Appendix A

Preliminaries

Some functional analytic material used in the chapters but not defined or explained there is being presented in Appendices A–D. This material is intended as an *aide-memoire*. For supplementary reading the reader is referred to Refs. [9, 21, 58, 282, 309, 353–356].

We start with the definition of a vector space as the concept of a vector space, also called a linear space, is fundamental to the functional analytic results which hold in vector spaces of some special type such as Hilbert spaces, Banach spaces and Sobolev spaces.

1. Abstract space. An *abstract space*, also called simply a 'space' is a set of (unspecified) elements satisfying certain axioms. By choosing different sets of axioms, a variety of abstract spaces can be defined.

2. Linear space. Let F be a field, generally taken to be R (real line) or C (complex plane). The elements of F are called 'scalars'. Let W be a nonempty set whose elements are called 'vectors'. If x, y belong to W, define a mapping from $W \times W$ into W as $(x, y) \to x + y$, where $x + y$ is a unique element of W. We call this mapping 'addition' of vectors which is a 'binary operation' on W. Let the addition of vectors satisfy the following axioms:

(i) $x + y = y + x$, for all $x, y \in W$. This axiom is called 'commutative law'.

(ii) $(x + y) + z = x + (y + z)$, for all $x, y, z \in W$. This axiom is called 'associative law'.

(iii) For each $x \in W$, there exists a unique element 0 in W such that $x + 0 = 0 + x = x$. This implies the existence of a 0 (zero) element in W.

(iv) For each $x \in W$, there exists a unique element $-x \in W$ such that $x+(-x) = 0 = (-x)+x$. This implies the existence of an additive 'inverse' for each x.

Scalar multiplication axioms. To every scalar $\alpha \in F$ and a vector $x \in W$, there corresponds a unique vector $\alpha\, x \in W$ such that

(v) $\alpha(\beta x) = (\alpha\beta)x$, for every $\beta \in F$,

(vi) $1x = x$ and $0x = 0$, for all $x \in W$,

(vii) $\alpha(x + y) = \alpha x + \beta y$ (distributive law),

(viii) $(\alpha + \beta)x = \alpha x + \beta x$ (distributive law),

The element '1' belonging to F is called the 'multiplicative identity'.

A set W with the operations of addition of vectors and the scalar multiplication of a scalar and a vector defined on it and satisfying axioms (i)–(vii) is called a *vector space* or a *linear space* over the field F.

The concept of a *metric* or 'distance' in the vector space W can be introduced through the notion of norm which is a real-valued function defined on W.

3. Norm. A norm (a length function) on a vector space W is a real-valued function defined for all $x \in W$, denoted by $\|x\|$. It is called *norm* of x if it satisfies the following axioms:

(i) $\|x\| \geq 0$; $\|x\| = 0$, if and only if $x = 0$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in W$,

(iii) $\|\alpha x\| = |\alpha|\,\|x\|$, $\alpha \in F$.

If the property, $\|x\| = 0$ implies $x = 0$ is not true, and all other properties described above hold, then we call the function $\|\cdot\|$, a *seminorm* on W.

A vector space with a norm defined on it is called *a normed space* and is denoted as $(W, \|\cdot\|)$. The notation $\|\cdot\|_W$ means the norm is defined on W. A vector space can be equipped with different norms. The *distance* $d(x, y)$ between $x, y \in W$ or a distance metric on W may be defined as

$$d(x, y) = \|x - y\|\,.$$

4. Completeness of a vector space. Let $\{x_n\}$ be a sequence of vectors in W. $\{x_n\}$ is said to be a *Cauchy sequence* if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that

$$d(x_n, x_m) < \varepsilon \text{ for every } m, n > N.$$

A linear space W is said to be a *complete linear space* if every Cauchy sequence in W converges to a vector in W or in other words it has a limit which is an element of W. A normed space which is complete in the distance metric defined by the norm is called a *Banach space*.

The concept of the length of a vector has been generalized in the definition of the norm of a function but what is missing is the analogue of the definition of familiar 'dot product' of two vectors in vector calculus. This is taken care by the definition of an inner product.

5. Inner product or scalar product. An *inner product* on W is a mapping from $W \times W$ into the scalar field F of the linear space W which associates with any two elements $x, y \in W$ a scalar which we denote by (x, y) such that

(i) $(x + y, z) = (x, z) + (y, z)$, for all $x, y, z \in W$,

(ii) $(\alpha x, y) = \alpha(x, y)$, and $(x, \alpha y) = \overline{\alpha}(x, y)$, $\alpha \in F$,

(iii) $(x, y) = \overline{(y, x)}$ (bar denotes the complex conjugate),

(iv) $(x, x) \geq 0$; $(x, x) = 0$ if and only if $x = 0$.

An inner product defines a norm as well as a metric on W which are given by

$$\|x\| = \sqrt{(x, x)}; \quad \text{and} \quad d(x, y) = \sqrt{(x - y, x - y)} = \|x - y\|\,.$$

An inner product space which is complete in the norm defined by the inner product is called a *Hilbert space*. All Hilbert spaces are Banach spaces.

If can be easily proved that: (i) $(x, y) \leq \|x\| \, \|y\|$, (ii) $| \, \|x\| - \|y\| \, | \leq \|y - x\|$. The following results hold good for normed spaces.

(1) On a finite-dimensional vector space W, any norm $\|\cdot\|$ is equivalent to any other norm $|||\cdot|||$. Two norms $\|\cdot\|$ and $|||\cdot|||$ are said to be equivalent if there exist positive numbers a and b such that for all $x \in W$, we have

$$a \, ||| x ||| \leq \|x\| \leq b \, ||| x ||| \, .$$

(2) Every finite-dimensional normed space is complete.

6. Compactness. A subset V in a normed space W is *compact* if every sequence in V has a convergent subsequence with a limit point in V.

The same definition holds for the normed space W. In a finite-dimensional normed space W, any subset V of W is compact if and only if V is closed and bounded. A compact subset V of W is closed and bounded but the converse of this statement is in general false for infinite-dimensional normed spaces. V is said to be *weakly compact* if every sequence of its elements contains a subsequence which converges weakly to an element of V (see **26** of this appendix for the definition of weak convergence).

7. Compact support. A function $f : W \to F$ (field F) has a *compact support* in W if it is zero outside a compact subset of W. The closure of the set $\{x \in W : F(x) \neq 0\}$ is called the support of F.

8. Denseness. A subspace V of a normed space W is *dense* or *everywhere dense* in W if its closure with respect to the norm is equal to W or contains W as a subset. W is said to be *separable* if it has a countable subset which is dense in W.

9. Linear operator. In the case of normed spaces, a mapping is called an 'operator'. A *linear operator* $P : D(P) \subset W_1 \to W_2$ is an operator such that: (i) the domain $D(P)$ of P is a normed vector space and the range $R(P)$ lies in a normed vector space over the same field F over which $D(P)$ is a vector space, (ii) for all $x, y \in D(P)$ and scalars $\alpha \in F$,

$$P(x + y) = Px + Py; \; P(\alpha x) = \alpha Px.$$

The *null space* or *kernel* of P is the set of all $x \in D(P)$ such that $Px = 0$. If P is one-to-one, then a mapping P^{-1}, called an *inverse mapping* of P, can be defined as $P^{-1} : R(P) \to D(P)$. If $Px_0 = y_0$, then $P^{-1}y_0 = x_0$. If P^{-1} exists, then it is linear. P^{-1} exists if and only if the kernel of P consists of only the zero element.

10. Bounded linear operator. Let W_1 and W_2 be two normed spaces and $P : D(P) \subset W_1 \to W_2$ be a linear operator. The operator P is said to be a *bounded linear operator* if there is a real number $\delta > 0$ such that for $x \in D(P)$,

$$\|Px\| \leq \delta \, \|x\| \, .$$

The norm on the left is on W_2 and the norm on the right is on W_1. A bounded linear operator P maps bounded sets in W_1 onto bounded sets in W_2. If W is a finite-dimensional normed space, then a linear operator on W is bounded.

11. Norm of a bounded linear operator. The norm $\|P\|$ of a bounded linear operator is defined as

$$\|P\| = \sup_{\substack{x\in D(P)\\ \|x\|=1}} \|Px\| .$$

By taking $\delta = \|P\|$, we have $\|Px\| \le \|P\|\,\|x\|$.

The vector space $\mathcal{L}(X,Y)$ of all bounded linear operators from a normed space X into a normed space Y is itself a normed space under the norm defined as

$$\|Q\| = \sup_{\substack{x\in X\\ \|x\|=1}} \|Qx\| , \quad Q \in L(X,Y) .$$

If Y is a Banach space, then $\mathcal{L}(X,Y)$ is a Banach space.

12. Continuity of an operator. Let W_1 and W_2 be two normed spaces and $P : D(P) \subset W_1 \to W_2$ be an operator not necessarily linear. The operator P is said to be continuous at a point $x_0 \in D(P)$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|Px - Px_0\| < \varepsilon$ for all $x \in D(P)$ satisfying $\|x - x_0\| < \delta$. P is continuous if P *is continuous operator* at every $x \in D(P)$.

The mapping $x \to \|x\|$ is continuous, i.e. the mapping $(W, \|\cdot\|_W) \to R$ is continuous. If P is linear, then it has some interesting properties.

(i) P is continuous if and only if P is bounded. If P is bounded, then $x_n \to x$ implies $Px_n \to Px$.

(ii) If P is continuous at a single point, then it is continuous.

13. Compact and completely continuous operators. Let W_1, W_2 be normed spaces and P an operator from W_1 into W_2. The operator P is called compact if $P(A)$ is *precompact* in W_2 whenever the set A is bounded in W_1. The term relatively compact is also used for precompact. A is precompact in W_1 if $\overline{A}$ is compact in W_1.

If P is continuous and compact, then it is called a *completely continuous operator*.

Every compact operator is bounded. A linear operator $P : W_1 \to W_2$ has the property that if W_1 is finite-dimensional, P is compact.

Every linear compact operator is continuous.

Let P be a linear operator from W_1 to W_2. Then P is compact if and only if it maps every bounded sequence $\{x_n\}$ in W_1 onto a sequence $\{Px_n\}$ in W_2 which has a convergent subsequence in W_2. If a sequence $\{T_n\}$ of compact linear operators from $W_1 \to W_2$ is uniformly operator convergent, i.e. $\|T_n - T\| \to 0$, then the limit operator T is compact provided W_2 is a Banach space.

14. Restriction (extension) of an operator. Let $P : D(P) \subset X \to Y$ and $A \subset D(P)$. Let $P|_A$ be a mapping from $A \to Y$ defined by

$$P|_A : A \to Y, \;\; P|_A x = Px \text{ for all } x \in A.$$

The operator $P|_A$ is called the *restriction of an operator P*. Let $M \supset D(P)$. An operator $\hat{P}$ is called an extension of P if $\hat{P} : M \to Y$ is such that $\hat{P}|_{D(P)} = P$, i.e. $\hat{P}x = Px$ for all $x \in D(P)$.

15. Orthogonal complement. In an *inner product space* (a space with a inner product defined on it) Z, two vectors x and y are said to be orthogonal, written as $x \perp y$, if $(x, y) = 0$.

The vector x is said to be orthogonal to a set $Y \subset Z$, written as $x \perp Y$, if x is orthogonal to every $y \in Y$. The subsets X and Y of Z are said to be orthogonal, written as $X \perp Y$, if $(x, y) = 0$ for all $x \in X$ and $y \in Y$. The set $Y^{\perp} = \{z \in Z : z \perp Y\}$ is called the *orthogonal complement* of Y.

16. Direct sum and projection operator. A vector space W is said to be the *direct sum* of two subspaces X and Y of W, written as $W = X \oplus Y$, if each $w \in W$ has a unique representation

$$w = x + y, \quad x \in X, \ y \in Y.$$

If Y is any closed subspace of a Hilbert space H, then $H = Y \oplus Z$, $Z = Y^{\perp}$ and for every $x \in H$ there exist a $y \in Y$ such that

$$x = y + z, \quad z \in Y^{\perp} \ \text{(direct sum)}.$$

The above equation defines a mapping $P : H \to Y$, $Px = y$. P is called *orthogonal projection* or *projection operator* of H onto Y. It can be proved that P is a bounded linear operator and P is *idempotent*, i.e. $P^2 = P$.

17. Functional. A *functional* is defined to be an operator whose range lies on the real line R or in the complex plane $\mathbb{C}$ or in other words a functional is a real- or complex-valued function. Some authors use the term functional for a continuous linear real-valued operator. In this volume, we have used the term functional as a real-valued function and whenever the term functional is used as a continuous linear real-valued function, it has been explicitly indicated.

18. Dual space. Let W be a normed space. Then the set of all functionals f (bounded linear real-valued functions on W) on W constitute a normed space with the norm defined by

$$\|f\| = \sup_{\substack{x \in W \\ \|x\|=1}} |f(x)|$$

This space of functionals is called the *dual space* of W and is denoted by W'.

The dual space W' of a normed space W is a Banach space (whether or not W is). We can consider $(W')'$, i.e. the dual space of the dual space W' denoted by W''. For each $x \in W$, we define a mapping g_x which is such that if $f \in W'$, then $g_x(f) = f(x)$, $g_x \in W''$ and to each $x \in W$ there exists a $g_x \in W''$. Thus we have defined a mapping $G : W \to W''$, $x \mapsto g_x$. G is called the *canonical mapping* or the canonical embedding of W into W''. G is linear.

19. Isomorphism. Let W_1, W_2 be two given vector spaces over the same field. We would like to know whether W_1 and W_2 are essentially identical, i.e. do they have the same structure in an abstract sense.

An *isomorphism* of a Hilbert space W_1 onto a Hilbert space W_2 over the same field is a *bijective* (one-to-one and onto) linear operator $P : W_1 \to W_2$ such that for $x, y \in W_1$,

$$(Px, Py) = (x, y).$$

W_1 and W_2 are called ‘isomorphic Hilbert spaces’. Isomorphisms in normed spaces preserve norms. If W_1 and W_2 are two vector spaces (not necessarily normed spaces), then P should preserve the two algebraic operations of a vector space.

20. Reflexivity. It can be proved that the canonical mapping $G : X \to X''$, $x \mapsto g_x$ is linear and one-to-one. G need not be onto. A normed space X is said to be *reflexive* if the canonical mapping G is onto. The following results hold.

(i) If a normed space X is reflexive, it is complete and hence a Banach space.

(ii) Every finite-dimensional normed space is reflexive.

(iii) Every Hilbert space is reflexive.

21. Riesz representation theorem for functionals. For any bounded linear functional f on a Hilbert space W there exists a unique vector $y \in W$ such that

$$f(x) = (x, y) \;\text{ for all } x \in W.$$

Here, y depends on f and is uniquely determined by f and has the norm

$$\|y\| = \|f\|.$$

22. Bilinear form. A mapping $a(u, v) : U \times V \to R$, where U and V are vector spaces over the field R of real numbers is called a *bilinear form* if the mapping $a(u, v)$ is linear in both the arguments, i.e.

$$a(\alpha_1 u_1 + \alpha_2 u_2, \beta_1 v_1 + \beta_2 v_2) = \sum_{i,J=1}^{2} \alpha_i \beta_J a(u_i, v_J),\; \alpha_i, \beta_J \in R,\; u_i \in U, v_J \in V;$$
$$i = 1, 2,\; J = 1, 2.$$

We can also consider a bilinear form from $U \times U \to R$. If U and V are normed spaces and there exists a real number δ such that

$$|a(u, v)| \le \delta \, \|u\|_U \, \|v\|_V, \quad u \in U,\; v \in V,$$

then $a(u, v)$ is said to be bounded. The norm $\|a\|$ of a bounded bilinear form is defined as

$$\|a\| = \sup_{\substack{\|u\|=1 \\ \|v\|=1}} |a(u, v)|.$$

It is easy to prove that

$$|a(u, v)| \le \|a\| \, \|u\|_U \, \|v\|_V.$$

A bilinear form is said to be continuous if it is bounded or if there exists a $\beta \in R$, $\beta > 0$ such that $|a(u, v)| \le \beta \, \|u\|_U \, \|v\|_V$.

23. Coercivity. Let $a(u, v) : U \times U \to R$, and U be a normed space. $a(u, v)$ is said to be *coercive* on U if there exists an $\alpha \in R$, $\alpha > 0$ and $a(u, u) \ge \alpha \, \|u\|^2$, for all $u \in U$.

Let $a(u, v) : U \times U \to R$, where U is a Hilbert space. Define $a(u, v) = (u, v)$. Then $a(u, v)$ is symmetric, i.e. $a(u, v) = a(v, u)$, is continuous (take $\beta = 1$ in **22**) and has *coercive property* (take $\alpha = 1$) as $a(u,u) = (u,u) = \|u\|^2 \geq \|u\|^2$. Note that a bilinear form is not symmetric in general.

24. Adjoint operator. Let W_1 and W_2 be two Hilbert spaces and P be a bounded linear operator, $P : W_1 \to W_2$. Then the *Hilbert-adjoint operator* P^* of P is an operator from $W_2 \to W_1$ such that for all $x \in W_1$, and $y \in W_2$,

$$(Px, y) = (x, P^* y).$$

It can be proved that P^* exists, is unique, is a bounded linear operator, and $\|P^*\| = \|P\|$.

25. Strong convergence of a sequence. A sequence $\{x_n\}$ in a normed space X is said to be *strongly convergent* or convergent in the norm if there exists an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. The strong convergence is indicated as $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

26. Weak convergence of a sequence. A sequence $\{x_n\}$ in a normed space X is said to be *weakly convergent* if there exists an $x \in X$ such that for every $f \in X'$ (dual space), we have $\lim_{n\to\infty} f(x_n) = f(x)$. Weak convergence is indicated as $x_n \rightharpoonup x$ or $x_n \xrightarrow{w} x$. The element x is unique and is called the weak limit of $\{x_n\}$. Note that in the weak convergence we are dealing with a sequence of numbers.

The following results hold:

(i) If $\{x_n\}$ converges weakly to x, then every subsequence of $\{x_n\}$ converges weakly to x.

(ii) If $x_n \rightharpoonup x$, then the sequence $\{\|x_n\|\}$ is bounded.

(iii) Strong convergence implies weak convergence with the same limit. The converse is generally not true.

(iv) If X is a finite-dimensional normed space, then weak convergence implies strong convergence.

A sequence of bounded linear operators $\{P_n\}$, $P_n \in \mathcal{L}(X,Y)$ (X and Y are normed spaces) can be considered for defining strong and weak operator limits. If an operator $P \in \mathcal{L}(X,Y)$ exists such that if, (i) $\|P_n - P\| \to 0$, then P is called the *uniform operator limit* of $\{P_n\}$, (ii) $\|P_n x - Px\| \to 0$ for all $x \in X$, then P is called the *strong operator limit* of $\{P_n\}$ and (iii) $\|f(P_n x) - f(Px)\| \to 0$ for all $x \in X$ and $f \in Y'$ (dual space of Y), then P is called the *weak operator limit* of $\{P_n\}$.