Chapter 8

The Hyperbolic Stefan Problem

8.1 INTRODUCTION

A *hyperbolic Stefan problem* is concerned with a phase-change problem in which the heat energy equation is hyperbolic. The change in the problem formulation introduced by the 'hyperbolic nature of the heat equation' is significant in many ways. The speed of heat propagation in the parabolic case is infinite which can be seen from the temperature solution given in Eq. (8.1.1) which is concerned with the problem of one-dimensional heat conduction in an unbounded medium with a constant initial temperature T_0 [24]:

$$T(x,t) = \frac{T_0}{2(\pi kt)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-x')^2/4kt} dx'. \tag{8.1.1}$$

For $t > 0$, howsoever small, and x howsoever large, $T(x,t)$ is different from T_0 which confirms the infinite speed of propagation of heat flow. In the parabolic case, heat flux is governed by the Fourier's law (cf. Eqs 1.3.8, 1.4.9). Thermal energy is transported by molecular motion which has finite speed and therefore Fourier's law seems to be a low order approximation to a more general constitutive law. Cattaneo [206] observed that the infinite speed of heat propagation in Fourier's law is due to diffusion and proposed that heat pulses ought to be transmitted by waves at finite but perhaps high speeds. This does not mean that Fourier's law leading to the diffusion equation should be discarded. *The relaxation time* τ in the hyperbolic model is generally very small in nearly all practical and exotic applications. As the heat flux equation (8.1.3) given later suggests that even on the shortest time scales of our daily lives, we get Fourier's law [207]. The use of hyperbolic heat equation in a mathematical model at very low temperatures is generally accepted, but its use at high temperatures has been debated in the literature and we refer the reader to an exhaustive review article [208] on the 'problem of second sound' (finite speed of heat propagation).

Our interest in the hyperbolic Stefan problems in this volume stems from the fact that models of these problems require many interesting physical concepts in addition to those associated with the Stefan problems. For example, in the hyperbolic Stefan problems, both temperature and flux can be taken to be discontinuous across the phase-change boundary. The delay in the release of latent heat and sensible heat can be considered. These, and several other variations in hyperbolic problems make these problems fairly interesting from the point of view of model and analysis. Many studies of hyperbolic Stefan problems have been conducted. There is still a need to collate the results of these studies in one source.

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8.1.1 Relaxation Time and Relaxation Models

In an idealized solid, for example, thermal energy is transported by two different mechanisms: (1) by quantized electronic excitations, which are called free electrons, and (2) by the quanta of lattice vibrations, which are called phonons. These quanta undergo collisions of a dissipative nature, giving rise to thermal resistance in the medium. A relaxation time τ is associated with the average communication time between these collisions for the commencement of the resistive flow. In essence τ is the relaxation parameter or the time which should elapse for the heat flow to take place after the temperature gradient is formed. There are different times of relaxation, so the mean relaxation time is generally not known. For such solids, it would be more important to know which modes carry the most heat and so we want the dominant rather than the mean mode of relaxation.

How does heat flux respond to the temperature gradient? This has been modelled in the literature in different ways. For example, the flux law in the hyperbolic model has been described as

$$\vec{q}\,(t+\tau) = -\nabla(KT), \quad \tau > 0, \tag{8.1.2}$$

where $\vec{q}$ is the heat flux vector and τ is the relaxation time [209]. In this model, if the temperature gradient is formed at time t then heat is released instantaneously at time $t+\tau$. The first-order approximation of Eq. (8.1.2) gives

$$\tau\vec{q}_t + \vec{q} = -\nabla(KT), \quad \tau > 0. \tag{8.1.3}$$

This model in Eq. (8.1.3) is commonly used in the place of Fourier's law if finite speed of heat propagation is considered. In the relaxation model (8.1.3) a delay of time τ in the heat flow is there but the heat release is not instantaneous at time $t+\tau$ and is distributed over a time interval. Eq. (8.1.3) can be derived as a particular case of the following linearized constitutive equation for the heat flux vector $\vec{q}$ [210]

$$\vec{q}(x,t) = \int_{-\infty}^{t} K(t-r)\nabla T(x,r)dr = -\int_{0}^{\infty} K(p)\nabla T(t-p)dp, \tag{8.1.4}$$

where $K : (0,+\infty) \to R$ is a given kernel called the 'heat flux relaxation function' and accounts for the memory of the material. It can be seen that if $K(t) = (K_0/\tau)\exp(-t/\tau)$ in Eq. (8.1.4), then Eq. (8.1.3) is obtained. In Eq. (8.1.4), the temperature gradient induces heat flow after a delay of time τ and heat is released during a period of time, whereas, in model (8.1.2), heat is released instantaneously after a lapse of time τ from the moment temperature gradient is formed.

We shall be dealing mostly with the heat flux model given in Eq. (8.1.3) which is called a *generalized Fourier's law* or a *non-Fourier's law*. Fourier's law cannot be derived from Eq. (8.1.4) unless ∇T is constant for all time and the thermal conductivity is defined as $\int_0^\infty K(p)dp$. It is clear that the flux law given in Eq. (8.1.3) is a particular case of a more general law stated in Eq. (8.1.2). Why do we not accept Eq. (8.1.2) as the generalized Fourier's law? This question has been discussed in [211] with the help of a simple experiment. Consider a piece of ice of unit volume at temperature $T < 0$. It is heated by a heat source of constant intensity $F > 0$. Let C_S and C_L be the specific heats of ice and water, respectively. The temperature of ice increases at the rate F/C_S until it reaches $T = 0$ (melting temperature).

The temperature remains at zero for time $C_S l/F$ which is also the time taken for l units of heat to be supplied to ice. In the parabolic Stefan problem, the specific energy $e(t)$ (or specific enthalpy) at time t is given by

$$e(t) = \beta(T(t)) + l\xi(t). \tag{8.1.5}$$

Here, $\beta(T) = C_S T$ for $T \leq 0$, $\beta(T) = C_L T$ for $T > 0$, ξ is the water-fraction and $\xi \in H_g(T)$, where H_g is the Heaviside graph (cf. Eq. 4.3.45). To be consistent with the assumption made in the constitutive law (8.1.2), the response of energy to the latent heat should also be delayed and Eq. (8.1.5) in the hyperbolic Stefan problem may be written as

$$e(t) = \beta(T(t)) + l\xi(t - \tau). \tag{8.1.6}$$

Let us assume that there is no delay in the release of sensible heat so that the energy conservation law $\rho C T_t = -\nabla \vec{q}$ holds. In the classical Stefan problem if ice attains the temperature zero at time $t = 0$ and is receiving heat, then it starts storing latent heat for $t > 0$. Relation (8.1.6) suggests that storing of latent heat does not take place during the time internal $(t, t + \tau)$. Therefore ice does not melt during time $(t, t + \tau)$ and its temperature goes on rising. At time $t + \tau$, this ‘superheated ice’ which has sucked in enough energy suddenly changes to water. This is a very unstable and unrealistic situation.

The constitutive equations for the heat flux vector have been considered in much more general form than the one considered in Eq. (8.1.4). For example in [210], a general theory of heat conduction with finite wave speeds has been built, in which the heat flux vector is taken as a functional of temperature, temperature gradient and temperature summed histories. Heat flux, like the entropy, is determined by the functional for the free energy. In [212], temperature, temperature gradient and the time rate of temperature are taken as independent variables. When the constitutive equations include rate of temperature, there arises local entropy production in addition to entropy production due to conduction. Within the framework of rate-dependent theory, thermal waves can occur in the finite theory but not in the corresponding linearized theory.

8.2 MODEL I: HYPERBOLIC STEFAN PROBLEM WITH TEMPERATURE CONTINUITY AT THE INTERFACE

8.2.1 The Mathematical Formulation

Several physical and mathematical models have been proposed in the literature for the hyperbolic Stefan problem. In model I, the response of flux to the temperature gradient is delayed but the storage or release of energy as latent heat is not delayed. Temperature is assumed to be continuous and known across the phase-change boundary and heat flux is given by Eq. (8.1.3) which is taken as a ‘local constitutive law’. The energy conservation equation is derived on the basis of energy conservation law (cf. Section 1.4.7) and therefore the energy equation in a medium which is not undergoing phase-change is given by

$$\rho C \frac{\partial T}{\partial t} = -\operatorname{div} \vec{q}, \tag{8.2.1}$$

where $\vec{q}$ is given by Eq. (8.1.3). On using Eq. (8.1.3), $\vec{q}$ can be eliminated from Eq. (8.2.1) and we obtain

$$\tau\rho C T_{tt} + \rho C T_t = K\nabla^2 T. \tag{8.2.2}$$

Eq. (8.2.2) is a hyperbolic equation and is commonly known as *telegrapher's equation*. In the one-dimensional case the wave speed of this telegrapher's equation is $(k/\tau)^{1/2}$ because the characteristic curves are given by (cf. [12])

$$\frac{k}{\tau}\left(\frac{dt}{dx}\right)^2 = 1, \tag{8.2.3}$$

and the lines $x \pm (k/\tau)^{1/2}t =$ constant, are the characteristics.

A One-Dimensional Two-Phase Hyperbolic Stefan Problem

For simplicity, we first consider a one-dimensional problem in the region $\Omega = \{x : 0 \le x \le 1\}$. $x = S(t)$, $0 < t < t_*$, is the phase-change boundary. Let the region $0 < x < S(t)$ be solid, and the region $S(t) < x < 1$, be liquid. It will be assumed that in both these regions equations of the type (8.2.2) hold good. The thermal properties and relaxation parameters of the two regions could be different but densities are taken to be equal.

Differential Equations

$$\tau_S\rho C_S(T_S)_{tt} + \rho C_S(T_S)_t = K_S\nabla^2 T_S, \quad 0 < x < S(t), \quad 0 < t < t_*, \tag{8.2.4}$$

$$\tau_L\rho C_L(T_L)_{tt} + \rho C_L(T_L)_t = K_L\nabla^2 T_L, \quad S(t) < x < 1, \quad 0 < t < t_*. \tag{8.2.5}$$

Initial Conditions

$$\text{(i)}\ \ T(x,0) = T_0(x), \quad x \in \Omega. \tag{8.2.6}$$

$$\text{(ii)}\ \ \frac{\partial T}{\partial t}(x,0) = g_0(x), \quad x \in \Omega. \tag{8.2.7}$$

On what physical basis can $\partial T/\partial t$ at $t = 0$ be prescribed? There is no clear answer to this. If $q(x,0)$ is continuously differentiable in the whole of the region Ω, then Eq. (8.2.7) can be replaced by

$$\text{(iii)}\ \ q(x,0) = q_0(x), \quad x \in \Omega. \tag{8.2.8}$$

Boundary Conditions at the Fixed Boundaries

If the temperature is prescribed on the fixed boundary, then the same boundary condition continues in the same form in the present model but if the boundary condition is of the type

Eq. (1.4.38), i.e. flux is prescribed then we have the following boundary condition on the boundary $\partial\Omega$ of Ω

$$q(x,t) = \beta E(T^4 - T_0^4) + (C_1 T - C_2 T_0), \quad \text{on } \partial\Omega. \tag{8.2.9}$$

Boundary Conditions at the Free Boundary

(i) Isotherm condition: $T = T_m$ (constant) on $x = S(t)$. Isotherm condition also implies continuity of temperature. We shall see later that in some formulations temperature could be discontinuous across the free boundary. This is not surprising because hyperbolic equation admits discontinuous solutions.

(ii) Energy balance condition: If the phase-change is taking place from liquid to solid, then the energy balance at $x = S(t)$ will have the form

$$\rho\hat{l}\frac{dS}{dt} = q_L(S(t),t) - q_S(S(t),t), \quad \hat{l} = l + (C_L - C_S)T_m. \tag{8.2.10}$$

Eq. (8.2.10) is not suitable for obtaining analytical and numerical solutions as it is in terms of fluxes. So it will be expressed in a different form. On differentiating Eq. (8.2.10) with respect to time and using Eqs (8.2.1) and (8.1.3), we get

$$\rho\hat{l}\frac{d^2S}{dt^2} = (q_L)_t + (q_L)_x\,\frac{dS}{dt} - (q_S)_t - (q_S)_x\,\frac{dS}{dt}$$

or

$$\rho\hat{l}\frac{d^2S}{dt^2} = \rho\left(C_S\,\frac{\partial T_S}{\partial t}\bigg|_{S(t)} - C_L\,\frac{\partial T_L}{\partial t}\bigg|_{S(t)}\right)\frac{dS}{dt} + \left(-K_L\frac{\partial T_L}{\partial x} - q_L\right)\bigg/\,\tau_L\bigg|_{S(t)}$$
$$+\left(K_S\frac{\partial T_S}{\partial x} + q_S\right)\bigg/\,\tau_S\bigg|_{S(t)}. \tag{8.2.11}$$

If isotherm conditions are differentiated with respect to time, we obtain

$$(T_L)_t = -\,(T_L)_x\,\frac{dS}{dt}, \quad \text{and} \quad (T_S)_t = -\,(T_S)_x\,\frac{dS}{dt}. \tag{8.2.12}$$

On substituting Eq. (8.2.12) in Eq. (8.2.11), we obtain the following equation:

$$\rho\hat{l}\frac{d^2S}{dt^2} = \rho\left(C_L\frac{\partial T_L}{\partial x} - C_S\frac{\partial T_S}{\partial x}\right)\left(\frac{dS}{dt}\right)^2 + (K_S/\tau_S)\frac{\partial T_S}{\partial x} - (K_L/\tau_L)\frac{\partial T_L}{\partial x}$$
$$+\,q_S/\tau_S - q_L/\tau_L, \ \text{on } x = S(t). \tag{8.2.13}$$

If $\tau_S = \tau_L$, then the last term in Eq. (8.2.13) can be written with the help of Eq. (8.2.10).

For a one-phase problem, condition (8.2.13) can be easily obtained, for example, take $T_L \equiv 0$, and $\tau_S = \tau_L = \tau$. In the above hyperbolic Stefan problem, there are three velocities, one that of the free boundary and two of the wave fronts in the two phases. Can the velocity of the free boundary exceed that of the wave front? This question has been studied in [213] with the help of an analytical solution of a one-phase problem. Consider a semiinfinite solid

$0 \leq x < \infty$ which is initially in a solid state at the melting temperature T_m. This solid is heated at $x = 0$ so that we have a one-phase melting problem. It can be easily seen that the pair $(S(t), T_L)$, where

$$S(t) = bt, \quad b > 0 \text{ and } b^2 \neq k/\tau \tag{8.2.14}$$

and

$$T_L(x,t) = T_m + (l/C_L)\left[\exp\left\{b(x-bt)/(\tau b^2 - k)\right\} - 1\right] \tag{8.2.15}$$

is a solution of the hyperbolic Stefan problem (8.2.16)–(8.2.18)

$$\tau\rho C_L(T_L)_{tt} + \rho C_L(T_L)_t = K_L\nabla^2 T_L, \quad 0 < x < S(t), \quad 0 < t < t_*, \tag{8.2.16}$$

$$T_L = T_m, \text{ on } x = S(t), \tag{8.2.17}$$

$$\frac{d^2S}{dt^2} + (1/\tau)\frac{dS}{dt} = (C_L/l)(T_L)_x\left[\left(\frac{dS}{dt}\right)^2 - k/\tau\right], \text{ on } x = S(t). \tag{8.2.18}$$

The constant b in Eq. (8.2.14) can be determined, for example, from the temperature prescribed condition at $x = 0$, i.e.

$$T_L(0,t) = T_m + (l/C_L)\left[\exp(b^2 t/(k - \tau b^2)) - 1\right]. \tag{8.2.19}$$

We shall now examine three cases: (i) $b^2 > k/\tau$, (ii) $b^2 < k/\tau$ and (iii) $b^2 = k/\tau$.

Case 1. $b^2 < k/\tau$. It can be seen from Eq. (8.2.15) that $T_L > T_m$ for $0 < x < S(t)$ and this is a physically realistic case.

Case 2. $b^2 > k/\tau$. From Eq. (8.2.15), $T_L(x,t) < T_m$ for $0 < x < S(t)$ and we have a supercooled water progressing on ice. If $(T_L)_x$ is calculated from Eq. (8.2.15) and substituted in Eq. (8.1.3) and the differential equation so obtained is integrated, then we obtain

$$q(x,t) = \rho l b \, \exp\left[b(x-bt)/(\tau b^2 - k)\right], \quad 0 \leq x \leq S(t). \tag{8.2.20}$$

From Eq. (8.2.20), flux is positive and so the heat is flowing from the supercooled water to ice which is not possible as it violates laws of thermodynamics. Therefore the solution in which $b^2 > k/\tau$ occurs is not physically realistic.

Case 3. $b^2 = k/\tau$. In this case, Eq. (8.2.18) cannot be used at the free boundary but Eq. (8.2.10) can be used. We shall try to obtain $q_L(S(t), t)$ occurring in Eq. (8.2.10). From the first equation in Eq. (8.2.12), $(T_L)_x$ can be written in terms of $(T_L)_t$ and from Eq. (8.2.1), $(T_L)_t$ can be expressed in terms of $(q_L)_x$. When this $(q_L)_x$ is substituted in Eq. (8.1.3), we get the following equation:

$$(q_L)_t + b\,(q_L)_x = -q_L/\tau, \text{ on } x = S(t) \tag{8.2.21}$$

or

$$\frac{d}{dt}(q_L) + (1/\tau)q_L = 0, \text{ on } x = S(t). \tag{8.2.22}$$

The solution of Eq. (8.2.22) is given by

$$q_L = Ae^{-t/\tau}, \quad A \text{ is arbitrary constant.} \tag{8.2.23}$$

q_L in Eq. (8.2.23) does not satisfy the interface condition $\rho l dS/dt = q_L(t)$.

The second law of thermodynamics requires that in any process there should be a positive entropy production. In the case of classical Stefan problem this implies that the condition

$$-T_x(x,t)q(x,t) \geq 0, \tag{8.2.24}$$

must hold at each point of solid and liquid regions. In the classical Stefan problem, condition (8.2.24) gets satisfied as the flux is defined through the Fourier's law. In the hyperbolic Stefan problem, condition (8.2.24) may not always hold, and an additional condition may have to be imposed to have a physically acceptable solution. If $b^2 < \tau/k$, then condition (8.2.24) is satisfied but not if $b^2 > \tau/k$.

Model I has been used in [214] to obtain a short-time analytical solution of the problem of melting of a semiinfinite solid $x \geq 0$ (one-dimensional problem). $\partial T_L/\partial x$ is prescribed at $x = 0$ in terms of a 'Dirac delta function'. An analytical solution has been obtained using a suitable Green's function and after making some assumptions about the form of the free boundary, a short-time solution has been obtained.

In a multidimensional hyperbolic Stefan problem, when the phase-change is taking place from liquid to solid, the energy balance condition at the interface is given by the equation

$$[H]\,\vec{V}\cdot\vec{n}_x = (\vec{q}_S - \vec{q}_L)\cdot\vec{n}_x. \tag{8.2.25}$$

Here, $\vec{V}$ is the velocity of the interface and $\vec{n} = (n_t, \vec{n}_x)$ is the unit normal to the interface which points into the liquid (see Eq. 1.4.17) and $[H]$ is the jump in the enthalpy.

8.2.2 Some Existence, Uniqueness and Well-Posedness Results

A One-Dimensional One-Phase Hyperbolic Stefan Problem

Consider the following problems:

$$\tau q_t + KT_x + q = 0, \text{ and } C\rho T_t + q_x = 0; \quad x_0 < x < S(t), \quad S(0) = 0, \quad t > 0, \tag{8.2.26}$$

$$T(x,t) = 0, \text{ and } \rho l dS/dt = q(x,t); \quad \text{on } x = S(t), \quad t > 0, \tag{8.2.27}$$

$$T(x,0) = T_0(x), \text{ and } q(x,0) = q_0(x). \tag{8.2.28}$$

The boundary condition could be any of the following types:

$$\text{(i)}\ \ T(x_0,t) = \hat{T}(t), \quad t > 0. \tag{8.2.29}$$

$$\text{(ii)}\ \ q(x_0,t) = \hat{q}(t), \quad t > 0. \tag{8.2.30}$$

$$\text{(iii)}\ \ q(x_0,t) = h\left[\hat{T}(t) - T(x_0,t)\right], \quad t > 0. \tag{8.2.31}$$

By considering the solution of a specific problem, it was shown in Section 8.2.1 that the velocity of the free boundary cannot exceed the wave front velocity. It has been argued in [215] that if $|\dot{S}(0)| > (k/\tau)^{1/2}$ then $x = S(t)$ would be a space-like curve for small t. Therefore to have a well-posed problem for (q, T), besides condition (8.2.27) on $x = S(t)$, one should impose two more conditions on $x = S(t)$ if $\dot{S}(0) > (k/\tau)^{1/2}$ and no condition when $\dot{S}(0) < (k/\tau)^{1/2}$. Otherwise one would lose either uniqueness or existence of the solution.

Depending on the value of x_0, two cases arise.

Case I: $x_0 < 0$. The following result has been proved in [215] which is valid for a short time.

Proposition 8.2.1. *Let T_0, $q_0 \in C^1(-\infty, 0)$. Suppose that at $(x, t) = (0, 0)$, the compatibility condition*

$$C\rho \frac{\partial}{\partial x} T_0(0) q_0(0) = \rho l \frac{\partial}{\partial x} q_0(0) \tag{8.2.32}$$

is satisfied and also

$$|q_0(0)| < \rho l (k/\tau)^{1/2}. \tag{8.2.33}$$

Then there exists $t_0 > 0$ such that in $[0, t_0]$, the problem (8.2.26)–(8.2.28) with any of the boundary conditions in Eqs (8.2.29)–(8.2.31) has a unique solution $(T, q, S) \in C^1 \times C^1 \times C^2$. If T_0, $q_0 \in C^\infty$ and C^∞ compatibility conditions are satisfied, then the solution $\in C^\infty$ $[0, t_0]$.

The compatibility condition (8.2.32) can be obtained if the time derivative of the isotherm condition is obtained and the derivatives occurring in it are calculated in terms of the known quantities at $x = 0$. Condition (8.2.33) implies that $\dot{S}(t)$ is less than the wave velocity at least initially.

Case II: $x_0 = 0$. In this case no initial condition is required. If $\hat{q}(t)$ and $\hat{T}(t) \in C^1$ $[0, \infty]$ and $q(x_0, t)$ is determined from Eq. (8.2.30) or (8.2.31) and satisfies the condition

$$0 < q(0, 0) < \rho l (k/\tau)^{1/2}, \tag{8.2.34}$$

then in $[0, t_0]$ a unique solution $(T, q, S) \in C^1 \times C^1 \times C^2$ exists when the boundary condition is either Eq. (8.2.30) or (8.2.31). In this case the problem consisting of Eqs (8.2.26)–(8.2.28), (8.2.29) is not well-posed.

Global Solution for the One-Phase Problem

If the following substitutions are made in Eqs (8.2.26)–(8.2.28), then the resulting equations become dimensionless and the thermophysical parameters do not appear in the changed equations. Let,

$$\left.\begin{aligned} t = \tau \bar{t}, \quad x = (K\tau/C\rho)^{1/2} \bar{x}, \quad \bar{T} = (C/l) T, \quad \bar{q} = (C\tau/l^2 K \rho)^{1/2} q, \\ \bar{S} = (C\rho/K\tau)^{1/2} S. \end{aligned}\right\} \tag{8.2.35}$$

For further discussion, bar over the changed variables will be dropped. For the analysis of hyperbolic Stefan problems it is more convenient to formulate the problems in terms of *Riemann invariants* A and B which are defined as

$$A = T + q, \quad \text{and} \quad B = T - q. \tag{8.2.36}$$

In terms of Riemann invariants the system of Eqs (8.2.26)–(8.2.28), (8.2.29) can be written as

$$\left.\begin{aligned} A_t + A_x + \tfrac{1}{2}(A - B) = 0 \\ B_t - B_x + \tfrac{1}{2}(B - A) = 0 \end{aligned}\right\}, \quad x_0 < x < S(t), \tag{8.2.37}$$

$$\left.\begin{aligned} (A + B)(x, t) = 0 \\ \dot{S}(t) = A(x, t) \end{aligned}\right\} \text{on } x = S(t), \tag{8.2.38}$$

$$(A + B)(x_0, t) = 2\hat{T}(t), \quad t > 0. \tag{8.2.39}$$

Let

$$A(x, 0) = A_0(x), \quad B(x, 0) = B_0(x), \quad S(0) = 0. \tag{8.2.40}$$

Proposition 8.2.2. *Assume the following: (i)* A_0, $B_0 \in C^\infty[x_0, 0]$, $x_0 < 0$, *and* $\hat{T}(t) \in C^\infty[0, \infty)$, *(ii)* $\hat{T}'(t) \geq 0$, $A_0' < 0$, $B_0' < 0$, $0 < A(0, 0) < 1$ *and (iii)* C^∞*-compatibility conditions are satisfied at* $(0, 0)$ *and at* $(x_0, 0)$. *Under these assumptions, the system* (8.2.36)–(8.2.40) *has a unique solution* $(A, B, S) \in C^\infty \times C^\infty \times C^\infty$ *for all* $t > 0$ *and the solution satisfies*

$$A(S, t) > 0, \quad A_x < 0, \quad B_x < 0, \quad 0 < \dot{S}(t) < 1. \tag{8.2.41}$$

From Proposition 8.2.1, a unique solution exists in some interval $[0, t_0]$ and if the condition $0 < \dot{S}(t_0) < 1$ is again satisfied then the time interval in which the solution exists can be extended. It has been proved in [215] that $0 < \dot{S}(t) < 1 - \delta$, $\delta = \delta(t_0) > 0$ for any fixed t_0 and so the Proposition 8.2.2 holds. For the proof of Propositions 8.2.1 and 8.2.2, the technique of integration along characteristics and linear iteration has been adopted.

A Two-Phase Problem

The following one-dimensional two-phase formulation has been considered in [215] in the regions $-\infty < x < \infty$ and $-x_0 \leq x \leq x_0$

$$\left.\begin{aligned} \tau_1(q_1)_t + K_1(T_1)_x + q_1 = 0 \\ C_1\rho(T_1)_t + (q_1)_x = 0 \end{aligned}\right\}, \quad x < S(t), \tag{8.2.42}$$

$$\left.\begin{aligned} \tau_2\,(q_2)_t + K_2\,(T_2)_x + q_2 = 0 \\ C_2\rho\,(T_2)_t + (q_2)_x = 0 \end{aligned}\right\}, \quad x > S(t), \tag{8.2.43}$$

$$\left.\begin{aligned} T_1(x, t) = T_2(x, t) = 0 \\ \rho l\dot{S}(t) = (q_1 - q_2)(x, t) \end{aligned}\right\}, \quad x = S(t), \quad S(0) = 0, \tag{8.2.44}$$

$$\left.\begin{aligned} T_1(x, 0) = T_{10}(x), \quad q_1(x, 0) = q_{10}(x), \quad x < 0, \\ T_2(x, 0) = T_{20}(x), \quad q_2(x, 0) = q_{20}(x), \quad x > 0. \end{aligned}\right\} \tag{8.2.45}$$

For the boundary conditions at the fixed boundaries, see Eq. (8.2.47). For a local-in-time solution of the above problem, the following proposition holds.

Proposition 8.2.3. *If T_{10}, q_{10}, T_{20} and $q_{20} \in C^1$, the compatibility conditions are satisfied at $(0,0)$ and*

$$|q_{20}(0) - q_{10}(0)| < \min\left\{\rho l(k_1/\tau_1)^{1/2}, \quad \rho l(k_2/\tau_2)^{1/2}\right\}, \tag{8.2.46}$$

then a unique solution $(T_1, T_2, q_1, q_2) \in C^1$, $S \in C^2$, of the problem (8.2.42)–(8.2.45) exists in some interval $[0, t_0]$, $t_0 > 0$.

The compatibility conditions at $(0,0)$ can be derived by calculating the total time derivatives of the isotherm conditions and replacing the partial derivatives so obtained by the known quantities. Condition (8.2.46) implies that the velocity of the phase-change interface should be less than the minimum of the two wave velocities in the two phases, at least initially.

For the global-in-time solution, the case $\tau_1 = \tau_2$ and $k_1 = k_2$ was considered in [215]. The region $[-x_0, x_0]$, $x_0 > 0$ has been considered. To complete the formulation in addition to Eqs (8.2.42)–(8.2.45), at the fixed boundaries, temperatures are prescribed as

$$T_1(-x_0, t) = \hat{T}_1(t), \quad t > 0; \quad T_2(x_0, t) = \hat{T}_2(t), \quad t > 0. \tag{8.2.47}$$

Proposition 8.2.4. *If A_{J0}, B_{J0}, $\hat{T}_J$, $j = 1, 2$, are all C^∞-functions, C^∞-compatibility conditions are satisfied at $(0,0)$ and at $(\pm x_0, 0)$, and*

$$(-1)^{J+1}\,\hat{T}_J > 0, \quad (-1)^{J+1}\,\hat{T}'_J \geq 0, \quad J = 1, 2, \tag{8.2.48}$$

$$A'_{J_0} < 0, \quad B'_{J_0} < 0, \quad J = 1, 2; \text{ and } |S(0)| < (k/\tau)^{1/2}, \tag{8.2.49}$$

then the two-phase problem considered in $[-x_0, x_0]$ has a unique C^∞- solution in $(0, \infty)$. A_{J_0}, B_{J_0} are the values of A_J and B_J at $t = 0$. The Riemann invariants A_J and B_J are defined as

$$q_J = \sqrt{(k_J/\tau_J)}(A_J - B_J), \quad J = 1, 2, \tag{8.2.50}$$

$$T_J = \sqrt{1/(C_J\rho)}(A_J + B_J), \quad J = 1, 2. \tag{8.2.51}$$

The existence and uniqueness of the solution of a one-phase problem similar to the one described in Section 8.2.1 in the region $0 \leq x \leq S(t)$, have been discussed in [216]. Using Riemann functions and hyperbolic-equation theory (cf. [216]) the solution of the problem can be expressed in terms of an integral and it can be shown by taking appropriate limits that if at $x = 0$ any temperature other than the melting temperature is prescribed then the problem is ill-posed. Flux prescribed boundary condition has been considered. The main result of [216] is as follows.

Proposition 8.2.5. *If $q = Lip(L_q, R_0)$, $R_0 > 0$ and $r_0 \in (0,1)$, then the one-phase hyperbolic problem with the initial condition $f(0) = 0$ and $f'(0) = r_0$ has a unique solution on some interval of time $[0, R_1]$, $R_1 > 0$. The solution can be continued to a maximum interval $[0, R_0]$ such that either $R_0 \to +\infty$ or $\min(f'(r), 1 - f'(r)) = 0$ as $r \to R_0 - 0$.*

In the above proposition $Lip(L_q, R_0)$ is the class of functions satisfying Lipschitz condition with constant L_q on $[0, R_0]$, $f(t/\tau) = S(t)/(k\tau)^{1/2}$ and r is defined by the equation

$r - f(r) = t/\tau$, r_0 is some value of r, and q is the prescribed flux at $x = 0$. In addition to other conditions, conditions $f(0) = 0$ and $f'(0) = r_0$ should also be satisfied. The correct choice of r_0 is dictated by extra physical laws [216].

8.3 MODEL II: FORMULATION WITH TEMPERATURE DISCONTINUITY AT THE INTERFACE

8.3.1 The Mathematical Formulation

In this model also the response of flux to the temperature gradient is delayed and the response of energy to the latent heat is not delayed but unlike model I, temperature could be discontinuous across the phase-change interface [217]. Consider a one-dimensional problem in which a solid material at time $\hat{t} = 0$ occupies the region $\hat{x} \geq 0$. The melting temperature is taken as $\hat{T} = 0$.

The constitutive or governing equations are

$$\rho \frac{\partial \hat{e}}{\partial \hat{t}} + \frac{\partial \hat{q}}{\partial \hat{x}} = 0; \quad \text{and} \quad \tau \frac{\partial \hat{q}}{\partial \hat{t}} + \hat{q} + K \frac{\partial \hat{T}}{\partial \hat{x}} = 0. \tag{8.3.1}$$

The boundary condition is taken as

$$\hat{T}\left(0, \hat{t}\right) = \hat{T}_0 > 0, \quad \hat{t} > 0, \tag{8.3.2}$$

and the initial conditions are taken as

$$\hat{T}(\hat{x}, 0) = \hat{q}(\hat{x}, 0) = 0, \quad \hat{x} > 0. \tag{8.3.3}$$

The first equation in Eq. (8.3.1) holds in the weak sense as $\hat{e}$ is discontinuous across the phase-change boundary and the second equation in Eq. (8.3.1) is now a constitutive equation and not a localized heat flow law which it was in the model I. $\hat{e} = \hat{e}(\hat{T})$ is the specific energy (enthalpy) and is defined by Eqs (2.1.22)–(2.1.24). Eqs (8.3.1)–(8.3.3) can be normalized by making the following substitutions:

$$\hat{t} = \tau t, \quad \hat{x} = (C_V K \tau)^{1/2}/(l \rho^{1/2}) x, \quad \hat{e} = l e, \tag{8.3.4}$$

$$\hat{q} = (C_V K \rho/\tau)^{1/2} q, \quad \hat{T} = \frac{C_V}{l} T. \tag{8.3.5}$$

On using Eqs (8.3.4), (8.3.5), the two equations in Eq. (8.3.1) are transformed into

$$\frac{\partial e}{\partial t} + \frac{\partial q}{\partial x} = 0, \quad \text{and} \quad \frac{\partial q}{\partial t} + q + \frac{\partial T}{\partial x} = 0. \tag{8.3.6}$$

We have

$$e = 1 + T, \text{ if } T > 0, \; e = T, \text{ if } T < 0, \quad \text{and} \quad e \in (0, 1), \text{ if } T = 0. \tag{8.3.7}$$

The initial and boundary conditions become

$$T(x, 0) = q(x, 0) = 0, \; x > 0; \quad \text{and} \quad T(0, t) = \hat{T}_0 l/C_V = T_0. \tag{8.3.8}$$

The hyperbolic system of equations in Eq. (8.3.6) are known to have discontinuous solutions (cf. Problem 1.1.6 and [12]) across shocks. If the shock is denoted by $x = S(t)$, then across the shock, the following Rankine–Hugoniot conditions are satisfied:

$$\dot{S}(e_- - e_+) = q_- - q_+; \quad \text{and} \quad \dot{S}(q_- - q_+) = T_- - T_+. \tag{8.3.9}$$

For notations used in Eq. (8.3.9), see Eqs (8.3.10), (8.3.11). The first condition in Eq. (8.3.9) is the familiar Stefan condition which arises from the conservation of energy at $x = S(t)$. If $T_- = T_+$, then we do not have the second condition at $x = S(t)$ and we have to necessarily impose the isotherm condition $T_- = T_+ = T = 0$ on $x = S(t)$:

$$(e_-, T_-, q_-) = \lim_{x \uparrow S(t)} (e, T, q)(x, t) \tag{8.3.10}$$

and

$$(e_+, T_+, q_+) = \lim_{x \downarrow S(t)} (e, T, q)(x, t). \tag{8.3.11}$$

In a pure solid, $e \le 0$ and in a pure liquid $e \ge 1$. It can be seen from Eq. (8.3.9) that the shocks propagate with speeds $\dot{S} = \pm 1$. In the mushy region $0 < e < 1$, $\dot{S} = \pm 0$, $T_- = T_+ = 0$ and $q_- = q_+$. When phase changes from solid to liquid, i.e. when $e_+ \le 0$ and $e_- \ge 1$ or from liquid to solid ($e_+ \ge 1$ and $e_- \le 0$), to pick up appropriate shocks, the following admissibility conditions should be satisfied [218].

Admissibility Conditions

A forward shock, $\dot{S} > 0$, with end states $e_- \ne e_+$ is admissible if and only if

$$T(e) - T(e_+) - \frac{T(e_-) - T(e_+)}{e_- - e_+}(e - e_+) \le 0, \quad e_+ < e < e_-, \tag{8.3.12}$$

$$T(e) - T(e_-) - \frac{T(e_-) - T(e_+)}{e_- - e_+}(e - e_-) \ge 0, \quad e_- < e < e_+. \tag{8.3.13}$$

A backward shock, $\dot{S} < 0$, with end states $e_- \ne e_+$ is admissible if and only if

$$T(e) - T(e_+) - \frac{T(e_-) - T(e_+)}{e_- - e_+}(e - e_+) \ge 0, \quad e_+ < e < e_-, \tag{8.3.14}$$

$$T(e) - T(e_-) - \frac{T(e_-) - T(e_+)}{e_- - e_+}(e - e_-) \le 0, \quad e_- < e < e_+. \tag{8.3.15}$$

The relevance of Eqs (8.3.12)–(8.3.15) in the present problem is that when $e_- > 1$ and $\dot{S} > 0$, then necessarily, $e_+ = T_+ = 0$ and we have the following one-phase melting problem:

$$T_t + q_x = 0, \text{ and } q_t + q + T_x = 0; \quad 0 < x < S(t), \quad t > 0, \tag{8.3.16}$$

$$q(S(t), t) = [T(1 + T)]^{1/2}(S(t), t), \text{ and } \frac{dS}{dt} = \left(\frac{T}{1 + T}\right)^{1/2}(S(t), t), \tag{8.3.17}$$

$$T(0, t) = T_0 > 0, \quad t > 0, \tag{8.3.18}$$

$$(e, T, q)(x, t) = (0, 0, 0), \quad x > S(t), \quad t > 0, \tag{8.3.19}$$

$$e(x, t) = 1 + T(x, t), \quad 0 < x < S(t), \quad t > 0. \tag{8.3.20}$$

At this point the necessity of considering the present new model can be questioned. Model I seems to be a good model as it is based on the conservation law of energy and incorporates generalized Fourier's law. As $\tau \to 0$ in the telegrapher's equation and in the Stefan condition (8.2.13) ($\tau_S = \tau_L = \tau$), we get back the classical Stefan problem formulation and for $\tau \to 0$ even the solution in Eqs (8.2.14)–(8.2.19) is the solution of the corresponding Stefan problem. The necessity of considering different models (we shall be considering some more models) for hyperbolic Stefan problems arose from the fact that we should be able to prove the existence and uniqueness of the solution and the well-posedness of the problem under general initial and boundary data and as $\tau \to 0$, the solution of the hyperbolic Stefan problem should converge to the solution of the classical Stefan problem. The solution given in Eqs (8.2.14)–(8.2.19) is the solution of a very special type of problem. There does not exist a general result for the hyperbolic phase-change problems which may tell us that as $\tau \to 0$, the solution of a problem formulated with the help of the model I, or for that matter any other model, will tend to the solution of the classical Stefan problem. Different authors have considered different mathematical models, showed the existence and uniqueness of solutions and tried to establish the convergence as $\tau \to 0$ by considering some specific problems. For model I, some results on the existence, uniqueness and regularity of the solutions of some hyperbolic Stefan problems have been discussed in Section 8.2.1 but no result has been reported in [213–216] on the convergence of the solution as $\tau \to 0$.

8.3.2 The Existence and Uniqueness of the Solution and Its Convergence as $\tau \to 0$

The existence of the weak solution of the problem (8.3.16)–(8.3.20) has been discussed in [217] and for this the functions T, q and e considered in $0 \le x \le S(t)$ and satisfying Eqs (8.3.16)–(8.3.20) are expressed in terms of Riemann invariants (cf. Eq. 8.2.36). $(T, q, e) = (0, 0, 0)$ for $x > S(t)$. Temperature is prescribed at $x = 0$. A family of problems parameterized by a time lag τ are considered and the desired solution is obtained as $\tau \to 0$. It has been shown that T, q and e defined by Riemann invariants satisfy the boundary conditions and for every simple closed curve C in $\{0 \le x \le S(t),\ t \ge 0\}$ surrounding a domain $R(C)$, we have

$$\int_C \{e(x, t)dx - q(x, t)dt\} = 0, \tag{8.3.21}$$

$$\int_C \{q(x, t)dx - T(x, t)dt\} + \int\int_{R(C)} q(x, t)dxdt = 0. \tag{8.3.22}$$

The integration along C is taken in the clockwise sense. The Rankine–Hugoniot conditions are satisfied in the sense that for every $0 \le t_1 < t_2 < \infty$

$$\int_{t_1}^{t_2} \left\{e(S(t), t)\dot{S}(t) - q(S(t), t)\right\} dt = 0 \tag{8.3.23}$$

and

$$\int_{t_1}^{t_2} \left\{q(S(t),t)\dot{S}(t) - T(S(t),t)\right\} dt = 0. \tag{8.3.24}$$

The short- and long-term asymptotics of the solution of Eqs (8.3.16)–(8.3.20) were also obtained.

The problem considered in [219] is also one-phase with temperature being discontinuous at the interface. The initial and boundary conditions considered are functions of space and time, respectively. In the problem formulation, to start with, energy $e(x,t)$ is taken as

$$e = e_0(T) + Z(T)q^2, \tag{8.3.25}$$

where $e_0(T)$ is the classical internal energy as considered in Eq. (8.3.7) and $Z(T)$ is defined as

$$Z(T) = -\frac{T^2}{2}\frac{d}{dT}\left(\frac{W(T)}{T^2}\right), \quad W(T) = \frac{\tau(T)}{K(T)}. \tag{8.3.26}$$

The function $Z(T)$ is a consequence of the second law of thermodynamics combined with the generalized Fourier's law [220]. The coefficients $\tau(T)$ and $K(T)$ depend both on the temperature and the phase-change material under consideration and they can have jump discontinuities at the phase-change temperature. If it is assumed that $\tau(T)/K(T)$ is differentiable at $T = 0$ (zero is the phase-change temperature) and e given in Eq. (8.3.25) is substituted in the first equation of Eq. (8.3.6), then we have

$$\frac{\partial e_0}{\partial t} + q_x = -\frac{\partial}{\partial t}\left[\left\{\frac{\tau(T)}{TK(T)} - \frac{1}{2}\frac{d}{dT}\left(\frac{\tau(T)}{K(T)}\right)\right\} q^2\right]. \tag{8.3.27}$$

The r.h.s. of Eq. (8.3.27) is too complicated and it is difficult to carry out further calculations on retaining it in the present form. If it is assumed that $\tau(T)/K(T) =$ constant for all T, $\tau(T) =$ constant and a small number, and $1/T$ is small, then the r.h.s. in Eq. (8.3.27) can be taken to be 0. Now we have the same formulation as in Eqs (8.3.16), (8.3.17). However, the initial and boundary conditions in Eqs (8.3.28), (8.3.29) considered in [219] are more general.

Dirichlet Problem

$$\left.\begin{aligned} &T(x,0) = \phi(x) > 0, \text{ and } q(x,0) = \psi(x); \quad 0 \le x < S(0), \\ &T(0,t) = f(t) > 0, \quad t \ge 0. \end{aligned}\right\} \tag{8.3.28}$$

Another type of boundary condition at $x = 0$ could be the 'Neumann boundary condition'.

Neumann Problem

$$\left.\begin{aligned} &T(x,0) = \phi(x) > 0, \text{ and } q(x,0) = \psi(x); \quad 0 \le x < S(0), \\ &q(0,t) = f(t), \quad t \ge 0. \end{aligned}\right\} \tag{8.3.29}$$

As reported earlier, it is convenient to work with the Riemann invariants A and B which in [219] are defined as

$$A = T + \sqrt{\tau}q; \quad \text{and} \quad B = T - \sqrt{\tau}q. \tag{8.3.30}$$

Substituting Eq. (8.3.30) in Eqs (8.3.16), (8.3.17), we get

$$\sqrt{\tau}A_t + A_x + (A - B)/(2\sqrt{\tau}) = 0, \qquad 0 < x < S(t), \quad t > 0, \tag{8.3.31}$$

$$\sqrt{\tau}B_t - B_x + (B - A)/(2\sqrt{\tau}) = 0, \qquad 0 < x < S(t), \quad t > 0, \tag{8.3.32}$$

$$B(S(t), t) = -\frac{A}{1 + 2A}(S(t), t), \quad t > 0, \tag{8.3.33}$$

$$\frac{dS}{dt} = \frac{A}{\sqrt{\tau}(1 + A)}(S(t), t), \quad t > 0. \tag{8.3.34}$$

For the Dirichlet problem Eq. (8.3.28), we have

$$A(x, 0) = A_0(x), \quad B(x, 0) = B_0(x), \quad 0 \le x < S(0), \tag{8.3.35}$$

$$A(0, t) + B(0, t) = 2f(t), \quad t > 0. \tag{8.3.36}$$

For the Neumann problem (8.3.29), we have

$$A(0, t) + B(0, t) + 2\sqrt{\tau}f(t), \quad t > 0, \tag{8.3.37}$$

and Eq. (8.3.35).

To discuss the existence, uniqueness and regularity results, the following assumptions have been made [219]:

$$f \in C^2[0, \infty], \quad \phi \in C^2[0, S(0)], \quad \psi \in C^2[0, S(0)], \tag{8.3.38}$$

$$\phi(x) > -\sqrt{\tau}\psi(x), \quad \phi(x) > \sqrt{\tau}\psi(x) - 1/4; \text{ if } 0 \le x < S(0). \tag{8.3.39}$$

If there exists a C^1 solution of Eqs (8.3.31)–(8.3.35), (8.3.37), then the following compatibility conditions hold good:

$$A_0(0) = B_0(0) + 2\sqrt{\tau}f(0), \tag{8.3.40}$$

$$A_{0,x}(0) + B_{0,x}(0) = -2(\tau f'(0) + f(0)) \tag{8.3.41}$$

and

$$S(0) = S_0, \quad S'(0) = S_1, \quad S''(0) = S_2, \tag{8.3.42}$$

where $S_0 > 0$ is given and

$$S_1 = \frac{1}{\sqrt{\tau}} \frac{A_0(S_0)}{1 + A_0(S_0)}, \tag{8.3.43}$$

$$S_2 = \frac{1}{2\tau^{3/2}} \frac{\left\{2\sqrt{\tau}(\sqrt{\tau}S_1 - 1)A_{0,x}(S_0) - (A_0(S_0) - B_0(S_0))\right\}}{(1 + A_0(S_0))^2}, \tag{8.3.44}$$

$$B_0(S_0) = -\frac{A_0(S_0)}{1+2A_0(S_0)}, \tag{8.3.45}$$

$$B_{0,x}(S_0) = \frac{\sqrt{\tau}A_{0,x}(S_0)(1+2A_0(S_0)) - 4A_0^2(S_0)(1+A_0(S_0))^3}{\sqrt{\tau}\,(1+2A_0(S_0))^4}. \tag{8.3.46}$$

Some of the results obtained in [219] are given below.

Proposition 8.3.1. *If Eqs (8.3.38)–(8.3.41), (8.3.45), (8.3.46) hold, then there exists a solution (A, B, S) of Eqs (8.3.31)–(8.3.35), (8.3.37) with $S \in C^{2,1}$ and A, B in $C^{1,1}$ up to the boundary.*

Proposition 8.3.2. *For any $t_1 > 0$ there exists at most one C^1 solution (A, B, S) of Eqs (8.3.31)–(8.3.35), (8.3.37) for $0 \le t \le t_1$.*

The above two results are for the Neumann problem. For the Dirichlet problem, the existence theorem requires that the data satisfy the assumptions

$$\left.\begin{array}{c} f'(t) \ge 0, \quad 0 \le t < \infty, \\ -\phi'(x) > \sqrt{\tau}\left|\psi'(x)\right|, \quad 0 \le x \le S_0, \quad \phi(S_0) + \sqrt{\tau}\psi(S_0) > 0. \end{array}\right\} \tag{8.3.47}$$

Proposition 8.3.3. *If Eqs (8.3.38)–(8.3.41), (8.3.45)–(8.3.47) hold and if one can establish an a priori estimate $A(S(t), t) \ge -1/2 + \varepsilon_\tau$ for some $\varepsilon_\tau > 0$, $0 \le t \le t_1 < \infty$, ε_τ depending on τ, and t_1, then there exists a solution (A, B, S) of the Dirichlet problem with $S \in C^{2,1}$ and A, $B \in C^{1,1}$ up to the boundary.*

A uniqueness result similar to that given in Proposition 8.3.2 holds good for the Dirichlet problem also.

Asymptotic behaviour of the one-phase problem as $\tau \to 0$ was also investigated in [219] and Proposition 8.3.4 given below was proved under some assumptions.

Proposition 8.3.4. *Consider the Dirichlet problem stated in Eq. (8.3.28). The solution T_τ (T depending on τ) of the Dirichlet problem with $T_\tau = 0$ in $x > S_\tau(t)$, tends to u as $\tau \to 0$ weakly in $(L^\infty\{0 \le x \le M,\ 0 \le t \le t_1\})^*$ for any $M > 0$, $t_1 > 0$, where u is the solution of the one-phase Stefan problem stated in Eqs (8.3.48)–(8.3.50)*

$$u_t - u_{xx} = 0, \quad 0 < x < S(t), \quad t > 0, \tag{8.3.48}$$

$$u(0,t) = f(t), \quad t > 0; \quad u(x,0) = \phi(x), \quad 0 < x < S(0), \tag{8.3.49}$$

$$u = 0, \quad u_x = -\frac{dS}{dt}; \ \text{on}\ x = S(t). \tag{8.3.50}$$

Here, $f(t)$ and $\phi(x)$ are the same as in Eq. (8.3.28).

A result similar to Proposition 8.3.4 could not be proved for the Neumann problem as the solution of Riemann invariants cannot be obtained by integration along characteristics.

A one-phase three-dimensional hyperbolic Stefan problem with discontinuous temperature was studied in [221]. The constitutive equations which hold throughout the region consisting of solid and liquid phases are taken as

$$\frac{\partial e}{\partial t} + \nabla \cdot \vec{q} = 0, \quad \text{and} \quad \tau\frac{\partial \vec{q}}{\partial t} + \vec{q} + \nabla(KT) = 0; \ \ \vec{q} = (q_1, q_2, q_3). \tag{8.3.51}$$

The first equation in Eq. (8.3.51) holds in the weak sense as the energy $e = e(T)$ is discontinuous across the phase-change boundary whose equation is given by $\phi(x_1, x_2, x_3, t) = x_3 - S(x_1, x_2, t) = x_3 - S(t) = 0$. Across the shock $x_3 = S(t)$, the following Rankine–Hugoniot conditions hold:

$$\left.\begin{aligned} [\tau q_1]\, S_t + [KT]\, S_{x_1} &= 0, \\ [\tau q_2]\, S_t + [KT]\, S_{x_2} &= 0, \\ [\tau q_3]\, S_t - [KT] &= 0, \end{aligned}\right\} \tag{8.3.52}$$

$$[e]S_t + [q_1]\, S_{x_1} + [q_2]\, S_{x_2} - [q_3] = 0. \tag{8.3.53}$$

Here, $[f]$ denotes the jump in f across the interface $\phi = 0$ and $C_S = C_L = C$ and $\rho_S = \rho_L$. It is easy to derive conditions (8.3.52) and (8.3.53) from equations of the type (8.3.9). Since the energy function is nonconvex, one cannot expect the two-phase problem to be well-posed even in the one-dimensional case and therefore only one-phase problem has been considered in [221]. The liquid occupies the region $0 \le x_3 < S(t)$ in R^3, $S(x_1, x_2, 0) = S_0(x_1, x_2)$ and the initial conditions for the liquid region are given by

$$\left.\begin{aligned} T(\bar{x}, 0) = T_0(\bar{x}) > 0, \quad \bar{x} = (x_1, x_{2,} x_3) \\ \vec{q}\,(\bar{x}, 0) = \vec{q}_0(\bar{x}) \end{aligned}\right\}, \quad 0 \le x_3 \le S_0. \tag{8.3.54}$$

$T = 0$ is the phase-change temperature. The initial conditions for the solid region are given by

$$T(\bar{x}, 0) = 0, \quad \vec{q}\,(\bar{x}, 0) = 0; \quad x_3 > S_0. \tag{8.3.55}$$

Let $U = (q_1, q_2, q_3, T)^{T_\gamma}$, where T_γ is the transpose of a matrix. The hyperbolic system consisting of Eq. (8.3.51) can be written in the form of the following matrix equation

$$\mathcal{L}U = U_t + \sum_{J=1}^{3} A_J U_{x_J} + BU = 0, \tag{8.3.56}$$

where A_J, $J = 1, 2, 3$ and B are 4×4 matrices which can be easily written. As we saw earlier, for the existence and uniqueness of the solution, some compatibility conditions are to be satisfied at S_0. These are the conditions imposed on the traces of an initial data $(T_0, \vec{q}_0)$ at $x_3 = S(0)$. The compatibility conditions can be obtained by comparing the two values of $U_t(x, 0)$ at $x_3 = S(0)$, one computed from the interior Eq. (8.3.56) and another from the interface conditions (8.3.52) and (8.3.53). Higher order compatibility conditions can be obtained similarly (cf. [222]).

The compatibility conditions can also be stated in an equivalent way as follows. The initial data in Eqs (8.3.52)–(8.3.56), are said to satisfy *kth order compatibility conditions* if there exist a C^∞- approximate solution $(\bar{U}, \bar{S})$ of Eqs (8.3.52)–(8.3.56) such that

$$\bar{U}(\bar{x}, 0) = U_0(\bar{x}), \quad \bar{S}(x_1, x_2, 0) = S_0(x_1, x_2), \tag{8.3.57}$$

$$\mathcal{L}\bar{U} = \mathrm{O}(t^k), \tag{8.3.58}$$

$$D(\bar{U}, \bar{S}) = \mathrm{O}(t^k), \tag{8.3.59}$$

where the system of Eqs (8.3.52), (8.3.53) is briefly denoted as $D(U,S)=0$. $U_0(\bar{x})$ is the value of U at $t=0$ and the operator $\mathcal{L}$ is the same as in Eq. (8.3.56).

The main result of [221] is the following proposition.

Proposition 8.3.5. *The hyperbolic Stefan problem* (8.3.52)–(8.3.56) *has a unique classical solution* $(U,S)\in H^k((0,t_0)\times R^3)\times H^{k+1}((0,t_0)\times R^2)$ *for some* $t_0>0$, *provided the following conditions are satisfied:*

1. $U_0(\bar{x})$, $S_0(x_1,x_2)\in C^\infty$.
2. *kth order compatibility conditions* (8.3.57)–(8.3.59) *are satisfied for* $k\geq 4$.
3. $\bar{S}_t(x_1,x_2,0)<\alpha=\sqrt{K/(\tau C)}$.

In the n-dimensional case, we take $k\geq (n+1)/2+2$. There are four main steps in the construction of the proof of Proposition 8.3.5. First, the free boundary is fixed by the transformation $x_3-S(t)=y_3$, $x_1=y_1$, $x_2=y_2$ so that in the new coordinates, Eq. (8.3.56) becomes

$$\mathcal{L}_1(S)U=0,\ \text{in } y_3<0, \tag{8.3.60}$$

where

$$\mathcal{L}_1(S)=\frac{\partial}{\partial t}+\sum_{J=1}^{2}A_J\frac{\partial}{\partial y_J}+\left(A_3-S_t-\sum_{J=1}^{2}A_J\frac{\partial S}{\partial y_J}\right)\frac{\partial}{\partial y_3}+B.$$

The free boundary conditions remain of the same form as stated in Eqs (8.3.52), (8.3.53). The second step is to formulate the given problem in terms of new variables (V,Ψ) which satisfy the homogeneous initial conditions. V and Ψ are defined by the relations

$$U=\bar{U}+V;\quad S=\bar{S}+\Psi. \tag{8.3.61}$$

The existence and uniqueness of the solution of the original problem can be established if the same can be proved for the problem formulated in terms of the new variables (V,Ψ).

The third step is to linearize about $(\bar{U},\bar{S})$ the nonlinear problem formulated in terms of (V,Ψ) and to prove that the linearized problem is uniformly stable. The linear boundary value problem for (V,Ψ) will be uniformly stable if the solution (V,Ψ) satisfies the estimate (cf. [223])

$$\eta\|V\|^2_{0,\eta}+|V|^2_{0,\eta}+|\Psi|^2_{1,\eta}\leq\beta\left(\frac{1}{\eta}\|f\|_{0,\eta}+|g|_{0,\eta}\right),\ \text{for } \eta\geq\eta_0, \tag{8.3.62}$$

where $\|V\|_{r,\eta}$ is the *hyperbolic η-weighted interior norm* defined as

$$\|V\|_{r,\eta}=\sum_{|\alpha|\leq r}\int_{R^3\times R^1_+}\left|\partial^\alpha_{t,x}(e^{-\eta t}V)\right|^2dx_1dx_2dx_3dt,\quad x\in R^3 \tag{8.3.63}$$

and $|V|_{r,\eta}$ is the *boundary norm* defined as

$$|V|_{r,\eta}=\sum_{|\alpha|\leq r}\int_{R^2\times R^1_+}\left|\partial^\alpha_{t,(x_1,x_2)}(e^{-\eta t}V)\right|^2dx_1dx_2dt. \tag{8.3.64}$$

Here, (f, g) are smooth functions which have zero traces at $t = 0$ up to the order k and are defined as follows:

$$f = \mathcal{L}_1(\bar{S} + \Psi)(\bar{U} + V) - \mathcal{L}_1(\bar{S})(\bar{U}), \quad y_3 < 0, \tag{8.3.65}$$

$$g = D(\bar{U} + V,\ \bar{S} + \Psi) - D(\bar{U}, \bar{S}). \tag{8.3.66}$$

It has been proved that under the assumptions about the initial data in Proposition 8.3.5, the smooth solution of the linearized problem satisfies Eq. (8.3.62) near $(\bar{y}, t) = (0, 0)$, $\bar{y} = (y_1, y_2, y_3)$.

Proposition 8.3.6. *If the linearized problem is uniformly stable, then for any $k \geq 0$ there is a unique solution $(V, \Psi) \in H^k \times H^{k+1}$ satisfying Eq. (8.3.62) near $(0, 0)$ with the subscript zero in Eq. (8.3.62) replaced by k and the subscript one replaced by $k + 1$; the constant β (depending on k) in Eq. (8.3.62) depends on the local H^m norm of the coefficients with $m = \max(k, 4)$.*

The fourth step is to use an iterative method to show the existence of a unique classical solution (V, Ψ) in the neighbourhood of $(0, 0)$. Only linear iterations can be employed as for the linearized problem the estimate of the boundary function Ψ is one order higher than the estimate of V. For the solution of the nonlinear problem formulated in terms of (V, Ψ), the following result holds good.

Proposition 8.3.7. *Consider $k \geq 4$ in the nonlinear problem formulated for (V, Ψ) and assume that the following conditions hold good:*

1. *$(f, g) \in H^k$ with zero traces at $t = 0$ up to the order $(k - 1)$.*
2. *At $(V, \Psi) = (0, 0)$, the boundary value problem is uniformly stable, i.e. the estimate in (8.3.62) is satisfied.*

Under the above conditions there exists a $t_0 > 0$ such that in $(0, t_0)$, the nonlinear problem for (V, Ψ) has a unique solution $\in H^k \times H^{k+1}$. Moreover,

$$\eta \, \|V\|^2_{k,\eta,t_0} + |V|^2_{k,\eta,t_0} + |\Psi|^2_{k+1,\eta,t_0} \leq \beta_k \left(\frac{1}{\eta} \, \|f\|^2_{k,\eta,t_0} + |g|^2_{k,\eta,t_0} \right), \text{ for } \eta \geq \eta_0. \tag{8.3.67}$$

Here, the norms $\|\cdot\|_{m,\eta,t_0}$ and $|\cdot|_{m,\eta,t_0}$ denote, respectively, the restriction in $[0, t_0]$ of the norms $\|\cdot\|_{m,\eta}$ and $|\cdot|_{m,\eta}$.

8.4 MODEL III: DELAY IN THE RESPONSE OF ENERGY TO LATENT AND SENSIBLE HEATS

8.4.1 The Classical and the Weak Formulations

In this model, it will be assumed that along the solid–liquid interface, the energy response to the latent heat release (storage) is delayed by the same amount of time as the delay in the response of flux to the temperature gradient. Independently, the specific heat may be delayed or advanced with respect to the total energy by an increment depending on the phase. This seems appropriate in order to match the wave speeds of the hyperbolic equations on either side of the phase-change interface. If it is assumed that the response of energy to the latent heat is instantaneous, following a delay of relaxation time τ, and if the response of energy to the specific heat is not delayed, then the total energy during the phase-change is given by Eq. (8.1.6). As discussed earlier this leads to an unrealistic situation.

To have a well-posed problem and render it consistent with the wave speeds in solid and liquid phases and also consistent with the delay in the energy response to the latent heat, the following constitutive relations have been considered in [211]:

$$-(\vec{q}\,(t)+\tau\vec{q}\,'(t)) = K\nabla T \tag{8.4.1}$$

and

$$\left(1+\tau\frac{d}{dt}\right)e(t) = \left(1+g(\tau)\frac{d}{dt}\right)\bar{C}(T)+l\xi(t), \quad \xi\in H_g(T). \tag{8.4.2}$$

Here, $g(\tau)=\tau_S$ if $T<0$ and $g(\tau)=\tau_L$ if $T>0$, $\bar{C}(T)=C_S T$ if $T<0$ and $\bar{C}(T)=C_L T$ if $T>0$. τ_S and τ_L are relaxation times for the response of energy to the specific heats in the pure solid and the pure liquid phases, respectively. ξ is the fraction of liquid present in any region. $\xi=0$ for the pure solid, $\xi=1$ for the pure liquid and $0<\xi<1$ for the mushy region whose temperature is $T=0$. H_g is the Heaviside graph (cf. Eq. 4.3.45). $(\vec{q}\,(t)+\tau\vec{q}\,'(t))$ and $(e(t)+\tau e'(t))$ are assumed to be piecewise smooth in the total region under consideration. The first term on the l.h.s. of Eq. (8.4.2) is the first-order approximation of $e(t+\tau)$ and the first term on the r.h.s. of Eq. (8.4.2) is the first-order approximation of $C(T(t+\tau_J))$, J $=S,L$. The essential assumptions in Eq. (8.4.2) are: (1) the latent heat affects the energy after a delay of time τ, and (ii) the specific heat affects the energy after a delay of $\tau-\tau_J$, J $=S$, L, in the corresponding phase. When $\tau_S=\tau_L=\tau$, Eq. (8.4.2) can be integrated and we get

$$e(t) = C(T(t))+(l/\tau)\int_0^\infty \exp(-p/\tau)\xi(t-p)dp, \quad \xi\in H_g(T). \tag{8.4.3}$$

Derivation of Energy Conservation Equation for the Two-Phase Problem

The energy conservation principle and the relation (8.4.2) will now be used to derive energy conservation equation in the phase-change region. We first obtain the classical formulation and consider the case in which only solid and liquid regions exist which are separated by a smooth interface $\Gamma(t)$. The densities of both the phases are taken to be the same . Let $G\subset R^3$, be a fixed open bounded region and $G=G_S(t)\cup G_L(t)\cup\Gamma(t)$, $0\le t<t_*$. The subscripts S and L stand for solid and liquid regions, respectively. Hyperbolic equations admit discontinuous solutions along shocks. Such shocks will be accommodated in the formulation. The time dependence of G_S, G_L and Γ will not be indicated in the further discussion of this problem. The regions G_S and G_L are disjoint. The external boundary of G is denoted by $\partial G^b=\partial G_S^b\cup\partial G_L^b$ where ∂G_S^b and ∂G_L^b are the external boundaries of solid and liquid regions. Let ∂G_S be the boundary of the solid region when the liquid region is removed from G and let ∂G_L be the boundary of the liquid region when the solid region is removed from G, i.e. $\partial G_S=\partial^b G_S\cup\Gamma$ and $\partial G_L=\partial^b G_L\cup\Gamma$. It will be assumed that ∂G^b and Γ are sufficiently smooth and the outward drawn normal can be defined uniquely on them. The energy conservation principle when applied to G gives

$$\frac{d}{dt}\int_G e(t)dx = -\int_{\partial G^b}\vec{q}\,(t)\cdot\vec{n}da+\int_G F(t)dx. \tag{8.4.4}$$

Here, $\vec{n}$ is the unit outward normal to ∂G^b and $F(t)$ is the intensity of the heat source. By virtue of the constitutive relation (8.4.2), $e(t)$ is differentiable in G and the time derivative on the l.h.s. of Eq. (8.4.4) can be taken inside the integral. According to Eq. (8.4.1), $\vec{q}\,(t)$ is also differentiable, therefore, on differentiating Eq. (8.4.4) again with respect to time, we obtain

$$\frac{d}{dt}\int_G e'(t)dx = -\int_{\partial G^b} \vec{q}\,'(t).\,\vec{n}da + \int_G F'(t)dx. \tag{8.4.5}$$

It will be assumed that G, ∂G^b and, $\vec{n}$ on ∂G^b do not change with time. Multiplying Eq. (8.4.5) by τ and adding Eq. (8.4.4) to it, we obtain

$$\frac{d}{dt}\int_G (e(t)+\tau e'(t))dx = -\int_{\partial G^b} (\vec{q}\,(t)+\tau\vec{q}\,'(t))\cdot\vec{n}da + \int_G (F(t)+\tau F'(t))dx. \tag{8.4.6}$$

In G, $e(t)+\tau e'(t)$ is only piecewise smooth but in G_S and G_L, $e(t)+\tau e'(t)$ is smooth. Therefore in Eq. (8.4.6) we consider $G_S \cup G_L$. The normal $\vec{n}$ should be defined in such a way that it is consistent with boundaries of both regions G_S and G_L. Therefore $\vec{n}$ will have opposite directions on Γ when Γ is approached from liquid and from solid regions. On Γ, $\vec{n}$ is time dependent and it will be denoted by $\vec{N}$ which points into the liquid. The boundary of G_S includes Γ which is moving with a velocity $\vec{V}(t)$ and we have (see Eq. 2.4.12)

$$\frac{d}{dt}\int_{G_S\cup G_L} (e(t)+\tau e'(t))dx = \int_{G_S\cup G_L} \frac{\partial}{\partial t}(e(t)+\tau e'(t))dx + \int_\Gamma \left[e+\tau e'\right](\vec{V}\cdot\vec{N})\,da. \tag{8.4.7}$$

In Eq. (8.4.7), we calculate d/dt from the first principle and assume that Γ does not have any surface energy.

Eq. (8.4.6) when written for $G_S \cup G_L$ will have the following form:

$$\begin{aligned}\int_{G_S\cup G_L} \frac{\partial}{\partial t}(e(t)+\tau e'(t))dx + \int_\Gamma \left[e+\tau e'\right](\vec{V}\cdot\vec{N})da &= -\int_{\partial G_S^b\cup\partial G_L^b} (\vec{q}\,(t)+\tau\vec{q}\,'(t))\cdot\vec{n}da \\ &- \int_\Gamma \left[\vec{q}\,(t)+\tau\vec{q}\,'(t)\right]\cdot\vec{N}da + \int_{G_S\cup G_L}(F(t)+\tau F'(t))dx.\end{aligned} \tag{8.4.8}$$

Here, $[e+\tau e']$ and $[\vec{q}+\tau\vec{q}\,']$ denote jumps in $e(t)+\tau e'(t)$ and $\vec{q}\,(t)+\tau\vec{q}\,'(t)$ across Γ. The surface integral in Eq. (8.4.8) can be converted to a volume integral. The direction of the normal on Γ from the solid side is opposite to its direction from the liquid side. Since Eq. (8.4.8) holds for any G and Γ and for any $t>0$, the following equations should hold:

$$\frac{\partial}{\partial t}(e(t)+\tau e'(t)) + \nabla\cdot(\vec{q}\,(t)+\tau\vec{q}\,'(t)) = F(t)+\tau F'(t) \text{ in } G\backslash\Gamma, \quad 0<t<t_*. \tag{8.4.9}$$

$$-\left[e(t)+\tau e'(t)\right]N_t = \left[\vec{q}\,(t)+\tau\vec{q}\,'(t)\right]\cdot\vec{N}_x, \text{ on } \Gamma, \quad 0<t<t_*. \tag{8.4.10}$$

Here, $\vec{N}=(N_t,\vec{N}_x)$, and $\vec{N}_x=(N_{x_1},N_{x_2},N_{x_3})$, $x=(x_1,x_2,x_3)$. If $\tilde{S}$ denotes the union of Γ and some other surfaces of discontinuities (shocks) of $e(t)+\tau e'(t)$ and $\vec{q}\,(t)+\tau\vec{q}\,'(t)$ in G other than Γ, then in Eqs (8.4.9), (8.4.10), Γ should be replaced by $\widetilde{S}$.

Next, we assume that G consists of solid, liquid and mushy regions and $G = G_S(t) \cup G_M(t) \cup G_L(t) \cup \Gamma_1(t) \cup \Gamma_2(t)$. Note that there could be surfaces of discontinuities other than Γ_1 and Γ_2 also. In $G_S(t)$, we have $T < 0$, and in $G_L(t)$ we have $T > 0$. $G_M(t)$ is the mushy region in which the temperature is $T = 0$. The liquid fraction $\xi = \xi(x,t)$ is equal to 1 and 0 in liquid and solid regions, respectively, and $0 < \xi(x,t) < 1$ in the mush and is assumed to be a smooth function of t in the mush. $\Gamma_1(t)$ is the solid–mush boundary and $\Gamma_2(t)$ is the liquid–mush boundary. $G_S(t)$, $G_M(t)$ and $G_L(t)$ are mutually disjoint regions. $\Gamma_1(t)$ separates the solid region from the mush and $\Gamma_2(t)$ separates the mush from the liquid region. The external boundary of G will be denoted by ∂G^b and $\partial G^b = \partial G_S^b \cup \partial G_M^b \cup \partial G_L^b$. Here, ∂G_S^b, ∂G_M^b and ∂G_L^b are the exterior boundaries of solid, mush and liquid regions, respectively. Let ∂G_S be the boundary of the solid region when liquid and mushy regions are removed from G, i.e. $\partial G_S = \partial G_S^b \cup \Gamma_1$ and similarly let $\partial G_M = \partial G_M^b \cup \Gamma_1(t) \cup \Gamma_2(t)$ and $\partial G_L = \partial G_L^b \cup \Gamma_2(t)$. The outward drawn normal $\vec{n}$ to ∂G_S, ∂G_M and ∂G_L is defined in such a way that it is continuous on the portions common to these boundaries. In order to obtain relations of the form Eq. (8.4.10) on $\Gamma_1(t)$ and $\Gamma_2(t)$ and the energy equation (8.4.9) in $G \backslash \Gamma_1 \cup \Gamma_2$, we follow a procedure similar to that used above to obtain Eq. (8.4.8). Repeating arguments which lead to Eqs (8.4.9), (8.4.10), we get the energy balance relations on Γ_1 and Γ_2 and the energy equation in $G \backslash \Gamma_1 \cup \Gamma_2$.

Let $\overset{\approx}{S} = \Gamma_1 \cup \Gamma_2 \cup$ (shocks other than Γ_1 and Γ_2). The classical formulation of the three region problem (take the densities of the three regions equal to one) is concerned with finding a pair (T, ξ), satisfying the system of Eqs (8.4.11)–(8.4.16) for $0 < t < t_*$:

$$\frac{\partial}{\partial t}\left(\tau_S C_S \frac{\partial T}{\partial t} + C_S T\right) - K_S \nabla^2 T = F + \tau F', \text{ in } G_S - \overset{\approx}{S}, \tag{8.4.11}$$

$$l\frac{\partial \xi}{\partial t} = F + \tau F', \text{ in } G_M - \overset{\approx}{S}, \tag{8.4.12}$$

$$\frac{\partial}{\partial t}\left(\tau_L C_L \frac{\partial T}{\partial t} + C_L T + l\right) - K_L \nabla^2 T = F + \tau F', \text{ in } G_L - \overset{\approx}{S}, \tag{8.4.13}$$

$$\left(l(1-\xi) + \tau_L C_L \frac{\partial T}{\partial t}\right) N_t = K_L \nabla T \cdot \vec{N}_x, \text{ on } \Gamma_2, \tag{8.4.14}$$

$$\left(-l\xi + \tau_S C_S \frac{\partial T}{\partial t}\right) N_t = K_S \nabla T \cdot \vec{N}_x, \text{ on } \Gamma_1, \tag{8.4.15}$$

$$\tau_i C_i \left[\frac{\partial T}{\partial t}\right] N_t = K_i [\nabla T] \cdot \vec{N}_x, \text{ on } \overset{\approx}{S} - (\Gamma_1 \cup \Gamma_2). \tag{8.4.16}$$

Here, $i = S$ or L depending on whether T < 0 or $T > 0$. $T = 0$ on Γ_1 and Γ_2 and $\vec{N} = (N_t, \vec{N}_x)$ is the normal on a surface of discontinuity and $\vec{N}$ points into the liquid. $[f]$ denotes the jump in f across a surface of discontinuity. Eqs (8.4.11), (8.4.13) are telegrapher's equations. Eq. (8.4.12) regulates the water fraction in the mush. Eqs (8.4.14), (8.4.15) are energy balance conditions at the phase-change boundaries (cf. Eq. 8.4.20 for their derivation). The propagation of wave fronts is described by Eq. (8.4.16). To complete the formulation, Eqs (8.4.11)–(8.4.16) should be supplemented with the initial and boundary conditions. For example,

$$T(x,t) = 0, \quad x \in \partial G^b, \quad 0 < t < t_*, \tag{8.4.17}$$

$$T(x,0) = T_0(x), \quad x \in G, \tag{8.4.18}$$

$$\left.\begin{aligned}(\tau_L C_L \partial T_L/\partial t + C_L T_L + l)|_{t=0} &= v_1(x), \text{ where } T_0(x) > 0,\\ (\tau_S C_S \partial T_S/\partial t + C_S T_S)|_{t=0} &= v_2(x), \text{ where } T_0(x) < 0,\\ l\,\xi(x, 0+) &= v_3(x), \text{ where } T_0(x) = 0, \quad x \in G.\end{aligned}\right\} \tag{8.4.19}$$

Let $u_0(x) = v_1(x)$ for $T_0(x) > 0$, $= v_2(x)$ for $T_0(x) < 0$, $= v_3(x)$ for $T_0(x) = 0$.

To derive energy balance condition (8.4.14) on Γ_2 and Eq. (8.4.15) on Γ_1 we consider two one-phase hyperbolic Stefan problems with the assumptions about the delay in the response of flux to the temperature gradient and the delay in the energy response to the sensible heat. At the liquid–mush boundary Γ_2, the energy balance condition without any approximation gives

$$-\vec{q}\,(t+\tau)\cdot \bar{N}_x + (l(1-\xi) + C_L T_L(t+\tau_L))\vec{V}\cdot \vec{N}_x = 0. \tag{8.4.20}$$

For simplicity we take densities of all the phases equal to unity. The first-order approximations of $\vec{q}\,(t+\tau)$ and $T_L(t+\tau_L)$ when substituted in Eq. (8.4.20) give Eq. (8.4.14). Similarly by considering the energy balance condition on Γ_1 and considering its approximation, we obtain Eq. (8.4.15).

The wave speeds in solid and liquid phases are given by $(k_S/\tau_S)^{1/2}$ and $(k_L/\tau_L)^{1/2}$, respectively. To have a global wave speed $(1/\tau_0)^{1/2}$ in G, $\tau_0 > 0$, independent of the phase, we define

$$\tau_S = \tau_0 k_S; \quad \text{and} \quad \tau_L = \tau_0 k_L. \tag{8.4.21}$$

To obtain a weak formulation of the classical problem stated in Eqs (8.4.11)–(8.4.16), we consider the integral

$$\int_{G_*} \left\{ \frac{\partial}{\partial t}\left(\tau_0 \frac{\partial K(T)}{\partial t} + \bar{C}(T) + l\xi \right) - \nabla^2(K(T)) \right\} \phi \; dxdt. \tag{8.4.22}$$

Here, $G_* = G \times (0, t_*)$, $0 < t < t_* < \infty$, $K(T) = K_S T$, if $T < 0$ and $K(T) = K_L T$ if $T > 0$, $\overline{C}(T) = C_S T$ if $T < 0$, $= C_L T$ if $T > 0$, and ϕ is the test function, $\phi \in C_0^\infty(G_*)$. By adopting the procedure indicated in Eqs (5.2.12)–(5.2.14) it can be shown that

$$\begin{aligned}&\int_{G_*} \left\{ \frac{\partial}{\partial t}\left(\tau_0 \frac{\partial K(T)}{\partial t} + \bar{C}(T) + l\xi \right) - \nabla^2(K(T)) \right\} \phi \; dxdt\\ &\quad = \int_{G_S} \left\{ \frac{\partial}{\partial t}\left(\tau_S C_S \frac{\partial T}{\partial t} + C_S T \right) - K_S \nabla^2 T \right\} \phi \; dxdt\\ &\qquad + \int_{G_M} l \frac{\partial \xi}{\partial t} \phi dxdt + \int_{G_L} \left\{ \frac{\partial}{\partial t}\left(\tau_L C_L \frac{\partial T_L}{\partial t} + C_L T \right) - K_L \nabla^2 T \right\} \phi \; dxdt\\ &\qquad + \int_{\approx\atop S} \left[\nabla K(T)\cdot \vec{N}_x - \left(l(1-\xi) + \tau_0 \frac{\partial}{\partial t} K(T) \right) N_t \right] \phi \; dxdt.\end{aligned} \tag{8.4.23}$$

On using Eqs (8.4.11)–(8.4.16) in Eq. (8.4.23), we get

$$\frac{\partial}{\partial t}\left(\tau_0 \frac{\partial K(T)}{\partial t} + \bar{C}(T) + \ell\xi \right) - \nabla^2(K(T)) = F + \tau F', \text{ a.e. in } G_*. \tag{8.4.24}$$

Eq. (8.2.24) is satisfied in G_* in the distributional sense. A pair of functions (T, ξ) such that $T \in W^{1,\infty}(0, t_*; L^2(G)) \cap L^\infty(0, t_*; H_0^1(G))$, $\xi \in L^\infty(G_*)$ is called a generalized solution of the problem (8.4.11)–(8.4.16) and (8.4.17)–(8.4.19) if Eq. (8.4.24) is satisfied for a.a. $t \in [0, t_*]$. $T(t) \in H_0^1(G)$ and $\xi(t) \in H_g(T(t))$ for a.a. $t \in [0, t_*]$; $T(0) = T_0$ and $\left(\tau_0 \partial K(T)/\partial t + \bar{C}(T) + l\xi\right)_{t=0} = u_0(x)$. If $F \in W^{1,\infty}(0, t_*; H^{-1}(G))$ (H^{-1} is the dual space of H^1), then $\tau_0 \partial K(T)/\partial t + \bar{C}(T) + l\xi(t) \in W^{1,\infty}(0, t_*; H^{-1}(G))$ and the initial conditions are meaningful. Under suitable assumptions on F, T_0, $\xi(x, 0)$ and u_0, existence of a unique generalized solution has been proved in [211].

A one-phase problem in which the melted ice first forms a mushy region with temperature zero and then becomes water with temperature greater than zero has also been discussed in [211]. The classical, weak and variational inequality formulations have been obtained for this one-phase problem. In terms of the freezing index the existence of the unique weak solution of the variational formulation has been discussed. An example of the nonexistence of the generalized solution of a one-phase problem formulated with the above delay assumptions has been constructed in [211].

The one-phase multidimensional hyperbolic Stefan problem discussed in [224] consists of finding a hypersurface $S(t) = \{x \in R^n : \phi(x, t) = 0\}$, $0 \leq t < \infty$, $n > 1$ and the temperature $T(x, t)$ in $G(t) = \{x \in R^n : \phi(x, t) > 0\}$, $0 \leq t < \infty$, such that

$$\tau C T_{tt} + C T_t - K \nabla T = 0, \text{ in } G(t), \tag{8.4.25}$$

$$\left.\begin{aligned} T(x,t) &= 0, \\ (l + \tau C T_t)\phi_t &= K \nabla T \cdot \nabla \phi. \end{aligned}\right\} \text{ on } S(t) \tag{8.4.26}$$

$$\left.\begin{aligned} T(x,0) &= T_0(x), \\ T_t(x,0) &= T_1(x), \\ S(0) = S_0 &= \{x : \phi(x,0) = \phi_0 = 0\}. \end{aligned}\right\} \tag{8.4.27}$$

This model in Eqs (8.4.25)–(8.4.27) is based on the delay assumptions of model III.

The existence and uniqueness of H^1-weak solutions were proved in [211]. The existence of the unique classical solution of the problem (8.4.25)–(8.4.27) has been proved in [224]. A solution (T, ϕ) of Eqs (8.4.25)–(8.4.27) is called classical if $(T, \phi) \in C^2(R^n \times R_+^1)$. The above problem is called *nondegenerate* if

$$\nabla T_0 \cdot \nabla \phi_0 \neq 0, \text{ on } S_0. \tag{8.4.28}$$

The condition (8.4.28) implies that the interface is really moving. We saw earlier that some compatibility conditions have to be satisfied at S_0 for studying existence and uniqueness of solutions. The zeroth-order compatibility conditions on S_0 are given by

$$\left.\begin{aligned} T_0(x) &= 0, \\ (l + \tau C T_1)T_1 - K |\nabla T_0|^2 &= 0. \end{aligned}\right\} \text{ on } S_0. \tag{8.4.29}$$

The first-order compatibility condition on S_0 can be obtained by comparing the two expressions for T_{tt} on S_0, one obtained from Eq. (8.4.25) and another from Eq. (8.4.26), and it is given by

$$(l + 2\tau C T_1)(T_{tt} + \nabla T_1 \cdot \vec{\eta}) = 2K \nabla T_0 \cdot \nabla T_1 + K \nabla |T_0|^2 \cdot \vec{\eta}, \tag{8.4.30}$$

where

$$\vec{\eta} = -T_1 \nabla T_0 \, |\nabla T_0|^{-2}. \tag{8.4.31}$$

The main result of [224] is the following proposition.

Proposition 8.4.1. *Assume that the following conditions are satisfied:*

(i) *The problem* (8.4.25)–(8.4.27) *is nondegenerate.*

(ii) $T_0(x)$, $T_1(x)$ *and* $\phi_0(x)$ *are sufficiently smooth.*

(iii) *The compatibility conditions are satisfied on* S_0 *up to the order* $\gamma : \gamma > (n+1)/2$.

(iv) *The stability condition*

$$|\nabla T_0|^2 > T_1^2 \frac{\tau C}{K} \left(1 + \frac{l}{2\tau C T_1 + l}\right) \tag{8.4.32}$$

is satisfied.

Then there exists a $t_0 > 0$, such that in $(0, t_0)$, there exists a unique classical solution (T, ϕ) of Eqs (8.4.25)–(8.4.27) belonging to the space $H^{\gamma+1} \times H^{\gamma+1}(0, t_0;\ R_+^n)$. The method of proof involves fixing the phase-change boundary by the *hodograph transformation* and then considering a linearized problem. By using the results of the linearized problem, the results for the nonlinear problem have been obtained. For $l = 0$, the stability condition (8.4.32) implies that the velocity of the interface should be less that the velocity of sound. For $l \neq 0$, condition (8.4.32) is stronger than the usual requirement that the phase interface moves slower that the sound speed. If condition (8.4.32) is not imposed, then generally, one cannot expect the one-phase problem to be well-posed globally because of the hyperbolic wave property. It has been remarked in [224] that this condition is not satisfied at the time specified in the example considered in [211] to show the nonexistence of the solution of the one-phase problem.