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001

ON THE MELTING OF ICE BALLS*

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Abstract. We consider here the problem of describing the melting of an ice ball surrounded by water. The corresponding mathematical model consists of the Stefan problem with radial symmetry. We obtain asymptotic expansions for the radius of the melting ball which turn out to be of a different nature according to the cases $N \geq 3$ and $N = 2$, N being the space dimension. The methods employed combine matched asymptotic expansion techniques, a priori estimates, and topological results.

Key words. Stefan problem, asymptotic behavior, matched asymptotic expansions, a priori estimates

AMS subject classifications. 35R55, 35B40, 35C20

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1. Introduction.

This paper is concerned with radial solutions of the following Stefan problem. To find functions $\theta(r,t)$ and $R(t)$ such that

$$\theta_t = \theta_{rr} + \left(\frac{N-1}{r}\right)\theta_r \quad \text{for } r > R(t) \quad \text{and} \quad t > 0 \tag{1.1a}$$

$$\theta(r,0) = \theta_0(r) \quad \text{for } r > R(0), \tag{1.1b}$$

$$\theta(r,t) = 0 \quad \text{if } r \leq R(t) \quad \text{and} \quad t > 0, \tag{1.1c}$$

$$\theta_r(R(t),t) = -\dot{R}(t) \quad \text{if } t > 0. \tag{1.1d}$$

Here $r = |x|$, $x \in \mathbb{R}^N$, and $N \geq 2$. As it stands, (1.1) is a model for describing the melting of a ball of ice surrounded by water. $\theta(r,t)$ denotes the temperature of the medium, which is assumed to be zero at the ice phase. We do not require $\theta_0(r)$ to be positive for every $r > R(0)$, so that the existence of regions where water is initially undercooled is not ruled out.

In view of classical results, one expects that under fairly general circumstances (for instance, if $\theta_0(r)$ is nonnegative or if undercooling does not affect the dynamics of the problem much) the ice ball will entirely melt at some finite time $t = T < \infty$. We shall address the following here.

Question. What is the speed at which ice balls collapse? In other words, what is the asymptotic behavior of $R(t)$ as $t \uparrow T$?

We shall show in what follows that there is a countable family of possible behaviors for $R(t)$ as the melting time $t = T$ is approached. To describe our results, it will be convenient to consider separately the cases $N = 2$ and $N \geq 3$. In the bidimensional situation we prove the following theorem.

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THEOREM 1.1. *Assume that $N = 2$. For any $T > 0$, there exist solutions of* (1.1) *such that the corresponding interfaces behave as $t \uparrow T$ in one of the following ways:*

$$R(t) = B(T-t)^{\frac{1}{2}} e^{-\frac{\sqrt{2}}{2}|\log(T-t)|^{\frac{1}{2}}} |\log(T-t)|^{\frac{1}{4\sqrt{\log(T-t)}}-\frac{1}{4}} \cdot (1+o(1)), \tag{1.2}$$

where B is a fixed positive constant, or

$$R(t) = C(T-t)^{\frac{l}{2}} |\log(T-t)|^{-\frac{l}{2(l-1)}} (1+o(1)), \tag{1.3}$$

where C is an arbitrary positive constant and l is any integer such that $l \geq 2$.

Concerning the case of higher dimensions, we obtain the following theorem.

THEOREM 1.2. *Assume that $N \geq 3$. For any $T > 0$, there exist solutions of* (1.1) *such that the corresponding interfaces behave as $t \uparrow T$ in one of the following ways:*

$$R(t) = B_N(T-t)^{\frac{1}{2}} |\log(T-t)|^{-\frac{1}{N-2}} (1+o(1)), \tag{1.4}$$

where B_N is a fixed positive constant depending on the dimension N,

$$\textit{or} \qquad R(t) = C(T-t)^{\frac{l}{2}} (1+o(1)), \tag{1.5}$$

where C is an arbitrary positive constant and l is any integer number such that $l \geq 2$.

Let us remark briefly on Theorems 1.1 and 1.2. To begin with, we do not preclude here the existence of other possible types of shrinking spheres besides those described in (1.2)–(1.5), although it seems very unlikely in view of the arguments leading to the proofs of these results. As a matter of fact, we expect (1.2) and (1.4) to provide the generic asymptotics for the case of the classical, not undercooled, Stefan problem. However, no proof of such a statement is given here. It will be apparent from the proofs that (1.3) and (1.5) correspond to problems with small undercooling, i.e., problems where temperature changes sign somewhere for any $t < T$.

We next observe that (1.2)–(1.5) imply that the ice radii $R(t)$ are such that

$$R(t) \ll (T-t)^{\frac{1}{2}} \quad \text{as } t \uparrow T,$$

and the contracting rates are therefore faster than those corresponding to the natural scales of the problem under consideration. In particular, the solutions obtained are not self-similar.

It is worth pointing out that our approach here allows us to obtain further information on the structure of the solutions involved. For instance, asymptotic expansions for the predicted water temperature near the melting ice ball can be obtained as $t \uparrow T$. The corresponding result reads as follows.

THEOREM 1.3. *Assume first that $N = 2$. Then the solutions referred to in Theorem* 1.1 *are such that the following expansions hold:*

If (1.2) *occurs, then*

$$\theta(x,T) = D\ e^{-2|\log|x||^{\frac{1}{2}}} |2\log|x||^{\frac{1}{2\sqrt{2|\log|x||}}} (1+o(1)) \quad \textit{as } x \downarrow 0, \tag{1.6}$$

where D is a fixed positive constant.

If (1.3) *occurs, then*

$$\theta(x,T) = D_1|x|^{2l-2}(|\log|x||)^{-\frac{1}{l-1}} (1+o(1)) \quad \textit{as } x \downarrow 0 \tag{1.7}$$

for some positive constant D_1.

Now suppose that $N \geq 3$. Then the solutions referred to in Theorem 1.2 *are such that the following expansions hold:*

If (1.4) *occurs, then*

$$\theta(x,T) = K_N(\log|x|)^{-\frac{2}{N-2}}(1+o(1)) \quad \textit{as } x \downarrow 0, \tag{1.8}$$

where K_N is a fixed positive constant depending on the dimension N.

If (1.5) *occurs, then*

$$\theta(x,T) = K_1|x|^{2l-2}(1+o(1)) \quad \textit{as } x \downarrow 0 \tag{1.9}$$

for some positive constant K_1.

Concerning previous related work, it is well known that for $N = 1$ the asymptotic shape of the vanishing ice phase is a space-time wedge which has its tip at $t = T$. We refer for such a case to the paper [AK], where disappearance of one of the phases in a one-dimensional, two-phase Stefan problem is studied by means of functional analysis methods. A different approach, based on matched asymptotics expansion techniques has been developed in [RSP] and [SW].

For instance, in [SW] a formal analysis of (1.1) with (1.1d) replaced by

$$\theta_r(R(t),t) = -\Lambda\dot{R}(t) \quad \text{for } t > 0 \tag{1.10}$$

is performed in the limit $\Lambda \to \infty$. In particular a boundary layer is then identified where an expansion similar to (1.2) and (1.4) takes place. The reader is referred to [DH], [HD1], [HD2], and [S] for related work, as well as to [R] and [M] for a general outline of results concerning Stefan problems.

We conclude this introduction by describing the plan of the paper. Some preliminary material is gathered in section 2. Section 3 is then devoted to deriving the results in Theorems 1.1–1.3 by means of matched asymptotic expansion techniques in a way which we believe to be conceptually simpler than the study done in [RSP], [SW] for (1.1a)–(1.1c) and (1.10). Besides its heuristic interest, this formal method detects a number of previously unnoticed patterns and provides the basic lines along which a rigorous proof is subsequently implemented. The arguments in section 3 are made rigorous in sections 4 to 7. To be precise, the sought-for solutions are obtained by means of a topological fixed point argument. This is a classical approach in the literature on partial differential equations (PDEs) which, to mention but a few examples, has been used recently to analyze singularity patterns arising in parabolic equations in the works [B1], [B2], [AV], and [HV3], among others. The basic aspects of our topological method are presented in section 4.

Sections 5 and 6 are then devoted to providing the various estimates required to yield (1.2) in Theorem 1.1 and (1.6) in Theorem 1.3. Once this has been achieved, we conclude by sketching in section 7 those modifications required to obtain (1.3)–(1.5) as well as (1.7)–(1.9).

2. Preliminaries.

Let $(\theta(r,t), R(t))$ be a solution of (1.1). It will be convenient for our purposes to introduce a new variable $u(r,t)$ given by

$$u(r,t) = \int_{R(t)}^{r} \xi^{1-N} d\xi \int_{R(t)}^{\xi} s^{N-1}(\theta(s,t)+1)\, ds \quad \text{if } r > R(t), \tag{2.1a}$$

$$u(r,t) = 0 \quad \text{if } r \leq R(t). \tag{2.1b}$$

We then readily see that $u(r,t)$ satisfies

$$u_t = u_{rr} + \left(\frac{N-1}{r}\right) u_r - H(u) \quad \text{for } x \in \mathbb{R}^N, \quad t > 0, \tag{2.2a}$$

where $H(u)$ is the standard Heaviside function; i.e., $H(u) = 1$ whenever $u > 0$ and $H(u) = 0$ otherwise. Equation (2.2a) is to be complemented with the initial condition

$$u(r,0) = \int_{R(0)}^{r} \xi^{1-N} d\xi \int_{R(0)}^{\xi} s^{N-1} (\theta(s,0) + 1)\, ds. \tag{2.2b}$$

For ease of notation, we shall often use the symbols Δ and ∇ instead of their radial counterparts when dealing with (2.2a) and related equations. Further, we introduce self-similar variables as follows:

$$u(r,t) = (T-t)\Phi(y,\tau), \quad y = r(T-t)^{-\frac{1}{2}}, \quad \tau = -\log(T-t); \tag{2.3}$$

we define a rescaled free boundary $\varepsilon(\tau)$ given by

$$\varepsilon(\tau) = R(t)e^{\frac{\tau}{2}} \equiv R(t)(T-t)^{-\frac{1}{2}}. \tag{2.4}$$

It is then readily seen that Φ satisfies the following equation:

$$\begin{aligned} \Phi_\tau &= \Phi_{yy} + \left(\frac{N-1}{y} - \frac{y}{2}\right)\Phi_y - H(\Phi) \\ &\equiv \Delta\Phi - \frac{1}{2} y \nabla\Phi + \Phi - (1-\chi_\varepsilon) \\ &\equiv A\Phi - (1-\chi_\varepsilon), \end{aligned} \tag{2.5}$$

where $\chi_\varepsilon(y) = 1$ for $y < \varepsilon(\tau)$ and $\chi_\varepsilon(y) = 0$ otherwise. The linear operator A will play a key role in our approach. Consider the weighted space

$$L^2_{w,r}(\mathbb{R}^+) = \left\{ f \in L^2_{\text{loc}}(\mathbb{R}^+) : \|f\|^2 = \int_0^\infty y^{N-1} |f(y)|^2 e^{\frac{-y^2}{4}} dy < \infty \right\}.$$

Clearly $L^2_{w,r}(\mathbb{R}^+)$ is a Hilbert space when endowed with the norm

$$\|f\|^2 = \langle f, f \rangle = \int_0^\infty y^{N-1} |f(y)|^2 e^{\frac{-y^2}{4}} dy,$$

where we have used the symbol $\langle\ ,\ \rangle$ to denote the corresponding scalar product. For any positive integer k, one may then define the Hilbert spaces $H^k_{w,r}(\mathbb{R}^+)$ in a straightforward way. By classical spectral theory, one then has that the radial operator A in (2.5) is self-adjoint in $L^2_{w,r}(\mathbb{R}^+)$ with domain $D(A) = H^2_{w,r}(\mathbb{R}^+)$. Furthermore, the eigenvalues of A consist of the sequence

$$\lambda_k = 1 - k, \quad k = 0, 1, 2, \ldots. \tag{2.6a}$$

The corresponding eigenfunctions can be written in the form

$$\varphi_k(y) = \begin{cases} c_k L_k\left(\frac{y^2}{4}\right) & \text{if } N = 2, \quad k = 0,1,2,\ldots, \\ c_{k,N} L_k^{\frac{N-2}{N}}\left(\frac{y^2}{4}\right) & \text{if } N \geq 3, \quad k = 0,1,2,\ldots, \end{cases} \tag{2.6b}$$

where $L_k(x)$ (resp. $L_k^{(N-2)/N}(x)$) denotes the standard kth Laguerre polynomial (resp. the modified $L_k^\alpha(x)$ Laguerre polynomial with $\alpha = \frac{N-2}{N}$), cf., for instance, [L] and [MF] for a review of properties of such functions. The normalization constants c_k and $c_{k,N}$ in (2.6b) are selected so that

$$\|\varphi_k\| = 1 \quad \text{for any } k. \tag{2.6c}$$

By classical results (cf., for instance, [MF]), we readily see that

$$c_{k,N}^2 = \frac{\Gamma(\frac{N}{2})\Gamma(k+1)}{2^N \left(\Gamma\left(\frac{N-2}{2}+k+1\right)\right)^2 \pi^{\frac{N}{2}}}, \tag{2.7a}$$

$$c_k^2 = \frac{1}{4\pi\Gamma(k+1)}. \tag{2.7b}$$

The following a priori bound on solutions of (2.5) is important for our purposes:

$$\Phi(y,\tau) \le C(y^2+1) \quad \text{for some } C > 0 \text{ and any } y > 0, \tau > 0. \tag{2.8}$$

Estimate (2.8) can be obtained, for instance, from the Bernstein-type bound

$$|\nabla\Phi(y,\tau)| \le C \quad \text{for any } y \text{ and any } \tau > 0,$$

which holds for solutions of (2.5) under rather loose assumptions on their initial values (cf., for instance, [HV1] for a related result). Arguing as in [HV2], we may deduce from (2.8) the following convergence result:

$$\begin{gathered} \Phi(y,t) \to \frac{y^2}{4} \quad \text{as } t \to \infty, \\ \text{uniformly on sets } y \le M < \infty. \end{gathered} \tag{2.9}$$

Since (2.9) plays an important role in what follows, we shall briefly sketch here the main ideas in its proof and refer to [HV2] for details. To begin with, an energy argument like that in [GK] shows that

$$\Phi(y,\tau) \to \Phi^*(y) \quad \tau \to \infty$$

uniformly on compact sets of $|y|$, where Φ^* is a stationary solution of (2.5). Arguing as in Lemma 4.3 of [HV2], we see that either $\lim_{\tau\to\infty}\varepsilon(\tau) = 0$ or $\lim_{\tau\to\infty}\varepsilon(\tau) = 1$. The second case would allow for a possible stationary solution $\Phi^*(y) = 1$, which would in turn yield that $\Phi(0,\tau) > 0$ for $\tau \gg 1$. This in particular implies that the ice ball has already disappeared for some time $t < T$, which is a contradiction. On the other hand, the case $\lim_{\tau\to\infty}\varepsilon(r) = 0$ gives rise to two possible stationary solutions satisfying (2.8), namely

$$\Phi^*(y) = 0, \qquad \Phi^*(y) = \frac{y^2}{4}.$$

To rule out the first possibility, we argue by contradiction as follows. Assume that $\lim_{\tau\to\infty}\Phi(y,r) = 0$ uniformly on sets $|y| \le R < \infty$. Then for fixed $A > 0$ and $\varepsilon > 0$ we may select $\tau \gg 1$ so that $\Phi(y,\tau) \le \varepsilon$ for $\tau \ge \tau_0$ and $|y| \le A$. A quick glance at equation (2.5) reveals then that $\Phi(y,\tau)$ is at most of order $O(\varepsilon e^{\tau-\tau_0})$ for $\tau > \tau_0$

at distances $y \sim Ae^{(\tau-\tau_0)/2}$. As a matter of fact, this estimate is readily suggested by dropping the absorption term $H(\Phi)$ in (2.5) and then checking how bounds on initial values propagate along characteristics for the resulting equation. In terms of the variable $u(x,t)$, one is thus led to a bound of the type

$$u(x,t) \leq \varepsilon x^2 \quad \text{for} \quad x \leq \delta, \quad t_0 < t < T, \tag{2.10}$$

where $\delta = \delta(\varepsilon) > 0$ is a small (but fixed) positive number and t_0 is close enough to T. On the other hand, our assumption $\Phi^*(y) = 0$ carries into

$$u(x,t_0) \leq \varepsilon(T - t_0) \quad \text{for } x \leq A. \tag{2.11}$$

From (2.10) and (2.11), a barrier argument as the one in [EK] or [HV1] yields that $u(x,T) = 0$ for some $x > 0$, thus contradicting the assumption that the ice ball collapses exactly at $t = T$. This concludes the proof.

3. The formal argument. This section is devoted to showing how to obtain the asymptotic results in Theorems 1.1 and 1.2 by means of formal perturbative methods. While the approach to be described is a nonrigorous one, it is in our opinion the crux of this work. The reason for this statement is that these heuristic methods not only provide deep insight into what to expect but also mark the path along which a rigorous argument can be implemented. This last task will be postponed until sections 4–7.

For definiteness, we shall consider first the case $N = 2$ and remark then about the differences which arise for $N \geq 3$. Our starting point is the convergence result (2.9). Bearing it in mind, we set

$$\psi(y,\tau) = \Phi(y,\tau) - \frac{y^2}{4} \tag{3.1}$$

so that the function $\psi(y,\tau)$ satisfies

$$\psi_\tau = A\psi + \chi_{\varepsilon(\tau)}. \tag{3.2}$$

We now introduce the following ansatz concerning the effect of the term $\chi_{\varepsilon(\tau)}$ in (3.2).

Assumption 3.1. For $|y| \gg \varepsilon(\tau)$ and $\tau \gg 1$, we may replace (3.2) by

$$\psi_\tau = A\psi + \gamma\varepsilon(\tau)^2\delta(y), \tag{3.3}$$

where constant γ is uniquely determined by imposing that

$$\int_{\mathbb{R}^2} \chi_{\varepsilon(\tau)} dy = \gamma\varepsilon(\tau)^2 \int_{\mathbb{R}^2} \delta(y) dy \quad \text{(i.e., } \gamma = \pi\text{)}. \tag{3.4}$$

We next proceed to derive (1.2) in Theorem 1.1. To this end, we set

$$\Psi(y,\tau) = \sum_{k=0}^{\infty} a_k(\tau)\varphi_k(y) \equiv a_0(\tau)\varphi_0(y) + a_1(\tau)\varphi_1(y) + Q(y,\tau). \tag{3.5}$$

The first two Fourier coefficients would then satisfy

$$\dot{a}_0 = a_0 + \gamma\varepsilon(\tau)^2\langle\varphi_0, \delta(y)\rangle \quad \text{for } \tau \gg 1, \tag{3.6a}$$

$$\dot{a}_1 = \gamma\varepsilon(\tau)^2\langle\varphi_1, \delta(y)\rangle \quad \text{for } \tau \gg 1, \tag{3.6b}$$

whereas the remainder term $Q(y,\tau)$ is such that

$$Q_\tau = AQ + \gamma\varepsilon(\tau)^2 \left(\delta(y) - \sum_{k=0}^{1} \langle\varphi_k, \delta(y)\rangle\varphi_k\right), \tag{3.6c}$$

$$\langle Q, \varphi_k \rangle = 0 \quad \text{for } k = 0, 1. \tag{3.6d}$$

We now introduce the following assumption.

Assumption 3.2. The leading term in (3.5) as $\tau \gg 1$ is $a_1(\tau)\varphi_1(y)$; i.e., evolution in time of $\psi(y,\tau)$ is driven by the eigenfunction corresponding to zero eigenvalue. Moreover, one then expects

$$|\dot{\varepsilon}(\tau)| \ll \varepsilon(\tau) \quad \text{as } \tau \to \infty, \tag{3.7a}$$

$$Q(y,\tau) \sim \gamma\varepsilon(\tau)^2 F(y) \quad \text{as } \tau \to \infty \tag{3.7b}$$

for a suitable function $F(y)$. It then turns out that $F(y)$ satisfies

$$AF + \left(\delta(y) - \sum_{k=0}^{1} \langle \varphi_k, \delta(y) \rangle \varphi_k \right) = 0, \tag{3.8a}$$

$$\langle F, \varphi_k \rangle = 0 \quad \text{for } k = 0, 1. \tag{3.8b}$$

We may now integrate (3.8a) and (3.8b) to obtain

$$F(y) = -\frac{1}{2\pi} \log y + B + O(y^2 |\log y|) \quad \text{for } \varepsilon(\tau) \ll y \le 1, \tag{3.9}$$

where constant B is detemined by the orthogonality conditions (3.8b). On the other hand, since we expect $\lim_{\tau\to\infty} a_k(\tau) = 0$ for $k = 0, 1$, we obtain from (3.6) that

$$a_k(\tau) \sim -\gamma\varphi_k(0) \int_\tau^\infty \varepsilon(s)^2 e^{(1-k)(\tau-s)} ds \quad \text{for } \tau \gg 1, \quad k = 0, 1. \tag{3.10}$$

Putting together (3.5), (3.9), and (3.10), we arrive at

$$\Phi(y, \tau \sim \frac{y^2}{4} - \gamma \sum_{k=0}^{1} \varphi_k(0)\varphi_k(y) \int_\tau^\infty \varepsilon(s)^2 e^{(1-k)(\tau-s)} ds \tag{3.11}$$
$$+ \gamma\varepsilon(\tau)^2 \left(B - \frac{1}{2\pi} \log y\right)$$

whenever

$$\varepsilon(\tau) \ll y \le C \quad \text{with } C > 0 \text{ and } \tau \gg 1.$$

Formula (3.11) provides an outer expansion for $\Phi(y,\tau)$ in regions sufficiently far from the free boundary. To analyze the set where $y \sim \varepsilon(\tau)$, we change variables as follows:

$$\Phi(y,\tau) = (\varepsilon(\tau))^2 w(\xi,\tau), \quad \xi = \frac{y}{\varepsilon(\tau)}. \tag{3.12}$$

Substituting (3.12) into (2.5) readily gives

$$\varepsilon\dot{\varepsilon}w - \dot{\varepsilon}\varepsilon\xi w_\xi + \varepsilon^2 w_\tau = \Delta w - \frac{\xi^2}{2}\xi w_\xi + \varepsilon^2 w - \tilde{\chi}, \tag{3.13}$$

where now $\Delta w = w_{\xi\xi} + \frac{w_\xi}{\xi}$ and $\tilde{\chi}(\xi) = 1$ whenever $\xi > 1$ and is zero elsewhere. After comparing the order of magnitude of the different terms in (3.13), we are led to guess that as $\tau \to \infty$, $w(\xi, \tau) \sim \bar{w}(\xi)$, where $\bar{w}(\xi)$ is the solution of

$$w_{\xi\xi} + \frac{w_\xi}{\xi} = 1 \quad \text{for } \xi > 1, \tag{3.14}$$

$$w(1) = w_\xi(1) = 0;$$

i.e.,

$$\bar{w}(\xi) = \frac{\xi^2}{4} - \frac{1}{2}\log\xi - \frac{1}{4} \quad \text{for } \xi > 1. \tag{3.15}$$

In view of (3.12) and (3.15), we expect that

$$\Phi(y,\tau) \sim \varepsilon^2(\tau)\bar{w}\left(\frac{y}{\varepsilon(\tau)}\right) = \frac{y^2}{4} - \frac{\varepsilon(\tau)^2}{2}\log\left(\frac{y}{\varepsilon(\tau)}\right) - \frac{\varepsilon(\tau)^2}{4} \tag{3.16}$$

for $y \sim \varepsilon(\tau)$ and $\tau \gg 1$.

Matching the inner and outer expansions (3.16) and (3.11), we obtain as a matching condition

$$B\varepsilon(\tau)^2 - \gamma\sum_{k=0}^{1}\varphi_k(0)^2\int_\tau^\infty \varepsilon(s)^2 e^{(1-k)(\tau-s)}ds = \frac{\varepsilon(\tau)^2}{2}\log\varepsilon(\tau) - \frac{\varepsilon(\tau)^2}{4}. \tag{3.17}$$

This is the basic integral equation that determines the position of the rescaled free boundary $\varepsilon(\tau)$. Actually, we claim that (3.17) yields

$$\varepsilon(\tau) \sim Ke^{-\frac{\sqrt{2\tau}}{2}}\tau^{\frac{1}{4\sqrt{\tau}}-\frac{1}{4}}\cdot(1+o(1)) \quad \text{as } \tau \to \infty, \tag{3.18}$$

where $K = e^{4B}$.

Taking into account (2.4), one readily checks that (3.18) gives (1.2) in Theorem 1.1. For the convenience of the reader we shall briefly sketch the way in which (3.18) can be derived from (3.17). We first observe that a dominated balance argument shows that the leading terms in (3.17) satisfy

$$\frac{\varepsilon(\tau)^2}{4}\log(\varepsilon(\tau)^2) \sim -\gamma\varphi_1(0)^2\int_\tau^\infty \varepsilon(s)^2 ds = -\frac{1}{4}\int_\tau^\infty \varepsilon(s)^2 ds. \tag{3.19}$$

Now set

$$G(\tau) = \int_\tau^\infty \varepsilon(s)^2 ds.$$

Then

$$G(\tau) = -\varepsilon(r)^2,$$

and it follows from (3.19) that

$$\log(\varepsilon(\tau)^2) \sim \log G(\tau) - \log(-\log G(\tau)), \tag{3.20}$$

where here and henceforth all asymptotic equivalences are understood to hold for $r \gg 1$. Hence

$$\varepsilon^2(\log G(\tau) - \log(-\log G(\tau))) \sim -G(\tau)$$

which in turn yields

$$G^{-1}\dot{G}\log G\left(1 - \frac{\log(-\log G)}{\log G} + \cdots\right) = 1,$$

and we obtain after integration

$$\frac{1}{2}(\log G)^2 = \tau\left(1 + \frac{\log(-\log G)}{\log G} + \cdots\right) + C$$

for some constant C. To the first term, the equality above gives $|\log G| \sim (2\tau)^{1/2}$, whence

$$G(\tau) \sim e^{-\sqrt{2\tau}} \quad \text{as } \tau \to \infty \tag{3.21}$$

up to some algebraic factor. From (3.21) and our choice of G, we deduce that

$$\int_\tau^\infty e^{\tau - s}\varepsilon(s)^2 ds \sim \varepsilon(\tau)^2. \tag{3.22}$$

it then follows from (3.22) and (3.17) that

$$\varepsilon(\tau)^2\log(\varepsilon(\tau)^2) = -G(\tau) + 4B\varepsilon(r)^2 + \cdots,$$

whence

$$\frac{\dot{G}}{G}(\log G - \log(-\log G) + 4B + \cdots) = 1.$$

Taking into account (3.21), we then see that

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}((\log G)^2) &= \left(1 + \frac{1}{2\sqrt{\tau}}\left(\log\sqrt{2} + \frac{1}{2}\log\tau\right) + \frac{4B}{\sqrt{2\tau}} + \cdots\right)^{-1} \\ &= 1 - \frac{1}{2\sqrt{2}}\frac{\log\tau}{\sqrt{\tau}} - \frac{(\log\sqrt{2} + 4B)}{\sqrt{2\tau}} + \cdots.\end{aligned}$$

Setting $\alpha = \sqrt{2}(\log\sqrt{2} + 4B)$, we obtain

$$\frac{1}{2}(\log G)^2 = \tau - \frac{\tau^{\frac{1}{2}}}{\sqrt{2}}\log\tau - \alpha\tau^{\frac{1}{2}} + \cdots$$

which at once yields

$$G(\tau) \sim K_0 e^{-\sqrt{2\tau}}\tau^{\frac{1}{2\sqrt{\tau}}} \quad \text{with } K_0 = \sqrt{2}e^{4B}. \tag{3.23}$$

Plugging (3.23) and (3.20) into (3.19) and (3.18) follows.

We next set out to describe the way in which (1.3)–(1.5) are obtained. For the ease of presentation, we shall merely sketch the points that give rise to the different behaviors involved. To begin with, we continue to suppose that $N = 2$ and that Assumption 3.1 is in force. We now recast (3.5) in the form

$$\psi(y,\tau) = \sum_{k=0}^{\infty} a_k(\tau)\varphi_k(y) = \sum_{k=0}^{l} a_k(\tau)\varphi_k(y) + Q(y,\tau). \tag{3.24}$$

It is readily seen that formulas (3.6) read in this case as

$$\dot{a}_k = (1-k)a_k + \gamma\varepsilon(\tau)^2\langle\varphi_k, \delta(y)\rangle \quad \text{for } \tau \gg 1, \quad k = 0, 1, \ldots, l, \tag{3.25a}$$

$$Q_\tau = AQ + \gamma\varepsilon(\tau)^2\left(\delta(y) - \sum_{k=0}^{l}\langle\varphi_k, \delta(y)\rangle\varphi_k\right), \tag{3.25b}$$

$$\langle Q, \varphi_k\rangle = 0 \quad \text{for } k = 0, 1, \ldots, l. \tag{3.25c}$$

We now replace Assumption 3.2 by the following.

Assumption 3.3. The leading term in (3.24) as $\tau \gg 1$ is $a_l(\tau)\varphi_l(y)$; i.e., evolution in time of $\psi(y,\tau)$ is driven by the eigenfunction corresponding to the lth eigenvalue. Moreover, one then expects

$$\frac{d}{d\tau}(\varepsilon^2(\tau)) \sim (1-l)\varepsilon^2(\tau) \quad \text{as } \tau \to \infty, \tag{3.26a}$$

$$Q(y,\tau) \sim \gamma\varepsilon(\tau)^2 F(y) \quad \text{as } \tau \to \infty, \tag{3.26b}$$

where $F(y)$ satisfies

$$A_l F + \left(\delta(y) - \sum_{k=0}^{l}\langle\varphi_k, \delta(y)\rangle\varphi_k\right) = 0, k = 0, 1, \ldots, l, \tag{3.27a}$$

$$\langle F, \varphi_k\rangle = 0 \quad \text{for } k = 0, 1, \ldots, l \tag{3.27b}$$

and the operator A_l is given by

$$A_l F \equiv F_{yy} + \left(\frac{1}{y} - \frac{y}{2}\right)F_y + lF. \tag{3.27c}$$

Integrating (3.27), we obtain

$$F(y) = -\frac{1}{2\pi}\log y + \cdots \quad \text{for } \varepsilon(\tau) \ll y \ll 1. \tag{3.28}$$

We point out that no further details on the expansion (3.28) are required to derive the sought-for result (1.3). Arguing as before, we now obtain the following outer expansion for $\Phi(y,\tau)$:

$$\Phi(y,\tau) \sim \frac{y^2}{4} - \gamma\varphi_l(0)^2\int_\tau^\infty \varepsilon(s)^2 e^{(1-l)(\tau-s)}ds + \varepsilon(\tau)^2\left(-\frac{1}{2\pi}\log y + O(1)\right) \tag{3.29}$$

whenever $\varepsilon(\tau) \ll y \ll 1$ and $\tau \gg 1$. The inner expansion for $\Phi(y,\tau)$ is exactly that already obtained in (3.16). From (3.29) and (3.16), we deduce the matching condition

$$-\gamma\varphi_l(0)^2 \int_\tau^\infty \varepsilon(s)^2 e^{(1-l)(\tau-s)} ds = \frac{1}{2}\varepsilon^2(\tau)\log\varepsilon(\tau), \tag{3.30}$$

which yields now the following asymptotic behavior for $\varepsilon(\tau)$:

$$\varepsilon(\tau) \sim C e^{\left(\frac{1-l}{2}\right)\tau} \tau^{-\frac{1}{l-1}} \quad \text{as } \tau \to \infty \tag{3.31}$$

for some positive constant C.

Estimate (3.31) can be obtained from (3.30) by means of an argument similar to that leading from (3.17) to (3.18). Indeed, setting $\tau(s) = \varepsilon^2(s)e^{-(1-l)s}$ and observing that $4\gamma\varphi_l(0)^2 = 1$ (cf. (2.8)), we may rewrite (3.20) in the form

$$-\int_\tau^\infty r(s)ds = r(\tau)\log(\varepsilon^2(\tau)) = r(\tau)\left((1-l)\tau + \log(r(\tau))\right).$$

Hence

$$\frac{1}{l-1}\int_\tau^\infty r(s)ds = r(\tau)\left(r + O(\log(r(\tau)))\right) \quad \text{for } \tau \gg 1,$$

which can be integrated to yield $r(\tau) = Cr^{-l/(l-1)}$, whence

$$\varepsilon^2(\tau) = Ce^{(1-l)\tau}\tau^{-\frac{l}{l-1}},$$

and (1.3) follows.

We conclude this section by sketching the formal derivation of (1.4) and (1.5) in Theorem 1.2. Consider first the case of (1.4). From Assumptions 3.1 and 3.2 (with $\varepsilon^2(\tau)$ replaced by $\varepsilon^N(\tau)$ where appropriate), we obtain the following outer expansion for $\Phi(y,\tau)$:

$$\Phi(y,\tau) \sim \frac{y^2}{2N} - \gamma\sum_{k=0}^{l}\varphi_k^2(0)\int_\tau^\infty \varepsilon^N(s)e^{(1-k)(\tau-s)}ds + O\left(\frac{\varepsilon^N(\tau)}{y^{N-2}}\right) \tag{3.32}$$

whenever $\varepsilon(\tau) \ll y \ll 1$ and $\tau \gg 1$. The corresponding inner expansion is also obtained in the form

$$\Phi(y,\tau) = \varepsilon^2(\tau)w(\xi,\tau) \quad \text{with } \xi = \frac{y}{\varepsilon(\tau)},$$

where $w(\xi,\tau) \sim \bar{w}(\xi)$ for large τ and $\bar{w}$ solves

$$\bar{w}_{\xi\xi} + \left(\frac{N-1}{\xi}\right)\bar{w}_\xi = 1 \quad \text{for } \xi > 1,$$

$$\bar{w}(1) = \bar{w}_\xi(1) = 0$$

(compare with (3.14)). This now yields

$$\bar{w}(\xi) = \frac{\xi^2}{2N} - \frac{1}{2(N-2)} + \frac{\xi^{2-N}}{N(N-2)},$$

whereupon the following inner expansion for Φ follows:

$$\Phi(y,\tau) \sim \frac{y^2}{2N} - \frac{\varepsilon^2(\tau)}{2(N-2)} + \frac{1}{N(N-2)}\left(\frac{\varepsilon(\tau)}{y}\right)^{N-2} \quad \text{for } y \sim \varepsilon(\tau) \text{ and } \tau \gg 1. \tag{3.33}$$

Matching (3.32) and (3.33) gives the following equation for $\varepsilon(\tau)$:

$$\gamma \sum_{k=0}^{1} \varphi_k(0)^2 \int_\tau^\infty \varepsilon(s)^N e^{(1-k)(\tau-s)} ds = \frac{\varepsilon(\tau)^2}{2(N-2)}, \tag{3.34}$$

which yields $\varepsilon(\tau) \sim C\tau^{-1/(N-2)}$ and hence (1.4). Finally, (1.5) is obtained by guessing an outer expansion of the form

$$\Phi(y,\tau) \sim \frac{y^2}{2N} - \alpha e^{(1-l)\tau}\varphi_l(y). \tag{3.35}$$

This follows by neglecting the term $\chi_{\varepsilon(\tau)}$ in (3.2) and assuming that the lth mode dominates in the Fourier expansion for $\psi(y,\tau)$. Matching (3.35) with (3.33), (1.5) follows.

4. The topological argument. In this section we shall describe the basic approach towards a rigorous derivation of Theorems 1.1–1.3. For simplicity, we shall concentrate on the case where $N = 2$ and (1.2) holds and remark briefly on the remaining situations afterwards.

4.1. Obtaining (1.2) in Theorem 1.1. Let us define $\bar{\varepsilon}(\tau)$ as follows:

$$\bar{\varepsilon}(\tau) = K e^{-\frac{\sqrt{2\tau}}{2}} \tau^{\frac{1}{4\sqrt{\tau}} - \frac{1}{4}}, \quad K \text{ given in (3.18).} \tag{4.1}$$

In another words, $\bar{\varepsilon}(\tau)$ is the leading part in the expected asymptotic behavior of the rescaled free boundary in this case. Fix now τ_0, τ_1 with $\tau_1 \geq \tau_0 \gg 1$ and consider functions $\varepsilon(\tau)$ such that the following estimates hold for some choice of $M > 1$.

$$\sup\left\{|\varepsilon(\tau) - \bar{\varepsilon}(s)|, \text{ where } \tau, s \in [\tau_0,\tau_1] \text{ and } |\tau - s| < \frac{1}{\tau}\right\} < M\varepsilon(\tau)\tau^{-\frac{3}{2}}, \tag{4.2a}$$

$$\frac{\bar{\varepsilon}(\tau)}{M} < \varepsilon(\tau) < M\bar{\varepsilon}(\tau) \quad \text{for } \tau \in [\tau_0,\tau_1]. \tag{4.2b}$$

We next recall that if $\Phi(y,\tau)$ is a (rescaled) solution of our problem (cf. (2.3)), then $\psi(y,\tau)$ given in (3.1) solves

$$\psi_\tau = A\psi + \chi_{\varepsilon(\tau)} \quad \text{for } y \in \mathbb{R}, \quad \tau > \tau_0, \tag{4.3a}$$

$$\psi(y,\tau_0) = \psi_0(y) \quad \text{at } \tau = \tau_0, \tag{4.3b}$$

where A is the linear operator in (2.5). We want to pick $\psi_0(y)$ above in a particular manner. Namely, we take

$$\psi_0(y) = \alpha_0\tilde{\varphi}_0(y) + \alpha_1\tilde{\varphi}_1(y) + \gamma\bar{\varepsilon}(\tau_0)^2 F(y), \tag{4.4}$$

where $F(y)$ is as in (3.9), and α_0, α_1, $\tilde{\varphi}_0$, and $\tilde{\varphi}_1$ will be selected presently. As a matter of fact, for $j = 0, 1$ functions $\tilde{\varphi}_j(y)$ will coincide with the eigenfunctions $\varphi_j(y)$ given in (2.6) for, say, $y \geq (\bar{\varepsilon}(\tau_0))^{1/2}$. The main point in selecting $\psi_0(y)$ in (4.4) is that we want it to match with the inner expansion (3.16) (with $\varepsilon(\tau)$ replaced by $\bar{\varepsilon}(\tau_0)$ there) at distances $y \sim \bar{\varepsilon}(\tau_0)^{1/2}$. This amounts essentially to imposing

$$\alpha_0\varphi_0(0) + \alpha_1\varphi_1(y) + \gamma\bar{\varepsilon}(\tau_0)^2\left(-\frac{1}{2\pi}\log y + B + \cdots\right)$$
$$= \frac{\bar{\varepsilon}(\tau_0)^2}{2}\log\bar{\varepsilon}(\tau_0) - \frac{\bar{\varepsilon}(\tau_0)^2}{2}\log y - \frac{\bar{\varepsilon}(\tau_0)^2}{4},$$

whence

$$\alpha_0\varphi_0 + \alpha_1\varphi_0(0) + \left(B + \frac{1}{4}\right)\gamma\bar{\varepsilon}(\tau_0)^2 = \frac{\bar{\varepsilon}(\tau_0)^2}{2}\log\bar{\varepsilon}(\tau_0) \tag{4.5}$$

so that

$$|\alpha_0| + |\alpha_1| = O(\bar{\varepsilon}(\tau_0)^2|\log\bar{\varepsilon}(\tau_0)|). \tag{4.6}$$

We have yet to determine what kind of modification is to be performed on the φ_k's near the origin for $k = 0, 1$. To ascertain this point, we observe that if no change were done at all, we would have that

$$\psi_0(y) \sim -\frac{\gamma\bar{\varepsilon}(\tau_0)^2}{2\pi}\log y \quad \text{as } y \to 0.$$

To remove such singularity, we just redefine the φ_k's near $y = 0$ as follows:

$$\alpha_0\tilde{\varphi}_0 + \alpha_1\tilde{\varphi}_1(y) = \frac{\gamma\bar{\varepsilon}(\tau_0)^2}{2\pi} + o(\bar{\varepsilon}(\tau)^2\log y) \quad \text{as } y \to 0. \tag{4.7}$$

Notice that relations (4.4)–(4.7) are compatible and allow for many possible choices of α_k, $\tilde{\varphi}_k$ for $k = 0.1$. Bearing in mind our previous arguments, we now introduce the following notation:

(4.8)

> Let τ_0, τ_1 be such that $\tau_1 \geq \tau_0 \gg 1$, and let μ be a given number such that $0 < \mu \leq 1$. We shall say that a solution $\psi(y, \tau)$ of (4.3a) which is defined for $\tau_0 \leq \tau \leq \tau_1$ belongs to the class $A(\tau_0, \tau_1, \mu)$ if there exists a constant M such that $|\psi(y, \tau)| < M(1 + y^2)$ for $y \in \mathbb{R}$ and $\tau \in [\tau_0, \tau_1]$ and conditions (4.2) are satisfied with M replaced by $M\mu$ there.

We shall say that $\psi(y, \tau) \in \overline{\mathcal{A}(\tau_0, \tau_1, \mu)}$ if it satisfies those conditions describing membership in the class $\mathcal{A}(\tau_0, \tau_s, \mu)$ when strict inequalities are replaced by the symbol $\leq$.

For $k = 0, 1$, let us now define

$$l_k(\alpha_0, \alpha_1; \tau) = \langle\psi(y, \tau; \alpha_0, \alpha_1), \varphi_k\rangle + \int_\tau^\infty e^{(1-k)(\tau - s)}\langle\chi_{\bar{\varepsilon}}, \varphi_k\rangle ds, \tag{4.9}$$

where $\psi(y, \tau; \alpha_0, \alpha_1)$ is the solution of (4.3a) such that $\psi(y, \tau_0; \alpha_0, \alpha_1) = \psi_0(y)$, $\psi_0(y)$ being a function satisfying (4.4)–(4.7) above.

The following result is a crucial ingredient in the proof of (1.2) in Theorem 1.1.

PROPOSITION 4.1. *Assume that* $M > 0$ *is large enough and let* $\psi(y,\tau)$ *be the solution of* (4.3)*, where* $\psi(y,\tau_0) \equiv \psi_0(y)$ *is such that* (4.4)*–*(4.7) *hold. Suppose also that*

$$\psi(y,\tau) \in \overline{\mathcal{A}(\tau_0,\tau_1,1)} \tag{4.10}$$

for some τ_1, τ_0 *so that* $\tau_1 > \tau_0 \gg 1$*. Then if*

$$l_k(\alpha_0,\alpha_1,\tau_1) = 0 \quad \textit{for } k = 0,1 \tag{4.11}$$

(*cf.* (4.9) *above*)*, one has that*

$$\psi(y,\tau) \in \mathcal{A}\left(\tau_0,\tau_1,\frac{1}{2}\right).$$

We shall prove Proposition 4.1 in sections 5 and 6, which contain most of the technical aspects of this paper. To keep the flow of the main arguments here, we will assume that the proposition holds true and continue with the derivation of (1.2). Let $\alpha = (\alpha_0,\alpha_1)$ be any pair of real numbers and set $l(\alpha_0,\alpha_1;\tau) = (l_0(\alpha_0,\alpha_1;\tau), l_1(\alpha_0,\alpha_1;\tau))$, where for $k = 0,1, l_k$ is defined as the right-hand side of (4.9). Let τ_1, τ_0 be such that $\tau_1 \geq \tau_0$ and define $\mathcal{U}(\tau_0,\tau_1) \subset \mathbb{R}^2$ as the open set consisting of all points $(\alpha_0,\alpha_1) \in \mathbb{R}^2$ such that the corresponding solution $\psi(y,\tau)$ of (4.3)–(4.8) satisfies that $\psi(y,\tau) \in \mathcal{A}(\tau_0,\tau_1,1)$. From our previous arguments, it follows that we may select an initial value $\psi(y,\tau_0) = \psi_0(y)$ in (4.3b) so that

$$\psi(y,\tau_0) \in \mathcal{A}\left(\tau_0,\tau_0,\frac{1}{2}\right)$$

and there exists a unique solution of $l(\alpha_0,\alpha_1,\tau_0) = 0$. Indeed, by (4.9) one has that

$$l_k(\alpha_0,\alpha_1;\tau_0) = \alpha_k + \delta(\alpha,\tau_0)(|\alpha_0| + |\alpha_1|) + O(\bar{\varepsilon}(\tau_0)^2) \quad \text{for } k = 0,1,$$

where $\delta(\alpha,\tau_0) \to 0$ as $\tau_0 \to \infty$, uniformly for $|\alpha| = |\alpha_0| + |\alpha_1|$ bounded, and the last term on the right above may be assumed to be independent on α. On the other hand, we may always suppose $l = (l_0,l_1)$ to be differentiable with respect to α_0,α_1 by means of a suitable choice of the initial value $\psi_0(y)$. We shall assume henceforth that $\varphi_0(y)$ satisfies such a condition. It then turns out that for $k = 0,1$ equation $l_k(\alpha_0,\alpha_1;\tau_0) = 0$ has a unique solution α_k such that

$$\alpha_k = O(\bar{\varepsilon}(\tau_0)^2).$$

As a matter of fact, one then has that

$$d(l,\mathcal{U}(\tau_0,\tau_0);0) = 1,$$

where for $\tau \geq \tau_0$, $d(l,\mathcal{U}(\tau_0,\tau);0)$ denotes the topological degree of the mapping l in the set $\mathcal{U}(\tau_0,\tau)$ at the value zero.

Now assume that $\mathcal{U}(\tau_0,\tau) \neq \phi$ for any $\tau \in [\tau_0,\tau_1]$ with $\tau_1 > 0$ and denote by $\partial\mathcal{U}(\tau_0,\tau)$ the boundary of the open set $\mathcal{U}(\tau_0,\tau)$. We notice that if $l \neq 0$ on $U(\partial\mathcal{U}(\tau_0,\tau))$ for $\tau_0 \leq \tau \leq \tau_1$, the $d(l,\mathcal{U}(\tau_0,\tau);0) = d(l,\mathcal{U}(\tau_0,\tau_0);0)$ for any such τ. It then follows from standard continuous dependence results that

$$\mathcal{U}(\tau_0,\tau_1) \neq \phi$$

and

$$d(l, \mathcal{U}(\tau_0, \tau_1); 0) = 1$$

for any $\tau_1 > \tau_0$ such that $(\tau_1 - \tau_0)$ is sufficiently small. We next claim that

$$d(l, \mathcal{U}(\tau_0, \tau); 0) = 1 \tag{4.12a}$$

for any $\tau > \tau_0$ as far as

$$\mathcal{U}(\tau_0, \tau) \neq \phi. \tag{4.12b}$$

Indeed, suppose that there exists a first time $\tau > \tau_0$ when (4.12a) fails but (4.12b) holds true. In view of our previous remark, there must be a point $\beta = (\beta_0, \beta_1) \in \partial\mathcal{U}(\tau_0, \tau)$, where $l(\beta) = 0$, and clearly $\psi(y, \tau; \beta_0, \beta_1) \in \overline{\mathcal{A}(\tau_0, \tau; 1)}$. We then use Proposition 4.1 to deduce that $\beta \in \mathcal{U}(\tau_0, \tau)$, which is a contradiction.

We further observe that

$$\mathcal{U}(\tau_0, \tau) \neq \phi \quad \text{for any } \tau > \tau_0, \tag{4.13}$$

$$\text{provided that } \tau_0 \gg 1.$$

To check (4.13), we define $\tau^* = \sup\{\tau : \mathcal{U}(\tau_0, \tau) \neq \phi\}$. We already know that $\tau^* > \tau_0$. Assume now that $\tau^* < \infty$. By (4.12), we may select a sequence of times $\{\tau_n\}$ increasing to τ^* and a sequence $\{\alpha_n\} = \{(\alpha_{0n}, \alpha_{1n})\}$ such that $l(\alpha_{0n}, \alpha_{1n}; \tau_n) = 0$ and $\alpha_n \in \mathcal{U}(\tau_0, \tau_n)$. Since $\mathcal{U}(\tau_0, \tau_{n+1}) \subset \mathcal{U}(\tau_0, \tau_n)$, one has that $\{\alpha_n\}$ is bounded. Therefore, a subsequence (still denoted by $\{\alpha_n\}$) exists which converges to some point $\alpha^* = (\alpha_0^*, \alpha_1^*)$. It then turns out that $l(\alpha_0^*, \alpha_1^*; \tau^*) = 0$ and hence by Proposition 4.1 the corresponding function $\psi(y, \tau; \alpha_0^*, \alpha_1^*)$ remains at the interior of $\mathcal{A}(\tau_0, \tau^*; 1)$; this is the point where restriction $\tau_0 \gg 1$ needs to be imposed on (4.13). By continuous dependence results, ψ would also remain at the interior of $\mathcal{A}(\tau_0, \tau^* + \delta; 1)$ for some $\delta > 0$, thus contradicting the definition of τ^*.

We are now prepared to detail the argument leading to the existence of the solutions referred to in Theorems 1.1 and 1.3. Take a sequence $\{\tau_n\}$ such that $\tau_1 > \tau_0$ and $\lim_{n\to\infty} \tau_n = \infty$. For any such n, $\mathcal{U}(\tau_0, \tau_n) \neq \phi$, and we may select $\alpha_n = (\alpha_{0n}, \alpha_{1n})$ such that $l(\alpha_{0n}, \alpha_{1n}; \tau_n\} = 0$.

Let $\psi_n(y, \tau) \equiv \psi_n(y, \tau; \alpha_{0n}, \alpha_{1n})$ be the solution of (4.3a) with initial value $\psi_n(y, \tau_0) = \psi_0(y; \alpha_{0n}, \alpha_{1n})$ satisfying (4.4)–(4.8). By Proposition 4.1, we have that $\psi_n(y, \tau) \in \mathcal{A}(\tau_0, \tau_n; \frac{1}{2})$. Since the sequence $\{\alpha_n\}$ is bounded, there exists a subsequence (still denoted by $\{\alpha_n\}$) and a value $\bar{\alpha} = (\bar{\alpha}_0, \bar{\alpha}_1)$ such that $\lim_{n\to\infty} \alpha_n = \bar{\alpha} \in \mathcal{U}(\tau_0, \tau_0)$. It then turns out that function $\psi(y, \tau; \bar{\alpha}_0, \bar{\alpha}_1)$, solution of (4.3a) with initial value $\psi(y, \tau_0; \bar{\alpha}_0, \bar{\alpha}_1)$, provides a sought-for solution satisfying (1.2), and the proof is concluded under our current assumptions.

4.2. The remaining cases. To derive (1.3) in Theorem 1.1, we just repeat our previous argument with the following modifications. First we replace $\bar{\varepsilon}(\tau)$ in (4.1) by

$$\bar{\varepsilon}(\tau) = Ce^{(1-l)\tau}\tau^{-\frac{1}{l-1}},$$

where, as in the statement of the theorem, l is any number larger than or equal to two and C is any positive constant. Instead of making use of (4.4), we now define $\psi_0(y)$ by

$$\psi_0(y) = \sum_{k=0}^{l} \alpha_k \tilde{\varphi}_k(y) - \frac{C}{2\pi} \log y$$

and replace condition (4.11) in Proposition 4.1 by

$$l_k(\alpha_0, \alpha_1, \dots, \alpha_0; \tau_1) = 0 \quad \text{for } k = 0, 1, \dots, l,$$

where the l_k's are defined as in (4.9), except that here we allow l_k to depend on all parameters $\alpha_0, \alpha_1, \dots, \alpha_l$.

The cases corresponding to dimensions $N \geq 3$ are similar. For instance, to obtain (1.4) (resp. (1.5)), we define $\bar{\varepsilon}(\tau)$ as follows:

$$\bar{\varepsilon}(\tau) = B\tau^{-\frac{1}{N-2}} \quad (\text{resp. } \bar{\varepsilon}(\tau) = Ce^{(1-\frac{l}{2})\tau}),$$

where $B > 0$ is a fixed constant which depends on N and can be determined from (3.34) and C is any given constant. We now have to redefine the φ_k's near $y = 0$ in order to remove singularities of the type $y^{-(N-2)}$ instead of logarithmic ones. A straightforward modification of the previous approach yields then the desired results.

5. Derivation of (1.2): Analysis of the outer region. We now set out to provide the details required to justify the picture given in sections 3 and 4. To this end, we shall concentrate on proving Proposition 4.1 to highlight those modifications required to obtain (1.3)–(1.5). From now on we shall thus assume that $N = 2$ and start by considering solutions to the equation satisfied by $\psi(y, \tau)$ given in (3.2); i.e.,

$$\psi_\tau = A\psi + \chi_{\varepsilon(\tau)} \quad \text{for } \tau > \tau_0, y \in \mathbb{R} \tag{5.1}$$

with initial condition (4.3b), where $\psi_0(y)$ satisfies (4.4)–(4.8) and operator A is given in (2.5). We shall compare solutions to (5.1) and (4.3b) with those to the auxiliary equation

$$W_\tau = AW + \gamma\varepsilon(\tau)^2\delta(y) \quad \text{for } \tau > \tau_0, y \in \mathbb{R} \tag{5.2}$$

with the same initial condition at $\tau = \tau_0$. Solutions of (5.1) can be represented in the form

$$\psi(y, \tau) = a_0(\tau)\varphi_0(y) + a_1(\tau)\varphi_1(y) + E(y, \tau), \tag{5.3}$$

where $E(y, \tau)$ satisfies

$$E_\tau = \Delta E - \frac{1}{2}y\nabla E + E + (\chi_{\varepsilon(\tau)} - \langle\varphi_0, \chi_{\varepsilon(\tau)}\rangle\varphi_0 - \langle\varphi_1\chi_{\varepsilon(\tau)}\rangle\varphi_1) \tag{5.4}$$

and $\langle E, \varphi_k\rangle = 0$ for $k = 0, 1$. Bearing in mind (5.2), we shall also consider solutions $Q(y, \tau)$ to the equation

$$\begin{aligned} Q_\tau = \Delta Q - \frac{1}{2}y\nabla Q + Q \\ + \gamma\varepsilon(\tau)^2(\delta(y) - \langle\varphi_0, \delta(y)\rangle\varphi_0 - \langle\varphi_1, \delta(y)\rangle\varphi_1) \end{aligned} \tag{5.5}$$

such that $\langle Q, \varphi_k\rangle = 0$ for $k = 0, 1$. We then have the following lemma.

LEMMA 5.1. *Let $\tau_0 > 0$ be fixed. Then the solution $Q(y, \tau_0)$ of* (5.5) *which is defined for $\tau > \tau_0$ and satisfies $Q_0(y, \tau_0) = 0$ is given by*

$$Q_0(y, \tau) = \gamma \int_{\tau_0}^{\tau} K(y, \tau - s)e^{\tau - s}\varepsilon(s)^2 ds, \tag{5.6a}$$

where

$$K(y,\tau) = (4\pi(1-e^{-\tau}))^{-1}\left(\exp\left(-\frac{y^2e^{-\tau}}{4(1-e^{-\tau})}\right)\right.$$
$$\left.-\sum_{j=0}^{1}\varphi_j\left\langle\varphi_j,\exp\left(-\frac{y^2e^{-\tau}}{4(1-e^{-\tau})}\right)\right\rangle\right). \tag{5.6b}$$

Proof. For convenience, we shall dispense with the subscript in $Q_0(y,\tau)$. Differentiating three times with respect to the y variables in (5.5) yields

$$\begin{aligned}\frac{\partial}{\partial\tau}Q_{i,j,k} &= \Delta Q_{i,j,k} - \frac{1}{2}y\nabla Q_{i,j,k} - \frac{1}{2}Q_{i,j,k} + \gamma\varepsilon(\tau)^2(\delta(y))_{i,j,k}\\ &= A_*Q_{i,j,k} - \frac{1}{2}Q_{i,j,k} + \gamma\varepsilon(\tau)^2(\delta(y))_{i,j,k}\end{aligned}$$

in an appropriate weak sense. Using a variation of constants formula in the equation above and denoting by S_* the semigroup generated by A_*, we obtain that

$$Q_{i,j,k}(y,\tau) = \gamma\int_{\tau_0}^{\tau} e^{-(\frac{\tau-s}{2})}S_*(\tau-s)\left(\varepsilon(s)^2\frac{\partial^3(\delta(\xi))}{\partial\xi_i\partial\xi_j\partial\xi_k}\right)ds = -\gamma\int_{\tau_0}^{\tau} e^{-\frac{(\tau-s)}{2}}\varepsilon(s)^2$$
$$\cdot\int_{\mathbb{R}^2}\frac{\partial^3}{\partial\xi_i\partial\xi_j\partial\xi_k}\left((4\pi(1-e^{-(\tau-s)}))^{-1}\exp\left(-\frac{(ye^{-(\frac{\tau-s}{2})}-\xi)^2}{4(1-e^{-(\tau-s)})}\delta(\xi)\right)d\xi\right)ds$$
$$= \gamma\frac{\partial^3}{\partial y_i\partial y_j\partial j_k}\int_{\tau_0}^{\tau} e^{\tau-s}\varepsilon(s)^2\left((4\pi(1-e^{-(\tau-s)}))^{-1}\exp\left(-\frac{(ye^{-(\frac{\tau-s}{2})})^2}{4(1-e^{-(\tau-s)})}\right)\right)ds.$$

Integrating now three times with respect to the y variables and imposing $\langle Q_1\varphi_0\rangle = \langle Q_1\varphi_1\rangle = 0$, the result follows. ∎

We shall elaborate a bit on the formulas in (5.6). To begin with, we observe that

$$\left\langle\varphi_0,\exp\left(-\frac{y^2e^{-(\tau-s)}}{4(1-e^{-(\tau-s)})}\right)\right\rangle = C(1-e^{-(\tau-s)})\quad\text{for some } C>0, \tag{5.7a}$$

$$\left\langle\varphi_1,\exp\left(-\frac{y^2e^{-(\tau-s)}}{4(1-e^{-(\tau-s)})}\right)\right\rangle = (a_0+a_1(1-e^{-(\tau-s)}))(1-e^{-(\tau-s)}) \tag{5.7b}$$

for some constants a_0 and a_1. To check (5.7a), we simply notice that

$$\begin{aligned}\left\langle\varphi_0,\exp\left(-\frac{y^2e^{-(\tau-s)}}{4(1-e^{-(\tau-s)})}\right)\right\rangle &= c_0\int_{\mathbb{R}^2}\exp\left(-\frac{y^2}{4}-\frac{y^2e^{-(\tau-s)}}{4(1-e^{-(\tau-s)})}\right)dy\\ &= c_0(1-e^{-(\tau-s)})\int_{\mathbb{R}^2}e^{-r^2}dr,\end{aligned}$$

where c_0 is given in (2.6b). The proof of (5.7b) is similar and will therefore be omitted.

Assume now that $\tau > \tau_0+1$. We may then split the integral term in (5.6a) in the form

$$Q_0(y,\tau) = \int_{\tau_0}^{\tau-1}(\ ) + \int_{\tau-1}^{\tau}(\ ) \equiv Q_{0,1}(y,\tau) + Q_{0,2}(y,\tau). \tag{5.8}$$

To estimate $Q_{0,2}$, we proceed to examine the quantity

$$J \equiv \gamma \int_{\tau-1}^{\tau} (4\pi(1-e^{-(\tau-s)}))^{-1} \exp\left((\tau-s) - \frac{y^2 e^{-(\tau-s)}}{4(1-e^{-(\tau-s)})}\right) \varepsilon^2(s)ds. \tag{5.9}$$

Setting $\eta = y^2 e^{-(\tau-s)}(4(1-e^{-(\tau-s)}))^{-1}$, it follows that $e^{\tau-s} = 1 + \frac{y^2}{4\eta}$ and $d\eta = 4^{-1}y^2 e^{-(\tau-s)}(1-e^{-(\tau-s)})^{-2}ds$. If we now write $f(y) = e\theta(y)$ with $\theta(y) = y^2e^{-1}(4(1-e^{-1}))^{-1}$, we easily see that

$$J = \frac{\gamma}{\pi}\int_{\theta(y)}^{\infty} \varepsilon(s)^2 \left(1+\frac{y^2}{4\eta}\right)^2 y^{-2}e^{-\eta}d\eta = \frac{\gamma}{4\pi}\int_{f(y)}^{\infty} \varepsilon(s)^2\left(1+\frac{y^2}{4\eta}\right)e^{-\eta}\eta^{-1}d\eta,$$

where $s = \tau - \log(1+\frac{y^2}{4\eta})$.

Now we shall pay attention to the term $S(\tau-\tau_0)Q(y,\tau_0)$, where $Q(y,\tau_0)$ is given by

$$\begin{aligned} Q(y,\tau_0) &= \gamma\bar{\varepsilon}(\tau_0)^2F(y) + \sum_{j=0}^{1}\alpha_j\left(\tilde{\varphi}_j - \sum_{k=0}^{1}\varphi_k\langle\varphi_k,\tilde{\varphi}_j\rangle\right) \\ &\equiv \gamma\bar{\varepsilon}(\tau_0)^2F(y) + R(y), \end{aligned} \tag{5.10}$$

where α_j, φ_j, and $\tilde{\varphi}_j$ are as in (4.4). Notice that $\langle Q(\cdot,\tau_0),\varphi_k\rangle = 0$ for $k = 0,1$. Moreover, one has that

$$R(y) = \sum_{j=0}^{1}\alpha_j(\tilde{\varphi}_j - \varphi_j) + \sum_{j=0}^{1}\alpha_j\left(\sum_{k=0}^{1}\varphi_k(\delta_{j,k} - \langle\varphi_k,\tilde{\varphi}_j\rangle)\right),$$

where, as customary, $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ otherwise. In view of (4.4)–(4.7), it holds that

$$|R(y)| \le C\left(\bar{\varepsilon}(\tau_0)^2\left|\log\left(\frac{y}{\bar{\varepsilon}(\tau_0)}\right)\right|\chi_{[y\le\bar{\varepsilon}(\tau_0)^{1/2}]} + \bar{\varepsilon}(\tau_0)^3\tau_0(1+y^2)\right).$$

Using the explicit kernel for the semigroup $S(\tau)$, we then derive

$$|S(\tau-\tau_0)R| \le C\bar{\varepsilon}(\tau_0)^3\tau_0^{\frac{3}{2}}(1+y^2) \quad \text{for } \tau_0 \le \tau \le \tau_0+1 \text{ and } y \ge \bar{\varepsilon}(\tau_0)^{\frac{1}{4}}. \tag{5.11a}$$

On the other hand, by regularizing properties of $S(\tau)$, we obtain

$$\begin{aligned} &|S(\tau-\tau_0)R| \le C\bar{\varepsilon}(\tau)^{3-\chi}(1+y^2) \quad \text{for } \tau \ge \tau_0+1, \\ &y \ge \bar{\varepsilon}(\tau_0)^{\frac{1}{4}}, \quad \text{and some } \chi \in (0,1). \end{aligned} \tag{5.11b}$$

Summing up, we have obtained the following lemma.

LEMMA 5.2. *Let $\tau^* = \max\{\tau_0, \tau-1\}$ and let $Q(y,\tau)$ be the solution of (5.5) for $\tau > \tau_0$ such that $Q(y,\tau_0)$ is given by (5.10). We then have that*

$$\begin{aligned} Q(y,\tau) &= \gamma\bar{\varepsilon}(\tau_0)^2S(\tau-\tau_0)F(y) + \gamma\int_{\tau_0}^{\tau^*}K(y,\tau-s)e^{\tau-s}\varepsilon(s)^2ds \\ &+ \gamma\int_{\tau^*}^{\tau}(A_1 + A_2(1-e^{-(\tau-s)}))e^{\tau-s}\varepsilon(s)^2ds + \frac{\gamma}{\pi}\int_{\Sigma}^{\infty}\varepsilon(s)^2e^{-\eta}\eta^{-1}d\eta \\ &+ \frac{\gamma}{\pi}\int_{\Sigma}^{\infty}\varepsilon(s)^2e^{-y}\eta^2(4\eta^2)^{-1}d\eta + O\left(\frac{\bar{\varepsilon}(\tau)^2}{\tau}\right)(1+y^2) \end{aligned} \tag{5.12}$$

in regions where $y \geq \bar{\varepsilon}(\tau_0)^{1/4}$ *and* $\tau > \tau_0$. *Here* A_1, A_2 *are some positive constants,* $\sum = y^2(4(e^\beta - 1))^{-1}$, *where* $\beta = \max\{\tau_0, \tau - 1\}$ *and in the last two integrals above,* $s = \tau - \log(1 + \frac{y^2}{4\eta})$.

We now proceed to estimate the difference $(E - Q)$, where E and Q are solutions of (5.4) and (5.5), respectively. To this end, we set

$$Z = E - Q, \quad g = \chi_{\varepsilon(\tau)} - \gamma\varepsilon(\tau)^2\delta(y) \tag{5.13}$$

so that Z satisfies

$$\begin{aligned} Z_\tau &= \Delta Z - \frac{1}{2}\eta\nabla Z + Z + (g - \langle\varphi_0, g\rangle\varphi_0 - \langle\varphi_1, g\rangle\varphi_1) \\ &\equiv AZ + (g - \langle\varphi_0, g\rangle\varphi_0 - \langle\varphi_1, g\rangle\varphi_1) \equiv AZ + h(y,\tau) \end{aligned} \tag{5.14a}$$

We shall consider equation (5.14a) for values $\tau > \tau_0 \gg 1$. At $\tau = \tau_0$, we impose $E(y, \tau_0) = Q(y, \tau_0)$ so that

$$Z(y, \tau_0) = 0. \tag{5.14b}$$

We then have that the solution to (5.14) can be written in the form

$$Z(y,\tau) = \int_{\tau_0}^{\tau} S(\tau - s)h(\cdot, s)ds \equiv \int_{\tau_0}^{\tau} L(y, \tau - s; s)ds, \tag{5.15a}$$

where

$$\begin{aligned} L(y, \tau - s; s) &= (4\pi(1 - e^{-(\tau - s)}))^{-1} \\ &\quad \cdot \int_{\mathbb{R}^2} \exp\left(-\frac{(ye^{-(\frac{\tau-s}{2})} - \xi)^2}{4(1 - e^{-(\tau-s)})}\right) h(\xi, s)d\xi. \end{aligned} \tag{5.15b}$$

Without loss of generality, we may assume $\tau > \tau_0 + 1$. We then split Z in the form

$$Z(y,\tau) = \int_{\tau_0}^{\tau-1} L\, ds + \int_{\tau-1}^{\tau} L\, ds. \tag{5.16}$$

Then the following lemma holds.

LEMMA 5.3. *There exists* $C > 0$ *such that*

$$\int_{\tau_0}^{\tau-1} |L|ds \leq C \int_{\tau_0}^{\tau-1} e^{-(\tau-s)}\varepsilon(s)^4 ds \tag{5.17}$$

uniformly on bounded sets $|y| \leq R < \infty$.

Proof. To begin with, we observe that

$$\begin{aligned} |\langle\varphi_0, g(\cdot, s)\rangle| &= \left|\int_{\mathbb{R}^2} \varphi_0(\chi_{\varepsilon(s)} - \gamma\varepsilon(s)^2\delta(y))e^{-y^2/4}dy\right| \\ &= \left|\int_{|y|\leq\varepsilon(s)} \varphi_0(e^{-y^2/4} - 1)dy\right| \leq C\int_{|y|\leq\varphi(s)} r^2dr \leq C\varepsilon(s)^4 \end{aligned}$$

and a similar bound is easily obtained for $|\langle\varphi_1, g\rangle|$. We thus have that

$$|\langle \varphi_0, g(\cdot, s)\rangle| + |\langle \varphi_1, g(\cdot, s)\rangle| \le C\varepsilon(s)^4 \tag{5.18}$$

for some $C > 0$.

Recalling that $\tau - s \ge 1$ under our current assumptions, we now consider the term

$$I = \int_{\tau_0}^{\tau-1} (4\pi(1 - e^{-(\tau-1)}))^{-1} \int_{\mathbb{R}^2} \exp\left(-\frac{(ye^{-(\frac{\tau-s}{2})} - \xi)^2}{4(1 - e^{-(\tau-s)})}\right) g(\xi, s) ds$$

$$\equiv \int_{\tau_0}^{\tau-1} S(\tau - s) g(\cdot, s) ds \equiv \int_{\tau_0}^{\tau-1} S\left(\frac{1}{2}\right) S\left(\tau - s - \frac{3}{4}\right) S\left(\frac{1}{4}\right) g(\cdot, s) ds. \tag{5.19}$$

We claim that

$$\left\| S\left(\frac{1}{4}\right) g(\cdot, s)\right\| \le C\varepsilon(s)^4 \quad \text{for some } C > 0. \tag{5.20}$$

Let us assume (5.20) for the moment and continue. One then may use classical regularizing effects to derive that

$$\left\| S\left(\tau - s - \frac{3}{4}\right)\left(S\left(\frac{1}{4}\right) g(\cdot, s)\right)\right\| \le Ce^{-(\tau-s)}\varepsilon(s)^4. \tag{5.21}$$

Finally, a standard Sobolev imbedding yields

$$\left| S\left(\frac{1}{2}\right)\left(S\left(\tau - s - \frac{3}{4}\right) S\left(\frac{1}{4}\right) g(\cdot, s)\right)\right| \le Ce^{-(\tau-s)}\varepsilon(s)^4. \tag{5.22}$$

Putting together (5.18)–(5.22), estimate (5.17) follows. The proof will thus be complete as soon as (5.20) has been obtained. To derive this last result, we make use of a duality argument. Let $\varphi(y)$ be any radial function in $L^2(\mathbb{R})$, and consider the integral

$$J = \int_{\mathbb{R}^2} g(\xi, s)\left(\int_{\mathbb{R}^2} e^{-(y-\xi)^2}\varphi(y)dy\right) d\xi \equiv \int_{\mathbb{R}^2} g(\xi, s) G(\xi) d\xi.$$

Recalling the arguments leading to (5.18), one readily sees that

$$\begin{aligned}\int_{\mathbb{R}^2} g(\xi, s) G(\xi) d\xi &= \int_{\mathbb{R}^2} (\chi_{\varepsilon(s)} - \gamma\varepsilon(s)^2\delta(y)) G(\xi) d\xi \\ &= \int_{|y| \le \varepsilon(s)} (G(\xi) - G(0)) d\xi \le C\varepsilon(s)^4,\end{aligned}$$

whereupon (5.20) follows. ∎

Our next result reads as follows.

LEMMA 5.4. *There exists $C > 0$ such that*

$$\int_{\tau-1}^{\tau} |L(y, \tau - s; s)| ds \le C\left(\int_{\tau-1}^{\tau} \varepsilon(s)^4 ds + |y| \int_0^1 \varepsilon(\tau - s)^3 s^{-2} e^{-\frac{Cy^2}{s}} ds\right). \tag{5.23}$$

Proof. Set

$$w(y,\tau-s;\xi)=\exp\left(-\frac{(ye^{-(\frac{\tau-s}{2})}-\xi)^2}{4(1-e^{-(\tau-s)})}\right). \tag{5.24a}$$

We can then readily check that

$$\int_{\tau-1}^{\tau} L(y,\tau-s;s)ds=\int_{\tau-1}^{\tau}(4\pi(1-e^{-(\tau-s)}))^{-1}e^{\tau-s}M(y,\tau-s)ds, \tag{5.24b}$$

where

$$\begin{aligned} M(y,\tau-s)=\int_{\mathbb{R}^2}&(w(y,\tau-s;\xi)-\langle\varphi_0,w(y,\tau-s;\xi)\rangle\varphi_0\\ &-\langle\varphi_1,w(y,\tau-s;\xi)\rangle\varphi_1)g(\xi,s)d\xi. \end{aligned} \tag{5.24c}$$

$$\frac{y^2}{4}+\frac{(ye^{-(\frac{\tau-s}{2})}-\xi)^2}{4(1-e^{-(\tau-s)})}=\frac{1}{4(1-e^{-(\tau-s)})}\left((y-\xi e^{-(\frac{\tau-s}{2})})^2+\xi^2(1-e^{-(\tau-s)})\right).$$

A quick computation then reveals that

$$\langle\varphi_0,w(y,\tau-s;\xi)\rangle\le Ce^{-\frac{y^2}{4}}(1-e^{-(\tau-s)}),$$

and a similar result holds when we replace φ_0 by φ_1 above. Recalling the argument leading to (5.18), we then have that

$$\int_{\mathbb{R}^2}|(\langle\varphi_0,w\rangle\varphi_0+\langle\varphi_1,w\rangle\varphi_1)|\,|g(\cdot,s)|ds\le C(1-e^{-(\tau-s)})\varepsilon(s)^4. \tag{5.25}$$

Now consider the integral

$$J\equiv\int_{\tau-1}^{\tau}(4\pi(1-e^{-(\tau-s)}))^{-1}e^{\tau-s}\left(\int_{\mathbb{R}^2}w(y,\tau-s;\xi)g(\xi,s)d\xi\right)ds. \tag{5.26}$$

Since $\int_{\mathbb{R}^2}g(\xi,s)d\xi=0$, it holds that

$$J=\int_{\tau-1}(4\pi(1-e^{-(\tau-s)}))^{-1}e^{\tau-s}\left(\int_{\mathbb{R}^2}(w(y,\tau-s;\xi)-w(y,\tau-s;0))g(\xi,s)d\xi\right)ds,$$

whence

$$\begin{aligned} |J|\le C&\textstyle\int_{\tau-1}^{\tau}(4\pi(1-e^{-1(\tau-s)}))^{-1}\\ &\cdot\left(\int_{|\xi|\le\varepsilon(s)}|w(y,\tau-s;\xi)-w(y,\tau-s;0)|d\xi\right)ds. \end{aligned} \tag{5.27}$$

We now observe that for any real numbers a and b,

$$|e^{-a^2}-e^{-(a-b)^2}|\le Ce^{-a^2/2}|a|\,|b|\quad\text{for some } C>0. \tag{5.28}$$

To check (5.28), we consider first the case where $|a|\ge|b|$. Then if $|a|\,|b|\le 1$, we have that $|1-e^{2ab-b^2}|\le C|a|\,|b|$, whereas if $|a|\,|b|>1$, $|e^{-a^2}-e^{-(a-b)^2}|\le$

$Ce^{-a^2/2} \leq Ce^{-a^2/2}|a|\,|b|$. When $|a| < |b|$, we simply select $\mu > 0$ large enough and observe that if $|a|\,|b| < \mu$, then $|1 - e^{2ab-b^2}| \leq C|a|\,|b|$ for some $C = C(\mu) > 0$, whereas for $|a|\,|b| > \mu$ one has that $2ab - b^2 \geq -\frac{b^2}{2}$ and hence $|1 - e^{2ab-b^2}| \leq C|a|\,|b|$. Having shown that (5.28) holds, we now take advantage of that inequality (with $a = ye^{-\frac{(\tau-s)}{2}}(4(1-e^{-(\tau-s)}))^{-\frac{1}{2}}$ and $b = \xi(4(1-e^{-(\tau-s)})^{\frac{1}{2}})$ and (5.27) to show that

$$|J| \leq C\int_{\tau-1}^{\tau}(1-e^{-(\tau-s)})^{-1}\int_{|\xi|\leq\varepsilon(s)}(1-e^{-(\tau-s)})^{-1}w(y,\tau-s;0)|y||\xi|d\xi$$

$$\leq C\int_{\tau-1}^{\tau}(1-e^{-s})^{2}\varepsilon(s)^3|y|\exp\left(-\frac{y^2e^{-s}}{(1-e^{-s})}\right)ds \leq C|y|\int_0^1\varepsilon(\tau-s)^3 s^{-2}e^{-\frac{Cy^2}{s}}ds. \tag{5.29}$$

Putting together (5.25) and (5.29), the result follows. □

For latter reference, we summarize the results obtained in Lemmas 5.2–5.4 as follows.

Corollary 5.5. *Let $E(y,\tau)$, $Q(y,\tau)$ be functions such that* (i) $E(y,\tau_0) = Q(y,\tau_0)$ *and* (ii) E *and* Q *solve, respectively,* (5.4) *and* (5.5) *for $\tau > \tau_0$. Assume also that* (4.2) *holds. Then for any $R > 0$, there exists $C > 0$ such that*

$$|E(y,\tau) - Q(y,\tau)| \leq C\left(\frac{\varepsilon(\tau)^3}{y}\right) \tag{5.30}$$

whenever $\bar{\varepsilon}(\tau_0)^{\frac{1}{4}} \leq y \leq R$ and $\tau > \tau_0$

Proof. The proof follows from (5.17), (5.23), and the bounds (4.2). □

6. Derivation of (1.2): Analysis of the inner region. Let $\sigma(\tau)$ be a function to be discussed later (cf. (6.9)). We now fix $\bar{\tau} \gg 1$ and define

$$\xi = \frac{y}{\sigma(\bar{\tau})}; \qquad w(\xi,\tau) = (\sigma(\bar{\tau}))^{-2}\Phi(\sigma(\bar{\tau})\xi,\tau), \tag{6.1}$$

where $\Phi(y,\tau)$ is given in (2.3). A quick computation reveals that $w(\xi,\tau)$ satisfies

$$(\sigma(\bar{\tau}))^2 w_\tau = \Delta w - H(w) + (\sigma(\bar{\tau}))^2\left(w - \frac{\xi\nabla w}{2}\right), \tag{6.2}$$

where the operators Δ and ∇ are now written with respect to the inner space variable ξ. When determining the asymptotics of solutions of (6.2), a key role is played by the stationary equation

$$\Delta\nu = H(\nu). \tag{6.3}$$

For any $\lambda > 0$, a radial solution of (6.3) is given by $\nu_\lambda(\xi) = \lambda^2\bar{\nu}(\frac{\xi}{\lambda})$, where

$$\bar{\nu}(r) = \frac{r^2}{4} - \frac{1}{4} - \frac{1}{2}\log r. \tag{6.4}$$

We can readily check that the radial, nontrivial solution of (6.3) which satisfies $\nu(\xi) = 0$ for $\xi \leq \lambda$ and $\nu(\lambda) = \nu'(\lambda) = 0$ is given by

$$\nu_\lambda(\xi) = \frac{\xi^2}{4} - \frac{\lambda^2}{2}\log\left(\frac{\xi}{\lambda}\right) - \frac{\lambda^2}{4} \quad \text{for } \xi > \lambda. \tag{6.5}$$

It will be convenient to compare the functions $\nu_\lambda(\xi)$ given in (6.5) with the stationary solution of (6.2) that takes off at $\xi = \lambda$. The corresponding result reads as follows.

LEMMA 6.1. *Let* $\tilde{w}_\lambda(\xi) \equiv \tilde{w}_\lambda(\xi;\bar{\tau})$ *be the stationary solution of* (6.2) *such that* $\tilde{w}_\lambda(\lambda) = \tilde{w}'_\lambda(\lambda) = 0$ *and* $\tilde{w}_\lambda(\xi) > 0$ *for* $\xi > \lambda > 0$. *Then there holds*

$$\tilde{w}_\lambda(\xi) = \nu_\lambda(\xi) + O(\sigma^2\lambda^2 \log\lambda) \quad \textit{for } \xi \le 1. \tag{6.6}$$

Proof. We set $\tilde{w}_\lambda = \nu_\lambda + \varphi$. A quick check reveals that φ solves

$$\varphi'' + \frac{\varphi'}{\xi} + \sigma^2\left(\varphi - \frac{\xi\varphi'}{2} - \frac{\lambda^2}{2}\log\frac{\xi}{\lambda} - \frac{\lambda^2}{4}\right) = 0,$$
$$\varphi(\lambda) = \varphi'(\lambda) = 0.$$

Consider first the case where ξ is close to λ. Standard ordinary differential equation (ODE) arguments yield that, in such a region

$$\varphi(\xi) \sim C\sigma^2\lambda^2\xi^2 \log\frac{\xi}{\lambda} \quad \text{for some real } C. \tag{6.7}$$

When $\lambda \ll \xi \le 1$, we introduce a new variable $\eta = \frac{\xi}{\lambda}$. Setting $\dot{\varphi} = \frac{d\varphi}{d\eta}$, we readily check that φ satisfies

$$\ddot{\varphi} + \frac{\dot{\varphi}}{\eta} + \sigma^2\lambda^2\left(\varphi - \frac{\eta\dot{\varphi}}{2}\right) = \frac{\sigma^2\lambda^4}{2}\log\eta + \frac{\sigma^2\lambda^2}{4}.$$

A dominated balance argument shows that the third term on the left is negligible with respect to the remaining ones. This in turn implies that $\varphi(\eta) \sim K^2\sigma^2\lambda^4\eta^2\log\eta$. Back to the original variables, we have derived

$$\varphi(\xi) \sim K\sigma^2\lambda^2\xi^2 \log\frac{\xi}{\lambda} \quad \text{for } \lambda \ll \xi \le 1. \tag{6.8}$$

Matching (6.7) and (6.8), we obtain $C = K$ and (6.6) follows. ◻

Let us now define

$$\sigma(\tau) = (\varepsilon(\tau))^\theta, \quad \text{where } \theta \text{ is a positive and small number}, 0 < \theta < \frac{1}{4}, \tag{6.9}$$

$$W(\tau) = \frac{1}{4} + \frac{1}{(\sigma(\bar{\tau}))^2}\left(a_0(\tau)\varphi_0 + a_1(\tau)\varphi_1(\sigma(\bar{\tau})) + E(\sigma(\bar{\tau}),\tau)\right), \tag{6.10}$$

where a_0, a_1, and E are as in (5.3), and

$$\lambda(\tau) = \frac{\varepsilon(\tau)}{\sigma(\tau)}. \tag{6.11}$$

We shall prove the following lemma.

LEMMA 6.2. *Assume that conditions* (4.2) *and* (4.11) *hold. Then there exists a constant* $C > 0$ *such that*

$$|W(\tau) - W(\bar{\tau})| \le \frac{C}{\tau}\left(\frac{\varepsilon(\bar{\tau})}{\sigma(\bar{\tau})}\right)^2, \tag{6.12}$$

provided that $|\tau - \bar{\tau}| \le \frac{1}{\tau}$ *and* $\bar{\tau} \ge \tau_0 \gg 1$.

Proof. We set

$$
\begin{aligned}
W(\tau) - W(\bar\tau) &= \varphi_0(\sigma(\bar\tau))^{-2}(a_0(\tau) - a_0(\bar\tau)) + \varphi_1(\sigma(\bar\tau))^{-2}(a_1(\tau) - a_1(\bar\tau)) \\
&\quad + (\sigma(\bar\tau))^{-2}(E(\sigma(\bar\tau),\tau) - E(\sigma(\bar\tau),\bar\tau)) \equiv W_1 + W_2 + W_3.
\end{aligned} \tag{6.13}
$$

Terms W_1 and W_2 in (6.13) are easily dealt with. For instance, since $\psi(y,\tau)$ satisfies (5.1), one sees that if $\tau > \bar\tau$,

$$
\begin{aligned}
W_2 &= \varphi_1(\sigma(\bar\tau))(\sigma(\bar\tau))^{-2}\int_{\bar\tau}^{\tau}\langle\chi_\varepsilon(s),\varphi_1\rangle ds \\
&= \varphi_1(\sigma(\bar\tau))(\sigma(\bar\tau))^{-2}\int_{\bar\tau}^{\tau}\int_{|y|\le\varepsilon(s)}\varphi_1(y)e^{-\frac{y^2}{4}}dyds;
\end{aligned}
$$

hence

$$
|W_2| \le \frac{C|\tau-\bar\tau|}{(\sigma(\bar\tau))^2}((\varepsilon(s)-\varepsilon(\bar\tau))^2 + \varepsilon(\bar\tau)^2) \le \frac{C}{\tau}\left(\frac{\varepsilon(\bar\tau)}{\sigma(\bar\tau)}\right)^2\left(1+\frac{M^2}{\tau^3}\right),
$$

where (4.2) has been used to obtain the last inequality above. A similar bound for W_1 is obtained by means of (4.11). To estimate W_3, we first observe that

$$
|Q(\sigma(\bar\tau),\tau) - Q(\sigma(\bar\tau),\bar\tau)| \le C\frac{\bar\varepsilon(\tau)^2}{\tau} \quad \text{for } |\tau-\bar\tau| \le \frac{1}{\tau}. \tag{6.14}
$$

To obtain (6.14), we consider first the case where $|\tau_0 - \bar\tau| \le \frac{1}{\tau_0}$. Setting

$$
D(y) = \delta(y) - \sum_{k=0}^{1}\langle\varphi_k,\delta(y)\rangle\varphi_k(y)
$$

it then turns out that

$$
\begin{aligned}
Q(y,\tau) &= \gamma\varepsilon(\tau_0)^2 S(\tau-\tau_0)F(y) + \gamma\int_{\tau_0}^{\tau}S(\tau-s)(\varepsilon(s)^2 D)ds \\
&= \gamma\varepsilon(\tau_0)^2 S(\tau-\tau_0)F(y) + \gamma\int_{\tau_0}^{\tau}S(\tau-s)(\varepsilon(\tau_0)^2 D)ds \\
&\quad + \gamma\int_{\tau_0}^{\tau}S(\tau-s)((\varepsilon(s)^2 - \varepsilon(\tau_0)^2)D)ds \\
&= \gamma\varepsilon(\tau_0)^2 F(y) + O(\varepsilon(\tau)^2\tau^{-\frac{3}{2}}|\log y|)
\end{aligned}
$$

for $|y|$ small, whereupon (6.14) follows. On the other hand, by (5.30) we have that

$$
|E(\sigma(\bar\tau),\tau) - E(\sigma(\bar\tau),\bar\tau)| \le C\frac{\varepsilon(\tau)^2}{\tau}|Q(\sigma(\bar\tau,\tau)) - Q(\sigma(\bar\tau),\bar\tau)|. \tag{6.15}
$$

We thus obtain from (6.14) and (6.15) that

$$
|W_3| \le C\left(\frac{\varepsilon(\tau)^2}{\tau}\right) \quad \text{for } |\tau_0 - \bar\tau| \le \frac{1}{\tau_0} \ \text{ and } \ \bar\tau \ge \tau_0 \gg 1,
$$

and putting together the bounds obtained for W_i $(i = 1, 2, 3)$ the proof is concluded in this case.

When $|\tau_0 - \bar{\tau}| > \frac{1}{\tau_0}$, we make use of (5.12) to check that (6.14) continues to hold. This is done by comparing the different terms appearing in the right-hand side of (5.12) when evaluated at $(\sigma(\bar{\tau}), \tau)$ and $(\sigma(\bar{\tau}), \bar{\tau})$, respectively. A typical argument in this direction goes as follows. For $i = 1, 2$ set $\sum_i = (\delta(\bar{\tau}))^2(4(e^{\beta_i} - 1))^{-1}$, where $\beta_i = \max\{\tau_i - 1, \tau_0\}$, and let us write $s_i = \tau_i - \log(1 - (\delta(\bar{\tau}))^2/4\eta)$. Then, if $|s_1 - s_2| \le \frac{1}{\bar{\tau}}$, we have that

$$
\begin{aligned}
&\left| \int_{\Sigma_1}^{\infty} \varepsilon(s_1)^2 e^{-\eta}\eta^{-1} d\eta - \textstyle\int_{\Sigma_2}^{\infty} \varepsilon(s_2)^2 e^{-\eta}\eta^{-1} d\eta \right| \\
&\qquad \le \int_{\Sigma_1}^{\infty} |\varepsilon(s_1) + \varepsilon(s_2)||\varepsilon(s_1) - \varepsilon(s_2)| e^{-\eta}\eta^{-1} d\eta + \int_{\Sigma_1}^{\Sigma_2} \varepsilon(s_2)^2 e^{-\eta}\eta^{-1} d\eta \\
&\qquad \le\ CM(\bar{\varepsilon}(\tau))^2 \tau^{-\frac{3}{2}} |\log \Sigma_1| + C(\bar{\varepsilon}(\tau))^2 (\delta(\bar{\tau}))^2 |\log \Sigma_1| \\
&\qquad \le\ CM\theta \frac{(\bar{\varepsilon}(\tau))^2}{\tau},
\end{aligned}
$$

where $\theta > 0$ can be selected arbitrarily small as $\tau_0 \to \infty$, and we have assumed for definiteness that $\Sigma_1 < \Sigma_2$. We omit further details. ∎

A key result in this section is the following.

LEMMA 6.3. *Under the assumptions of Lemma* 6.2*, there exists a constant* $C > 0$ *such that*

$$
|W(\tau) - \nu_{\lambda(\tau)}(1)| \le \frac{C}{\tau} \left(\frac{\varepsilon(\bar{\tau})}{\sigma(\bar{\tau})} \right)^2 \tag{6.16}
$$

provided that $|\tau - \bar{\tau}| \le \frac{1}{\tau}$*, where* ν_λ *is given in* (6.5) *and* $\bar{\tau} \ge \tau_0 \gg 1$.

Proof. We shall argue by contradiction and therefore assume that for any $K > 0$ there exists $\bar{\tau} \gg 1$ and $\tilde{\tau} \in (\bar{\tau} - \frac{1}{\tau}, \bar{\tau} + \frac{1}{\bar{\tau}})$ such that

$$
|W(\tilde{\tau}) - \nu_{\lambda(\tilde{\tau})}(1)| > \frac{K}{\tilde{\tau}} \left(\frac{\varepsilon(\bar{\tau})}{\sigma(\bar{\tau})} \right)^2. \tag{6.17}
$$

Now let τ be any time in the interval $(\bar{\tau} - \frac{1}{\bar{\tau}}, \bar{\tau} + \frac{1}{\bar{\tau}})$. In view of (6.12) and (6.17), it holds that

$$
|W(\tau) - \nu_{\lambda(\tilde{\tau})}(1)| > \frac{K}{2\tilde{\tau}} \left(\frac{\varepsilon(\tilde{\tau})}{\sigma(\tilde{\tau})} \right)^2. \tag{6.18a}
$$

Assume for definiteness that

$$
|W(\tau) - \nu_{\lambda(\tilde{\tau})}(1)| = v_{\lambda(\tilde{\tau})}(1) - W(\tau). \tag{6.18b}
$$

We now claim the following:

$$
\begin{aligned}
&\text{There exists } \mu > 0 \quad \text{such that if} \quad \lambda_0(\tilde{\tau}) = \lambda(\tilde{\tau})(1 + \mu(\tilde{\tau})^{-\frac{3}{2}}), \\
&\qquad\qquad\qquad\qquad \text{then} \quad \nu_{\lambda_0}(1) > W(\tau).
\end{aligned} \tag{6.19}
$$

To check (6.19), we observe that since $\tilde{\tau} \gg 1$,

$$
\begin{aligned}
\nu_{\lambda_0}(1) &= \frac{1}{4} + \frac{\lambda_0^2}{2}\log\lambda_0 - \frac{\lambda_0^2}{4} \\
&\sim \frac{1}{4} + \frac{\lambda^2}{2}\left(1+\mu(\tilde{\tau})^{-\frac{3}{2}}\right)^2 \log\lambda_0 - \frac{\lambda_0^2}{4} \\
&\sim \frac{1}{4} + \frac{\lambda(\tilde{\tau})^2}{2}\left(1+2\mu(\tilde{\tau})^{-\frac{3}{2}}\right)\left(\log\lambda(\tilde{\tau}) + \log\left(1+\mu(\tilde{\tau})^{-\frac{3}{2}}\right)\right) \\
&\quad - \frac{\lambda(\tilde{\tau})^2}{4}\left(1+2\mu(\tilde{\tau})^{-\frac{3}{2}}\right) = \frac{1}{4} + \frac{\lambda(\tilde{\tau})^2}{2}\log\lambda(\tilde{\tau}) \\
&\quad + \mu\lambda(\tilde{\tau})^2(\tilde{\tau})^{-\frac{3}{2}}\log\lambda(\tilde{\tau}) - \frac{\lambda(\tilde{\tau})^2}{4}
\end{aligned} \tag{6.20}
$$

whereas by (6.18),

$$
W(\tau) < \nu_{\lambda(\tilde{\tau})}(1) - \frac{K}{2\tilde{\tau}}\left(\frac{\varepsilon(\tilde{\tau})}{\sigma(\tilde{\tau})}\right)^2 = \frac{1}{4} + \frac{\lambda(\tilde{\tau})^2}{2}\log\lambda(\tilde{\tau}) - \frac{\lambda(\tilde{\tau})^2}{4} - \frac{K}{2\tilde{\tau}}\lambda(\tilde{\tau})^2. \tag{6.21}
$$

From (6.20) and (6.21), it follows that (6.19) holds provided that

$$
\frac{K}{2\tilde{\tau}} > \frac{\mu}{\tilde{\tau}^{\frac{3}{2}}}|\log\lambda(\tilde{\tau})|,
$$

and this last inequality is satisfied by selecting $\mu > 0$ small enough since

$$
\tau^{-\frac{3}{2}}\log\lambda(\tau) \sim (1-\theta)\tau^{-1} \quad \text{as } \tau \to \infty.
$$

We now set

$$
z(\xi,\tau) = (w(\xi,\tau) - \tilde{w}_{\lambda_0}(\xi))_+, \quad \text{where } s_+ = \max\{s,0\}.
$$

Since $(H(s)-H(t))(s-t)^{-1} \geq 0$ whenever $s \neq t$, we can readily check that z satisfies

$$
z_\tau \leq (\sigma(\tilde{\tau}))^{-2}\Delta z + \left(z - \frac{\xi\nabla z}{2}\right) \quad \text{for } \tau > \tilde{\tau} - \frac{1}{\tilde{\tau}}, \quad 0 < \xi < 1, \tag{6.22a}
$$

whereas by (6.19),

$$
z = 0 \quad \text{when } \xi = 0, 1 \text{ and } \tau > \tilde{\tau} - \frac{1}{\tilde{\tau}} \tag{6.22b}
$$

and

$$
z = O\left(\frac{1}{\sigma(\tilde{\tau})^2}\right) \quad \text{at } \tau = \tilde{\tau} - \frac{1}{\tilde{\tau}}. \tag{6.22c}
$$

By classic parabolic theory, it follows from (6.22) that

$$
z(\xi,\tau) \leq A(\sigma(\tilde{\tau}))^{-2}\exp\left(-\frac{A(\tau-\tilde{\tau})}{(\sigma(\tilde{\tau})^2}\right) \quad \text{in } Q,
$$

where

$$Q = \left\{ (\xi, \tau) : |\xi| \le 1, \quad \tau \in \left(\bar{\tau} - \frac{1}{\tilde{\tau}}, \tilde{\tau} + \frac{1}{\tilde{\tau}} \right) \right\}.$$

In particular, $w(\xi,\tau) \sim \tilde{w}_{\lambda_0}(\xi)$ at $\tau = \tilde{\tau}$. Recalling (6.6) (with σ, λ replaced by $\sigma(\tilde{\tau})$, $\lambda_0(\tilde{\tau})$, respectively), we see that the (rescaled) free boundary of $w(\xi, \tilde{\tau})$ is very close to $\lambda_0(\tilde{\tau})$ and in particular is larger than $\lambda(\tilde{\tau})$, which is a contradiction. The case where (6.18b) is replaced by $|W(\tau) - \nu_{\lambda(\tilde{\tau})}(1)| = W(\tau) - \nu_{\lambda(\tilde{\tau})}(1)$ is similar and will be omitted. ◻

We now point out the following consequence of Lemmas 6.2 and 6.3.

Corollary 6.4. *There holds*

$$|\varepsilon(\tau) - \varepsilon(\tilde{\tau})| \le C\tau^{-\frac{3}{2}}\varepsilon(\tilde{\tau}) \quad \textit{for some } C > 0 \tag{6.23}$$

whenever $|\tau - \tilde{\tau}| < \frac{1}{\tau}$ *and* $\tilde{\tau} \ge \tau_0 \gg 1$.

Proof. From (6.12) and (6.16) we readily see that

$$|\nu_{\lambda(\tau)}(1) - \nu_{\lambda(\tilde{\tau})}(1)| \le \frac{2C}{\tau}(\lambda(\tilde{\tau}))^2 \quad \text{for } |\tau - \tilde{\tau}| < \frac{1}{\tau}$$

and the result follows at once in view of the explicit formula (6.5). ◻

We shall conclude the proof of (1.2) by means of a careful analysis of (6.16), which can be thought of as an integral equation for the unknown rescaled free boundary $\varepsilon(\tau)$. Assume now that (4.2) holds. Then in view of (5.3), (5.30), and (6.16), we have that

$$\left| a_0(\tau)\varphi_0 + a_1(\tau)\varphi_1(0) + Q(y,\tau) - \frac{\varepsilon^2(\tau)}{2}\log\left(\frac{\varepsilon(\tau)}{\sigma(\tilde{\tau})}\right) + \frac{\varepsilon(\tau)^2}{4} \right| = O\left(\frac{\varepsilon(\tau)^2}{\tau}\right). \tag{6.24}$$

Note that the error involved in replacing $\varphi_1(y)$ by $\varphi_1(0)$ is already accounted for in the right-hand side of (6.24). We now take advantage of (5.12) to estimate $Q(y,\tau)$ in (6.24). Recalling (6.23), we have that for $(z(\tau))^0 \le y \le 1$ and Σ as in (5.12),

$$\begin{aligned} &\left| \int_\Sigma^\infty \varepsilon(s)^2 e^{-\eta}\eta^{-1} d\eta - \varepsilon(\tau)^2 \int_\Sigma^\infty e^{-\eta}\eta^{-1} d\eta \right| \\ &\quad \le\ C\bar{\varepsilon}(\tau)^2 \tau^{-\frac{1}{2}} \int_\Sigma^\infty e^{-\eta}\eta^{-1} d\eta \\ &\quad \le\ C\bar{\varepsilon}(\tau)^2\tau^{-\frac{1}{2}}(1 - |\log\Sigma|)e^{-\Sigma} \le C\bar{\varepsilon}(\tau)^2\tau^{-\frac{1}{2}}. \end{aligned} \tag{6.25a}$$

Notice that (6.23) provides a factor $\tau^{-\frac{3}{2}}$ in the right-hand side of the first inequality above in sets where $|s - \tau| < \frac{1}{\tau}$. Extending such a bound to the interval $|s - \tau| \le 1$ required by our choice of Σ yields then the final factor $\tau^{-\frac{1}{2}}$. A similar argument gives

$$\left| \int_\Sigma^\infty \varepsilon(s)^2\eta^2 e^{-\eta}(4\eta^2)^{-1} d\eta - \varepsilon(\tau)^2 \int_\Sigma^\infty \eta^2 e^{-\eta}(4\eta^2)^{-1} d\eta \right| \le C\bar{\varepsilon}(\tau)^2\tau^{-\frac{1}{2}} \tag{6.25b}$$

and

$$\begin{aligned} &\left| \int_{\tau-1}^\tau (A_1 + A_2(1 - e^{-(\tau-s)}))e^{\tau-s}\varepsilon(s)^2 ds \right. \\ &\quad \left. - (\varepsilon(\tau))^2 \int_{\tau-1}^\tau (A_1 + A_2(1 - e^{-(\tau-1)}))e^{\tau-s} ds \right| \le C\bar{\varepsilon}(\tau)^2\tau^{-\frac{1}{2}}. \end{aligned} \tag{6.25c}$$

Recalling (5.12), we have obtained the following estimate:

$$\left|Q(y,\tau)-\gamma-\varepsilon(\tau_0)^2S(\tau-\tau_0)F(y)-\gamma\varepsilon(\tau_0)^2\int_{\tau_0}^{\tau}S(\tau-s)\left(\delta(y)-\sum_{k=0}^{1}\langle\varphi_k,\delta(y)\rangle\varphi_k\right)ds\right| \le C(\bar{\varepsilon}(\tau))^2\tau^{-\frac{1}{2}}. \tag{6.26}$$

Keeping in mind the definition of $F(y)$ (cf. (3.8)), we see that

$$S(\tau-\tau_0)F(y)+\int_{\tau_0}^{\tau}S(\tau-s)\left(\delta(y)-\sum_{k=0}^{1}\langle\varphi_k,\delta(y)\rangle\varphi_k\right)ds=F(y).$$

It then turns out that (6.26) can be recast in the form

$$\begin{aligned}&|Q(y,\tau)-\gamma\varepsilon(\tau)^2F(y)-\gamma(\varepsilon(\tilde{\tau})^2-\varepsilon(\tau)^2)S(\tau-\tau_0)F(y)|\\ &\qquad\le C\bar{\varepsilon}(\tau)^2\tau^{-\frac{1}{2}}\quad\text{for } (\bar{\varepsilon}(\tau))^\theta\le y\le 1.\end{aligned}$$

Since the term $S(\tau-\tau_0)F(y)$ decays exponentially on sets where $|y|$ is bounded, we may take advantage again of (6.23) to obtain that

$$|Q(y,\tau)-\gamma\varepsilon(\tau)^2F(y)|\le C\bar{\varepsilon}(\tau)^2\tau^{-\frac{1}{2}}\quad\text{whenever } (\bar{\varepsilon}(\tau))^\theta\le y\le 1. \tag{6.27}$$

Using now the explicit representation for $F(y)$ (cf. (3.9)), we deduce from (6.24) and (6.27) that

$$\left|a_0(\tau)\varphi_0+a_1(\tau)\varphi_1(0)+\left(B\gamma+\frac{1}{4}\right)\varepsilon(\tau)^2-\frac{\varepsilon(\tau)^2}{2}\log\varepsilon(\tau)\right|\le C\bar{\varepsilon}(\tau)^2\tau^{-\frac{1}{2}}. \tag{6.28}$$

Let us now define $\varepsilon^*(\tau)$ as follows

$$\varepsilon^*(\tau)=\begin{cases}\varepsilon(\tau) & \text{if } \tau_0\le\tau\le\tau_1,\\ \bar{\varepsilon}(\tau) & \text{if } \tau\ge\tau_1.\end{cases}$$

We next observe that if (4.11) is satisfied there holds

$$a_k(\tau)=-\int_\tau^\infty e^{\lambda_k(\tau-s)}\langle\chi_{\varepsilon^*(\tau)},\varphi_k\rangle ds.$$

Substituting this into (6.28), we finally arrive at

$$\begin{aligned}&\left|\left(B\gamma+\frac{1}{4}\right)\varepsilon^*(\tau)^2-\frac{\varepsilon^*(\tau)^2}{2}\log\varepsilon^*(\tau)\right.\\ &\left.\quad-\sum_{k=0}^{1}\varphi_k(0)\int_\tau^\infty e^{(1-k)(\tau-s)}\langle\chi_{\varepsilon^*(s)},\varphi_k\rangle ds\right|\\ &\quad= 0(\varepsilon^*(\tau)^2\tau^{-1/2})\quad\text{for } \tau\gg 1.\end{aligned} \tag{6.29}$$

This is essentially the integral equation that has been studied in detail in section 3 (cf. the argument following (3.17)). In view of our previous analysis in section 3, we may summarize our discussion in the following.

LEMMA 6.5. *Assume that* (4.2) *and* (4.11) *hold. We then have that*

$$\varepsilon(\tau) = \bar{\varepsilon}(\tau)(1 + o(1)) \quad \textit{for } \tau \gg 1 \tag{6.30}$$

uniformly on $\tau_0 \leq \tau \leq \tau_1$.

End of the proof for Proposition 4.1. Having obtained (6.30) under assumption (4.11), it merely remains to show that condition $|\psi(y,\tau)| < M(1+y^2)$ in (4.8) holds for some constant M which is independent of the size of the interval $[\tau_0, \tau_1]$. To check this point, we argue as follows. We have just seen that

$$\psi(y,\tau) \sim a_0(\tau)\varphi_0 + a_1(\tau)\varphi_1(y) + O(\varepsilon(\tau)^2) \quad \text{for } \tau \gg 1 \text{ and } y = O(1). \tag{6.31}$$

We now claim that we may formally differentiate twice with respect to y in both sides of (6.31) and the corresponding expansion still holds. To wit, we set $z(y,\tau) = \psi(y,\tau) - a_1(\tau)\varphi_1(y)$ and remark that in regions where $y = O(1)$, one has that $z = O(\varepsilon^2(\tau))$ and satisfies

$$Lz = 0(\varepsilon^2(\tau)\varphi_1(y)),$$

where L denotes the parabolic operator in (5.1). It then follows that for $\tau \geq \tau_0 \gg 1$

$$z(y,\tau) = \exp\left(\int_{\tau_0}^{\tau} D(s)ds\right)\varphi_1(y) \quad \text{with } D(s) = O(\varepsilon(s)^2)$$

whereupon the desired bound for ψ follows. ∎

Proof of (1.6) *in Theorem* 1.3. It has been shown above that

$$\psi_{yy}(y,\tau) \sim Ca_1(\tau) + O(\varepsilon^2(\tau)) \quad \text{with } C = \frac{d^2}{dy^2}(\varphi_1(y)) \tag{6.32}$$

for, say, $y = O(1)$ and $\tau \gg 1$. Since $\theta(r,t) = \psi_{yy} + \frac{\psi_y}{y} = Ca_1(\tau)$ for some $C_1 > 0$, it follows that, setting $r = Ae^{-\tau/2}$ with $A > 0$,

$$\theta(r,T) \sim C_1 a_1 \left(-2\log\frac{r}{A}\right)(1 + o(1)) \quad \text{as } \tau \to 0. \tag{6.33}$$

Since

$$a_1(\tau) \sim \frac{\varepsilon^2(\tau)\log\varepsilon(\tau)}{2} \quad \text{as } \tau \to \infty,$$

the result follows at once from (6.33) and (3.18). ∎

$$f(x,T) \sim Ca_1 \left(-2\log\frac{x}{A}\right)(1 + o(1)) \quad \text{as } x \to 0. \tag{6.34}$$

Since $a_1(\tau) \sim \frac{1}{2}\varepsilon^2(\tau)\log\varepsilon(\tau)$ as $\tau \to \infty$, the result follows at once from (6.34) and (3.18). ∎

7. The remaining cases. In this final section we shall merely sketch those modifications of the arguments developed in sections 4–6 which are required to obtain (1.3)–(1.5) and (1.7)–(1.9), thus concluding the proofs of Theorems 1.1–1.3.

7.1. Obtaining (1.3) in Theorem 1.1. In the topological argument described in section 4, we must replace $\bar{\varepsilon}(\tau)$ in (4.1) by

$$\bar{\varepsilon}(\tau) = Ce^{(1-\frac{l}{2})\tau}\tau^{-1/l-1},$$

where C is an arbitrary constant and l is an integer such that $l \geq 2$. We then substitute (4.2a) by

$$\sup\left\{|\varepsilon(\tau) - \bar{\varepsilon}(s)|, \text{where } \tau, s \in [\tau_0, \tau_1] \text{ and } |\tau - s| < \frac{1}{\tau}\right\} < M\bar{\varepsilon}(\tau)\tau^{-1},$$

which can be rephrased in an informal way as requiring that $|\frac{d}{d\tau}\bar{\varepsilon}(\tau)| < M\bar{\varepsilon}(\tau)$. Condition (4.2b) is then kept as before. As for (4.11), it is to be replaced by

$$a_k(\tau) = -\int_\tau^\infty e^{(1-\frac{k}{2})(\tau-s)}\langle \chi_{\bar{\varepsilon}(s)}, \varphi_k\rangle ds$$

for $k = 0, 1, 2, \ldots, l$.

With these modifications in mind, the analogue of Proposition 4.1 is readily stated. To analyze the outer region in this case, one writes $\psi(y,\tau)$ in the form

$$\psi(y,\tau) = \sum_{k=0}^{l-1} a_k(\tau)\varphi_k(y) + a_l(\tau)\varphi_l(y) + E(y,\tau). \tag{7.1}$$

The term $E(y,\tau)$ in (7.1) will be approximated as before by $Q(y,\tau)$, where Q satisfies now (3.25b) instead of (5.5). The solution of such an equation for $\tau > \tau_0$ such that $Q(y,\tau_0) = 0$ is given by

$$Q_0(y,\tau) = \gamma \int_{\tau_0}^{\tau} K_l(y, \tau - s)e^{\tau-s}\varepsilon(s)^2 ds, \tag{7.2a}$$

where

$$K_l(y,\tau) = (4\pi(1-e^{-\tau}))^{-1}\left(\exp\left(-\frac{y^2e^{-\tau}}{4(1-e^{-\tau})}\right) - \sum_{k=0}^{l}\varphi_k\left\langle\varphi_k, \exp\left(-\frac{y^2e^{-\tau}}{4(1-e^{-\tau})}\right)\right\rangle\right). \tag{7.2b}$$

Notice that $|K_l(y,\tau)| \leq Ce^{-l\tau}$ uniformly on sets $|y| \leq R < \infty$ when $\tau \geq 1$. Arguing as for Lemma 5.2, we then obtain the corresponding version of that result in our case. This last is obtained by making a few modifications in (5.12):

(a) Replace $S(\tau)$ by $S_l(\tau)$ there, where $S_l(\tau)$ is the semigroup associated to operator A_l in (3.27c).

(b) Replace $F(y)$ given in (3.8) by the corresponding solution of (3.27).

(c) Substitute $K(y,\tau)$ by $K_l(y,\tau)$ given in (7.2b).

(d) Replace the factor $(A_1 + A_2(1 - e^{-(\tau-1)}))$ in the integral where it appears in (5.12) by $(A_1 + A_2(1 - e^{-(\tau-1)}) + \cdots + A_l(1 - e^{-(\tau-s)})$.

The analysis of the inner region is then performed as in section 6. The integral equation (6.16) is now to be replaced by

$$|W(\tau) - \nu_{\lambda(\tau)}(1)| \leq C\varepsilon(\tau)^2,$$

provided that $|\tau - \bar{\tau}| \leq \frac{1}{\tau}, \bar{\tau} \gg 1$.

Arguing as in section 6, we then arrive at an integral equation which is similar to (3.30). We shall omit further details.

7.2. The case where $N \geq 3$: End of the proof of Theorem 1.2. We first describe the main steps toward obtaining (1.4). To begin with, we replace $\bar{\varepsilon}(\tau)$ in (4.1) by

$$\bar{\varepsilon}(\tau) = C\tau^{-\frac{1}{N-2}}, \quad C > 0,$$

and replace (4.2a) by

$$|\varepsilon(\tau) - \bar{\varepsilon}(s)| < M\bar{\varepsilon}(s)\tau^{-(1-\theta)} \quad \text{for some } 0 < \theta \ll 1$$

whenever $|\tau - s| \leq \tau^{\theta}$.

The main novelty now with respect to the previous cases is that we may estimate directly the error term $E(y,\tau)$ in (5.3) to obtain

$$|E(y,\tau)| \leq C\bar{\varepsilon}(\tau)^N(1 + y^{-(N-2)}) \quad \text{for } \tau \gg 1$$

uniformly on sets $y \leq R < \infty$. We are thus led to a version of the integral equation (6.16) which reads now as follows:

$$|W(\tau) - \nu_{\lambda(\tau)}(1)| \leq C\bar{\varepsilon}(\tau)^2\tau^{-(1-\theta)} \quad \text{for } |\tau - \bar{\tau}| \leq \tau^{\theta}, \bar{\tau} \gg 1,$$

whence the statement in Lemma 6.5 follows in this case.

Finally, (1.5) corresponds to the situation where $\chi_{\varepsilon(\tau)}$ can be asymptotically neglected in (5.1), so that the analysis sketched at the end of section 3 can be carried out in a straightforward way.

7.3. End of the proof of Theorem 1.3. To obtain (1.7), we can argue as in the last part of section 6 to obtain that

$$\psi(y,\tau) \sim a_l(\tau)\varphi_l(y) \quad \text{for } y = O(1) \text{ and } \tau \gg 1,$$

where the expansion above also holds when differentiated twice with respect to y. In view of (6.33), one then has that at points $x = Ae^{-\tau/2}$,

$$\theta(x,T) \sim a_l\left(-2\log\frac{x}{A}\right)A^{2l-2} = a_l\left(-2\log\frac{x}{A}\right)|x|^{2l-2}e^{(l-1)\tau},$$

whence (1.7) follows in view of (3.30) and (3.31).

As to (1.8), we make use of (3.33) and (3.34) to observe that

$$\psi_{yy}(y,\tau) \sim C\tau^{-\frac{2}{N-2}} \quad \text{for } y = O(1) \text{ and } \tau \gg 1.$$

This readily gives that for $x = Ae^{-\tau/2}$

$$\theta(x,T) \sim C(|\log|x||)^{-\frac{2}{N-2}} \quad \text{as } x \downarrow 0.$$

Finally, to obtain (1.9) we merely recall that, in view of (3.35), one has that for $x = Ae^{-\tau/2}$

$$\psi_{yy}(A,\tau) \sim e^{(1-l)\tau}A^{2l-2},$$

whereupon $\theta(x,T)$ is shown to be such that

$$\theta(x,T) \sim C|x|^{2l-2} \quad \text{as } x \downarrow 0.$$

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