

Chapter 1

The Stefan Problem and Its Classical Formulation

1.1 SOME STEFAN AND STEFAN-LIKE PROBLEMS

The term *Stefan problem* can be best understood with the help of an example for which the reader is referred to [Section 1.3](#). Our concern in the present section is to understand the notion of a *free boundary* which is a typical feature of the Stefan and Stefan-like problems. Therefore this chapter begins with some examples of Stefan and Stefan-like problems which demonstrate the existence of an unknown boundary, commonly known in the literature as a ‘free boundary’ or a *moving boundary*. In the context of solidification/melting problems, with which Stefan problems are commonly associated, the free boundary is also called a *phase-change boundary* or a *melting front* or a *freezing front*. Some authors use the term free boundary when the unknown boundary is static and moving boundary when it is time dependent. In this volume we do not make any distinction between ‘free’ and ‘moving’ boundaries. The term boundary is used for a surface also. In many of the examples considered in this section, the identification of the free boundary and the mathematical formulation of the problem are rather easy but in some of them even the identification of the free boundary is difficult. The problems given here are from various fields of mathematics, physics and engineering and demonstrate the existence of free boundaries. Our interest in examples given in this section is more on demonstrating the existence of a free boundary and its typical characteristics than justifying the formulation.

Problem 1.1.1 (Steady-State Heat Conduction With a Free Boundary). Find the steady-state temperature $T(x, y)$ satisfying the equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad (1.1.1)$$

in an open bounded region $D \subset \mathbb{R}^2$. The boundary ∂D of D consists of two disjoint parts R_1 and R_2 , i.e. $\partial D = R_1 \cup R_2$, where R_1 is unknown and R_2 is known. On R_2 , the temperature is prescribed as

$$T(x, y)|_{R_2} = f(x, y). \quad (1.1.2)$$

If $f(x, y)$ is known throughout the plane, then one boundary conditions on R_1 will be

2 The Classical Stefan Problem

$$T(x, y)|_{R_1} = f(x, y), \quad (1.1.3)$$

and another boundary condition can be imposed as [1]

$$\nabla(T - f)|_{R_1} = 0. \quad (1.1.4)$$

The problem is to determine the temperature $T(x, y)$ in D , and the unknown boundary R_1 . Two boundary conditions are to be prescribed on R_1 . One, because it is a boundary and one more boundary condition is required to determine an unknown boundary. Note that R_1 can be determined only by solving the system (1.1.1)–(1.1.4). Such an unknown boundary is called a *free boundary* or a *moving boundary*. In ‘boundary value problems’ of mathematical physics, the boundary of the region under consideration is completely known. Conditions (1.1.3) and (1.1.4) are called *free boundary conditions* and (1.1.2) is a *fixed boundary condition*. Although Eq. (1.1.1) is linear, free boundary problems are nonlinear problems because of the nonlinearity of the boundary conditions at the free boundary.

Problem 1.1.2 (Steady-State Free Surface Flow With Surface Tension). Consider a two-dimensional steady, incompressible, irrotational flow in a long channel. x and y axes are taken along the length and depth of the channel, respectively, with the bottom of the channel taken as $y = 0$ and the upper surface of water, as a free surface or a free boundary denoted by $y = \eta(x)$, where $\eta(x)$ is unknown. If $u(x, y)$ and $v(x, y)$ are the velocity components in x and y directions, respectively, then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (\text{equation of continuity}), \quad (1.1.5)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad (\text{irrotationality condition}). \quad (1.1.6)$$

The pressure p is given by the following Bernoulli equation [2]

$$H = y + p/(\rho g) + |\vec{q}|^2/(2g), \quad (1.1.7)$$

where H is the given total water head of water, ρ is the density, g the acceleration due to gravity and \vec{q} is the velocity vector. If the bottom of the channel is a rigid boundary, then

$$v = 0, \quad \text{on } y = 0. \quad (1.1.8)$$

At the free boundary $y = \eta(x)$,

$$H = \eta(x) + p/(\rho g) + |\vec{q}|^2/(2g), \quad p = \bar{p} - \sigma/R_c \quad (1.1.9)$$

and

$$\vec{q} \cdot \vec{n} = 0. \quad (1.1.10)$$

Here, \bar{p} is the atmospheric pressure (known), σ is the surface tension (known) and R_c is the radius of curvature of the free surface, taken as positive when the centre of curvature is situated above the free surface and \vec{n} is the unit outward normal to the free surface. R_c can be expressed in terms of derivatives of $\eta(x)$. Free boundary has been taken as static in

this problem. Eq. (1.1.10) implies that the normal component of the fluid velocity at the free boundary is zero.

Problem 1.1.3 (Free Surface Flow With Time Dependent Free Surface). Let the free surface in Problem 1.1.2 be time dependent and represented by $y = \eta(x, t)$ with $\eta|_{t=0}$ being given. Surface tension effects will be neglected. If the velocity field \vec{q} is expressed as

$$\vec{q} = \nabla \phi, \quad (1.1.11)$$

then from Eqs (1.1.5), (1.1.6) it is easy to conclude that $\phi(x, t)$ satisfies the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad b(x) < y < \eta(x, t). \quad (1.1.12)$$

Here, $y = b(x)$ is the equation of the bottom of the channel. The momentum equation can be written as (cf. [3])

$$\frac{\partial \vec{q}}{\partial t} + \frac{1}{2} \text{grad} |\vec{q}|^2 - \vec{q} \wedge \text{curl} \vec{q} = \vec{F} - \text{grad}(p/\rho), \quad (1.1.13)$$

\vec{F} represents body forces. On substituting \vec{q} from Eq. (1.1.11) in Eq. (1.1.13) and integrating with respect to x , we obtain

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + gy + p/\rho = \psi(t), \quad (1.1.14)$$

provided the density is taken as constant and the gravitational field is the only force field. The arbitrary function $\psi(t)$ can be absorbed in $\phi(t)$ and Eq. (1.1.14) becomes

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + gy + p/\rho = 0. \quad (1.1.15)$$

If $y = b(x)$ is taken as a rigid boundary, then

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } y = b(x), \quad (1.1.16)$$

where \vec{n} denotes the unit outward normal to $y = b(x)$. On the unknown free boundary $y = \eta(x, t)$, the two conditions are given by

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + g\eta(x, t) + p/\rho = 0, \quad (1.1.17)$$

and

$$\frac{\partial \phi}{\partial n} = \vec{V} \cdot \vec{n}, \quad (1.1.18)$$

where \vec{V} is the velocity of the free boundary and \vec{n} is the unit outward drawn normal on it. Eq. (1.1.18) can be expressed in terms of quantities already defined. Let,

$$F(x, y, t) = y - \eta(x, t) = 0. \quad (1.1.19)$$

4 The Classical Stefan Problem

Then

$$DF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial t} dt = 0, \quad (1.1.20)$$

or

$$-\frac{\partial F}{\partial t} = -\frac{\partial \eta}{\partial x} \frac{dx}{dt} + \frac{dy}{dt} = \vec{V} \cdot \nabla F = (\vec{V} \cdot \vec{n}) |\nabla F|, \quad \vec{n} = \nabla F / |\nabla F|. \quad (1.1.21)$$

From Eqs (1.1.18), (1.1.21), we get

$$-\frac{\partial F}{\partial t} = \frac{\partial \phi}{\partial n} |\nabla F| = (\nabla \phi \cdot \vec{n}) |\nabla F| = \left(\nabla \phi \cdot \frac{\nabla F}{|\nabla F|} \right) |\nabla F| = (\nabla \phi \cdot \nabla F) \quad (1.1.22)$$

or

$$\frac{\partial \eta}{\partial t} = -\frac{\partial \phi}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y}. \quad (1.1.23)$$

Eq. (1.1.12) is to be solved using the boundary conditions (1.1.17) and (1.1.23), the fixed boundary condition (1.1.16) and the prescribed $\eta(x, t)$ at $t = 0$. In this problem the velocity of the fluid is not time dependent but the free boundary is time dependent. Such problems are called *quasi-steady state free boundary problems or degenerate free boundary problems*. We shall see later that the term degenerate free boundary problem is used for other types of problems also.

Linearization of the Above Problem

Let $y = y_0$ be the flat upper surface of water. When the deviation of the free boundary from the flat surface is small, i.e. if $|\partial \eta / \partial x| \ll 1$, then the above problem can be linearized as follows. Let

$$y = \eta(x, t) = y_0 + \varepsilon f(x, t), \quad \phi = \varepsilon u, \quad \varepsilon \ll 1. \quad (1.1.24)$$

$\varepsilon f(x, t)$ can be considered as the disturbance at the flat surface $y = y_0$. Substituting Eq. (1.1.24) in Eqs (1.1.12), (1.1.16), (1.1.17), (1.1.23), and in the changed equations retaining only linear terms in ε and dropping higher order terms of ε , we obtain

$$\nabla^2 u = 0, \quad b(x) < y < y_0, \quad (1.1.25)$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } y = b(x), \quad (1.1.26)$$

$$\frac{\partial u}{\partial t} + g f = 0 \quad \text{on } y = y_0, \quad (1.1.27)$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial t} \quad \text{on } y = y_0. \quad (1.1.28)$$

In Eq. (1.1.27), we have taken $p = -\rho g y_0$ which comes from the contribution of the zeroth-order terms in Eq. (1.1.17).

On eliminating f from Eqs (1.1.27), (1.1.28), we obtain

$$\frac{\partial^2 u}{\partial t^2} + g \frac{\partial u}{\partial y} = 0 \quad \text{on } y = y_0. \quad (1.1.29)$$

Since $y = y_0$ is not a free boundary, only one boundary condition is to be prescribed on it. For further information about Problems 1.1.2 and 1.1.3, see [3–5].

Problem 1.1.4 (A Problem of Reproductive Toxic Mass Diffusion). Let $u(x, t)$ be the concentration of a toxic mass which is diffusing in a region Ω , where

$$\Omega = \{x \in R : 0 \leq x \leq 1\}. \quad (1.1.30)$$

If the concentration exceeds a certain value u_v in a portion of Ω , then it is called a toxic region. Let the reproduction rate of toxic mass in the toxic region be P and in the nontoxic region αP , $0 < \alpha < 1$. The toxic and nontoxic regions are separated by a surface S , where

$$S = \{(x, t) \in \Omega_{t_*} \mid x = \phi(t), u = u_v\} \quad (1.1.31)$$

and

$$\Omega_{t_*} = \{(x, t) \mid x \in \Omega, 0 < t \leq t_*\}. \quad (1.1.32)$$

$u(x, t)$ and the free boundary $x = \phi(t)$ are to be obtained by solving the following system of equations. For $u > u_v$

$$u_t - u_{xx} + d_0 + d_1 u = P, \quad (x, t) \in \Omega_{t_*}, \quad 0 < x < \phi(t), \quad 0 < t \leq t_*. \quad (1.1.33)$$

For $u < u_v$

$$u_t - u_{xx} + d_0 + d_1 u = \alpha P, \quad (x, t) \in \Omega_{t_*}, \quad \phi(t) < x < 1, \quad 0 < t \leq t_*. \quad (1.1.34)$$

The term $d_0 + d_1 u$ accounts for the mass loss due to the bottom leakage, and other similar factors. d_0, d_1, α and P are positive constants. Mass diffusion coefficient has been taken to be unity which is possible by suitably defining the time and/or length scales.

At the free boundary

$$u_x(\phi(t)-, t) = u_x(\phi(t)+, t) \quad (\text{continuity of mass flux}) \quad (1.1.35)$$

and

$$u(\phi(t), t) = u_v. \quad (1.1.36)$$

It may be noted that the velocity of the free boundary is not explicitly occurring in Eq. (1.1.35) which was so in Problems 1.1.2 and 1.1.3. The boundary conditions at the free boundary in which the velocity of the free boundary is not occurring explicitly are known as *implicit free boundary conditions*. Eqs (1.1.33)–(1.1.36) are to be supplemented with a suitable initial condition at $t = 0$ and with boundary conditions at the fixed boundaries $x = 0$ and $x = 1$. Some results on the existence of solution of the above problem and the regularity of the free boundary can be found in [6].

6 The Classical Stefan Problem

Problem 1.1.5 (Gas Flow Through Porous Media). The equation of state for an isentropic (constant entropy) flow of an ideal gas in a homogeneous porous media is given by [7]

$$\rho(x, y) = \rho_0 p^\alpha \geq 0, \quad (1.1.37)$$

where $\rho(x, y)$ is the density and $p(x, y)$ is the pressure. $\rho_0 \in R^+$ and $\alpha \in (0, 1]$ are constant. The conservation of mass gives

$$\operatorname{div}(\rho \vec{V}) = -\gamma \frac{\partial \rho}{\partial t}, \quad (1.1.38)$$

where γ is the porosity of the medium. According to Darcy's law [8], the velocity \vec{V} of the gas flow in a porous medium is given by

$$\vec{V} = -(\beta/\eta) \operatorname{grad} p, \quad (1.1.39)$$

$\beta \in R^+$ is the permeability of the medium and $\eta \in R^+$ is the viscosity of the gas. \vec{V} and p can be eliminated from Eqs (1.1.38), (1.1.39) and we obtain

$$\frac{\partial \rho}{\partial t} = \frac{\beta}{\eta \gamma \rho_0^{1/\alpha} (1 + \alpha)} \nabla^2(\rho^m), \quad \rho \geq 0, \quad (1.1.40)$$

where $m = 1 + 1/\alpha$. The diffusion is called 'fast' if $m > 1$, and 'slow' if $m < 1$.

By suitably choosing the time and/or length scales, the following equation can be obtained from Eq. (1.1.40):

$$\frac{\partial \rho}{\partial t} = \nabla^2(\rho^m), \quad \rho \geq 0. \quad (1.1.41)$$

If $\alpha \in (0, 1]$, then $m \in [2, \infty)$. Eq. (1.1.41) is the porous media equation and it arises also in other contexts such as population dynamics and plasma physics [7]. In order to calculate the mass flux of the gas, the r.h.s. of Eq. (1.1.41) can be written as

$$\nabla^2(\rho^m) = \operatorname{div}(m\rho^{m-1} \operatorname{grad} \rho), \quad (1.1.42)$$

$m\rho^{m-1} \operatorname{grad} \rho$ is the mass flux and $m\rho^{m-1}$ can be taken as diffusivity. Diffusivity vanishes as the density ρ tends to zero. Therefore Eq. (1.1.41) is a *nonlinear degenerate equation* in the neighbourhood of any point where $\rho = 0$ but is *nondegenerate and uniformly parabolic* (see [9] and Eq. (7.3.26)) in the neighbourhood of any point at which ρ is away from zero. Such problems are called *degenerate parabolic-elliptic* problems. An important consequence of nonlinear degeneracy is that there is a finite speed of propagation of a disturbance from rest which is in contrast to the parabolic heat equation in which the speed of heat propagation is infinite. The finite speed of propagation may give rise to *waiting time solutions*. Eq. (1.1.41) is to be supplemented with an initial condition if the region is infinite and with both initial and boundary conditions if the region considered is finite. The existence of a free boundary in such problems can be illustrated with the help of the following example. Let

$$\frac{\partial \rho}{\partial t} = \nabla^2(\rho^m), \quad -\infty < x < \infty, \quad t > 0, \quad (1.1.43)$$

$$\rho(x, 0) = \begin{cases} > 0 \text{ for } x \in R_I = (a_1, a_2), -\infty < a_1 < a_2 < \infty \\ = 0 \text{ for } x \in R \setminus R_I. \end{cases} \quad (1.1.44)$$

This problem, generally, does not have a *classical solution*. The classical solution of a problem can be roughly stated to be a solution in which the dependent variable possesses continuous derivatives of the order required in the problem formulation. The mathematical definition of a classical solution will be discussed later but at present it would suffice to state that the solution $\rho(x, t)$ of Eqs (1.1.43), (1.1.44) may not possess the required continuous derivatives. For $t > 0$, gas will be diffusing to the right of $x = a_2$ and to the left of $x = a_1$ and thus giving rise to two moving boundaries $x = S_i(t)$, $i = 1, 2$. Let S_1 be moving towards $+\infty$ and S_2 moving towards $-\infty$. Using a *weak formulation* of the above problem in Eqs (1.1.43), (1.1.44), several interesting results on the behaviour of $S_i(t)$, $i = 1, 2$ have been obtained in [10, 11]. The following proposition indicates that in some cases, the interface $S_i(t)$ starts moving only after an elapse of time $t^* > 0$.

Proposition 1.1.1. *There exist numbers $t_i^* \in [0, +\infty)$ for $i = 1, 2$ such that $S_i(t)$ is strictly monotone for $t \in (t_i^*, +\infty)$ and*

$$S_i(t) = a_i, \quad i = 1, 2 \text{ for } t \in [0, t_i^*]. \quad (1.1.45)$$

If $t_i^* > 0$, then $S_i(t)$, $i = 1, 2$ remain stationary for t_i^* units of time [11].

In this case t_i^* is called a *waiting time*. It has been proved in [11] that the interface is *Hölder continuous* under certain conditions and if the interface is in motion, then one expects it to move with the velocity \vec{V} of the gas, and

$$\frac{dS_i}{dt} = - \lim_{x \rightarrow S_i(t) \pm} W_x(x, t), \quad i = 1, 2; \quad W = \frac{m}{m-1} \rho^{m-1}. \quad (1.1.46)$$

The limits in Eq. (1.1.46) are taken as x approaches the boundary of the region from within the region in which $\rho(x, t) > 0$. Eq. (1.1.46) can also be obtained from the mass balance condition at the interface which states that the jump in the density at the interface multiplied by the velocity of the interface is equal to the jump in the mass flux across the interface.

In a problem symmetrical with respect to x , it is sufficient to consider the region $0 \leq x < \infty$ with a single interface $x = S_1(t)$ and $a_1 = a_2 > 0$. $S_1(t)$ should satisfy Eq. (1.1.46) for $i = 1$, and another condition to determine the unknown $S_1(t)$ may be prescribed as

$$\int_0^{S_1(t)} \rho(x, t) dx = \int_0^{a_1} \rho(x, 0) dx. \quad (1.1.47)$$

Conditions of the type Eq. (1.1.47) are called *nonlocal boundary conditions* at the free boundary. $x = 0$ is now a fixed boundary and the boundary condition on it is given by

$$\left. \frac{\partial \rho}{\partial x} \right|_{x=0} = 0. \quad (1.1.48)$$

Note that we have two conditions prescribed at the free boundary, viz., Eqs (1.1.46), (1.1.47).

Problem 1.1.6 (Shock Propagation). The solution of Burger's equation (1.1.49) with boundary condition (1.1.50) has been discussed in [12]

$$u_y + uu_x = u_y + \frac{1}{2} \frac{\partial}{\partial x}(u^2) = 0, \quad (x, y) \in R^2, y \geq 0. \quad (1.1.49)$$

On $y = 0$,

$$u(x, y)|_{y=0} = f(x), \quad x \in R, \quad (1.1.50)$$

where

$$\left. \begin{aligned} f(x) &= 1, & x &\leq 0, \\ &= 1 - x, & 0 \leq x \leq 1, \\ &= 0, & x &\geq 1. \end{aligned} \right\} \quad (1.1.51)$$

The characteristic equations of Eq. (1.1.49) in parametric form in terms of a parameter t are given by

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 1 \quad \text{and} \quad \frac{du}{dt} = 0. \quad (1.1.52)$$

Let $x(s, 0) = s$, $y(s, 0) = 0$ and $u(s, 0) = f(s)$.

The solution of Eq. (1.1.52) is given by

$$x(s, t) = \begin{cases} s + t, & s \leq 0, \\ (1 - s)t + s, & 0 \leq s \leq 1, \\ s, & s \geq 1. \end{cases} \quad y = t \quad (1.1.53)$$

$$u(s, t) = \begin{cases} 1, & s \leq 0, \\ (1 - s), & 0 \leq s \leq 1, \\ 0, & s \geq 1. \end{cases} \quad (1.1.54)$$

The characteristic curves and $u(x, y)$ are shown in Fig. 1.1.1. It is clear from the figure that the characteristic curves intersect in the region $x \geq 1$, $y \geq 1$ and therefore $u(x, t)$ is not defined (is not single valued) in this region. $u(x, y)$ defined by Eq. (1.1.54) ceases to be the classical solution of Eq. (1.1.49) in the region $x \geq 1$, $y \geq 1$ and Eq. (1.1.49) is satisfied in this region only in the weak sense. It may be noted that the solution $u(x, y)$ could be discontinuous in some portion of the region even if $f(x)$ is a continuous and differentiable function throughout the region. If Eq. (1.1.49) is written as

$$\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0, \quad P = u, \quad Q = \frac{1}{2}u^2, \quad (1.1.55)$$

then it can be shown (cf. [12]) that the weak solution is discontinuous across a curve which is called a *shock* and whose equation is given by

$$[P]dx - [Q]dy = 0. \quad (1.1.56)$$

The notation $[f]$ denotes the difference between the limits of a function f as the shock or a surface of discontinuity of f is approached from both sides. Conditions of the type Eq. (1.1.56) are called *Rankine–Hugoniot conditions*. In the present problem $[P] = 1$ and $[Q] = 1/2$, and therefore the shock is given by $y = 2x + d$. The constant d can be determined if we know the point from which the shock is emanating, for example, in the present problem this point is (1,1) (see Fig. 1.1.2). Shock is the oldest form of the free boundary and its origin lies in the study of gas dynamics [13].

If Eq. (1.1.49) is multiplied by u^n , $n \geq 1$, then P and Q in Eq. (1.1.55) will change. This change will result in the nonuniqueness of the shock and infinitely many shocks can be obtained. For nonlinear hyperbolic equations of the type Eq. (1.1.49), further information in the form of physical arguments involving stability, entropy, dissipation or continuous dependence on the initial data is needed to ensure uniqueness (cf. [14, 15]).

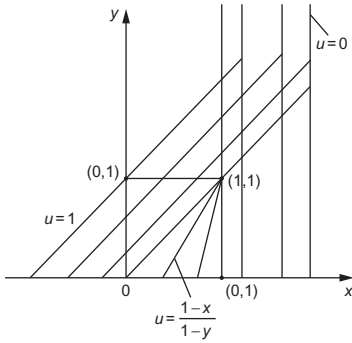


Fig. 1.1.1 Characteristic curves and $u(x, y)$.

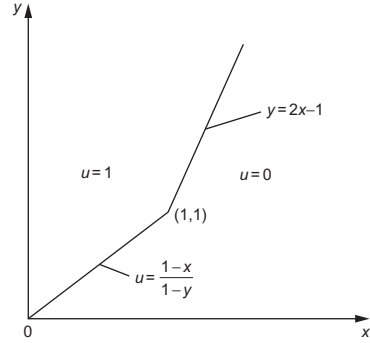


Fig. 1.1.2 Geometry of the shock.

There are some important differences between the present problem and the free boundary Problems 1.1.1–1.1.5. In the previous five problems, the free boundaries can be identified without much difficulty and the boundary conditions on them can be imposed after some thought. It is a different matter that the classical solution may or may not exist. In the solution of Eqs (1.1.49)–(1.1.50), there is no indication of a free boundary. It is only through the construction of the solution that we come to know about the magnitudes of jumps in P and Q . Eq. (1.1.56) can be obtained only through the weak solution and not through the classical approach, which was followed in other problems.

Problem 1.1.7 (Free Boundary Associated With a Frictional Oscillator Problem). This interesting free boundary problem has been reported in [15]. As shown in Fig. 1.1.3, a block of mass m rests on a conveyor belt moving with a constant velocity V . The forces acting on the mass are: (1) spring force with a spring modulus S_m , (2) prescribed force $F(t)$ assumed to be sufficiently smooth and (3) Coulomb frictional force with coefficient of friction μ . The motion of the block with mass m is governed by the equation

$$\frac{d^2x}{dt^2} + \mu mg \left\{ \operatorname{sgn} \left(\frac{dx}{dt} - V \right) \right\} + S_m x = F(t), \quad (1.1.57)$$

where

$$\begin{aligned} \operatorname{sgn}(z) &= 1 \text{ if } z > 0, \\ &= -1 \text{ if } z < 0, \end{aligned} \quad (1.1.58)$$

and g is the acceleration due to gravity.

At time $t = 0$, x and dx/dt are prescribed. Both analytical and numerical solutions of Eq. (1.1.57) are extremely difficult as at each instant of time information is required whether $dx/dt > V$, $= V$ or $< V$. The unknown instants of time $t = t_i$, $i = 1, 2, \dots$ or the points $x_i = x(t_i)$, $i = 1, 2, \dots$ at which $dx/dt = V$ can be regarded as free boundaries. When $dx/dt = V$, $d^2x/dt^2 = 0$. The second derivative becomes discontinuous at $t = t_i$. Once dx/dt becomes V at $t = t_i$, it will remain so in some interval $t_i \leq t \leq t_i^*$ until at $t = t_i^*$, $|F(t) - S_m x| > \mu mg$. Again at $t = t_{i+1}$, dx/dt may be equal to V . If so, $t = t_{i+1}$ is a free boundary. If it is assumed that mass does not jump and there is no abrupt change in the velocity of the mass, then it can be assumed that $x(t)$ and dx/dt are continuous functions of time and $dx/dt = V$ at the free boundary. These assumptions provide boundary conditions at each free boundary.

There is a fundamental difference between this problem and the free boundary Problems 1.1.1–1.1.6. In the earlier problems, free boundaries arose due to the physical situations or the mathematical nature of the solution such as a weak solution, but in the present problem free boundaries arise due to discontinuities in the physical properties of the solution.

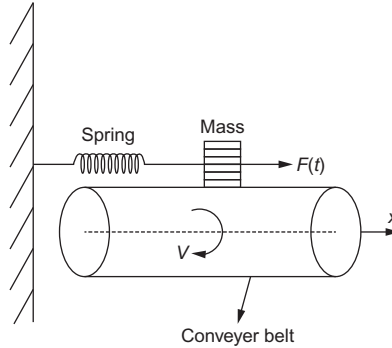


Fig. 1.1.3 Frictional oscillator.

Problem 1.1.8 (Impact of a Visco-Plastic Bar on a Rigid Wall). A bar of length b made up of visco-plastic incompressible material moving with constant velocity $-V_0$ hits a rigid wall at time $t = 0$. We consider a one-dimensional problem in which x -axis is taken opposite to the initial motion of the bar and the rigid wall is taken at $x = 0$. After the bar hits the wall, compressional stresses develop in the bar giving rise to visco-plastic flow of the material in the region. If $V(x, t)$ is the velocity of the bar for $t > 0$, then the gradient of $V(x, t)$ or the velocity of deformation can be expressed as follows (cf. [16]):

$$\frac{\partial V}{\partial x} = \begin{cases} \frac{\tau + \tau_0}{\mu}, & |\tau| \geq \tau_0, \quad \tau_0 > 0, \quad \tau \leq 0, \\ 0, & |\tau| \leq \tau_0. \end{cases} \quad (1.1.59)$$

Here, τ represents compressional stress, which is negative as the x -axis is oriented opposite to the direction of motion of the bar, τ_0 is the stress at the limit point and μ is the coefficient of viscosity of the material.

It will be assumed that any disturbance is propagated over the whole bar instantaneously. The equation of motion in the visco-plastic region can be easily obtained by using Newton's second law of motion. We have

$$\rho \frac{\partial V}{\partial t} = \frac{\partial \tau}{\partial x} = \mu \frac{\partial^2 V}{\partial x^2}, \quad 0 \leq x \leq S_0(t), \quad 0 < t < t_*, \quad S_0(0) = 0, \quad (1.1.60)$$

where ρ is the density of the material and $x = S_0(t)$ is the interface between elastic and visco-plastic regions. In the elastic region

$$\frac{\partial V}{\partial x} = 0, \quad |\tau| \leq \tau_0, \quad S_0 < x < b. \quad (1.1.61)$$

Integrating Eq. (1.1.61), we obtain

$$V = -X(t), \quad X(0) = V_0. \quad (1.1.62)$$

In obtaining Eq. (1.1.62) it has been assumed that the elastic portion of the rod moves like a rigid body.

Using again Newton's second law of motion at $x = S_0$, we obtain

$$-M \frac{dX}{dt} = \tau|_{S_0} + F_0, \quad (1.1.63)$$

where M is the mass of the elastic part and F_0 is the area of cross-section of the rod assumed to be uniform. Since $M = F_0 \rho (b - S_0(t))$, Eq. (1.1.63) becomes

$$\frac{dX}{dt} = -\frac{\tau_0}{\rho(b - S_0(t))}. \quad (1.1.64)$$

In obtaining Eq. (1.1.64) it has been assumed that stress is continuous at the interface $x = S_0(t)$. Using the condition that $V(x, t)$ is also continuous at $x = S_0(t)$, we get

$$V(S_0, t) = -X(t). \quad (1.1.65)$$

Also

$$\frac{\partial}{\partial x} V(S_0, t) = 0. \quad (1.1.66)$$

The initial and boundary conditions for $V(x, t)$ are

$$V(x, 0) = -V_0, \quad 0 \leq x \leq b; \quad V(0, t) = 0, \quad t > 0. \quad (1.1.67)$$

The functions $V(x, t)$, $S_0(t)$ and $X(t)$ are to be determined. An approximate solution of the above problem has been discussed in [16].

The free boundary in this problem arises due to the change in the physical properties of the system.

Problem 1.1.9 (A Problem With Discontinuous Moving Boundary). Discontinuity of the free boundary in multidimensional problems can be easily demonstrated geometrically. For example, a piece of ice floating in water breaks into two pieces after melting for some time. A discontinuous moving boundary in a one-dimensional problem does not commonly occur. In [17] an interesting problem of discontinuous moving boundary which is associated with the diffusion of moisture in a porous capillary tube of length unity is discussed. At time $t = 0$, the portion of the tube $0 \leq x \leq x_0$, $x_0 < 1$ is filled with moisture and the remaining portion is dry. The temperature $T(x, t)$ of the moisture is less than the boiling temperature $T = 0$, except at $x = x_0$ where $T = 0$. The temperature of the dry air in some neighbourhood of $x = x_0$ on the right is greater than zero and in the remaining portion, the temperature of the air is less than zero. There is a continuous flow of moisture into the tube at $x = 0$. At $x = 1$, the dry air is getting heated, causing evaporation to take place and making the moisture advance into the dry air. Let $T_1(x, t)$ and $T_2(x, t)$ be the temperatures of the moisture and the dry air respectively, $W_1(x, t)$ the concentration of the moisture, and $x = S(t)$, the equation of the free boundary which is the interface between moisture and the dry air.

Under certain initial conditions it may happen that at some instant of time, say $t = m$, $m > 0$, the temperature in some neighbourhood δ_m of the moving boundary becomes less than or equal to zero. In this case the moisture will advance with a jump into the dry part, i.e.

$$S(m+) - S(m-) = \delta_m > 0. \quad (1.1.68)$$

The problem is concerned with finding T_1 , T_2 , W_1 and $S(t)$. The formulation of this problem is as follows:

Differential equations

$$\frac{\partial T_i}{\partial t} = a_i^2 \frac{\partial^2 T_i}{\partial x^2}, \quad (x, t) \in D_i, \quad i = 1, 2, \quad (1.1.69)$$

$$\frac{\partial W_1}{\partial t} = d \frac{\partial^2 W_1}{\partial x^2}, \quad (x, t) \in D_1, \quad (1.1.70)$$

$$D_1 = \{(x, t) : 0 < x < S(t), 0 < t < m\} \cup \{(x, t) : 0 < x < S(t), m < t < t_*\},$$

$$D_2 = \{(x, t) : S(t) < x < 1, 0 < t < m\} \cup \{(x, t) : S(t) < x < 1, m < t < t_*\}.$$

Here, a^2 is the thermal diffusivity and d is the mass diffusivity. Eq. (1.1.69) is the heat conduction equation and Eq. (1.1.70), the mass diffusion equation (Fick's law).

Initial conditions

$$T_1(x, 0) = \phi_1(x) \leq 0, \quad W_1(x, 0) = \psi_1(x) \geq 0; \quad 0 \leq x \leq x_0, \quad (1.1.71)$$

$$T_2(x, 0) = \phi_2(x), \quad x_0 \leq x \leq 1; \quad \phi_2(x_0) > 0 \text{ and } \phi_2(1) > 0. \quad (1.1.72)$$

Boundary conditions at $x = 0$ and $x = 1$

$$T_1(0, t) = f_1(t) \leq 0, \quad W_1(0, t) = g_1(t) \geq 0; \quad 0 < t \leq t_*, \quad (1.1.73)$$

$$T_2(1, t) = f_2(t) \geq 0. \quad (1.1.74)$$

Boundary conditions at the moving boundary $x = S(t)$, $t \neq m$

$$\{T_1(x, t), W_1(x, t)\}_{x=S(t)} = 0, \quad t > 0, \quad S(0) = x_0, \quad (1.1.75)$$

$$K_2 \left. \frac{\partial T_2}{\partial x} \right|_{x=S(t)} = \alpha(T_2(S(t) + 0, t) - T_1(S(t) - 0, t)), \quad t > 0. \quad (1.1.76)$$

Here, K is the thermal conductivity and α is the heat transfer coefficient. The convective boundary condition in Eq. (1.1.76) arises because of the discontinuity of temperatures at $x = S(t)$, $t \neq m$. Heat balance at $x = S(t)$ implies

$$\begin{aligned} C_2 \rho_2 T_2(S(t) + 0, t) \frac{dS}{dt} + K_2 \frac{\partial T_2}{\partial x}(S(t) + 0, t) - K_1 \frac{\partial T_1}{\partial x}(S(t) - 0, t) \\ = -d q_1 \frac{\partial W}{\partial x}(S(t) - 0, t). \end{aligned} \quad (1.1.77)$$

Here, C is the specific heat, ρ the density and q_1 the latent heat of evaporation. Definitions of different parameters have been given later in Section 2.1.3. The derivation of Eq. (1.1.77) is based on the law of conservation of energy (see Section 1.4.7).

A sufficiently small real number $\varepsilon > 0$ exists such that

$$T_2(x, t) > 0 \text{ for } S(t) < x < S(t) + \varepsilon, \quad t \neq m. \quad (1.1.78)$$

For $t = m$,

$$\left. \begin{aligned} T_1(x, m+) = T_2(x, m-), \text{ and } W_1(x, m+) = 0; S(m-) \leq x \leq S(m+), \\ m \pm = m \pm 0, \end{aligned} \right\} \quad (1.1.79)$$

$$T_2(x, m) \leq 0, \quad S(m-) \leq x \leq S(m+). \quad (1.1.80)$$

The existence and uniqueness of this problem under suitable assumptions have been discussed in [17] and sufficient conditions for the existence of a discontinuous moving boundary are given in [18]. Discontinuity in the temperature and in the free boundary is a typical feature of this problem.

Problem 1.1.10 (Penetration of Solvents in Polymers). Consider a slab of a glassy polymer, such as methyl methacrylate in contact with a solvent, n -alkyl alcohol [19]. If the solvent concentration exceeds a threshold value, say, $q \geq 0$, then the solvent moves into the polymer, creating a swollen layer in which the solvent diffuses according to Fick's law for mass diffusion. If $W(x, t)$ is the concentration of the solvent in the polymer and $x = S(t)$ is the free boundary representing the penetration depth of the solvent in the polymer, then W and S satisfy the following equations:

$$W_t - W_{xx} = 0, \quad (x, t) \in D_{t_*} = \{(x, t) : 0 < x < S(t), 0 < t < t_*\}, \quad (1.1.81)$$

$$W(0, t) = 1, \quad 0 < t < t_*, \quad (1.1.82)$$

$$S(t) = f\{W(S(t))\}, \quad S(0) = 0, \quad 0 < t < t_*, \quad (1.1.83)$$

$$W_x(S(t), t) = -\dot{S}(t)[W(S(t), t) + q], \quad 0 < t < t_*. \quad (1.1.84)$$

In these equations, normalized solvent concentration is represented by $W + q$. W represents the excess concentration, normalized to 1 at $x = 0$. Eq. (1.1.83) describes the prescribed penetration law and Eq. (1.1.84) arises from the mass conservation at the free boundary. The well-posedness, existence, uniqueness, etc., of the solution of the above problem have been shown in [19]. A numerical method for its solution has also been suggested in [19]. Some more mathematical models describing the crystallization of polymers and their mathematical analysis are presented in [20]. In Eq. (1.1.83), $S(t)$ is a function of the concentration.

Problem 1.1.11 (Filtration of Water Through Oil in a Porous Medium). Consider a one-dimensional problem in a semiinfinite porous medium $x \geq 0$ of porosity m . At time $t = 0$, the region $0 \leq x \leq b$ is filled with oil and the region $b \leq x < \infty$ is filled with water. Water percolates into oil, and so for $t > 0$ there will be three regions. Water-filled region, $b < x < \infty$, will have 100% saturation of water; oil-bearing region, $0 \leq x < S(t)$, will have 100% saturation of oil; in the region $S(t) < x < b$, both water and oil mixture will be present. This region can be called an intermediate zone. $S(t)$ is the free boundary. The oil content of this intermediate zone changes with time, and tends to reach a limiting state called the residual oil saturation state. This stage is characterized by the fact that if time is counted from the moment of passage of the oil–water contact boundary past a fixed element of volume, the degree of oil saturation of this volume will not depend on time and the flow in this region will be of one phase. It will be assumed that water filters through the intermediate zone while the oil in this zone remains stationary [21]. However the rate of penetration of water in the intermediate zone is lower than in the water-filled zone as the oil concentration in the intermediate zone is greater than zero. Under appropriate assumptions, the following one-dimensional model is obtained:

$$u_0 = -\frac{\lambda_0}{\mu_0} \frac{\partial p_0}{\partial x}; \quad \alpha_0^2 \frac{\partial^2 p_0}{\partial x^2} = \frac{\partial p_0}{\partial t}, \quad b < x < \infty, \quad t > 0, \quad (1.1.85)$$

$$u_1 = -\frac{\lambda_1}{\mu_1} \frac{\partial p_1}{\partial x}; \quad \alpha_1^2 \frac{\partial^2 p_1}{\partial x^2} = \frac{\partial p_1}{\partial t}, \quad S(t) < x < b, \quad t > 0, \quad (1.1.86)$$

$$u_2 = -\frac{\lambda_2}{\mu_2} \frac{\partial p_2}{\partial x}; \quad \alpha_2^2 \frac{\partial^2 p_2}{\partial x^2} = \frac{\partial p_2}{\partial t}, \quad 0 < x < S(t), \quad t > 0. \quad (1.1.87)$$

The subscripts 0, 1, 2 stand for the water-filled, intermediate and oil-rich zones, respectively. u_i and p_i , $i = 0, 1, 2$, denote the filtration velocities and pressures in the three different zones. The constants λ_i , α_i and μ_i , $i = 0, 1, 2$, denote the coefficients of permeability, piezoconductivity and viscosity of the i th zone. The equation of velocity in terms of pressure gradient is the result of Darcy's law [8] and the pressure obeys the equation of piezoconductivity.

Boundary conditions at $x = b$ and $x = S(t)$ follow from the continuity of velocities and pressures and are given by

$$u_0 = u_1, \quad p_0 = p_1; \text{ at } x = b, \quad (1.1.88)$$

$$u_1 = u_2, \quad p_1 = p_2; \text{ at } x = S(t). \quad (1.1.89)$$

At the free boundary, an additional condition is required which is obtained from the 'mass balance' consideration

$$m(1 - \delta_1 - \delta_2) \frac{dS}{dt} = u_1. \quad (1.1.90)$$

Here, δ_1 and δ_2 are the contents of water and the residual oil saturation, respectively in the transition zone, and m is the porosity of the stratum. To complete the formulation of the problem, suitable initial and boundary conditions for pressure should be prescribed. For example in [21], the initial pressure in the entire stratum is assumed to be constant. Similarly it is assumed that pressure has a constant value at the boundary $x = 0$. The zero reference point for calculation of pressure is so chosen that $p_2(0, t) = 0$. With the above assumptions the initial and boundary conditions can be written as

$$p_i|_{t=0} = P = \text{constant}, \quad i = 0, 1, 2; \quad p_2|_{x=0} = 0. \quad (1.1.91)$$

An approximate solution under the quasi-steady approximation, valid at the initial stages has been obtained in [21] with constant parameter values. The boundary conditions (1.1.89) do not contain the velocity of the free boundary.

Problem 1.1.12 (Obstacle Problem for a String). This problem belongs to a class of problems which have a *variational inequality* formulation. Three different types of formulations of this obstacle problem are possible and their equivalence is discussed in Chapter 7. A new notion of codimensionality-two of the free boundary is associated with this problem and this will be discussed briefly in Section 1.2. Consider a weightless elastic string which is held tight between two fixed points $P_1 = (0, 0)$ and $P_2 = (b, 0)$, $b > 0$ in R^2 -plane. This string is displaced upwards by a rigid body called an obstacle (cf. [15, 22]). A view of the cross-section of this system of string and obstacle from above in R^2 -plane is shown in Fig. 1.1.4. Let $y = \psi(x)$ be the equation of the cross-section of the obstacle in R^2 , i.e. the equation of the curve Q_1ABQ_2 . $\psi(x)$ is assumed to be sufficiently smooth. We shall relax this smoothness condition later (see Section 7.2.5). Let $y = u(x)$ be the equation of the string in the equilibrium position or the equation of the curve P_1ABP_2 . The problem is to find the function $u(x)$ and the arc AB of the string which is in contact with the obstacle and on which $u(x) = \psi(x)$. The points A and B are free boundaries and in the present case we can call them 'free points'. Once A and B are determined, the arc AB is also determined because $\psi(x)$ is known.

The formulation of the problem is as follows:

$$u(0) = u(b) = 0 \quad (\text{fixed end conditions}), \quad (1.1.92)$$

$$u(x) = \psi(x) \text{ on } AB; \quad u(x) \geq \psi(x) \text{ on } P_1ABP_2, \quad (1.1.93)$$

$$u''(x) \leq 0, \quad (1.1.94)$$

$$u(x) > \psi(x) \Rightarrow u''(x) = 0. \quad (1.1.95)$$

The second equation in Eq. (1.1.93) implies that the string does not penetrate the obstacle. The concavity of the string as viewed from the x -axis implies Eq. (1.1.94) and the tightness of the elastic string implies that the portions P_1A and BP_2 are straight lines and so $u''(x) = 0$. The two boundary conditions at the free boundary can be obtained from the continuity of u and du/dx , i.e.

$$[u]_A = [u]_B = \left[\frac{du}{dx} \right]_A = \left[\frac{du}{dx} \right]_B = 0, \quad (1.1.96)$$

where $[f]$ stands for the jump in the quantity under consideration at the given point. Eq. (1.1.95) is equivalent to the following condition

$$[u(x) - \psi(x)]u''(x) = 0. \quad (1.1.97)$$

In view of Eqs (1.1.93)–(1.1.95), Eq. (1.1.97) is valid. When $u(x) = \psi(x)$, Eq. (1.1.97) is satisfied and when $u(x) \neq \psi(x)$ then the second equation in Eq. (1.1.93) implies $u(x) > \psi(x)$ and from Eq. (1.1.95), $u''(x) = 0$ and therefore Eq. (1.1.97) is satisfied. If Eq. (1.1.97) holds and if $u(x) > \psi(x)$, then $u''(x) = 0$, which is Eq. (1.1.95). The formulation (1.1.92)–(1.1.95) is equivalent to the formulation given by Eqs (1.1.92)–(1.1.94), (1.1.97). Let us call the formulation (1.1.92)–(1.1.95) as Problem (R_1) and the formulation equivalent to Problem (R_1) as Problem (R_2) . Consider the following minimization problem which is concerned with the minimization of the energy of the above string:

$$\min_{u \geq \psi} \int_{P_1}^{P_2} \left(\frac{du}{dx} \right)^2 dx, \quad \forall u(x) \in \mathcal{D}, \quad (1.1.98)$$

where $\mathcal{D} = \{u : u(0) = u(b) = 0, u \text{ and } \partial u/\partial x \text{ are continuous, and } u \text{ satisfies relations (1.1.94) and (1.1.95)}\}$.

We shall call this formulation in Eq. (1.1.98) as Problem (R_3) . This is a fixed domain formulation as it does not consider separate formulations on different portions of P_1ABP_2 . In view of the equivalence of Problems (R_2) and (R_3) which will be established in Chapter 7, the formulation (R_2) is also called ‘variational formulation’ of the obstacle problem. A variational inequality formulation of this problem has been given in Chapter 7.

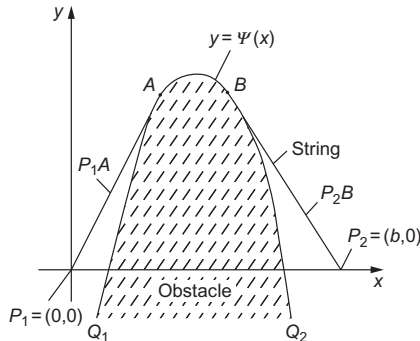


Fig. 1.1.4 Obstacle and the stretching of the string.

1.2 FREE BOUNDARY PROBLEMS WITH FREE BOUNDARIES OF CODIMENSION-TWO

In [Problem 1.1.12](#), the free boundary consists of just two points A and B . Therefore the dimension of the free boundary is zero, whereas the spacial dimension of the problem is two. Another example of a free boundary of codimensional-two (or free points) is the tip of a propagating line crack in an elastic medium. We shall see later in [Section 3.2.2](#) that in the problems concerning initiation of solidification/melting along a known surface, codimension-two free boundaries could be time dependent. If the spacial dimension of the problem is n (the problem could be transient) and the dimension of the free boundary is $n - 2$, then we say that the free boundary is of codimension-two. Several problems with free boundaries of codimension-two have been discussed in [\[23\]](#). In problems [\(1.1.1\)–\(1.1.11\)](#), the dimension of the free boundary is only one less than the spacial dimension of the problem. The distinction between the free boundaries of codimension-two and the free boundaries of codimension-one is not superficial but it raises some serious questions [\[23\]](#), some of which are as follows:

1. Can these problems with free boundaries of codimension-two, in some sense such as linearization, be modelled as limits of free boundary problems of codimension-one?
2. What can be said about the existence, uniqueness and regularity of solutions to models as stated? In particular, is any of the information redundant?
3. What methodology is available to solve these problems explicitly? No systematic methodology has yet been proposed for nonlinear problems of this type.
4. If solutions exist, are they stable to perturbations in the direction parallel to the free curve?
5. Is there a possibility of generating the models either to make their analysis or numerical solutions easier?

1.3 THE CLASSICAL STEFAN PROBLEM IN ONE-DIMENSION AND THE NEUMANN SOLUTION

Stefan problems are free boundary problems with some special features which can be easily explained with the help of the following one-dimensional solidification problem. At time $t = 0$, a pool of liquid at constant temperature $T_0 > T_m$, occupies a semiinfinite region $x \geq 0$. T_m is the temperature at constant pressure at which the liquid solidifies. It is called *freezing temperature* or *phase-change temperature*. In the case of pure metals such as copper, T_m is called *melting temperature* also because in the case of pure metals freezing and melting temperatures are the same. T_m is also called *equilibrium phase-change temperature* as both solid and liquid phases can stay together in thermodynamic equilibrium at $T = T_m$ (cf. [Section 2.1.4](#)). The fixed boundary $x = 0$ is being cooled, for example, temperature $T_b < T_m$ is prescribed at $x = 0$. In this case solidification starts instantaneously at $x = 0$. For any $t > 0$, the region $0 \leq x < \infty$ will consist of solid and liquid phases with the solid phase occupying the region $0 \leq x < S(t)$ and the liquid phase the region $S(t) < x < \infty$. $S(t)$ is the free boundary or the phase-change boundary/interface. This solidification problem can be formulated mathematically as follows:

In the solid region

$$C_S \rho_S \frac{\partial T_S}{\partial t} = K_S \frac{\partial^2 T_S}{\partial x^2}, \quad 0 < x < S(t), \quad t > 0, \quad (1.3.1)$$

$$T_S(x, t)|_{x=0} = T_b < T_m, \quad t > 0. \quad (1.3.2)$$

In the liquid region

$$C_L \rho_L \frac{\partial T_L}{\partial t} + C_L \rho_L u_x \frac{\partial T_L}{\partial x} = K_L \frac{\partial^2 T_L}{\partial x^2}, \quad S(t) < x < \infty, \quad t > 0, \quad (1.3.3)$$

$$T_L(x, t)|_{t=0} = T_0, \quad T_L(x, t)|_{x \rightarrow \infty} = T_0. \quad (1.3.4)$$

At $x = S(t)$

$$T_S = T_L = T_m, \quad (1.3.5)$$

$$K_S \frac{\partial T_S}{\partial x} - K_L \frac{\partial T_L}{\partial x} = \{l\rho_S + (\rho_L C_L - \rho_S C_S)T_m\} \frac{dS}{dt}, \quad (1.3.6)$$

$$S(0) = 0. \quad (1.3.7)$$

Eqs (1.3.1)–(1.3.6) are not dimensionless. Temperatures T_b , T_0 , T_m and thermophysical parameters ρ , C , K and l are known quantities. The subscripts S and L stand for solid and liquid regions.

The symbols used in Eqs (1.3.1)–(1.3.6) are explained in the ‘List of Notations’. The definitions of thermophysical parameters C , K , l , etc., will be discussed later in [Section 2.1.3](#). At present, our interest is to highlight some basic features of Stefan problems. Eq. (1.3.1) is the Fourier heat conduction equation derived on the basis of Fourier’s law given in Eq. (1.3.8) for heat conduction in a homogeneous isotropic medium:

$$\vec{q} = -K \text{ grad } T. \quad (1.3.8)$$

Here, \vec{q} is called a heat flux vector and gives the heat flow at a spacial point at any instant of time t . $\vec{q} \cdot \vec{n}$ gives heat flow per unit area, per unit time in the direction of the outward drawn unit normal \vec{n} to the surface of the region under consideration. The derivation of heat conduction equation is given in many books such as [24] and is obtained on the basis of principle of conservation of energy (cf. [Section 1.4.7](#)). The heat equation (1.3.1) is based on a *macroscopic model* but it can also be derived on the basis of a *microscopic model*. The transport of heat energy takes place due to the random motion of molecules. The parabolic heat equation as well as hyperbolic heat equations can also be derived by considering discrete random walks of molecules [25].

If the densities of solid and liquid phases are not equal, then liquid may acquire motion whose velocity u_x in the one-dimensional case can be obtained from the mass balance condition at the interface given below

$$\rho_L u_x = (\rho_L - \rho_S) \frac{dS}{dt}. \quad (1.3.9)$$

If $\rho_S > \rho_L$, then there will be a shrinkage of total volume in solidification, and u_x will be negative as dS/dt is positive in the above problem. If $\rho_S < \rho_L$, then there will be an expansion of the volume and u_x will be positive as dS/dt is positive. Eq. (1.3.3) is the Fourier's heat conduction equation for the liquid in motion. Boundary conditions at the fixed boundary $x = 0$ could differ from that in Eq. (1.3.2) (see Section 1.4.4) but in the 'Neumann solution', a constant temperature is prescribed at $x = 0$. At $t = 0$ it was only the liquid phase therefore the initial condition has been prescribed only for the liquid. In the present problem T_0 is a constant. If T_m is a constant, then the conditions in Eq. (1.3.5) are called *isotherm conditions*. Eq. (1.3.6) is the *dynamic energy balance condition*. Both Eqs (1.3.5), (1.3.6) are based on the assumption of instantaneous achievement of local thermodynamic equilibrium at the phase-change interface. $-K_L \partial T_L / \partial x$ is the heat flux from the liquid outwards (the outward normal to liquid at $S(t)$ points into the solid) and $-K_S \partial T_S / \partial x$ is the heat flux from the solid outwards (the outward normal to solid at $S(t)$ points into the liquid).

Eq. (1.3.6) can be derived by using the energy conservation law. It gives the energy balance at the phase-change boundary. In Section 1.4.1, the derivation of Eq. (1.3.6) has been obtained in a more general context. When the phase-change takes place from liquid to solid, the latent heat l released by the system is removed by conduction. Eq. (1.3.6) is a dynamic compatibility condition and is commonly known as the *Stefan condition*. It is called a condition of Rankine–Hugoniot type. Eq. (1.3.6) can also be stated in the form

$$\left[K \frac{\partial T}{\partial x} \right]_{\text{liquid}}^{\text{solid}} = -[H]_{\text{liquid}}^{\text{solid}} V_n. \quad (1.3.10)$$

Here, H is the enthalpy per unit volume, which is the sum of the latent heat plus the sensible heat, and V_n is the normal component of the velocity of the interface with the unit normal \vec{n} pointing into the liquid.

1.3.1 Melting Problem

The problem discussed in Eqs (1.3.1)–(1.3.6) is a solidification problem as the phase changes from liquid to solid. The melting problem in which the phase changes from solid to liquid can be easily formulated by making appropriate changes in Eqs (1.3.1)–(1.3.6). For example, Eq. (1.3.3) is still valid for liquid but liquid region is now $0 < x < S(t)$. Similarly other equations can be modified. During solidification, the system releases latent heat but during melting the system 'absorbs' latent heat. Since we take $l > 0$, a minus sign should be put for heat absorbed. Eq. (1.3.10) is still valid for the problem of melting provided the changed directions of normals to the two regions at the free boundary are taken into account. Since the formulation of both solidification and melting problems has the same mathematical structure, they are said to be mathematically analogous. If a solution to one of them is obtained, then the solution to the other problem can be obtained by making some minor changes in it. It is for this reason that it suffices to study either of them.

1.3.2 Neumann Solution

The method of analytical solution of Eqs (1.3.1)–(1.3.7) which is discussed below is based on some special type of solutions of heat equations (1.3.1), (1.3.3), which can be written in terms of error and complementary error functions [26]. An exact analytical solution of the

Stefan problem even in the one-dimensional case is possible only in few simple cases. If in Eq. (1.3.2), T_0 is time dependent or if a constant flux is prescribed at $x = 0$ then the method discussed below fails to provide a solution. The solution of Eq. (1.3.1) can be written as

$$T_S = A \operatorname{erf} \frac{x}{2(k_S t)^{1/2}} + T_b, \quad (1.3.11)$$

$$T_L = T_0 - B \operatorname{erf} c \left\{ \frac{x}{2(k_L t)^{1/2}} + \frac{\delta(\rho_S - \rho_L)k_S^{1/2}}{\rho_L k_L^{1/2}} \right\}. \quad (1.3.12)$$

A and B are constants to be determined. It is easy to see that T_S satisfies Eq. (1.3.1) and T_L satisfies Eq. (1.3.3) with u_x given in Eq. (1.3.9) and the initial condition given by the first condition in Eq. (1.3.4). Let $S(t)$ be given by

$$S(t) = 2\delta(k_S t)^{1/2}, \quad \delta \text{ an unknown constant.} \quad (1.3.13)$$

On $x = S(t)$ (given in Eq. 1.3.13), both T_S and T_L are constant, and $S(0) = 0$. Now we have three unknowns, namely, A , B and δ , and three conditions, i.e. Eq. (1.3.6) and isotherm conditions for T_S and T_L . After some algebraic manipulations, the unknown constants can be obtained thus

$$A = (T_m - T_b) / \operatorname{erf}(\delta), \quad (1.3.14)$$

$$B = (T_0 - T_m) / \operatorname{erf} c \left(\delta \rho_S k_S^{1/2} / (\rho_L k_L^{1/2}) \right), \quad (1.3.15)$$

$$\frac{e^{-\delta^2}}{\operatorname{erf} \delta} - \frac{(T_0 - T_m) K_L k_S^{1/2} e^{-(\delta^2 \rho_S^2 k_S / \rho_L^2 k_L)}}{T_m K_S k_L^{1/2} \operatorname{erfc} \left((\delta \rho_S k_S^{1/2} / (\rho_L k_L^{1/2})) \right)} = \frac{\delta \hat{l} \pi^{1/2}}{C_S T_m}, \quad (1.3.16)$$

$$\hat{l} = l + (C_L \rho_L / \rho_S - C_S) T_m. \quad (1.3.17)$$

This completes the solution of Eqs (1.3.1)–(1.3.7). When $\rho_S = \rho_L = \rho$, this solution is called *Neumann solution* which was given by Neumann [27] in 1860. The first published discussion of such problems seems to be that by Stefan [28] in a study of the thickness of polar ice.

The problem discussed above is called a *two-phase Stefan problem* as there are two distinct phases. If the liquid is initially at the equilibrium temperature T_m , then heat conduction will take place only in the solid phase. In principle two phases will be present, but since there is no temperature gradient in the liquid, such problems are called *one-phase Stefan problems*. If there are n (> 2) distinct phases separated by $(n - 1)$ distinct phase-change boundaries, then we have an n -phase Stefan problem. It is not possible to give a precise definition of a Stefan problem but in the light of the above discussion some of the characteristic features of Stefan problems can be described as follows: (1) the transport phenomenon is that of heat transfer governed by parabolic equations, (2) any two distinct phases are separated by a sharp phase-change interface, (3) at the phase-change boundary, temperature is prescribed which is known and (4) heat flux is discontinuous across the phase-change boundary and the latent heat is released or absorbed at the interface. Phase changes in which the heat flux is discontinuous are known as *first-order phase transitions*. We shall learn more and more about these characteristic features of Stefan problems as we proceed further. In many physical problems although a free boundary exists, there is some deviation of features in them from the characteristic features of

the Stefan problem. It has become almost customary to indicate such problems by a suitable nomenclature. For example, if a hyperbolic equation is considered instead of a parabolic equation, then it is called ‘hyperbolic Stefan problem’. Similarly, we have a ‘Stefan problem with kinetic condition’ if a kinetic condition is introduced at the free boundary or ‘Stefan problem with supercooling’ if the liquid is supercooled.

The term ‘classical’ is generally used to distinguish the classical formulation from the *weak formulation* (see Section 5.2 for the weak formulation of the classical Stefan problem). In a classical formulation it is assumed that there exists a sufficiently regular sharp interface, which separates two distinct phases. Two phases will be considered different if they differ in their composition, structure, or properties. At the phase-change interface, two boundary conditions are prescribed, one in the form of temperature and the other arising from the energy balance at the interface.

1.4 CLASSICAL FORMULATION OF MULTIDIMENSIONAL STEFAN PROBLEMS

1.4.1 Two-Phase Stefan Problem in Multiple Dimensions

We shall now extend the mathematical formulation given in Eqs (1.3.1)–(1.3.7) to multidimensional problems. For simplicity we consider R^3 . Consider a heat-conducting open-bounded region $G \subset R^3$ whose boundary ∂G is smooth, for example $\partial G \in C^{2+\eta}$, for some $\eta > 0$. Let $G(t) = G \times \{t\}$ and $\partial G(t) = \partial G \times \{t\}$, $0 \leq t \leq t_* < \infty$ for some suitable real number t_* . $G^{t_*} = \cup_{0 \leq t \leq t_*} G(t)$ is a cylinder over G (see Figs 1.4.1–1.4.4). G consists of two phases, for example, solid and liquid which for the sake of convenience will be denoted by subscripts 1 (solid) and 2 (liquid). Let $G_i(t)$, $i = 1, 2$ be the region occupied by the i th phase at time $t \in [0, t_*]$. $G(t)$ admits disjoint decomposition as

$$G(t) = G_1(t) \cup S(t) \cup G_2(t), \quad \forall t \in [0, t_*], \quad (1.4.1)$$

where

$$S(t) = \{(x, t) \in \bar{G}^{t_*} \mid \Phi(x, t) = 0, x \in G \text{ and } t \in [0, t_*]\}, \quad (1.4.2)$$

$\Phi \in C^1(\bar{G}^{t_*})$ and $\nabla_x \Phi(x, t) \neq 0$ on $S(t)$. $\Phi(x, t) < 0$ is the phase 1 and $\Phi(x, t) > 0$ is the phase 2. \bar{G} is the closure of G .

We assume that G^{t_*} also admits disjoint decomposition

$$G^{t_*} = G_1^{t_*} \cup \Gamma(t) \cup G_2^{t_*},$$

where

$$G_1^{t_*} = \cup_{0 \leq t \leq t_*} G_1(t), \quad G_2^{t_*} = \cup_{0 \leq t \leq t_*} G_2(t), \quad \Gamma(t) = \cup_{0 \leq t \leq t_*} S(t).$$

Here, $\Gamma(t)$ and $S(t)$ are both called free boundaries. $\Gamma(t)$ is the free boundary in $G \times [0, t]$ and $S(t)$ is the free boundary in G . It may be noted that we are still dealing with the same solidification process discussed in Eqs (1.3.1)–(1.3.7). The only difference is that

for the solid and the liquid regions and the phase-change boundary, some set theoretic notations have been introduced for rigorous mathematical analysis. In problem (1.3.1)–(1.3.7), $\Phi(x, t) = x - S(t)$.

Let $\mathcal{P}_i, i = 1, 2$ be the elliptic operators [9] defined as

$$\mathcal{P}_i = \sum_{J,k=1}^3 a_{Jk}^i(x, t) \frac{\partial^2}{\partial x_J \partial x_k} + \sum_{J=1}^3 b_J^i(x, t) \frac{\partial}{\partial x_J} + r^i(x, t), \quad i = 1, 2. \quad (1.4.3)$$

Here, $a_{Jk}^i, \nabla_x a_{Jk}^i, \nabla_x^2 a_{Jk}^i, b_J^i, \nabla_x b_J^i$ and r^i are continuous in \bar{G}^{t_*} ; $x \in G$ and $x = (x_1, x_2, x_3)$. Consider the following solidification problem in $G(t)$.

Conservation of energy in $G_i(t)$

$$C_i \rho_i \frac{\partial T_i}{\partial t} = \mathcal{P}_i T_i, \quad (x, t) \in G_i(t), \quad i = 1, 2, \quad 0 < t < t_*. \quad (1.4.4)$$

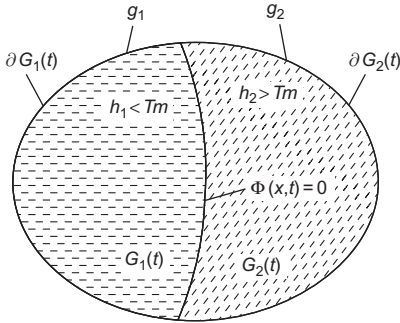


Fig. 1.4.1 Solid and liquid regions at a fixed time in a 2-D problem. Geometry number 1.

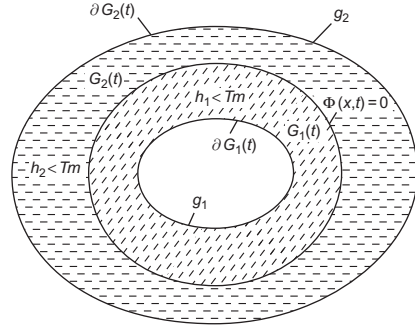


Fig. 1.4.2 Solid and liquid regions at a fixed time in a 2-D problem. Geometry number 2.

Boundary conditions on $\partial G_i(t)$

$$T_i(x, t) = g_i(x, t), \quad x \in \partial G_i(t), \quad i = 1, 2; \quad 0 < t < t_*, \quad g_1 < T_m \text{ and } g_2 > T_m. \quad (1.4.5)$$

Initial conditions

$$T_i(x, 0) = f_i(x), \quad x \in G_i(0), \quad i = 1, 2; \quad f_1 < T_m \text{ and } f_2 > T_m. \quad (1.4.6)$$

Isotherm conditions on $\Phi(x, t) = 0$

$$T_i = T_m, \quad i = 1, 2. \quad (1.4.7)$$

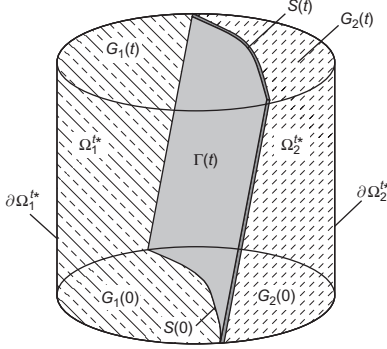


Fig. 1.4.3 Cylinder over $G(0)$ in Fig. 1.4.1.

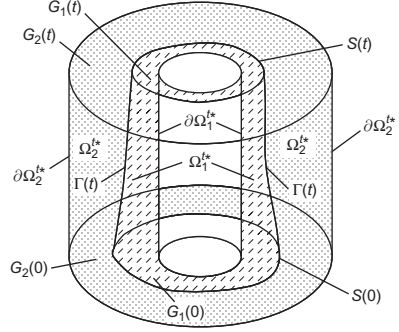


Fig. 1.4.4 Cylinder over $G(0)$ in Fig. 1.4.2.

Energy balance at $\Phi(x, t) = 0$

$$[-\vec{q}]_2^1 \cdot \vec{n} = \rho_1 \hat{l} \vec{V} \cdot \vec{n}, \quad (1.4.8)$$

where $\vec{q} = (q_1, q_2, q_3)$ is the heat flux vector for anisotropic bodies and is defined as

$$q_i = - \sum_{j=1}^3 K_{ij} \frac{\partial T}{\partial x_j}, \quad i = 1, 2, 3. \quad (1.4.9)$$

Here, K_{ij} are called thermal conductivity coefficients, \vec{n} is the unit outward normal to the solid phase at $S(t)$, and \vec{V} is the velocity of the free boundary. Eq. (1.4.8) is a generalization of Eq. (1.3.10) to a multidimensional case.

If the heat equation (1.4.3) is written for anisotropic bodies, then $a_{jk}^i(x, t)$ are known functions which can be identified with $K_{ij}(x, t)$ in Eq. (1.4.9). In a more general case K_{ij} could be functions of temperature also and in that case the operator \mathcal{P}_i will change (see Eq. 1.4.29). The coefficients $b_{jk}^i(x, t)$ could be the space derivatives of $K_{ij}(x, t)$ and/or convective terms arising due to the motion of the i th phase. In the case of motion of phases, total derivative of T_i with respect to time should be considered. If so,

$$C_i \rho_i \frac{DT_i}{Dt} = C_i \rho_i \frac{\partial T_i}{\partial t} + C_i \rho_i \sum_{j=1}^3 u_j^i \frac{\partial T_i}{\partial x_j}, \quad i = 1, 2, \quad (1.4.10)$$

where $u_j^i, j = 1, 2, 3$ are components of the velocity of the i th phase. Motion in the liquid phase may arise due to natural convection and/or shrinkage or expansion of the volume, etc. In Eq. (1.4.4), $b_{jk}^i(x, t)$ could be taken to be either known or unknown quantities. r^i is the rate at which heat is generated/annihilated per unit volume. If $b_{jk}^i(x, t)$ are unknown and arise due to the liquid motion, then Eqs (1.4.4)–(1.4.8) have to be supplemented with other equations to determine them. The motion of the solid phase is generally neglected but the motion of

the melt plays an important role in solidification/melting problems. In the one-dimensional case, u_x could be easily determined with the help of mass balance condition Eq. (1.3.9). In the multidimensional case, the velocity of the melt can be determined only by supplementing Eqs (1.4.4)–(1.4.8) with the equations of fluid dynamics such as Navier–Stokes equations discussed in Section 1.4.7.

Instead of temperature-prescribed boundary conditions on $\partial G_i(t)$, $i = 1, 2$, other types of boundary conditions can also be prescribed provided solidification takes place in such a way that only one phase-change boundary separates the two phases. This restriction is inherent in the classical formulation. Solidification/melting problems could be much more complicated. For example, if ice pieces are put in a glass of water, then a free boundary exists at the boundary of each ice piece. Two free boundaries which may be disjoint at some instant of time may join at a later instant. A piece of ice on melting, may break into two. Such problems are extremely difficult to investigate. In Eq. (1.4.7) it has been assumed that there is thermodynamic equilibrium at the phase-change boundary.

Eq. (1.4.8) can be expressed as

$$\sum_{j,k=1}^3 a_{jk}^1 \frac{\partial T_1}{\partial x_j} \frac{\partial \Phi}{\partial x_k} - \sum_{j,k=1}^3 a_{jk}^2 \frac{\partial T_2}{\partial x_j} \frac{\partial \Phi}{\partial x_k} = -\rho \hat{l} \frac{\partial \Phi}{\partial t}. \quad (1.4.11)$$

Eq. (1.4.11) can be obtained as follows. Let \vec{q}^i be the heat flux vector in the i th phase given by Eq. (1.4.9) with K_{jk} replaced by a_{jk} . Then

$$\vec{q}^i \cdot \vec{n} = \vec{q}^i \cdot \text{grad } \Phi / |\text{grad } \Phi| = \left(q_1^i \frac{\partial \Phi}{\partial x} + q_2^i \frac{\partial \Phi}{\partial y} + q_3^i \frac{\partial \Phi}{\partial z} \right) / |\text{grad } \Phi|, \quad i = 1, 2 \quad (1.4.12)$$

$$= \left(- \sum_{j,k=1}^3 a_{jk}^i \frac{\partial T^i}{\partial x_j} \frac{\partial \Phi}{\partial x_k} \right) / |\text{grad } \Phi|, \quad i = 1, 2. \quad (1.4.13)$$

we have

$$\frac{d\Phi}{dt}(x, y, z, t) = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x} \frac{dx}{dt} + \frac{\partial \Phi}{\partial y} \frac{dy}{dt} + \frac{\partial \Phi}{\partial z} \frac{dz}{dt} = 0 \quad (1.4.14)$$

or

$$(\vec{V} \cdot \vec{n}) |\text{grad } \Phi| = -\frac{\partial \Phi}{\partial t}. \quad (1.4.15)$$

Substitution of Eqs (1.4.13), (1.4.15) into Eq. (1.4.8) results in Eq. (1.4.11). The negative sign on the r.h.s. of Eq. (1.4.11) will arise in Eq. (1.3.6) also if we take $\Phi(x, t) = x - S(t)$ in Eq. (1.3.6). The energy balance condition Eq. (1.4.8) has been obtained on the basis of the principle of conservation of energy stated in Section 1.4.7. The detailed derivation of Eq. (1.4.8) using energy conservation principle is given below.

Consider the three-dimensional problem of solidification of a melt in which we take densities of solid and liquid phases to be the same and consider an isotropic case for the sake of simplicity. Let P be a point on the interface $S(t)$ at some given time t , and Q be a point on $S(t + \Delta t)$. Q lies along the vector normal to $S(t)$ at P as shown in Fig. 1.4.5. Solidification progresses from P to Q . Take \vec{PQ} to be the direction of the vector \vec{n}_x normal to the surface $S(t)$

at a fixed time t where $\vec{n}_x = (n_{x_1}, n_{x_2}, n_{x_3})$, $n_{x_i} = \cos(\vec{n}, \vec{x}_i)$, $i = 1, 2, 3$, \vec{n} is the unit vector normal to the interface $S(t)$ and $\vec{n} = (\vec{n}_x, n_t)$. $n_t = \cos(\vec{n}, \vec{t})$, \vec{t} is the unit vector in the direction of t and \vec{x}_i , $i = 1, 2, 3$ are unit vectors in the x_1, x_2 and x_3 directions, respectively. Here, x_1, x_2 and x_3 are Cartesian coordinates of a point in R^3 . Let Δw be a small area on $S(t)$ enclosing the point P . Construct a cylinder with base Δw , height $|\vec{PQ}|$ and axis PQ . The latent heat released during the solidification of this volume element is $\rho l \Delta w |\vec{PQ}|$. The rate at which latent heat is released during solidification of this volume element is

$$\lim_{\Delta t \rightarrow 0} \rho l \left| \frac{\vec{PQ}}{\Delta t} \right| \Delta w = \rho l \vec{V} \cdot \vec{n}_x \Delta w. \quad (1.4.16)$$

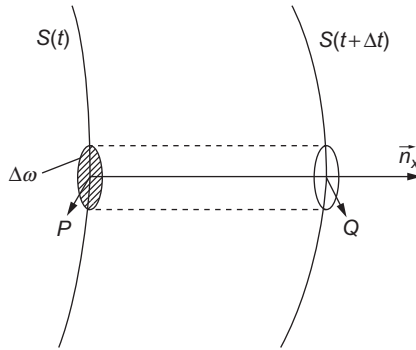


Fig. 1.4.5 Geometry in the derivation of the Stefan condition.

Here, \vec{V} is the velocity of the interface. On using the energy conservation principle in the cylinder of volume $\Delta w |\vec{PQ}|$ which requires calculating the difference in the rates at which heat flows into the cylinder across the left face and leaves through the face on the right and adding to it the rate at which heat is released and cancelling Δw , we obtain

$$K_S \frac{\partial T_S}{\partial n_1} + K_L \frac{\partial T_L}{\partial n} = -\rho l \vec{V} \cdot \vec{n}_x, \quad (1.4.17)$$

where $\vec{n}_1 = -\vec{n}$ and \vec{n}_1 is the outward normal to the base of the cylinder at P .

1.4.2 Alternative Forms of the Stefan Condition

The Stefan condition (1.4.11) in multiple dimensions can be expressed alternatively in a form more suitable for analytical and numerical treatment. It will be assumed that $\Phi(x, y, z, t) = 0$ can be written as

$$\Phi(x, y, z, t) = z - S(x, y, t) = 0. \quad (1.4.18)$$

Alternative expressions like $x - f(y, z, t)$ or $y - g(x, z, t)$ can also be used for Φ . Sometimes the physics of the problem requires the free boundary to be expressed in the form $t = w(x, y, z)$ (see Eq. 3.3.46). Differentiation of Eq. (1.4.7) with respect to x and y gives

$$\frac{\partial T_i}{\partial x} = -\frac{\partial T_i}{\partial z} \frac{\partial z}{\partial x}, \quad \frac{\partial T_i}{\partial y} = -\frac{\partial T_i}{\partial z} \frac{\partial z}{\partial y}, \quad i = 1, 2. \quad (1.4.19)$$

From Eq. (1.4.18), we have

$$\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial z} \frac{\partial z}{\partial x} = 0, \quad \frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial z} \frac{\partial z}{\partial y} = 0, \quad \frac{\partial \Phi}{\partial z} = 1. \quad (1.4.20)$$

Eq. (1.4.11) can be rewritten as

$$\sum_{J=1}^3 \left(\sum_{k=1}^3 a_{Jk}^1 \frac{\partial \Phi}{\partial x_k} \right) \frac{\partial T_1}{\partial x_J} - \sum_{J=1}^3 \left(\sum_{k=1}^3 a_{Jk}^2 \frac{\partial \Phi}{\partial x_k} \right) \frac{\partial T_2}{\partial x_J} = \rho \hat{l} \frac{\partial S}{\partial t}. \quad (1.4.21)$$

On using Eqs (1.4.19), (1.4.20) in Eq. (1.4.21) the latter can be written as

$$\left\{ \sum_{J=1}^3 \left(\sum_{k=1}^3 a_{Jk}^1 \frac{\partial \Phi}{\partial x_k} \right) \frac{\partial \Phi}{\partial x_J} \right\} \frac{\partial T_1}{\partial z} - \left\{ \sum_{J=1}^3 \left(\sum_{k=1}^3 a_{Jk}^2 \frac{\partial \Phi}{\partial x_k} \right) \frac{\partial \Phi}{\partial x_J} \right\} \frac{\partial T_2}{\partial z} = \rho \hat{l} \frac{\partial S}{\partial t}. \quad (1.4.22)$$

Note that in Eq. (1.4.22), the derivative of temperature is with respect to z but not with respect to x and y . Eq. (1.4.22) looks like the Stefan condition in a one-dimensional problem.

In an isotropic case,

$$a_{Jk}^i = K_i \delta_{Jk}, \text{ where } \delta_{Jk} \text{ is the 'Kronecker delta function'}. \quad (1.4.23)$$

Using Eq. (1.4.23) in Eq. (1.4.22), the Stefan condition can be expressed as

$$\left\{ 1 + \left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 \right\} \left\{ K_1 \frac{\partial T_1}{\partial z} - K_2 \frac{\partial T_2}{\partial z} \right\} = \rho \hat{l} \frac{\partial S}{\partial t}. \quad (1.4.24)$$

On using Eqs (1.4.19), (1.4.20), we have

$$\vec{n} = \frac{\nabla \Phi}{|\nabla \Phi|} = \frac{\nabla T_i}{|\nabla T_i|}. \quad (1.4.25)$$

Also on the free boundary

$$\vec{V} \cdot \vec{n} = \vec{V} \cdot \nabla T_i / |\nabla T_i| = -\partial T_i / \partial t / |\nabla T_i|, \quad i = 1, 2. \quad (1.4.26)$$

Isotherm conditions (1.4.7) can be differentiated with respect to t to obtain the second relation in Eq. (1.4.26). Substituting Eq. (1.4.26) in Eq. (1.4.8), we obtain

$$|K_1 \nabla T_1| - |K_2 \nabla T_2| = -\rho \hat{l} \frac{\partial T_1}{\partial t} / |\nabla T_1| = -\rho \hat{l} \frac{\partial T_2}{\partial t} / |\nabla T_2|. \quad (1.4.27)$$

1.4.3 The Kirchhoff's Transformation

Consider the case in which the thermal conductivity varies with the temperature, but is independent of position and time. The heat equation for isotropic material in this case can be written as

$$\begin{aligned}
\rho C \frac{\partial T}{\partial t} &= \operatorname{div}(K(T) \operatorname{grad} T) + A(T, x, y, z, t) \\
&= K \nabla^2 T + \frac{\partial K}{\partial T} \left\{ \left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 + \left(\frac{\partial T}{\partial z} \right)^2 \right\} + A,
\end{aligned} \tag{1.4.28}$$

A is the rate at which heat is supplied per unit volume.

Kirchhoff's transformation is defined as [24],

$$\theta = \frac{1}{K_0} \int_0^T K(\alpha) d\alpha, \tag{1.4.29}$$

where K_0 is the value of $K(T)$ when $T = 0$. It can be seen that

$$\frac{\partial \theta}{\partial t} = \frac{K}{K_0} \frac{\partial T}{\partial t}, \quad \frac{\partial \theta}{\partial x} = \frac{K}{K_0} \frac{\partial T}{\partial x}, \quad \frac{\partial \theta}{\partial y} = \frac{K}{K_0} \frac{\partial T}{\partial y}, \quad \frac{\partial \theta}{\partial z} = \frac{K}{K_0} \frac{\partial T}{\partial z} \tag{1.4.30}$$

and

$$\nabla^2 \theta - \frac{\rho C}{K} \frac{\partial \theta}{\partial t} = -\frac{A}{K_0}, \tag{1.4.31}$$

where K and C are now functions of θ . In many cases, for example, in metals near absolute zero temperature, both K and C are proportional to the temperature and $\rho C/K$ can be taken to be a constant.

If A is independent of the temperature, then there will be no difference between the structure of the Fourier's heat conduction equation and Eq. (1.4.31). However if we use Kirchhoff's transformation (1.4.29), the choice of boundary conditions will be limited. When T is prescribed as $T = f(x, y, z, t)$ on the boundary, the expression

$$\theta = \frac{1}{K} \int_0^{f(x,y,z,t)} K(\alpha) d\alpha, \tag{1.4.32}$$

gives the boundary condition for θ . Similarly, if the normal derivative of temperature T is prescribed, the normal derivative of θ can also be prescribed.

The extension of the formulation (1.4.4)–(1.4.8) to R^n , $n > 3$ is simple. Many generalizations of this formulation are possible and some of them will be discussed later.

1.4.4 Boundary Conditions at the Fixed Boundary

(A) Standard Boundary Conditions

(AI) Type I Boundary Condition

In the boundary condition of this type temperature is prescribed. Let ∂G be the boundary of the region $G \subset R^3$. Type I boundary condition is of the form

$$T|_{\partial G} = f(x, y, z, t), \quad t > 0, \quad (x, y, z) \in \partial G. \tag{1.4.33}$$

If $f < T_m (> T_m)$ then solidification (melting) starts instantaneously, i.e. there is no waiting time for the phase-change to take place. Since energy can neither be produced nor disappear at infinity, if G is unbounded, we have

$$\lim_{|x| \rightarrow \infty} T(x, t) = T(x, 0), \quad t > 0. \quad (1.4.34)$$

In order to obtain a physically meaningful solution to the diffusive heat equation for unbounded domains, it is required that

$$\lim_{|x| \rightarrow \infty} \frac{\partial T}{\partial n}(x, t) = 0, \quad t > 0. \quad (1.4.35)$$

Temperature-prescribed boundary conditions are known as *Dirichlet type boundary conditions*.

Sometimes, while constructing approximate solutions more than the prescribed number of boundary conditions are required to match the number of unknowns. In such a case additional boundary conditions are generated. Let us consider a radially symmetric one-dimensional heat conduction problem in which the temperature at $r = a$ is prescribed as

$$T(r, t)|_{r=a} = f(t). \quad (1.4.36)$$

If Eq. (1.4.36) is differentiated with respect to time and the Fourier's heat equation is used, then

$$\left. \frac{\partial T}{\partial t} \right|_{r=a} = k \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right)_{r=a} = f'(t). \quad (1.4.37)$$

Eq. (1.4.37) provides one more boundary condition for the problem.

(All) Boundary Condition of Radiative-Convective Type

This boundary condition can be written as

$$-K \left. \frac{\partial T}{\partial n} \right|_{\partial G} = \beta E (T^4 - T_0^4)|_{\partial G} + (C_1 T - C_2 T_0)|_{\partial G}. \quad (1.4.38)$$

Here, \vec{n} stands for the unit outward normal, β is Stefan-Boltzmann constant, E is the emissivity of the surface which is defined as the ratio of the heat emitted by the body to the black body at the same temperature, T_0 is the ambient temperature, and C_1 and C_2 are functions of time. β is positive and has dimensions $[\text{watt}]/[\text{meter}]^2 \times [\text{temperature}]^4$, $0 \leq E \leq 1$. From this boundary condition, three types of well-known boundary conditions can be obtained as particular cases. If $\beta = 0$ and $C_1 = C_2 = h_t$ then we have a *convective type boundary condition*. $h_t (> 0)$ is called 'heat transfer coefficient' and has dimensions $[\text{power}]/[\text{temperature}] [\text{length}]^2$. If $\beta = 0$, $C_1 = 0$ and $C_2 T_0 = g(t)$, then we have a *flux-prescribed boundary condition*. Flux prescribed boundary conditions are known as *type II boundary condition*. Convective type boundary conditions are called *type III boundary conditions* and radiative-convective boundary conditions are called *type IV boundary conditions*.

More details about these boundary conditions can be found in [24]. These four types of boundary conditions are commonly prescribed in heat transfer problems.

(B) Nonstandard Boundary Conditions

(BI) Nonlocal Boundary Condition

To explain the nature of a nonlocal boundary condition, we consider a simple one-dimensional problem of fluidized-bed coating. Our interest in this problem is more on pointing out the characteristic features of a nonlocal boundary condition than its rigorous formulation. A thin metallic plate of thickness $2b$, area A_p and the initial temperature T_0 ($T_0 > T_m$) is immersed vertically in a pool of plastic coating material [29] which is maintained at temperature T_∞ which is also the temperature away from the boundary layer. The softening temperature of the coating material is $T_m > T_\infty$. Because of the high-thermal conductivity and finite heat capacity of the metallic plate, the plate can be treated as a lumped parameter system in which the temperature can be regarded as a function of time only. Let $T_p(t)$ be the temperature of the plate. In the one-dimensional problem considered here, let the metallic plate be situated at $x = 0$. The problem is that of finding the temperature $T(x, t)$ of the softened plastic, temperature $T_p(t)$ and the extent $x = \delta(t)$ up to which the plastic has softened. The unknown quantities can be obtained by solving the following system of equations:

$$\rho_f C_f \frac{\partial T}{\partial t} = K_f \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < \delta(t), \quad t > 0, \quad (1.4.39)$$

$$T(0, 0) = T_0, \quad T(0, t) = T_p(t), \quad T(\delta, t) = T_m, \quad (1.4.40)$$

$$\frac{m_p C_p}{A_p} (T_0 - T_p(t)) = h_t t (T_m - T_\infty) + \rho_f C_f \int_0^{\delta(t)} (T - T_\infty) dx, \quad (1.4.41)$$

$$-K_f \frac{\partial T}{\partial x} \Big|_{x=\delta} = h_t (T_m - T_\infty) + \rho_f C_f (T_m - T_\infty) \frac{d\delta}{dt}. \quad (1.4.42)$$

The subscripts f and p stand for fluid plastic and plate, respectively, m stands for mass, ρ for the density and h_t is the heat transfer coefficient. Eq. (1.4.41) is the boundary condition at $x = 0$ which because of the integration over $0 \leq x \leq \delta$ in the second term on the r.h.s. is a nonlocal boundary condition. The l.h.s. in Eq. (1.4.41) is the heat given out by the plate per unit area over the time interval $[0, t]$ which is equal to the sum of the heat transported by convection into the coating material and the heat received by the fluid plastic. Eq. (1.4.42) is the energy balance equation at $x = \delta(t)$.

(BII) Boundary Condition of the Fifth Type

Boundary condition of the fifth kind arises in the following context. Suppose the boundary of a region is in contact with a thermostat. The continuity of the temperature and of the flux at the surface of contact, say $x = 1$, in the case of a one-dimensional slab $0 \leq x \leq 1$ can be expressed as

$$T(1, t) = \theta(1, t), \quad \bar{C} \frac{\partial \theta}{\partial t} = -K \frac{\partial T}{\partial x}, \quad \text{at } x = 1, \quad (1.4.43)$$

where T and θ are temperatures of the slab and thermostat, respectively, and \bar{C} is the ‘heat capacity’ of the thermostat. The thermostat maintains a uniform temperature. On combining the two equations of Eq. (1.4.43), we obtain a boundary condition of the fifth type

$$\bar{C} \frac{\partial T}{\partial t} = -K \frac{\partial T}{\partial x}, \text{ at } x = 1. \quad (1.4.44)$$

(C) Boundary Conditions With Multivalued Functions

In this boundary condition the prescribed quantities are multivalued functions. So it is not a classical boundary condition in which the prescribed functions are assumed to be sufficiently regular functions. Such boundary conditions are suitable for weak formulations and can be described with the help of ‘subdifferentials’ (see Eq. 4.3.44 for its definition). The boundary condition derived below is an extension of the fifth type of boundary condition (1.4.44) in the weak form.

Let Ω be a bounded open domain in R^3 which is occupied by both solid and liquid regions. The boundary $\partial\Omega$ of Ω is smooth, and is surrounded by a high-conducting material of negligible thickness so that the temperature of the surrounding material is taken to be a function of time alone. Let the time interval be given by $0 \leq t \leq t_*$, for some positive number t_* , and $Q = \Omega \times (0, t_*)$, $\Sigma = \partial\Omega \times (0, t_*)$, $Q = Q_S \cup Q_L$ where Q_S and Q_L are the regions occupied by the solid and the liquid, respectively. $\Sigma = \Sigma_L \cup \Sigma_S \cup \Sigma_{LS}$ where Σ_L and Σ_S are the boundaries of solid and liquid regions and Σ_{LS} is the portion of Σ which is neither solid nor liquid. We assume that the measure of the set Σ_{LS} is zero. Because of the perfect thermal contact between $\partial\Omega$ and the outside material, the temperature is taken to be continuous and so

$$T|_{\Sigma} = \theta|_{\Sigma}, \quad (1.4.45)$$

where $T(x, t)$ is the temperature of Ω and $\theta(t)$ is the temperature of the outside material.

The heat balance on Σ_L gives (see [30])

$$\hat{C} \frac{\partial \theta}{\partial t} + K_L \frac{\partial T}{\partial n} + g(x, t, \theta) = 0, \quad (1.4.46)$$

where \vec{n} is the outward normal to $\partial\Omega$, \hat{C} is the heat capacity of the surrounding material and $g(\cdot)$ is the rate of heat flow from outside of Ω to $\partial\Omega$.

The heat balance on Σ_S gives

$$\hat{C} \frac{\partial \theta}{\partial t} + K_S \frac{\partial T}{\partial n} + g(x, t, \theta) = 0. \quad (1.4.47)$$

Define a multivalued function ξ as

$$\xi(r) = \begin{cases} \frac{1}{K_L}, & r > 0, \\ \left(\frac{1}{K_S}, \frac{1}{K_L} \right), & r = 0, \\ \frac{1}{K_S}, & r < 0. \end{cases} \quad (1.4.48)$$

If $K_S > K_L$, then ξ is a *monotone graph*. Eqs (1.4.46), (1.4.47) can be combined into a single equation

$$\xi \hat{C} \frac{\partial \theta}{\partial t} + \frac{\partial T}{\partial n} + \xi g(x, t, \theta) = 0, \text{ on } \partial \Omega. \quad (1.4.49)$$

In Eq. (1.4.49) normal derivative of the temperature on $\partial \Omega$ is prescribed in terms of a multivalued function. $T < 0$ is solid, $T > 0$ is liquid and $T = 0$ corresponds to the mushy region. Conditions of the type (1.4.49) are also called *dynamical boundary conditions* [31]. Weak formulation is a continuum model and in this formulation, solid, liquid and mushy regions (both solid and liquid phases are present in the mush) are modelled with the help of a single equation which is valid in the region Q in the *distributional sense*. Therefore the boundary conditions over different portions of the boundary should be combined into a single equation which results in Eq. (1.4.49). Stefan problems with dynamical boundary conditions have been investigated by several authors (cf. [31, 32]).

1.4.5 Conditions at the Free Boundary

In the Stefan problem two boundary conditions are prescribed at the free boundary. One specifies the temperature and another accounts for the energy balance. The isotherm condition in Eq. (1.4.7) is the simplest type of temperature-prescribed boundary condition and is justified only if the free boundary is planar and is in thermodynamic equilibrium. In this case, isotherm temperature is the equilibrium temperature T_m . The equilibrium temperature is affected by the curvature and the surface tension in the free boundary. This effect is known as *Gibbs–Thomson effect*. Similarly equilibrium temperature may depend on the velocity of the free boundary and in this case the free boundary condition is known as a *kinetic condition*. An explanation of curvature effects and kinetic condition requires thermodynamical and metallurgical concepts and so these effects will be discussed a little later in Section 2.2.3.

One form of generalization of Eq. (1.4.8) is to consider the thermophysical parameters as general functions of space, time and temperature and include a heat flux acting at the free boundary. Such generalizations, at least in the one-dimensional problems, have been considered extensively and analyzed mathematically. Some of these extensions will be discussed later with respect to specific problems. Another type of generalization of free boundary condition is to consider the transport and accumulation of different forms of energies in the phase-change interface such as by heat conduction along the interface. This generalization of the Stefan condition, called *generalized Stefan condition*, will be discussed in Section 2.4.

The notion of Stefan condition is not restricted to conditions of the form (1.4.8) which arise due to heat balance during phase changes. Stefan condition can be considered in a functional form, for example, the velocity of the free boundary can be prescribed as a function of temperature, temperature derivatives, time and $S(t)$ (see Eq. 3.1.5).

Implicit Free Boundary Condition

In the Stefan condition (1.4.11), the velocity of the free boundary occurs explicitly. A boundary condition of this type is called an *explicit free boundary condition*. There are several problems in which the boundary conditions at the free boundary do not involve the normal velocity of the free boundary. For example, the condition $\partial T / \partial x|_{x=S(t)} = 0$ does not involve the term

$\partial S/\partial t$. Such boundary conditions are called *implicit free boundary conditions*. In some one-dimensional problems, by using suitable transformations, implicit boundary conditions can be easily converted to an equivalent explicit boundary condition. Implicit boundary conditions will be discussed further in [Section 3.3](#).

1.4.6 The Classical Solution

The definition of a classical solution of problem Eqs (1.4.4)–(1.4.8) is given below. Classical solutions of other problems to be discussed later can be defined on similar lines.

Definition The classical solution of Eqs (1.4.4)–(1.4.8) is a quadruple (T_1, T_2, Φ, t_*) such that $T_i, \nabla_x T_i$ are continuous in $\cup_{0 \leq t < t_*} \bar{G}_i(t)$ and $\nabla_x^2 T_i$ and $\partial T_i/\partial t$ are continuous in $\cup_{0 < t < t_*} G_i(t)$; $\Phi(x, y, z, t)$ has continuous first-order partial derivatives in \bar{G}^{t_*} and satisfies conditions imposed in Eq. (1.4.2) in [Section 1.4.1](#). The triple $(T_1, T_2, S(t))$ satisfies Eqs (1.4.4)–(1.4.8) in which the coefficients satisfy the constraints in Eq. (1.4.3).

Here, $\nabla_x T_i = (\partial T_i/\partial x, \partial T_i/\partial y, \partial T_i/\partial z)$, $i = 1, 2$ and $\nabla_x^2 T_i$ can be similarly defined.

It may be noted that the existence of a classical solution is not guaranteed merely by prescribing sufficiently regular initial and boundary data satisfying the sign constraints, such as in Eqs (1.4.5), (1.4.6). Some additional compatibility conditions discussed in later sections should also be satisfied. t_* cannot be arbitrarily taken and is related to the existence of the solution. In R^n , $n \geq 2$, existence and uniqueness of classical solutions have been mostly proved only locally-in-time, this too under suitable constraints.

1.4.7 Conservation Laws and the Motion of the Melt

There is a clear and overwhelming experimental evidence [33] that except in very early times, convective heat transfer dominates over conductive heat transfer during solidification/melting. Motion of the liquid which may arise due to natural convection, forced convection or shrinkage/expansion of the volume becomes more significant when the phase-change is of relatively large volumes of material. Motion of the melt can be accounted for by solving Navier–Stokes equations [34] together with the energy equation.

Most equations of mathematical physics are derived on the application of the following conservation laws:

1. *Conservation of mass*: The time rate of increase of mass of a system is equal to the difference between the rate at which mass enters into the system, and the rate at which mass leaves the system (disregarding relativity effects).
2. *Conservation of momentum*: The time rate of change of linear momentum of the mass of a system is equal to the sum of all the forces acting on the system.
3. *Conservation of energy*: The time rate of change of energy (internal energy + kinetic energy + potential energy) stored in the system is equal to the sum of the time rate of work of the external forces, the rate at which energy is transported into the system (heat energy, electrical energy, etc.) or leaving the system across its boundaries and the, rate at which energy is produced or consumed in the system.

The fourth conservation law which deals with the conservation of moment of momentum is not required in the context of this volume. When a conservation law is valid in any arbitrarily small neighbourhood of each material point, we say that the conservation law holds locally.

If the melt is in motion, then in order to determine the motion of the melt, energy equation should be solved in conjunction with Navier–Stokes equations. We derive below the complete system of equations briefly.

Control Volume and Applications of Conservation Laws

A ‘control volume’ refers to a fixed region in space which encloses a fixed volume V_0 of the fluid in space. Its surface A_0 is called control surface, which could be of any shape. In the *Eulerian coordinate system*, the fluid is imagined to pass through this arbitrary control volume. At the same position we observe different particles at different times. In the *Lagrangian coordinate system*, the coordinate system is attached to the moving particle. Therefore the control volume consists of the same particles all the time. Let ϕ be some ‘scalar-specific property’ (per unit mass) such as mass, momentum, energy of the fluid. We shall consider the conservation of a scalar property ϕ in the control volume. The time rate of change of the total property ϕ contained in V_0 is given by

$$\frac{\partial}{\partial t} \int_{V_0} \rho \phi dV_0. \quad (1.4.50)$$

The conservation equation for ϕ is

$$\frac{\partial}{\partial t} \int_{V_0} \rho \phi dV_0 = - \int_{A_0} (\rho \vec{V} \phi) \cdot \vec{n} dA_0 - \int_{A_0} \vec{J} \cdot \vec{n} dA_0 + \int_{V_0} \dot{f} dV_0, \quad (1.4.51)$$

where \vec{V} is the velocity of the fluid, \vec{n} is the unit outward drawn normal to A_0 , \vec{J} is the surface flux of ϕ by diffusion from inside to outside across the surface A_0 , f is the production or annihilation of ϕ per unit volume in the control volume, ρ is the density and dot denotes the time derivative. dA_0 is the elementary surface area and dV_0 is the elementary volume. Using Gauss’s theorem, surface integrals can be converted to volume integrals and the local conservation of ϕ implies

$$\frac{\partial}{\partial t} (\rho \phi) + \nabla \cdot (\rho \vec{V} \phi) = -\nabla \cdot \vec{J} + \dot{f}. \quad (1.4.52)$$

Conservation of Mass: Equation of Continuity

Take $\phi = 1, f = 0$ and $\vec{J} = 0$ in Eq. (1.4.52). We obtain the equation of continuity as

$$\frac{\partial \rho}{\partial t} + \text{div} (\rho \vec{V}) = 0. \quad (1.4.53)$$

Equation of Conservation of Linear Momentum

In order to obtain the equation of conservation of linear momentum in the x -direction, take $\phi = V_1, \vec{J} = -\vec{p}_x = -(p_{xx}\vec{i} + p_{xy}\vec{j} + p_{xz}\vec{k})$ and take F_1 instead of \dot{f} in Eq. (1.4.52) where F_1 is the body force in the x -direction. V_1 is the x -component of the fluid velocity \vec{V} and p_{ij} are stress components

34 The Classical Stefan Problem

$$p_{ij} = -p\delta_{ij} + \tau_{ij}, \quad i = 1, 2, 3 \text{ and } j = 1, 2, 3, \quad (1.4.54)$$

$$p = -\frac{1}{3}(p_{xx} + p_{yy} + p_{zz}). \quad (1.4.55)$$

Here, τ_{ij} represent viscous stress components, and p is the fluid pressure. The r.h.s. of Eq. (1.4.52) in this present conservation law should represent the sum of the forces acting on the control volume which are surface forces and body forces. If g_x is the x -component of the body force per unit mass, then the equation of conservation of linear momentum in the x -direction for constant ρ is given by

$$\rho \frac{DV_1}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x. \quad (1.4.56)$$

DV_1/Dt is the material time derivative of V_1 in Eq. (1.4.56). Two more equations in y and z directions can be similarly obtained by taking $\phi = V_2$ and $\vec{J} = -\vec{p}_y = -(p_{yx}\vec{i} + p_{yy}\vec{j} + p_{yz}\vec{k})$ for the second equation and $\phi = V_3$ and $\vec{J} = -\vec{p}_z$ for the third equation. We have

$$\rho \frac{DV_2}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y, \quad (1.4.57)$$

$$\rho \frac{DV_3}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho g_z, \quad (1.4.58)$$

where g_y and g_z are the y and z components of the body force per unit mass.

For Newtonian fluids [34]

$$\tau_{ij} = 2\mu e_{ij} + \mu^1 \delta_{ij}(e_{xx} + e_{yy} + e_{zz}), \quad i = 1, 2, 3; \quad j = 1, 2, 3, \quad (1.4.59)$$

where e_{ij} are the strain components. From Eq. (1.4.54) it is clear that

$$\tau_{xx} + \tau_{yy} + \tau_{zz} = 0, \quad (1.4.60)$$

which implies that $\mu^1 = \frac{-2}{3}\mu$, where μ is the ‘coefficient of viscosity’.

Eqs (1.4.56)–(1.4.58) can be written as a single equation in the vector notation:

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{F} - \text{grad } p + \mu \nabla^2 \vec{V} + \frac{\mu}{3} \text{grad div } \vec{V}, \quad \vec{F} = (g_x, g_y, g_z), \quad \vec{V} = (V_1, V_2, V_3). \quad (1.4.61)$$

Eq. (1.4.61) is called *Navier–Stokes equation* for Newtonian fluids. This equation is independent of the temperature.

The Equation of Conservation of Energy

A general expression of specific (per unit mass) energy E is given by

$$E = e + \text{mechanical energy} + \text{potential energy} + \text{chemical energy}. \quad (1.4.62)$$

Here, e is the specific internal energy or specific internal heat energy. Mechanical energy, potential energy and chemical energy are generally omitted in the formulation of Stefan problems and e is taken as specific enthalpy which is the sum of the latent heat and the sensible heat (see Section 2.1.3 for their definitions). Let $E = e = h$, where h is the specific enthalpy. Taking $\phi = h$ and $\vec{J} = -K\nabla T$ in Eq. (1.4.52), where K is the thermal conductivity in the isotropic case, we obtain

$$\frac{\partial}{\partial t}(\rho h) + \nabla \cdot (\rho \vec{V} h) = \nabla \cdot (K \nabla T) + \dot{f}. \quad (1.4.63)$$

Here, f is the heat generated or absorbed per unit volume. \vec{V} is to be obtained by solving Navier–Stokes equation and substituted in Eq. (1.4.63). To take into account mechanical and potential energies in Eq. (1.4.63), we take body force $\vec{F} = -\text{grad } \psi$, $\vec{J} = -K\nabla T - \vec{P}_V$, where \vec{P}_V is given by

$$\vec{P}_V = \left(\sum_{k=1}^3 p_{ik} V_k, \sum_{k=1}^3 p_{2k} V_k, \sum_{k=1}^3 p_{3k} V_k \right). \quad (1.4.64)$$

Here, p_{ij} is the stress tensor defined in Eq. (1.4.54). We present below the final form of the energy equation for constant ρ case (cf. [34]):

$$\rho \frac{D}{Dt} \left(e + \frac{1}{2} |\vec{V}|^2 + \psi \right) = \nabla \cdot (K \nabla T) - \text{div}(\rho \vec{V}) + W_T + \dot{f}, \quad (1.4.65)$$

$$W_T = \frac{\partial}{\partial x} (\vec{V} \cdot \vec{\tau}_x) + \frac{\partial}{\partial y} (\vec{V} \cdot \vec{\tau}_y) + \frac{\partial}{\partial z} (\vec{V} \cdot \vec{\tau}_z), \quad (1.4.66)$$

$$\vec{\tau}_i = i\tau_{ix} + j\tau_{iy} + k\tau_{iz}, \quad i = x, y, z. \quad (1.4.67)$$

Here, $\frac{1}{2} \rho |\vec{V}|^2$ is the kinetic energy and for simplicity we take the body force to be $\vec{F} = -\text{grad } \psi$.