Chapter 6

Steady-State and Degenerate Classical Stefan Problems

6.1 SOME STEADY-STATE STEFAN PROBLEMS

Steady-state free boundary problems occur more frequently in Stefan-like problems such as seepage through dam and free surface flows (cf. Eqs 1.1.5–1.1.10). Only a few studies on steady-state Stefan problems have been reported. These problems can be studied from the point of view of the conditions of their origin, and the existence, uniqueness and regularity of their solutions. In [162] some conditions which lead to steady-state solutions have been discussed.

Consider a bounded region $\Omega \subset R^n$, $n \geq 1$, with a sufficiently regular boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, $\Omega = \Omega_1 \cup \Omega_2 \cup S$ where Ω_1 is the solid region, Ω_2 is the liquid region and S is the sharp interface separating these regions. The portion Γ_1 of the boundary $\partial\Omega$ is kept at temperature $T = b > 0$, heat flux is prescribed on Γ_2 and the boundary Γ_3 is insulated. A volumetric heat sink $g(x)$, $x \in \Omega$, per unit volume is acting in Ω. A steady state will be reached if the outflow of heat through Γ_2 is large, and g is small. The temperature $T(x)$ in Ω is defined in the following way:

$$\left.\begin{aligned} T(x,t) &= T_1(x) < 0, \quad \text{if } x \in \Omega_1, \\ &= 0, \quad \text{if } x \in S, \\ &= T_2(x) > 0, \quad \text{if } x \in \Omega_2. \end{aligned}\right\} \tag{6.1.1}$$

The formulation of the steady-state problem consists of the following equations:

$$\nabla^2 T_i = -g, \quad \text{in } \Omega_i, \ i = 1,2; \ 0 \leq g < \infty, \tag{6.1.2}$$

$$T_1 = T_2 = 0, \quad K_1 \frac{\partial T_1}{\partial n} = K_2 \frac{\partial T_2}{\partial n}; \quad \text{on } x \in S, \tag{6.1.3}$$

$$T_2|_{\Gamma_1} = b > 0 \quad \text{and} \quad \left.\frac{\partial T}{\partial n}\right|_{\Gamma_3} = 0, \tag{6.1.4}$$

$$\begin{aligned} &-K_2 \left.\frac{\partial T_2}{\partial n}\right|_{\Gamma_2} = q, \quad \text{if } T_2 > 0; \ q > 0, \\ &-K_1 \left.\frac{\partial T_1}{\partial n}\right|_{\Gamma_2} = q, \quad \text{if } T_1 > 0, \end{aligned} \tag{6.1.5}$$

The Classical Stefan Problem. https://doi.org/10.1016/B978-0-444-63581-5.00006-3

where $\vec{n}$ is the unit normal vector on S pointing into the liquid.

If

$$\theta = K_2 T^+ - K_1 T^-, \tag{6.1.6}$$

where T^+ is the positive part of T and T^- is the negative part of T, then we have the following problem to be solved:

$$\nabla^2\theta = -g \quad \text{in} \;\; \Omega, \quad \theta|_{\Gamma_1} = B, \quad -\frac{\partial\theta}{\partial n}\bigg|_{\Gamma_2} = q, \quad \frac{\partial\theta}{\partial n}\bigg|_{\Gamma_3} = 0, \tag{6.1.7}$$

where $B = K_2 b > 0$.

The main result in [162] is concerned with the existence of a critical flux $q = q_c(B, g)$ such that:

(i) (q, g) with $q \leq q_c(B, g)$ implies $T > 0$ in Ω. This means that only one-phase will be present in the steady state,

(ii) (q, g) with $q > q_c(B, g)$ implies T has both negative and positive values in Ω. This in turn implies that both solid and liquid phases will be present in the steady state.

First, Eq. (6.1.7) is formulated as a variational problem. Then with the help of the properties of the solution of the variational problem, q_c has been obtained. Some steady-state problems have been considered whose explicit solutions verify the theoretical results. The problem considered in [162] is a generalization of the problem in [163] in which $g \equiv 0$. It has been proved in [163] that there exists a $q_1 > 0$ such that for all $q > q_1$, we have a steady-state two-phase problem in Ω, and

$$q_1 = \frac{K_2 b}{a_0} \text{meas} \;\; (\Gamma_2), \tag{6.1.8}$$

where a_0 is some constant having dimensions of (length)^n.

6.2 DEGENERATE STEFAN PROBLEMS

The term *degenerate Stefan problem* is used for a Stefan problem in which the mathematical nature of the differential equation or the boundary condition changes if some parameter associated with the problem varies within its admissible range of values. Some of the commonly studied degenerate Stefan problems are:

(i) *Quasi steady-state Stefan problems* or quasi-static Stefan problems (QSSPs) in which the heat equation is elliptic but the free boundary is time dependent. Such problems may arise if the specific heat $C \equiv 0$ or the temperature has attained a steady state. The Hele-Shaw problem (HSP) described in Eqs (3.3.45)–(3.3.46) is a quasi steady-state degenerate Stefan problem.

(ii) *Parabolic–elliptic Stefan problems* in which the specific heat $C = C(T)$ is such that

$$\begin{aligned} C(T) &=> 0, \quad \text{if} \;\; T > 0, \\ &= 0, \qquad \text{if} \;\; T = 0. \end{aligned} \tag{6.2.9}$$

Quasi Steady-State Stefan Problems

Using the theory of conformal mappings, a two-dimensional quasi-static moving boundary problem can be described by a nonlinear *Löwner–Kufarev equation* [164] and a functional relation $\mathcal{F}$ between the shape of the free boundary and its velocity can be established. Together with the initial data this leads to an initial-value problem. Assuming that $\mathcal{F}$ satisfies certain conditions, the existence of a local-in-time solution of this initial value problem has been proved in [165]. This method is mainly applicable to those free boundary problems in which the domain is shrinking. Continuity conditions on $\mathcal{F}$ are also not easy to satisfy. The proof is based on the convergence of Picard iterative method.

In the Stefan problem considered in [166], the free boundary conditions are

$$T = 0 \quad \text{and} \quad \tilde{R}(x)V_n = (n_t, A(x,t)\nabla_x T); \quad \text{on} \;\; \Gamma(t). \tag{6.2.10}$$

Here $\Gamma(t)$ is the free boundary, $\Gamma(t) \cap \partial\Omega = \emptyset$ and $\partial\Omega(t) = \Gamma(t) \cup \partial\Omega$, $\forall t \in [0, t_*]$, $\vec{n}$ is the unit normal outward to $\Gamma(t)$, $\Omega(t) = \Omega \times \{t\}$, $0 \le t < t_*$, $\Omega \subset R^n$, $n \ge 1$. $A(x,t)$ is a uniformly elliptic matrix associated with the parabolic operator $\mathcal{P}$ defined as

$$\mathcal{P}T \equiv \left(\frac{\partial}{\partial t} - \nabla_x(A(x,t)\nabla_x)\right) T = F, \quad \text{in} \;\; \Omega(t), \;\; 0 < t < t_*. \tag{6.2.11}$$

Suppose

$$\tilde{R}(x) = 0, \quad x \in \Gamma(0), \quad \tilde{R}(x) > 0, \quad \text{in} \;\; \Omega(0) \tag{6.2.12}$$

and $\partial\tilde{R}/\partial\lambda_n < 0$ in some neighbourhood N_0 of $\Gamma(0)$ in Ω. Let $(\lambda_n(x), \omega(x))$ be the local coordinates in N_0, with $\omega(x)$ the local coordinates of the projection $P(x)$ of a point $x \in N_0$ on $\Gamma(0)$ and $\lambda_n(x)$ the distance from x to $P(x)$. For the conditions in Eq. (6.2.12), we have a degenerate Stefan problem. If $\tilde{R}(x) \equiv 0$ then we have the oxygen-diffusion problem in R^n. Generally the existence, uniqueness and regularity of solution of degenerate Stefan problems are discussed in the context of their weak solutions [167–172], but the weak solutions are not sufficiently regular to be called classical solutions. In some cases the existence of the classical solutions of degenerate Stefan problems has been investigated which are discussed later.

Using Hanzawa transformation [133] so as to consider a problem in a fixed domain and an analogue of *Moser–Nash theorem* [173] and assuming some compatibility and other conditions, the existence of the unique solution of the degenerate problem (6.2.10)–(6.2.12) and the solution of oxygen-diffusion problem for $R(x) \equiv 0$ has been established in [166] on a sufficiently small time scale.

Degenerate Parabolic–Elliptic Problems

The parabolic–elliptic Stefan problems have been analysed mostly for their weak solutions. Only few studies have been reported on the analysis of the classical solutions of these problems. The regularity of the classical solution of the two-phase one-dimensional degenerate Stefan problem described in Eqs (6.2.13)–(6.2.17) has been discussed in [174]. Let the regions $-1 < x < S(t)$ and $S(t) < x < 1$ be denoted in the following equations by superscripts 1 and 2, respectively.

$$\beta^1(T)T_t^1 - T_{xx}^1 = 0, \quad -1 < x < S(t), \;\; 0 < t < t_*, \tag{6.2.13}$$

$$\beta^2(T)T_t^2 - T_{xx}^2 = 0, \quad S(t) < x < 1, \;\; 0 < t < t_*, \tag{6.2.14}$$

$$T^{1,2}(x,0) = T_0^{1,2}(x,0), \quad S(0) = 0, \tag{6.2.15}$$

$$T^1(-1,t) = g^1(t) < 0, \quad T^2(1,t) = g^2(t) > 0, \ 0 < t < t_*, \tag{6.2.16}$$

$$T^{1,2}(S(t),t) = 0; \quad S(t) = T_x^1(S(t)-0,t) - T_x^2(S(t)+0,t). \tag{6.2.17}$$

The basic assumptions are:

(P_1) $\beta^{1,2} \in C^\infty(\bar{R}_\mp)$, $\beta^{1,2}(T) \geq 0$ and $\beta^{1,2}(T) = 0$ if and only if $T = 0$,

(P_2) $\partial_x T_0^{1,2}(0) > 0$, $\mp g^{1,2}(t) \geq \delta > 0$ and $T_0^{1,2}(x)\,x \geq 0$ where the inequality holds if and only if $x = 0$.

The main result of [174] is the following proposition.

Proposition 6.2.1. *Under the assumptions* (P_1) *and* (P_2), *the following results hold good.*

(1) If the data $T_0^{1,2}(x) \in C^4[\mp 1, 0]$ *and* $g^{1,2}(t) \in C^2[0, t_*]$ *satisfy second-order compatibility conditions, then the unique weak solution has the following regularity:*

$$S(t) \in C^\infty(0, t_*]; \quad T^{1,2}(x,t) \in C^\infty\left((\mp 1, S(t)] \times [0, t_*]\right). \tag{6.2.18}$$

(2) If the data $T_0^{1,2}(x) \in C^\infty[\mp 1, 0]$ *and* $g^{1,2}(t) \in C^\infty[0, t_*]$ *satisfy the* C^∞ *compatibility conditions, then the unique weak solution is also* C^∞*-smooth up to the boundaries* $t = 0$ *and* $x = \pm 1$.

Definition 6.2.1 (C^∞-compatibility conditions). The data $T_0^{1,2}(x)$ and $g^{1,2}(t)$ are called *m-order compatible* at $(x,t) = (0,0)$ and $(\mp 1, 0)$ if $T_0^{1,2}(x) \in C^{2m}[\mp 1, 0]$, $g^{1,2}(t) \in C^m[0, t_*]$ and there exist functions $\tilde{S}(t) \in C^{2m}[0, t_*]$ and $\tilde{T}^{1,2}(x,t) \in C^{2m,m}[\tilde{S}(t), \mp 1] \times [0, t_*]$ such that

$$\tilde{T}^{1,2}(\mp 1, t) = g^{1,2}(t), \quad \tilde{T}^{1,2}(\tilde{S}(t), t) = 0, \tag{6.2.19}$$

$$\tilde{f}(x,t) = \beta^{1,2}(\tilde{T}^{1,2})\tilde{T}_t^{1,2} - \tilde{T}_{xx}^{1,2} = \mathrm{O}(t^m), \tag{6.2.20}$$

$$\tilde{S}(0) = 0, \quad \tilde{g}(t) \equiv \tilde{S}'(t) - \tilde{T}_x^1(\tilde{S}(t)-0,t) + \tilde{T}_x^2(\tilde{S}(t)+0,t) = \mathrm{O}(t^m). \tag{6.2.21}$$

If $T_0^{1,2}(x) \in C^\infty[\mp 1, 0]$, $g^{1,2}(t) \in C^\infty[0, t_*]$ and Eqs (6.2.19)–(6.2.21) are satisfied for any m, then the data are called C^∞*-compatible*.

A weak variational formulation of the multidimensional degenerate parabolic–elliptic Stefan problem has been presented in [167]. By applying the parabolic regularization technique, the existence, uniqueness and stability of the solution with respect to the data have been analysed. Boundary control aspects have also been discussed. By considering the weak formulation of a two-phase degenerate Stefan problem, the Lipschitz continuity of the free boundary (under suitable assumptions) has been proved in [169] in some small time interval. These results have been further extended in [170] and under suitable assumptions the free boundary is Lipschitz continuous and temperatures satisfy Eqs (6.2.13), (6.2.14) classically.

6.2.1 A Quasi Steady-State Problem (QSSP) and Its Relation to the HSP

A one-phase QSSP (see Section 6.2) in the region $\Omega(t) = \Omega \times \{t\}$, $\Omega \subset R^n$, $n \geq 2$ can be formulated by taking the specific heat $C \equiv 0$ in the heat equation in the one-phase Stefan problem. Temperature is static but the free boundary is time dependent. Let $\partial\Omega(t)$ be the free boundary which is also the outer boundary of $\Omega(t)$. $\Omega(t)$ could be expanding or shrinking for $t > 0$. Shrinking region problems or suction problems are generally ill-posed. The HSP (see Section 6.2) in R^2 given in Eqs (3.3.45)–(3.3.46) can be identified with a Stefan problem if the pressure p of the fluid is regarded as the temperature of the liquid in QSSP. If $p > 0$, then p is the temperature of the warm liquid which is expanding on ice. If $p < 0$, then p is the temperature of the supercooled liquid, and ice is expanding on this supercooled liquid. The pressure of the liquid can be increased (decreased) by injecting (withdrawing) fluid though the inner fixed boundary of $\Omega(0)$ if $\Omega(0)$ is considered to be a doubly connected region. In this case the fluid is surrounding a region $G \subset R^n$, $n \geq 2$ and the inner static boundary of $\Omega(0)$ is ∂G. The boundary conditions on ∂G could be

$$\frac{\partial p}{\partial n} = -Q, \quad Q > 0 \ \text{(pressure increases)}, \tag{6.2.22}$$

$$\frac{\partial p}{\partial n} = Q, \quad Q > 0 \ \text{(pressure decreases)}, \tag{6.2.23}$$

where $\vec{n}$ is the unit outward normal to ∂G.

Fluid can also be injected (withdrawn) though point sources (sinks) or distributed sources (sinks) situated in $\Omega(0)$. $\Omega(t)$ could be a simply connected region with outer boundary as the free boundary. When sources or sinks are present, the governing differential equation will have singularities. If the two-dimensional Hele-Shaw cell has porous plates though which a uniform suction is applied, then the formulation for pressure can be written as

$$\nabla^2 p + F(t) = 0, \ \ F > 0, \quad x \in \Omega(t); \quad p = 0 \quad \text{and} \quad \partial p/\partial n = V, \quad \text{on} \ \ \partial\Omega(t). \tag{6.2.24}$$

Here V is the outward normal component of the velocity of the free boundary.

By using the transformation

$$\hat{p} = p + F(t)U, \tag{6.2.25}$$

where U is the solution of the problem

$$\nabla^2 U = 1, \quad \text{in} \ \ \Omega(t); \ \ U = 0 \quad \text{and} \quad \partial U/\partial n = 0, \quad \text{on} \ \ \partial\Omega(t), \tag{6.2.26}$$

this uniform suction problem can be converted to a standard HSP except at the singularities of F.

For a two-dimensional HSP, complex variable methods can be used to find some exact analytical solutions. For example, it has been shown in [161, 175] that due to suction from a point sink, a limacon can become a cardioid with a cusp. For HSPs with shrinking regions a variety of cusp may occur.

For both expanding and contracting one-phase Stefan problems, the free boundary has been shown in [176] to depend continuously and monotonically on the specific heat C taken to be greater than or equal to zero. In some cases, temperature has also continuous dependence on C (specific heat). In particular, taking $C \to 0$, the free boundary in the Stefan problem approaches that of the HSP and it follows that even well-posed Stefan problems can have free boundaries which can get arbitrarily close to forming cusps. If the free boundary in an expanding HSP has cusps as $t \to \infty$, then Stefan problem in the limit $t \to \infty$ also develops cusps [177].

For $n \geq 2$, Hele-Shaw suction problem with boundary conditions (6.2.22), (6.2.23) and one-phase supercooled water problem were discussed in [177]. Existence of weak solutions for both the problems can be proved if and only if the initial domain belongs to a certain class of domains. Uniqueness does not hold in general. In the ill-posed Hele-Shaw suction problem, fingering configuration can arise from a suitable initial domain whose boundary is smooth and nearly spherical. In the supercooled water problem the initial domain should belong to a certain class which depends on the initial temperature. Regularity of the free boundary has also been discussed.