_0^1 (v-c)dxdt = \int_0^{t_*}\int_0^1 \left(\frac{\partial c}{\partial t} - \frac{\partial^2 c}{\partial x^2} + 1\right) v dxdt. \tag{7.4.69}$$

We have $c_t - c_{xx} + 1 = 0$ if $c > 0$ and $c_t - c_{xx} + 1$ is equal to 1 if $c = 0$. Therefore $c_t - c_{xx} + 1 \geq 0$ for $c \geq 0$ and $c(c_t - c_{xx} + 1) = 0$ in Ω. As $v \geq 0$, the last term on the r.h.s. of Eq. (7.4.69) is greater than or equal to zero and we get

$$\int_0^{t_*}\left\{(c_t, v-c) + \left(\frac{\partial c}{\partial x}, \frac{\partial}{\partial x}(v-c)\right) + (1, v-c)\right\} dt \geq 0. \tag{7.4.70}$$

This completes the variational inequality formulation of ODP. If there exists a function $c(x,t) \in E$, where $E = \{v \in L^2(0, t_*; H^1(\Omega)); v \geq 0,\ \text{a.e. in } \Omega_{t_*},\ v(x,0) = c(x,0)\}$, and $c(x,t)$ satisfies Eq. (7.4.70) for all $v \in E$ then $c(x,t)$ is called a solution of the variational inequality (7.4.70). In the multidimensional case when $\Omega \subset R^n$, $n \geq 1$, the variational inequality formulation can be obtained following the procedure adopted in Eqs (7.4.68)–(7.4.70). The integration by parts will then require the use of Green's formula.

If oxygen diffusion takes place in a subset of R^n, $n > 1$ then the initial concentration has to be suitably prescribed as in this case it is difficult to obtain it from the solution of a steady-state problem which was possible in the one-dimensional case. The existence, uniqueness and regularity of the solution of Eq. (7.4.70) were studied in [74]. Some regularity results for the one-phase Stefan problems have been reported earlier in this section and for further information, see [197–199].

Is it possible to obtain a one-phase Stefan problem from its variational inequality formulation? Consider the problem of finding a function $u(x,t)$ satisfying the following complementarity problem:

$$u_t - u_{xx} \geq f, \quad u \geq \psi(x,t), \quad \text{and} \quad (u_t - u_{xx} - f)(u - \psi) = 0, \quad \text{a.e. in } \Omega_{t_*},$$

$$\Omega_{t_*} = \Omega \times (0 < t < t_*), \quad \Omega = \{x : 0 \leq x \leq 1\}, \tag{7.4.71}$$

$$u(x,0) = u_0(x) \geq \psi(x,0), \quad x \in \Omega,$$

$$u(0,t) = \hat{u}(t) \geq \psi(0,t), \quad 0 < t < t_*. \tag{7.4.72}$$

One can study the above variational inequality problem by employing standard methods for parabolic equations involving well known a priori L^p-estimates [9]. The solution is generally smooth in the sense that the distributional derivatives u_t, u_x, u_{xx} are in L^p, $p > 1$. u_t and u_{xx} are not continuous in general.

Now suppose that $u(x,t) > \psi(x,t)$ if and only if $0 < x < S(t)$, $0 < t < t_*$, $S(0) = 1$. In this case, we have $u_t - u_{xx} = f$ in $0 < x < S(t)$, $u(S(t),t) = \psi(S(t),t)$ (cf. Section 7.2 and Eqs 7.2.38–7.2.39). If a new function $T(x,t)$ is defined as

$$T(x,t) = \frac{\partial}{\partial t}(u - \psi), \tag{7.4.73}$$

then $T(x,t)$ satisfies the following equations:

$$T_t - T_{xx} = F_t = (f - \psi_t + \psi_{xx})_t, \quad 0 < x < S(t), \quad t > 0, \tag{7.4.74}$$

$$T(x,0) = T_0(x) = u_0''(x) + (f - \psi_t)_{t=0}, \quad 0 \leq x \leq 1, \tag{7.4.75}$$

$$T(0,t) = \hat{T}(t) = \hat{u}_t - \psi_t|_{x=0}, \quad t > 0, \tag{7.4.76}$$

$$T(S(t),t) = 0, \quad t > 0, \quad S(0) = 1, \tag{7.4.77}$$

$$T_x(S(t),t) = \lambda(S(t),t)\dot{S}(t) = (f + \psi_{xx} - \psi_t)|_{x=S(t)}\dot{S}(t), \quad t > 0. \tag{7.4.78}$$

By choosing F, $T_0(x)$, $\hat{T}(t)$ and $\lambda(S(t),t)$ suitably in Eqs (7.4.74)–(7.4.78), a one-phase Stefan problem or an ODP can be obtained. On inverting Eq. (7.4.73), we get

$$u(x,t) = \int_{\phi(x)}^{t} T(x,\eta)d\eta + \psi(x,t) \tag{7.4.79}$$

on the free boundary $t = S^{-1}(x) = \phi(x)$. From Eq. (7.4.79), $u(S(t),t) = \psi(S(t),t)$. In order to derive Eq. (7.4.78), calculate $u_t - u_{xx}$ from Eq. (7.4.79) at $x = S(t)$.

The equivalence of the weak formulation of a one-phase Stefan problem and its variational inequality formulation has been discussed in Section 11.2.2.

7.4.2 A Stefan Problem With a Quasi-Variational Inequality Formulation

Our interest in this section is only in introducing the *quasi-variational inequality*. For a rigorous discussion of the quasi-variational inequality, see [22] and the references mentioned there in this connection.

One of the forms of quasi-variational elliptic inequality considered by Bensoussan–Goursat–Lions (cf. [200]), and further discussed in [22], can be defined as follows. Let H_1 and H_2 be two Hilbert spaces and $v = (v_1, v_2)$ be the generic element of $H = H_1 \times H_2$. Define $a : H \times H \to R$ and $g : H \to R$ as follows:

$$a(w,z) = a_1(w_1,z_1) + a_2(w_2,z_2) + b_1(w_1,z_2) + b_2(z_2,w_1), \tag{7.4.80}$$

$$g(w) = g_1(w_1) + g_2(w_2). \tag{7.4.81}$$

Here $a_1 : H_1 \times H_1 \to R$ and $a_2 : H_2 \times H_2 \to R$ are continuous bilinear forms, symmetric and nonnegative along the diagonal, $b_1 : H_1 \times H_2 \to R$ and $b_2 : H_2 \times H_1 \to R$ are continuous bilinear forms, $g_1 : H_1 \to R$ and $g_2 : H_2 \to R$ are continuous linear functionals. Let $D \subset H = H_1 \times H_2$ be a nonempty closed convex set. If $v \in D$, then define $D(v) = D_1(v) \times D_2(v)$ where

$$D_1(v) = \{z_1 \in H_1 : (z_1, v_2) \in D\} \quad \text{and} \quad D_2(v) = \{z_2 \in H_2 : (v_1, z_2) \in D\}\,. \tag{7.4.82}$$

An inequality

$$a(u, u - v) \le g(u - v), \quad \forall v \in D(u), \ \ u \in D \tag{7.4.83}$$

is an example of a quasi-variational inequality.

$D(v)$ is a nonempty closed convex set of $D_* = P_{H_1}(D) \times P_{H_2}(D)$, where P_{H_1} and P_{H_2} are 'projection operators'. If $u \in H_1$ and $W \subset H_1$ is a nonempty closed convex set, then $P_W(u)$ is that element of W which is nearest to u. The inequality (7.4.83) is a variational inequality when for $\forall u \in D$, $D(u) = Q = Q_1 \times Q_2$ with Q_1 being a nonempty closed convex set of H_1 and Q_2 being a nonempty closed convex set of H_2. $v \in D(u)$ suggests that quasi-variational inequality entails a convex set of competing functions which may depend on the possible solution whereas in the variational inequality formulation convex set is fixed.

A family of variational inequalities can be associated with the quasi-variational inequality (7.4.83). If u is fixed in D, then along u, we can consider a variational inequality concerned with finding a $w \in D(u)$ such that

$$a(w, w - v) \le g(w - v), \quad \forall v \in D(u). \tag{7.4.84}$$

A quasi-variational inequality formulation for the following one-phase Stefan problem has been studied in [201, 202]. Find a function $T(x,t)$ and the curve $\Gamma : x = S(t)$, $0 < t < t_*$ such that

$$T_t - T_{xx} = 0, \quad 0 < x < S(t), \ \ 0 < t < t_*, \tag{7.4.85}$$

$$T = \lambda(x,t), \quad x = S(t), \ \ 0 < t < t_*, \tag{7.4.86}$$

$$T_x = 0, \quad x = S(t), \ \ 0 < t < t_*, \tag{7.4.87}$$

$$T(x,0) = T_0(x), \quad 0 \le x < S_0, \ \ S(0) = S_0, \tag{7.4.88}$$

$$T(0,t) = r(t), \quad 0 < t < t_*. \tag{7.4.89}$$

If the transformation $u = -T_x$ (see Section 3.3) is used, then the boundary condition (7.4.87) becomes $u = 0$ and Eq. (7.4.86) becomes $u_x = -\lambda_x \dot{S} - \lambda_t$. The coefficient $-\lambda_x$ can be identified with the latent heat and λ_t with the displacement of the free boundary. If appropriate conditions are imposed on the initial and boundary data, then the problem (7.4.85)–(7.4.89) can be identified with a phase-change problem. Global existence of the classical solution of the transformed system has been discussed in [203] when $\lambda_x \equiv 1$, $\lambda_t \le 0$, $-T_0'(x) \ge 0$ and $r'(t) \ge 0$. It turns out that in this case $\dot{S}(t) \ge 0$ and $u(x,t) > 0$ if $0 < x < S(t)$.

Let $\lambda(x,t)$, $T_0(x)$ and $r(t)$ be given smooth functions for $0 \le x \le R_0$, $0 \le t \le t_*$ and $S_0 \in (0, R_0)$. A quasi-variational inequality formulation of problem (7.4.85)–(7.4.89) will now be considered in the region $D = \{(x,t), 0 < x < R_0, 0 < t < t_*\}$ in terms of a new variable $W(x,t)$ defined as

$$W(x,t) = \int_x^{R_0} (T(\xi,t) - \lambda(\xi,t))d\xi, \quad (x,t) \in D. \tag{7.4.90}$$

It is understood that if $0 < t < t_*$, then $S(t) < R_0$. We extend $T(x,t)$ as $\lambda(x,t)$ in the region $S(t) \le x \le R_0$, $0 \le t \le t_*$ and thus $W(x,t) = 0$ if $S(t) \le x \le R_0$, $0 < t < t_*$. Let $\Omega = \{(x,t) : 0 < x < S(t), 0 < t < t_*\}$. From Eq. (7.4.90), $\partial W/\partial t$ can be calculated as follows:

$$\begin{aligned} W_t &= \int_x^{R_0} \{T_t(\xi,t) - \lambda_t(\xi,t)\}d\xi, \quad (x,t) \in D \\ &= \int_x^{S(t)} \{T_{\xi\xi}(\xi,t) - \lambda_t(\xi,t)\}d\xi \\ &= T_x(S(t),t) - T_x(x,t) - \int_x^{S(t)} \lambda_t(\xi,t)d\xi \\ &= -T_x(x,t) - \int_x^{S(t)} \lambda_t(\xi,t)d\xi. \end{aligned} \tag{7.4.91}$$

For $(x,t) \in \Omega$, we have

$$W_x(x,t) = -(T(x,t) - \lambda(x,t)) \tag{7.4.92}$$

and

$$W_{xx}(x,t) = -T_x(x,t) + \lambda_x(x,t). \tag{7.4.93}$$

$W(x,t)$ satisfies the differential equation

$$-W_{xx}(x,t) + W_t(x,t) = f(x,t), \quad \text{for } (x,t) \in D, \tag{7.4.94}$$

where

$$\begin{aligned} f(x,t) &= -\lambda_x(x,t) - \int_x^{S(t)} \lambda_t(\xi,t)d\xi, \quad (x,t) \in \Omega, \\ &= -\lambda_x(x,t) \quad \text{if } (x,t) \in D\backslash\Omega. \end{aligned} \tag{7.4.95}$$

$f(x,t)$ depends on $S(t)$, i.e. on the solution. It may be noted that $f(x,t)$ has been suitably extended to the region $D\backslash\Omega$ as it is required afterwards in the formulation. It will be assumed that $\lambda_x > 0$, $\lambda_t \le 0$ and

$$T_0(x) > \lambda(x,0), \quad 0 \le x < S_0 \quad \text{and} \quad T(S_0,0) = \lambda(S_0,0). \tag{7.4.96}$$

We define a function $g(x)$ as follows:

$$\begin{aligned} g(x) &= \int_x^{S_0} (T(\xi,0) - \lambda(\xi,0))d\xi, \quad \text{if } 0 \le x \le S_0 \\ &= 0, \quad \text{if } \ S_0 \le x \le R, \end{aligned} \tag{7.4.97}$$

so that $W(x,0) = g(x)$ and $g'(x) \le 0$. Because of the assumption that $T_0(x) > \lambda(x,0)$ in $(0,S_0)$, the free boundary Γ starts at S_0. The boundary condition for $W(x,t)$ at $x = 0$ is given by

$$W_x(0,t) = \Psi(t), \tag{7.4.98}$$

where

$$\Psi(t) = -r(t) + \lambda(0,t), \quad 0 \le t \le t_*.$$

Assume that $\Psi(t) \le 0$ for $0 < t < t_*$. It can be shown that W is greater than zero in Ω. On differentiating Eq. (7.4.94) with respect to x, we obtain

$$-W_{xxx} + W_{xt} = f_x = -\lambda_{xx} + \lambda_t, \quad \text{in } \Omega. \tag{7.4.99}$$

It will be assumed that

$$-\lambda_{xx} + \lambda_t \le 0, \quad \text{in } D. \tag{7.4.100}$$

Since $W_x = \lambda(x,t) - T(x,t) = 0$ for $(x,t) \in \Gamma$, and $\Psi(t) \le 0$, we obtain

$$-W_{xxx} + W_{xt} \le 0, \quad \text{in } \Omega \tag{7.4.101}$$

and

$$W_x \le 0, \quad \text{on } \partial_p\Omega, \tag{7.4.102}$$

where $\partial_p\Omega$ is the parabolic boundary of Ω. Assuming W_x to be continuous in $\bar{\Omega}$, the maximum principle (see Appendix C) implies that $W_x < 0$ in Ω and on integrating W_x, it can be concluded that $W(x,t) > W(S(t),t) = 0$.

The variational inequality formulation of the problem (7.4.85)–(7.4.89) can now be obtained in terms of $W(x,t)$ as follows:

Find $W(x,t) \in M = \{v(x,t) \in H^1(D), v \ge 0\}$ such that

$$(-W_{xx} + W_t)(v - W) \ge f(v - W), \quad \text{a.e. in } D \quad \text{and} \quad \forall v \in M, \quad W \in M, \tag{7.4.103}$$

$$W(x,0) = g(x), \quad 0 < x < R_0,$$

$$W_x(0,t) = \Psi(t), \quad 0 < t < t_*,$$

$$W(R_0,t) = 0, \quad 0 < t < t_*.$$

For $W \geq 0$ and $v \geq 0$, we have

$$\begin{aligned}(-W_{xx} + W_t - f)(v - W) &= -(-W_{xx} + W_t - f)W + (-W_{xx} + W_t - f)v \\ &= (-W_{xx} + W_t - f)v \\ &= 0, \quad \text{if } W > 0 \ \text{ and } \ = -fv \quad \text{if } W = 0 \\ &\geq 0, \quad \text{for } \forall v \in M, \quad W \in M,\end{aligned} \tag{7.4.104}$$

provided f is negative in $D \backslash \Omega$, which it is, under the assumption that $\lambda_x > 0$ (see Eq. 7.4.95). It is clear from Eq. (7.4.95) that f is a function of $(x,t;S(t))$ and $S(t) = \min\{x, W(x,t) = 0\}$. Therefore f is a discontinuous function of $W(x,t)$ and Eq. (7.4.104) is not a variational inequality in the usual sense. In [201], authors call it a *quasi-variational inequality*. The existence of the solution of Eq. (7.4.103) has been proved in [201] by considering a sequence of variational inequality problems in which the approximations of the free boundary are taken as known. Fixed point theorem for monotone mappings was used in [201] for the existence proof and in [202] existence of the solution of the above problem has been proved using finite-difference approximations and this resulted in a smoother solution. Under appropriate smoothness assumptions on λ, g, Ψ, etc. (cf. [202]) it has been shown that a solution of Eq. (7.4.103) exists and is such that: (1) W_t, W_{xx} belong to $L^\infty(D)$, (ii) W_x is Hölder continuous (exponent $1/2$) in $\bar{D}$, (iii) $W_x(x,t) < 0$ for $0 < x < S(t)$, $0 < t < t_*$, (iv) $S(t)$ is Hölder continuous and monotone decreasing, and $S(0) = S_0$, $S(t) > 0$.

7.4.3 The Variational Inequality Formulation of a Two-Phase Stefan Problem

In the one-phase Stefan problems, the freezing index $u(x,t)$ proved to be very useful as the gradient of u is continuous in Ω (the region under consideration). The temperature constraint $T \geq 0 (\leq 0)$ also holds throughout Ω. In the two-phase problem although freezing index can be defined, the constraint $T \geq 0(\leq 0)$ does not hold throughout the two-phase region. An approach different from the one adopted for the one-phase problems is required for the variational inequality formulation. If the weak enthalpy formulation is considered for the two-phase Stefan problem, then the enthalpy, which is a multivalued function of temperature, can be written as (for simplicity take specific heats and densities of the two phases to be unity)

$$h(T) = h_0(T) + l\, sgn^+(T), \tag{7.4.105}$$

where $h_0(T)$ is the sensible heat and sgn^+ represents Heaviside graph (cf. Eq. 4.3.45). In terms of the freezing index u,

$$u(x,t) = \int_0^t T(x,t)dt, \quad (x,t) \in \Omega_{t_*} = \Omega \times (0,t_*), \quad \Omega \subset R^3, \tag{7.4.106}$$

the heat energy equation (4.2.2) can be written as (see [204] for the derivation of Eq. 7.4.107)

$$-h_0(u_t)+\nabla^2 u+h_1(x)\in l\partial\Psi(u_t),\quad \text{a.e. in } \Omega_{t_*}. \tag{7.4.107}$$

Here $h_1(x)$ is the initial enthalpy of the material, and $\partial\Psi$ is the subdifferential of a convex function $\Psi(z)=\max(0,z)$ which coincides with the multivalued mapping $sgn^+(T)$. Using the definition of the subdifferential of a function (see Eq. 4.3.44), the parabolic variational inequality for the enthalpy formulation can be obtained (cf. [204] and the references mentioned there). Our interest here is in the classical formulation.

A variational inequality formulation for a two-phase Stefan problem in the region $\Omega\subset R^3$ has been studied in [205]. The phase-change boundary $\Phi(t): t=S(x), x\in R^3$, divides Ω into two disjoint regions Ω_1 and Ω_2, representing the solid and the liquid regions, respectively. The temperature in the interior $\widetilde{\Omega}_1$ of Ω_1 is negative and in the interior $\widetilde{\Omega}_2$ of Ω_2 is positive. The cases of regions of zero temperature within Ω_1 and Ω_2 can also be considered. Initially Ω is occupied by ice. On the portion $\partial\Omega_b$ of the boundary $\partial\Omega$ of Ω, temperature $\hat{b}(t)$ is prescribed and the remaining boundary of Ω is insulated. Let x be a point in Ω which is initially at $t_0=0$ in the solid phase, and $t_1(x)$ be the first time that x is in the liquid phase, and $t_2(x)>t_1(x)$ be the next time at which x is again in the solid phase. In this way a sequence $\{t_n\}$ can be defined such that $t_0(=0), t_1, t_2, \ldots$, are such that in (t_i, t_{i+1}), x is in the liquid (solid) phase if i is odd (even). Since the normal derivative of ∇T is not continuous throughout Ω we define another dependent variable $V(x,t)$ by the equation

$$V(x,t)=K_1\int_{t_0}^{t_1}T(x,\tau)d\tau+K_2\int_{t_1}^{t_2}T(x,\tau)d\tau+\cdots+K_J\int_{t_i}^{t}T(x,\tau)d\tau, \tag{7.4.108}$$

where for $t\in(t_i,t_{i+1})$, $J=1(2)$ if i is even (odd). K_1 and K_2 are the thermal conductivities of solid and liquid regions, respectively and are taken as constant. Densities of both phases are taken to be equal to unity. If $T^+=\sup(T,0)$ and $T^-=\sup(-T,0)$ where 0 is the phase-change temperature then $V(x,t)$ can be written as

$$V(x,t)=\int_0^t\left\{-K_1T^-(x,\tau)+K_2T^+(x,\tau)\right\}d\tau. \tag{7.4.109}$$

Since T is continuous throughout the interior of Ω,

$$\frac{\partial V}{\partial t}=K_J T(x,t), \tag{7.4.110}$$

where $J=1(2)$ if i is even (odd) for $t\in(t_i,t_{i+1})$.

We are now interested in obtaining a differential equation for $V(x,t)$ which holds throughout Ω. On differentiating Eq. (7.4.108), we obtain

$$\begin{aligned}\nabla V(x,t)=K_1&\left\{\int_0^{t_1(x)}\nabla T d\tau+T(x,t_1(x))\right\}\\&+K_2\left\{\int_{t_1(x)}^{t_2(x)}\nabla T d\tau+T(x,t_2(x))-T(x,t_1(x))\right\}+\cdots+K_J\left\{\int_{t_i}^{t}\nabla T d\tau-T(x,t_i)\right\}.\end{aligned} \tag{7.4.111}$$

At $t_1(x), t_2(x), \dots, t_i(x)$, phase change takes place and, $T(x, t_i) = 0$, $i = 1, 2 \dots$. Therefore

$$\nabla V(x,t) = K_1 \int_0^{t_1} \nabla T d\tau + K_2 \int_{t_1}^{t_2} \nabla T d\tau + \cdots + K_J \int_{t_i}^{t} \nabla T d\tau. \tag{7.4.112}$$

Note that V, $\partial V/\partial t$ and ∇V are continuous in Ω. ∇T is discontinuous only across $t = t_i(x)$ and not within the interval (t_i, t_{i+1}). When Eq. (7.4.112) is differentiated, we get

$$\begin{aligned}\nabla^2 V(x,t) = {} & K_1 \int_0^{t_1} \nabla^2 T d\tau + K_2 \int_{t_1}^{t_2} \nabla^2 T d\tau \\ & + \{K_1 \nabla T(x,t_1) \cdot \nabla t_1(x) - K_2 \nabla T(x,t_1) \cdot \nabla t_1(x)\} + \cdots \\ & + K_J \int_{t_i}^{t} \nabla^2 T d\tau - K_J \nabla T(x,t_i) \cdot \nabla t_i(x).\end{aligned} \tag{7.4.113}$$

If the Stefan condition (1.4.17) is used in Eq. (7.4.113), then we obtain (take $\rho_1 = \rho_2 = 1$)

$$\nabla^2 V(x,t) = C_1 \int_0^{t_1} \frac{\partial T}{\partial \tau} d\tau + C_2 \int_{t_1}^{t_2} \frac{\partial T}{\partial \tau} d\tau + C_J \int_{t_i}^{t} \frac{\partial T}{\partial \tau} d\tau + l - l + \cdots. \tag{7.4.114}$$

Performing the integration in Eq. (7.4.114) and remembering whether J is odd or even, we get

$$\begin{aligned}\nabla^2 V(x,t) &= -C_1 T(x,0) + l + \frac{C_2}{K_2}\frac{\partial V}{\partial t}, \quad \text{if } T(x,t) > 0 \\ &= -C_1 T(x,t) + \frac{C_1}{K_1}\frac{\partial V}{\partial t}, \quad \text{if } T(x,t) < 0.\end{aligned} \tag{7.4.115}$$

Since $T(x,t)$ has the same sign as $\partial V/\partial t$ (see Eq. 7.4.110), Eq. (7.4.115) can be written as

$$\nabla^2 V(x,t) = -C_1 T(x,0) + l\hat{H}\left(\frac{\partial V}{\partial t}\right) + \frac{C_2}{K_2}\left[\frac{\partial V}{\partial t}\right]^+ - \frac{C_1}{K_1}\left[\frac{\partial V}{\partial t}\right]^-, \tag{7.4.116}$$

where

$$\hat{H}(\partial V/\partial t) = 1, \quad \text{if } \partial V/\partial t > 0,$$
$$\hat{H}(\partial V/\partial t) = 0, \quad \text{if } \partial V/\partial t < 0.$$

In Eq. (7.4.116), $[x]^{\pm}$ are the positive and negative parts of x.

What happens when $\partial V/\partial t = 0$ or $T(x,t) = 0$? If there exists a sharp boundary Γ separating solid and liquid phases at which $T = 0$, then the meas$(\Gamma) = 0$ in Ω and Eq. (7.4.116) holds a.e. in Ω. Two other cases may arise. For example, there may exist regions of nonzero measures in $\Omega_1(t)$ or $\Omega_2(t)$ whose temperatures are zero. It has been argued in [205] that if such a region exists in the solid region then we define $\hat{H}(\partial V/\partial t) = 0$ when $\partial V/\partial t = 0$ and if such a region exists in the liquid region then we define $\hat{H}(\partial V/\partial t) = 1$ if $\partial V/\partial t = 0$. In all the three cases Eq. (7.4.116) holds in Ω.

To obtain a variational inequality for the above problem, the procedure adopted is similar to the one used for obtaining the weak enthalpy formulation in Eq. (7.4.107). We write

$$\hat{H}\left(\frac{\partial V}{\partial t}\right) \in \partial g_0\left(\frac{\partial V}{\partial t}\right), \tag{7.4.117}$$

where $g_0(x) = x^+$ defined as $\sup(0, x)$ and ∂g_0 is the subdifferential of g_0. Now our objective is to look for an appropriate convex function $g_1(x) : R \to R$ so that we can write

$$\nabla^2 V(x,t) + C_1 T(x,0) \in \partial g_1\left(\frac{\partial V}{\partial t}\right). \tag{7.4.118}$$

If $g_1(x)$ is defined as

$$g_1(x) = \frac{1}{2}(C_1/K_1)(x^-)^2 + \frac{1}{2}(C_2/K_2)(x^+)^2 + lx^+, \tag{7.4.119}$$

then using the definition of the *subgradient of a convex function*, i.e. $f(x) \in \partial g(x) \Leftrightarrow g(\xi) - g(x) \geq (\xi - x)f(x)$ for all $\xi \in R$, a relation of the form (7.4.120) can be obtained. The relation (7.4.118) can be expressed as

$$g_1(\xi) - g_1\left(\frac{\partial V}{\partial t}\right) \geq (\nabla^2 V + C_1 T(x,0))\left(\xi - \frac{\partial V}{\partial t}\right), \quad \forall \xi \in R. \tag{7.4.120}$$

The boundary condition to be satisfied by $V(x,t)$ on $\partial\Omega_b$ can be expressed as

$$\begin{aligned} V(x,t) &= K_1 \int_0^{t_1} \hat{b}(\tau)d\tau + \cdots + K_J \int_{t_i}^{t} \hat{b}(\tau)d\tau, \quad \text{on } \partial\Omega_b \\ &= \int_0^t \left\{K_2(\hat{b}(\tau))^+ - K_1(\hat{b}(\tau))^-\right\} d\tau, \quad \text{on } \partial\Omega_b. \end{aligned} \tag{7.4.121}$$

On the remaining boundary, we take

$$\frac{\partial V}{\partial n} = 0, \quad \text{on } \partial\Omega \backslash \partial\Omega_b. \tag{7.4.122}$$

Let the initial condition for V be given by

$$V(x,0) = 0, \quad x \in \Omega. \tag{7.4.123}$$

On taking $\xi = \Psi(x), x \in \Omega$ in Eq. (7.4.120) and integrating over Ω, we obtain

$$G(\Psi) - G\left(\frac{\partial V}{\partial t}\right) \geq \int_\Omega \nabla^2 V\left(\Psi - \frac{\partial V}{\partial t}\right) dx + C_1 \int_\Omega T(x,0)\left(\Psi - \frac{\partial V}{\partial t}\right) dx, \tag{7.4.124}$$

where

$$G(\Psi) = \int_\Omega g_1(\Psi(x))dx. \tag{7.4.125}$$

If the spaces to which Ψ, V and $\partial V/\partial t$ belong are defined suitably such as in Eq. (7.4.128), then Eq. (7.4.124) can be identified with a variational inequality formulation of the above two-phase problem. On integrating by parts the first term on the r.h.s. of Eq. (7.4.124) and assuming that Ψ and $\partial V/\partial t$ satisfy the same boundary conditions, we obtain

$$G(\Psi) - G\left(\frac{\partial V}{\partial t}\right) + a\left(V, \Psi - \frac{\partial V}{\partial t}\right) \geq C_1 \int_\Omega T(x,0)\left(\Psi - \frac{\partial V}{\partial t}\right) dx, \tag{7.4.126}$$

where

$$a(V,Z) = \int_\Omega \nabla V \cdot \nabla Z dx. \tag{7.4.127}$$

Let

$$M(t) = \left\{Z \in H^1(\Omega), Z|_{\partial\Omega_b} = b(t)\right\}, \quad \text{a.a. } t \in (0,t_*), \tag{7.4.128}$$

$$b(t) = K_2(\hat{b}(t))^+ - K_1(\hat{b}(t))^-. \tag{7.4.129}$$

If there exists a V and the following conditions hold:

1. V, $\partial V/\partial t \in L^2(0,t_*;H^1(\Omega))$,
2. $\partial V/\partial t \in M(t)$ (defined in Eq. 7.4.128),
3. The inequality (7.4.126) is satisfied for all $\Psi \in M(t)$ for a.a. $t \in (0,t_*)$,
4. $V = 0$ when $t = 0$,

then V is called a solution of the variational inequality (7.4.126). Under the assumptions that $T(x,0) \in L^2(\Omega)$ and $b(t) \in L^2(0,t_*)$, a unique solution of Eq. (7.4.126) exists (cf. [205]). The numerical solution of the above variational inequality can be obtained with the help of a finite-difference scheme.