Chapter 3

Extended Classical Formulations of n-Phase Stefan Problems With $n \geq 1$

3.1 ONE-PHASE PROBLEMS

3.1.1 An Extended Formulation of One-Dimensional One-Phase Problem

One-phase Stefan problems are mainly of three types. In the first type, one of the phases is considered at the melting temperature so that there is temperature gradient present only in one of the phases. Such problems could arise either in melting or solidification. Problems of the second kind are *ablation problems* in which a solid is melting and the liquid drains out instantaneously. In the third type, problems have to do with solidification of supercooled liquids, in which the temperature of the liquid is less than the equilibrium temperature leading to some changes in the energy balance equation at the interface. The melting problems in which the solid is superheated also belong to this class of problems.

One-dimensional one-phase problems with different formulations have been studied from the perspectives of mathematical analysis and computation. Instead of giving complete formulations of problems arising in different contexts, we give here a general formulation of the one-phase problem so that in later sections other one-phase problems can be treated as particular cases of this general problem. This will avoid unnecessary repetitions of some details common in different formulations.

A fairly general one-phase Stefan problem with nonlinear parabolic equation and nonlinear free boundary conditions has been discussed in [55] and is being presented here. Consider a time interval $(0, \hat{t}_*)$. For each $t \in (0, t_*)$, consider the set $\sum(t)$ of functions $p(\tau)$ which are continuously differentiable in $[0, t)$, continuous in [0,t], and are such that $p(\tau) \in (b_0, b_1)$ for $\tau \in (0, t)$, $p(0) = b$, $b_1 > b > b_0 > 0$. For $S(\tau) \in \sum(t_*)$ and $t \in (0, t_*)$ define the set $\Omega(t) \equiv \{(x, \tau) : 0 < x < S(\tau), 0 < \tau < t\}$ and let $C^{1,0}(\bar{\Omega}(t))$ be the set of functions $T(x, t)$ which are continuous in $\bar{\Omega}(t)$ together with their x-first derivatives.

Assume that $a(x, t, y_0, y_1, p_0, p_1)$ is a positive function for $0 \leq x < \infty$, $0 \leq t \leq t_*$, $-\infty < y_0, y_1 < \infty$, $p_0 \geq 0$, $-\infty < p_1 < \infty$. The function $q(x, t, y_0, y_1, p_0, p_1)$ is defined

The Classical Stefan Problem. https://doi.org/10.1016/B978-0-444-63581-5.00003-8

in the same domain as the function a, and $\tilde{h}(x), f(t)$ are defined for $0 \leq x \leq b$, $0 \leq t \leq \hat{t}_*$, respectively, and $\psi(x,t)$ is defined for $x \geq 0$, $0 \leq t \leq \hat{t}_*$.

The problem is to find a triple $(t_*, S(t), T(x,t))$ such that

(i) $0 < t_* \leq \hat{t}_*$,

(ii) $S(\tau) \in \sum(t_*)$,

(iii) $T(x,t) \in C^{1,0}(\bar{\Omega}(t_*))$, T_{xx} and T_t are continuous in $\Omega(t_*)$,

(iv) the following equations are satisfied:

$$T_t - a^2(x,t,T,T_x,S,\dot{S})T_{xx} = q(x,t,T,T_x,S,\dot{S}), \quad (x,t) \in \Omega(t_*), \tag{3.1.1}$$

$$T(x,0) = \tilde{h}(x), \quad x \in [0,b], \tag{3.1.2}$$

$$T(0,t) = f(t), \quad t \in (0,t_*), \tag{3.1.3}$$

$$T(S(t),t) = \psi(S(t),t), \quad t \in (0,t_*), \tag{3.1.4}$$

$$\dot{S}(t) = \phi(S(t),t,T_x(S(t),t)), \quad t \in (0,t_*). \tag{3.1.5}$$

Here $\phi(p,t,y_1)$ is a function defined for $p > 0$, $0 < t < t_*$, $-\infty < y_1 < \infty$, a^2 is the thermal diffusivity which has been considered as a function of independent and dependent variables, q is the heat source per unit time per unit volume. The temperature $T(x,t)$ can be made dimensionless by dividing it by T_m so that at the free boundary $T = 1$. By redefining the temperature as $T - 1$, the temperature at the free boundary can be taken to be zero. In the literature, both 0 and 1 are used as dimensionless isotherm temperatures. If we make the transformation $\hat{T} = T - \psi(x,t)$ in Eqs (3.1.1)–(3.1.5), then $\hat{T}$ becomes zero at the free boundary.

A mere formulation of the problem as given in Eqs (3.1.1)–(3.1.5) does not guarantee that its solution exists. Even if the solution exists, $T(x,t)$ and $S(t)$ may not satisfy the regularity conditions mentioned previously. The appropriate conditions to be satisfied for the existence, uniqueness, well-posedness, etc., of the solution of Eqs (3.1.1)–(3.1.5) and solutions of some other Stefan problems will be discussed in Chapter 10. Presently we are concerned with the formulation.

If the region $0 < x < S(t)$, $S(0) = b$ is taken as solid (could be identified as ice) and the region $x > S(t) (S(t) < x \leq b_1)$ is taken as liquid at the equilibrium temperature 0 (or 1) (identified as water) and there is cooling at $x = 0$, then we have a one-phase solidification problem. The growth of the solid in the region $x > b$ depends on some appropriate conditions which the prescribed data has to satisfy. In particular the data has to satisfy some *compatibility conditions* at $x = b$ for $\dot{S}(t)$ to be positive. If $b = b_0 = 0$ then the data has to satisfy some compatibility conditions at $x = 0$. If the region $0 < x < S(t)$ is liquid (could be identified as warm water) and the region $x > S(t)$ is solid (identified as ice) at the equilibrium temperature and if there is heating at $x = 0$ then we have a one-phase melting problem. In the melting problem also some compatibility conditions are to be satisfied by the data for $\dot{S}(t)$ to be positive. These compatibility conditions have been discussed in Chapter 10 for several problems. Any of the boundary conditions (A) discussed in Section 1.4.4 can be prescribed at $x = 0$ instead of Eq. (3.1.3).

Often the mathematical formulations of physical problems are presented in the form of a set of differential equations and boundary and initial conditions without defining the relevant function spaces in which a solution is sought, the spaces to which the known functions belong, ranges of dependent variables and domains of independent variables. In a rigorous mathematical formulation all such details should be mentioned as done for problem (3.1.1)–(3.1.5). However, due to length constraints it is not possible to do so for every problem discussed in this volume. These details may differ for different problems.

A physical problem can be generalized to any extent but it may not be possible to throw any light on of its solution. We shall discuss only those generalizations of Stefan problems which have been rigorously explored. When dealing with the classical formulation of a Stefan problem, its solution should satisfy some regularity conditions given earlier in Section 1.4.6.

In the place of Eqs (3.1.4), (3.1.5) more general boundary conditions of the type

$$T(S(t), t) = Z(\partial T/\partial x, S(t), \dot{S}(t)), \tag{3.1.6}$$

and

$$W(T, \partial T/\partial x, S(t), \dot{S}(t)) = 0, \quad \text{on } x = S(t), \tag{3.1.7}$$

can be prescribed. Here Z and W could be functionals and need not be pointwise functions of their arguments (cf. Eq. 3.3.33).

3.1.2 Solidification of Supercooled Liquid

Consider the following problem:

$$T_t = T_{xx}, \quad \text{in } D_{t_*} = \{(x,t) : 0 < x < S(t), 0 < t < t_*, S(0) = 1\}, \tag{3.1.8}$$

$$T(x, 0) = T_0(x) \leq 0, \quad 0 \leq x \leq 1, \tag{3.1.9}$$

$$T_x(0, t) = g(t) \geq 0, \quad 0 < t < t_*, \tag{3.1.10}$$

$$T(S(t), t) = 0, \quad 0 < t < t_*, \tag{3.1.11}$$

$$T_x(S(t), t) = -\dot{S}(t), \quad 0 < t < t_*. \tag{3.1.12}$$

By appropriate scaling of time and/or length, the various parameters have been taken to be unity in Eqs (3.1.8)–(3.1.12). This problem is concerned with the solidification of a supercooled liquid which initially occupies the region $0 \leq x \leq 1$ and ice at $T = 0$ occupies the region $1 < x < \infty$. It can be argued that the region $0 \leq x \leq 1$ cannot be solid. For if it were so then the region $x > 1$ would have to be liquid at the temperature zero and solidification would begin in the liquid. This would further imply that $T_x(S(t), t)$ and $\dot{S}(t)$ are positive, violating Eq. (3.1.12). This problem will be referred as *supercooled Stefan problem* or in short SSP. The analysis of such problems has been presented in Section 4.4.1.

3.1.3 Multidimensional One-Phase Problems

The formulation given in Eqs (3.1.1)–(3.1.5) can be extended to multidimensional one-phase problems if the free boundary is defined by the equation $\Phi(x, t) = 0, x \in R^n$ as in Section 1.4.1. Multidimensional problems have not been investigated as exhaustively as one-dimensional problems and often the problems studied are not as general as described in formulations (3.1.1)–(3.1.5). A typical multidimensional problem is the *ablation problem* described in the following section.

A Three-Dimensional Ablation Problem

During melting of a solid, if the melt is removed as soon as it is formed, heat flux has to be prescribed at the phase-change interface for further melting to take place. Melting of a piece of ice when water formed is removed instantaneously and the melting of the surface of a spacecraft during reentry into earth's atmosphere are examples of one-phase ablation problems.

Consider a half space $z \geq 0$ which at time $t = 0$ is in a solid state. Heat input $Q(x, y, z, t) > 0$ is prescribed at the free boundary $z = S(x, y, t) = S(t)$ and it will be assumed that melting starts instantaneously at $t = 0$ (this is a minor assumption, see Section 3.2.2). The melt is drained out as soon as it is formed. The problem is to find the temperature $T(x, y, z, t)$ and the phase-change boundary which we shall denote by a short notation as $z = S(t)$. The following dimensionless equations are to be satisfied:

$$\frac{\partial T}{\partial t} = k\nabla^2 T, \quad z > S(t), \quad t > 0; \quad S(0) = 0, \tag{3.1.13}$$

$$T(x, y, z, t)|_{t=0} = f(x, y, z); \quad f(x, y, 0) = 1, \tag{3.1.14}$$

$$T(x, y, z, t)|_{z=S(t)} = 1, \quad S(0) = 0. \tag{3.1.15}$$

The melting temperature has been taken to be unity. The energy balance at $z = S(t)$ is given by the equation

$$Q(x, y, z, t)|_{z=S} + K\left\{1 + \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right\} \frac{\partial T}{\partial z}\bigg|_{z=S} = l\rho \frac{\partial S}{\partial t}. \tag{3.1.16}$$

A Signorini-Type Boundary Condition

Suppose in the previous ablation problem melting starts at $t = t_0 > 0$ (instead of $t = 0$) where t_0 is unknown. Then we have two types of boundary conditions on $z = S(t)$, $t > 0$

$$T(x, t, S(t), t) < 1, \quad \left\{Q + K\frac{\partial T}{\partial n}\right\}_{z=S(t)} = 0, \quad \frac{\partial S(t)}{\partial t} = 0; \quad t > 0, \tag{3.1.17}$$

$$T(x, y, S(t), t) = 1, \quad \left\{Q + K\frac{\partial T}{\partial n}\right\}_{z=S(t)} = l\rho\frac{\partial S}{\partial t}, \quad \frac{\partial S}{\partial t} > 0; \quad t \geq t_0, \tag{3.1.18}$$

where $\vec{n}$ is the unit normal pointing into the solid. There is another way of expressing Eqs (3.1.17), (3.1.18)

$$\left.\begin{array}{l} T(x, y, z, t) \leq 1, \\ \{Q + K\partial T/\partial n\}_{z=S(t)} \geq 0, \\ \{T(x, y, z, t) - 1\}\{K\partial T/\partial n + Q\} = 0. \end{array}\right\} t > 0, \quad z = S(t), \quad S(0) = 0, \tag{3.1.19}$$

$$Q + K\frac{\partial T}{\partial n} = \rho l \frac{\partial S}{\partial t}, \quad z = S(t), \quad t > 0; \quad \frac{\partial S}{\partial t} \geq 0. \tag{3.1.20}$$

The flux prescribed condition at the free boundary is now an inequality and not an equation. Such boundary conditions are called *Signorini-type boundary conditions* [56]. When melting or solidification does not start at $t = 0$, then up to time $t = t_0 > 0$ or in other words till the time the temperature at the boundary $z = 0$ becomes the melting temperature, a pure heat conduction problem without phase-change occurs. At $t = t_0$, the temperature of the half space is to be calculated, which serves as the initial temperature for the phase-change problem.

In essence we are solving two heat conduction problems. If a Signorini-type boundary condition is prescribed, then we are solving one problem but the boundary condition is specified in terms of an inequality. This type of formulation is more suitable for mathematical analysis of ablation problems and their numerical computations. It is possible to obtain variational inequality formulations for some problems with Signorini-type boundary conditions (cf. [22]). In problem (3.1.13)–(3.1.16), the melting starts over the whole boundary $z = 0$. However, depending on the boundary conditions it is possible that the melting begins only over a portion of $z = 0$. Such problems will be discussed in Section 3.3.2.

3.2 EXTENDED CLASSICAL FORMULATIONS OF TWO-PHASE STEFAN PROBLEMS

3.2.1 An Extended Formulation of the One-Dimensional Two-Phase Problem

An extended formulation of problem (1.4.4)–(1.4.8) in the one-dimensional case was discussed in [57]. Some results on the existence, uniqueness and regularity of the solution to problem (3.2.1)–(3.2.10) will be presented in Chapter 10.

$$\mathcal{P}^{(1)}(T^{(1)}(x,t)) \equiv T^{(1)}_{xx} - \delta^{(1)}T^{(1)}_t = q^{(1)}(x,t), \quad \text{in } D^{(1)}, \tag{3.2.1}$$

$$D^{(1)} = \{(x,t) : 0 < x < S(t), 0 < t < t_*\},$$

$$T^{(1)}(x,0) = h^{(1)}(x), \quad 0 \leq x \leq S(0), \; 0 < S(0) = b < 1, \tag{3.2.2}$$

$$T^{(1)}(0,t) = \phi^{(1)}(t), \quad 0 < t < t_*, \tag{3.2.3}$$

$$\mathcal{P}^{(2)}(T^{(2)}(x,t)) \equiv T^{(2)}_{xx} - \delta^{(2)}T^{(2)}_t = q^{(2)}(x,t), \quad \text{in } D^{(2)}, \tag{3.2.4}$$

$$D^{(2)} = \{(x,t) : S(t) < x < 1, 0 < t < t_*\},$$

$$T^{(2)}(x,0) = h^{(2)}(x), \quad S(0) \leq x \leq 1, \tag{3.2.5}$$

$$T^{(2)}(1,t) = \phi^{(2)}(t), \quad 0 < t < t_*, \tag{3.2.6}$$

$$T^{(1)}(S(t),t) = T^{(2)}(S(t),t) = f(S(t),t), \quad 0 < t < t_*, \tag{3.2.7}$$

$$\chi^{(1)}(S(t),t)T^{(1)}_x(S(t),t) - \chi^{(2)}(S(t),t)T^{(2)}_x(S(t),t) = \dot{S}(t) + \mu(S(t),t), \quad 0 < t < t_*. \tag{3.2.8}$$

Here $\delta^{(1)}$ and $\delta^{(2)}$ are inverse of thermal diffusivities.

The boundary conditions (3.2.3), (3.2.6) can be replaced by other boundary conditions such as

$$T^{(1)}_x(0,t) = g^{(1)}(T^{(1)}(0,t),t), \quad 0 < t < t_*, \tag{3.2.9}$$

$$T^{(2)}_x(1,t) = g^{(2)}(T^{(2)}(1,t),t), \quad 0 < t < t_*. \tag{3.2.10}$$

Boundary conditions of mixed type can also be prescribed.

The regularity conditions assumed in [57] for the analysis of the problem are: (1) $\phi^{(1)}(t)$, $\phi^{(2)}(t) \in C^1[0,t_*]$, (2) $h^{(1)}(x)$, $h^{(2)}(x)$ belong to $H^1(0,1)$ where $H^1(0,1)$ is the Sobolev space [58] endowed with the norm $\|\bar{h}\|_{H^1} = \|\bar{h}\|_{L^2} + \|\bar{h}'\|_{L^2}$; $\bar{h}(x) = h^{(1)}(x)$ for $0 \le x \le b$ and $\bar{h}(x) = h^{(2)}(x)$, $b \le x \le 1$, (3) the compatibility conditions for the initial and boundary data are satisfied, i.e. $\phi^{(1)}(0) = h^{(1)}(0)$ and $\phi^{(2)}(1) = h^{(2)}(1)$. $\delta^{(1)}$ and $\delta^{(2)}$ are constants, (4) the functions $q^{(1)}$ and $q^{(2)}$ are Hölder continuous in $D^{(1)}$ and $D^{(2)}$, respectively, with respect to x (or t) and $|q^{(i)}(x,t)| \le Q$ (constant), $(x,t) \in D^{(i)}$, $i = 1,2$ and (5) $f(x,t)$ and f_x are continuous and bounded in $\bar{D} = \bar{D}^{(1)} \cup \bar{D}^{(2)}$ and $f_{xx} - f_t$ is bounded and locally Hölder continuous in $\bar{D}$ with respect to x (or t). By redefining the temperature as $(T - f(x,t))$, Eq. (3.2.7) can be transformed to $T^{(i)} = 0$, $i = 1,2$. This transformation can be used without redefining the assumptions about the changed data and the coefficients in Eqs (3.2.1)–(3.2.8). This transformation is the physical motivation for the term $\mu(S(t),t)$ in Eq. (3.2.8). Some of these regularity conditions can be relaxed, see Chapters 10 and 11.

The initial data and $f(x,0)$ should also satisfy some compatibility conditions given in the following:

$$|h^{(1)}(x) - f(b,0)| \le \gamma_1 (b - x)^{\alpha}, \quad 0 \le x \le b, \tag{3.2.11}$$

$$|h^{(2)}(x) - f(b,0)| \le \gamma_1 (b - x)^{\alpha}, \quad b \le x \le 1, \tag{3.2.12}$$

where γ_1 and α satisfy conditions

$$\gamma_1 b^{\alpha} \ge d \quad \text{and} \quad \gamma_1 (1 - b)^{\alpha} \ge d. \tag{3.2.13}$$

Here d is such that $|\phi^{(i)}(t)| \le d$, $t \ge 0$. $\chi^{(i)}$, $\chi_x^{(i)}$, $\chi_{xx}^{(i)}$, $\chi_t^{(i)}$, $i = 1,2$ are continuous in D and

$$|\chi^{(i)}(x,t)| \le \beta, \quad (x,t) \in \bar{D}, \tag{3.2.14}$$

for some suitable $\beta \ge 0$. If the boundary conditions are of the type (3.2.9), (3.2.10), then $g^{(i)}$, $i = 1,2$ have to satisfy some conditions (cf. [59]).

For given $q^{(i)}$, $\phi^{(i)}$, $h^{(i)}$, $\chi^{(i)}$, $i = 1,2$, and $f(x,t)$ and $\mu(x,t)$ satisfying the conditions imposed on each of them mentioned previously; find t_*, $T^{(1)}(x,t)$, $T^{(2)}(x,t)$ and $S(t)$ such that Eqs (3.2.1)–(3.2.8) are satisfied. In principle, the boundary conditions (3.2.7), (3.2.8) on $x = S(t)$ can be replaced by conditions of the form

$$T^{(1)}(S(t),t) = T^{(2)}(S(t),t) = Z_1 \left(\partial T^{(1)}/\partial x, \partial T^{(2)}/\partial x, S(t), \dot{S}(t) \right), \tag{3.2.15}$$

$$W_1(T^{(1)}, T^{(2)}, \partial T^{(1)}/\partial x, \partial T^{(2)}/\partial x, S(t), \dot{S}(t)) = 0, \quad \text{on } x = S(t). \tag{3.2.16}$$

Here Z_1 and W_1 could be functionals and need not be pointwise functions of their arguments (cf. Eq. 3.3.33).

3.2.2 Multidimensional Stefan Problems of Classes II and III

According to the change of phase initiating along the fixed boundary of the region originally occupied by the melt, classical Stefan problems can be divided into three classes [60]. In *class I problems* solidification or melting starts simultaneously at all points of a fixed surface of the region under consideration. In *class II problems* solidification begins at a 'portion' of the fixed surface. In *class III problems* it begins at a 'point' of the fixed surface. Classes II and III problems essentially differ from class I problems as solidification (melting) in them both spreads along the surface and grows towards the interior. Under suitable assumptions, class I problems can be formulated as one-dimensional problems but classes II and III problems are necessarily multidimensional. The position of the interface in Fig. 1.4.2 is shown after solidification for a long time. It could be due to solidification at a point or over a portion of $\partial G_1(t)$ or over whole of $\partial G_1(t)$. After sufficiently long time, when the spread of solidification over $\partial G_1(t)$ is complete, then solidification growth will be only towards the interior. In Fig. 1.4.1, even for longer times, a class II problem persists. At least in the analytical solutions and the analysis of Stefan problems, this distinction between different classes affects both procedures and solutions.

A formulation of a multidimensional classical Stefan problem has already been given in Section 1.4. Some more formulations will be discussed later. Because of their complexity, classes II and III problems have not been studied widely. They can be formulated as one- or two-phase problems. A three-dimensional one-phase ablation problem in a semiinfinite region with melting due to a 'hot spot' was considered in [61] and a short-time asymptotic solution was obtained. We present below a class III two-phase Stefan problem discussed in [62] in which solidification initiates at a point due to a 'cold spot'.

Consider an axisymmetric cylindrical problem in which R and Z are cylindrical polar coordinates. A superheated melt at time $t = 0$ occupies the region $0 < R_0 \leq R < \infty$, $|Z| < \infty$. This melt is being cooled and a known flux $Q_S(Z, t)$ which is symmetric in Z is prescribed on the surface $R = R_0$ of the cylinder. The temperature $f_L(R, Z)$ of the melt at $t = 0$ is a known quantity which is taken to be a symmetric function in Z and is such that

$$
\begin{aligned}
f_L(R, Z) &= T_m, \quad Z = 0, \;\; R = R_0 \\
&> T_m, \quad Z \neq 0, \;\; R_0 \leq R < \infty,
\end{aligned} \tag{3.2.17}
$$

$$
\frac{\partial f_L}{\partial Z} = 0, \;\; Z = 0, \;\; R = R_0; \quad \frac{\partial^2 f_L}{\partial Z^2} \neq 0, \;\; Z = 0, \;\; R = R_0. \tag{3.2.18}
$$

The solidification will start instantaneously at the point $Z = 0$, $R = R_0$ and with time it will spread along the surface $R = R_0$ and also towards the interior $R > R_0$ of the cylinder. The equation of the solidification front can be written as

$$
R = S(Z, t); \quad |Z| \leq b(t), \;\; t > 0. \tag{3.2.19}
$$

$|Z| = b(t)$ is the equation of the spread of solidification along $R = R_0$, and $R = S(Z, t) > R_0$ is the growth of the solidified layer towards the interior of the cylinder.

The dimensionless formulation of the problem is as follows:

In the liquid region

$$
2\alpha^2 \frac{\partial T_L}{\partial V} = V\left(\frac{\partial^2 T_L}{\partial r^2} + \frac{1}{r}\frac{\partial T_L}{\partial r} + \frac{\partial^2 T_L}{\partial Z^2}\right), \quad S(z, V) \leq r < \infty, \;\; |z| < \infty, \;\; V > 0, \tag{3.2.20}
$$

$$
T_L(r, z, V)|_{V=0} = \hat{f}_L(r, z), \quad 1 \leq r < \infty, \;\; |z| < \infty, \tag{3.2.21}
$$

$$
\left.\frac{\partial T_L}{\partial r}\right|_{r=1} = \hat{Q}_L(z, V), \quad |z| \geq B(V), \;\; V > 0. \tag{3.2.22}
$$

In the solid region

$$2\frac{\partial T_S}{\partial V} = V\left(\frac{\partial^2 T_S}{\partial r^2} + \frac{1}{r}\frac{\partial T_S}{\partial r} + \frac{\partial^2 T_S}{\partial z^2}\right), \quad 1 \le r \le S(z,V), \quad |z| \le B(V), \quad V > 0, \tag{3.2.23}$$

$$\left.\frac{\partial T_S}{\partial r}\right|_{r=1} = \beta \hat{Q}_L(z,V), \quad |z| < B(V), \quad V > 0. \tag{3.2.24}$$

At the solid–liquid interface

$$T_L(r,z,V)|_{r=S(z,V)} = 1, \quad |z| \le B(V), \quad V > 0, \tag{3.2.25}$$

$$T_S(r,z,V)|_{r=S(z,V)} = 1, \quad |z| \le B(V), \quad V > 0, \tag{3.2.26}$$

$$\left[1 + \left(\frac{\partial S}{\partial z}\right)^2\right]\left.\left(\frac{\partial T_S}{\partial r} - \beta\frac{\partial T_L}{\partial r}\right)\right|_{r=S(z,V)} = \frac{2\lambda}{V}\frac{\partial S}{\partial V}, \quad |z| \le B(V), \quad V > 0, \tag{3.2.27}$$

$$S(z,V)|_{V=0} = 1, \quad B(V)|_{V=0} = 0, \quad S(z,V)|_{|z|=B(V)} = 1. \tag{3.2.28}$$

The following dimensionless variables have been used in Eqs (3.2.20)–(3.2.28):

$$\begin{aligned} &z = Z/R_0, \quad r = R/R_0, \quad V = 2(k_S t/R_0^2)^{1/2}, \quad \alpha^2 = k_S/k_L, \\ &\lambda = l/C_S T_m, \quad \beta = K_L/K_S, \quad \hat{Q}_L(z,V) = Q_L(z,t)\cdot R_0/K_S T_m, \\ &S(z,V) = S(Z,t)/R_0, \quad \hat{f}_L(r,z) = f_L(R,Z)/T_m, \quad B(V) = b(t)/R_0. \end{aligned} \tag{3.2.29}$$

If the melt is superheated, i.e. if its temperature is greater than the melting temperature then for solidification to begin it is necessary (not sufficient), that the prescribed flux be infinite at $t = 0$. If at $t = 0, \hat{f}_L(1,0) > 1$, then we shall have a heat conduction problem (without phase-change) up to time $t = t_m > 0$ such that at $t = t_m$, conditions (3.2.17), (3.2.18) are satisfied. Time can be rescaled and this instant of time can be taken as $t = 0$ with the initial temperature changed to the temperature at $t = t_m$. A short-time analytical solution of a class II problem has been obtained in [63] in which solidification/melting starts at a portion of an edge in a plate. The prescribed flux need not be symmetrical. It is interesting to note that the growth of the free boundary for a short time along $r = 1$ in the previous cold spot problem could be of unusual type such as $O(t^{1/4})$ (similar results were obtained in [63]) but the growth of the free boundary along the interior is of well-known type such as of the order of $t^{1/2}$ or t or $t^{3/2}$.

Although classes II and III problems appear to be interesting and distinct from class I problems there are only some short-time analytical solutions to them in the literature. There is no report available on the existence, uniqueness and regularity of the solutions of these problems.

3.2.3 Classical Stefan Problems With n-Phases, $n > 2$

Some One-Dimensional Problems With More Than Two Phases

Classical Stefan problems with more than two phases are much more difficult to study than two-phase problems because of interactions among phases. Several types of generalizations of two-phase formulations to n-phase problems are possible. We shall report here some of the formulations studied in the literature. In principle, phase-change boundaries could be intersecting but such formulations have not be studied in detail. Some of these n-phase formulations are simple extensions of two-phase problems reported earlier in Sections 1.3 and 3.2.1. In an n-phase problem, it is difficult to prescribe the exact number of distinct phase-change boundaries without knowing the number of disjoint subregions and the physical situation. In several problems this has been taken to be $(n - 1)$. In each phase a suitable parabolic heat equation is satisfied and at each phase-change boundary, two boundary conditions are prescribed (cf. Section 1.3).

The formulation of an n-phase one-dimensional problem in which the boundary conditions at the phase-change boundaries could be of type (1.3.5), (1.3.6) has been considered in [64]. Odd and even numbered phases are identical so that there are only two different initial temperatures and two different temperatures. The existence of a global solution to this problem which is nearly classical (the Stefan condition is satisfied in the integrated form) has been discussed in [64]. An analytical solution to a one-dimensional n-phase solidification problem in the region $0 \leq x < \infty$ has been discussed in [65]. At each free boundary, the temperature has a specified, constant, limiting value from each side. These limiting values may differ by a finite jump at different phase-change boundaries. Initially all the free boundaries coincide at $x = 0$ and the temperature takes the constant value T_m for $x > 0$ where T_m is the limiting value of the temperature as $x \to \infty$. The Stefan condition is considered at the free boundaries. Densities of different phases could be different, giving rise to the movement of phases. Using local coordinates which are fixed in each phase, a similarity solution is presented.

The formulation of the one-dimensional multiphase problem reported in [21] is a little different. At time $t = 0$, an unbounded plate of thickness $2d$ and temperature T_0 is dipped into a melt maintained at constant temperature $T_1 > T_0$. It is assumed that a known constant heat flux q is maintained from within the melt to the plate and in the 'boundary layer' the temperature changes linearly in the direction normal to the surface of the plate. Let T_k be the freezing temperature of the melt and T_n be the melting temperature of the plate. Depending on the magnitudes of different temperatures, the following three cases arise.

Case I

$$T_0 < T_k < T_1 < T_n. \tag{3.2.30}$$

Since $T_0 < T_k$, crust forms on the plate till some time $t_0 > 0$ after which the crust begins to melt as the temperature at the surface of the plate rises to T_k and at the free boundary heat flux q is acting. Melting will continue till time $t_1 > t_0 > 0$ when the crust has completely melted. After that, heating of the plate without a phase-change continues. The plate cannot melt because the melting temperature T_n of the plate is greater than T_1. For $0 \leq t \leq t_1$, there exists a two-phase problem with only one free boundary and for $t > t_1$ the problem is that of heat conduction in the plate without phase-change. Because of the assumption of symmetry it is sufficient to consider the region $0 \leq x \leq \infty$.

Case II

$$T_0 < T_k < T_n < T_1. \tag{3.2.31}$$

At any given time there will be only one phase-change boundary. The freezing temperature of the melt is lower than the melting temperature of the plate. First the crust forms and then the crust starts melting and when the plate attains temperature T_n, the plate starts melting.

Case III

$$T_0 < T_n < T_k < T_1. \tag{3.2.32}$$

There are three possibilities.

(P1) The plate melts and the melt crystallizes instantaneously with the immersion of the plate in the melt.

(P2) The melt crystallizes instantaneously but the melting of the plate starts late.

(P3) The plate begins to melt instantaneously but the crystallization of the melt is delayed.

Problem (P1) corresponding to the possibility (P1) can be formulated as (not made dimensionless) follows:

$$a^2(x)\frac{\partial^2 T_i}{\partial x^2} = \frac{\partial T_i}{\partial t}, \quad i = 1, 2, 3, \;\; x \neq S_1(t), \;\; x \neq d; \;\; x \neq S_2(t), \;\; t > 0, \tag{3.2.33}$$

$$\begin{aligned} a^2(x) &= k_1^2, \quad 0 < x < S_2(t),\\ &= k_2^2, \quad S_2(t) < x < d,\\ &= k_3^2, \quad d < x < S_1(t). \end{aligned} \tag{3.2.34}$$

Here $x = S_1(t)$ is the equation of the freezing front in the solidification of the melt and $x = S_2(t)$ is the equation of the melting front in the melting of the plate. The subscripts 1, 2 and 3 refer to the regions $0 < x < S_2(t)$, $S_2(t) < x < d$ and $d < x < S_1(t)$, respectively.

$$\left.\frac{\partial T_1}{\partial x}\right|_{x=0} = 0; \quad T_1|_{x=S_2(t)} = T_n = T_2|_{x=S_2(t)}; \;\; t > 0, \tag{3.2.35}$$

$$T_3|_{x=S_1(t)} = T_k; \;\; t > 0; \quad T_1|_{t=0} = T_0, \;\; 0 < x < d, \tag{3.2.36}$$

$$T_2|_{x=d-0} = T_3|_{x=d+0}; \quad K_2 \left.\frac{\partial T_2}{\partial x}\right|_{x=d-0} = K_3 \left.\frac{\partial T_3}{\partial x}\right|_{x=d+0}; \;\; t > 0, \tag{3.2.37}$$

$$K_1\frac{\partial T_1}{\partial x} - K_2\frac{\partial T_2}{\partial x} = \rho_1 l_1 \dot{S}_2(t), \quad \text{for } x = S_2(t), \;\; t > 0; \;\; S_2(0) = d, \tag{3.2.38}$$

$$-q + K_3\frac{\partial T_3}{\partial x} = \rho_3 l_3 \dot{S}_1(t), \quad \text{for } x = S_1(t), \;\; t > 0; \;\; S_1(0) = d. \tag{3.2.39}$$

The motion of the liquid due to the difference in densities has been neglected. Problems (P2) and (P3) can be formulated on the same lines as Problem (P1), but the time interval $t > 0$ has to be divided into several ones and in each interval an appropriate problem is to be formulated. For example in Problem (P2), if the melting of the plate begins at $t = t_0$ then for $0 \le t < t_0$, we have a problem of pure heat conduction in the plate and crust formation in the melt. For $t_0 < t < t_1$, where t_1 is the time at which the temperature at $x = d$ becomes T_k, there will be both melting of the plate and freezing of the melt. For $t > t_1$, crust starts melting and melting of the plate continues or plate might have completely melted by that time. At each stage, temperature and the position of the free boundary/boundaries in the previous stage are to be ascertained.

In [21] the main interest in the study of Case III is to examine the possibilities of occurrences of three cases (P1), (P2) and (P3). This requires short-time ($t \to 0$) analytical solution of Problem (P1) which was obtained by using fundamental solutions of the heat equation for a double layer and taking the limit as $t \to 0$. Both necessary and sufficient conditions for the occurrence of the case (P1) have been obtained in [21]. The solution to Problem (P3) will exist if $S_1(t) < d$ for $t > 0$ and therefore Problem (P3) does not have a solution. It has also been shown in [21] that the conditions under which both (P1) and (P2) are possible cannot coincide.

A three-phase problem with two free boundaries has been discussed in [66] which is concerned with both melting and evaporation. Consider a solid occupying the region $0 \leq x \leq a$ (one-dimensional problem). The boundary $x = 0$ is insulated and the solid is heated at $x = a$. First the solid melts at $x = a$ and for some time, solid and liquid regions separated by a free boundary occupy the region $0 \leq x \leq a$. With further heating, when vaporization temperature is attained at $x = a$, the liquid starts evaporating. There will now be two phase-change boundaries, viz., liquid–vapour and solid–liquid. If the solid is enclosed in a container, then heat transfer in the vapour is to be considered and there will be three phases and two free boundaries. If the vapour is allowed to escape, then there are two phases and two free boundaries. Using finite-difference and finite-element methods, numerical solution of this problem has been obtained by several workers (cf. [66]).

3.2.4 Solidification With Transition Temperature Range

A very pure metal has a fixed melting temperature, which is also its freezing temperature. In the case of alloys or metals with impurities, melting and freezing temperatures are not the same and phase-change takes place over a temperature range. Let freezing and melting temperatures be denoted by T_1 and T_2, respectively. For metals with impurities, both heat and mass transfer should be considered but if the concentration of impurity is small, then only heat transfer can be considered with phase-change taking place over a temperature range $T_1 \leq T \leq T_2$. The region whose temperature lies between T_1 and T_2 is called a mushy region. There are two phase-change boundaries. The solid–mush boundary separates the solid region from the mushy region and the liquid–mush boundary separates the liquid region from the mushy region. In [67] an analytical solution of a one-dimensional solidification problem in cylindrical symmetry with an extended freezing temperature range has been obtained. The finite-difference numerical solution of a one-dimensional solidification problem in a finite slab with an extended freezing temperature range has been presented in [68]. We present later a two-dimensional formulation of an extended freezing temperature range problem whose finite-difference numerical solution is reported in [69].

A two-dimensional region $0 \leq X \leq 1$, $-a \leq Y \leq a$ (X and Y are dimensionless coordinates of a point in a plane) at time $t = 0$ is occupied by a superheated melt. Solidification takes place over a temperature range $T_1 \leq T \leq T_2$. For $T < T_1$, the material is in a stable solid phase and for $T > T_2$, the material is in a stable liquid phase and for $T_1 < T < T_2$, a mushy region exists. Cooling is done at the boundary of the rectangular region in such a way that the mushy region is sandwiched between stable solid and stable liquid regions. It will be assumed that the solid–mush boundary can be expressed in the form $X = R_1(Y,t)$ and the liquid–mush boundary can be expressed in the form $X = R_2(Y,t)$. Without loss of generality it can be assumed that the solidification starts at $X = 0$ at time $t = 0$. Even if there is some delay in the starting of solidification, for example, if the solidification starts at $t = t_0 > 0$, then during the time $0 \leq t < t_0$, there exists only a heat conduction problem without phase change whose numerical solution is generally considered to be simple and so assumed to be known by using well known methods. By redefining the time scale it can be assumed that the solidification starts at $t = 0$.

The mathematical formulation of this three-phase problem in the dimensionless form as considered in [69] is as follows:

In the solid region

$$\frac{\partial T_S}{\partial t} = \alpha_S \left(\frac{\partial^2 T_S}{\partial X^2} + \frac{\partial^2 T_S}{\partial Y^2} \right), \quad 0 < X < R_1(Y,t), \quad -a < Y < a; \quad t > 0, \tag{3.2.40}$$

$$\left.\frac{\partial T_S}{\partial X}\right|_{X=0} = F_p(Y,t) \quad \text{or} \quad T_S|_{X=0} = T_p(Y,t); \quad t > 0, \tag{3.2.41}$$

$$\left.\frac{\partial T_S}{\partial Y}\right|_{Y=a} = 0, \quad 0 < X < R_1(a,t), \quad t > 0, \tag{3.2.42}$$

$$\left.\frac{\partial T_S}{\partial Y}\right|_{Y=-a} = 0, \quad 0 < X < R_1(-a,t), \quad t > 0. \tag{3.2.43}$$

In the mushy region

$$\frac{\partial T_M}{\partial t} = \alpha_M \left(\frac{\partial T_M}{\partial X^2} + \frac{\partial T_M}{\partial Y^2} + \frac{\lambda}{\beta_1} \frac{\partial f_S}{\partial t} \right), \quad R_1(Y,t) < X < R_2(Y,t), -a < Y < a; \quad t > 0, \tag{3.2.44}$$

$$\left.\frac{\partial T_M}{\partial Y}\right|_{Y=a} = 0, \quad R_1(a,t) < X < R_2(a,t), \quad t > 0, \tag{3.2.45}$$

$$\left.\frac{\partial T_M}{\partial Y}\right|_{Y=-a} = 0, \quad R_1(-a,t) < X < R_2(-a,t), \quad t > 0, \tag{3.2.46}$$

$$T_M|_{t=0} = g(X,Y) \le T_{ml}, \quad 0 \le X \le R_2(Y,0), \quad g(0,Y) = 1.0. \tag{3.2.47}$$

In the liquid region

$$\frac{\partial T_L}{\partial t} = \alpha_L \left(\frac{\partial^2 T_L}{\partial X^2} + \frac{\partial^2 T_L}{\partial Y^2} \right), \quad R_2(Y,t) < X < 1, \quad t > 0, \tag{3.2.48}$$

$$T_L|_{t=0} = g(X,Y) \ge T_{ml}, \quad X \ge R_2(Y,0), \tag{3.2.49}$$

$$\left.\frac{\partial T_L}{\partial X}\right|_{X=1} = 0, \quad t > 0, \tag{3.2.50}$$

$$\left.\frac{\partial T_L}{\partial Y}\right|_{Y=a} = 0, \quad R_2(a,t) < X < 1, \quad t > 0, \tag{3.2.51}$$

$$\left.\frac{\partial T_L}{\partial Y}\right|_{Y=-a} = 0, \quad R_2(-a,t) < X < 1, \quad t > 0. \tag{3.2.52}$$

At the solid–mush boundary $X = R_1(Y,t)$

$$T_S = T_M = 1.0, \tag{3.2.53}$$

$$\left\{ 1 + \left(\frac{\partial R_1}{\partial Y} \right)^2 \right\} \left\{ \frac{\partial T_S}{\partial X} - \beta_1 \frac{\partial T_M}{\partial X} \right\} = \lambda d_1 \frac{\partial R_1}{\partial t}. \tag{3.2.54}$$

At the liquid–mush boundary $X = R_2(Y,t)$

$$T_M = T_L = T_{ml}, \tag{3.2.55}$$

$$\left\{1 + \left(\frac{\partial R_2}{\partial Y}\right)^2\right\}\left\{\beta_1 \frac{\partial T_M}{\partial X} - \beta_2 \frac{\partial T_L}{\partial X}\right\} = \lambda d_2 \frac{\partial R_2}{\partial t}, \tag{3.2.56}$$

$$g(R_2(Y,0),Y) = T_{ml}. \tag{3.2.57}$$

In the previous formulation, d_1 and d_2 are solid fractions present at solid–mush and liquid–mush boundaries, respectively; f_S is the solid fraction in the mushy region, $g(X,Y)$ is the initial temperature/T_1, b is the length of the plate, a is the breadth of the plate/b, T is the temperature/T_1, T_{ml} is the temperature of the liquid–mush boundary/T_1. Other parameters are defined in the following:

$$\alpha = \text{thermal diffusivity} \cdot t_m/b^2, \quad \beta_1 = K_M/K_S, \quad \beta_2 = K_L/K_S, \quad \lambda = \rho l b^2/(K_S t_m T_1), \tag{3.2.58}$$

t_m is the time taken for the liquid to attain the temperature T_1 at $X = 0$. Densities of all the three regions have been taken to be equal and are denoted by ρ.

The constant d_1 in Eq. (3.2.54) is generally taken as unity. But in the case of 'eutectics' it can be taken to be less than unity. At the liquid–mush boundary, $d_2 = 0$. If in Eq. (3.2.56), $d_2 = 0$ is taken then we have an implicit free boundary condition which is not convenient for the numerical solution. By taking d_2 very small, an explicit free boundary condition can be generated. In Section 3.3 some transformations to convert an implicit free boundary condition to an explicit free boundary condition are given but this treatment may give rise to some other difficulties in the numerical schemes. Although it is not a rigorous mathematical convergence proof, a reasonably accurate solution can be obtained if the numerical results converge as d_2 is gradually decreased. The accuracy of the numerical solution can be further checked by some methods such as integral heat balance calculation (cf. [69]).

The solid fraction in the mush depends on various physical parameters, such as the temperature of the mush and the width of the mushy region. An exact mathematical expression for the solid fraction in the mush cannot be given and in its absence some approximate mathematical models are proposed. Two such models have been considered in [69] for the numerical solution of the problem.

Model I

$$f_S(X,Y,t) = \{d_1(T_{ml} - T_M(X,Y,t)) - d_2(1 - T_M(X,Y,t))\}/(T_{ml} - 1.0). \tag{3.2.59}$$

Model II

$$f_S(X,Y,t) = [d_1\{R_2(Y,t) - X\} - d_2\{R_1(Y,t) - X\}]/\{R_2(Y,t) - R_1(Y,t)\}. \tag{3.2.60}$$

In the first model, f_S is a linear function of the temperature of the mush and in the second model f_S is a linear function of the 'width' of the mushy region. Thermodynamically, the heat extracted from the system at the fixed boundaries should be equal to the heat given out by the system during solidification. If this balancing of heat is done in an integrated way over the whole region and over a period of time, then it is called *integral heat balance* verification. Integral heat balance is satisfied for model I but not for model II. In the first model we can calculate f_S from the temperature (calculated temperature) of the mush and this procedure is thermodynamically consistent but in the second model corresponding to the calculated f_S, the temperature is calculated in the numerical scheme and therefore integral heat balance is not satisfied.

3.3 STEFAN PROBLEMS WITH IMPLICIT FREE BOUNDARY CONDITIONS

If in the place of Eq. (3.1.12), we have the condition

$$\left.\frac{\partial T}{\partial x}\right|_{x=S(t)} = 0, \tag{3.3.1}$$

then velocity of the free boundary is not explicitly prescribed. As stated earlier, free boundary conditions of the previous type in which $\dot{S}(t)$ is not prescribed explicitly are known as implicit free boundary conditions. Free boundary condition (3.1.12) is an explicit boundary condition as $\dot{S}(t)$ is prescribed in it. Our main aim in this section is to present some transformations which convert a Stefan problem with an implicit free boundary condition to a Stefan problem with an explicit free boundary condition. An extensively studied problem with an implicit free boundary condition known as oxygen-diffusion problem will also be discussed later in detail.

3.3.1 Schatz Transformations and Implicit Free Boundary Conditions

Problem 3.3.1. Find t_*, $T(x,t)$ and $S(t)$ satisfying the following system of dimensionless equations:

$$T_{xx} - T_t = F(x,t), \quad a < x < S(t), \;\; 0 < t < t_*, \tag{3.3.2}$$

$$\alpha T_x(a,t) + \beta T(a,t) = f(t), \quad 0 < t < t_*, \tag{3.3.3}$$

$$T(x,0) = \phi(x), \quad -\infty < a \le x \le b = S(0), \tag{3.3.4}$$

$$T(S(t),t) = g(S(t),t), \quad 0 < t < t_*, \tag{3.3.5}$$

$$\gamma(S(t),t)\dot{S}(t) = -T_x(S(t),t) + \hat{h}(S(t),t); \quad \gamma \ne 0, \;\; 0 < t < t_*. \tag{3.3.6}$$

All the thermophysical parameters in this one-phase problem have been taken to be unity. This is possible by suitably choosing time and/or length scales. In those Stefan problems in which the effect of thermophysical parameters is not to be investigated, it is convenient to take parameters to be unity. α and β are constants. Depending on the nature of the data, problems (3.3.2)–(3.3.6) could be either a melting problem or a solidification problem. We report here some of the assumptions and for complete details see [70].

- **(i)** $F(x,t)$, $g(x,t)$, $\hat{h}(x,t)$ and $F_x \in C$ (continuous) for $a \le x < \infty$, $0 < t < t_*$.
- **(ii)** $f(t) \in C$, $0 \le t < t_*$.
- **(iii)** If $a < b$, then $\phi \in C^1$ for $a \le x \le b$ and $\phi(b) = g(b,0)$, $\phi'(b) = \hat{h}(b,0)$. These are compatibility conditions to be satisfied by the initial temperature and the functions g and $\hat{h}$ at the free boundary.
- **(iv)** In addition to the regularity conditions to be satisfied by $S(t)$ and $T(x,t)$ in the classical solution of a Stefan problem mentioned in Section 1.4.6 it will be assumed that T_{xxx},

$T_{xt} \in C$ for $a < x < S(t)$, $0 < t < t_*$. The significance of this assumption will be made clear later.

Problem 3.3.2. Consider problem (3.3.2)–(3.3.6) with some changes. Let $\gamma(S(t), t) = 0$ in Eq. (3.3.6), so that

$$T_x(S(t), t) = \hat{h}(S(t), t). \tag{3.3.7}$$

Make some additional assumptions as follows:

$$g_x(x, t) \neq \hat{h}(x, t), \quad a < x < \infty, \quad 0 < t < t_* \tag{3.3.8}$$

and

$$\alpha = 1, \quad \beta = 0. \tag{3.3.9}$$

Problem 3.3.3. Let $\gamma(x, t) = g(x, t) = \hat{h}(x, t) = 0$ and $\alpha = 0$, $\beta = 1$ in Problem 3.3.1. In this case the free boundary conditions are given by

$$T(S(t), t) = 0, \quad 0 < t < t_*, \tag{3.3.10}$$

$$T_x(S(t), t) = 0, \quad 0 < t < t_*. \tag{3.3.11}$$

The equivalence of Problems 3.3.1–3.3.3 is established by the following propositions.

Proposition 3.3.1. *If (S, T) is the solution of Problem 3.3.2, then (S, v) where $v = T_x$ and S is the same as in (S, T), is the solution of the following Stefan problem:*

$$v_{xx} - v_t = F_x(x, t), \quad 0 < x < S(t), \quad 0 < t < t_*, \tag{3.3.12}$$

$$v(a, t) = f(t), \quad 0 < t < t_*, \tag{3.3.13}$$

$$v(x, 0) = \phi'(x), \quad a \leq x \leq b = S(0), \tag{3.3.14}$$

$$v(S(t), t) = \hat{h}(S(t), t), \quad 0 < t < t_*, \tag{3.3.15}$$

$$[\hat{h}(S(t), t) - g_x(S(t), t)]\dot{S}(t) = -v_x(S(t), t)) + F(S(t), t) + g_t(S(t), t), \quad 0 < t < t_*. \tag{3.3.16}$$

Eq. (3.3.12) can be easily derived if the partial derivative of Eq. (3.3.2) with respect to x is taken. If the material time derivative of Eq. (3.3.5) is taken, then we obtain

$$T_x(S(t), t)\dot{S}(t) + T_t = g_x(S(t), t)\dot{S} + g_t. \tag{3.3.17}$$

On using Eqs (3.3.7), (3.3.2) in Eq. (3.3.17), Eq. (3.3.16) can be obtained. Eq. (3.3.17) suggests that we impose the condition $g \in C^1$, $b \leq x < \infty$. Other conditions in Eqs (3.3.13)–(3.3.15) can be easily derived.

Proposition 3.3.2. *Conversely, if the pair (S, v) is the solution of Eqs (3.3.12)–(3.3.16) then the pair (S, T) where S is the same as in (S, v) and $T(x, t)$ is defined by*

$$T(x, t) = \int_x^{S(t)} v(p, t)dp + g(S(t), t), \quad a \leq x \leq S(t), \quad 0 < t < t_* \tag{3.3.18}$$

is the solution of Problem 3.3.2.

On repeated differentiations of Eq. (3.3.18) with respect to x, we get

$$T_x = v(x,t) \quad \text{and} \quad T_{xx} = v_x. \tag{3.3.19}$$

Differentiation of Eq. (3.3.18) with respect to time gives

$$T_t = -\int_x^{S(t)} v_t(p,t)dp - v(S(t),t)\dot{S}(t) + g_x(S(t),t)\dot{S} + g_t(S(t),t). \tag{3.3.20}$$

Also

$$\int_x^{S(t)} v_t(p,t)dp = \int_x^{S(t)} (v_{xx} - F_x)dp = v_x(S(t),t) - v_x(x,t) - F(S(t),t) + F(x,t). \tag{3.3.21}$$

In order to obtain Eq. (3.3.2) from the solution of Eqs (3.3.12)–(3.3.16), we use Eq. (3.3.21) in Eq. (3.3.20) and then substitute v_x from Eq. (3.3.16). Derivation of other conditions in Problem 3.3.1 is straightforward. Even if $\alpha = 0$ and $\beta = 1$ in Eq. (3.3.3) the substitution $v = T_x$ works. Differentiation of Eq. (3.3.3) ($\alpha = 0$) with respect to t gives

$$T_t(a,t) = T_{xx}(a,t) - F(a,t) = f'(t) \quad \text{or} \quad v_x(a,t) = f'(t) + F(a,t). \tag{3.3.22}$$

In this case it has to be assumed that T_t and T_{xx} are continuous at $x = a$ and $f(t) \in C^1$. We conclude that Problem 3.3.1 and the problem defined by Eqs (3.3.12)–(3.3.16) are equivalent. Eq. (3.3.16) is an explicit free boundary condition.

Proposition 3.3.3. *If (S,T) is the solution of Problem 3.3.3, then (S,v), where $v = T_t$, is the solution of the following problem:*

$$v_{xx} - v_t = F_t(x,t), \quad 0 < x < S(t), \quad 0 < t < t_*, \tag{3.3.23}$$

$$v(a,t) = f'(t), \quad 0 < t < t_*, \tag{3.3.24}$$

$$v(x,0) = \phi''(x) - F(x,0), \quad a \le x \le b = S(0), \tag{3.3.25}$$

$$v(S(t),t) = 0, \quad 0 < t < t_*, \tag{3.3.26}$$

$$F(S(t),t)\dot{S} = -v_x(S(t),t), \quad 0 < t < t_*. \tag{3.3.27}$$

Eqs (3.3.23)–(3.3.25) suggest that we impose the conditions that F_t and $f'(t) \in C$ and $\phi(x) \in C^2$.

The derivation of Eqs (3.3.23)–(3.3.26) is simple. To obtain Eq. (3.3.27), differentiate Eq. (3.3.11) with respect to t and use Eq. (3.3.2).

Proposition 3.3.4. *Conversely, if (S,v) is the solution of Eqs (3.3.23)–(3.3.27), then (S,T) where S is the same as in (S,v), $\dot{S}$ is of one sign, and $T(x,t)$ is given by*

$$T(x,t) = \int_x^{S(t)} \int_\eta^{S(t)} \{v(p,t) + F(p,t)\}dpd\xi, \quad a \le x \le S(t), \quad 0 < t < t_*, \tag{3.3.28}$$

is a solution of Problem 3.3.3.

To derive Eq. (3.3.2), differentiate Eq. (3.3.28) twice with respect to x so that

$$\frac{\partial T}{\partial x} = -\int_x^{S(t)} \{v(p,t) + F(p,t)\} dp \tag{3.3.29}$$

and

$$\frac{\partial^2 T}{\partial x^2} = v(x,t) + F(x,t) = \frac{\partial T}{\partial t} + F(x,t). \tag{3.3.30}$$

On substituting $x = S(t)$ in Eqs (3.3.28), (3.3.29), we obtain Eqs (3.3.10), (3.3.11). If it is assumed that $f(0) = \phi(a)$ and $\phi(b) = 0$ then Eqs (3.3.3), (3.3.4) can be obtained.

Although our main concern in this section is to show how the transformations $v = T_x$ and $v = T_t$ transform Stefan problems with implicit free boundary conditions to Stefan problems with explicit free boundary conditions, some remarks on the analysis of Problems 3.3.1–3.3.3 will be in order (see also Chapter 10). If T_x is continuous in $a \leq x \leq S(t)$, then the relation $\dot{S}(t) = -T_x(S(t),t)$ (for simplicity take $\gamma = 1$ and $\hat{h} = 0$ in Eq. 3.3.6) implies that $\dot{S}(t)$ is continuous. This is true for the more general boundary condition (3.3.16) also if continuity assumptions are made for other functions involved. If the free boundary condition is of the form (3.3.7) or (3.3.11), then even if T_x is continuous, it cannot be directly concluded that $\dot{S}(t)$ is continuous.

Existence of unique solutions of Problems 3.3.2 and 3.3.3 has been discussed in [70] under suitable data assumptions. If $a < b$ and the data satisfy appropriate assumptions, then it can be proved that $0 \leq \dot{S}(t) \leq A$, $0 < t \leq t_0 < \infty$ for some constant A (see Proposition 10.1.20). The method of proof is the same as in [71]. In the place of Eq. (3.3.2) a quasi-linear heat equation with some constraints can also be considered.

Conversion of a Stefan-Type Problem to a Stefan Problem

We consider a one-dimensional Stefan-type problem with phases 1 and 2 in which phase 1 occupies the region $0 \leq x < S(t)$ and phase 2 occupies the region $S(t) < x \leq 1$. This problem differs from a Stefan problem only in the free boundary conditions. Let $T^{(1)}$ and $T^{(2)}$ be temperatures of phases 1 and 2, respectively. $T_x^{(1)}$ and $T_x^{(2)}$ are prescribed on $x = S(t)$ and another boundary condition on $S(t)$ is given in the form of a relation

$$\dot{S} = f(T^{(1)}(S(t),t), \quad T^{(2)}(S(t),t)). \tag{3.3.31}$$

If the transformations $v^{(1)} = T_x^{(1)}$ and $v^{(2)} = T_x^{(2)}$ are used in the Stefan-type problem (Eq. 3.3.31 is not a Stefan condition) mentioned previously and the problem is formulated in terms of $v^{(1)}(x,t)$ and $v^{(2)}(x,t)$, then at the free boundary $v^{(1)}$ and $v^{(2)}$ will be prescribed. If the heat equations in the two phases are of the form (3.3.2), then

$$T^{(i)}(S(t),t) = \int_0^t v_x^{(i)}(S(\tau),\tau)d\tau - \int_0^t F^{(i)}(S(\tau),\tau)d\tau, \quad i = 1,2. \tag{3.3.32}$$

In view of Eq. (3.3.32), Eq. (3.3.31) can be written as

$$\dot{S} = P_t(v_x^{(1)}, v_x^{(2)}, S(t)), \tag{3.3.33}$$

where for any $t \in [0, t_*]$, P_t is a functional (a real valued function) acting on functions $S(t)$, $v_x^{(1)}(x,t)$, $v_x^{(2)}(x,t)$, $x \in [0, S(t)]$ in $v_x^{(1)}(x,t)$ and $x \in [S(t), 1]$ in $v_x^{(2)}(x,t)$ and $0 < t \leq t_* < \infty$. As explained earlier, the Stefan condition (3.3.33) has been considered in the functional form and not as a heat balance condition.

3.3.2 Unconstrained and Constrained Oxygen-Diffusion Problem

We shall first discuss a one-dimensional oxygen-diffusion problem, and use a shorter notation ODP for it. This problem was first formulated in [72] and studied later by several authors from various view points such as the existence and uniqueness, analytical and numerical solutions. Oxygen is fed to a tissue at the boundary $x = 0$ at which a constant concentration c_0 of oxygen is maintained. It is assumed that oxygen diffuses through the tissue and is absorbed at a constant rate α per unit volume. After some time, a steady state is reached. Suppose that in the steady state oxygen has penetrated up to a distance x_0 in the tissue. Then, at $x = x_0$, both the oxygen concentration and the flux are zero. Steady-state concentration and unknown x_0 can be easily obtained. If suitable dimensionalization is carried out as in [72], then the steady-state concentration can be obtained as $0.5(1-x)^2$, $0 \leq x \leq 1$. Once the steady state is reached, the boundary $x = 0$ is sealed. Oxygen diffusion and absorption starts again and the penetration depth of oxygen starts receding giving rise to a free boundary problem whose dimensionless formulation is given in the following:

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} - 1, \quad \text{in } D_{t_*} = \{(x,t) : 0 < x < S(t), 0 < t < t_* < \infty\}, \tag{3.3.34}$$

$$c(x,0) = 0.5(1-x)^2, \quad 0 \leq x \leq 1, \tag{3.3.35}$$

$$\left.\frac{\partial c}{\partial x}\right|_{x=0} = 0, \tag{3.3.36}$$

$$c(S(t),t) = 0, \;\; 0 < t < t_*; \quad S(0) = 1, \tag{3.3.37}$$

$$\frac{\partial c}{\partial x}(S(t),t) = 0, \quad 0 < t < t_*. \tag{3.3.38}$$

Here $c(x,t)$ is the concentration and $x = S(t)$ is the free boundary. It may be noted that this problem is similar to Problem 3.3.3 and with the help of the transformation $T = c_t$, it can be converted to a Stefan problem with an explicit free boundary condition. A typical feature of this problem is the noncompatibility of the initial and boundary data at $x = 0$. Since concentration is a physical quantity, $c(x,t)$ should be greater than or equal to zero. It can be proved rigorously that the classical solution (c, S) of Eqs (3.3.34)–(3.3.38) exists with $c(x,t) \geq 0$, $0 \leq x \leq S(t)$, $0 \leq t \leq t_*$. If the substitution $T = c_x(x,t)$ is used in Eqs (3.3.34)–(3.3.38), then by using the *maximum principle* [9] it can be easily concluded that the maximum value of $T(x,t)$ is zero and therefore $c_x(x,t) < 0$ for $0 < x < S(t)$. This implies $c > 0$ for $0 < x < S(t)$ or $c \geq 0$ for $0 \leq x \leq S(t)$. We shall discuss the existence of the solution of ODP a little later. It may be pointed out here that ODP is closely related to the SSP formulated in Eqs (3.1.8)–(3.1.12).

It can be easily checked that if the transformation $T = c_t$ is used in ODP then SSP formulation (see Section 3.1.2) is obtained. If the concentration is defined by the relation

$$c(x,t) = \int_{S(t)}^{x} \int_{S(t)}^{\xi} (T(\eta,t) + 1) d\eta d\xi, \tag{3.3.39}$$

then ODP can also be obtained from SSP. Because of this connection between ODP and SSP, ODP can be regarded as a Stefan problem with an implicit free boundary condition even though ODP is concerned with the diffusion of mass and not with the heat transfer. The existence and uniqueness of ODP has been discussed in [73–75]. In [73], the existence and uniqueness of the solution of ODP was in essence proved by extending the domain of $c(x,t)$ to the interval $0 \leq x \leq 1$ and taking $c = 0$ for $S(t) < x \leq 1$. The solution of this extended domain problem was obtained as the limit of a one-parameter family of problems. If a constraint $c \geq 0$ is added in the extended domain ODP formulation in $0 \leq x \leq 1$, then it can be identified with an obstacle problem (see Chapter 7) whose variational inequality formulation exists. The existence and uniqueness of this obstacle problem has been proved in [74]. It has been proved that the obstacle problem never exhibits *blow-up*, in the sense that either (i) $c(x,t) \geq 0$, $c(x,t) \not\equiv 0$ for all $t > 0$ or (ii) $c(x,t) \equiv 0$ for t greater than some finite time $\hat{t}$ (extinction time). In [75], the existence and uniqueness were proved using fixed point arguments. It may be noted that if the initial concentration is given by Eq. (3.3.35), then it is not necessary to add the constraint $c \geq 0$ to the problem formulation of ODP as in this case it has been proved that the unique solution exists and $c \geq 0$.

In the place of Eq. (3.3.35), let us take the initial concentration as

$$c(x,0) = g(x), \quad 0 \leq x \leq 1, \quad g \not\equiv 0. \tag{3.3.40}$$

We shall still call this changed problem as ODP.

If it is assumed that $g(x)$ is nonnegative, nonincreasing, sufficiently regular such as $g \in C^2$, g'' is Hölder continuous at $x = 1$ and $g(x)$ satisfies compatibility conditions $g'(0) = 0$, $g(1) = g'(1) = 0$, $g''(1) = 1$, then ODP with Eq. (3.3.40) in the place of Eq. (3.3.35) possesses a very smooth solution with $c \geq 0$ (cf. [75]). It is interesting to note that if $g(x) = 0.5(1-x)^2$ then $g'(0) \neq 0$ but still the existence and uniqueness of ODP can be established [75].

Constrained and Unconstrained ODP

We shall now consider ODP with initial concentration $\hat{g}(x)$ where $\hat{g}(x)$ can have any sign for $x \in [0,1]$. Add the constraint $c \geq 0$ in the formulation. This problem will be called *constrained oxygen-diffusion problem* or in short CODP. If the constraint $c \geq 0$ is not imposed in the formulation, then we have an *unconstrained ODP* or in short UODP. CODP is equivalent to solving the following equation with a suitable initial and boundary data:

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} - H_v(c), \tag{3.3.41}$$

where H_v is the Heaviside function

$$\left.\begin{aligned} H_v(c) &= 1, \quad c > 0 \\ &= 0, \quad c \leq 0. \end{aligned}\right\} \tag{3.3.42}$$

The solution of CODP and UODP coincides as long as $c \geq 0$. The solution of CODP generally exists, is unique, and remains well behaved even for a sufficiently long time. The global behaviour of UODP is a delicate matter as a *blow-up* may occur in which $\dot{S}$ becomes infinite in finite time. Blow-up will be dealt with further in Section 4.4. The solution $T(x,t)$ of SSP is equal to c_t where $c(x,t)$ is the solution of UODP but in general $T(x,t)$ is not equal to $\hat{c}_t$ where $\hat{c}(x,t)$ is the solution of CODP. The nonconsideration of the negativity set of $c(x,t)$ in

CODP deprives its solution of many interesting features which the solutions of UODP have, for example, the approach of the free boundary to a negativity set [75].

ODP in a Radially Symmetric Domain

The existence of the solution of ODP in a cylindrical domain $r_0 \leq r \leq r_1$ was considered in [76]. ODP in cylindrical symmetry can be easily formulated and this formulation is not being given here. The steady-state concentration $c_0(r)$ in cylindrical symmetry can be obtained as

$$c_0(r) = \frac{r^2 - r_1^2}{4} - \frac{r_1^2}{2} \ln \frac{r}{r_1}, \tag{3.3.43}$$

where r_1 solves the equation

$$\frac{r_0^2 - r_1^2}{4} - \frac{r_1^2}{2} \ln \frac{r_0}{r_1} = c_1, \quad c_1 = c_0(r_0). \tag{3.3.44}$$

Here c_1 and r_0 are given, r_0 is the inner radius of the cylinder. A unique root $r_1 > r_0$ of Eq. (3.3.44) can be obtained. The existence of a global solution has been proved by extending the existence of the local-in-time solution. The proof of the results in the cylindrical geometry are not on the same lines as proofs developed for ODP given in Eqs (3.3.34)–(3.3.38). It has been proved in [76] that under suitable assumptions there exists a t_* such that $S(t_*) = r_0$.

Quasi-Static Two-Dimensional ODP and the Hele-Shaw Problem

A link between *quasi-static two-dimensional ODP* and the *Hele-Shaw problem* has been discussed in [77]. Let $p(x, y, t)$, where t is a parameter, be the pressure in the well-posed Hele-Shaw problem in which the free boundary is blown outwards. The pressure in the liquid satisfies the equation

$$\nabla^2 p = 0, \quad \text{in } \Omega \subset R^2. \tag{3.3.45}$$

Let $\partial\Omega$, the boundary of the simply connected region Ω, be a free boundary.

On $\partial\Omega$, we have

$$p = 0, \quad \frac{\partial p}{\partial n} = -V_n \quad \text{or} \quad \nabla p \cdot \nabla w = -1, \quad \text{on } t = w(x, y), \tag{3.3.46}$$

where $t = w(x, y)$ is the equation of the free boundary and $\vec{n}$ is the unit outward normal to the free boundary.

Eq. (3.3.45) can be obtained by taking $\vec{V} = -\text{grad}\, p$ in the equation of continuity of an incompressible fluid whose flow is considered in a narrow channel [78, 79]. Free boundary is the surface of an expanding fluid blob. The continuity of pressure at the free boundary after appropriate scaling gives $p = 0$. The steady-state diffusion equation for concentration and the boundary conditions at the free boundary in a two-dimensional ODP can be written as follows:

$$\nabla^2 c = 1, \quad \text{in } \Omega \subset R^2, \tag{3.3.47}$$

$$c = \frac{\partial c}{\partial n} = 0, \quad \text{on } t = w(x, y). \tag{3.3.48}$$

The Hele-Shaw problem (3.3.45)–(3.3.46) can be converted to a steady-state two-dimensional ODP described in Eqs (3.3.47)–(3.3.48) by using the transformation given in Eq. (6.2.25).

The problem defined in Eqs (3.3.47)–(3.3.48) is generally well-posed and so is the Hele-Shaw problem in which the boundary blows outwards [77]. However the Hele-Shaw 'suction problem' is an *ill-posed problem* and the quasi-static ODP has recently been found to be a very effective tool for revealing some unexpected regularity properties of the free boundary in the solution of unstable Hele-Shaw suction problem [77].