Chapter 11

Regularity of the Weak Solutions of Some Stefan Problems

11.1 REGULARITY OF THE WEAK SOLUTIONS OF ONE-DIMENSIONAL STEFAN PROBLEMS

A weak formulation (also called a weak enthalpy formulation) of the one-dimensional classical Stefan problem was first developed in [329] which was further explored in [330]. Since the publication of these two fundamental works, the theory of weak solutions (also called generalized solutions) of Stefan problems and Stefan-like problems had a phenomenal growth. Since the emphasis in this volume is on the classical solutions, our interest in this chapter in dealing with weak solutions is limited to the extent of exhibiting the regularity of some weak solutions under suitable assumptions. By considering a one-dimensional Stefan problem it has been shown earlier in Section 5.2 that a classical solution is a weak solution. Some conditions under which a weak solution becomes a classical solution were also discussed. We shall state in this chapter some properties of the weak solutions and discuss under what conditions a weak solution becomes a classical solution. Some questions related to the behaviour of the mush will also be examined.

The weak formulation of a classical problem was given in Section 5.2. For the analysis of weak solutions, the following formulation is more convenient. We consider the two-phase melting problem (10.2.9)–(10.2.16) and for simplicity take densities and specific heats of the two phases equal to unity. Eqs (10.2.9)–(10.2.16) will now be considered with these changes. Let enthalpy $H(T)$ and functions $\phi(x)$ and $f(t)$ be defined as follows (cf. [331]).

Set

$$\Omega = \{x : 0 \le x \le d\}, \text{ and } \Omega_{t_*} = \Omega \times \{0 < t < t_*\}, \tag{11.1.1}$$

and let $T(x,t)$ be the temperature in Ω_{t_*}, where $T = K_1 T_1$ at points where $T_1 > 0$ and $T = K_2 T_2$ at points where $T_2 < 0$. Let,

$$\left.\begin{aligned} H(T) &= T/K_1, &&\text{if} \quad T > 0,\\ &= T/K_2 - l, &&\text{if} \quad T \le 0,\\ &= \beta(x,t), &&\text{if} \quad T = 0, \quad -l \le \beta(x,t) \le 0, \end{aligned}\right\} \tag{11.1.2}$$

The Classical Stefan Problem. https://doi.org/10.1016/B978-0-444-63581-5.00011-7

where $\beta(x,t)$ is an arbitrary function,

$$\left.\begin{aligned} T(x,0) = \phi(x) &= K_1\phi_1, \quad \text{for} \quad 0 \le x \le b, \\ &= K_2\phi_2, \quad \text{for} \quad b \le x \le d, \end{aligned}\right\} \tag{11.1.3}$$

$$\left.\begin{aligned} T(x,t)|_{x=0,d} &= K_1 f_1(t), \quad \text{on} \quad x = 0, \\ &= K_2 f_2(t), \quad \text{on} \quad x = d. \end{aligned}\right\} \tag{11.1.4}$$

In Eq. (11.1.2), the enthalpy of the mushy region belongs to the interval $[-l, 0]$, whereas in Section 5.2, the enthalpy of the mushy region belongs to the interval $[0, l]$. This is possible by choosing the reference enthalpy differently in different problems but the jump in the enthalpy across the free boundary should remain l.

Definition 11.1.1. A pair of bounded measurable functions $(T(x,t), H(T(x,t)))$, on Ω_{t_*} is called a weak solution (or a generalized solution) of Eqs (10.2.9)–(10.2.16) if the equality

$$\iint_{\Omega_{t*}} [T\psi_{xx} + H(T)\psi_t]\, dxdt = \int_0^{t_*} [\phi\psi_x]_{x=0}^{x=d}\, dt - \int_\Omega H(\phi(x))\psi(x,0)dx, \tag{11.1.5}$$

holds for any $\psi(x,t)$ such that ψ_x, ψ_{xx}, ψ_t are continuous in $\overline{\Omega}_{t_*}$ and $\psi = 0$ at $x = 0, d$ and at $t = t_*$.

It may be noted that if no sign restrictions are imposed on the initial temperature, then the uniqueness of the weak solution is guaranteed only if the initial enthalpy is known exactly. This implies that if in the classical formulation the initial temperature is equal to the melting temperature in any region, then it should be specified whether it is a solid region or a liquid region.

The energy equation has the form

$$\frac{\partial H(T)}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad (x,t) \in \Omega_{t_*}, \tag{11.1.6}$$

which is satisfied in the distributional sense as at $T = 0$, $H(T)$ is not differentiable. It has been shown in Section 5.2 that a classical solution is a weak solution. The converse that a weak solution is a classical solution was discussed in Section 5.2 under the assumptions that a smooth interface $x = S(t)$ having the temperature $T = 0$ exists which separates the solid region from the liquid region. Examining the conditions under which the mushy region in a weak solution in R^n, $n \ge 1$ has measure zero, and it coincides with the phase-change interface and the Stefan condition holds on the interface, is the main consideration in this chapter.

The definition of a weak solution in R^n, $n > 1$ and some properties of the weak solution will be discussed in Section 11.2. In [331], the existence and the continuity of the free boundary in the weak solution of Eqs (10.2.9)–(10.2.16) were established and several other results were also proved. On using some of the results established in [332], the continuous differentiability of the free boundary was proved in [333]. Some of the results which were used in [333] to prove the continuous differentiability of the free boundary are given in Proposition 11.1.1.

Proposition 11.1.1. *Assume that ϕ' belongs to $L^2(0,d)$ and there exists a function $\Psi(x,t)$ in $\overline{\Omega}_{t_*}$ with Ψ_x, Ψ_{xx}, Ψ_t continuous in $\overline{\Omega}_{t_*}$ and $\Psi(0,t) = f_1(t)$, $\Psi(d,t) = f_2(t)$, $\Psi(x,0) = \phi(x)$ for $0 \le t \le t_*$ and x close to 0 or 1. Then the following results hold.*

(1) *There exists a unique weak solution $T(x,t)$.*

(2) *$T(x,t)$ is continuous on $\overline{\Omega}_{t_*}$, smooth in $\Omega_{t_*} \backslash \{T = 0\}$, and satisfies $T_t = k_1 T_{xx}$ ($T_t = k_2 T_{xx}$) in $\Omega_{t_*} \cap \{T > 0\}$ (resp. $\Omega_{t_*} \cap \{T < 0\}$).*

(3) *There exists a constant $p > 0$, which depends only on the data, such that $T > 0$ ($T < 0$) for $|x| \le p$ (resp. $|1 - x| \le p$).*

(4) *There exist a constant q such that*

$$\left(\int_0^1 |T_x(x,t)|^2 \, dx\right)^{1/2} \le q, \quad 0 \le t \le t_*.$$

(5) *For each $t \in [0,t_*]$ and x_1, $x_2 \in [0,d]$, there exists a constant q' such that*

$$|T(x_1,t) - T(x_2,t)| \le q' \, |x_1 - x_2|^{1/2}.$$

(6) *For each $t \in [0,t_*]$, there exist a unique $S(t)$ such that $T(S(t),t) = 0$.*

(7) *$S(t)$ is a continuous function of t for $0 \le t < t_*$.*

(8) *The energy balance condition at $x = S(t)$ is satisfied for each $t \in [0,t_*]$ in a weak form of Eq. (10.2.16) as follows.*

$$\begin{aligned} l(S(t) - S(0)) = {} & -(1/k_1)\int_0^{S(t)} T_1(x,t)dx - (1/k_2)\int_{S(t)}^d T_2(x,t)dx \\ & + (1/k_1)\int_0^b \phi_1(x)dx + (1/k_2)\int_b^d \phi_2(x)dx \\ & + \int_0^t \frac{\partial T_2}{\partial x}(d,\sigma)d\sigma - \int_0^t \frac{\partial T_1}{\partial x}(0,\sigma)d\sigma. \end{aligned} \tag{11.1.7}$$

Eq. (11.1.7) represents the energy balance in the region $0 \le x \le d$ (apply energy conservation law) if the parameters are defined suitably (take $K_S = K_L = 1$ as in [331]). Let $W_\sigma = \{x \in \Omega : T(x,\sigma) = 0\}$ and $W = \bigcup_{0 \le \sigma < t_*} W_\sigma$. W is called a mushy region or a cloud. The main result of [333] is the following proposition.

Proposition 11.1.2. *Let $T(x,t)$ and $S(t)$ satisfy the conditions given earlier in (2)–(7). Assume (without loss of generality) that $S(t)$ is continuous for $0 \le t \le t_*$ for some t_*. Then $S(t)$ is continuously differentiable for $0 < t \le t_*$, $T_x(S(t) \pm 0, t)$ are well defined, bounded and continuous for $0 < t \le t_*$ and*

$$l\dot{S}(t) = -K_1 T_x(S(t) - 0, t) + K_2 T_x(S(t) + 0, t), \quad 0 < t \le t_*. \tag{11.1.8}$$

In particular, Propositions 11.1.1 and 11.1.2 guarantee the existence of a unique classical solution without size restrictions on the data which have been imposed in some other studies, see [292, 318]. As discussed earlier, continuous differentiability of the free boundary in the classical solution of Eqs (10.2.9)–(10.2.16) implies that $S(t)$ is a C^∞-function. The first step in the proof of Proposition 11.1.2, as discussed in [334], is to prove that if the results of Proposition 11.1.1 hold, then there exists a constant M such that (Hölder continuity with exponent 3/4)

$$|S(t_1) - S(t_2)| \leq M\,|t_1 - t_2|^{3/4}, \quad 0 \leq t_1, t_2 \leq t_*. \tag{11.1.9}$$

On using the uniform continuity of $S(t)$ on $[0, t_*]$ and the weak form of the energy equation (11.1.7), an estimate of the maximum oscillation of $S(t)$ over a time interval of variable length can be obtained. The following proposition asserts the behaviour of the derivative of the temperature in the solution of the heat equation considered in the region $0 \leq x \leq S(t)$, $0 < t \leq t_*$. Proposition 11.1.3 when used in Eq. (11.1.8) establishes that $\dot{S}(t)$ is continuous.

Proposition 11.1.3. *Let $S(t)$ be such that $S(t) \geq \hat{d} > 0$ $(0 \leq t \leq t_*)$, $S(0) = b$ and*

$$|S(t_1) - S(t_2)| \leq M\,|t_1 - t_2|^{\lambda}, \quad 0 \leq t_1, t_2 \leq t_*, \tag{11.1.10}$$

where $1/2 < \lambda \leq 1$. Let $V(x,t)$ be the solution of the following problem: (i) $V_t = V_{xx}$, $0 < x < S(t)$, $0 < t \leq t_*$, $S(t)$ is known, (ii) $V(x,0) = \phi(x)$, $0 \leq x \leq b$, (iii) $V(0,t) = f(t)$, $0 \leq t \leq t_*$, (iv) $V(S(t),t) = 0$, $0 \leq t \leq t_*$ and (v) $f(t)$ and $\phi(x)$ are continuous with $f(0) = \phi(0)$ and $|\phi(x)| \leq \alpha(b - x)$, $0 \leq x \leq b$, α constant. Then $V_x(x,t)$ converges as $x \to S(t) - 0$ to a limit $V_x(S(t)-0, t)$ which is a bounded continuous function of t for $0 < t \leq t_*$. Moreover, the convergence is uniform on $[\delta, t_*]$ for any $\delta > 0$.

The proof of Proposition 11.1.3 has been developed in [334] by using the results which follow from Proposition 11.1.1, and the integrated (with respect to time over the interval $[t_1, t_2] \subset (0, t_*]$) form of the Stefan condition. As the following proposition suggests, the infinite differentiability of the free boundary can be proved under weaker conditions also (cf. [45]).

Proposition 11.1.4. *Assume that $\phi_i(x)$, $f_i(t)$ $(i = 1, 2)$ are bounded piecewise continuous functions. Then there exists a $t_* > 0$ such that problem (10.2.9)–(10.2.16) possesses a unique solution in $(0, t_*)$ and $t_* = \sup(t; 0 < S(t) < d, -\infty < \dot{S}(t) < +\infty)$. Moreover $S(t) \in C^{\infty}(0, t_*)$ and the solution exhibits continuous and monotone dependence upon the data.*

To prove Proposition 11.1.4 firstly a smooth chain of data (e.g. C^{∞}-functions) with appropriate compatibility conditions at $(0,0)$ and $(1,0)$ is considered. The existence of a classical solution in some small time interval is guaranteed. By showing the convergence of a suitable sequence of weak solutions with the assumed data, the Lipschitz coefficient of the classical temperature solution, its derivative and $S(t)$ can be estimated. It has been shown that

$$|T(x,t)| \leq M\,|x - S(t)|, \quad \tau < t \leq t_*,\ \tau \in (0, t_*/2),\ 0 \leq x \leq d, \tag{11.1.11}$$

and

$$\left|S(t') - S(t'')\right| \leq \nu M\left|t' - t''\right|, \quad \tau \leq t',\ t'' \leq t_*, \tag{11.1.12}$$

for some constant M and $\nu = (k_1 + k_2)/l$ which depend only on the L_∞ norm of the data. Next, consider sequences $\{f_i^{(k)}\}$, $\{\phi_i^{(k)}\}$ forming a smooth chain of data which approximate f_i, ϕ_i and preserve L_∞ norms and signs. In this way we can construct sequences of classical solutions $\{S^{(k)}(t), T^{(k)}(x,t)\}$. It has been shown that these sequences converge to the classical solution of the Stefan problem in which the Stefan condition (11.1.8) is satisfied in the integrated (w.r.t. time) form.

The regularity of the weak solutions in the degenerate Stefan problems has also been considered in some references. For the classical formulation of parabolic–elliptic degenerate Stefan problem, we make some changes in Eqs (10.2.9) and (10.2.10) and consider the following equations in their place.

$$\alpha_1(T)T_t = T_{xx} + q(T), \quad 0 < x < S(t),\ 0 < t < t_*, \tag{11.1.13}$$

$$\alpha_2(T)T_t = T_{xx} + q(T), \quad S(t) < x < d,\ 0 < t < t_*, \tag{11.1.14}$$

$$\left.\begin{aligned} T &= T_1,\ \text{if } T_1 > 0, \\ &= T_2,\ \text{if } T_2 \le 0. \end{aligned}\right\} \tag{11.1.15}$$

Here, $\alpha_i(T) \ge 0$ and the equality holds if and only if $T = 0$ ($i = 1, 2$). For the formulation of a parabolic–elliptic Stefan problem, we shall consider Eqs (11.1.13)–(11.1.15) and Eqs (10.2.11)–(10.2.16). To define a weak formulation of this degenerate problem, we introduce the following notations.

$$H(T) \in Q(T),\ \text{for a.e. } (x,t) \in \Omega_{t_*} \quad (\text{see Eq. 11.1.1 for } \Omega_{t_*}), \tag{11.1.16}$$

$$\left.\begin{aligned} Q(T) = \int_0^T \alpha_1(\xi)d\xi &= C_+(T), \quad T > 0, \\ &= [-l, 0], \quad T = 0, \\ &= -l + \int_0^T \alpha_2(\xi)d\xi = C_-(T), \quad T < 0. \end{aligned}\right\} \tag{11.1.17}$$

Let

$$\left.\begin{aligned} V_0(x) &= C_+(\phi(x)), \quad \text{if} \quad x < S(0) = b, \\ &= C_-(\phi(x)), \quad \text{if} \quad x > b. \end{aligned}\right\} \tag{11.1.18}$$

Here, $\phi(x)$ is the initial temperature (see Eq. 11.1.3). Let $\psi(x,t)$ be a test function as defined in Eq. (11.1.5). A weak solution of the degenerate problem (11.1.13)–(11.1.15) and (10.2.11)–(10.2.16) is defined as a pair of bounded measurable functions (H, T) such that the following equation

$$\iint_{\Omega_*} (H\psi_t + T\psi_{xx} + \psi q)dt = \int_0^{t_*} (f_2(t)\psi_x(1,t) - f_1(t)\psi_x(0,t))dt - \int_0^{t_*} V_0(x)\psi(x,0)dx, \tag{11.1.19}$$

is satisfied for all test functions $\psi(x,t)$. Note that in the degenerate case, the temperature is not a well-defined function of enthalpy as at $T = 0$, $\alpha_i(T) = 0$, $i = 1, 2$.

The uniqueness of the weak solution of the above degenerate problem has been proved in [335]. The Lipschitz continuity of the free boundary in the degenerate problem has been proved in [336] under the following assumptions.

(1) $\alpha_i(T) \in C^{\infty}(R)$, $\alpha_i(T) \geq 0$ and $\alpha_i(T) = 0$ if and only if $T = 0$ ($i = 1, 2$).

(2) $f_1(t), f_2(t) \in C^2[0, t_*]$, $\phi(x) \in C^3[0, 1]$.

(3) $f_1(t)$ is strictly positive, while $f_2(t)$ is strictly negative.

(4) $\phi(x)(x - b) \leq 0$ and the equality holds only at $x = b$.

(5) The compatibility conditions hold at $x = 0$ and at $x = d$, i.e. $f_1(0) = \phi(0), f_2(0) = \phi(1)$.

(6) $\phi'(b) < 0$.

If in addition to assumptions (1)–(6), we have also

(7) $f_1(t), f_2(t)$ and $\phi'(x)$ piecewise monotone,

then using the lap number theory [337] it has been proved in [338] that for each $t \in [0, t_*]$, the limits $T_x(S(t)\pm, t)$ exist and $S(t)$ is differentiable.

A weak formulation of an n-phase ($n > 2$) classical Stefan problem (10.2.59)–(10.2.65) with $(n - 1)$ free boundaries can be easily obtained if enthalpy and temperature are suitably defined (see Eqs 11.1.2–11.1.4) in the union of regions occupied by different phases. It has been shown in [331] that under suitable assumptions, the free boundaries are C^{∞}-functions.

Using a weak formulation, the criterion for the disappearance of a mushy region in finite time or its existence for all times has been studied in many references. In Section 5.2, this aspect has been discussed for some one-dimensional problems, and in Section 11.2 this aspect will be further discussed for multidimensional problems. It may be noted that in the weak formulation, mushy region came into existence because of the formulation of the classical problem in terms of enthalpy defined in Eq. (11.1.2). We can say that mushy region in Eq. (11.1.2) is a mathematical novelty. But in classical enthalpy formulation (CEF) discussed in Section 5.1, the physics tells us that the mush is formed before the liquid phase comes into existence. The mushy region in the weak formulation of the classical problem has been introduced artificially, in the sense that in the classical formulation there was no mushy region. No energy equation is written for the mushy region introduced in Eq. (11.1.2). In CEF discussed in Section 5.1 (Eqs 5.1.1–5.1.12), an energy equation is to be solved in the mushy region and energy balance is done at the boundaries of the mushy region, i.e. at the solid-mush and liquid-mush boundaries.

We shall now briefly discuss the characterization of mushy regions which may arise in the weak formulation or in CEF. It may be noted that the solution of CEF (we call this solution CES) is not a solution of the classical problem in which a single phase-change boundary exists but CES could be a weak solution. A weak solution will be CES provided smooth interfaces $S_1(t)$ and $S_2(t)$ exist in the weak solution and energy balance conditions are satisfied on them. The relationship amongst these three solutions was briefly discussed in Eq. (5.2.16).

The nonexistence of a classical solution, i.e. the existence of a mushy region in the two-phase formulation of the Joule heating problem (see Section 5.1) has been discussed in [339]. If a constant heat source is present, then the infinite differentiability of the solid-mush boundary $S_1(t)$ and of the liquid-mush boundary $S_2(t)$ has been established in [146]. Several results on the structure of the weak solution have been presented in [309] and we present some of them here. When a volumetric heat source $f(x,t)$ is present, the differentiability of $S_1(t)$ and $S_2(t)$ has been proved in a small time interval $(0, t_*)$ in [309, Chapter VI] under the assumption that at $t = 0$, $S_1(0) = S_2(0) = x_0 \in [-1, 1]$ and $f(x_0, 0) > 0$.

The existence of the weak solution of a phase-change problem considered in $\Omega = \{x : |x| \leq 1\}$ in which an initial specific internal energy is arbitrarily prescribed has been proved in [309]. Let $T(\pm 1, t) = T^{\pm}(t)$ for $t \in (0, \hat{t})$. Then for $t > t_*$ (see [309] for the definition of t_* as it involves many quantities) there exists only one phase (solid or liquid) if $T^+T^- > 0$. If $T^+T^- < 0$, then for $t > t_*$ both solid and liquid phases are present and the weak solution coincides with the classical solution with only one free boundary. For some more results about the disappearance of the mushy region see [309] and Section 5.1. The evaluation of the lifetime of a mush has been studied in [340]. If a suitable energy criterion holds, then the mushy zone disappears in finite time. Otherwise, it exists for all times.

11.2 REGULARITY OF THE WEAK SOLUTIONS OF MULTIDIMENSIONAL STEFAN PROBLEMS

As mentioned earlier, we are in principle interested only in those weak solutions which under suitable assumptions are as good as classical solutions, i.e. the phase-change boundary exists and is differentiable. This requirement is not met by the weak solutions of multiphase multidimensional Stefan problems studied in the references. Therefore multidimensional problems are being discussed here for an expository reason. We first present a weak formulation of a simple two-phase Stefan problem. Weak formulation of a one-phase Stefan problem can be easily obtained as a particular case of the formulation discussed later.

11.2.1 Weak Solutions of Some Two-Phase Stefan Problems in R^n, $n > 1$

The weak solution of a two-phase Stefan problem is being discussed first. For the one-phase problems stronger results are available than for two-phase problems. The geometry and notations of the two-phase Stefan problem formulated in Section 1.4.1 will be retained. Let ∂G_1 lie in the interior of ∂G_2. For any t_*, $0 < t \leq t_* \leq \infty$, set $\Omega_{t_*} = G \times (0, t_*)$. For simplicity, the parabolic heat equations in the two-phases will be taken as

$$\alpha_i \frac{\partial T_i}{\partial t} = \mathcal{L}_i T_i = \nabla^2 T_i + q^{(i)}(x,t) T_i, \quad (x,t) \in \Omega_{t_*}^{(i)} = G_i(0) \times (0, t_*); \; i = 1, 2, \tag{11.2.1}$$

where α_i is a positive constant. We introduce the following notations.

$$\left.\begin{aligned} H(T) &= \alpha_1 T, && \text{if} \quad T > 0, \\ &= [-l, 0], && \text{if} \quad T = 0, \\ &= \alpha_2 T - l, && \text{if} \quad T \leq 0. \end{aligned}\right\} \tag{11.2.2}$$

$$\left.\begin{aligned} T &= T_1/\alpha_1, && \text{if} \quad T > 0, \\ &= T_2/\alpha_2, && \text{if} \quad T < 0. \end{aligned}\right\} \tag{11.2.3}$$

$$T = g = g_i/\alpha_i, \text{ on } \partial_i G(0) \times (0, t_*), \; i = 1, 2. \tag{11.2.4}$$

$$T = f = f_i/\alpha_i, \text{ on } G_i(0), \; i = 1, 2. \tag{11.2.5}$$

$$q = q^{(i)}, \quad (x, t) \in \Omega_{t_*}^{(i)}, \; i = 1, 2. \tag{11.2.6}$$

With the help of Eqs (11.2.2)–(11.2.6), the two equations in Eq. (11.2.1) can be written as a single equation

$$\frac{\partial H}{\partial t} = \nabla^2 T + qT, \quad (x, t) \in \Omega_{t_*}, \tag{11.2.7}$$

which holds in the distributional sense in Ω_{t_*}. Following the procedure indicated in Section 5.2 to obtain a weak solution in the one-dimensional case, a weak formulation for the problem (11.2.2)–(11.2.6) can also be obtained. In the present case, integration by parts is to be done using Green's formulas [82]. A pair of bounded measurable functions (H, T) is called a weak or a generalized solution of the equation

$$\iint_{\Omega_{t_*}} \left[T\mathcal{L}^*\phi + H(T)\frac{\partial \phi}{\partial t} \right] dxdt = \int_0^{t_*} \int_{\partial G} g\frac{\partial \phi}{\partial n} ds_x dt - \int_{G(0)} T(f)\phi dx, \tag{11.2.8}$$

if Eq. (11.2.8) is satisfied for all test functions $\phi(x, t)$ with $\nabla_x \phi$, $\nabla_x^2 \phi$, $D_t \phi$ continuous in $\overline{\Omega}_{t_*}$ and $\phi = 0$ on $G(t_*)$ and on $\partial G \times (0, t_*)$. $\vec{n}$ is the unit outward normal on the lateral surface and ds_x is the elementary surface area. $\mathcal{L}^* = \mathcal{L}_1$ at points where $T > 0$ and $\mathcal{L}^* = \mathcal{L}_2$ at points where $T < 0$. The set W defined as

$$W = \{(x, t) \in \Omega_{t_*} : -l < H(T) < 0\}, \tag{11.2.9}$$

is called a mushy region or a cloud. It has been proved in [332] that under suitable assumptions such as appropriate sign restrictions, continuity (in some cases smoothness is also required) of initial and boundary data, smoothness of ∂G, the condition $q = q(x) \leq 0$, and holding of compatibility conditions, there exists a unique weak solution of Eq. (11.2.8) which belongs to $W^{1,2}(\Omega_{t_*})$ (see [332] for the complete set of assumptions).

Following the procedure indicated in Section 5.2 for a one-dimensional Stefan problem, it can be proved that a classical solution of the present multidimensional problem is also its weak solution. If a smooth function $\Phi(x, t)$ or a smooth surface $\Gamma(t)$ exists (see Section 1.4.1) which satisfies the conditions mentioned in Section 1.4.1, then it can be proved that a weak solution is also a classical solution. In Section 5.2, it was assumed that the weak solution satisfies initial and boundary conditions and $T_i = 0$, $i = 1, 2$, on $\Gamma(t)$. These assumptions are not necessary. If Φ and $\Gamma(t)$ satisfy appropriate smoothness assumptions together with other assumptions mentioned in Section 1.4.1, then by choosing test functions suitably, it can be proved that a weak solution satisfies initial and boundary conditions in addition to the Stefan condition.

The following result plays an important role in the justification of numerical solutions of the classical Stefan problems obtained with the help of weak formulations. Since weak solutions are fixed domain formulations without phase-change boundaries, it is easier to obtain these solutions numerically.

Proposition 11.2.1. *Assume that a unique weak solution of Eq. (11.2.8) exists in Ω_{t_*}. In any open subset M of Ω_{t_*} where $T \geq 0$ and $H(T) \geq 0$ (resp. $T \leq 0$ and $H(T) \leq -l$), T is a classical solution of $\alpha_1 \partial T_1/\partial t = \mathcal{L}_1 T_1 = \nabla^2 T_1$ (resp. $\alpha_2 \partial T/\partial t = \mathcal{L}_2 T_2 = \nabla^2 T_2$).*

In essence, Proposition 11.2.1 tells us that under certain assumptions, away from the mushy region, the weak solution is as good as a classical solution. More general parabolic operators can also be considered in Proposition 11.2.1. The stability of the weak solution and some properties of the weak solution have also been discussed in [332].

In several references, more general parabolic equations have also been considered. For further discussion, we shall continue with the notations given in Eqs (11.2.2)–(11.2.6). In [341], parabolic equations of the following form have been considered in both solid and liquid phases.

$$C(x,t,T)\frac{\partial T}{\partial t} - \nabla(K(x,t,T)\nabla T) = q(x,t,T), \quad (x,t) \in \Omega_{t_*}. \tag{11.2.10}$$

On using Kirchhoff's transformation given in Eq. (1.4.29) and notations used in Eqs (11.2.2)–(11.2.6), an equation of the form Eq. (11.2.7) can be obtained in terms of θ (see Eq. 1.4.29 for θ). The existence and uniqueness of the weak solution and its continuous and monotone dependence on the initial and boundary data has been proved in [341] under weaker conditions and piecewise smoothness of ∂G is sufficient. It has been proved that

$$\theta \in H^1(\Omega_{t_*}) \cap L^\infty(0, t_*;\ H^1(G)).$$

See Appendix D for the definition of $L^\infty(0, t_*; H^1(G))$.

Instead of a classical formulation of the two-phase Stefan problem, the following singular nonlinear partial differential equation has been considered in [342].

$$\frac{\partial H(T)}{\partial t} - \operatorname{div} \vec{a}(x,t,T,\nabla T) + b(x,t,T,\nabla T) \ni 0. \tag{11.2.11}$$

The weak derivatives have to be considered in Eq. (11.2.11). By a weak solution of Eq. (11.2.11), we mean a function $T(x,t) \in W_2^{1,1}$ defined by $T = H^{-1}(w)$, where w is a function defined in Ω_{t_*} such that $w \subset H(T)$, the inclusion being intended in the sense of graphs and w and T satisfy the equation

$$\int_G w(x,\tau)\phi(x,\tau)\Big|_{t_0}^{t} dx + \int_{t_0}^{t}\int_G \left\{-w(x,\tau)\frac{\partial \phi}{\partial \tau} + \vec{a}\cdot\nabla_x\phi + b(x,\tau,T,\nabla_x T)\phi\right\} dx d\tau = 0, \tag{11.2.12}$$

for all test functions $\phi \in W_2^{1,1}(\Omega_{t_*})$, whose trace is zero on $\partial G \times (0, t_*)$ and on all intervals $[t_0, t] \subset (0, t_*]$. Under suitable assumptions (cf. [342]), the continuity of $T(x,t)$ has been established which can be extended up to $\overline{\Omega}_{t_*}$. By considering a singular partial differential equation of the form

$$\frac{\partial H(T)}{\partial t} \ni \nabla^2 T(x,t), \tag{11.2.13}$$

which holds in Ω_{t_*} in the sense of distributions, continuity of the temperature in the weak solution of a two-phase problem has been proved in [343]. The method of proof in [343] relies strongly on the properties of the Laplacian operator and the absence of lower order terms in the energy equation. The approach in [342] is different from [343] and the method of proof consists of a suitable modification of the parabolic version of De Giorgi estimates reported in [295]. For further references on weak solutions, see the bibliography in [342].

The characterization of the mushy region in CEF in R^n, $n > 1$ and in the weak solution of the classical Stefan problem will now be discussed briefly. The first attempt to investigate the behaviour of the mushy region in a multidimensional two-phase problem was made in [344]. The nonincrease of the mushy region in the homogeneous Stefan problem (no heat sources) with constant Dirichlet data and almost uniformly continuous initial data was proved. The results of [344] have been generalized in [345] by considering a nonhomogeneous Stefan problem with heat sources. Let $(T, H(T))$ be a bounded generalized solution of the following problem.

$$\left.\begin{aligned} \frac{\partial H}{\partial t} - \nabla^2 T &= f(H), \text{ in } \Omega_{t_*}, \\ T|_{\partial G \times (0,t_*)} &= \overline{T}(x,t), \ x \in \partial G, \\ H|_{t=0} &= H_0(x), \ x \in G. \end{aligned}\right\} \tag{11.2.14}$$

Let the mushy region $W(x,t)$ be defined by Eq. (11.2.9) and let $W(t_0) = W \cap (t = t_0)$. If the function $f(H)$ in Eq. (11.2.14) is uniformly Lipschitz continuous, $f(0) \geq 0$ and $f(-l) \leq 0$, then for every $\overline{T} \in L_\infty(\partial G \times (0, t_*))$, $H_0 \in L_\infty(G)$, the mushy region $W(x,t)$ in the bounded generalized solution of problem (11.2.14) is nonincreasing in time. $W(t_2) \subset W(t_1)$ for every $t_2 > t_1$, in the sense that $|W(t_2) \backslash W(t_1)| = 0$. Furthermore, the mushy region can be described in the following way: There exists a nonnegative function $P : G \to R \cup \{+\infty\}$ such that

$$W = \{(x,t) : x \in W(0), \ 0 \leq t < P(x)\}. \tag{11.2.15}$$

For every $x \in W(0)$ on the interval $t \in [0, P(x))$, the function $H(T)$ is a solution of the Cauchy problem

$$H_t = f(H), \quad H|_{t=0} = H_0(x). \tag{11.2.16}$$

Criteria for the disappearance of the mushy region after some finite time in the one-phase and two-phase Stefan problems in the absence of heat sources in domains with C^1 boundary have been discussed in [346]. The behaviour of the mushy region in the corner points of a domain from different angles has also been discussed.

A CEF in R^n, $n \geq 1$ has been considered in [157] and under the assumption that both regular and weak solutions exist, the behaviour of the solution has been investigated. A weak formulation in the form of the following singular equation has been considered.

$$\frac{\partial H}{\partial t} - \operatorname{div}\vec{K}(x,t,T,\nabla_x T) = Q(x,t,H), \text{ in } \Omega_{t_*}. \tag{11.2.17}$$

Here, the function $Q \in C(\Omega_{t_*} \times R)$. The main interest in the weak solution in [157] is in the behaviour of the solution near the free boundaries and in the growth of the mushy region. To investigate the behaviour of the solution near solid-mush and liquid-mush boundaries, it has been assumed that the classical enthalpy solution exists. It has been further assumed that the mush keeps expanding into the solid phase, until, eventually, it is invaded by a new liquid phase. To investigate the behaviour of the mush near the points where $H = 0$ ($H \geq 0$ in the liquid region), it is assumed that a weak solution exists with $T \geq 0$ in Ω_{t_*} (temperature of the solid is taken as zero). Let (T, H), where $H = T + w$ with $T \in L^\infty(\Omega_{t_*}) \cap W_2^{1,1}(\Omega_{t_*})$, $w \in H_\nu(T)$ a.e. in Ω_{t_*} and $w \in L^\infty(\Omega_{t_*})$ be a local solution of Eq. (11.2.17) in Ω_{t_*} (see [157] for the definition of a local solution). Here, H_ν is Heaviside graph in which $H_\nu(T) = 0, T > 0$, $H_\nu(T) = -l, T < 0$ and $H_\nu(T) \in [0, -l]$ for $T = 0$. If $\vec{K}$ and Q satisfy some assumptions (cf. [157]), then

$$w(x,t) - w(x,\tau) \geq \int_\tau^t Q(x,p,w(x,p))\chi_{\{T=0\}}(x,p)\,dp, \tag{11.2.18}$$

a.e. in Ω_{t_*}, $\tau < t$. Also for $0 < \tau < t < t_*$, we have

$$\text{meas}\,\{x \in G | T(x,t) = 0\} \leq \text{meas}\,\{x \in G | T(x,\tau) = 0\}\,. \tag{11.2.19}$$

Similar results can be obtained near the points where $H = -l$ (see [157] for other results). An example has been constructed in which the mushy region is enclosed by two solid phases. The mushy region disappears and reappears immediately after extinction.

11.2.2 Regularity of the Weak Solutions of One-Phase Stefan Problems in R^n, $n > 1$

In view of Eq. (11.2.2), we shall call a Stefan problem one-phase, if either $H \leq -l$ ($H \geq 0$) in one of the phases and in another phase $H = 0$ ($H = -l$). We consider the second case and let $G_1(t)$ be the liquid phase at any time $t \in (0, t_*)$ with temperature $T(x,t) \geq 0$ and $H(T) \geq 0$. In the solid phase $H(T) = -l$. $G_1(t)$ is the set of points (x,t) with x outside $\partial_1 G$. Note that ∂G_1 lies inside ∂G_2. Take $\phi_2 = 0, f_2 = 0$, $H(\phi_2) = -l$ and $H(\phi_1) > 0$. The existence and uniqueness of the weak solution (under suitable assumptions) of the one-phase Stefan problem has been proved in [332]. The continuity of the temperature in Ω_{t_*} in the weak solution can also be proved under suitable assumptions some of which are: (i) $\partial_1 G \in C^{2+\eta}$, $\eta > 0$, (ii) f_1 and ϕ_1 are continuous functions on $\partial_1 G \times [0,\infty)$ and $\overline{(G_1(0))}$ respectively, coinciding on $\partial_1 G$ and (iii) $\phi_1 > 0$ in $G_1(0)$ (see [332] for some other assumptions). The nonoccurrence of the mushy region in the one-phase problem has also been proved.

Let $W = \{(x,t) \in \Omega_{t_*}; -l < H(T) < 0\}$ and $W(\sigma) = W \cap \{t = \sigma\}$. Then $W(\sigma) \subset \overline{(G_1(\sigma))} - G_1(\sigma)$. W can be called a *weak free boundary*. It has been proved that: (i) a weak free boundary has no interior points in Ω_{t_*}, and (ii) W is determined only up to a set of measure zero.

The measure of the mushy region in the weak solution of a one-phase multidimensional Stefan problem has been estimated in [347] with the help of a refined method of isoperimetric inequalities. There are two disjoint connected components of the boundary. On one part of the boundary, the temperature $T = 0$ is prescribed and on the other part of the boundary, a constant temperature $T > 0$ is prescribed. The results obtained in [347] hold till the liquid phase reaches the second component of the boundary. The weak solution of a spherically symmetric one-phase Stefan problem in R^n has been considered in [348]. The main goal is to obtain conditions on the prescribed boundary temperature and initial enthalpy for the disappearance of the mushy region after a finite time. A sketch of the proof has been given for a one-phase problem considered in a regular bounded open set of R^n.

The behaviour of the mush in a one-phase Stefan problem has been discussed in [349]. A one-phase problem in a region $\Omega \subset R^n$ with a piecewise-smooth boundary has been considered. It has been shown that the lifetime of the transient phase is uniformly bounded for every $\hat{t}$ and $\gamma > 0$ on the Lebesgue set $\{G_1(\hat{t}, x) < -\gamma\}$ for some specific function G_1 (cf. [349]). If the boundary of Ω is C^2-smooth and a suitable smooth boundary temperature is prescribed, then the necessary and sufficient conditions for the disappearance of the mushy region in some finite time have been obtained. The behaviour of the mush near corner points of a two-dimensional domain has also been investigated.

The equivalence of the weak enthalpy formulation of a one-phase Stefan problem in R^n and its weak variational inequality formulation has been established in [197]. It may be noted that a weak enthalpy formulation of a one-phase Stefan problem can be extended to a two-phase problem but the variational inequality formulation given in Eq. (7.4.36) has no natural extension to an inequality formulation of a two-phase problem. Let Ω be a bounded domain in R^n, $n \geq 1$, whose boundary consists of two smooth connected hypersurfaces Γ_1 and Γ_2 with Γ_1 lying inside Γ_2 and bounding a simply connected domain G. Let B be a large ball with centre 0 containing Ω and set $D = B - G$. The one-phase Stefan problem consists of finding the free boundary $t = S(x)$ and the temperature $T(x, t)$, $0 < t < t_*$, $x \in D$, such that

$$T_t - \nabla^2 T = 0, \text{ in } \{(x,t) : x \in D,\ t > S(x)\}\,;\ S(x) = 0, \text{ if } x \in \Omega, \tag{11.2.20}$$

$$T = 0, \text{ and } \nabla T \cdot \nabla S = -l;\ t = S(x),\ x \in D - \Omega, \tag{11.2.21}$$

$$T = \phi(x) > 0,\ x \in \Omega,\ t = 0, \text{ and } T = g(x,t) > 0,\ x \in \Gamma_1,\ 0 < t < t_*. \tag{11.2.22}$$

Here, $l > 0$ is a constant, and $g(x, t)$ and $\phi(x)$ are C^2-functions in $\Gamma_1 \times (0, t_*]$ and $\overline{\Omega}$, respectively. To define a weak formulation of the problem (11.2.20)–(11.2.22), we define enthalpy $H(T)$ as:

$$\left.\begin{aligned} H(T) &= T, & T > 0 \\ &= T - l, & T \leq 0 \end{aligned}\right\}, \quad \text{and} \quad \left\{\begin{aligned} H(T(x,0)) = f &= \phi(x),\ x \in \Omega, \\ &= -l,\ x \in D - \Omega. \end{aligned}\right. \tag{11.2.23}$$

The region $D - \Omega$ is occupied by ice at temperature $T = 0$. The weak formulation of the problem (11.2.20)–(11.2.22) in D can be easily obtained and is given by

$$\int_0^{t_*} \int_D \left(T\nabla^2 v + H(T) v_t\right) dx dt = \int_0^{t_*} \int_{\partial D} g \frac{\partial v}{\partial n} ds_x dt - \int_D H(T(x,0)) v(x,0) dx. \tag{11.2.24}$$

Here, $v(x, t)$ is a test function in $\overline{D} \times (0, t_*)$ with a definition similar to that given in Eq. (11.1.5) (make appropriate changes). The unique solution of Eq. (11.2.24) exists. On using the transformation

$$u(x,t) = \int_0^t T(x,\tau) d\tau, \quad T \geq 0,\ (x,t) \in D_{t_*} = D \times (0, t_*), \tag{11.2.25}$$

we have $u_t = T(x,t)$. Writing the energy equation in terms of enthalpy and then integrating Eq. (11.2.20) with respect to time, we get

$$u_t - \nabla^2 u = \gamma(u_t) + f, \text{ a.e. in } D_{t_*}. \tag{11.2.26}$$

Here, we have set $H(T) = T - \gamma(T)$, γ is a monotone graph, and $\gamma(T) \geq 0$ (cf. [332]).

For all test functions $v \geq 0$, a.e. in D_{t_*} with appropriate initial and boundary values (make appropriate changes in Eq. (7.4.19) to define $v(x,t)$), we have

$$(u_t - \nabla^2 u)(v - u_t) \geq f(v - u_t), \text{ a.e. in } D_{t_*},\ u_t \geq 0. \tag{11.2.27}$$

Note that $(u_t - \nabla^2 u - f)v = \gamma(T)v \geq 0$ and $(u_t - \nabla^2 u - f)u_t = 0$, in D_{t_*}. Conversely, if u satisfies Eq. (11.2.27) and $T(x,t)$ is defined through Eq. (11.2.25) and satisfies the boundary and initial conditions ($T(x,0) = \phi(x)$, $x \in \Omega$ and $T(x,0) = 0$, $x \in D - \Omega$), then enthalpy equation (11.2.7) with $q = 0$ can be recovered from Eq. (11.2.27) and $(T(x,t), H(T))$ is a solution of Eq. (11.2.24). In other words, $T(x,t)$ is a weak solution if and only if $u(x,t)$ in Eq. (11.2.25) is a solution of the 'variational inequality' (11.2.27). Note that the variational inequality (7.4.36) is different from the 'variational inequality' (11.2.27) as we have $(v - u_t)$ in Eq. (11.2.27) and not $(v - u)$ as in Eq. (7.4.36) and u is also defined differently in Eq. (7.4.36). The variational inequality corresponding to the problem described earlier is given by Eq. (7.4.36) whose unique solution exists (under suitable assumptions). It can be proved that the unique solution of Eq. (7.4.36) satisfies Eq. (11.2.27). Therefore $T(x,t)$ is a solution of Eq. (11.2.24) if and only if it is a solution of Eq. (7.4.36). Using the variational inequality formulation of the one-phase problem, it has been proved in [197] that the free boundary arises as a boundary of a set and so it has no interior points. It has also been shown that under suitable hypotheses, the domain occupied by water in an ice–water system is star shaped and the free boundary is star shaped with the representation $\rho = \rho^*(\theta, t)$, (ρ, θ) are polar coordinates; ρ^* is a continuous function of θ and t and uniformly Lipschitz continuous in the angles θ, and is monotonically increasing in t.

A two-phase two-dimensional Stefan problem in a rectangular region $\Omega_{t_*} = \Omega \times (0, t_*)$, $\Omega = \{(x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < b\}$ has been considered in [350]. Linear homogeneous parabolic equations are considered in the two phases occupying the regions Ω_1 and Ω_2, and $\Omega = \Omega_1 \cup \Omega_2 \cup S(x,0)$; $x_2 = S(x_1, t)$ is the equation of the free boundary. $\Omega_1 = \{(x_1, x_2, t) : 0 < x_2 < S(x_1, t)\}$ and $\Omega_{t_*} \cap \partial\Omega_1 \times (0, t_*)$ is the free boundary. Let $T(x_1, x_2, t)$ be the temperature in Ω_{t_*} defined as in Eq. (11.2.3) with appropriate changes (to formulate this problem see Section 3.2.4 and make appropriate changes). Prescribe suitable boundary conditions on the edges so that the free boundary divides Ω_{t_*} into two disjoint regions, each consisting of a single phase. Let,

$$T(x_1, x_2, 0) = \psi(x_1, x_2), \text{ in } \Omega; \quad \text{and} \quad S(x_1, 0) = y_0(x_1),\ 0 < x_1 < 1. \tag{11.2.28}$$

Here, ψ and y_0 are given functions. It has been proved that if the classical Stefan problem described earlier is formulated in terms of the temperature $T(x_1, x_2, t)$ in Ω_{t_*}, then $T(x_1, x_2, t) \in W_2^{1,1}(\Omega_{t_*}) \cap H^{\delta, \delta/2}(\overline{\Omega}_{t_*})$, $0 < \delta < 1$, the free boundary is a Lipschitz surface and the Stefan condition written in terms of the temperature $T(x_1, x_2, t)$ is satisfied in almost everywhere sense. The Stefan condition has been considered in the form

$$\left[\alpha(T)\frac{\partial T}{\partial x_i}\right]\cos(\vec{n}, x_i) + l\cos(\vec{n}, t) = 0, \quad x_2 = S(x_1, t),\ i = 1, 2, \tag{11.2.29}$$

$\vec{n}$ is the unit outward normal to the free boundary, $\alpha(T) = K_1$ in Ω_1 and $\alpha(T) = K_2$ in Ω_2, the notation $[v]$ stands for the jump in v across the free boundary. The next step is to obtain a weak formulation by multiplying the energy equation formulated in terms of $T(x_1, x_2, t)$ by a suitable test function $\eta(x_1, x_2, t)$ and carry out the integration by parts. We obtain

$$\int_{\Omega_{t_*}} \{\alpha(T)\nabla T \cdot \nabla\eta + T_t\eta + l\chi_t\eta\}\, dxdt + \beta \int_{\Gamma_{1t_*}} T\eta ds = 0. \tag{11.2.30}$$

Here, $\chi(x,t)$ is the characteristic function of the region Ω_1, $\Gamma_{1t_*} = \{(1, x_2); 0 < x_2 < b\} \times (0, t_*)$ and $K_i \partial T/\partial n + \beta T = 0$ on $\Gamma_{1t_*} \cap \partial\Omega_i$. Other boundary conditions have been taken to be either temperature prescribed or no flux conditions. The second term in Eq. (11.2.30) arises during the integration by parts and using the boundary condition, $\eta = 0$ on the boundary on which temperature is prescribed. When Eq. (11.2.30) is discretized in time and not in space (cf. [307, 351]), a sequence of elliptic free boundary problems is obtained. The solution of each free boundary problem has been obtained as the minimum of a suitable functional. To obtain existence, uniqueness and regularity results some conditions on the initial temperature $\psi(x_1, x_2)$ are imposed such $\psi(x_1, x_2) \in W_2^1(\Omega) \cap C^\nu(\overline{\Omega})$, $0 < \nu < 1$, $\psi_{x_1} \leq 0$, $\psi_{x_2} \geq 0$ (cf. [350] for further details).

A regularity theory for the weak free boundary which is defined as $\partial(T > 0)$ for the parabolic two-phase free boundary problems ($T > 0$ in one of the phases and $T \leq 0$ in another phase) has been developed in [352]. The regularity theory has several approaches:

(i) Lipschitz minimal surfaces are smooth.

(ii) 'Flat' minimal surfaces (in some 'Lebesgue' differentiability sense) are smooth.

(iii) Generalized minimal surfaces are smooth except on some small set.

In the approach (i) it has been proved that the free boundary which is Lipschitz in space and time is also regular, see [352]. *Viscosity solutions* have been considered (weak solution is a viscosity solution but every viscosity solution is not a weak solution) whose free boundaries are given (locally) by a Lipschitz graph. In this case they enjoy further regularity and other properties such as the free boundary is a C^1 graph in space and time and temperature is a classical solution. The regularity of the free boundary is possible for those two-phase problems in which the two fluxes from both sides at the free boundary are not vanishing simultaneously. An example has been constructed which shows that a Lipschitz free boundary may remain Lipschitz for an interval of time and may not regularize instantaneously although both the phases may have nonzero temperatures.