Chapter 10

Analysis of the Classical Solutions of Stefan Problems

Some results of the analysis (existence, uniqueness, stability and regularity results) of classical solutions of Stefan problems will be presented in this chapter. The analysis aspect of one-dimensional classical Stefan problems has been thoroughly investigated and it may not be possible to add any significant result to the existing literature in the future. This can also be felt from the results reported in Sections 10.1 and 10.2. For analysis, we shall mainly consider the formulations (1.4.3)–(1.4.9) for multidimensional problems and formulations such as those given in Eqs (3.1.1)–(3.1.5), (3.2.1)–(3.2.10) for one-dimensional problems. In essence, it will be assumed in the classical formulation that the solid and the liquid phases are separated by a smooth free boundary, the temperatures in the two regions satisfy heat equations and appropriate sign constraints (see Section 1.4.6). The analysis of the problem of solidification of a supercooled liquid and some related problems has already been presented in Chapter 4.

In the multidimensional Stefan problems, one can easily think of geometries in which the free boundary is initially regular but becomes discontinuous after some time. The results of analysis of classical solutions of multidimensional Stefan problems are available mostly for short-time (local-in-time) solutions. However, in principle, it may be possible to study these solutions till the time the free boundary becomes discontinuous. Weak (generalized) solutions of multidimensional Stefan problems have been extensively studied under fairly general assumptions about the input data and the coefficients occurring in the formulation. Due to reasons of emphasis, weak solutions of only some selected Stefan problems in which the differentiability or the Lipschitz continuity of the free boundary has been established, will be discussed in Chapter 11.

10.1 ONE-DIMENSIONAL ONE-PHASE STEFAN PROBLEMS

The analysis of one-dimensional one-phase problems has a long history and a brief account of it can be found in [21]. There are several factors that affect the results of analysis of Stefan problems, such as, formulation of the problem, the definition of the solution, assumptions about the input data and coefficients, and the mathematical tools used in proving the results. The proof of existence, uniqueness, etc. requires some estimates of temperature derivatives, which involve lengthy calculations. The main result can be proved only after proving several lemmas and propositions. It is not possible to give complete details of the analysis here. As a suitable compromise on length, only some basic approaches used in the proofs will be discussed together with some of the main results. Taking a contemporary perspective, we can

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consider many of the problems studied earlier as particular cases of the problems studied later. It does not seem necessary to discuss all the particular cases. The reader is requested to look into cross references mentioned in the bibliography.

10.1.1 Analysis Using Integral Equation Formulations

We present some results since 1959. Several references of works prior to 1959 with comments on the methods used in proving the results are reported in [21, 279–281]. The one-phase melting problem considered in [279] is to find the temperature $T(x,t)$ and the free boundary $S(t)$ satisfying the following system of equations:

$$(p(x,t,T,T_x)T_x)_x = f(x,t,T)T_t, \text{ in } D_{t_*} = \{(x,t) : 0 < x < S(t), 0 < t < t_* < \infty\}, \quad (10.1.1)$$

$$p_0(t)T_x(0,t) = -g(t), \quad 0 < t < t_*; \quad p_0(t) = p|_{x=0}, \quad (10.1.2)$$

$$T(S(t),t) = 0, \quad 0 < t < t_*, \quad (10.1.3)$$

$$\dot{S}(t) = b - (pT_x)|_{x=S(t)}, \quad 0 < t < t_*; \quad S(0) = 0. \quad (10.1.4)$$

At time $t = 0$, the region $x > 0$ is occupied by ice at the melting temperature which is taken as zero. No initial condition is required as $S(0) = 0$. The coefficients in Eqs (10.1.1)–(10.1.4) satisfy the following conditions:

(1) b is a nonnegative constant. The functions $g(t)$, f, p have continuous second order derivatives.

(2) $0 < a_0 \leq g(t)/p_0(t) \equiv b(t) < a^0$.

(3) $0 < f_0 \leq f(x,t,T)$.

(4) $0 < p_1 \leq p(x,t,T,T_x) = p_0(t) +$ terms which vanish at $x = 0$.

(5) $p_x \geq 0, p_T \leq 0, p_z = p_{T_x} \leq 0$.

(6) $S(t)$ is monotonic and it is possible to express $x = S(t)$ as $t = S^{-1}(x)$.

The fifth assumption implies that Eq. (10.1.1) can be written as a linear differential equation with nonnegative coefficients, i.e.

$$(p + p_z T_x)T_{xx} + (p_x + p_T T_x)T_x = fT_t. \quad (10.1.5)$$

It can be proved that $T_x \leq 0$. Introducing the notations

$$F(x,t,T) = \int_0^T f(x,t,q)dq \text{ and } G(x,t,T) = -\int_0^T f_t(x,t,q)dq, \quad (10.1.6)$$

and integrating Eq. (10.1.1) over the region $0 \leq x \leq S(t')$, $0 \leq t \leq t'$, and writing t in the place of t', the following integral equation is obtained:

$$S(t) = \int_0^t (b + g(q))dq - \int_0^{S(t)} F(x,t,T)dx - \int_0^t \int_0^{S(q)} G(x,q,T(x,q))dxdq. \quad (10.1.7)$$

By using fixed point argument, the existence and uniqueness of the classical solution (T,S) of Eqs (10.1.1)–(10.1.4) has been proved in [279]. Let $T(x,t;S(t))$ be the solution of the following ‘reduced problems’:

$$(pT_x)_x = fT_t, \text{ in } D_{t_*}, \tag{10.1.8}$$

$$p_0(t)T_x(0,t) = -g(t), \quad 0 < t < t_*, \tag{10.1.9}$$

$$T(S(t),t) = 0, \quad 0 < t < t_*. \tag{10.1.10}$$

Here, $S(t)$ is a given continuous monotonic function which vanishes at $t = 0$. The dependence of the temperature on the solution of the reduced problem for a given $S(t)$ will be expressed in the form $T(x,t;S)$. It may be noted that the given $S(t)$ may not be the solution of the original Stefan problem as Eq. (10.1.4) is not satisfied. The following propositions which help in applying fixed point argument have been proved in [279].

Proposition 10.1.1. *If $T(x,t;S)$ is the solution of the reduced problem, then there exist numbers $a_1 > 0$ and $a_2 > 0$, independent of $S(t)$, such that*

(a) $T(x,t;S) \le a_1(S(t) - x)$, $0 \le x \le S(t)$, $0 < t < t_*$,

(b) $-a_2 \le p(x,t,T,T_x)T_x \le 0$.

Proposition 10.1.2. *There exists a unique solution of the reduced problem.*

By integrating Eq. (10.1.4) also an integral equation for $S(t)$ can be obtained but it will involve integral of T_x over a boundary and this is not suitable for using fixed point argument. If $T(x,t)$ and $S(t)$ represent the solution of Eqs (10.1.1)–(10.1.4), then Eq. (10.1.7) is satisfied exactly. For approximate values of T and S, it is appropriate to write Eq. (10.1.7) as

$$z(t) = \int_0^t (b + g(q)dq - \int_0^{S(t)} F(x,t,T)dx - \int_0^t \int_0^{S(q)} G(x,q,T(x,q))dxdq. \tag{10.1.11}$$

Eq. (10.1.11) defines a mapping

$$Z = W(S). \tag{10.1.12}$$

Here, W is defined on a set of real valued, differentiable boundary curves which vanish at $t = 0$. These conditions on S are enough because

$$\frac{dz}{dt} = b - p(S(t),t,0,T_x)T_x(S(t),t;S(t)), \tag{10.1.13}$$

and

$$b \le \frac{dz}{dt} \le b + a_2. \tag{10.1.14}$$

The function $z(t)$ is a differentiable and monotone function which vanishes at the origin, and it can therefore serve as a boundary curve for the reduced problem.

Proposition 10.1.3. *Let V be a set of continuously differentiable functions, defined on some finite time interval, which vanish at $t = 0$ and whose derivatives satisfy Eq. (10.1.14). If W is the transformation defined by Eq. (10.1.12), then W is defined on V and maps V into itself.*

Proposition 10.1.4. *Under the uniform norm (see Eq. 9.7.12), V is a subset of a Banach space of continuous functions defined on $[0,t_*]$. The set V is convex and equicontinuous. The closure of V denoted by $\overline{V}$ is also convex. Every infinite subset of $\overline{V}$ has an accumulation point in $\overline{V}$, i.e. $\overline{V}$ is compact.*

Proposition 10.1.5. *W is continuous on the closure $\overline{V}$ of V. W maps $\overline{V}$ into itself.*

Since $\overline{V}$ is a compact and convex subset of a Banach space, and W is a continuous mapping of $\overline{V}$ into itself, by *Schauder's fixed point theorem* [282], there exists at least one element of $\overline{V}$ which is left invariant under W, i.e. $S^0(t) = W(S^0(t))$. The function $S^0(t)$ is the solution of Eq. (10.1.11) and $T(x, t; S^0)$ is the solution of reduced problem (10.1.8)–(10.1.10).

It can be proved that if $S(t)$ is in $\overline{V}$ and $S(t) = W(S(t))$, then $S(t)$ is differentiable, i.e. $S(t)$ is in V. Further

$$\frac{dS}{dt} = b - (pT_x)|_{x=S(t)},$$

and therefore $(S(t), T(x, t; S(t)))$ is a solution of the problem (10.1.1)–(10.1.4). Uniqueness of the problem (10.1.1)–(10.1.4) has also been proved in [279] but it is not based on *contraction mapping* argument [282].

The problem considered in [280] can be obtained by making some changes in Eqs (10.1.1)–(10.1.4), such as, take $p = 1, f = 1, b = 0$, $S(0) = A$ and $T(x, 0) = a(x)$, where $0 \leq a(x) \leq d(A - x)$, $0 \leq x \leq A$ and d is some positive constant. The functions $a(x)$ and $g(t)$ are continuous, $g(t) \leq d$. An integral equation of the type (10.1.7) has been obtained in this case also and now a reduced problem of the type (10.1.8)–(10.1.10) will have a prescribed initial temperature also as $S(0) = A > 0$. The existence of the solution on some finite time interval $[0, t_*]$ has been proved using fixed point theorem as in [279]. However, the uniqueness of the solution has been proved by showing that the iterations done in the numerical solution for calculating the free boundary are converging, i.e. if $S_0 = A$ and $S_{n+1} = F(S_n)$, then F is a contraction.

For the numerical solution of the problem considered in [280], the time interval $[0, t_*]$ is divided into n small time intervals, each of length Δt and in each one of them iterations are done to get better values of $S(t)$. For this purpose an integral equation of the form (10.1.11) is used. $T(x, t; S(t))$ is obtained from the solution of the 'reduced problem' formulated for this problem with appropriate changes. The first iterative process will converge to the solution if the time interval is small (existence holds and uniqueness is proved by contraction argument). Then another iterative process is carried out in the time interval $(0, 2\Delta t]$. The solution of the previous time step is used to obtain $S(t)$ and $T(x, t; S(t))$ in $(0, 2\Delta t]$ by using a suitable iterative process (cf. [280]). The initial temperatures in the 'reduced problems' will go on changing. This procedure is repeated in other time intervals also till the solution is obtained in the time interval $[0, t_*]$. It has been shown that $\lim_{n\to\infty} S_{n+1} = S(t)$ and $S(t)$ is invariant under the transformation of the form (10.1.12) for the present problem also . The subscript n refers here to the nth iterative process.

The problem considered in [281] can be obtained from Eqs (10.1.1)–(10.1.4) if we take $p = 1$ and $f = 1$ in Eq. (10.1.1), $b = 0$ in Eq. (10.1.4), $S(0) = d$ and replace Eq. (10.1.2) by a temperature prescribed boundary condition, e.g.

$$T(0, t) = v(t) \geq 0, \quad 0 < t < t_* < \infty. \tag{10.1.15}$$

Since $S(0) = d$, initial temperature is to be prescribed and let

$$T(x, 0) = T_0(x) \geq 0, \quad 0 \leq x \leq d. \tag{10.1.16}$$

We shall refer to this problem in [281] as Problem (F). The main result of [281] is the following proposition:

Proposition 10.1.6. *Assume that $v(t)$ $(0 \le t < \infty)$ and $T_0(x)$ $(0 \le x \le d)$ are continuously differentiable functions. Then there exists one and only one solution $(T(x,t), S(t))$ of Problem (F) for all $t < \infty$. Furthermore, the function $x = S(t)$ is monotone nondecreasing in t.*

By solution we mean here the classical solution discussed in Section 1.4.6.

If $S(t)$ exists in any time interval $0 \le t < \sigma$, then on using maximum principles for parabolic operators [9], it can be proved that $S(t)$ is nondecreasing in this interval. The Problem (F) can be reduced to a problem of solving a nonlinear integral equation. Following the procedure indicated in Eqs (9.6.5)–(9.6.8), the temperature can be expressed as in Eq. (9.6.8) for the present problem also (substitute d in place of b). We use the notations used in Eq. (9.6.8). If both sides of Eq. (9.6.8) are differentiated with respect to x and the limit $x \to S(t) - 0$ is taken, then we get the following integral equation:

$$\begin{aligned} y(t) &= 2\,[T_0(0) - v(0)]\,N(S(t), t; 0, 0) + 2\int_0^d \frac{\partial T_0}{\partial \xi} N(S(t), t; \xi, 0) d\xi \\ &\quad - 2\int_0^t \dot{v}(\tau) N(S(t), t; 0, \tau) d\tau + 2\int_0^t y(\tau) G_x(S(t), t; S(\tau), \tau) d\tau, \end{aligned} \tag{10.1.17}$$

$$y(\tau) = T_\xi(S(\tau), \tau), \tag{10.1.18}$$

and

$$N(x, t; \xi, \tau) = Q(x, t; \xi, \tau) + Q(-x, t; \xi, \tau). \tag{10.1.19}$$

In Eq. (10.1.17), G is given by Eq. (9.6.6) and in Eq. (10.1.19), Q is given by Eq. (9.6.7). In obtaining Eq. (10.1.17), the following result has been used.

Proposition 10.1.7. *Let $\rho(t)$, $0 \le t \le \sigma$, be a continuous function and let $S(t)$, $0 \le t \le \sigma$, satisfy a Lipschitz condition. Then for every $0 < t \le \sigma$*

$$\lim_{x \to S(t)-0} \frac{\partial}{\partial x} \int_0^t \rho(\tau) Q(x, t; S(\tau), \tau) d\tau = \frac{1}{2}\rho(\tau) + \int_0^t \rho(\tau) \left[\frac{\partial Q}{\partial x}(x, t; S(\tau), \tau)\right]_{x = S(t)} d\tau. \tag{10.1.20}$$

On integrating the Stefan condition, we get

$$S(t) = d - \int_0^t y(\tau) d\tau. \tag{10.1.21}$$

It can be proved that Problem (F) is equivalent to the problem of finding a continuous function $y(t)$ which is the solution of Eq. (10.1.17) where $S(t)$ is defined by Eq. (10.1.21) and $S(t)$ is positive. For the set of continuous functions $y(t)$ defined for $0 \le t \le \sigma$, σ sufficiently small, Eq. (10.1.17) defines a mapping which can be expressed as

$$w(t) = P(y(t)). \tag{10.1.22}$$

The function $w(t)$ is the l.h.s. of Eq. (10.1.17) for some $y(t)$ which need not be the value of T_x at the exact solution $x = S(t)$. By using fixed point argument, the existence of $y(t)$, i.e. the solution of Eq. (10.1.17) has been proved in [281] and by showing that P is a contraction, uniqueness of $y(t)$ has been proved.

The validity of the solution obtained above for a short time can be extended to longer times. It has been proved in [281] that there exists an $\varepsilon > 0$ such that if the continuous solution $y(t)$ of the integral equation (10.1.17) exists and is unique for $0 \le t \le t_0$, then it exists and is unique for $0 \le t < t_0 + \varepsilon$. Note that the continuity of $y(t)$ implies the continuity of $\dot{S}(t)$. The existence of the solution in the time interval $[0, t_0 + \varepsilon]$ requires that $T_x(x,t)$ is bounded by a constant which is independent of x and t in the interval $0 \le x \le S(t)$, $t_0 - \delta \le t < t_0$ for some δ, $0 < \delta < t_0$. The solution can then be started with some very small $\eta > 0$ and we get a classical solution for $t_0 - \eta \le t < t_0 + \varepsilon$. This solution coincides with the classical solution for $t_0 - \eta < t < t_0$ as the solution is unique for $0 \le t < t_0$.

The one-phase Stefan problem considered in [21, 283] has stronger nonlinearity than in [279] and no sign restrictions have been imposed on the boundary and initial data. The problem studied is to find a classical solution (see Section 1.4.6 and [21]) $(T(x,t), S(t))$ of problem (10.1.23)–(10.1.27)

$$a^2 \frac{\partial^2 T}{\partial x^2} - \frac{\partial T}{\partial t} + F(x,t,T,T_x,S(t),\dot{S}(t)) = 0, \quad (x,t) \in D_{t_*},$$

$$D_{t_*} = \{(x,t) : 0 < x < S(\tau), 0 < \tau \le t < t_*\}, \quad a > 0, \tag{10.1.23}$$

$$\frac{\partial T}{\partial x} = f(t,T), \quad x = 0; \quad t > 0, \tag{10.1.24}$$

$$T = \phi(x), \quad t = 0; \quad 0 \le x \le S(0) = d, \tag{10.1.25}$$

$$T = \psi(x), \quad x = S(t); \quad t > 0, \tag{10.1.26}$$

$$\dot{S} = Z(t,T,T_x,x), \quad x = S(t); \quad t > 0. \tag{10.1.27}$$

Here, F, f, ϕ, ψ and Z are known functions which are defined for $0 \le x \le 1$, $0 \le t \le 1$ and $0 \le S(t) \le 1$. The functions F, f and Z are differentiable, ϕ is thrice differentiable and ψ is twice differentiable with respect to their arguments everywhere in the domain of definition. The known functions and some of their derivatives are bounded by suitable constants (see [21, pp. 95–140]). The compatibility conditions are satisfied, i.e.

$$f(0,\phi(0)) = \phi_x(0); \quad \phi(d) = \psi(d). \tag{10.1.28}$$

We introduce the following notations:

$$T|_{x=0} = w; \quad T_x = q, \quad 0 < x < S(t); \quad T_x(S(t),t) = v; \quad \dot{S}(t) = p. \tag{10.1.29}$$

On using a method similar to that used in obtaining the temperature in Eq. (9.6.8), the temperature given below which is the solution of problem (10.1.23)–(10.1.27) can also be obtained

$$\begin{aligned} T(x,t) = & -a^2 \int_0^t f(\tau,x) G_1(x,0,t-\tau) d\tau + \int_0^t \phi(\xi) G_1(x,\xi,t) d\xi \\ & + \int_0^t d\tau \int_0^{S(t)} F(\xi,\tau,\ldots,p) G_1(x,\xi,t-\tau) d\xi - a^2 \int_0^t \psi(S(\tau)) \frac{\partial}{\partial \xi} G_1(x,S(\tau),t-\tau) d\tau. \\ & + \int_0^t \left\{ a^2 v(\tau) + \psi(S(\tau)) p(\tau) \right\} G_1(x,S(\tau),t-\tau) d\tau. \end{aligned} \tag{10.1.30}$$

Here, the Green's function $G_1(x,\xi,t) = Q(x,t;\xi,\tau) + Q(-x,t;\xi,\tau)$ and Q is given by Eq. (9.6.7). Using Eq. (10.1.30), the functions w, q and v can be expressed in terms of appropriate integrals. On integrating $\dot{S}(t) = p$, we get

$$S(t) = d + \int_0^t p(\tau)d\tau. \tag{10.1.31}$$

The results given below in Propositions 10.1.8 and 10.1.9 have been proved in [21] under suitable assumptions.

Proposition 10.1.8 (Existence and uniqueness). *The unique solution $(T(x,t), S(t))$ of the systems (10.1.23)–(10.1.28) can be constructed by Picard iteration method which can be started with any set of functions $\{T_0, w_0, q_0, v_0, S_0, p_0\}$ having bounded partial derivatives with respect to each of their arguments. If the starting solution satisfies the conditions*

$$\phi_x(0) = f(0, w_0(0)); \quad \phi_x(d) = q_0(d,0) = v_0(0); \quad \dot{S}_0(t) = p_0, \tag{10.1.32}$$

then each of the functions $T, w, \ldots, p$ obtained as solutions of Picard iteration satisfy a Lipschitz condition of the form

$$|f(t) - f(\tau)| < A(\sqrt{t} + \sqrt{\tau})^{-1}|t-\tau|, \ A \text{ constant}, \tag{10.1.33}$$

with respect to each of their arguments.

Here, T_0, w_0, q_0, v_0, S_0 and p_0 are some initial approximations of the functions $T, w, \ldots, p$, respectively.

Proposition 10.1.9. *The solution (T, w, q, v, S, p) obtained by Picard iteration is stable relative to small variations of all the 'data' of the problem.*

The existence of a local-in-time solution has been proved by establishing the convergence of iterations and uniqueness has been proved by showing that the sequences $\{T_n\}, \{w_n\}, \ldots, \{p_n\}$ obtained by iterations are not only uniformly bounded and equicontinuous but are also uniformly convergent. This will imply convergence of the entire iteration process to a solution of the system of integral equations of the problem.

The local-in-time (short-time) solution cannot be extended to any arbitrarity preassigned time interval without assuming the monotonicity of the free boundary and imposing additional conditions of boundedness on the data. The additional assumptions (cf. [21]) imply that if the solution is valid in the time interval $[0, t_0]$, then it can be extended to $[t_0, t_1]$, where $\Delta t = t_1 - t_0$ is dependent only on the magnitude of $T(x,t)$ and its first three derivatives with respect to x at $t = t_0$ and not on the time t, for which the solution has already been constructed.

The approach to prove existence, uniqueness and stability of the solution of the free boundary problem considered in [59], consists of proving the convergence of a sequence of approximating solutions. The formulation of the problem considered in [59] is as follows:

$$\mathcal{L}T = T_{xx} - T_t = q(x,t), \text{ in } D_{t_*} = \{(x,t) : 0 < x < S(t),\ 0 < t < t_*\}, \tag{10.1.34}$$

$$T(x,0) = \phi(x), \quad 0 \le x \le S(0) = d > 0, \tag{10.1.35}$$

$$T(0,t) = u(t), \quad 0 < t < t_*, \tag{10.1.36}$$

$$T(S(t),t) = f(S(t),t), \quad 0 < t < t_*, \tag{10.1.37}$$

$$T_x(S(t),t) = \lambda(S(t),t)\dot{S}(t) + \mu(S(t),t), \quad 0 < t > t_*. \tag{10.1.38}$$

The problem (10.1.34)–(10.1.38) will be called Problem (P1). If instead of Eq. (10.1.36), we have

$$T_x(0,t) = g(T(0,t),t), \quad 0 < t < t_*, \tag{10.1.39}$$

and all other equations remain the same as in Problem (P1), then we shall call it Problem (P2). The boundary condition (10.1.37) is of 'Cauchy-type'. Several particular cases of Problems (P1) and (P2) have been studied in the literature which have been briefly reviewed in [59]. Let Ω be the quarter-plane $\{(x,t) : 0 < x < \infty, 0 < t < \infty\}$. The input data and the coefficients in Eqs (10.1.34)–(10.1.39) satisfy the following assumptions.

(A) $q(x,t)$ is locally Hölder continuous in $\bar{\Omega}$ with respect to x (or t), and

$$|q(x,t)| \leq Q, \quad (x,t) \in \overline{\Omega}.$$

(B) $f(x,t)$ is continuous and bounded in $\overline{\Omega}$ together with f_x, and the difference $f_{xx} - f_t$ is bounded and locally Hölder continuous in $\bar{\Omega}$ with respect to x (or t).

(C_1) $u(t)$ is (piecewise) continuous for $t \geq 0$ and

$$|u(t)| \leq \Phi, \quad t \geq 0.$$

(C_2) $g(y,t)$ is Lipschitz continuous with respect to y, i.e.

$$|g(y_1,t) - g(y_2,t)| \leq m_g\, |y_1 - y_2|,$$

uniformly with respect to $t \geq 0$ and it satisfies one of the conditions (α_1), (α_2) and one of the conditions (β_1), (β_2) listed below.

(α_1) There exists a constant $Y_1 > \max\left\{Md, \sup_{\Omega} f(x,t)\right\}$ such that

$$g(Y_1,t) \geq 0, \quad t \geq 0.$$

For M, see (F) given below.

(α_2) There exist two constants Y' and G' such that

$$g(y,t) \geq G', \text{ for } y \geq Y', \quad t \geq 0.$$

(β_1) There exists a constant $Y_2 < \min\{-Md, \inf_{\Omega} f(x,t)\}$ such that

$$g(Y_2,t) \leq 0, \quad t \geq 0.$$

(β_2) There exist two constants Y'' and G'' such that

$$g(y,t) \leq G'', \text{ for } y \leq Y'', \quad t \geq 0.$$

(D) $\lambda(x,t)$ is continuous in $\overline{\Omega}$ together with its first derivatives and

$$|\lambda(x,t)| \geq \lambda_0 > 0, \quad (x,t) \in \overline{\Omega}.$$

(E) $\mu(x,t)$ is continuous in $\overline{\Omega}$, uniformly Lipschitz continuous w.r.t. x in bounded sets, and there exists a constant P such that

$$|\mu(x,t)| \le P, \quad (x,t) \in \overline{\Omega}.$$

(F) $\phi(x)$ is (piecewise) continuous in $[0,d]$ and a positive constant M exists such that

$$|\phi(x) - f(d,0)| \le M(d-x), \quad x \in [0,d],$$

with (for Problem P1)

$$Md \ge \Phi + |f(d,0)|.$$

By redefining the temperature as $T(x,t) - f(x,t)$ in Eqs (10.1.34)–(10.1.39), Eq. (10.1.37) can be written as $T(S(t),t) = 0$. The new temperature will also be denoted by $T(x,t)$. The assumptions on the various functions have been made in such a way that they still remain valid after this transformation.

If $\lambda \equiv 0$, then we have an implicit free boundary condition and either this condition can be converted to an explicit condition (cf. Section 3.3.1) which can then be studied by the method suggested in [59] or $\lambda = 0$ may yield nonexistence, nonuniqueness or instability of the solution on the input data. The main existence result of [59] is that either the solution exists globally in time, i.e. $t_* = +\infty$ or one of the following cases must occur for some $\hat{t} < \infty$:

$$\text{(i)} \lim_{t \to \hat{t}-0} S(t) = 0, \quad \text{(ii)} \lim_{t \to \hat{t}-0} |\dot{S}(t)| = +\infty. \tag{10.1.40}$$

The free boundary condition (10.1.38) can be reformulated as

$$T_x(S(t),t) = (d/dt)\{V(S(t),t) - W(t;S)\}, \tag{10.1.41}$$

where

$$V(x,t) = \int_0^x \lambda(\xi,t)d\xi, \quad (x,t) \in \overline{\Omega}, \tag{10.1.42}$$

and for any $S(t) \in C^1(0,t_*) \cap C[0,t_*]$,

$$W(t;S) = \int_0^t \{V_\tau(S(\tau),\tau) - \mu(S(\tau),\tau)\}\, d\tau, \quad t \in [0,t_*]. \tag{10.1.43}$$

If in Eq. (4.4.20), we take P as the heat operator, $P^* = \partial^2/\partial x^2 + \partial/\partial\tau$ and $v(x,t) = V(x,t)$, then on integrating Eq. (4.4.20), we get the following equation for Problem (P1):

$$\begin{aligned}\frac{1}{2}V^2(S(t),t) - \frac{1}{2}V^2(d,0) &= \int\int_{D_{t_*}} \{V(x,\tau)q(x,\tau) - T(x,\tau)\,[\lambda_x(x,\tau) + V_\tau(x,\tau)]\}\, dxd\tau \\ &+ \int_0^{S(t)} V(x,t)T(x,t)dx - \int_0^d V(x,0)\phi(x)dx \\ + \int_0^t V(S(\tau),\tau)\,\{V_\tau(S(\tau),\tau) &- \mu(S(\tau),\tau)\}\, d\tau - \int_0^t \lambda(0,\tau)u(\tau)d\tau, t \in [0,t_*].\end{aligned} \tag{10.1.44}$$

For Problem (P2), take $v = 1$ in Eq. (4.4.20) and P and P^* as above. On integrating, we get

$$V(S(t),t) - V(d,0) = \int\int_{D_{t_*}} q(x,\tau)dxd\tau + \int_0^{S(t)} T(x,t)dx - \int_0^d \phi(x)dx$$
$$+ \int_0^t g(T(0,\tau),\tau)d\tau + \int_0^t \{V_\tau(S(\tau),\tau) - \mu(S(\tau),\tau)\}\, d\tau,\; 0 \le t \le t_*. \tag{10.1.45}$$

Each solution of Problem (P1) satisfies the integral equation (10.1.44). Similarly each solution of Problem (P2) satisfies the integral equation (10.1.45). If $S(t)$ is Lipschitz continuous and $T_x(x,t)$ is continuous up to $x = S(t)$ for $t > 0$, then Eq. (10.1.38) is also satisfied by the solution (T,S) of Eqs (10.1.34)–(10.1.37), (10.1.44), (10.1.45).

Approximate solutions to Problem (P1) (and similarly for Problem P2) can be obtained by induction starting with $S_1(t) = d$ and the temperature determined by the solution of the following system of equations:

$$\mathcal{L}T_k = q(x,t),\text{ in } D_{t_*^{(k)}}^{(k)} = \{(x,t) : 0 < x < S_k(t), 0 < t < t_*^{(k)}\}, \tag{10.1.46}$$

$$T_k(x,0) = \phi(x), \quad 0 \le x \le S_k(0) = d, \tag{10.1.47}$$

$$T_k(0,t) = u(t), \quad 0 < t < t_*^{(k)}, \tag{10.1.48}$$

$$T_k(S_k(t),t) = 0, \quad 0 < t < t_*^{(k)}, \tag{10.1.49}$$

$$\lambda(S_k(t),t)\dot{S}_{k+1}(t) = T_{k,x}(S_k(t),t) - \mu(S_k(t),t), \quad 0 < t < t_*^{(k+1)} \le t_*^{(k)}, \tag{10.1.50}$$

$$S_{k+1}(0) = d, \quad k = 1,2,\ldots \tag{10.1.51}$$

The time $t_*^{(k)}$ is the supremum of the values of τ for which $S_k(\tau) > 0$, $S_k(\tau) \in C^1(0,\tau)$. Note that, for any k such that $t_*^{(k)} > 0$, the problem (10.1.46)–(10.1.51) has a unique solution (cf. [284]) and $T_{k,x}$ is continuous up to the curve $x = S_k(t)$.

The next step in the existence proof is to show the convergence of the sequence of approximate solutions. It can be shown that there exist two constants, t_0 and A_0 such that

$$t_*^{(k)} \ge t_0, \quad (t_0 < d/A_0) \text{ and } |\dot{S}_k| \le A_0, \quad 0 < t \le t_0; \quad k = 1,2,\ldots, \tag{10.1.52}$$

and the sequence $\{S_k(t)\}$ converges uniformly in $[0,t_0]$ to a positive function $S(t)$ with $S(0) = d$ such that

$$|S(t_1) - S(t_2)| \le A_0\,|t_1 - t_2|\,,\text{ for } t_1,t_2 \in [0,t_0]\,. \tag{10.1.53}$$

Moreover, the sequence $\{T_k(x,t)\}$ converges uniformly in $R_{t_0} = [0, d + A_0 t_0] \times [0,t_0]$ to a function $T(x,t)$ satisfying Eqs (10.1.34)–(10.1.37) with the above $S(t)$. If this pair $(T(x,t),S(t))$ satisfies Eq. (10.1.38) or (10.1.44), then the existence of the solution is established for Problem (P1). This has been done in [59] by substituting (T_k,S_k) in Eq. (10.1.44) and taking the limit as $k \to \infty$. For Problem (P2) also the same procedure can be followed.

It can be shown that for $0 \le t \le t_0$, we have

$$|T(x,t_0)| \le \alpha_1(S(t_0) - x), \quad \alpha_1 S(t_0) \ge \Phi, \quad S(t_0) \ge d - A_0 t_0 > 0. \tag{10.1.54}$$

Here, α_1 is a constant independent of the index k. In view of Eq. (10.1.55) (given below), the solution can be extended to a time interval $t_0 \le t \le t_1$. For this, we take $S(t_0) = d_1 > 0$ and a new initial temperature in $0 \le x \le d_1$. This process can be extended to other larger time intervals.

Let F_0 be the class of functions which are Lipschitz continuous to which $S(t)$ also belongs. It has been shown that if T and S belong to the class F_0, then the solution depends continuously on the data and the coefficients. The uniqueness of the solution in the class F_0 is an immediate consequence of the stability of the solution.

The local-in-time solution $(t_*, S(t), T(x,t))$ can be extended to larger time intervals provided some further assumptions are made such as: (1) $S(t)$ is nondecreasing in $(0, t_*)$, and (2) the data satisfies some sign constraints such as $\lambda < 0, \phi \ge 0, u \ge 0, \mu \ge 0$ and $q \le 0$ which hold in some time interval $(0, t_0)$, $t_0 < +\infty$. If these assumptions together with the assumptions (A)–(F) mentioned above hold, then the solution of Problem (P1) exists for $t > t_0$ and

$$0 \le \dot{S}(t) \le \hat{A}_0, \quad 0 \le t \le t_0 < \infty. \tag{10.1.55}$$

Here, $\hat{A}_0$ can be determined in terms of the data.

An ablation problem concerning melting of a solid in which the melt is instantaneously removed has been considered in [285, 286]. In order to obtain the formulation of this ablation problem, we take $q(x,t) = 0$ in Eq. (10.1.34), $g = g(t)$ in Eq. (10.1.39), $f(S(t),t) = 0$ in Eq. (10.1.37), retain Eq. (10.1.35) as it is, denote μ by $q(t)$ and $\lambda(x,t) = \lambda < 0$ in Eq. (10.1.38). The existence, uniqueness and stability results have been proved in [285, 286] using the approaches followed in [279–281]. Such results have been obtained for a more general problem in [59]. Some physical aspects of the ablation problem such as the time required for complete melting of the solid have been discussed in [286]. The conditions required for the uniqueness of the solution in a one-dimensional ablation problem with the heat input dependent on time and on melted depth have been studied in [287]. The formulation of this problem is similar to problem (3.1.13)–(3.1.16) except that the problem in [287] is one-dimensional. The heat flux $Q = Q(x,t)$, and the thermal conductivity and the specific heat are functions of temperature. Some examples have been constructed in which the solution is nonunique. The one-phase problem considered in [288] is concerned with a solidification problem in which the region $0 \le x \le d$ is initially ice with temperature $\phi(x) \le 0$ and a liquid at the melting temperature occupies the region $x > d$. The problem formulation is as follows:

$$\left.\begin{aligned} &T_{xx} = T_x,\ 0 < x < S(t),\ t > 0;\ T(x,0) = \phi(x),\ 0 \le x \le d, \\ &-\lambda\dot{S}(t) + T_x(S(t),t) = 0,\ 0 < t \le t_*,\ \lambda > 0;\ T(S(t),t) = 0,\ S(0) = d. \end{aligned}\right\} \tag{10.1.56}$$

At $x = 0$, either

$$T(0,t) = -f(t), \quad 0 < t \le t_*, \quad f(t) > 0, \tag{10.1.57}$$

is prescribed, or we have

$$T_x(0,t) = f(t) \quad 0 < t \le t_*. \tag{10.1.58}$$

To indicate the dependence of the solution on λ the solution will be written as $(T(x,t;\lambda), S(t;\lambda))$.

The main interest in [288] is to investigate the behaviour of the solution $(T(x,t;\lambda), S(t;\lambda))$ as $\lambda \to 0$ (latent heat tends to zero). By using appropriate Green's functions, the solution for the temperature derivative or the temperature can be written as in Eq. (10.1.17) or as in Eq. (10.1.30) depending on the boundary condition (10.1.57) or (10.1.58) (in this case we take Eq. 10.1.30). $S(t,\lambda)$ can be obtained from the integral equation given below (cf. [71])

$$\lambda(S(\lambda,t)^2 - d^2) = 2\int_0^t f(\tau)d\tau - 2\int_0^d x\phi(x)dx + 2\int_0^{S(t,\lambda)} xT(x,t,\lambda)dx. \tag{10.1.59}$$

It has been shown that for a fixed $t > 0$, $S(t,\lambda)$ obtained as a solution of Eqs (10.1.56)–(10.1.57) or of Eqs (10.1.56), (10.1.58), behaves as

$$S(t,\lambda) \sim [4t\log(1/\lambda)]^{1/2}, \quad \lambda \to 0. \tag{10.1.60}$$

The above result has been proved using monotone dependence theorem and some other results obtained in [289]. For a fixed $t > 0$, estimates of $S(\lambda,t)$ have also been obtained.

The problem considered in [290] can be described in terms of Eqs (10.1.34)–(10.1.38) (Problem P1) or Eqs (10.1.34)–(10.1.37), (10.1.39) (Problem P2) provided we take $d = 0$, i.e. $S(0) = 0$. The condition $S(0) = d > 0$ and the hypothesis of Lipschitz continuity of $\phi(x)$ at $x = b$ played a major role in [59] in proving the well-posedness of Problems (P1) and (P2). Both these assumptions essentially resulted in ensuring that $S(t)$ is Lipschitz continuous in $[0, t_*]$. If $d = 0$, then a different approach is needed. Some additional regularity conditions on q, f, λ, μ and u or g and sign constraints on the data and coefficients are needed so that the free boundary actually starts from $x = 0$ at $t = 0$. For the proof of existence of the solution, in addition to the assumptions (A), (B), (C_1), (D) and (E) mentioned earlier in this section, the following assumptions have been made.

Suppose a real number θ exists such that for $t \in [0,\theta]$ the following conditions are fulfilled:

(i) $$-\lambda^{(\theta)} \le \lambda(x,t) \le -\lambda_0 < 0, \quad x \ge 0, \quad 0 \le t \le \theta, \tag{10.1.61}$$

for some positive constants $\lambda^{(\theta)}$ and λ_0.

(ii) $$\mu(x,t) \ge 0, \text{ and } q(x,t) \le 0, \quad x \ge 0, \quad 0 < t \le \theta. \tag{10.1.62}$$

(iii) $$u(t) \ge 0, \quad u(t) \ne 0 \text{ in each neighbourhood of } t = 0. \tag{10.1.63}$$

Conditions (10.1.61)–(10.1.63) are sufficient to ensure that $T(x,t) \ge 0$ in $D_\theta = \{(x,t) : 0 < x < S(t), 0 < t < \theta\}$ and $\dot{S}(t)$ is monotonically increasing.

Let (S^d, T^d) be the solution of Eqs (10.1.34)–(10.1.38) in Problem (P1) or in Problem (P2) with $\phi(x) = 0$ in $0 \le x \le d$, for each $d \in (0,1)$. It has been shown in [290] that as $d \to 0$, the solution (S^d, T^d) converges to a solution of the system consisting of Eqs (10.1.34), (10.1.36)–(10.1.38) with $d = 0$. For the existence proof an approach similar to the one used in [59] has been used in [290] also. The case $d > 0$ considered in Eqs (10.1.34)–(10.1.38) was reconsidered in [290] and it was observed that it is possible to relax the condition (F) (given earlier in this section) and replace it by the condition (F') given below.

(F') $\phi(x)$ is piecewise continuous in $(0,d)$ and two positive constants M and α $(0 \le \alpha \le 1)$ exist such that

$$|\phi(x) - f(d,0)| \le M(d-x)^{\alpha}, \quad 0 \le x \le d, \tag{10.1.64}$$

in Problem (P1). The constant M is such that $Md^{\alpha} \ge \Phi + |f(d,0)|$.

The assumption (F') can be further relaxed and replaced by (F'') which is more general.

(F'') $\phi(x)$ is piecewise continuous and bounded in $[0,d]$.

Assume that in addition to Eqs (10.1.61)–(10.1.63), we have

$$\phi(x) \ge 0, \quad x \in [0,d]\,. \tag{10.1.65}$$

These restrictions ensure the monotonicity of $S(t)$.

It was observed that for proving the stability of the solution when $d = 0$, Lipschitz continuity of $S(t)$ (proved in the case $d > 0$) is not enough. When $d = 0$, another estimate for $\dot{S}(t)$ has been obtained as follows. There exist two constants δ and β $(\beta < 1/2)$ such that

$$|\dot{S}(t)| < \delta t^{-\beta}. \tag{10.1.66}$$

The one-phase problem (3.1.1)–(3.1.5) has been studied in [55] by reformulating it as two different problems. Let the problem discussed in Eqs (3.1.1)–(3.1.5) be called Problem (FM). If Eq. (3.1.1) is replaced by the equation

$$T_t - a^2(x,t,T)T_{xx} = q(x,t,T,T_x), \quad (x,t) \in \Omega(t_*), \tag{10.1.67}$$

$\psi(x,t) = 0$ in Eq. (3.1.4), and conditions (3.1.2), (3.1.3), (3.1.5) are retained, then the problem so obtained will be called Problem (FMP).

If only Eq. (3.1.5) is replaced by

$$\dot{S}(t) = P_t(S,T), \tag{10.1.68}$$

and Eqs (3.1.3)–(3.1.4) are retained, then we get a generalization of Problem (FM) and this more general problem with Eq. (10.1.68) will be called Problem (GM) in which P_t is a functional. Eq. (10.1.68) is more general than Eq. (3.1.5). Except the coefficient a, which has to be greater than zero, no other sign restriction has been imposed on the data for proving the existence, uniqueness and stability of the local-in-time solution. For bounds on the various functions, compatibility conditions and spaces to which various functions belong, the reader is referred to [55] as they will occupy considerable space.

The existence proof of the local-in-time solution of Problem (FMP) is based on a method of successive approximations whose convergence has been shown by an argument of contractive type. Under suitable assumptions (cf. [55]), there exists a solution $(t_*, S(t), T(x,t))$ of Problem (FMP) in which $S(t) \in H_{1+\alpha/2}(0,t_*)$ for any $\alpha \in (0,1)$. For $\nu \in (0,1)$, the space $H_{\nu}[b,d]$ is the space of all functions which are Hölder continuous with exponent

ν in $[b,d]$. If for $\nu \in (0,1)$, $G(x) \in H_\nu[b,d]$, then for some constant A and all ξ_1, $\xi_2 \in [b,d]$

$$|G(\xi_1) - G(\xi_2)| \leq A\,|\xi_1 - \xi_2|^\nu\,. \tag{10.1.69}$$

The norm of $G(x)$ in $H_\nu[b,d]$, $\nu \in (0,1)$ is defined as:

$$\|G(x)\|_{H_\nu} = \sup_{x\in[b,d]} |G(x)| + \sup_{x_1,x_2\in[b,d]} |G(x_1) - G(x_1)|\,/\,|x_1 - x_2|^\nu\,. \tag{10.1.70}$$

For the definition of a suitable norm in $H_{N+\nu}[b,d]$, $N > 0$, see Appendix B.

The construction of approximating solutions in [55] is similar to that described in Eqs (10.1.46)–(10.1.51). The problem corresponding to an approximating solution for any index k, can be easily formulated as a fixed domain problem, for example, if we put $y = x/S(t)$, then for $0 \leq x \leq S(t)$, we have $0 \leq y \leq 1$. By obtaining estimates for S_k, $\dot{S}_k$, v_k, $v_{k,y}$ (the subscript k stands for the kth approximating solution and v_k is the temperature in the new coordinate y), uniform interior Schauder estimates for v_k can be derived and it can be proved that the limit function v of v_k is the solution of the problem. Stability and uniqueness of the solution have also been established. Under suitable assumptions, the methods employed for proving results for Problem (FMP) can be extended to Problem (GM). By using the method discussed in [59] and obtaining uniform estimates, the local-in-time solution of Problem (FMP) can be extended to a solution valid in larger time intervals.

10.1.2 Infinite Differentiability and Analyticity of the Free Boundary

The differentiability of the free boundary has been discussed in [281] under assumptions of continuous differentiability of the initial and boundary data. Infinite differentiability of the free boundary in the one-dimensional Stefan problems has also been discussed in [291–293]. A simple proof of the infinite differentiability of $S(t)$ which is widely referred has been given in [294] and we give here the main steps in this proof.

The problem formulation is as follows. Let $S(t)$ be a continuous function in $0 < t \leq t_*$ with $S(0) = d$ and let

$$\left.\begin{aligned} &T_t = T_{xx} \text{ in } D_S; \quad T(x,0) = \phi(x),\ 0 \leq x \leq d; \quad T(0,t) = f(t),\\ &T(S(t),t) = 0; \qquad T_x(S(t),t) = -\lambda\dot{S}(t), \qquad \lambda > 0, \end{aligned}\right\} \tag{10.1.71}$$

$$D_S = \left\{(x,t) : 0 < x < S(t),\ 0 < t < t_{*,}\right\}. \tag{10.1.72}$$

It is not necessary to impose any specific conditions on ϕ and f as we are concerned with the solution in a neighbourhood of $x = S(t)$. The main result of [294] is the following proposition.

Proposition 10.1.10. *If the pair (S,T) satisfies Eq. (10.1.71), then $(S,T) \in C^\infty(\varepsilon, t_*)$ for any $\varepsilon > 0$.*

By using the transformation

$$\xi = x/S(t), \quad \tau = t, \tag{10.1.73}$$

a formulation of the Stefan problem (10.1.71) on the fixed domain $Q = (0,1) \times (0, t_*)$ can be obtained in which $x = S(t)$ becomes $\xi = 1$. This transformation is C^∞ with respect to x and C^1 with respect to t. If $v(\xi, t) = T(x, t)$, then

$$v_\tau = (1/(\hat{\sigma})^2) v_{\xi\xi} + (\sigma/\hat{\sigma})(\xi v_\xi), \text{ in } Q, \tag{10.1.74}$$

where

$$\sigma = \dot{S} \quad \text{and} \quad \hat{\sigma} = S(t). \tag{10.1.75}$$

The initial and boundary conditions for v can be easily written. At $x = S(t)$, we have $v(1,t) = 0$. The Stefan condition is transformed into the condition

$$\sigma(t) = -(1/\hat{\sigma}(t))(v_\xi(1,t))/\lambda, \quad 0 < t < t_*. \tag{10.1.76}$$

The following proposition has been proved in [295] and with its help, the proof of Proposition 10.1.10 has been completed in [294].

Proposition 10.1.11. *If $T(x,t)$ is a bounded solution of the equation $T_t = aT_{xx} + bT_x + eT$, $(a > 0)$ in the region Q such that $T(1,t) = 0$ for $0 \le t \le t_*$ and if the coefficients a, b and e belong to $H_\alpha(Q)$ $(\alpha > 0)$, then for any $\varepsilon > 0$, we have $T(x,t) \in H_{\alpha+2}(Q_\varepsilon)$, where $Q_\varepsilon = (\varepsilon, 1) \times (\varepsilon, T)$. Here H_α stands for an anisotropic Hölder space (cf. [295]).*

Anisotropic Hölder spaces enter in the a priori estimates of Schauder type for the parabolic equations. If for some $\alpha > 0$, $\sigma \in H_\alpha(Q_\delta)$, $\delta \ge 0$, then from Eq. (10.1.76) $\hat{\sigma} \in H_{\alpha+1}(Q_\delta)$ (σ can be taken as a function on Q_δ although it is not a function of x). From Proposition 10.1.11, we conclude that $v \in H_{\alpha+2}(Q_{\delta+\epsilon})$ as coefficients in Eq. (10.1.74) belong to $H_\alpha(Q_\delta)$. Further, $v_\xi \in H_{\alpha+1}(Q_{\delta+\epsilon})$. From Eq. (10.1.76), $\sigma \in H_{\alpha+1}(Q_{\delta+\varepsilon})$. If $\sigma \in H_{\alpha+1}(Q_{\delta+\varepsilon})$, then from Eq. (10.1.74) and Proposition 10.1.11 $v_\xi \in H_{\alpha+2}(Q_{\delta+\varepsilon'})$ and if $v_\xi \in H_{\alpha+2}(Q_{\delta+\varepsilon'})$, then $\sigma \in H_{\alpha+2}(Q_{\delta+\varepsilon'})$ from Eq. (10.1.76). Proceeding inductively in this way it can be proved that $\sigma \in C^\infty(\varepsilon, t_*)$, for any $\varepsilon < t_*$. To start the induction process one needs an initial estimate for the Hölder continuity (see Eq. 10.1.69) of v_ξ which has been obtained in [295]. The infinite differentiability of the temperature follows from the Stefan condition. It has been mentioned in [55] that if a, q and ϕ in Eqs (3.1.1)–(3.1.5) are infinitely differentiable, then $S(t)$ is infinitely differentiable.

One of the main results in [296] is that if $f(t)$ in Eq. (10.1.71) is an analytic function in $0 \le t \le t_*$, then $S(t)$ is also analytic in $0 \le t \le t_*$. Assume that in Eq. (10.1.71), $f \ge 0, f(0) = \phi(0)$, $\phi \ge 0$, $\phi(b) = 0$, $\lambda = 1$, and $f(t)$ is analytic in $0 < t < t_*$. The sign constraints ensure that a unique solution exists in some time interval. If f and ϕ are continuously differentiable, then it can be proved that $\dot{S}(t)$ is continuous. Some of the steps in the proof of analyticity in [296] are: (1) Converting the free boundary problem into a fixed domain problem in which $S(t)$ is fixed at $y = 1$ (see Eq. 10.1.77) and then an application of Proposition 10.1.11. This will ensure that the temperature $v(y, \tau)$ is a C^∞-function in $0 \le y \le 1$, $0 \le \tau \le \tau_0$ and $S(\tau)$ is C^∞ $[0, \tau_0]$. (2) Obtaining appropriate estimates for the derivatives of v and S of all orders.

We state below the transformations used in [296] which converts the Stefan problem into an appropriate parabolic problem on a fixed domain for which Proposition 10.1.11 is applicable. Let

$$y = x/S(t), \quad \tau = \int_0^t \frac{d\alpha}{S^2(\alpha)}, \quad \tau_0 = \int_0^{t_*} \frac{d\alpha}{S^2(\alpha)}, \tag{10.1.77}$$

$$v(y,\tau) = T(x,t) - (1 - x/S(t))f(t). \tag{10.1.78}$$

It can be seen that

$$\frac{dt}{d\tau} = S^2(t), \text{ and } S(t)\dot{S}(t) = f(t) - v_y(1,\tau), \tag{10.1.79}$$

$$\dot{S}(t) = -T_x(S(t),t) = \left[-v_y(1,\tau) + f(t)\right]/S(t), \tag{10.1.80}$$

$$v_{yy} - v_\tau = y\left[v_y(1,\tau) - f(t)\right]\left[v_y - f(t)\right] + (1-y)f'(t)S^2(t), \quad 0 < y < 1, \quad 0 < \tau < \tau_0, \tag{10.1.81}$$

$$v(0,\tau) = v(1,\tau) = 0, \quad 0 < \tau < \tau_0. \tag{10.1.82}$$

All the coefficients in Eq. (10.1.81) are not known but we know their behaviour. By the application of Proposition 10.1.11, it can be concluded that $v(y,\tau)$ and $S(\tau)$ are C^∞-functions for $0 \le y \le 1, 0 \le t \le t_*$. The proof of analyticity of $S(t)$ requires estimates of the derivatives of all orders of several quantities such as $v(y,\tau)$, $v_y(y,\tau)$, $v_{yy}(y,\tau)$ and $S(t(\tau))$. It has been proved that if $f(t)$ is analytic in $0 \le t \le t_*$, then $S(t)$ is analytic in $0 < t \le t_*$ (cf. [296]). Firstly, Schauder type interior-boundary estimates are obtained for the heat equation $T_t - T_{xx} = 0$ in the rectangle $-1 < x < 1, 0 < t < t_*$. The way in which negative powers of t (as $t \to 0$) enter into the estimates is crucial. These estimates are then used in proving further results concerning analyticity of $S(t)$. The estimates for $S(t(\tau))$ imply that $S(t(\tau))$ is analytic in τ and so is $S^2(t)$. The first of Eq. (10.1.79) implies that $t = t(\tau)$ is analytic in τ and $\tau = \tau(t)$ is analytic in t. Writing $S(t) = S(t(\tau(t)))$, we conclude that $S(t)$ is analytic in t.

The analyticity of $S(t)$ at $t = 0$ has been studied in [297]. At $t = 0$, the region $x > 0$ is at the melting temperature zero and the region $x < 0$ is occupied by a warm liquid. We consider the following problem formulation

$$\left.\begin{array}{l} T_{xx} - T_t = 0, \ -\infty < x < S(t), \ t > 0; \ T(x,0) = f(x), \ -\infty < x < 0, \\ \quad T(S(t),t) = 0, \ t > 0; \ \dot{S}(t) = T_x(S(t),t) - \phi(t), \ t > 0; \ S(0) = 0. \end{array}\right\} \tag{10.1.83}$$

In [21], the problem formulated in Eq. (10.1.83) has been named as *Cauchy–Stefan problem* and the problem formulated in Eq. (10.1.71) has been named *Dirichlet–Stefan problem*. It has been proved in [297] that if $\phi(t)$ in Eq. (10.1.83) is analytic in $(0,t_0)$, $t_0 > 0$, and if the solution of Eq. (10.1.83) exists in $(0,t_1)$, $t_1 > 0$, then $S(t)$ is analytic in $(0,\hat{t})$, $\hat{t} = \min(t_0,t_1)$. For many practical purposes, we require polynomial approximations of $S(t)$ in terms of the initial data. The following results about the analyticity of $S(t)$ have been established in [297].

Proposition 10.1.12. *Assume that $\phi(t)$ is analytic with respect to $\sqrt{t}$ in $[0,t_0]$, for some $t_0 > 0$ and f is an entire function of x such that*

$$f(0) = 0, \text{ and } \left|f'(x)\right| \le \mathrm{M}\exp(\alpha x^2), \tag{10.1.84}$$

for some positive constants M and α. Then there exists $t_1 \in (0,t_0]$ such that Eq. (10.1.83) has a unique solution and the free boundary $x = S(t)$ has the following series representation

$$S(t) = \sum_{n=2}^{\infty} Y_n t^{n/2}/n!, \quad t \in [0, t_1). \tag{10.1.85}$$

In Eq. (10.1.85), Y_n is defined as follows:

$$y = \sqrt{t},\ Y(y) = S(y^2), \text{ and } Y(y) = \sum_{n=1}^{\infty} \frac{Y_n}{n!} y^n. \tag{10.1.86}$$

To prove Proposition 10.1.12, integral representations of $v(t) = T_x(S(t), t)$ and $S(t)$ (see Eqs 10.1.17, 10.1.21) have been used in conjunction with complex variable techniques. The transformation $y = \rho e^{i\phi}$ introduces a complex variable. Let $C_R = \{y : |y| < \rho_0\}$. The integral representations of $v(y)$ and $Y(y)$ define a mapping $\mathcal{P}$ on a set $M(\rho_0, N)$ of functions $w(y)$, $|w(y)| \le N$, $y \in C_R$ which are analytic in C_R and continuous in $\overline{C}_R$. It has been proved that $\mathcal{P}$ maps $M(\rho_0, N)$ into itself. This mapping is a contraction with respect to the distance metric. If $\overline{V}$ is the unique fixed point of the mapping $\mathcal{P}$, then the restriction of $\overline{V}$ to the real axis is an analytic function in $[0, \rho_0]$. Recursive relations have been developed to determine Y_n. It may be noted that for proving the analyticity, estimates of Y_n are not required. The uniqueness of the one-phase Stefan problem is well-known. The domain of analyticity can be extended under certain assumptions.

If $f(0) \neq 0$, then the following result holds good.

Proposition 10.1.13. *Assume that $\sqrt{t}\phi(t)$ is analytic with respect to $\sqrt{t}$ in $[0, t_0]$, that $\left|\phi(t)\sqrt{t}\right| \le \phi_0$, and $\lim_{x \to 0-} f(x) = f_0$. Then, if the other assumptions of Proposition 10.1.12 are satisfied, two positive constants ϕ_0^*, f_0^* can be found such that for any $\phi_0 \le \phi_0^*$, $|f_0| \le f_0^*$, the problem described in Eq. (10.1.83) has a unique solution in the class of solutions whose free boundary is analytic in $[0, t_1)$, for some $t_1 \in (0, t_0]$, with respect to $\sqrt{t}$. If $t^{1/2}\phi(t) \to 0$ as $t \to 0+$ and if $f_0 \ge 1$, then there is no solution of Eq. (10.1.83) such that $S(t)/t^{1/2}$ has a bounded limit as $t \to 0+$.*

In [297], the problem (10.1.71) has been considered in the region $-d \le x \le 0$ $(\lambda = -1)$ also. For an unbounded region it was assumed earlier that $f(0) = 0$ but for the problem in a bounded domain this assumption has been relaxed and the analyticity of the free boundary has been proved in $(0, t_0)$ under the assumption that $|f(0)| < f_0^*, f_0^* > 0$ and $f(t)$ is analytic for $t > 0$. The proof of this result is on the same lines as the proof of Proposition 10.1.12.

The analyticity of the free boundary in the one-phase Stefan problem, with strong nonlinearity formulated in Eqs (10.1.23)–(10.1.27), has been discussed in [298]. The main result of [298] is as follows.

Proposition 10.1.14. *The free boundary $S(t)$ is a holomorphic function in some neighbourhood $\cup(t_*) \subset D(\hat{t})$ of the interval $0 < t < t_*$, $\forall\, t_* < \hat{t}$, where $D(t_*) = \{t = \rho e^{i\alpha} \in \mathbb{C}; 0 < \rho < t_*, |\alpha| < \pi/32\}$, and $\hat{t}$ is the supremum of all $t_* > 0$ such that there exists a solution to the problem on $[0, t_*]$.*

The proof of this proposition is based on the application of Banach contraction mapping theorem to the system of integral equations obtained in the method of Picard iteration (see Eqs 10.1.29–10.1.31) and extended into the complex plane. The possibility of applying this principle follows from a priori estimates of the heat potentials and their variations in the complex plane. These estimates which are crucial for the proofs developed in [298] also provide generalization of the results obtained in [299]. In order to prove the analyticity of the free boundary in the nonlinear problem (10.1.23)–(10.1.27), the analyticity of Poisson's integral and of the volume heat potentials as well as their variations is needed along the free boundary. If in the complex (x, t) plane, x and t are independent, then the analyticity of Poisson's integral and of other quantities, generally does not hold up to the free boundary. However, the substitution $x = \lambda S(t)$, $\lambda \in [0, 1]$, helps in proving that the integrals are holomorphic in the sector $D(t_*)$, ($t_* > 0$ and small enough) when the integrals are considered as functions of t.

By representing temperature in an infinite series of integrals of error functions, and the free boundary in an infinite series in positive integral powers of $\sqrt{t}$, series solutions of temperatures and the free boundary in a one-dimensional Stefan problem have been obtained in [300]. Proof of the convergence of series expansions considered have been developed in [300] and also in several other works of the same author mentioned in [300]. It has been remarked in [297, 298] that the analyticity proofs are unconvincing. A simple proof has been developed in [301] which shows that there are infinitely many temperature solutions satisfying all the equations of the problem considered in [300]. All these temperatures give the same free boundary which has been obtained in [300]. In view of the nonuniqueness of the solution of the temperature in [300], any proof about the convergence of the series does not seem to be important.

10.1.3 Unilateral Boundary Conditions on the Fixed Boundary: Analysis Using Finite-Difference Schemes

The one-phase problem considered in [302] is concerned with the melting of a solid and its formulation can be obtained if some changes are made in the formulation given in Eq. (10.1.56). Take $\lambda = -1$ and prescribe the following unilateral boundary condition at $x = 0$:

$$T_x(0, t) \in \gamma(T(0, t)), \quad t > 0. \tag{10.1.87}$$

Here, γ is a maximal monotone graph (see Appendix B) in R^2 with $\gamma(\alpha) \ni 0$ for some nonnegative constant α. We give below an example of a unilateral boundary condition.

Unilateral boundary condition at $x = 0$ Let $T(0, t) \geq 2$; $T_x(0, t) \leq -3$, ($T = 2$); $T_x(0, t) = -3$, ($2 < T < 3$); $T_x(0, t) = T - 6$, ($T \geq 3$). Define $\gamma(T)$ as:

$$\left.\begin{aligned} \gamma(T) &= \emptyset, \ \ T < 2, \\ &= (-\infty, -3], \ \ T = 2, \\ &= -3, \ \ 2 < T < 3, \\ &= T - 6, \ \ T \geq 3. \end{aligned}\right\} \tag{10.1.88}$$

Signorini-type boundary condition at $x = 0$

$$T(0, t) \geq 2, \ T_x(0, t) \leq 0 \text{ and } T_x(0, t)\ (T(0, t) - 2) = 0. \tag{10.1.89}$$

The unilateral boundary conditions model several physical situations in controlling the temperature of a body during heating or cooling at the fixed boundary. The problem described in Eq. (10.1.56) with $\lambda = -1$ together with the boundary condition (10.1.87) at $x = 0$ will be called Problem (Y). The existence and uniqueness of the global-in-time solution of Problem (Y) have been investigated in [302]. For the analysis it will be assumed that

$$\phi(x) \geq 0, \text{ is bounded and continuous a.e. for } x \in [0, d]. \tag{10.1.90}$$

We introduce the following notations:

$$\left.\begin{aligned} D &= \{(x,t) : 0 < x < S(t), 0 < t \leq t_*\}, \; \overline{D} = \text{the closure of } D, \\ D^S &= \{(x,t) : 0 < x < S(t),\; 0 < t \leq t_*\}, \\ Z &= \{x \in [0,d] : x \text{ is a point of the discontinuity of } \phi\} \times \{0\}. \end{aligned}\right\} \tag{10.1.91}$$

Definition 10.1.1. The pair (T, S) is a solution of Problem (Y) if the following conditions are satisfied.

(1) $S(0) = d$, $S(t) > 0$ for $t > 0$, $S \in C[0, t_*] \cap C^\infty(0, t_*)$.

(2) T is bounded on $\overline{D}$, $T \in C^\infty(D^S) \cap C(\overline{D} - Z)$,

$$\int_\tau^{t_*} \int_0^{S(t)} T_{xx}(x,t)^2 dxdt < \infty, \text{ for each } \tau \in (0, t_*).$$

(3) The pair (T, S) satisfies the heat equation, initial condition and the free boundary conditions in Eq. (10.1.56) with appropriate changes as described above.

(4) For a.a. $t \in [0, t_*)$, Eq. (10.1.87) is satisfied.

The term 'solution' has been used in the above sense in [302]. The main result of [302] is the following proposition.

Proposition 10.1.15. *If $d > 0$, and ϕ satisfies Eq. (10.1.90), then there exists a unique solution (S, T) of Problem (Y) satisfying*

$$\int_0^{t_*} \int_0^{S(t)} t T_{xx}^2 \, dxdt + t_* \int_0^{S(t_*)} T_x^2 dx < +\infty. \tag{10.1.92}$$

Let $d > 0$. The existence proof in this case consist of the following steps.

(1) First an 'implicit' finite-difference discretization of equations in Problem (Y) is done in which the mesh size is of uniform width Δx and time steps $\{\Delta t_n\}$, $n = 1, 2, \ldots$ are of variable size such that the free boundary after time $t_n = \cup_{k=1}^n \Delta t_k$, $n = 1, 2, \ldots$, is at $x_n = n\Delta x$, $n = 1, 2, \ldots$, i.e. $S(t_n) = x_n$. Thus the position of the free boundary is always known. The variable time step can be obtained approximately from the Stefan condition $\dot{S} = -T_x(S(t), t)$.

(2) The unilateral boundary condition is handled as follows. The discretization of the unilateral boundary condition will have the form

$$\frac{T_1^n - T_0^n}{\Delta x} \in \gamma(T_1^n). \tag{10.1.93}$$

Here, the superscript n stands for the temperature at time t_n and the subscript 0 and 1, stand for temperatures at $x = 0$ and $x = \Delta x$, respectively. Since γ is a maximal monotone graph, $(I + \Delta x\gamma)^{-1}$, $\Delta x > 0$, is a contraction mapping from R to itself with $D((I + \Delta x\gamma)^{-1}) = R$ (D stands for the domain and I for the identity map) and $(I + \gamma)(u) = \{u + f, f \in \gamma(u)\}$ (cf. [303]). It can be shown that Eq. (10.1.93) is equivalent to $T_0^n = (I + \Delta x\gamma)^{-1}(T_1^n)$. If instead of Eq. (10.1.93), we take

$$T_0^n = \xi \in R, \tag{10.1.94}$$

and study the discretized problem with Eq. (10.1.94), then this problem has a unique solution (cf. [304]). We define a mapping P as:

$$P : R \ni \xi \mapsto (I + \Delta x\gamma)^{-1}(T_1^n(\xi)) \in R. \tag{10.1.95}$$

It can be shown that P is a contraction. Therefore, for each J, $1 \leq J \leq m$ (m is the total number of mesh points in space), T_J^n is the unique solution of the discretized Problem (Y) with condition (10.1.93).

(3) The next step is to obtain suitable estimates of T_J^n and $S_n = S(t_n)$. Further conditions are imposed on $\phi(x)$, and $\phi(x)$ satisfies one of the two conditions given below.

(A1) Let $\phi(x)$ satisfy the condition (10.1.90). Further, for some positive constant β, let $\phi(x) \leq \beta(d - x)$, for $x \in [0, d]$, $d > 0$. Note that (A1) $\Rightarrow$ (10.1.90).

(A2) Let $\phi(x)$ satisfy the condition (A1). Further, $-\beta x + \delta^1 \leq \phi(x) \leq \beta x + \delta^2$, $\delta^1, \delta^2 \in D(\gamma)$. $(A2) \Rightarrow (A1) \Rightarrow$ (10.1.90).

(4) Extend the discretization in space suitably to the region $\overline{G}$, where $G = (0, \infty) \times (0, t_*]$. By using *Ascoli–Arzela theorem* [58], it has been proved that $T_J(x, t)$ (see [302] for the construction of $T_J(x, t)$ on G) converges uniformly on compact subsets in G to a function $T(x, t) \in C(\overline{G})$. The numerical solution should converge to the solution of Problem (Y). The convergence of the numerical solution to $T(S(t), t)$ and $T(x, 0)$ (for those x at which $\phi(x)$ is continuous) has been proved by using estimates of the absolute values of T_J^n and S_n and imposing a further condition on $\phi(x)$. In addition to (A1), $\phi(x)$ is such that

$$\phi \in C^3\left[d', d\right] \text{ for some } d',\ 0 \leq d' < d. \tag{10.1.96}$$

Existence of the solution under the condition (10.1.90) has also been discussed. The proof of the convergence of the numerical solution to the solution of Problem (Y) satisfying the unilateral boundary condition and the Stefan condition requires L^2-estimates of the finite difference solution. It has been shown that $S(t) \in C^1(0, t_*)$ and $\dot{S}(t) = -T_x(S(t), t)$. As discussed in [294], under these conditions $S(t) \in C^\infty(0, t_*]$ and $T \in C^\infty(D^S)$.

To prove the uniqueness of Problem (Y), first the existence of the unique solution of temperature in an auxiliary problem which we call Problem (M) is proved. In Problem (M), $S(t)$ is assumed to be known and $S(t)$ is a nondecreasing function, positive for $t > 0$, and $S(t) \in C[0, t_*] \cap C^{0,1}(0, t_*]$. $C^{0,1}(0, t_*]$ denotes the space of Lipschitz continuous functions on $(0, t_*]$. In Problem (M), all the equations of Problem (Y) are present except the Stefan condition ($S(t)$ is known). By using an integral equation formulation as in Eq. (10.1.59), it can be proved that if $T(x,t)$ is the solution of Problem (M), and $S(t) \in C[0,t_*] \cap C^{0,1}(0,t_*]$, then $T_x(S(t),t) = -\dot{S}(t)$, $t \in (0,t_*]$.

The following comparison results have been proved.

Proposition 10.1.16. *If $\phi_1 \le \phi_2$ and $0 \le d_1 \le d_2$, then $S_1 \le S_2$ where (S_i,T_i), $i = 1,2$ are the solutions of Problem (Y) corresponding to the data $\{d_i,\phi_i\}$.*

The uniqueness of $S(t)$ can be easily proved by using the argument that $\phi_1 = \phi_2 \Rightarrow \phi_1 \ge \phi_2$ and it also implies $\phi_1 \le \phi_2$ and applying Proposition 10.1.16.

Proposition 10.1.17. *Let T and $\hat{T}$ be two solutions of Problem (M) corresponding, respectively, to the data $\{S(t),T(x,0)\}$ and $\{\hat{S}(t),\hat{T}(x,0)\}$. Let $d > 0$ and ϕ satisfies Eq. (10.1.90). Further, let $S(t) \le \hat{S}(t)$, $T(x,0) \le \hat{T}(x,0)$ and $\hat{T}(S(t),t) \ge T(S(t),t) = 0$. Then we have $0 \le \hat{T} - T \le \max\{\|\hat{T}(\cdot,0) - T(\cdot,0)\|_{L^\infty}, \|\hat{T}(S(t),t)\|_{L^\infty(0,t_*)}\}$ in $\overline{D} = \{(x,t) : 0 \le x \le S(t), 0 \le t \le t_*\}$, where $L^\infty = L^\infty(0,d)$, $S(0) = d$. When $d = 0$, we take $\|\hat{T}(\cdot,0) - T(\cdot,0)\|_{L^\infty} = 0$.*

In the problem considered in [302] (in which $d > 0$) take $d = 0$ and this case has been considered in [305]. The existence of a unique solution has been proved. The asymptotic behaviour of the solution as $t \to \infty$ has also been investigated. The following results have been established.

Proposition 10.1.18. *Let $d = 0$ and γ satisfies the assumptions (a) and (b) given below.*

(a) $\gamma^{-1}(0) \cap [0,\infty)$ *is not an empty set.*

(b) $\gamma(0) \subset (-\infty,0)$.

Then there exists a unique solution (S,T) of Problem (Y) with $d = 0$ and the following results hold.

(i) $S \in C[0,\infty) \cap C^\infty(0,\infty)$, and $S(t)$ is nondecreasing in t,

(ii) $0 < T(x,t) \le \alpha$, in $\overline{D}$,

(iii) $\left|T(x',t) - T(x,t)\right| \le C_\sigma \left|x' - x\right|$, on $\overline{D} \cap \{t \ge \sigma\}$.

Here, $\alpha = \mathrm{Proj}_{\gamma^{-1}(0)}(0)$, i.e. $\alpha \ge 0$ is an element of $\gamma^{-1}(0)$ which has minimum absolute value. The assumption (b) implies that $\alpha > 0$.

Proposition 10.1.19. *Let $\alpha > 0$ and $d \ge 0$ and $\phi(x)$ satisfies assumptions in Eq. (10.1.90). Then we have*

(iv) $\lim_{t\to\infty} T(x,t) = 0$ *uniformly on any compact subset of* $[0,\infty)$,

(v) $\lim_{t\to\infty} S(t)/\sqrt{t} = \beta$,

where β is the unique solution of

$$\sum_{n=1}^{\infty} (n!/2n!)\beta^{2n} = \alpha. \tag{10.1.97}$$

When $\alpha = 0$ and $d > 0$, $\lim_{t\to\infty} T(x,t) = 0$ uniformly on $(0,t_*)$ and $\lim_{t\to\infty} S(t) = S^*$ exists such that $d \le S^* \le d + \int_0^d \phi(x)dx$.

10.1.4 Cauchy-Type Free Boundary Problems

In the Cauchy-type free boundary problems, the temperature and its normal derivative at the free boundary are prescribed as functions of t and $S(t)$. Such problems have been discussed earlier in Section 3.3.1. We shall discuss now some more general Cauchy-type problems considered in [306].

Problem (C) Find a triple (t_*, S, T), where (S, T) is a classical solution of the following problem:

$$a(x,t,T,T_x,S)T_{xx} - T_t = q(x,t,T,T_x,S), \text{ in } D_{t_*} = \{(x,t) : 0 < x < S(t),\ 0 < t < t_*\}, \tag{10.1.98}$$

$$T(x,0) = \phi(x),\ 0 \le x \le b = S(0); \quad T(0,t) = u(t), \quad 0 < t < t_*, \tag{10.1.99}$$

$$T(S(t),t) = f(S(t),t), \quad 0 < t < t_*, \tag{10.1.100}$$

$$T_x(S(t),t) = g(S(t),t), \quad 0 < t < t_*. \tag{10.1.101}$$

Instead of the temperature, flux can also be prescribed at $x = 0$. Our concern here is to obtain a free boundary condition in which $\dot{S}(t)$ is explicitly appearing.

Problem (C1) Let

$$f_x(x,t) - g(x,t) \neq 0. \tag{10.1.102}$$

On differentiating Eq. (10.1.100) with respect to time, we get

$$\frac{dT}{dt} = (T_x\dot{S} + T_t) = g\dot{S} + T_t = f_x\dot{S} + f_t, \text{ at } x = S(t), \tag{10.1.103}$$

or

$$((f_x - g)\dot{S} + T_t - f_t)_{x=S(t)} = 0. \tag{10.1.104}$$

If Eq. (10.1.102) holds, then from Eq. (10.1.104), $\dot{S}$ can be expressed as

$$\dot{S} = \{(aT_{xx} - q - f_t)/(g - f_x)\}_{x=S(t)}\,. \tag{10.1.105}$$

The Problem (C1) consists of Eqs (10.1.98), (10.1.99), (10.1.101), (10.1.105). From Eqs (10.1.103), (10.1.104), we get

$$\frac{dT}{dt} = T_t + T_x\dot{S} = T_t + g\dot{S} = f_x\dot{S} + f_t = \frac{df}{dt}, \text{ at } x = S(t), \tag{10.1.106}$$

which is the condition (10.1.100).

Problem (C2) Let

$$f_x(x,t) = g(x,t). \tag{10.1.107}$$

If Eq. (10.1.107) holds, then from Eq. (10.1.104), we have

$$T_t = f_t, \text{ at } x = S(t). \tag{10.1.108}$$

From Eqs (10.1.100), (10.1.101), (10.1.107), we have

$$\frac{d}{dt}(T_x) = T_{xx}\dot{S} + T_{xt} = g_x\dot{S} + g_t = f_{xx}\dot{S} + f_{xt}, \text{ at } x = S(t). \tag{10.1.109}$$

On using Eqs (10.1.98), (10.1.108) in Eq. (10.1.109), we get

$$a^{-1}(q + f_t)\dot{S} + T_{xt} - f_{xx}\dot{S} - f_{xt} = 0, \text{ at } x = S(t) \tag{10.1.110}$$

If

$$af_{xx} - f_t - q \neq 0, \tag{10.1.111}$$

then from Eq. (10.1.110), we get

$$\dot{S} = a(af_{xx} - f_t - q)^{-1}(T_{xt} - f_{xt}), \text{ at } x = S(t). \tag{10.1.112}$$

The r.h.s. of Eq. (10.1.112) involves T_{xxx}. The Problem (C2) consists of Eqs (10.1.98)–(10.1.100), (10.1.112). Several differentiation operations have been done in Eqs (10.1.102)–(10.1.112) which will be valid under the following assumptions. These assumptions are also required to prove some results concerning analysis of these problems.

(A1) ϕ, u, a and q are continuous functions of their arguments, $f(x, t)$ is continuous for $x > 0$, $t \geq 0$, and $g(x, t)$ is continuous for $x > 0$, $t > 0$. Further,

$$\phi(0) = u(0), \quad \phi(b) = f(b, 0). \tag{10.1.113}$$

(A2) The function f is continuously differentiable for $t > 0$.

(A3) g is continuously differentiable for $t > 0$ and

$$\phi'(b) = g(b, 0). \tag{10.1.114}$$

If assumptions (A1) and (A2) are satisfied and Eq. (10.1.102) holds, then any solution of Problem (C1) solves the Problem (C) (only Eq. (10.1.100) is to be satisfied which has been done in Eq. 10.1.106). Under assumptions (A1), (A2), (A3) and the conditions Eqs (10.1.107), (10.1.111), the solution of Problem (C2) will be the solution of Problem (C) if it can be proved that Eq. (10.1.101) is satisfied. It can be proved that the solution of Problem (C2) satisfies

$$T_x(S(t), t) = f_x(S(t), t). \tag{10.1.115}$$

From Eq. (10.1.103), we have

$$(T_x - f_x)\dot{S}(t) + (T_t - f_t) = 0, \text{ at } x = S(t). \tag{10.1.116}$$

On differentiating Eq. (10.1.101), we get

$$\frac{d}{dt}T_x|_{x=S(t)} = T_{xx}\,\dot{S} + T_{xt}. \tag{10.1.117}$$

On substituting T_{xt} from Eq. (10.1.110) and T_{xx} from Eq. (10.1.98) in Eq. (10.1.117), we get

$$\frac{d}{dt}T_x(S(t),t) = \left[a^{-1}(T_t - f_t)\right]_{x=S(t)} \dot{S}(t) + \frac{d}{dt}f_x(S(t),t). \tag{10.1.118}$$

If $X(t) = (T_x - f_x)_{x=S(t)}$, then from Eqs (10.1.116), (10.1.118), we have

$$\dot{X}(t) = -\left[a^{-1}\right]_{x=S(t)} \dot{S}^2 X(t), \quad X(0) = 0, \quad a > 0. \tag{10.1.119}$$

The solution of Eq. (10.1.119) is $X(t) = 0$ which implies Eq. (10.1.115). The well-posedness of Problem (C1) has been proved in [306] under the assumption that $T_x(S(t),t) = 0$ and that of Problem (C2) under the assumption that $T_t(S(t),t) = 0$. Since these problems are strongly nonlinear, several assumptions are required to prove the results and the reader is referred to [306] for other assumptions.

Under suitable assumptions about the data and some appropriate compatibility conditions, Problem (C1) has a solution $(\hat{t}, T, S)$ which is unique in $(0,\hat{t})$. Moreover, $S(t) \in H_{1+\nu}[0,\hat{t}]$, $\nu \in (0,\alpha]$, $\alpha \in (0,1)$ and $T_x \in C_{1+\nu}(D_{\hat{t}})$ (see Appendix B for the definition of $C_{1+\nu}$). Continuous dependence of the solution on the data has also been proved. Under slightly different assumptions, similar results have been proved for Problem (C2).

An implicit free boundary problem has been considered in [307] and to obtain this problem formulation we take $a = 1$ and $q = 0$ in Eq. (10.1.98), $f(S(t),t) = f(t)$ in Eq. (10.1.100), and $g(S(t),t) = g(t)$ in Eq. (10.1.101). Such problems are usually reduced to problems in which $\dot{S}(t)$ occurs explicitly and then for analysis either Schauder fixed point theorem or contraction mapping theorem is applied. In [307], the implicit free boundary condition is retained and the existence and uniqueness of the solution has been proved by showing the convergence of a numerical solution obtained by finite-difference discretization of the parabolic heat equation in time but not in space. The discretized equation at $n\Delta t$, $n = 1, 2, \ldots$, can be written as a second order differential equation in space variable whose explicit solution has been constructed. This solution can be used to determine S_n. In this way we obtain a sequence of interrelated free boundary problems. In each time interval, an appropriate free boundary problem is to be solved in the region $0 \le x \le S_n$, $n = 1, 2, \ldots$; $S_n = S(n\Delta t)$, where Δt is the time step. In this way, we construct a two-dimensional function $\hat{T}(x,t)$ and a polygonal path $\hat{S}(t)$ which approximate the solution (T,S) of the original Stefan problem. The convergence of the approximate solution as $\Delta t \to 0$ ($n\Delta t$ remains finite) has been shown under appropriate assumptions.

In the problem studied in [308], instead of an implicit condition at the free boundary, the Stefan condition is prescribed and the problem has been studied with the help of its numerical solution. The numerical solution involves only finite-difference discretization in time as in [307]. In each time interval of length Δt, the free boundary is assumed to be known approximately and the temperature is determined by solving a parabolic heat conduction problem. The free boundary in the next time interval of length Δt is obtained with the help of its earlier value and the temperature derivative already calculated. By solving a sequence of time-independent free boundary problems we obtain a polygonal path $\hat{S}(t)$ which approximates $S(t)$ and also a two-dimensional temperature function. The convergence of the numerical solution has been proved and error estimates have been obtained.

Problem 3.3.2 described in Section 3.3.1 is a particular case of the problem (10.1.98)–(10.1.101). In this case because of the simplicity of the problem the following existence result can be proved (cf. [70]) under some simple assumptions about the data.

Proposition 10.1.20. *Suppose in Problem 3.3.2 we have: (i)* $0 = a < b$, $t_* = \infty$, $\beta = 0$ *and* $\alpha = 1$, *(ii)* $F(x,t) = \hat{h}(x,t) = 0$, *(iii)* $g(x,t) = g(x) \in C^1$ *and there exist constants* g_* *and* g^* *such that* $0 < g_* \le -g'(x) \le g^*$ *for* $b \le x < \infty$, *(iv)* $0 \le f(t) \in C$ *for* $0 \le t < \infty$, *(v)* $\phi(x) \in C^1$, $\phi(b) = g(b)$, *and there exists a constant* N_1 *such that* $0 \le \phi'(x) \le N_1(b - x)$ *for* $0 \le x \le b$. *If the above data assumptions hold, then there exists a unique classical solution* (T, S) *of Problem 3.3.2. Furthermore, for each fixed* $0 < t_0 < \infty$ *there exists a positive constant A depending only on b,* g_*, g^*, N_1 *and M such that* $0 \le \dot{S}(t) \le A$ *for* $0 \le t \le t_0$. *Here*

$$M = \max\left\{ \sup_{0 \le t \le t_0} f(t), \quad \sup_{0 \le x \le b} \phi(x) \right\}.$$

10.1.5 Existence of Self-Similar Solutions of Some Stefan Problems

Some results about the existence and uniqueness of the classical solutions of one-dimensional one-phase Stefan problems have been reported in [309]. These results follow from some theorems proved in the analysis of Stefan problems in R^n, $n \ge 1$. In Section 10.4, Meirmanov's [309] method of introducing local coordinates to prove the existence and uniqueness of solutions has been briefly described. Very general results about the analysis of one-dimensional one-phase problems have already been reported in this section. Therefore, instead of the existence and uniqueness results, we present here some results reported in [309] about the asymptotic behaviour of the solution of a Stefan problem whose formulation is as follows:

$$\frac{\partial \Phi(T)}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < S(t), \quad S(0) = d, \quad 0 < t < t_*. \tag{10.1.120}$$

Here, the thermal diffusivity $k = (dt/d\Phi)$. Other conditions are:

$$T = 0, \text{ and } \frac{dS}{dt} = -\frac{dT}{dx}, \text{ on } x = S(t); \quad T(x,0) = T_0(x), \quad 0 \le x \le d, \tag{10.1.121}$$

$$T = f(t) \text{ or } \partial T/\partial x + b(t)T = g(t) \text{ at } x = 0, \quad 0 < t < t_*. \tag{10.1.122}$$

Proposition 10.1.21. *Let* $\Phi \in C^2[0,\infty)$, $\Phi'(T) \ge a_0 > 0$ *and nonnegative functions* $f(t)$ *and* $T_0(x)$ *be such that for some constant M, we have*

$$f(t) \le Md \quad \text{and} \quad T_0(x) \le M(d - x). \tag{10.1.123}$$

Then a unique global classical solution (S, T) *exists for all finite* $t > 0$ *with* $\dot{S}(t)$ *Hölder continuous for* $t > 0$ *and* $T(x,t) \in H^{r,r/2}(\overline{G_{\delta,t_*}})$, $r > 2$ *in each bounded domain* $G_{\delta,t_*} = \{(x,t) : x > \delta, t > \delta, 0 < x < S(t), 0 < t < t_*\}$. *If* $\Phi(T)$ *is infinitely differentiable, then* $S(t)$ *is also infinitely differentiable for* $t > 0$. *If* $\lim_{t\to\infty} f(t) = \beta$, $0 \le \beta < \infty$ *and the above assumptions hold, then* $\lim_{t\to\infty} t^{-1/2} S(t) = D_*(\beta)$, $\beta > 0$, *where* $D_*(\beta)$ *is given by Eq. (10.1.125) (given below).*

Proposition 10.1.22. *Let* $\Phi \in C^2[0,\beta]$, $\Phi'(T) > 0$ *for* $T \in [0,\beta]$, $f(t) = \beta =$ *constant* > 0, $d = 0$, *and for* $x \in (0,\infty)$, $T = 0$ *and the specific energy* $\mathrm{U} = -1$. *Then problem* (10.1.120)–(10.1.122) *has a unique solution such that*

$$S(t) = D_*(\beta)t^{1/2}, \quad \text{and} \quad T_*(x,t) = T(xt^{-1/2}, \beta). \tag{10.1.124}$$

Here, $D_*(\beta)$ depends continuously on β, and lim $D_*(\beta) = 0$ as $\beta \to 0$. $D_*(\beta)$ is given by the equation

$$\frac{1}{2}D_*^2(\beta) + \int_0^{D_*(\beta)} \xi\Phi(T(\xi,\beta))d\xi = \beta. \tag{10.1.125}$$

10.1.6 The Effect of Density Change

If in the phase-change problems, the density changes sharply, then the effect of the change has to be included in the formulation, although this may make the analysis of such problems more difficult. In the one-phase problem considered in [310], the effect of change in the density on the change of phase has been considered. The formulation given in Eqs (10.1.56)–(10.1.57) can be used for the problem considered in [310] provided the Stefan condition is replaced by the following condition:

$$\frac{dS}{dt} - \varepsilon\left(\frac{dS}{dt}\right)^3 = (K_S/(\rho_S l))T_x(S(t),t), \quad \varepsilon = (1/2l)\,(1-\rho_S/\rho_L)^2\,. \tag{10.1.126}$$

Under suitable assumptions about the data (cf. [310]), global existence and stability of the classical solution has been proved using Schauder's fixed point theorem and it has been shown that

$$T(x,t) \in C^{1+1,0+1}(\overline{\Omega}_{t_*}) \cap C^\infty(\Omega_{t_*}), \quad S(t) \in C^1[0,t_*];$$
$$\Omega_{t_*} = \{(x,t) : 0 < x < S(t), 0 < t < t_*\}\,. \tag{10.1.127}$$

As $\varepsilon \to 0$, the convergence of the solution of the present problem to the solution of the classical one-phase Stefan problem has been shown.

10.2 ONE-DIMENSIONAL TWO-PHASE STEFAN PROBLEMS

10.2.1 Existence, Uniqueness and Stability Results

The classical formulation considered in [57] (see Section 3.2.1, Eqs 3.2.1–3.2.10) is fairly general. In some problems discussed in this section, nonlinear heat equations have been considered but the initial and interface conditions are not as general as in [57]. The results obtained in [57] are generalizations of the results obtained in [311, 312]. If we take $q^{(1)} = q^{(2)} = 0, f(x,t) = 0$, $\chi_1^{(1)} = K_1$ and $\chi_2^{(2)} = K_2$ in Eqs (3.2.1)–(3.2.10), then the problem considered in [311] and [312] is obtained. General results about the existence of the classical solutions of a two-phase Stefan problem have been obtained in [311] under the main regularity assumptions: (i) $\phi^{(1)}(t)$, $\phi^{(2)}(t) \in C^1[0,t_*]$, (ii) $h^{(1)}(x)$, $h^{(2)}(x)$ are Hölder continuous with exponent $\in (0,1]$, and (iii) compatibility conditions are satisfied at $S(0)$ and at $x = 0, 1$. The proofs in [311] are based upon potential theoretic arguments coupled with Schauder's fixed point theorem. It has been proved that if the data for the Stefan problem belong to a certain class (cf. [311]), then there exists $S(t) \in C^{3/4}[0, t_*] \cap C^1(0, t_*)$ and $T(x, t)$ is continuous on $\overline{\Omega}_{t_*}$ satisfying all the equations of the problem. If also $\phi \in C^{\beta}[0, 1]$, $1/2 \le \beta \le 1$, then $S(t) \in C^{\hat{\beta}}[0, t_*]$ and $\dot{S} \in C_{\varepsilon}^{\hat{\beta}}(0, t_*]$, where $\hat{\beta} = (1 + \beta)/2$ and $\varepsilon < \beta/2$. $C_{\varepsilon}^{\nu}(\mathrm{I})$ $(\varepsilon > 0)$ is the subspace of $C_0^{\nu}(\mathrm{I})$ and $C_0^{\nu}(\mathrm{I})$ is the subspace of C(I). For appropriate norms in these spaces see [311] (the norm of functions in $C_{\varepsilon}^{\nu}(\mathrm{I})$ depends on ε). By using a weak formulation, the existence of a classical solution has been established in [312] which will be discussed in Section 11.2.

In [57], the existence of a unique solution has been proved under weaker conditions and differentiability conditions are not imposed on $\phi^{(1)}$ and $\phi^{(2)}$. Only Hölder continuity of $h^{(1)}$ and $h^{(2)}$ is required at $x = b$ as stated in Eqs (3.2.11)–(3.2.13). Other assumptions required in [57] for proving well-posedness are (see assumptions (A)–(F) made for problem 10.1.34–10.1.38) as follows:

- **(i)** $q^{(i)}(x, t)$, $i = 1, 2$ satisfy condition (A) in which Q will be an upper bound of $|q^{(i)}(x, t)|$ in $\Omega_{t_*} = \{(x, t) : 0 < x < 1, 0 < t < t_* < \infty\}$.
- **(ii)** $f(x, t)$ satisfies assumption (B) and $\mu(x, t)$ satisfies (E).
- **(iii)** The assumption (C_1) is satisfied by both $\phi^{(1)}$ and $\phi^{(2)}$ and $|\phi^{(i)}| \le \Phi$, $i = 1, 2$ and $t \ge 0$ (note that notations in Eqs 3.2.11–3.2.13 are different from those in Eqs 10.1.34–10.1.38)
- **(iv)** $\chi^{(i)}$, $i = 1, 2$ and their derivatives satisfy conditions given in Eq. (3.2.14).
- **(v)** All the assumptions (A)–(F) hold.

As mentioned in the context of Eqs (10.1.34)–(10.1.38), with no loss of generality we can set $f \equiv 0$. To develop an existence proof, an integral equation of the form Eq. (10.1.7) is required for the present problem also (see Eq. 10.2.2 given below). As done in the context of Eqs (10.1.34)–(10.1.38), a sequence of approximating solutions can be defined for the present problem also. It has been shown that the limiting solution (S, T) of the sequence of approximating solutions exists (for approximating solutions see the discussion after Eq. 10.2.3). S is Lipschitz continuous and T_x is continuous up to $x = S(t)$ and satisfies the integral equation for $S(t)$ given in Eq. (10.2.2). Hence, the Stefan condition is satisfied. The derivation of an integral equation for $S(t)$ of the form Eq. (10.1.7) for the boundary conditions (3.2.3) and (3.2.6) is lengthy so we explain the derivation of this integral equation for the boundary conditions (3.2.9) and (3.2.10). In Eq. (4.4.20), a Green's identity has been given. In the present problem, we consider the following Green's identity

$$\int\int_D \{V(T_{xx} - \alpha T_\tau) - T(V_{xx} + \alpha V_\tau)\}\, dx d\tau$$
$$= \oint_{\partial D} \{(VT_x - TV_x) d\tau + \alpha TV dx\},\ \alpha \text{ constant.} \tag{10.2.1}$$

This identity is valid for sufficiently smooth functions T and V and for sufficiently regular domains. We take $T = T^{(1)}$, $V = \chi^{(1)}$, $\alpha = \delta^{(1)}$ and $D = \{(x, \tau) : \varepsilon < x < S(\tau), \varepsilon' < \tau < t\}$. ∂D is the boundary of D. Next we use the identity Eq. (10.2.1) by taking $T = T^{(2)}$, $V = \chi^{(2)}$, $\alpha = \delta^{(2)}$ and $D = \{(x, \tau) : S(\tau) < x < 1 - \varepsilon, \varepsilon' < \tau < t\}$. Adding the two results and taking the limits as $\varepsilon \to 0$ and $\varepsilon' \to 0$, we get

$$
\begin{aligned}
S(t) - b = & \int_0^t \int_0^1 (\chi q - \chi^* T) dx d\tau - \int_0^1 \delta\chi(x,0)\hat{h}(x)dx + \int_0^1 \delta\chi(x,t)T(x,t)dx \\
& + \int_0^t \{\chi^{(1)}(0,\tau)g^{(1)}(T^{(1)}(0,\tau),\tau) - T^{(1)}(0,\tau)\chi_x^{(1)}(0,\tau)\}d\tau \\
& - \int_0^t \{\chi^{(2)}(1,\tau)g^{(2)}(T^{(2)}(1,\tau),\tau) - T^{(2)}(1,\tau)\chi_x^{(2)}(1,\tau)\}d\tau \\
& - \int_0^t \mu(S(\tau),\tau)d\tau, \quad 0 < t < t_*.
\end{aligned} \tag{10.2.2}
$$

Here, χ, q, δ and $\hat{h}$ take appropriate values in the regions $0 < x < S(t)$ and $S(t) < x < 1$,

$$
\left.
\begin{aligned}
\chi^*(x,t) &= \chi_{xx}^{(1)}(x,t) + \delta^{(1)}\chi_t^{(1)}(x,t),\ 0 \le x \le S(t), \\
&= \chi_{xx}^{(2)}(x,t) + \delta^{(2)}\chi_t^{(1)}(x,t),\ S(t) < x \le 1.
\end{aligned}
\right\} \tag{10.2.3}
$$

The triple $(S(t), T^{(1)}, T^{(2)})$ satisfies all Eqs (3.2.1)–(3.2.10) except Eqs (3.2.3), (3.2.6) as the boundary conditions are taken as Eqs (3.2.9), (3.2.10).

A sequence of approximating solutions $(t_*^{(k)}, S_k, T_k^{(1)}, T_k^{(2)})$ can be defined as in Eqs (10.1.46)–(10.1.51) provided the approximating problems for $T_k^{(2)}$ are appropriately defined in $S_k(t) < x < 1,\ 0 < t < t_*^{(k)}$ $(S_1(t) = b)$ and $0 < t < t_*^{(k+1)} \le t_*^{(k)}$. $S_{k+1}(0) = b$, $k = 1, 2, \ldots$

$$
\dot{S}_{k+1}(t) = \chi^{(1)}(S_k(t),t)T_{k,x}^{(1)}(S_k(t),t) - \chi^{(2)}(S_k(t),t)T_{k,x}^{(2)}(S_k(t),t) + \mu(S_k(t),t). \tag{10.2.4}
$$

The following uniform estimate for $t > 0$ has been obtained for $\dot{S}(t)$ and

$$
|\dot{S}_k(t)| \le M_1 t^{-(1-\alpha)/2},\ 0 < \alpha < 1,\ M_1 \text{ is some constant;} \quad k = 1, 2, \ldots \tag{10.2.5}
$$

Further, a constant M_2 exists such that

$$
|T_k^{(i)}(x,t)| \le M_2 t^{-(1-\alpha)/2}|x - S_k(t)|; \quad i = 1, 2, \quad k = 1, 2, \ldots \tag{10.2.6}
$$

and

$$
|T^{(i)}(x,t)| \le M_2 t^{-(1-\alpha)/2}|x - S(t)|, \quad i = 1, 2. \tag{10.2.7}
$$

First, the existence of the solution (S, T) is established in a small time interval $(0, t_0)$. Then using the estimates obtained in Eqs (10.2.5)–(10.2.7), the solution can be extended to larger time intervals. Thus we get a sequence $t_0 < t_1 < t_2 < \cdots < t_n$. Since the sequence $\{t_n\}$ is monotonically increasing, we have either $t_n \to \infty$ or $\lim t_n = t_* < \infty$. In the latter case if $\dot{S}(t)$ is finite, then by using the estimates given in Eqs (10.2.5)–(10.2.7), the solution can be extended beyond the time t_* and we have a contradiction.

Proposition 10.2.1. *Under assumptions (i)–(v) mentioned above in this section, a solution of problem* (3.2.1)–(3.2.8) *exists in the time interval* $[0, t_*]$ *and estimates* (10.2.5)–(10.2.7) *hold in* $[0, t_*]$. *If* $t_* < \infty$, *then*

$$\lim_{t \to t_*-} \{S(t), 1 - S(t)\} = 0 \quad \text{and/or} \quad \lim_{t \to t_*-} \sup \dot{S}(t) = +\infty. \tag{10.2.8}$$

The following simple two-phase Stefan has been studied by many authors with or without some changes in the initial and boundary conditions:

$$k_1 \frac{\partial^2 T_1}{\partial x^2} = \frac{\partial T_1}{\partial t}, \quad 0 < x < S(t), \quad S(0) = b, \quad t > 0, \tag{10.2.9}$$

$$k_2 \frac{\partial^2 T_2}{\partial x^2} = \frac{\partial T_2}{\partial t}, \quad S(t) < x < d, \quad t > 0, \tag{10.2.10}$$

$$T_1(0,t) = f_1(t) \geq 0, \quad t > 0, \tag{10.2.11}$$

$$T_2(d,t) = f_2(t) \leq 0, \quad t > 0, \tag{10.2.12}$$

$$T_1(x,0) = \phi_1(x) \geq 0, \quad 0 \leq x \leq b, \quad \phi_1(b) = 0, \tag{10.2.13}$$

$$T_2(x,0) = \phi_2(x) < 0, \quad b \leq x \leq d, \quad \phi_2(b) = 0, \tag{10.2.14}$$

$$T_1(S(t),t) = T_2(S(t),t) = 0, \quad t \geq 0, \tag{10.2.15}$$

$$\lambda \dot{S}(t) = -K_1 \partial T_1/\partial x + K_2 \partial T_2/\partial x, \quad \text{at } x = S(t). \tag{10.2.16}$$

The method proposed in [283] for the analysis of a one-phase problem described earlier in Section 10.1, can be suitably extended to the above two-phase problem (10.2.9)–(10.2.16). The problems described below which can be obtained by making some changes in the above two-phase problem have been investigated in [21, Part two, Chapter II].

Problem (A1) Temperature boundary conditions are prescribed at $x = 0$ and at $x = 1$ (take $d = 1$), $S(0) = 0$. $T_2(x,0) = \phi(x) > 0$, $0 \leq x \leq 1$, $f_1(t) < 0$, $f_2(t) > 0$ and $\lim_{t \to \infty} |f_i(t)| = \alpha_i > 0$, $i = 1, 2$. Note that there is only one initial temperature for the two phases.

Problem (A2) Radiation type boundary condition is prescribed at $x = 0$ and temperature is prescribed at $x = 1$ (see Eq. 10.2.18). Other conditions are the same as in Problem (A1).

Problem (B1) Consider the problem described in Eqs (10.2.9)–(10.2.16) with $f_1(t)$ and $f_2(t)$ as in Problem (A1), $S(0) = b$, $0 < b < d$. $\phi_1(x) < 0$ in $0 \leq x \leq b$ and $\phi_2(x) > 0$ in $b \leq x \leq d$.

Problem (B2) Radiation type boundary condition is prescribed at $x = 0$ and temperature is prescribed at $x = 1$ as in Problem (A2). Other conditions are the same as in Problem (B1).

Problem (C) (Cauchy–Stefan problem). Consider the region $-\infty < x < \infty$ with $T_1(x,0) = \phi_1(x)$, $-\infty < x < 0$ and $T_2(x,0) = \phi_2(x)$, $0 < x < \infty$. $T_1(x,t)$ and $T_2(x,t)$ satisfy heat equations, isotherm conditions and the Stefan condition.

For the regularity and compatibility conditions to be satisfied by the data, the reader is referred to [21]. By using heat potentials, the temperatures $T_1(x,t)$ and $T_2(x,t)$ and their first derivatives w.r.t. x can be expressed in the form of integrals similar to those obtained in Eqs (10.1.17), (10.1.30). For Problem (B1), which is easier to handle, the existence, uniqueness and stability of the local-in-time solution has been proved under suitable assumptions using Picard iteration which was used earlier for the problem

(10.1.23)–(10.1.27). By assuming suitable a-priori uniform estimates of the temperature derivatives and other quantities, it is possible to extend this solution to longer times. The constant b in Problem (B1) cannot be taken to be zero as one of the phases degenerates to a point which destroys the convergence of Picard iteration which is the main step in the proof of the existence of the solution in [21]. By imposing additional conditions (see [21]) on the initial and boundary conditions, the existence and uniqueness of solutions of Problems (A1) and (A3) have been discussed. Since fairly general results related to the existence and uniqueness have already been reported, we discuss the asymptotic behaviour of the solution. For the asymptotic behaviour of the solution of Problem (B1), the following result holds good.

Proposition 10.2.2. *If* $\lim_{t\to\infty} f_1(t) = \alpha_1 < 0$, *and* $\lim_{t\to\infty} f_2(t) = \alpha_2 > 0$, *then as* $t \to \infty$, *the solution of Problem (B1) tends uniformly in x to the following limits:*

$$\lim_{t\to\infty} T_i(x,t) = -\alpha_1 + (\alpha_1+\alpha_2)x,\ i=1,2; \quad \lim_{t\to\infty} S(t) = \frac{\alpha_1}{\alpha_1+\alpha_2}. \tag{10.2.17}$$

The solution in Eq. (10.2.17) is also the solution of the corresponding stationary problem. In Problems (A2) and (B2), we take

$$\left(\frac{\partial}{\partial x} - g_0\right) T_1 = -g_0 f_1(t), \text{ at } x=0, \quad \text{and} \quad T_2|_{x=1} = f_2(t);\ t>0. \tag{10.2.18}$$

The existence, uniqueness and stability of the solution of Problem (B2) have been proved locally-in-time as well as globally-in-time using Picard iteration method (see [311] for a direct proof of the existence of the classical solution). Concerning asymptotic behaviour of the solution of Problem (B2), the following results have been proved:

$$\lim_{t\to\infty} T_1(x,t) = -g_0\alpha_1(\lambda - x)/(1+\lambda g_0),\ g_0 \text{ is constant}, \tag{10.2.19}$$

$$\lim_{t\to\infty} T_2(x,t) = \alpha_2(x-\lambda)/(1-\lambda), \tag{10.2.20}$$

$$\lim_{t\to\infty} S(t) = \lambda = (g_0\alpha_1 - \alpha_2)/(g_0(\alpha_1+\alpha_2)). \tag{10.2.21}$$

The limiting solution in Eqs (10.2.19)–(10.2.21) is also a stationary solution of Problem (B2). In Problem (C), if $\phi_1(0) = \phi_2(0) = 0$; $S(0) = 0$, $\lim_{x\to-\infty} \phi_1(x) = -\alpha_1 < 0$, $\lim_{x\to+\infty} \phi_2(x) = \alpha_2 > 0$, $\lim \dot{\phi}_i(x) = 0$ as $|x| \to \infty$ and $\phi_i(x)$ are thrice differentiable, then existence, uniqueness and stability of the solution have been established for large time. If $\alpha_2 - a\alpha_1 > 0$, where $a^2 = k_2$ $(k_1 = 1)$, then $\lim_{t\to\infty} S(t) = -\infty$. If $\alpha_2 - a\alpha_1 < 0$, then $\lim_{t\to\infty} S(t) = +\infty$.

The initial velocity of the free boundary in Problems (B1), (B2) and (C) is known if the compatibility condition

$$\phi_i(b) = 0,\ i=1,2, \tag{10.2.22}$$

holds. In this case

$$\dot{S}(0) = \dot{\phi}_1(b) - \dot{\phi}_2(b). \tag{10.2.23}$$

If $\beta = \dot{S}(0) \neq 0$, then

$$S(t) = b + \beta t(1 + S^*(t)); \lim_{t \to 0+} S^*(t) = 0. \tag{10.2.24}$$

But for $\beta = 0$, the order of contact of the curves $x = b$ and $x = S(t)$ at $t = 0$ becomes unknown. In addition, it is unknown when the condition (10.2.22) is omitted, or in the case of Problems (A1) or (A2) when the domain of existence of one of the phases degenerates to a point.

A detailed study of the interface initial velocity, has been done in [21]. We present here just one result. In Problem (A1), assume that as $t \to 0$,

$$\dot{f}_1(t) = -\psi_1(t)t^{m_o}, \quad m_o > -1; \quad \dot{\phi}(x) = \psi_2(x)x^{n_o}, \quad n_o > -1. \tag{10.2.25}$$

Further, for any $\varepsilon > 0$, let

$$\lim_{t \to \infty} \psi_i(t)t^{\varepsilon} = 0 \text{ and } \lim \psi_i(t)t^{-\varepsilon} = \infty, \quad i = 1, 2. \tag{10.2.26}$$

If $f_1(0) \neq 0$ and $\sqrt{t}\dot{S}(t)$ is continuous for $t \geq 0$, then

$$\dot{S}(t) = \beta t^{-/2}(1 + S_1^*(t)); \quad \lim_{t \to 0} S_1^*(t) = 0, \tag{10.2.27}$$

where β is the root of a transcendental equation (cf. [21]). In most of the practical problems, $S(t) \sim \mathrm{O}(t^{1/2})$ or $\mathrm{O}(t)$ or $\mathrm{O}(t^{3/2})$ (cf. [60]).

The existence and uniqueness of Problem (A1) or Problem (A2) cannot be investigated easily as the convergence of Picard iteration process is destroyed by the degeneracy of one of the phases. Some additional conditions are to be imposed on the data to prove the existence of the solution of Problem (A1) by Picard iteration (cf. [21]). To construct the existence proof the solution of Problem (B1), $\left(T_{1,n}, T_{2,n}, S_n\right)$ is obtained for $t > t_n$ by choosing an arbitrary monotonically decreasing sequence

$$t_0 > t_1 > \cdots > t_n > \cdots; \quad \lim_{n \to \infty} t_n = 0; \ t_0 < t^*. \tag{10.2.28}$$

The definition of t^* is complicated as it depends on the time for which the solution of an auxiliary problem exists and simultaneously some other conditions are satisfied and for this information the reader is referred to [21]. We now consider a sequence of Problems of the type B_1 in the regions, $0 < x < S_n(t)$, (S_n, $n = 1, 2, \ldots$ are known) and $S_n(t) < x < d$, $t > t_n$, in which temperature at t_n is substituted from the solution of an auxiliary problem in which $S(t_n)$ is used. It has been shown that the sequence of solutions $\left\{T_{1,n}, T_{2,n}, S_n\right\}$ of these problems converges to the unique solution of Problem (A1). Problem (A2) can be similarly studied.

By using Picard iteration method, the existence and uniqueness of a one-dimensional two-phase problem in a region with cylindrical symmetry has also been studied in [21]. In this case the Green's function used to construct the solution of the heat equation is given by

$$E_0(r, \xi, a^2(t - \tau)) = \frac{\exp\left(-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right)}{2a^2(t - \tau)} I_0\left(\frac{r\xi}{2a^2(t - \tau)}\right). \tag{10.2.29}$$

Here, r is the radial coordinate in a cylinder of radius a and $I_0(x)$ is the modified Bessel's function of the first kind of zeroth order.

A two-phase Stefan problem in an unbounded domain has been considered in [313]. The heat equations in the regions $D_{t_*}^{(i)}$, $i = 1, 2$, have been considered as follows:

$$\mathcal{L}^{(1)}T^1(x,t) = q^{(1)}(x,t), \text{ in } D_{t_*}^{(1)} = \{(x,t) : -\infty < x < S(t),\ S(0) = d > 0,\ 0 < t < t_*\}, \tag{10.2.30}$$

$$\mathcal{L}^{(2)}T^2(x,t) = q^{(2)}(x,t), \text{ in } D_{t_*}^{(2)} = \{(x,t) : S(t) < x < \infty,\ 0 < t < t_*\}, \tag{10.2.31}$$

$$\mathcal{L}^{(i)}T^{(i)}(x,t) = a^{(i)}(x,t)T_{xx}^{(i)} + b^{(i)}(x,t){T_x}^{(i)} + e^{(i)}(x,t)T^{(i)} - {T_t}^{(i)},\ i = 1, 2. \tag{10.2.32}$$

The problem formulation consists of Eqs (10.2.30)–(10.2.32), (3.2.2), (3.2.5), (3.2.7) with $f(x,t) = 0$, and the equation

$$\begin{aligned} a^{(1)}(S(t),t)X^{(1)}(S(t),t)T_x^{(1)}(S(t),t) - a^{(2)}(S(t),t)X^{(2)}(S(t),t)T_x^{(2)}(S(t),t) \\ = \dot{S}(t) + \mu(S(t),t), \quad 0 < t < t_*, \quad S(0) = d. \end{aligned} \tag{10.2.33}$$

This problem will be called Problem (YA). It is assumed that: (i) $q^{(i)}(x,t)$ and other coefficients in Eqs (10.2.31), (10.2.32) are Hölder continuous in $\Omega = \{(x,t) : -\infty < x < \infty, 0 < t < \infty\}$ with $q^{(1)}(x,t) \geq 0$ and $q^{(2)}(x,t) \leq 0$ in Ω, (ii) $h^{(1)}(x)$ and $h^{(2)}(x)$ are continuous and $h^{(1)}(x) \geq 0$ and $h^{(2)}(x) \leq 0$, (iii) $X^{(i)}(x,t)$, $X_x^{(i)}$ and $X_t^{(i)}$ are continuous and $-N_1 \leq X^{(1)}(x,t) \leq 0$ and $N_2 \geq X^{(2)}(x,t) \geq 0$, (iv) $\mu(x,t)$ is continuous and differentiable in (x,t), $-\mu_0 \leq \mu(x,t) \leq 0$ $(\mu_0 > 0)$, $\mu(d,0) = X^{(i)}(d,0) = 0$ $(i = 1,2)$, (v) $e^{(i)} \leq 0$, $i = 1, 2$.

For the complete set of assumptions, the reader is referred to [313]. It has been shown that a unique classical solution $\{t_*, S(t), T^{(1)}, T^{(2)}\}$ exists, $S(t)$ is differentiable and $0 \leq \dot{S}(t) \leq \beta$, β constant. To prove this result, solutions of a sequence of auxiliary problems are constructed (by retarding the argument) as follows. For an arbitrary θ, $0 < \theta < d$, let,

$$h^{(1)\theta}(x) = \begin{cases} h^{(1)}(x) & \text{for} \quad x \leq d - \theta, \\ 0 & \text{for} \quad x > d - \theta. \end{cases} \tag{10.2.34}$$

Similarly $h^{(2)\theta}(x)$ can be defined for $x \geq d - \theta$ and $x < d - \theta$. We take $S^\theta(t) \equiv d$ on the time interval $[0, \theta]$ and solve Problem (YA) without the condition (10.2.33) (free boundary is now known). Let $T^{(1)\theta}$ and $T^{(2)\theta}$ be the temperature solutions in the two regions $-\infty < x < S^\theta(t)$ and $S^\theta(t) < x < \infty$ for $0 \leq t \leq \theta$. To construct the solution for $0 \leq t \leq 2\theta$, $S^\theta(t)$ is determined in the time interval $[\theta, 2\theta]$ from Eq. (10.2.33) after integrating it w.r.t. to time and using the value of $S^\theta(t)$ in $[0, \theta]$ in the temperature derivatives at $x = S(t)$. Now this $S^\theta(t)$ for $\theta \leq t \leq 2\theta$ is used to determine temperatures $T^{(1)\theta}$ and $T^{(2)\theta}$ for $0 \leq t \leq 2\theta$. It may be noted that temperatures are uniquely determined as they are solutions of parabolic heat equations with boundary conditions prescribed at the known boundaries and with known initial temperatures. By induction we obtain $S^\theta(t)$, $n\theta \leq t \leq (n+1)\theta$ and continue till we get a solution in $[0, t_*]$ The conditions required for the existence of a unique solution are met by the assumptions already made about the data. Proceeding inductively in this way, we get a sequence of approximate solutions $\{S^\theta, T^{(1)\theta}, T^{(2)\theta}\}$ in $0 \leq t \leq n\theta$, $n = 1, 2, \ldots$ It can be proved that the approximate solutions S^θ are such that $0 \leq \dot{S}^\theta(t) \leq \alpha$ (constant) and the sequence $\{S^\theta\}$ is uniformly bounded and equicontinuous on a compact set. On using Ascoli–Arzela theorem it can be concluded that a subsequence converges to $S(t)$ (the conditions required for the application of Ascoli–Arzela theorem have been established in [313]). With this known $S(t)$, we determine the temperatures $T^{(1)}(x,t)$ and $T^{(2)}(x,t)$ by solving heat equations. To show that this limiting solution is the solution of Problem (YA), it should be proved that the limiting solution satisfies an equation of the form (10.2.2). For the present problem, an integral equation for $S(t)$ can be obtained by using the following Green's formula:

$$\int\int_{\Omega_{t_*}} (v\mathcal{L}T - T\mathcal{L}^* v)\,dx\,dt = \int_{\partial\Omega_{t_*}} Tv\,dx + \int_0^{t_*} \{a\,(x,t)\,(vT_x - Tv_x) + \left(b\,(x,t) - \frac{\partial a}{\partial x}\right) Tv\Big\}\,dt. \tag{10.2.35}$$

Here, Ω_{t_*} could be $D_{t_*}^{(1)}$ or $D_{t_*}^{(2)}$ and v is a smooth function in Ω_{t_*}.

For the operator $\mathcal{L}$ in Eq. (10.2.32), the operator $\mathcal{L}^*$ (conjugate) is given by

$$\mathcal{L}^* v = \frac{\partial^2}{\partial x^2}(av) - \frac{\partial}{\partial x}(bv) + e(x,t)v + v_t. \tag{10.2.36}$$

For the application of Green's formula and deriving an integral equation for $S(t)$ in Problem (YA), the reader is referred to [313].

In [314], instead of the Stefan condition in Eqs (10.2.9)–(10.2.16), the following condition is prescribed at the free boundary:

$$\partial T_1(x)/\partial x = F_t(S, T_2, T_{2,x}), \text{ on } x = S(t). \tag{10.2.37}$$

Here, F_t is a functional acting on the triple $\{S(\tau), T_2(x,\tau), T_{2,x}(x,\tau)\}$, $x \in [S(\tau), 1]$, $\tau \in [0,t]$. Let $k_1 = k_1(T_1)$, and $k_2 = k_2(T_2)$. Introduce heat source or sink terms $q^{(1)}(T_1)$ and $q^{(2)}(T_2)$ in Eqs (10.2.9), (10.2.10), respectively. In the place of Eq. (10.2.16), prescribe Eq. (10.2.37) and let other conditions be the same as in Eqs (10.2.11)–(10.2.15) but without sign constraints. This problem will be called Problem (FAP). If instead of Eq. (10.2.37), we take

$$\left.\frac{\partial T_1}{\partial x}\right|_{x=S(t)} = g(t), \tag{10.2.38}$$

in which $g(t)$ is known, then we have the formulation of a one-phase Stefan problem in $0 < x < S(t)$, $0 < t < t_*$ with an implicit free boundary condition (see Section 10.1.4, Eq. 10.1.101). Under suitable assumptions, this one-phase problem is well-posed and the unique $S(t)$ obtained as the solution of this problem can be used to solve a parabolic heat conduction problem to determine $T_2(x,t)$ in the region $S(t) \le x \le 1$, $0 < t < t_*$. The unique solution of $T_2(x,t)$ can also be obtained provided appropriate assumptions are made about the data in the region $S(t) \le x \le 1$. Here, $S(t)$ is known for determining $T_2(x,t)$. Now the relationship between $g(t)$ and the functional F_t in Eq. (10.2.37) should be explored. Note that we have taken $g(t)$ to be some known function but it may not satisfy Eq. (10.2.37) exactly and in this situation it is appropriate to write

$$G(t) = F_t(S, T_2, T_{2,x}). \tag{10.2.39}$$

We express $G(t)$ as

$$G(t) = Mg. \tag{10.2.40}$$

The operator M is acting on a suitable function space to which g belongs. It has been proved in [314] that a $t_0 > 0$ exists such that M has at least one fixed point on $(0, t_0)$. If $g \in H_{1/2+\varepsilon}$, $\varepsilon \in (0, 1/2)$, then a unique classical solution (S, T_1, T_2) of Problem (FAP) exists in the interval $(0, t_0)$. Moreover, $S(t) \in H_{1+\varepsilon}[0, t_0]$ with $\varepsilon \in (0, 1/2)$.

A two-phase Stefan problem in the region $0 \le x \le 1$ with unilateral boundary conditions at $x = 0$ and $x = 1$ has been considered in [315, 316]. The formulation of the problem considered in these references can be obtained from Eqs (10.2.9)–(10.2.16) if instead of Eqs (10.2.11), (10.2.12) (take $d = 1$), the following boundary conditions are considered:

$$T_x(0, t) \in \gamma_0(T(0, t)), \quad 0 < t < t_*, \tag{10.2.41}$$

and

$$-T_x(1, t) \in \gamma_1(T(1, t)), \quad 0 < t < t_*, \tag{10.2.42}$$

and all the thermophysical parameters are taken to be unity. γ_0 and γ_1 are maximal monotone graphs in R^2 such that both $\gamma_0^{-1}(0) \cap [0, \infty)$ and $\gamma_1^{-1}(0) \cap (-\infty, 0]$ are nonempty sets. This implies that there is a kind of heater at $x = 0$ and a kind of cooler at $x = 1$. The time t_*, $0 < t^* < \infty$, is defined to be the first time that the free boundary $x = S(t)$ touches $x = 0$ or $x = 1$.

We shall call this Stefan problem with Eqs (10.2.41), (10.2.42) and other conditions in Eqs (10.2.9)–(10.2.16) (except Eqs 10.2.11, 10.2.12), Problem (YU). The existence and uniqueness of the solution of Problem (YU) has been proved in [315] under the following assumptions:

$$\left.\begin{aligned} &\phi_1(x) \ge 0 \ (0 < x < b), \ \phi_2(x) \le 0 \ (b < x < 1), \\ &\phi_1 \text{ and } \phi_2 \text{ are bounded and continuous for a.e. } x \in [0, 1]. \end{aligned}\right\} \tag{10.2.43}$$

Several properties of $S(t)$ and of temperatures in the two-phases have been discussed. For example, it has been proved that:

(1) $S \in C^{0,1/3}[0, t_*] \cap C^{0,2/3}(0, t_*] \cap C^{\infty}(0, t_*)$, (10.2.44)

(2) $0 \le T_1 \le \max\{\|\phi_1\|_{L^\infty(0,b)}, \alpha_1\}$ on $\overline{D}_{t_*}^{(1)} = \{(x, t) : 0 \le x \le S(t),\ 0 \le t \le t_*\}$,

$$\alpha_1 = \min\left\{\alpha_1 \ge 0;\ \alpha_1 \in \gamma_0^{-1}(0)\right\}, \tag{10.2.45}$$

(3) $\min\{-\|\phi_2\|_{L^\infty(b,1)}, \alpha_2\} \le T_2 \le 0$ on $\overline{D}_{t_*}^{(2)} = \{(x, t) : S(t) \le x \le 1,\ 0 \le t \le t_*\}$,

$$\alpha_2 = \max\{\alpha_2 \le 0;\ \alpha_2 \in \gamma_1^{-1}(0)\}. \tag{10.2.46}$$

The space $C^{0,\alpha}$, $\alpha \in (0, 1)$ is the space of Hölder continuous functions.

The existence and uniqueness of Problem (YU) have been proved using the finite-difference discretization of equations. Several assumptions about the data and, estimates of the nodal values in the finite-difference numerical solutions of $S(t)$ and temperatures, are required which cannot be given here and the reader is referred to [315] for them. The convergence of the numerical solution has been established. The free boundary in a two-phase problem cannot be attached to mesh points which could be easily done in the one-phase problem discussed in [302]. For the existence of local-in-time solution no sign constraints on $\phi(x)$ are required.

The Problem (YU) has been considered in [316] also but the emphasis in [316] is on the asymptotic behaviour of its solution. The stationary solution of Problem (YU) is a solution of the following elliptic problem:

$$\left.\begin{array}{c} w_{xx}(x) = 0,\ 0 < x < 1, \\ w_x(0) \in \gamma_0(w(0)),\ \text{and } -w_x(1) \in \gamma_1(w(1)), \\ w(\mu) = 0 \text{ for some } \mu \in [0, 1]. \end{array}\right\} \tag{10.2.47}$$

It has been shown that if a stationary solution does not exist, then $t_* < \infty$, and if Eq. (10.2.47) has a solution, then there exist a minimum solution $\underline{w}(x)$ and a maximum solution $\overline{w}(x)$. Let

$$\underline{S} = \begin{cases} 0, & \text{if } \underline{w}(x) \equiv 0 \\ \text{the unique zero of } \underline{w}(x), & \text{if } \underline{w}(x) \neq 0. \end{cases}$$

and

$$\overline{S} = \begin{cases} 1, & \text{if } \overline{w}(x) \equiv 0 \\ \text{the unique zero of } \overline{w}(x), & \text{if } \overline{w}(x) \neq 0 \end{cases}$$

The following proposition describes some properties of the asymptotic solution.

Proposition 10.2.3. *Suppose that $t_* = \infty$. Then there exists a real number S^* with $\underline{S} \leq S^* \leq \overline{S}$, and a solution $w^*(x)$ of Eq. (10.2.47) such that*

(i) $\lim_{t\to\infty} S(t) = S^*$, (ii) $\lim_{t\to\infty} T(x,t) = w^*(x)$ in $C[0,1]$, (iii) $w^*(x) = w_x(0)(x - S^*)$.

In the multiphase phase-change problems, the density change in the phases gives rise to mass transport. The formulation of problems with both heat and mass transfer is not our concern here. Our interest in the problem discussed in [317] which is described below in Eqs (10.2.48)–(10.2.54), arises from the fact that after suitable transformation, the formulation becomes an extended formulation of the types (3.2.1)–(3.2.8). The local-in-time existence of the solution and well-posedness of the freezing problems (10.2.48)–(10.2.54) has been established in [317]. The following equations are to be satisfied:

$$\beta_S(T_S)(T_{S,t} + V_S T_{S,x}) - (K_S(T_S)T_{S,x})_x = 0, \quad 0 < x < S(t), \quad 0 < t < t_*, \tag{10.2.48}$$

$$T_S(x,0) = \phi_1(x), \quad 0 \leq x \leq S(0) = b; \quad T_S(0,t) = f_1(t), \quad 0 < t < t_*, \tag{10.2.49}$$

$$\beta_L(T_L)(T_{L,t} + V_L T_{L,x}) - (K_L(T_L)T_{L,x})_x = 0, \quad S(t) < x \leq d, \quad 0 < t < t_*, \tag{10.2.50}$$

$$T_L(x,0) = \phi_2(x), \quad b \leq x \leq d; \quad T_L(0,t) = f_2(t), \quad 0 < t < t_*, \tag{10.2.51}$$

$$T_S(S(t),t) = T_L(S(t),t) = T_m(P), \quad 0 < t < t_*, \tag{10.2.52}$$

$$[\rho_S l + (\rho_L C_L - \rho_S C_S) T_m(P)]\,\dot{S}(t) = (\rho_L C_L V_L - \rho_S C_S V_S) T_m(P)$$

$$+ K_S T_{S,x} - K_L T_{L,x}, \text{ at } x = S(t), \quad 0 < t < t_*, \quad \dot{S}(t) > 0, \tag{10.2.53}$$

$$(\rho_S - \rho_L)\dot{S}(t) = \rho_S V_S - \rho_L V_L, \text{ at } x = S(t), \quad 0 < t < t_*. \tag{10.2.54}$$

Here, $P = P(x,t)$ is the pressure and the phase-change temperature $T_m(P)$ in Eq. (10.2.52) is a pressure dependent known quantity. Both solid and liquid phases are compressible and V_S and V_L are velocities of the two phases and are assumed to be known from the hydrodynamical and thermoelastic considerations. $\beta_i(T_i) = C_i + a_i\gamma_i + (-1)^i T_i \gamma_i (\rho_i \alpha_i)^{-1}$, $i = S, L$ and equal to 1, 2 in $(-1)^i$; a_i are some specified positive constants, α_i is the compressibility of the ith phase and γ_i is the volumetric thermal expansion coefficient. Eq. (10.2.52) describes the local thermodynamic equilibrium temperature of the two phases, Eq. (10.2.53) describes the condition of dynamical compatibility for heat transfer and Eq. (10.2.54) is the mass balance condition at the interface. The coefficient of $\dot{S}(t)$ in Eq. (10.2.53) is the jump in the enthalpy across $S(t)$ and the first term on the right represents the difference in rates at which heat enters and leaves across $S(t)$ by convection. The second term on the r.h.s is the difference in the fluxes. Flux prescribed boundary conditions can also be considered at the fixed boundaries.

On using the transformation (1.4.29) in formulations of both phases, making use of some thermodynamical relations and adopting suitable notations, Eqs (10.2.48)–(10.2.54) get reduced to equations similar to Eqs (3.2.1)–(3.2.8) (provided velocity terms are added in Eqs 3.2.1–3.2.8 and the coefficients are redefined suitably). The small-time existence of the classical solution has been proved in [317] using the method of approximating solutions (see Eqs 10.2.46–10.2.51) and making use of some results given in [295]. It has been proved that $S(t) \in H^{1+\alpha/2}(0, t_*)$ for any $\alpha \in (0,1)$. The continuous dependence of the solution on the data and coefficients has been established from which the uniqueness of the solution also follows.

10.2.2 Differentiability and Analyticity of the Free Boundary in the One-Dimensional Two-Phase Stefan Problems

We assume that a classical solution of the two-phase problem (10.2.9)–(10.2.16) exists and the Stefan condition (10.2.16) holds for $< t \leq t_*$. For the one-phase problem formulated in Eq. (10.1.71), the proof of the infinite differentiability of $S(t)$ for $0 < t \leq t_*$, depends mainly on the application of Proposition 10.1.11 which tells us about the function space to which the temperature belongs. We assert from Eq. (10.1.76) that $\dot{S}(t) \in H_\alpha(Q_\delta)$, $\delta \geq 0$, $0 < \alpha < 1$. If appropriate assumptions on coefficients in the heat equations after transformations (see Eq. 10.1.73) in both the two-phases are made, then Proposition 10.1.11 can be used to conclude that the second order temperature derivatives in the two-phases are Hölder continuous with exponent α, $0 < \alpha < 1$. Arguments similar to those used in Section 10.1.2 for proving the infinite differentiability of $S(t)$ and of the temperature in the one-phase problem can be put forward for proving the infinite differentiability of $S(t)$ and of temperatures in both phases in a two-phase problem. The infinite differentiability of the free boundary in a two-phase problem under unilateral boundary conditions has been discussed in [315, 316].

The analyticity of the free boundary in a two-phase ice-water system described by Eqs (10.2.9)–(10.2.16) (with all thermophysical parameters taken as unity) has been described in [296]. Let $f_1(t) > 0, f_2(t) < 0, \phi_1(x) \geq 0, \phi_2(x) \leq 0$; f_i and ϕ_i, $i = 1, 2$, be continuously differentiable functions. Further, $f_1(0) = \phi_1(0), f_2(0) = \phi_2(d), \phi_1(b) = \phi_2(b) = 0$. Making use of the transformation $y = x/S(t)$ and using Schauder estimates for parabolic equations satisfied by $V_1(y,t) = T_1(x,t)$ and $V_2(y,t) = T_2(x,t)$, one can deduce (as done in the one-phase problem (see Section 10.1.2 and [296]) that if a classical solution of Eqs (10.2.9)–(10.2.16) exists, then $S(t)$ is a C^∞-function. Concerning analyticity of $S(t)$, the following proposition has been proved in [296].

Proposition 10.2.1. *If $f_1(t)$ and $f_2(t)$ are analytic functions for $0 \leq t \leq t_*$, in Eqs (10.2.9)–(10.2.16) then $S(t)$ is analytic for $0 < t \leq t_*$.*

To prove the above proposition, in addition to quantities defined in Eqs (10.1.77)–(10.1.79), we define the following transformations:

$$Z = \frac{(d-x)}{(d-S(t))} \quad \text{and} \quad \sigma = \int_0^t \frac{d\lambda}{(d-S(\lambda))^2}, \tag{10.2.55}$$

$$V(y,\tau) = T_1(x,t) - (1 - x/S(t))f_1(t), \tag{10.2.56}$$

$$W(y,\sigma) = T_2(x,t) - (x - S(t))/(d - S(t))f_2(t). \tag{10.2.57}$$

With the help of the above transformations, the heat equation in each phase is reduced to a form suitable for the application of Proposition 10.1.11 and various estimates for $S(t)$ and derivatives of temperatures can also be obtained. The local-in-time unique classical solution can be extended to a global-in-time solution provided the a priori estimates as given below can be obtained:

$$\left|\frac{\partial T_1}{\partial x}(x,t)\right| \leq A_t, \quad 0 < x < S(t) \quad \text{and} \quad \left|\frac{\partial T_2}{\partial x}(x,t)\right| \leq A_t, \quad S(t) < x < d, \tag{10.2.58}$$

where A_t is a bounded function of t. It has been proved in [318] that A_t is bounded under suitable assumptions.

The analyticity of the free boundary in a strongly nonlinear two-phase Stefan problem has been discussed in [298] by using the method of proof (expressing temperatures in terms of heat potentials and treating the problem in a complex plane by defining a suitable complex variable) discussed earlier in Section 10.1.2 concerning the analyticity of a one-phase problem. The heat source terms have been taken in the form $F_i = F_i(t, x, T_1, T_2, T_{1x}, T_{2x}, S, \dot{S})$, $i = S, L$. The one-phase formulation (10.1.23)–(10.1.27) can be easily generalized to obtain the formulation of a two-phase problem defined in $D_{t_*} = \{(x,t) : (0 \leq x \leq S(t)) \cup (S(t) \leq x \leq 1), 0 \leq t \leq t_*\}$ and is not being given here. Under suitable assumptions it has been proved that $S(t)$ is an analytic function of $t^{1/2}$ in $0 < t < \hat{t}$, and $T_i(\lambda^* S(t), t)$ and $T_{i,x}(\lambda^* S(t), t)$ are analytic in $t^{1/2}$ in $0 < t < \hat{t}$, for each $\lambda \in [0, 1]$. Here $\lambda^* = \lambda$ if $i = 1$ $(x < S(t))$ and $\lambda^* S(t) = 1 + \lambda(S(t) - 1)$ if $i = 2$ $(x > S(t))$.

10.2.3 One-Dimensional n-Phase Stefan Problems With $n > 2$

In this subsection, an n-phase problem refers to a problem with n phases and $(n-1)$, $n > 2$ distinct free boundaries. Some n-phase problems have already been discussed in Sections 3.2.3 and 3.2.4. We present below a simple formulation of an n-phase Stefan problem. Find $(n-1)$ free boundaries $S_i(t)$, $i = 1, 2, \ldots, (n-1)$ and temperatures $T_r(x,t)$, $r = 1, 2, \ldots, n$, such that

$$0 < S_1(t) < \cdots < S_{n-1}(t) < d; \quad S_i(0) = b_i, \quad i = 1, 2, \ldots, (n-1), \tag{10.2.59}$$

$$k_r T_{r,xx} - T_{r,t} = 0, \text{ if } S_{r-1}(t) < x < S_r(t), \quad 0 < t < t_*; \quad 1 \le r \le n, \tag{10.2.60}$$

$$S_0(t) = 0, \quad \text{and} \quad S_n(t) = d, \tag{10.2.61}$$

$$T_1(0,t) = f_1(t), \quad \text{and} \quad T_n(d,t) = f_2(t); \ 0 < t < t_*, \tag{10.2.62}$$

$$T_r(x,0) = \phi_r(x), \quad b_{r-1} \le x \le b_r, \ r = 1,2,\ldots,n; \ b_0 = 0, \ b_n = d, \tag{10.2.63}$$

$$T_r(S_i(t),t) = 0, \quad 0 < t < t_*; \quad r = 1,2,\ldots,n; \quad i = 1,2,\ldots,(n-1) \\ (\text{for } i = m, \ r = m \text{ and } (m+1)), \tag{10.2.64}$$

$$-K_r \frac{\partial T_r}{\partial x}(S_i(t) - 0) + K_{r+1}\frac{\partial T_{r+1}}{\partial x}(S_i(t) + 0) = (-1)^{i-1}\frac{dS_i(t)}{dt}, \quad 0 < t < t_*; \\ r = 1,2,\ldots,(n-1), \quad i = 1,2,\ldots,(n-1) \ (\text{for } i = m, \ r = m \text{ and } (m+1)). \tag{10.2.65}$$

The above formulation can be easily generalized to more complicated problems. When the number of phases is more than two, in addition to the study of those aspects of the analysis discussed earlier, the questions related to the disappearance of phases should also be addressed. A three-phase problem in which regions $-\infty < x < S_1(t)$, $S_1(t) < x < S_2(t)$ and $S_2(t) < x < \infty$ are initially occupied by water, ice, and water, respectively, has been studied in [319]. Thermal properties of the two water regions are the same but their initial temperatures are taken be different. Depending on the initial temperatures, the piece of ice can melt away entirely at some finite time t_* or water can freeze at each ice-water interface. Let $T_1(x,t)$, $T_2(x,t)$ and $T_3(x,t)$ be temperatures of water, ice and water, respectively, with $T_1(x,0) = \phi_1(x) \ge 0$, $-\infty < x \le b_1$, $T_2(x,0) = \phi_2(x) \le 0$, $b_1 \le x \le b_2$, and $T_3(x,0) = \phi_3(x) \ge 0$, $b_2 \le x < \infty$. Here the latent heat has been taken as unity. Suppose that after time $t > t_*$, the ice phase disappears and $T(x,t)$ be the temperature of water for $t \ge t_*$ with $T(x,t_*) = T_1(x,t_*)$ for $-\infty < x < S_1(t_*) = S_2(t_*)$ and $T(x,t_*) = T_3(x,t_*)$ for $S_2(t_*) < x < \infty$. The global existence and uniqueness of the solution $(S_1, S_2, T_1, T_2, T_3, T)$ has been proved. It has been shown that S_1 and S_2 depend continuously and monotonically on the data. The main tool used in the proofs is the maximum principle, both in its strong form [320] and in the form of the parabolic version of Hopf's lemma [321]. The constructive element in the approach in [319] is based on the idea of retarding the argument in the Stefan conditions at the free boundary (see the discussion concerning Eq. 10.2.34).

It has been shown that under suitable assumptions, free boundaries are continuously differentiable and $|\dot{S}_i(t)| \le$ a constant, for $0 \le t < t_*$, $i = 1,2$. Let $E(t) = H(t) - (S_2(t) - S_1(t))$, where $H(t)$ is the sensible heat of the system at time t. It can be proved that $E(t) = E(0)$. The following proposition holds.

Proposition 10.2.2. *Suppose that* $\lim_{t\to\infty} H(t) = 0$. *If* $E(0) > 0$, *then* t_* *is finite. If* $E(0) = 0$, *then* t_* *is plus infinity and* $\lim_{t\to\infty}(S_2 - S_1) = 0$. *If* $E(0) < 0$, *then* $\lim_{t\to\infty}(S_2 - S_1) = -E(0) < 0$ *and there does not exist a finite* t_*. *If* $t_* = \infty$, *and if the initial temperatures are bounded and have compact support in* $-\infty < x < \infty$, *then* $\lim_{t\to\infty} H(t) = 0$.

A three-phase problem similar to that considered in [319] has been considered in [322] in the bounded region $-d \le x \le d$. A piece of ice with temperature $T_2(x,t)$ occupies the region $S_2(t) \le x \le S_1(t)$ and $T_2(x,0) = \phi_2(x) < 0$. The region $-d \le x \le S_2(t)$ is water having the temperature $T_1(x,t)$ and $T_1(x,0) = \phi_1(x) > 0$. The region $S_1(t) \le x \le d$ is also water having the temperature $T_3(x,t)$ and $T_3(x,0) = \phi_3(x) > 0$. $T_1(-d,t) = f_1(t) > 0$ and $T_3(d,t) = f_3(t) > 0$. The classical formulation of this problem can be easily written. Let this problem be called Problem (AK). The main result of [322] is the following proposition.

Proposition 10.2.3. *Let* $\{T_1, T_2, T_3, S_1, S_2\}$ *be the classical solution of Problem (AK). There exist constants* $\alpha \in R^+$, $\beta \in R^+$, $\lambda_1 \in R^-$, $\lambda_2 \in R^+$, $\hat{t} \in R^+$ *and* $t_* \in R^+$ *such that*

$$|T_2(x,t)| \leq \alpha e^{\beta/(t-t_*)} \text{ for } t \in (0, \hat{t}); \quad S_2(t) \leq x \leq S_1(t), \tag{10.2.66}$$

$$\lim_{t \to t_* - 0} \left. \frac{\partial T_2}{\partial x} \right|_{x = S_i(t)} = 0, \quad i = 1, 2; \quad S_1(t_*) = S_2(t_*), \tag{10.2.67}$$

and

$$\lim_{t \to t_* - 0} \frac{dS_i}{dt} = \lambda_i, \quad i = 1, 2; \quad -\infty < \lambda_1 < 0, \quad 0 < \lambda_2 < \infty. \tag{10.2.68}$$

The temperature in the solid phase decreases to zero as t approaches t_*.

The existence and uniqueness of similarity solutions of a one-dimensional multiphase Stefan problem has been discussed in [323]. The formulation of the problem is the same as in [65] (see Section 3.2.3) except that phases are not in motion. The sufficient conditions for the existence of the similarity solution have been obtained in [323] but are not being discussed here as they can be described only after describing the complete solution of the problem.

Analyticity of $(n-1)$ nonintersecting phase-change boundaries in an n-phase problem has also been discussed in [296]. The formulation of the problem is similar to that given in Eqs (10.2.59)–(10.2.65) except that all the thermophysical parameters have been taken equal to unity. It has been assumed that $(-1)^{m-1}\phi_m(x) \geq 0$, $f_1 > 0$, $(-1)^n f_2 < 0$, $\phi_m(b_m + 0) = \phi_{m+1}(b_m - 0) = 0$; $1 \leq m \leq n$, $\phi_{n+1} \equiv 0$, $\phi_1(0) = f_1(0)$, $\phi_n(d) = f_2(0)$; $\phi_r (1 \leq r \leq n)$, f_1 and f_2 are continuously differentiable, and b_i $(1 \leq i \leq n-1)$ are given constants. Under suitable assumptions, using the arguments given in the case of one-phase and two-phase problems the existence of the unique solution of this n-phase problem in a small time interval can be proved. If a suitable transformation is used in each phase, a fixed domain formulation in each phase can be obtained. On making suitable assumptions and using Proposition 10.1.11, it can be shown that free boundaries are C^∞-curves (cf. [296]). To prove the existence of a global classical solution it suffices to establish a priori bounds in (10.2.58) for the temperature $T(x,t)$ defined in $0 \leq x \leq d$, $x \neq S_i(t)$. For the present problem, A_t (see Eq. 10.2.58) is given by

$$A_t = \max \left\{ \sup_{0<\tau<t} |T_x(0,\tau)|, \sup_{0<\tau<t} |T_x(d,\tau)|, \max_{1 \leq r \leq n} \sup_{b_{r-1} \leq x \leq b_r} \left|\phi_r'(x)\right| \right\}, \tag{10.2.69}$$

for all t for which $S_i(\tau)$, $1 \leq i \leq n-1$, $\tau < t$ do not intersect each other.

Proposition 10.2.4. *If* $f_1(t)$, $f_2(t)$ *are analytic functions for* $0 \leq t \leq t_*$ *in Eqs (10.2.59)–(10.2.65), then* $S_1(t), \ldots, S_{n-1}(t)$ *are analytic functions for* $0 < t \leq \sigma$, *where* σ *is the first time such that*

$$\lim_{t \to \sigma -} \min_i [S_i(t) - S_{i-1}(t)] = 0. \tag{10.2.70}$$

To prove the above proposition, functions V_r have been introduced in [296] which are defined as follows:

$$V_r(y_i, \tau_i) = T(x,t), \quad 1 \leq r \leq n, \tag{10.2.71}$$

where $T(x,t)$ is the temperature in the region $0 \le x \le d$, and y_i and τ_i are related to x, t in the ith phase by the relations

$$\tau_{i+1} = \int_0^t \frac{d\lambda}{[S_{i+1}(\lambda) - S_i(\lambda)]^2}, \quad y_{i+1} = \frac{x - S_i(t)}{S_{i+1} - S_i}; \quad 1 \le i \le n-1. \tag{10.2.72}$$

To prove the analyticity, the inductive estimates are then obtained for derivatives of V_r and $S(t)$ with respect to τ.

10.3 ANALYSIS OF THE CLASSICAL SOLUTIONS OF MULTIDIMENSIONAL STEFAN PROBLEMS

10.3.1 Existence and Uniqueness Results Valid for a Short Time

The earliest results of general nature available on the existence and uniqueness of the classical solutions of multidimensional Stefan problems seem to be those reported in [324]. A two-dimensional two-phase Stefan problem has been considered in a rectangular region Ω, where

$$\Omega = \{(x_1, x_2) : 0 \le x_1 \le d_1, 0 \le x_2 \le d_2\} \subset R^2; \quad \text{and} \quad \Omega_{t_*} = \Omega \times (0, t_*). \tag{10.3.1}$$

The classical formulation of this problem will not be presented here as it can be obtained by making appropriate changes in the formulation given in Section 3.2.4. In the present problem, it has been assumed that a single free boundary exists which has a parametric representation

$$x_1 = x_1(\lambda, t), \quad x_2 = x_2(\lambda, t); \quad x_{1,\lambda}^2 + x_{2,\lambda}^2 \ne 0; \quad \lambda_0 \le \lambda \le \lambda_1. \tag{10.3.2}$$

The functions $x_{1,t}$, $x_{2,t}$, $x_{1,\lambda}$ and $x_{2,\lambda}$ are continuous and all the interior points of the surface obtained after eliminating λ in Eq. (10.3.2) are the interior points of $\overline{\Omega}_{t_*}$.

On eliminating λ from the two equations in Eq. (10.3.2), we get an equation of the free boundary S as $\Phi(x_1, x_2, t) = 0$. In view of the parametric representation in Eq. (10.3.2), instead of a Stefan condition of the type (1.4.8), the following equations have been considered at the free boundary:

$$\frac{\partial x_1}{\partial t} = K_1 \frac{\partial T_1}{\partial x_1} - K_2 \frac{\partial T_2}{\partial x_1}, \tag{10.3.3}$$

$$\frac{\partial x_2}{\partial t} = K_1 \frac{\partial T_1}{\partial x_2} - K_2 \frac{\partial T_2}{\partial x_2}. \tag{10.3.4}$$

We now consider a classical two-phase Stefan problem in Ω_{t_*} in which the Stefan condition is replaced by the conditions (10.3.3) and (10.3.4). Let the free boundary S be given by $\Phi(x_1, x_2, t) = 0$ and temperatures $T_1(x_1, x_2, t)$ and $T_2(x_1, x_2, t)$ in the two-phases satisfy all the conditions mentioned in the definition of the classical solution given in Section 1.4.6. Then the solution (T_1, T_2, S) in which S has the parametric representation and the Stefan condition is replaced by Eqs (10.3.3)–(10.3.4) also forms a classical solution of the Stefan problem (substitute Eqs 10.3.3, 10.3.4 in the Stefan condition).

Homogeneous parabolic heat equations have been considered in [324] in the two phases and heat fluxes have been prescribed on the fixed boundaries. The fluxes and their derivatives, initial temperature and the function representing the initial position of the free boundary have been taken to be continuous functions. On using suitable Green's functions, temperatures at the free boundary together with the temperature derivatives, can be expressed in terms of nonlinear Volterra equations of the second kind (see Eq. 9.6.32). It has been reported that under suitable assumptions a unique classical solution exists for a short time. However no details of the proof are available. The classical solution is stable in the norm $C(\Omega_{t_*})$ with respect to perturbations of the data in $C(\Omega_*)$. It has been proposed that similar results about the existence, uniqueness and stability will hold good for problems considered in $\Omega \subset R^n$, $n > 2$.

A problem more general than the one considered in [324] has been considered in [325]. There are m-phases in the region $0 \le x_2 \le d$, $-\infty < x_1 \le S^{(m)}$, where $S^{(1)}, S^{(2)}, \ldots, S^{(m-1)}$ are simple continuous, piecewise smooth, nonintersecting free boundaries and $S^{(m)}$ is a given surface. The parabolic equations considered in different phases are nonhomogeneous and temperatures at the free boundaries are unequal and are not constant. Each free boundary has a parametric representation of the form (10.3.2). At each free boundary, instead of the Stefan condition, conditions given in Eqs (10.3.3), (10.3.4) have been considered in which thermal conductivities are functions of (x_1, x_2, t). On obtaining solutions of temperatures with the help of appropriate heat potentials and using contraction mapping argument, the existence, uniqueness and stability of the solution have been investigated for a short time. It has been reported that the method and the results obtained can be extended to problems in R^n, $n > 2$.

The multidimensional multiphase problem considered in [325] has been further generalized in [326]. A general linear parabolic equation of second order has been considered in a noncylindrical region occupied by each phase. The boundary of the region could be piecewise smooth. Instead of a Stefan condition, conditions of the forms (10.3.3) and (10.3.4) have been considered at the free boundaries. On using suitable heat potentials, at the free boundaries, temperatures and temperature derivatives can be obtained in terms of Volterra integral equations of the second kind. Under suitable assumptions (see [326]), contraction mapping theorem can be applied to the system of integral equations to which the problem is reduced and it has been proved that the existence, uniqueness and stability of the solution holds for a short time.

The existence and uniqueness proofs have been constructed in [309] using local coordinates and parametrization of the free boundary. This requires 'regularization' of the Stefan condition as after transformation in the local coordinates the compatibility conditions at the free boundary are not satisfied. The classical solutions of both one- and two-phase problems have been investigated and the existence of solutions has been studied for both short time and long times. We begin with the main steps in the proof of the existence of short-time solutions of one-phase Stefan problems. Let $\Omega_{t_*} = \Omega \times (0, t_*) \subset R^n_{t_*} = R^n \times (0, t_*)$, $n \ge 1$ and $\Omega(t) \subset R^n$, $t \in (0, t_*)$, $\Omega(0) = G \subset R^n$. Ω_{t_*} is a domain lying between two hypersurfaces $F_{t_*} = F \times (0, t_*)$, $F \subset R^n$, $n \ge 1$ and $\Gamma_{t_*} = \{(x, t); x \in \Gamma(t) \subset R^n, t \in (0, t_*)\}$. F_{t_*} is a known surface and Γ_{t_*} is the unknown free boundary. The classical solution of one-phase Stefan problem consists of finding the temperature $T(x, t)$, $x = (x_1, x_2, \ldots, x_n) \in R^n$, $n \ge 1$, $0 < t < t_*$ and the free boundary $\Gamma(t)$, $0 < t < t_*$, satisfying the following system of equations:

$$a(T)\frac{\partial T}{\partial t} = \sum_{i=1}^{n} \frac{\partial^2 T}{\partial x_i^2} + f(x, t), \text{ in } \Omega_{t_*}, \tag{10.3.5}$$

$$T = T_1(x,t), \text{ or } \sum_{i=1}^{n} b_i(x,t)\frac{\partial T}{\partial x_i} + g(x,t)T = T_2; \text{ on } F_{t_*}, \tag{10.3.6}$$

$$T(x,t) = 0, \text{ on } \Gamma_{t_*}, \tag{10.3.7}$$

$$\vec{V}\cdot\vec{\nu} = -\frac{\partial T}{\partial t}/\left|\nabla T\right| = \sum_{i=1}^{n}\nu_i\frac{\partial T}{\partial x_i}, \text{ on } \Gamma(t),\ t\in(0,t_*);\quad |\nabla T|\neq 0,\ (x,t)\in\Gamma_{t_*}, \tag{10.3.8}$$

$$\Gamma(0) = S;\quad T(x,0) = T_0(x),\quad x\in\Omega(0) = G. \tag{10.3.9}$$

Here, $\vec{\nu} = (\nu_1, \nu_2, \ldots, \nu_n)$ is the unit vector normal to $\Gamma(t)$, $\vec{\nu} = \nabla T/\left|\nabla T\right|$ and $\vec{V}$ is the velocity of the free boundary. The surface $S\,(=\Gamma(0))$ is a C^2-surface and does not intersect F. The compatibility conditions of a suitable order which follow from Eqs (10.3.5)–(10.3.9) are satisfied. The basic step in the proof of the existence of the solution of multidimensional Stefan problems consists of establishing some a priori estimates for the solution in a neighbourhood of the free boundary $\Gamma(t)$. Therefore, with the help of a scalar function $R(x_0,t)$, $x_0\in S$, $\Gamma(t)$ is defined by the equality

$$x = x_0 + R(x_0,t)\vec{\nu}(x_0),\quad \vec{\nu}(x_0) = D\,T_0(x_0)/\left|DT_0(x_0)\right|,\quad \vec{\nu} = (\nu_1,\nu_2,\ldots,\nu_n). \tag{10.3.10}$$

Here, $D = (\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)$ and $|R(x_0,t)| \leq 2B$, B is a constant which is chosen small enough so that $\Gamma(t)$ defined in Eq. (10.3.10) does not intersect F. In view of Eq. (10.3.10), the isotherm condition (10.3.7) becomes

$$T(x_0 + R(x_0,t)\vec{\nu}(x_0),t) = 0. \tag{10.3.11}$$

On differentiating Eq. (10.3.11) w.r.t. t and using Eq. (10.3.8), the Stefan condition becomes

$$\frac{\partial}{\partial t}R(x_0,t) = -\left|DT(x,t)\right|^2/(DT(x,t)\,\vec{\nu}(x_0)). \tag{10.3.12}$$

The main result of [309] concerning the existence and uniqueness of the solution of problem (10.3.5)–(10.3.9) is the following proposition.

Proposition 10.3.1. *Assume the following hypotheses:*

(i) *Closed surfaces F and S bound the sets $W_F\subset R^n$ and $W_S\subset R^n$, respectively, such that $W_F\subset W_S$, $F\cap S=\emptyset$, and F and S belong to the class H^{2r}, $[r] = m+1 = n+5$, $[r]$ stands for the greatest integer in r (for m see the next assumption).*

(ii) *$a\in C^{m+2}[0,\infty)$, and functions T_1, g and b_i ($i = 1,2,\ldots,n$) are in the space $H^{2r,r}(F_\infty)$ and*

$$a \geq M_0^{-1} = \text{constant} > 0,\quad \sum_{i=1}^{n} b_i q_i \geq M_0^{-1}, \tag{10.3.13}$$

where $\vec{q} = (q_1, q_2, \ldots, q_n)$ is the normal to the surface F at the point x.

(iii) *$f\in H^{2r,r}(R^n_\infty)$, and $T_0\in H^{2r}(\overline{G})$. On the surfaces F and S the compatibility conditions up to the order $m+1$ which follow from Eqs (10.3.5)–(10.3.7), (10.3.12) hold. Moreover,*

$$T_0(x) > 0,\quad x\in G;\quad \left|\log\left|DT_0(x)\right|\right| \leq N_0,\quad x\in S. \tag{10.3.14}$$

(iv) *Norms of the functions f, T_0, T_1, T_2 defined in appropriate spaces are bounded by a common constant N_0, and norms of a, g and b_i $(i = 1, 2, \ldots, n)$ and norms of functions defined in local coordinates of surfaces F and S are bounded by a constant M_0.*

Then there exists a sufficiently small $t_* > 0$, dependent only on M_0, N_0 and B, such that problem (10.3.5)–(10.3.9) admits a unique solution $(R(x_0, t), T(x, t))$ with $R(x_0, t) \in C^{2,1}(S \times (0, t_*))$ and $T \in C^{2,1}(\overline{\Omega}_{t_*})$.

It may be noted that if $f(x, t) = 0$, then it follows from the condition $T_0(x) > 0$ and the maximum principle that $T(x, t)$ is strictly positive at least in a sufficiently small time interval $(0, t_*)$. If $f(x, t) \neq 0$, then $T_0(x) > 0$ implies the positivity of the short-time solution outside a small neighbourhood of the free boundary Γ_{t_*}. The second inequality in Eq. (10.3.14) allows us to use local coordinates. In a simple problem setting in which the surface F is given by $x_n = 1$ and S is given by $x_n = 0$; $f = 0$, $T_1(x) = 1$, $T_0(x)$ is 1-periodic with respect to $x' = (x_1, x_2, \ldots, x_{n-1})$ and

$$|\log|D_n T_0(x)|| \leq N_0 \text{ for } x \in G, \quad D_n = \partial/\partial x_n, \tag{10.3.15}$$

the following local coordinates (von Mises variables) can be used:

$$\tau = t, \quad y' = (y_1, y_2, \ldots, y_{n-1}) = x', \quad y_n = T(x, t). \tag{10.3.16}$$

The surface F_{t_*} is invariant under the mapping $(x, t) \to (y, \tau)$ and the surface $S_{t_*} = S \times (0, t_*)$ corresponds to the free boundary Γ_{t_*}. The region $G_{t_*} = G \times (0, t_*)$ corresponds to Ω_{t_*}. A new dependent variable $u(y, \tau) = x_n$, 1-periodic with respect to x' is defined and the Stefan problem (10.3.5)–(10.3.9) can be formulated in terms of $u(y, \tau)$. Compatibility conditions satisfied earlier according to Eqs (10.3.5)–(10.3.7), (10.3.12) are lost in the new formulation which can be restored by an appropriate 'regularization' of the Stefan condition transformed in terms of $u(y, \tau)$.

The regularization of the Stefan condition can be explained only with the help of the problem formulated in terms of von Mises variables and therefore the reader is referred to [309] for further details. Let u^ε be the solution of the regularized problem. It has been proved in [309] that on a sufficiently small time interval $(0, t_*^\varepsilon)$, a unique solution of the regularized problem exists and $u \in H^{2r,r}(\overline{G}_{t_*^\varepsilon})$, $[r] = m + 1 = n + 5$.

For any arbitrary surface S, a unique representation of the free boundary $\Gamma(t)$ in the form $x_n = S(x', t)$ is generally not available and the regularization of the compatibility conditions which was possible in a simple problem setting discussed above, is not possible in a general case. However, on a sufficiently small time interval, a regularized problem can be defined in a different way as follows.

We look for a scalar function $R^\varepsilon(x_0, t)$ defined on the surface S_{t_*} which determines the surface $\Gamma_{t_*}^\varepsilon$ by the equation

$$x = x_0 + R^\varepsilon(x_0, t)\vec{\nu}(x_0), \text{ for } x_0 \in S. \tag{10.3.17}$$

Here, $\vec{\nu}(x_0)$ is the normal as defined in Eq. (10.3.10). The temperature $T^\varepsilon(x, t)$ and R^ε satisfy the following system of equations:

$$a(T^\varepsilon)\frac{\partial T^\varepsilon}{\partial t} = \sum_{i=1}^{n} \frac{\partial^2(T^\varepsilon)}{\partial x_i^2} + f, \text{ for } (x, t) \in \Omega_{t_*}^\varepsilon, \tag{10.3.18}$$

$$T^\varepsilon = T_1^\varepsilon \text{ or } \sum_{l=1}^{n} b_i \frac{\partial T^\varepsilon}{\partial x_i} + gT^\varepsilon = T_2^\varepsilon, \text{ for } (x,t) \in F_{t_*}, \tag{10.3.19}$$

$$T^\varepsilon(x,t) = 0, \text{ for } (x,t) \in \Gamma_{t_*}^\varepsilon, \tag{10.3.20}$$

$$T^\varepsilon(x,t) = T_0^\varepsilon(x), \text{ for } x \in G; \quad R^\varepsilon|_{t=0} = 0, \tag{10.3.21}$$

$$\frac{\partial R^\varepsilon}{\partial t} - \varepsilon \nabla_S^2 R^\varepsilon = -\left|DT^\varepsilon\right|^2 / (DT^\varepsilon \vec{\nu}(x_0)), \quad (x_0,t) \in S_{t_*}. \tag{10.3.22}$$

Here, $\Omega_{t_*}^\varepsilon$ is the region bounded by the surfaces F_{t_*}, $\Gamma_{t_*}^\varepsilon$ and the planes $t = 0$ and $t = t_*$. The derivative on the r.h.s. in Eq. (10.3.22) is calculated at (x,t) with x given by Eq. (10.3.17) and ∇_S^2 is the *Laplace–Beltrami operator* on the surface S (cf. [309]).

At $\varepsilon = 0$, the problem (10.3.18)–(10.3.22) coincides with the original Stefan problem and it is expected that for any $\varepsilon > 0$, the solution of Eqs (10.3.18)–(10.3.22) gives an approximate solution to the Stefan problem. The condition (10.3.22) is no more a Stefan condition and $T_0^\varepsilon(x)$ may not satisfy the compatibility conditions. Therefore, the initial temperature for T_ε should be changed to satisfy the compatibility conditions which follow from Eqs (10.3.18), (10.3.20), (10.3.22). It has been proved in [327] that for each $T_0(x) \in H^{2r}(\overline{G})$ satisfying the compatibility conditions up to the order $[r] = m + 1$ that follow from Eqs (10.3.5), (10.3.7), (10.3.8) on the surface S, a $T_0^\varepsilon(x) \in H^{2r}(\overline{G})$ exists. For each $\varepsilon > 0$, $T_0^\varepsilon(x)$ coincides with $T_0(x)$ outside a small neighbourhood of S and satisfies the compatibility conditions up to the order $m + 1$ that follow from Eqs (10.3.18), (10.3.20), (10.3.22). Further, $\lim\limits_{\varepsilon \to 0} \|T_0 - T_0^\varepsilon\|_G^{(2r)} = 0$, $R^\varepsilon \in H^{2r,r}(S_{t_*})$ and $T^\varepsilon \in H^{2r,r}(\overline{\Omega}_{t_*}^\varepsilon)$.

Using a variational inequality formulation of the classical one-phase multidimensional problem, the regularity of the free boundary has been proved in [328]. If for a fixed time t_0, the point x_0 is a density point for the coincidence set (ice) in a ice-water system, then in a neighbourhood in space and time of (x_0, t_0), the free boundary is a surface of class C^1 in space and in time and all the second derivatives (in space and in time) of the solution are continuous up to the free boundary. The solution is hence classical in that neighbourhood.

All the results on the solvability of the one-phase Stefan problem (10.3.5)–(10.3.9) apply without any change to the problem in a two-phase setting (cf. [309]). The formulation of the two-phase problem can be easily written and it will not be presented here. If the free boundary $\Gamma(t)$ is expressed as in Eq. (10.3.10), then the Stefan condition on the surface Γ_{t_*} becomes

$$\frac{\partial}{\partial t} R(x_0,t) = X^-(x_0, DT) - X^+(x_0, DT), \text{ for } (x_0,t) \in S_{t_*}. \tag{10.3.23}$$

In Eq. (10.3.23), DT is to be calculated at (x,t) and

$$X^\pm = \lim_{r \to 0} \left|DT(x_r^\pm, t)\right|^2 / (DT(x_r^\pm, t)\vec{\nu}(x_0)), \tag{10.3.24}$$

$$x_r^\pm = x_0 + \vec{\nu}(x_0)(R(x_0,t) \pm r) \in \Omega_{t_*}^\pm, \quad \Omega_{t_*} = \Omega_{t_*}^+ \cup \Omega_{t_*}^- \cup \Gamma_{t_*}. \tag{10.3.25}$$

Here, $\Omega_{t_*}^+$ and $\Omega_{t_*}^-$ are regions occupied by the two phases. $T_0(x)$ is strictly positive in G^+ and strictly negative in G^- and the second condition in Eq. (10.3.14) should be taken as

$$\lim_{r \to 0} |\log|DT_0(x \pm r\nu(x))|| < \infty, \text{ for } x \in S. \tag{10.3.26}$$

10.3.2 Existence of the Classical Solution on an Arbitrary Time Interval

How can a short-time solution be extended in time? It has been remarked in [324] that following the arguments proposed in [9] (see also [281]), the solution of the problem considered in [324] can be extended till the time the free boundary reaches $x_2 = d_2$. It may be noted that to construct a regular solution even on a short time interval $(0, t_*)$, the initial temperature $T_0(x)$ in problem (10.3.5)–(10.3.9) has to be in $H^{2r}(\overline{G})$, where $[r] = m+1 = n+5$. Therefore to extend the solution $T(x,t)$ beyond $t > t_*$, $T(x, t_*)$ should belong to $H^{2r}(\overline{\Omega}_{t_*})$. Such a regularity of the solution cannot be expected in general. However, the following statement can be made. The existence interval $(0, t_*^{\infty})$ of the classical solution of the one-phase Stefan problem is characterized by the relations

$$\left.\begin{aligned}
&J_0(t_*) = |\log|D_n T||_{\Omega_{t_*}}^{(0)} < \infty,\\
&J_1(t_*) = |T(x,t)|_{\Omega_{t_*}}^{(2r)} < \infty,\ t_* < t_*^{\infty},\\
&\quad \lim_{t_* \to t_*^{\infty}} \{J_0(t_*) + J_1(t_*)\} = \infty.
\end{aligned}\right\} \tag{10.3.27}$$

It turns out that under some restrictions on the data of the problem which ensure that the first equality in Eq. (10.3.27) is satisfied on an infinite time interval, the classical solution of the reference Stefan problem exists for all positive times.