# Chapter 5

# Superheating due to Volumetric Heat Sources: The Formulation and Analysis

Some amount of superheating of the solid occurs with the melting of pure metals. When superheating is present, a 'mushy region' may develop in front of the phase front, separating the stable solid phase from the stable liquid phase. Note that a superheated solid is in a solid state but with time degenerates into mush. The temperature of the mushy region is taken to be equal to $T_m$ if the phase-change is taking place at $T_m$. The degree of superheating is small and it is generally neglected in the formulation but when melting takes place due to strong volumetric heat sources, there is clear evidence of the superheating of the solid. The theoretical analysis of a sharp melting front model in the presence of strong volumetric sources indicated 'super-heated regions' in the solid just in front of the interface [144]. Superheating or supercooling indicates the instability of the sharp interface and requires corrective measures in the modelling of classical Stefan problems. Some of the corrective measures are: (1) consider an enthalpy model in the place of the classical model, (2) introduce modified Gibbs–Thomson relation in the temperature relation at the phase-change interface in the classical formulation and (3) consider a different type of formulation such as a 'classical enthalpy formulation' (CEF).

## 5.1 THE CLASSICAL ENTHALPY FORMULATION (CEF)

We shall first discuss CEF in the context of one-dimensional melting due to volumetric heat sources in a slab $(-1 \leq x \leq 1)$. This physical problem arises in the 'spot welding' of two large metal sheets of equal thickness of some heat and electric conducting material. Two circular electrodes are placed opposite to each other with the two sheets held tightly in between the electrodes [145]. A high-electric current is passed through the system for a short time. The energy produced by the current raises the temperature of the sheets. It will be assumed that the heating takes place due to *Joule heating* throughout the volume of the conducting material. If the conditions are symmetrical about $x = 0$ then it is sufficient to consider heat transfer in only one sheet with no flux condition at $x = 0$. Under certain assumptions, a one-dimensional problem can be formulated in the region $0 \leq X \leq 1$ ($X$ is dimensionless). At $X = 0$, the two sheets are joined and at $X = 1$, the electrodes are cooled. According to the maximum principle, the temperature will be maximum at $X = 0$. The temperature at $X = 0$ will rise only due to the heat received by it due to Joule heating and not by conduction as other parts are at a lower

temperature and because $\partial T/\partial X = 0$ at $X = 0$ (because of symmetry). Liquid will form at $X = 0$ when the specific enthalpy at $X = 0$ becomes $C_L T_m + l$. It is assumed that $t = 0$ is the time when the temperature at $X = 0$ rises to the melting temperature $T_m$. For $t > 0$ the specific enthalpy at $X = 0$ will rise from $C_S T_m$ to $C_L T_m + l$ (we assume the specific heat of the mushy region also to be $C_S$). During this time the enthalpy of the points in some neighbourhood of $X = 0$ will have risen. By the time melting starts at $X = 0$, i.e. the enthalpy at $X = 0$ rises to $C_L T_m + l$, a mushy region will have developed in some neighbourhood of $X = 0$. This clearly suggests a three-region formulation, given first in [146], and then extended in [147]. The dimensionless formulation described in [147] is given below.

*In the solid region*

$$\frac{\partial T_S}{\partial t} = \alpha_S^2 \frac{\partial^2 T_S}{\partial X^2} + A_S + B_S T_S; \quad A_S > 0, \ A_S + B_S T_S > 0,$$
$$S_1(t) < X < 1, \ 0 < t < t_e < \infty, \tag{5.1.1}$$

$$T_S(X,0) = f_S^{(1)}(X), \quad 0 \le X \le 1, \tag{5.1.2}$$

$$f_S^{(1)}(0) = 1, \quad \left.\frac{\partial f_S^{(1)}}{\partial X}\right|_{X=0} = 0, \quad \left.\alpha_S^2 \frac{\partial^2 f_S^{(1)}}{\partial X^2}\right|_{X=0} + A_S + B_S > 0, \tag{5.1.3}$$

$$\frac{\partial T_S}{\partial X} + d_1 T_S = d_2(t), \quad \text{on } X = 1, \tag{5.1.4}$$

$$\left.\begin{aligned} T_S &= 1, \\ \partial T_S/\partial X &= 0 \end{aligned}\right\} \quad \text{on } X = S_1(t); \ S_1(0) = 0, \ t = S_1^{-1}(X), \ S_1^{-1}(0) = 0. \tag{5.1.5}$$

In the dimensionless form $T_m = 1$ and $A_S$, $B_S$ and $d_1$ are known constants. The equation of the solid–mush boundary is $X = S_1(t)$ or $t = S_1^{-1}(x)$, $t_e$ is the time at which the mushy region becomes extinct and $\alpha$ is the dimensionless diffusivity.

*In the mushy region*

$$\frac{\partial H_M}{\partial t} = Q(H_M); \quad S_2(t) < X < S_1(t), \ 0 < t < t_e, \tag{5.1.6}$$

$$H_M = 1, \quad \text{at } t = S_1^{-1}(X), \tag{5.1.7}$$

$$\left.\begin{aligned} Q(H_M) &= A_S + B_S, \quad H_M = 1, \\ &= D_1 + D_2 H_M, \quad 1 < H_M < C_L/C_S + \lambda, \\ &= A_L + B_L, \quad H_M = C_L/C_S + \lambda. \end{aligned}\right\} \tag{5.1.8}$$

Here $H_M$ is the enthalpy per unit volume of the mushy region, $\lambda = l/(C_S T_m)$. $D_1$ and $D_2$ are unknown constants. An exact form of $Q$ cannot be guessed. A simple assumption that $Q$ is a linear function of enthalpy in the mushy region has been made in Eq. (5.1.8).

*In the liquid region*

$$\frac{\partial T_L}{\partial t} = \alpha_L^2 \frac{\partial^2 T_L}{\partial X^2} + A_L + B_L T_L; \quad A_L > 0, \ A_L + B_L T_L > 0,$$
$$0 < X < S_2(t) < S_1(t), \ t \ge t_*, \tag{5.1.9}$$

$$T_L|_{X=S_2(t)} = 1, \tag{5.1.10}$$

$$(C_L/C_S)\alpha_L^2 \frac{\partial T_L}{\partial X} = -\frac{dS_2}{dt}\{(C_L/C_S) + \lambda - H_M\}, \quad \text{on } X = S_2(t), \ t \geq t_*, \ S_2(t_*) = 0,$$

$$(5.1.11)$$

$$\left.\frac{\partial T_L}{\partial X}\right|_{X=0} = 0. \tag{5.1.12}$$

Here $X = S_2(t)$ is the equation of the liquid–mush boundary and $t = t_*$ is the time at which $S_2(t)$ starts growing, $S_2(t_*) = 0$.

The above formulation requires some elaboration. The conditions in Eq. (5.1.3) have interesting physical interpretations. For example, $f_S^{(1)}(0) = 1$ implies that the temperature at $X = 0$ is the melting temperature. It is generally understood that if more volumetric heat is supplied, then the enthalpy at $X = 0$ should rise. But this does not always happen. For example, if the last condition in Eq. (5.1.3) is not satisfied then the temperature at $X = 0$ for $t > 0$ first becomes less than unity. After some time $\partial T/\partial t$ becomes greater than zero so that the temperature starts rising, and again the temperature at $X = 0$ becomes unity and then the third condition in Eq. (5.1.3) is satisfied. $t = 0$ has been taken as the instant of time at which this happens. In view of the boundary condition (5.1.12), we should have the second condition in Eq. (5.1.3). As mentioned earlier, the melting does not start at $t = 0$ but at $t = t_*$. This is the time at which the dimensionless enthalpy at $X = 0$ becomes $C_L/C_S + \lambda$. The instant $t^*$ can be calculated by obtaining the solution of Eq. (5.1.6) subject to Eqs (5.1.7), (5.1.8) (cf. [147]). All the temperatures have been made dimensionless by dividing them by the melting temperature $T_m$ (K). The dimensionless time $t$ is obtained by dividing the real time by $t_d$, which is an appropriate time with dimensions and $t = 0$ is time at which all the conditions in Eq. (5.1.3) are satisfied.

Till $t = t_*$, there exists only solid–mush boundary $X = S_1(t)$ but for $t > t_*$, we have liquid–mush boundary $X = S_2(t)$ also. The assumption of the continuity of enthalpy at $X = S_1(t)$ implies $C_M = C_S$ which for the sake of simplicity has been assumed to be so in the whole of the mushy region. The condition $\partial T_S/\partial X = 0$ at $X = S_1(t)$ in Eq. (5.1.5) can be derived independently of the assumption of continuity of the enthalpy.

The energy balance at $X = S_1(t)$ gives

$$\dot{S}_1[H]_{S_1(t)+0}^{S_1(t)-0} = -\left[K\frac{\partial T}{\partial X}\right]_S^M, \quad \dot{S}_1 > 0. \tag{5.1.13}$$

$H_M \geq 1$ as $X \uparrow S(t) - 0$ and $H_S = 1$ as $X \downarrow S(t) + 0$. Since $T_M = 1$ in the mushy region $\partial T_M/\partial X = 0$ as $X \uparrow S(t) - 0$. It can be easily concluded from Eq. (5.1.13) that $\partial T_S/\partial X \geq 0$. In view of $T_S(S_1(t), t) = 1$ and $T_S(X, t) < 1$ for $X > S_1(t)$, $\partial T_S/\partial X \leq 0$ as $X \rightarrow S(t) + 0$. Therefore $\partial T_S/\partial X = 0$ at $X = S_1(t)$. This also implies that enthalpy is continuous at $X = S_1(t)$ if $\dot{S}_1$ is not zero in Eq. (5.1.13).

The problem in the solid region is independent of liquid and mushy regions and therefore the solid region problem is a one-phase Stefan problem with an implicit boundary condition at $X = S_1(t)$. Under appropriate assumptions, the existence and uniqueness of the solution of this one-phase problem is known [108]. It can be proved that $S_1(t)$ is nondecreasing and sufficiently smooth and $S_1(t)$ exists till the time $S_1(t) = 1$. If, instead of a heat source, a heat sink is considered, then the problem in the solid region will be similar to oxygen-diffusion problem (ODP). In [147], by considering series expansions of both initial temperature distribution

$f_S^{(1)}(X)$ and $S_1(t)$, short-time asymptotic behaviour of $S_1(t)$ has been investigated in terms of coefficients in the series expansion of $f_S^{(1)}(X)$ and some physical parameters. Numerical results were also obtained for $T_S(X,t)$, $S_1(t)$ and $S_2(t)$ in classical enthalpy solution (CES) and ODP. It was not possible to carry out numerical computations till $X = 1$ in ODP. When $S_1(t)$ is almost nearing unity, $\dot{S}_1(t)$ becomes large and the numerical scheme which uses moving grid points failed. We are not aware of better results.

The problem for the liquid region is not the standard Stefan problem in which latent heat is nonzero and maintains its sign. Because of the boundary condition (5.1.11), the quantity $C_L/C_S + \lambda - H_M$ (call it latent heat) is zero at $X = 0$. Also when $S_2(t)$ crosses $S_1(t)$, the sign of 'latent heat' changes. Therefore the usual methods of proving existence results do not work here directly. However adopting a different procedure in the form of constructing approximating problems [146], existence of the solution of Eqs (5.1.9)–(5.1.12) has been proved and it has been shown that $S_2(t) < S_1(t)$. If the heat source does not vanish in the liquid region, then the mushy region will become extinct in finite time. The mushy region may become extinct before $S_1(t)$ reaches $X = 1$. In this case $S_2(t)$ will meet $S_1(t)$ after some finite time, say, $t = t_e$. For $t > t_e$, we shall have a two-phase classical Stefan problem. Initially $S_1(t)$ moves very slowly as $S_1(t) \sim O(e^{-1/t})$ (cf. [147]).

For a short-time, asymptotic behaviour of $S_2(t)$ which depends on $S_1(t)$ can also be calculated. When $S_1(t) \approx E_1 t^{1/2}$ and $S_2(y) \approx E_2 y^{1/2}$, $y = t - t_*$, $y \geq 0$ then $E_2$ is given by

$$E_2 = \left\{ 2E_1^2 - \alpha_L^2 + \sqrt{(2E_1^2 - \alpha_L^2)^2 + 24E_1^2\alpha_2^2} \right\} \Big/ 4, \tag{5.1.14}$$

where $E_1$ is the root of the equation

$$\sqrt{\pi}E_1 \exp(E_1^2/\alpha_S^2)(1 - \text{erf}(E_1/\alpha_S)) - \alpha_S = \alpha_S^3 \left. \frac{\partial^2 f_S^{(1)}}{\partial X^2}\right|_{X=0} \Big/ (A_S + B_S). \tag{5.1.15}$$

By taking initial temperature of the solid as constant, viz., $-P$, several results concerning the solution of CEF were obtained in [148] which include the asymptotic behaviours of $S_1(t)$ and $S_2(t)$, finite-time extinction of the mushy region, and the result that $\dot{S}_1(t_e) < \dot{S}_2(t_e)$.

In CEF, superheating in the solid does not occur. CEF regularizes the classical two-phase sharp interface problem in the presence of heat sources in which superheating would otherwise occur. Weak formulation also provides regularization. Is there any connection amongst the solution of CEF, weak solution (WS) and the classical solution? This question has been analysed in [149, 150] for a short time by considering a one-dimensional problem in the region $-\infty < x < \infty$ with the free boundary starting at $x = 0$. Since superheating, if present, will be restricted to a small neighbourhood of $x = 0$, it is sufficient to consider the behaviour of the initial temperature $f(x)$. To understand the connection between the classical solution and the WS, we should first define a 'WS'.

## 5.2    THE WEAK SOLUTION (WS)

### 5.2.1    The WS and Its Relation to Classical Solution

The definition of the WS of a Stefan problem is based on the classical formulation of the Stefan problem. For simplicity, first a weak formulation of a one-dimensional two-phase Stefan problem will be defined in the region $0 \leq x \leq 1$. For multidimensional Stefan problems, weak

formulations can be defined similarly (see Section 11.2.1). In terms of the enthalpy $H(T)$ (per unit volume), the two heat equations in the solid and liquid regions can be written as a single equation as follows:

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(K(T)\frac{\partial T}{\partial x}\right) + Q(T), \quad 0 \le x \le 1, \ 0 < t < t_*, \tag{5.2.1}$$

where $K(T)$ and $Q(T)$ are defined as

$$\begin{aligned} K &= K_1(T), \quad \text{in the solid}, \quad 0 < x < S(t), \\ &= K_2(T), \quad \text{in the liquid}, \quad S(t) < x < 1, \end{aligned} \tag{5.2.2}$$

$$\left.\begin{aligned} Q(T) &= A_S + B_S T, \quad \text{in the solid}, \\ &= A_L + B_L T, \quad \text{in the liquid}. \end{aligned}\right\} \tag{5.2.3}$$

More general expressions of thermal conductivities and heat generation terms can be considered. Let the phase-change temperature be denoted by $T_m$. Enthalpy $H(T)$ is defined by Eq. (4.2.1) (take $\phi = 0$ for $T < T_m$ and $\phi = 2$ for $T > T_m$). By inverting Eq. (4.2.1), we obtain

$$T = \beta(H), \tag{5.2.4}$$

where

$$\left.\begin{aligned} \beta(H) &= H/(\rho_S C_S), \quad H < \rho_S C_S T_m, \\ &= 0, \quad \rho_S C_S T_m < H < \rho_L C_L T_m + \rho_L l, \\ &= H/\rho_L C_L - l/C_L, \quad H > \rho_L C_L T_m + \rho_L l. \end{aligned}\right\} \tag{5.2.5}$$

Eq. (5.2.1) should be supplemented with the initial and boundary conditions such as

$$T(x,0) = T_0(x), \ 0 \le x \le 1; \quad T(0,t) = g_1(t) < 0, \quad T(1,t) = g_2(t) \ge 0. \tag{5.2.6}$$

Eq. (5.2.1) is satisfied in the distributional sense as $\partial H/\partial t$ does not exist at $T = T_m$ in the classical sense. The classical formulation of the above two-phase Stefan problem is simple and will not be given here. Multiply Eq. (5.2.1) by a suitable test function $\psi(x,t) \in C^{2,1}([0,1] \times [0,t_*])$ such that

$$\psi(0,t) = \psi(1,t) = 0, \ 0 < t \le t_* \quad \text{and} \quad \psi(x,t_*) = 0, \quad \text{in } 0 \le x \le 1 \tag{5.2.7}$$

and integrate over the $(x,t)$ region. We obtain

$$\int_0^1 \int_0^{t_*} \left[\psi \frac{\partial}{\partial x}\left(K(T)\frac{\partial T}{\partial x}\right) + \psi Q - \psi \frac{\partial H}{\partial t}\right] dx dt = 0. \tag{5.2.8}$$

On integrating by parts the first term in the integrand in Eq. (5.2.8), we obtain

$$\int_0^{t_*} \left[\psi K \frac{\partial T}{\partial x}\right]_{x=0}^{x=1} dt - \int_0^1 \int_0^{t_*} K \frac{\partial T}{\partial x}\frac{\partial \psi}{\partial x} dx dt + \int_0^1 \int_0^{t_*} \psi Q dx dt$$

$$- \int_0^1 [\psi H]_0^{t_*} dx + \int_0^1 \int_0^{t_*} H \frac{\partial \psi}{\partial t} dx dt = 0. \tag{5.2.9}$$

The first term on the l.h.s. of Eq. (5.2.9) in zero in view of Eq. (5.2.7). Integrating the second term in Eq. (5.2.9) by parts, we obtain

$$
- \int_0^{t_*} \left[ TK \frac{\partial \psi}{\partial x} \right]_0^1 dt + \int_0^1 \int_0^{t_*} T \frac{\partial}{\partial x} \left( K \frac{\partial \psi}{\partial x} \right) dx dt + \int_0^1 \int_0^{t_*} \psi Q dx dt
$$
$$
+ \int_0^1 H(T_0) \psi(x, 0) dx + \int_0^1 \int_0^{t_*} H \frac{\partial \psi}{\partial t} dx dt = 0. \tag{5.2.10}
$$

On using Eq. (5.2.6) in the first term in Eq. (5.2.10), we get

$$
\int_0^1 \int_0^{t_*} \left( T \frac{\partial}{\partial x} \left( K \frac{\partial \psi}{\partial x} \right) + H \frac{\partial \psi}{\partial t} + \psi Q \right) dx dt = \int_0^{t_*} g_2(t) K(g_2(t)) \frac{\partial \psi}{\partial x} (1, t) dt
$$
$$
- \int_0^{t_*} g_1(t) K(g_1(t)) \frac{\partial \psi}{\partial x} (0, t) dt - \int_0^1 H(T_0) \psi(x, 0) dx. \tag{5.2.11}
$$

If a pair of bounded measurable functions $(T, H)$ exists such that Eq. (5.2.11) is satisfied for all test functions $\psi(x, t)$ satisfying Eq. (5.2.7), then the pair $(T, H)$ is called a WS of the classical two-phase one-dimensional Stefan problem in the region $0 \le x \le 1$, $0 \le t \le t_*$. Note that in the definition of a WS, the boundary conditions at the interface $x = S(t)$ are not required. So they do not appear in Eqs (5.2.1)–(5.2.6). A WS for multidimensional two-phase classical Stefan problem can be defined similarly. In this case integration by parts is to be done with the help of Green's formula. Other types of boundary conditions can also be considered in Eq. (5.2.6) and in that case test functions have to satisfy appropriate boundary conditions. The boundary conditions for test functions are chosen in such a way that all those terms appearing in the integration by parts, which are to be evaluated at the boundary and which are not prescribed, are eliminated from Eq. (5.2.11) through which the WS is defined. For example, in Eq. (5.2.9) the boundary conditions $\psi(0, t) = \psi(1, t) = 0$ eliminate the term $[\partial T/\partial x]_0^1$ as these boundary conditions are not prescribed in the present problem. Similarly weak formulations of more complicated problems and their WSs can be defined.

A classical solution is a WS. This proposition holds for a general Stefan problem but it will be discussed here for a simple one-dimensional problem given in Eqs (5.2.1)–(5.2.6). To avoid repetition, we give here only outlines (see Eqs 5.2.12–5.2.14 for details). If a classical solution exists, then we know that Eq. (5.2.1) holds separately in solid and liquid regions. $T$ is continuous in $0 \le x \le 1$, and a bounded measurable function $H$ defined in Eq. (5.2.1) also exists. We have to show that the pair $(T, H)$ satisfies Eq. (5.2.11). To prove this, we start with Eq. (5.2.8) and write the double integral as the sum of two integrals, $I_1 + I_2$. In $I_1$ the limits of integration for the $x$-variable are taken from 0 to $S(t) - 0$ (solid region) and in $I_2$ the limits for $x$ are taken from 0 to $S(t) + 0$ (liquid region) (the limits for $t$ remain the same). The procedure given in Eqs (5.2.12)–(5.2.14) is to be followed for both the integrals. On adding the two integrals, we get Eq. (5.2.11) with an additional term which is the last term in Eq. (5.2.13). The line integral along $x = S(t)$ in Eq. (5.2.13) is zero because the classical solution exists and Stefan condition is satisfied. $I_1 + I_2 = 0$ as both $I_1$ and $I_2$ are zero, the remaining terms give Eq. (5.2.11).

In the weak formulation, the phases are 'pointwise' characterized by enthalpy, whereas, in the classical formulation the phases are 'globally' determined by the interface. Therefore a WS of the Stefan problem will be a classical solution provided:

**(i)** initially no mushy region exists,

**(ii)** distributed heat sources (sinks) are absent as in their presence mushy region may develop,

**(iii)** a smooth surface exists which divides the region under consideration into two disjoint regions, in each of them parabolic heat equations are satisfied in the sense of Section 1.4.6. Temperature is less than or equal to the melting temperature in one of the regions (solid region) and in another region temperature is greater than or equal to the melting temperature (liquid region) with continuity of temperature across the interface,

**(iv)** the Stefan condition is satisfied at the interface.

These assumptions are not rigorous, but they serve our present purpose, which is to show that the Stefan condition can be recovered from the WS. We shall continue with the formulation given in Eqs (5.2.1)–(5.2.7). Assume that a WS satisfying the initial and boundary conditions (5.2.6), (5.2.11) exists. A smooth interface $x = S(t)$ exists. Let the region $\Omega(x,t) = \{(x,t) : 0 \le x \le 1, 0 \le t \le t_*\}$ be divided by the curve $x = S(t)$ into two disjoint parts $\Omega_1(x,t)$ and $\Omega_2(x,t)$. $\Omega_1$ lies to the left of $x = S(t)$ and $\Omega_2$ lies to the right. $T = 0$ on $x = S(t)$ (under suitable assumptions on the data, this and some other assumptions can be proved; also, see Section 11.1), $\Omega_1(x,t) = \{(x,t) : 0 \le x < S(t) - 0, 0 \le t < t_*\}$ and $\Omega_2(x,t) = \{(x,t) : S(t) + 0 < x \le 1, 0 \le t < t_*\}$. If Eq. (5.2.1) with $Q(T) = 0$ is multiplied by a test function $\psi(x,t)$ and integrated over $\Omega_1(t)$, then we obtain

$$\int_0^{t_*} \int_0^{S(t)-0} \left[ \psi \frac{\partial}{\partial x}\left(K(T)\frac{\partial T}{\partial x}\right) - \psi\frac{\partial H}{\partial t} \right] dx dt = I_1. \tag{5.2.12}$$

If integrations are performed in Eq. (5.2.12) using integration by parts and the *Stoke's theorem* is applied, then we arrive at the following equation:

$$I_1 = \int_0^{t_*} \int_0^{S(t)-0} \left\{ T\frac{\partial}{\partial x}\left(K\frac{\partial \psi}{\partial x}\right) + H(T)\frac{\partial \psi}{\partial t} \right\} dx dt$$
$$+ \int_0^{S(t)-0} H(T_0)\psi(x,0)dx + \int_0^{t_*} g_1(t)K(g_1)\frac{\partial \psi}{\partial x}(0,t)dt$$
$$+ \int_{x=S(t)-0} \left[ \left(\psi K \frac{\partial T}{\partial x}\right)dt + H\psi dx \right]. \tag{5.2.13}$$

On using Stoke's theorem [82], the area integral is converted to the boundary integral and for this we use the relation $\psi \partial f/\partial x = \partial(\psi f)/\partial x - f \partial \psi/\partial x$ in Eq. (5.2.12).

On integrating Eq. (5.2.1) with $Q(T) = 0$ again over the region $\Omega_2(t)$ (call this integral $I_2$) we obtain an equation similar to Eq. (5.2.13) (with appropriate changes) but the curve $x = S(t)$ is traversed in a direction opposite to that in Eq. (5.2.13). Adding the two equations of the type (5.2.13), one each for $\Omega_1(t)$ and $\Omega_2(t)$ and subtracting the sum from Eq. (5.2.11) ($Q(T) = 0$) and remembering that $I_1 + I_2 = 0$ as Eq. (5.2.1) holds, we obtain

$$\int_{x=S(t)} \psi \left\{ \left[ K\frac{\partial T}{\partial x} \right] dt + [H]dx \right\} = 0. \tag{5.2.14}$$

Here $[f]$ means the jump in the quantity $f$ across $x = S(t)$ as we move in the positive $x$-direction. Since Eq. (5.2.14) holds for any arbitrary test function $\psi$ and at an arbitrary time, we have

$$\frac{dS}{dt} = \left[K\frac{\partial T}{\partial x}\right]\bigg/[H], \quad x = S(t).$$ (5.2.15)

Eq. (5.2.15) is the Stefan condition. A similar procedure can be followed for multidimensional problems also.

   For a solution of the Stefan problem to be a classical solution, in addition to the regularity of the temperature and the phase-change boundary, the temperature of the solid region should be less than the melting temperature and the temperature of the liquid region should be greater than the melting temperature.

## 5.2.2   Structure of the Mushy Region in the Presence of Heat Sources

In the presence of heat sources or if melting temperature is a function of $x$ and $t$, or if mushy region exists initially, a WS will not be a classical Stefan solution (CSS). However, if the solution of CEF denoted by CES exists and WS is sufficiently regular, then WS $\equiv$ CES but WS $\not\equiv$ CSS. This is because in CEF, a mushy region is present initially. By considering different cases in terms of the sign of the initial temperature $f(x)$, and the derivatives of $f(x)$ at $x = 0$, CES, WS and CSS have been compared in [149, 150]. The phase-change starts at $x = 0$ in the region $-\infty < x < \infty$. The existence of solutions has also been discussed. We present only one result here which is valid only for a short time.

   If $f(x) > 0$ for $x < 0$ and $f(x) < 0$ for $x > 0$, $h(0) = 0$ and

$$f'(0+) = 0 \quad \text{and} \quad K_S f''(0+) + Q = 0, \quad \begin{cases} f'''(0+) - \dfrac{\beta Q}{K_S^2} < 0, & \text{then } CSS \equiv WS, \\[3mm] f'''(0+) - \dfrac{\beta Q}{K_S^2} > 0, & \text{then } CSS \not\equiv WS, \end{cases}$$ (5.2.16)

where $Q$ is the constant heat source in $-\infty < x < \infty$, and

$$\beta = \dot{S}(0) = -K_L h'(0-)/l.$$ (5.2.17)

   By considering a one-dimensional problem in $0 \le x \le 1$, it has been shown in [151] that if the melting temperature is space dependent, then a mushy region can appear spontaneously even in the absence of volumetric heat sources. The following weak formulation has been considered in [152]:

$$H_t = (\phi(H))_{xx} + f(H), \quad (x,t) \in (0,1) \times R^+,$$ (5.2.18)

$$\phi(H)(0,t) = (\phi(H))_x(1,t) = 0, \quad t \in R^+,$$ (5.2.19)

$$H(x,0) = H_0(x), \quad x \in [0,1].$$ (5.2.20)

Here $\phi(H)$ can be identified with the temperature and $H$ with the enthalpy per unit volume. $f, \phi$ and $H_0$ satisfy some assumptions (cf. [152]). The source term $f(H)$ and the initial enthalpy $H_0(x)$ are so chosen that enthalpy is increasing in $x$ and $t$. Initially the region [0,1] is solid. At $x = 1$ it passes to mush in time and then to liquid. Interfaces appear first between solid and mush and then between mush and liquid. These recede monotonically towards $x = 0$. Under

suitable assumptions on $\phi, f, H_0$ it has been shown that solid–mush and liquid–mush interfaces are continuous, enthalpy is continuous across the solid–mush boundary but not across the mush–liquid interface and the mushy region disappears in finite time.

In Eq. (5.1.13) it has been assumed that the mush is progressing on the solid. If the solid is progressing on mush, then it cannot be proved that the enthalpy is continuous at the solid–mush boundary as the Stefan condition (5.1.13) with a positive sign on the r.h.s. is to be considered. If $\dot{S}$ changes sign at both solid–mush and mush–liquid boundaries, then the regularity of the free boundary and of the velocity satisfying the Stefan condition has been examined in [153]. The enthalpy has been assumed to be an increasing function of $x$. An example has been constructed in [154] in which the velocity of the free boundary changes sign infinitely many times.

CEF arises in thermal switching problems also (cf. [155, 156]). Certain materials, such as some types of glasses and vanadium dioxide, change phase at relatively low temperature, such as, at 70°C. In the liquid state the electrical conductivity of such materials increases by a factor of $10^4$. For such materials, in Joule heating, the heat source is given by $\sigma(H)E^2$, where $\sigma$ is the electrical conductivity, $H$ is the enthalpy and $E$ is the electrical field which depends on $\sigma$ and some other parameters. In essence, heat source may depend on the free boundary and the latent heat $l$ depends on the history of the free boundary [156]. The existence of the solution to a one-dimensional CEF of the thermal switching problem have been proved in [156]. The emphasis is on the solution for the solid region and the solid–mush boundary. The nonhomogeneous term in the heat equation contains the free boundary due to the dependence of the heat source on it.

The behaviour of mushy regions in spacial dimension, $n \geq 1$, under the action of volumetric heat source depending on $x$, $t$ and enthalpy has been studied in [157]. Conditions on the heat source necessary for the appearance of a mushy region inside a purely liquid or a purely solid phase have been discussed. As far as regular solutions are concerned, the behaviour of the solution at the free boundary separating a mushy region from a pure phase is related to the behaviour of the source on the free boundary. Under some Lipschitz-continuity conditions on heat sources, a mushy region can expand into a pure phase only if the heat source is nonzero and is of a suitable sign at the interface. Hence, both energy and heat flux are continuous across the interface, unlike the case when a pure phase expands into a mushy region. An example has been given in which a mushy region in a pure phase disappears and again reappears after some time. Both weak and classical solutions have been discussed. For some results pertaining to WSs of this problem, see Section 11.2.1.

By considering a one-dimensional two-phase classical Stefan melting problem in the region $0 \leq x \leq 1$ with volumetric heat sources and writing its solution in terms of Green's functions, it was shown in [144] that there exists a region $S(t) \leq x \leq S(t) + d$, for some $d > 0$ in which the solid is superheated. This happens provided the heat source is large and the prescribed heat flux on the solid at $x = 1$ is small. Linear stability analysis of the classical solution of this one-dimensional two-phase Stefan problem was also carried out for two-dimensional perturbations (three-dimensional perturbations can also be considered). If the heat flux towards the free boundary from the superheated solid side is greater than from the liquid side, then this situation suggests instability.

Superheating occurs generally for very pure metals. Its occurrence suggests that for such problems we should have a mathematical model different from the classical two-phase formulation as in the classical two-phase model, the solid cannot be superheated. One of the ways to present the superheating of the solid is to introduce the mushy region in the formulation as done in CEF. Another way is to consider enthalpy formulation which is a weak formulation.

A grain or dendrite model has been proposed in [158] to describe the microstructure of the mushy region and discuss its stability. The solid is considered in the form of a sequence of uniform one-dimensional grains of very small length which are part solid and part liquid in the mushy region. An explicit solution for the one-dimensional version of the model is obtained. It has different forms near the pure solid boundary, in the mush and near the pure liquid boundary. An appropriate average of this solution for the microstructure formulation reduces to the WS proposed in [145].

The simple microscopic model considered in [158] for the mushy region has been further extended in [159] by including modified Gibbs–Thomson relation that results from the curvature of the interface and the kinetic condition (the normal velocity of the interface is taken away from the liquid). The fine structure of the mush consists of regularly spaced nucleation sites (grains) in one-dimension and a lattice of squares in two-dimension. A method of multiple scales is employed and a classical formulation of the free boundary problem has been used to model the evolution of the two-phase microstructure. Then a microscopic model for the mush is obtained by an averaging procedure. Emphasis is not put on volumetric heating in [159].

## 5.3   BLOW-UP AND REGULARIZATION

As mentioned earlier, a blow-up may occur during the solidification of supercooled liquid if regularization is not done. In the superheating of the solid also, in the absence of regularization, i.e. noninclusion of surface tension effect and/or kinetic condition at the interface or absence of mushy region in the formulation, we can raise the three questions (A), (B) and (C) (Section 4.4.1) that arose in the context of supercooling.

A heat source term of the form $T^{1+\alpha}$, $\alpha > 0$, has been considered in [160] in the one-phase melting Stefan problem with the superheated solid occupying the region $0 \leq x \leq S(t)$, $S(0) = S_0 > 0$. The initial temperature of the solid is $T_0(x) > 0$, $x \in [0, S_0]$, 0 is the phase-change temperature, and $T(0, t) = 0$, $0 < t \leq t_*$. The temperature $T_0(x)$ satisfies some assumptions such as: (1) $T_0(0) = T_0(S_0) = 0$, (2) $T_{0,x} < 0$ on $(x_0, S_0)$ for some $x_0 \in (0, S_0)$, (3) $T_{0,xx} + T_0^{1+\alpha} \geq 0$ for $x \in (0, S_0)$. The formulation of this problem can be easily written. Let $(t_*, T(x, t), S(t))$ be the classical solution. The main result of [160] can be stated as follows.

**Proposition 5.3.1.** *If $t_* < \infty$, then one (and only one) of the following will always happen.*

(I) $S(t) \rightarrow S_\infty \in (0, \infty)$ *as* $t \uparrow t_*$ *and there exists only one blow-up point* $x^* \in (0, S_\infty)$.

(II) $S(t) \rightarrow +\infty$ *as* $t \uparrow t_*$ *and the set* $\{T(x, t)\}$ *is bounded on* $\{(t, x) : 0 \leq t < t_*, 0 \leq x \leq \min(a, S(t))\}$ *for each* $a > 0$.

It was pointed out earlier that questions (A), (B) and (C) (see Section 4.4.1) can be answered for the supercooled problem (3.1.8)–(3.1.12) with the help of the quantity $Q$ defined by Eq. (4.4.6). A superheated one-dimensional problem can be formulated on the same lines as the formulation of the supercooled problem (3.1.8)–(3.1.12) and so complete formulation will not be given here (cf. [161]). But, for the sake of clarity, it may be mentioned here that the region $0 \leq x < S(t)$, $S(0) = 1$, is the superheated solid and $x > S(t)$ is the liquid region with $T_L \equiv 0 \cdot S(t)$ is progressing towards $x = 0$. For simplicity we take $\partial T/\partial x = 0$ at $x = 0$. The quantity $P_0$ given by

$$P_0 = \int_0^1 (T_0(x) - 1)dx, \tag{5.3.21}$$

plays an important role in answering the questions concerning blow-up in superheating. $T_0(x) \geq 0$ is the initial temperature of the solid, $0 \leq x \leq 1$, and the latent heat of fusion has been taken as unity in Eq. (5.3.21) after scaling. In melting, latent heat is negative and $P_0$ is the energy required for the complete melting of the superheated solid $0 \leq x \leq 1$. If $P_0 > 0$, then

$$\int_0^{S(t)} (T(x, t) - 1)dx, \tag{5.3.22}$$

will remain positive. The statements (A1), (B1) and (C1) given below hold.

(A1) If the solution exists for all time, $S(t) \to S_\infty > 0$ and $T(x, t) \to 0$ as $t \to \infty$ then $-\int_0^{S_\infty} dx$ is negative. (B1) If there exists some finite time $t = t_*$ such that the solution exists for $0 \leq t \leq t_*$ and as $t \to t_*$, $S(t) \to 0$ and $dS/dt \to -\infty$, then $P_0 = 0$. (C1) If the solution exists only for $0 \leq t \leq t_*$ but as $t \to t_*$, $dS/dt \to -\infty$ and $S \to S_* > 0$ then the solution blows-up. In the supercooled problem, roughly speaking, when the free boundary touches the negativity set $\{T(x, t) < -1\}$ then blow-up occurs. In the superheating problem, if the free boundary touches the positivity set $\{T(x, t) > 1\}$ then blow-up occurs. The heat in the solid near the free boundary is too much and the advancing free boundary cannot remove the heat and bring down the temperature to the equilibrium level which is taken as zero.

It is possible to give more general sufficient conditions for a blow-up. If there exists a function $f(x)$ defined for $0 \leq x \leq S(0)$ such that: $f(0) \geq 0$, $f'(0) \geq 0$, $f''(x) \geq 0$ for $0 < x < S(0)$ and

$$\int_0^1 (T_0(x) - 1)fdx > 0, \tag{5.3.23}$$

then blow-up occurs.

The questions (A), (B) and (C) (Section 4.4.1) were discussed in [161] for a two-dimensional one-phase melting of a superheated solid also. If there exists some $f(x)$, $x \in R^2$ with $f \geq 0$ and $\nabla^2 f \geq 0$ in $\Omega(0) = \Omega_0 \subset R^2$, $\Omega(t) = \Omega \times \{t\}$, $\Omega \subset R^2$, $0 \leq t \leq t_*$, such that

$$\int_{\Omega_0} f(T_0 - 1)dx > 0, \tag{5.3.24}$$

then case (C) must occur. If case (B) is to occur, then the integral in Eq. (5.3.24) should be zero for all the harmonic functions $f$. If $\int_{\Omega_0} (T_0 - 1)dx = 0$, but there is some harmonic function $f$ such that $\int_{\partial\Omega_0} f(T_0 - 1)dx \neq 0$, then blow-up occurs.