Appendix D

Sobolev Spaces

One of the several reasons to enlarge the family of spaces discussed in the earlier appendices is the incompleteness of some of the linear spaces in some norms. For example, the space of functions $C[a,b]$ is a normed space in the norm

$$\|f\| = \left(\int_a^b f^2 dx \right)^{1/2}.$$

However, in this norm $C[a,b]$ is not complete. If the norm on $C[a,b]$ is defined as $L^2[a,b]$ norm, then $C[a,b]$ becomes a Banach space. This is called 'completion' of an incomplete normed space. The sequence of functions $\{f_m(x)\}$, $f_m(x) = \sin \mathrm{mx}/\sqrt{\mathrm{m}}$, $x \in R$, converges uniformly to $f = 0$ in the 'distance norm' but $\{f'_m(x)\}$ does not converge in the distance norm. If we want $\{f'_m(x)\}$ to converge, then we cannot consider derivative as the classical derivative. The sequence does converge if a *weak derivative* (defined below), also called a *distributional derivative* or a *generalized derivative*, is considered. The space of those functions whose weak derivatives exist should also contain all those functions whose classical derivatives exist. We shall first define a weak derivative and then define a suitable norm on the space of functions whose weak derivatives exist and discuss the completeness of this normed space.

1. The weak derivative. Let Ω be a bounded domain in R^n. If $f \in L^p(\Omega)$, then the weak derivative $D^\alpha f$ of order $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$, where $D_i = \partial/\partial x_i$, $i = 1, 2, \ldots n$, is defined as follows.

A function $g \in L^p(\Omega)$ is said to be the *αth weak derivative* of $f \in L^p(\Omega)$, $1 \leq p < \infty$ in the sense of $L^p(\Omega)$, if

$$\int_\Omega g(x)\phi(x)dx = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha \phi(x)dx,$$

for all $\phi \in C_0^{|\alpha|}(\Omega)$. $C_0^{|\alpha|}(\Omega)$ is the subspace of $C^{|\alpha|}(\Omega)$ and consists of functions which have compact support in Ω.

The weak derivative if it exists is unique. If for all $|\alpha| \leq m$ the weak derivatives $D^\alpha f$ exist, then we say that f possesses weak derivatives of all orders $|\alpha| \leq m$.

2. The strong derivative in $L^p(\Omega)$. A function $f \in L^p(\Omega)$ is said to be the *αth strong derivative* of $u \in L^p(\Omega)$, if there exists a sequence $\{u_m\}$ in $C^{|\alpha|}(\overline{\Omega})$ such that $\{u_m\} \to u$ in $L^p(\Omega)$, and $D^\alpha\{u_m\} \to f$ in $L^p(\Omega)$. If the strong derivative exists, then the weak derivative also exists and is unique.

3. The space $V(\Omega)$ of testing functions. The space $C_0^m(\Omega)$ is a subspace of $C^m(\Omega)$ and $C^m(\bar{\Omega})$ is complete in the norm defined earlier in Appendix C. The space $C_0^m(\Omega)$ is a topological vector space in the induced topology but is not complete. We consider $C_0^\infty(\Omega)$, the space of all C^∞-functions ϕ such that: (i) ϕ has a compact support in Ω, and (ii) ϕ has continuous derivatives of all orders. The convergence of a sequence $\{\phi^{(n)}\}$ in $C_0^\infty(\Omega)$ to a function ϕ in $C_0^\infty(\Omega)$ is defined by demanding that for every nonnegative integer r, $D^r\phi^{(n)} \rightarrow D^r\phi$ uniformly. Let $V(\Omega)$ be the space of functions in $C_0^\infty(\Omega)$ with the above convergence criterion. $V(\Omega)$ is a complete topological vector space but is not a normed space. $V(\Omega)$ is reflexive. The elements of $V(\Omega)$ are called *testing functions* or *test functions.*

4. Distributions and the space of distributions. The continuity of a linear real-valued function v on $V(\Omega)$ is defined by specifying that v is continuous if and only if $v(\phi_n) \rightarrow v(\phi)$ whenever $\phi_n \rightarrow \phi$ in the sense of $V(\Omega)$. A functional (continuous linear real-valued function) on $V(\Omega)$ is a called a *distribution.* The space of distributions on $V(\Omega)$ is the dual space denoted by $V(\Omega)'$. If $v \in V(\Omega)'$, then for any $\phi \in V(\Omega)$, the value of v at ϕ is denoted by $v(\phi)$. There is no natural norm in $V(\Omega)'$ and this space can only be given a weak-star topology as dual of $V(\Omega)$. $V'(\Omega)$ is a locally convex topological vector space (cf. [58]) with this topology.

We shall not dwell on the space $V(\Omega)'$, the space of distributions, as it is too big for our purpose. As mentioned earlier bigger spaces could be an advantage for the study of existence results, but they are disadvantageous for the study of uniqueness and stability results. We need some complete normed subspaces of the space of distributions. These subspaces are called *Sobolev spaces*. Sobolev spaces were introduced in the analysis earlier than the space $V(\Omega)'$ (also called the ‘space of Schwartz distributions’).

5. Sobolev space $W^{m,p}(\Omega)$. We shall consider only those spaces in which m is a nonnegative integer p, $1 \leq p \leq \infty$, and Ω is a bounded domain in R^n. The space $W^{m,p}(\Omega)$ is the space of all functions f in $L^p(\Omega)$ whose weak derivatives $D^\alpha f$ of order $|\alpha| \leq m$ belong to $L^p(\Omega)$. $W^{m,p}(\Omega)$ is a Banach space in the norm

$$\|f\|_{m,p} = \|f\|_{p,\Omega}^{(m)} = \left(\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha f|^p \, dx \right)^{1/p}, \quad \|f\|_p = \|f\|_{0,p}\,.$$

For $p = \infty$, we define the norm of f as

$$\|f\|_{m,\infty} = \sum_{|\alpha|\leq m} \left\|D^\alpha f\right\|_{L^\infty(\Omega)} = \sum_{|\alpha|\leq m} \operatorname*{ess\,sup}_{\Omega} \left|D^\alpha f\right|.$$

$W_{loc}^{l,p}(\Omega)$ denotes the space of functions which belong to $W^{l,p}(Q)$, where $Q \subset \Omega$ is an arbitrary bounded domain. Note that $C^\infty(\Omega) \cap W^{m,p}(\Omega)$, is dense in $W^{m,p}(\Omega)$.

6. The spaces $H^m(\Omega)$ and $H_0^m(\Omega)$. It is customary to use the notation $H^{m,p}(\Omega)$ for $W^{m,p}(\Omega)$ and $H^m(\Omega) = W^{m,2}(\Omega)$. The closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ is denoted by $W_0^{m,p}(\Omega)$ and $H_0^m(\Omega) = W_0^{m,2}(\Omega)$. The dual space of $H^m(\Omega)$ is denoted by $H^{-m}(\Omega)$.

7. Sobolev space $W_p^{2m,m}(\Omega_{t_*})$. In the parabolic problems, functions are defined on the set $\Omega_{t_*} = \Omega \times (0, t_*) = \{(x,t) : x \in \Omega \subset R^n,\ 0 < t < t_*\}$. $W_p^{2m,m}(\Omega_{t_*})$, where m is a nonnegative integer and $p \geq 1$, is the closed subspace of $L^p(\Omega_{t_*})$ consisting of functions whose weak derivatives $D_t^r D_x^q$ with r and q satisfying $2r + q \leq 2m$ have finite norms. If $f \in W_p^{2m,m}(\Omega_{t_*})$, then

$$\|f\|^{2m}_{p,\Omega_{t*}} = \sum_{2r+q=0}^{2m} \left\| D_t^r D_x^q \right\|_{p,\Omega_{t*}} .$$

If $f \in L^p(\Omega_{t_*})$, then

$$\|f\|_{p,\Omega_{t*}} = \left(\int_{\Omega_{t*}} |f|^p \, dxdt \right)^{1/p} .$$

The space $W_2^{1,0}(\Omega_{t_*})$ is a Hilbert space with the scalar product

$$(f,g)_{W_2^{1,0}(\Omega_{t*})} = \int_{\Omega_{t*}} (fg + \nabla f \cdot \nabla g) dxdt,$$

and the space $W_2^{1,1}(\Omega_{t_*})$ is the Hilbert space with the scalar product

$$(f,g)_{W_2^{1,1}(\Omega_{t*})} = \int_{\Omega_{t*}} (fg + D_t f D_t g + \nabla f . \nabla g) dxdt.$$

8. The space $L^p(0, t_*; H^{m,q}(\Omega))$. Functions dependent on x and t, where $x \in \Omega$ and $t \in (0, t_*)$ are often treated as functions of t with values in some Banach space for a.a. $t \in (0, t_*)$. For example, the space $L^p(0, t_*; H^{m,q}(\Omega))$ consists of functions $u(t)$ (u is in fact a function of x and t) with values in $H^{m,q}(\Omega)$ for a.a. $t \in (0, t_*)$ and is equipped with the norm

$$\|u\| = \left(\int_0^{t_*} \left(\|u(t)\|_{q,\Omega}^{(m)} \right)^p dt \right)^{1/p}, \quad 1 \le p \le \infty, \;\; 1 \le q < \infty.$$

The space $L^p(0, t_*; W)$, where W is a Banach space, is a Banach space in the norm defined above.

9. The space $L^2\left(\Omega_{t_*}\right)$. We write $L^2\left(0, t_*;\; L^2(\Omega)\right) = L^2\left(\Omega_{t_*}\right)$.

10. Gâteaux derivative or G-derivative. Let $f : H \to R$ be a linear continuous real-valued function, and H a Hilbert space. If there exists an element $f'(u) \in H'$ (dual of H), $u \in H$, such that for all $v \in H$, we have

$$\frac{f(u + \lambda v) - f(u)}{\lambda} \to {}_{H'}\langle f'(u), v \rangle_H \text{ when } \lambda \to 0$$

then $f'(u)$ is called the *Gâteaux derivative* of f at u or *G-derivative* of f at u.

11. Fréchet derivative. The function f as defined in **10**. is said to have a *Fréchet derivative* or a *F-derivative* of f at $u \in H$ if there exists a $\phi \in H'$ such that

$$f(u + v) = f(u) + \phi(v) + O(\|v\|)\, \|v\| .$$

If the F-derivative exists then the G-derivative also exists and both are equal.

12. Sobolev imbedding theorem. The elements of $W^{m,p}(\Omega)$ are strictly speaking not functions defined everywhere on Ω. Equivalence classes of such functions are defined and the functions are equal up to a set of measure zero. When we say that $W^{m,p}(\Omega) \hookrightarrow C^J(\overline{\Omega})$, it means that each $f \in W^{m,p}(\Omega)$ when considered as a function can be redefined on a set of measure zero in Ω such that the modified function $\hat{f}$, which equals f in $W^{m,p}(\Omega)$, belongs to $C^J(\overline{\Omega})$ and satisfies $\|\hat{f};\ C^J(\overline{\Omega})\| \leq M\,\|f\|_{p,\Omega}^{(m)}$, with M independent of f.

We give here just two results pertaining to Sobolev imbedding (also called embedding). For further details, see [22, 58]. Let m, n and r be nonnegative integers.

(1) If $m > n/2 + r$, $\Omega \subset R^n$, then $H^m(\Omega) \subset C^r(\overline{\Omega})$ with continuous injection. Hence, if $\Omega \subset R^2$, then $f \in H^2(\Omega) \Rightarrow f \in C(\overline{\Omega})$ and f is almost everywhere equal to a unique function in $C(\overline{\Omega})$.

(2) If $mp > n$, then $W^{J+m,p}(\Omega) \hookrightarrow C^J(\Omega)$. In particular, if $f \in W^{1,p}$, $\Omega \subset R^n$, $p > n$, then f is almost everywhere equal to a unique function in $C(\overline{\Omega})$.

13. Trace operator. For an arbitrary function $f \in L^p(\Omega)$, $\Omega \subset R^n$, $1 \leq p < \infty$, how to define the values of f on $\partial\Omega$. If $f \in L^p(\Omega)$ is continuous up to the boundary $\partial\Omega$ of Ω, then one can say that the value f takes on $\partial\Omega$ is the restriction to $\partial\Omega$ of the function f. In general, however, the elements of $W^{m,p}(\Omega)$ are defined except on a set of measure zero and it is meaningless therefore to speak of their restrictions to $\partial\Omega$ which has an n-dimensional measure zero. As mentioned earlier $H^1(\Omega)$ is the closure of $C^\infty(\overline{\Omega})$ with respect to the norm

$$\|f\|^2 = \int_\Omega (|f|^2 + |\nabla f|^2)dx.$$

If $f \in C^\infty(\bar{\Omega})$, then it can be proved that there exists a unique continuous linear operator γ_0 from $C^\infty(\bar{\Omega})$ to $L^2(\partial\Omega)$ (provided $\partial\Omega$ is Lipschitz continuous) such that $\gamma_0(f(x)) = f(x)$ for $x \in \partial\Omega$. If $f \in H^1(\Omega)$, we call γ_0, a *trace operator*, denoted here by T_r, such that

$$T_r : H^1(\Omega) \to L^2(\partial\Omega);\ \ T_r f(x) = f(x) \text{ for } x \in \partial\Omega,\ f \in C^\infty(\overline{\Omega}).$$

In particular $H_0^1(\Omega) = \{f \in H^1(\Omega);\ f = 0 \text{ on } \partial\Omega\}$.