

## Chapter 12

# Quasi-Analytical Solutions and Methods

### 12.1 INTRODUCTION

Stefan problems form a subset of a bigger class of problems known as free boundary problems (FBPs). Although both FBPs and Stefan problems have many characteristic features in common, the later class of problems is known by their typical mathematical formulations and physical concepts, some of which are apparent in the Neumann problem formulation in [Section 1.3](#). In the first 11 chapters of this book, Stefan problems were discussed concerning their theoretical background, modelling, analysis, etc., in relation to some existing topics of their study. Analytical and numerical solutions were only marginally discussed as the aims and objectives of the earlier book were different. Although the sole interest in the earlier book was to discuss classical Stefan problems, it was unavoidable to study weak formulations in the context of variational inequalities, inverse problems, analysis, etc. The existing literature on Stefan problem is so vast that it can be said, only a little effort was made to discuss the existing literature in the earlier book. Enormous literature exists concerning weak formulations of Stefan and Stefan-like problems and reporting and discussing the material would require big volumes.

In this chapter, the study of Stefan problems has been extended further to Quasi-Analytical Solutions and Methods (in short Q.An.S.M.) of classical formulations of some Stefan and Stefan-like problems. Purely numerical solutions have not been included in this chapter for the simple reason that vast literature exists on numerical methods and numerical solutions of Stefan problems which for presentation would require big volumes. The prime interest in this chapter is on methods of analytical solutions. Here analytical solutions include exact analytical solutions, series solutions, solutions by approximate methods, perturbation methods and analytical-numerical or semianalytical solutions, etc. We prefer to call semianalytical solutions as analytical-numerical solution to indicate that analytical solution part is dominant. These analytical-numerical solutions are different from purely numerical solutions. Stefan and Stefan-like problems are highly nonlinear. Apart from some exact analytical solutions which have been obtained for some typical formulations of problems, invariably the analytical method of solution can be continued only up to certain stage of solution and after that solution can be obtained only numerically with the help of numerical schemes. In many problems these schemes could be of text book type. For the purpose of this chapter such solutions have been called analytical-numerical solutions.

There is very large number of physical problems whose characteristic features and formulations are to some extent similar to Stefan problems. Such problems are called Stefan-like problems. Some authors prefer to call them generalized Stefan problems or FBPs. It is difficult to find areas of Science and Engineering in which Stefan-like problems do not occur. The problem of solidification/melting is much more complicated than just treating it as a heat transfer or a heat and mass transfer problem. At very high temperatures such as in casting technology, the mechanical behaviour of solidifying ingot should be treated as temperature-dependent elastic, elastic-plastic or nonlinear viscoelastic. Stresses and strains in the solidifying mass should be accounted in the formulation of problems. To include the motion of the melt which arises due to forced convection, natural convection and other effects, Navier-Stokes equations should be coupled with heat and mass transfer equations. The discussion of quasianalytical solutions of such and many other Stefan-like problems is beyond the scope of this chapter. Even to discuss mere formulation of such a problem would require several pages as it requires background information, assumptions, identification of parameters, etc. However, there are some simple or simplified with the help of assumptions, Stefan-like problems, whose formulations are similar to those of Stefan problems. The methods of analytical solutions of Stefan problems are also applicable to the solutions of some Stefan-like problems with some modifications. Some such Stefan-like problems have been included in this chapter and, in some sections of this chapter separate subsections are devoted to their solutions.

The information available about the Stefan problems in the earlier chapters has been successfully and positively used in this chapter. The reader gets a broader view of different classes of Stefan problems, their different characteristic features which helps in choosing topics and appropriate methods for study and for obtaining solutions. A method or a solution discussed in a book on Stefan problems cannot be confined only to the solution of a Stefan problem. It is a thought, a procedure, which may have applications in different branches of Science and Engineering including the problems discussed in the earlier chapters.

In the earlier book, attempts were made towards comprehensive treatment of the material presented. The same thinking continues for the presentation of the material in this chapter. The reporting of material concerning any research work has been done here only after going through the full length paper and not merely by looking at the abstract or the reviews published somewhere. Wherever it was felt and was also possible, further explanations about the methods and solutions have been provided for better understanding. Although purely numerical solutions have not been considered, the highlights of numerical solutions in analytical-numerical solutions have been indicated briefly. It is neither feasible nor practical to report full solutions or present all steps in the solution procedure or steps in the proofs mentioned in research articles. However, attempts have been made here that the reader is able to understand the problem considered, the application of the method and the solution highlights. If interested, the reader may see the original papers and look at other references connected with the work. In this respect a pertinent question arises. How much discussion of a particular research work is necessary? The interest of readers may vary which cannot be taken care by any author and it seems appropriate to leave this matter at the discretion of the author who is aware of all the practical constraints. Invariably the idea of obtaining a solution comes first and then the method is thought of. This is why in the title, solution comes first and then the method. Keeping in mind the current trends in the study of FBPs, a list of supplementary references has been given which includes variety of topics in FBPs, which are connected with the Stefan problem. A brief review of some of the supplementary references has been included in [Section 12.10](#).

The notations used in the previous chapters continue for this chapter also. Occasionally, for better understanding, when the notations play an important role in grasping the physics of the problem, notations have been further explained in the text. Not much attention has been paid in the text to describe the lengthy process of making different parameters dimensionless, deriving differential equations, explaining boundary conditions, etc. Most of the time method of solution and the process of obtaining solution can be explained without explaining dimensionless quantities. This has been done to save space, which is needed for presentation of solution procedure with clarity. However, if a dimensionless quantity has a bearing on the solution then the process of making it dimensionless has been explained clearly.

In every research paper a big list of assumptions is given. If these assumptions are mentioned, then there will be very little space left to discuss any other material. However, if an assumption is crucial for the solution then it has been mentioned. The presentation of the material in this chapter is a mixture of old and recent research works. It is the earlier works, say before 1995, which contain rich information about various quasianalytical methods and solution procedures. We shall see later that most of the earlier concepts, formulations, etc., have been further extended citing the earlier works. Many of the extensions and variations of the earlier proposed methods can be better implemented now because of the fastness of numerical computing. Variety of computing software are readily available now. This has given rise to many analytical-numerical methods and solution procedures, which can handle generalizations of problems discussed earlier.

To discuss wide variety of methods and solution procedures, some compromises have been done in the presentation. Some such compromises are apparent in accommodating several equations within a single equation number and taking thermo-physical parameters as unity in many problems. However, minimum compromise has been done in terms of clarity in understanding the presentation.

## **12.2 EXACT ANALYTICAL SOLUTIONS OF SOME STEFAN AND STEFAN-LIKE PROBLEMS**

### **12.2.1 Some Preliminaries, Problem Formulations and Solution of Various Types**

The following topics will be discussed in this subsection.

1. Neumann solution revisited; Green's function and similarity solutions in various geometries including ellipsoidal and paraboloidal coordinates.
2. Solutions of problems in which thermo-physical parameters are not constant.
3. Solutions of problems with kinetic conditions at the phase-change interface.
4. Some exact solutions obtained in particular cases of general formulations and their solution highlights have been included in [Section 12.2.4](#).

Some Stefan-like problems resembling Stefan problems will be discussed in [Section 12.2.2](#).

#### *Neumann Solution Revisited*

A well-known and widely referred example of exact solution is Neumann solution which can be easily obtained from Eqs (1.3.11)–(1.3.17) by neglecting velocity term in the liquid which

arises due to difference in densities  $\rho_S$  and  $\rho_L$  of solid and liquid regions, respectively. Many of the mathematical and physical aspects of Stefan problems can be easily discussed with the help of Neumann solution. Although names of Stefan and Neumann are generally associated with these solid-liquid phase-change problems, such problems were discussed earlier by Lamé and Clapeyron [357] also. Neumann solution is an exact solution as it satisfies all the equations (1.3.1–1.3.7) of the stated problem exactly. We shall see later that it is very difficult to construct analytical solutions of Stefan problems and in large number of problems the conditions imposed in the formulation are satisfied approximately by the solution. Sometimes the problem formulation is suitably modified to obtain an explicit or some approximate solution.

In the present section only exact solutions will be considered. Other types of solutions such as series solutions which cannot be summed up in closed form, or solutions which are presented in integral form and the integrals cannot be evaluated analytically will be discussed in separate sections. In Neumann solution and many other solutions of explicit type, a parameter  $\delta$  such as in Eq. (1.3.16) occurs which is to be obtained as a real positive root of a transcendental equation. Associated with finding the real positive root of a transcendental equation, there are questions of existence and uniqueness of this root which should be addressed in the context of the ranges of other parameters. It has been observed in [358] that when this real positive root is very large or very small, its numerical computation becomes difficult.

To emphasize the importance of discussing some other aspects of the solution, given below is an illustrative example and its solution in which solution is unique but the problem is ill-posed [278]. Some comments about this solution were made in Section 9.7 but neither the problem nor its solution was reported there. We consider the following problem.

$$\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} = f(x, t), \quad 0 < x < S(t), \quad t > 0; \quad T|_{t=0} = \phi(x), \quad 0 \leq x \leq S(0), \quad (12.2.1)$$

$$\left. \begin{aligned} T|_{x=0} = g_0(t); \quad T|_{x=S(t)} = f_1(t); \quad \lambda(t) \frac{dS(t)}{dt} = - \left. \frac{\partial T}{\partial x} \right|_{x=S(t)} \\ + g_1(t), \quad t \geq 0, \quad S(0) = S_0. \end{aligned} \right\} \quad (12.2.2)$$

As we are following the notations and formulations as used in the earlier chapters, the heat equation and imposed boundary conditions need not be explained here. For convenience some of the parameters have been taken to be unity in Eqs (12.2.1), (12.2.2). It is assumed that the prescribed functions  $f(x, t)$ ,  $\phi(x)$ ,  $g_0(t)$ ,  $f_1(t)$ ,  $g_1(t)$  and  $\lambda(t) \geq 0$  are all sufficiently smooth so that a unique solution exists. Let the prescribed functions be given as follows. For  $p > 0$  and  $t \geq 0$

$$\left. \begin{aligned} f(x, t) &= p(1+p) \exp\{-p[2+t(1+p)]\} - p(1+p) \exp\{-p(t+x+1)\}, \\ \phi(x) &= \exp[-p(1+x)] - \exp(-2p); \quad g_0(t) = -p \exp\{-p(1+t)\}, \\ \lambda(t) &= \exp\{-p[2+t(1+p)]\}; \quad g_1(t) = f_1(t) = 0; \quad S(0) = S_0 = 1. \end{aligned} \right\} \quad (12.2.3)$$

It can be checked that for  $p > 0$  and  $t \geq 0$

$$S(t)|_p = 1 + tp; \quad T(x, t)|_p = \exp[-p(1+x+t)] - \exp\{-p[2+t(1+p)]\}, \quad (12.2.4)$$

satisfy Eqs (12.2.1), (12.2.2). Take  $p_1 = p + 1$ , then

$$S(t)|_{p_1} = 1 + tp_1; \quad T(x, t)|_{p_1} = \exp[-p_1(1+x+t)] - \exp\{-p_1[2+t(1+p_1)]\}. \quad (12.2.5)$$

Let  $d = S(t)|_{p_1} - S(t)|_p = t(p_1 - p) > 0$ . As  $t$  and  $p_1 - p$  increase, the solution goes on becoming large and it does not hold for large values of  $t$  and  $p$ . Hence the problem is ill-posed. In conclusion, if the perturbation of parameters takes the solution beyond the limits then it is an ill-posed problem. This example clearly indicates that merely constructing a solution is not sufficient but other aspects of the solution such as existence and uniqueness, well-posedness, etc., must also be studied.

If the densities of solid and liquid regions are very different then because of the shrinkage or expansion of the region under consideration, the liquid acquires a velocity which should be accounted in the formulation. This appears as a convective term in Eq. (1.3.3) in the liquid. This phenomenon is called natural convection. More general problems with natural convection will be dealt later. At present it may be mentioned that to include this convection term in the analytical solution a simple transformation in Eq. (1.3.3) can be used. Let

$$y = -(1 - \rho_S/\rho_L)S(t) + x. \quad (12.2.6)$$

In terms of  $y$  coordinate, the transformed heat equation is now free of convective term and the solution can be easily obtained as in Eq. (1.3.11) in terms of  $y$  and then  $y$  can be replaced by  $x$ .

### *Green's Functions, Similarity Transformation and Exact Solutions in Planar, Cylindrical, Spherical, Ellipsoidal and Paraboloidal Coordinates*

*Green's function* for the linear parabolic heat equation in the unbounded region  $R^3$  ( $-\infty < x < \infty$ ,  $-\infty < y < \infty$ ,  $-\infty < z < \infty$ ) and  $t \geq t'$  is defined as

$$G(x, y, z, t; x', y', z', t') = \frac{1}{8(\pi k(t - t'))^{3/2}} e^{-\{(x - x')^2 + (y - y')^2 + (z - z')^2\} / 4k(t - t')}, \quad (12.2.7)$$

$k$  is thermal diffusivity and  $(x', y', z')$  could be any point in  $R^3$ . Note that if  $x \neq x'$ ,  $y \neq y'$ ,  $z \neq z'$  and  $t \rightarrow t'$  then  $G = 0$  but when  $x = x'$ ,  $y = y'$ ,  $z = z'$  and then  $t \rightarrow t'$  then  $G$  is infinite. For  $t < t'$ , we take  $G = 0$ .  $G$  is not a function in the usual sense of definition of a function. It is a distribution (see Appendix D). Further property of  $G$  is that

$$\left\{ \int_{-\infty}^{\infty} e^{-\frac{(x - x')^2}{4k(t - t')}} dx \int_{-\infty}^{\infty} e^{-\frac{(y - y')^2}{4k(t - t')}} dy \int_{-\infty}^{\infty} e^{-\frac{(z - z')^2}{4k(t - t')}} dz \right\} / 8(\pi k(t - t'))^{3/2} = 1. \quad (12.2.8)$$

$G$  satisfies heat equation in the following sense.

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} - \frac{1}{k} \frac{\partial G}{\partial t} = \delta(x - x')\delta(y - y')\delta(z - z')\delta(t - t'), \quad (12.2.9)$$

$\delta(x)$  is the Dirac delta function which is zero at all  $x$ ,  $-\infty < x < \infty$ , except at  $x = 0$  where it is infinite. The following function  $T(x, y, z, t)$  in Eq. (12.2.10) can be taken as the temperature distribution in  $R^3$  as it satisfies the heat equation  $\nabla^2 T = \frac{1}{k} \frac{\partial T}{\partial t}$  everywhere in  $R^3$  except when we take the limit at  $(0, 0, 0)$  and limit  $t \rightarrow 0$  simultaneously or in other words four-dimensional limit is taken at  $(0, 0, 0, 0)$ .

$$T(x, y, z, t) = \frac{Q}{8(\pi k t)^{3/2}} e^{-\frac{(x^2 + y^2 + z^2)}{4k t}}, \quad t > 0. \quad (12.2.10)$$

At  $t = 0$ ,  $T(x, y, z, 0) = 0$  except at  $(0, 0, 0)$  in the above-mentioned sense. Further

$$\iiint_{R^3} \rho C T(x, y, z, t) dx dy dz = Q \rho C. \quad (12.2.11)$$

$Q \rho C$  is the quantity of heat in  $R^3$  which is generated instantaneously at  $t = 0$  by a heat source of strength  $Q$  applied at  $(0, 0, 0)$  in  $R^3$  as without any loss of generality we can take, if we like,  $(x', y', z') = (0, 0, 0)$  and  $t' = 0$  in Eq. (12.2.7). Note that Dirac-delta function may not be a function but it has some nice properties. For example,  $f(x)\delta(x)$  for any continuous function  $f(x)$  on integration over  $-\infty < x < \infty$  gives  $f(0)$ .

$G$  is often called a *source solution* or a *fundamental solution*. *Green's function* is also commonly used for it. If the initial temperature in  $R^3$  is  $f(x, y, z)$  then  $\rho C f(x', y', z') dx' dy' dz'$  is the amount of heat present in the elementary volume  $dx' dy' dz'$  at the point  $(x', y', z')$ . It is clear from Eqs (12.2.7), (12.2.10), (12.2.11) that if the initial temperature in  $R^3$  is  $f(x, y, z)$  then the temperature at any point  $(x, y, z, t)$  is given by

$$T(x, y, z, t) = \iiint_{R^3} f(x', y', z') G(x, y, z, t; x', y', z', 0) dx' dy' dz'. \quad (12.2.12)$$

It can be checked that the above temperature satisfies the heat equation and initial temperature is obtained by taking an appropriate limit. Green's function is just the temperature in  $R^3$  for  $t > t'$  due to a heat source of unit strength placed at  $(x', y', z')$  at time  $t = t'$  in an infinite medium whose initial temperature is zero.

It is interesting that Green's function  $G$  is not even a function but it has a nice property that as in Eq. (12.2.12) if it is multiplied by a suitable function and integrated then it gives a function  $T(x, y, z, t)$  which is twice differentiable with respect to space coordinates and once with respect to time.

By using Green's function  $G(x, y, z, t; x', y', z', 0)$ , Green's functions for several other regions can be obtained. To obtain source solution of unit strength in  $(x, y)$  plane  $R^2$ , we integrate  $G(x, y, z, t; x', y', z', 0) dz'$  over  $R^2$ , where  $dz'$  is an infinitesimal element, on a line parallel to  $z$ -axis and passing through the point  $(x', y')$  in the  $(x, y)$  plane. On integration, we get

$$G(x, y, t; x', y', 0) = \frac{1}{4\pi kt} e^{-\{(x-x')^2 + (y-y')^2\}/4kt}, \quad t > 0, -\infty < x, y < \infty. \quad (12.2.13)$$

This is also the temperature distribution in  $R^2$  due to an instantaneous line heat source of strength unity acting at  $t' = 0$ , parallel to  $z$ -axis and passing through the point  $(x', y')$  with initial temperature 0 in  $R^2$ . In an one-dimensional planar case, we consider an instantaneous plane heat source of strength unity applied at  $t' = 0$  which passes through the point  $x = x'$  and is parallel to  $x = 0$  plane. In this case we multiply  $G(x, y, z, t; x', y', z', 0)$  by  $dy' dz'$  and integrate the product over  $-\infty < y' < \infty, -\infty < z' < \infty$ . We get

$$G(x, t; x', 0) = \frac{1}{2\sqrt{\pi kt}} e^{-(x-x')^2/4kt}, \quad t > 0, -\infty < x < \infty. \quad (12.2.14)$$

The quantity of heat liberated per unit area of the plane parallel to  $x = 0$  plane is  $\rho C$ .

To get the source solution in one-dimensional spherical polar coordinates, consider a spherical ball of radius  $r'$  at the point  $(r' = 0, \theta', \phi')$  where  $(r, \theta, \phi)$  are spherical polar

coordinates of a point in  $R^3$ . Now consider a point source of strength  $r'^2 \sin \theta' d\theta' d\phi'$  on the surface of this ball of radius  $r'$  at the point  $(r', \theta', \phi')$  at time  $t' = 0$ . If such point sources are put over the whole surface of the sphere then the strength of the heat source will be obtained by integrating  $r'^2 \sin \theta' d\theta' d\phi'$  over the surface. The temperature  $T(r, t)$  at the point  $(r, 0, 0)$  can be obtained by first obtaining  $G(r, \theta, \phi, t; r', \theta', \phi', 0)$  from Eq. (12.2.7), then putting  $\phi = 0, \theta = 0$  in it. The temperature  $T(r, t)$  is obtained as follows.

$$T(r, t) = \frac{r'^2}{8(\pi kt)^{3/2}} \int_{-\pi}^{\pi} d\phi' \int_0^{\pi} e^{-(r^2+r'^2-2rr' \cos \theta')/4kt} \sin \theta' d\theta', \quad (12.2.15)$$

$$\begin{aligned} &= \frac{r'^2}{4\sqrt{\pi}(kt)^{3/2}} e^{-(r^2+r'^2)/4kt} \int_0^{\pi} e^{rr' \cos \theta'/2kt} \sin \theta' d\theta', \\ &t > 0, 0 < r, r' < \infty, \\ &= \frac{Q'}{8\pi rr'(\pi kt)^{3/2}} \left[ e^{-(r-r')^2/4kt} - e^{-(r+r')^2/4kt} \right], \quad Q' = 4\pi r'^2. \end{aligned} \quad (12.2.16)$$

Note that  $T(r, t) = 4\pi r'^2 G(r, t; r', 0)$  is the temperature in Eq. (12.2.16) due to heat source of strength  $4\pi r'^2$  and not of strength unity.  $4\pi r'^2$  is the surface area of the ball of radius  $r'$ . The Green's function  $G(r, t; r', 0)$  is given by

$$G(r, t, r', 0) = T(r, t)/4\pi r'^2; \int_0^{\infty} 4\pi r^2 G dr = 1; \int_0^{\infty} 4\pi r'^2 G dr' = 1. \quad (12.2.17)$$

$$\left. \frac{\partial G}{\partial r} \right|_{r=0} = 0, \left. \frac{\partial G}{\partial r'} \right|_{r'=0} = 0; G|_{r \rightarrow \infty} = 0, G|_{r' \rightarrow \infty} = 0; G(r, t)|_{r=0} = 0, \quad t > 0. \quad (12.2.18)$$

Green's function  $G(r, t; r', 0)$  satisfies the following equation

$$\frac{\partial^2 G}{\partial r^2} + \frac{2}{r} \frac{\partial G}{\partial r} - \frac{1}{k} \frac{\partial G}{\partial t} = -\frac{\delta(r-r')\delta(t)}{4\pi rr'}, \quad t \geq 0, 0 < r, r' < \infty. \quad (12.2.19)$$

$T(r, t)$  and  $T(r', t)$  both satisfy the equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} - \frac{1}{k} \frac{\partial T}{\partial t} = 0, \quad t > 0, 0 < r, r' < \infty, \quad (12.2.20)$$

$r$  can be replaced by  $r'$  in Eq. (12.2.20). Eq. (12.2.20) is satisfied by the following function given in Eq. (12.2.21) also. Hence it is also temperature and is denoted by  $T(r, t)$ .

$$T(r, t) = \frac{(kt)^{1/2}}{r} \exp(-r^2/4kt) - \frac{1}{2} \pi^{1/2} \operatorname{erfc} \frac{r}{2\sqrt{kt}}, \quad t > 0, 0 \leq r < \infty. \quad (12.2.21)$$

The above solution does have singularity at  $r = 0$ .  $T(r, t)$  in Eq. (12.2.21) is suitable for obtaining analytical solutions when the region does not include  $r = 0$ . This is the only solution of Eq. (12.2.20) of its kind.

To obtain temperature or Green's function in cylindrical polar coordinates  $(r, 0, 0)$ , that is, in terms of a single radial coordinate  $r$  which is obtained by taking  $\theta = 0$  and  $z = 0$  in the cylindrical coordinates  $(r, \theta, z)$ , we look for the Green's function  $G$  given in Eq. (12.2.13)

(take  $z = 0$  in Eq. 12.2.10) and obtain it in terms of cylindrical coordinates  $(r, \theta; r', \theta')$  and put  $\theta = 0$  in  $G(r, \theta, t; r', \theta', 0)$ . At  $t' = 0$ , apply an instantaneous heat source of strength unity at some point along a cylindrical surface of radius  $r'$  and axis along the  $z$ -axis. If a heat source of unit strength is applied at each point of an arc of length  $r' d\theta'$  taken on a circle of radius  $r'$  then the strength of the heat source will be  $r' d\theta'$ . If the heat source is applied along the whole circular length then the strength of the source will be  $2\pi r'$ . The temperature  $T(r, t)$  is given by

$$T(r, t) = 2\pi r' G(r, 0, t; r', \theta', 0)$$

$$= \frac{2\pi r'}{4(\pi kt)} \int_0^{2\pi} e^{-(r^2 + r'^2 - 2rr' \cos \theta')/4kt} d\theta', \quad (12.2.22)$$

$$= \frac{2\pi r'}{4\pi kt} e^{-\frac{(r^2 + r'^2)}{4kt}} \int_0^{2\pi} e^{2rr' \cos \theta' / 4kt} d\theta',$$

$$= 2\pi r' / (4\pi kt) e^{-(r^2 + r'^2)/4kt} I_0 \left( \frac{rr'}{2kt} \right) = 2\pi r' G(r, t; r', 0). \quad (12.2.23)$$

In Eq. (12.2.22),  $I_0$  is the Bessel function of second kind of order zero.

The solutions of the heat equations in Neumann solution have been constructed as linear combinations of source solutions. The functions  $\operatorname{erf} \eta$  and  $\operatorname{erfc} \eta$ ,  $\eta = x/(2\sqrt{kt})$ , are both source solutions and satisfy linear heat parabolic equation in one-dimensional planar case. More general solutions of heat equation Eq. (1.3.1) also exist. For each integer  $n$ ,  $n \geq 0$ , the functions

$$t^{n/2} i^n \operatorname{erfc}(\eta) = t^{n/2} \int_{\eta}^{\infty} i^{n-1} \operatorname{erfc}(\xi) d\xi, \quad i^0 \operatorname{erfc} \eta = \operatorname{erfc} \eta, \quad \eta = x/(2\sqrt{kt}), \quad (12.2.24)$$

satisfy Eq. (1.3.1). These functions are called *iterated error functions* or *repeated integrals of error functions*.

If Eq. (1.3.1) is written in terms of the variable  $\eta = x/(2\sqrt{kt})$  which is *Boltzman transformation* then it is transformed to

$$\frac{d^2 T_S}{d\eta^2} + 2\eta \frac{\partial T_S}{\partial \eta} = 0. \quad (12.2.25)$$

Eq. (12.2.25) is an ordinary differential equation whose solution can be easily obtained. Other boundary conditions in Neumann problem can also be written in terms of a single variable  $\eta$ . There were two independent variables  $x$  and  $t$ , which are reduced to one variable  $\eta$ . The functional form of  $S(t)$  can be easily thought of as isotherm conditions on  $x = S(t)$  are to be satisfied. Suppose the boundary condition at  $x = 0$  in Neumann problem is changed to nonzero flux prescribed boundary condition or temperature at  $x = 0$  is prescribed as a function of time then such type of boundary conditions cannot be prescribed in terms of a single variable  $\eta$ . Note that even if all the equations and boundary conditions except at  $x = 0$  can be expressed in terms of a single variable  $\eta$ , the solution of the problem cannot be expressed in terms of a single variable  $\eta$ . If all the equations in the problem formulation can be expressed in terms of a single variable, say  $\eta$ , and the solution can also be obtained in terms of this single variable then we say  $\eta$  is a *similarity variable* and a similarity solution exists for the problem or a *similarity transformation* exists for the solution of the problem.

This definition of similarity variable and of similarity solution is not restricted to functions of only two variables. Suppose  $W = f(x_1, x_2, x_3, t)$  is a function of four independent variables



and a variable  $\eta = g(x_2, t)$  is defined so that the complete mathematical formulation of the problem can be expressed in terms of only three variables  $x_1$ ,  $x_3$  and  $\eta$  then we can say that a similarity transformation exists for the problem and  $\eta$  is a similarity variable. It seems pertinent to remark here that if independent variables are more than two then whilst defining a new variable in terms of any two independent variables, it is not necessary that a function or equation is reduced to less number of independent variables. In such a situation more number of similarity variables are to be defined to reduce the number of independent variables (cf. [359]). This process can continue for further reduction of independent variables, if possible. If the solution also exists in terms of  $x_1$ ,  $x_3$  and  $\eta$  then a similarity solution exists under the similarity transformation. We shall see later that similarity variables other than  $\eta$  given in Eq. (12.2.24) also exist.

Let us consider the following heat equation in one-dimensional cylindrical polar coordinates  $(r, 0, 0)$ ,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} = \frac{1}{k} \frac{\partial T}{\partial t}, \quad 0 \leq r \leq r_0, \quad t > 0, \quad r_0 < \infty. \quad (12.2.26)$$

The only solution of its kind which satisfies Eq. (12.2.26) is

$$T(r, t) = \int_{r^2/4kt}^{\infty} \frac{e^{-p}}{p} dp = -Ei\left(\frac{-r^2}{4kt}\right) = -Ei(-\eta^2), \quad \eta = \frac{r}{2\sqrt{kt}}. \quad (12.2.27)$$

If at  $r = r_0$ , where  $r_0$  is a constant, some boundary condition such as temperature  $T(r, t)$  is prescribed then temperature cannot be prescribed at  $r = r_0$  as some value of  $\eta$ ,  $\eta$  as defined in Eq. (12.2.27). Although the heat equation (12.2.26) on using  $\eta$  is transformed to

$$\eta \frac{d^2 T}{d\eta^2} + (1 + 2\eta^2) \frac{dT}{d\eta} = 0, \quad (12.2.28)$$

$\eta$  is not a similarity variable for the problem with temperature prescribed at  $r = r_0$  and a similarity solution does not exist. However, we shall see later that in some special types of Stefan problems it is possible to obtain similarity solutions in cylindrical regions. Most of the exact solutions of Stefan problems are similarity solutions in which a systematic method of solution is involved. However, exact solutions of Stefan problems in which similarity variables have not been used for solution also exist and invariably no systematic method of solution can be assigned to them such as in the solution described in Eqs (12.2.3)–(12.2.5) and many other solutions. Only those solutions in which a systematic method of analytical solution is involved, the solution could be exact or not, will be presented in this chapter as our emphasis is on methods.

Having discussed some preliminaries, some methods of solutions of Stefan problems will now be discussed. Exact analytical solutions will now be discussed and problems in Cartesian coordinates will be taken up first. In a small 11th chapter in [24] which is devoted to phase-change problems, the solutions of Neumann problem and many other Stefan problems have been given in closed form. Each of these solutions is important and useful in its own way. Since the Neumann solution has been discussed at length, it seems reasonable to leave these solutions to the readers to look into as the method of solution is similar to that discussed earlier.

The integral equation method suggested by Lightfoot and reported in [24] has some novelty. The problem being considered now is similar to Neumann problem considered in Eqs (1.3.1)–(1.3.7) in which all the thermo-physical parameters in both the solid and liquid

phases are taken the same as the Lightfoot's method of solution is applicable for this case only. The latent heat released in solidification at moving boundary is at the rate  $\rho l \frac{dS}{dt}$  and can be thought of as liberated due to a moving heat source at the moving boundary with strength of the source  $\rho l \frac{dS}{dt}$ . The temperature  $W(x, t)$  due to this moving heat source in the half space  $x > 0$  whose initial temperature is zero and surface  $x = 0$  is maintained at zero temperature is given by

$$W(x, t) = \frac{l}{2C(\pi k)^{1/2}} \int_0^t \frac{\frac{dS}{dt} \big|_{t'}}{(t-t')^{1/2}} \left\{ e^{-\frac{(x-S(t'))^2}{4k(t-t')}} - e^{-\frac{(x+S(t'))^2}{4k(t-t')}} \right\}. \quad (12.2.29)$$

The second term in Eq. (12.2.29) has been included to get  $W(0, t) = 0$ . It can be checked that  $W(x, t)$  satisfies linear heat equation. Here it may be mentioned that corresponding to Green's function  $G(x, t; x', t')$  there is a Green's function which is the mirror image of  $G$  in  $x = 0$ . The second term in Eq. (12.2.29) is the mirror image of the first term in  $x = 0$ . There is also an adjoint Green's function. If the *adjoint Green's function* is denoted by  $G_1$  then

$$G_1(x, t; x', t') = \frac{1}{2\sqrt{\pi k(t'-t)}} e^{-\frac{(x-x')^2}{4k(t'-t)}}, \quad t' > t; \quad G_1 = 0 \text{ for } t' < t, \quad (12.2.30)$$

and

$$\frac{\partial^2 G_1}{\partial x^2} + \frac{1}{k} \frac{\partial G_1}{\partial t} = \delta(x' - x) \delta(t' - t), \quad -\infty < x, \quad x' < \infty. \quad (12.2.31)$$

Let the initial temperature of the liquid be  $V > T_m$ , where  $T_m$  is the melting temperature, and at  $x = 0$  temperature  $T_0 < T_m$  is prescribed. Then  $U(x, t)$ , the temperature of the liquid in  $x \geq 0$  for a heat conduction problem without phase change is given by

$$U(x, t) = T_0 + V \operatorname{erf} \left( \frac{x}{2\sqrt{kt}} \right). \quad (12.2.32)$$

If  $T(x, t)$  is the temperature of the original solidification problem then it is given by  $T(x, t) = W(x, t) + U(x, t)$ . At  $x = S(t)$ , we have

$$\{W(x, t) + U(x, t)\}_{x=S(t)} = T_m, \quad t > 0. \quad (12.2.33)$$

If the moving boundary is taken as  $x = S(t) = 2\delta(kt)^{1/2}$  and substituted in Eq. (12.2.29) together with the substitutions

$$y = x/S(t), \quad t' = t \left( \frac{1-z^2}{1+z^2} \right)^2, \quad (12.2.34)$$

then the integral in Eq. (12.2.29) can be evaluated for  $y < 1$  and  $y > 1$ .  $\delta$ , which is the unknown constant, can be determined by satisfying Eq. (12.2.33) and solving a transcendental equation. This solution for  $W(x, t)$  is valid for all  $x$ ,  $0 \leq x < \infty$ , so the parameters in both the solid and liquid phases have to be the same. This moving heat source method of Lightfoot has been used by other authors also such as in [360].

Some exact analytical solutions of Stefan problems in planar problems in which heat generation terms have been included in the formulation will now be discussed. Exact analytical solutions of Stefan-like problems in which heat generation terms have been included in the formulation will be discussed in [Section 12.2.2](#). Exact analytical solutions of Stefan problems in which a suitable method of solution could be ascribed are mostly similarity solutions in which the moving boundary is of the form  $S(t) = \delta t^{1/2}$ , where  $\delta$  is an unknown positive real constant or  $S(t)$  is just a constant. The thermo-physical parameters, boundary conditions at the fixed boundaries, initial conditions and conditions at the phase-change front are chosen in such a way that a similarity solution can be obtained.

In [361], volumetric heat sources in both the solid and liquid phases are present in the formulation of Stefan problem considered. The complete formulation of the problem is not being given here as it is similar to the formulation of Neumann problem with some changes which are mentioned here. The region  $0 \leq x < \infty$  is initially occupied by a solid with a constant initial temperature  $T_0$  less than the melting temperature which is taken as zero. Melting of the solid starts at  $x = 0$  and this is due to temperature  $T_1$  prescribed at  $x = 0$  which is greater than the melting temperature. Analytical solutions have been obtained for temperature prescribed condition at  $x = 0$  and also for flux prescribed at  $x = 0$  in which flux is of the form

$$K_L \frac{\partial T_L}{\partial x}(0, t) = -q_0/\sqrt{t}, \quad t > 0, \quad q_0 \text{ is a given constant } > 0. \quad (12.2.35)$$

Heat sources in the two solid and liquid regions have been taken in the form

$$g_{S,L}(x, t) = \frac{1}{t} \beta_{S,L} \left( \frac{x}{2\sqrt{t}} \right), \quad S \text{ and } L \text{ stand for the solid and liquid, respectively.} \quad (12.2.36)$$

In Eq. (12.2.36), all the occurring parameters (cf. [361]) have been taken here to be unity. This is done presently just for convenience and this does not in any way affect the method of solution which is of prime importance in this chapter. In the subsequent discussion of the solutions also parameters will be taken as unity except where they affect the nature of solution or the method of solution. In Eqs (12.2.35), (12.2.36), the functional forms of prescribed flux and source terms are typically chosen for obtaining similarity solutions. To obtain a similarity solution, we take

$$S(t) = 2k_L \lambda t^{1/2}, \quad \lambda > 0 \text{ is a real unknown positive constant.} \quad (12.2.37)$$

When the heat equations are considered in terms of the similarity variable  $\eta$ , we get

$$\frac{d^2 T_{S,L}}{d\eta^2} + 2\eta \frac{\partial T_{S,L}}{\partial \eta} = \beta_{S,L}(\eta), \quad \eta = x/(2\sqrt{t}). \quad (12.2.38)$$

All the thermo-physical parameters have been taken as unity in Eq. (12.2.38) for convenience. The l.h.s. in Eq. (12.2.38) is the same as in Eq. (12.2.25). Eq. (12.2.38) can be obtained easily in the same way as the heat equation (1.3.1) is transformed in terms of the similarity variable. The solution of the nonhomogeneous linear equation (12.2.38) is simple to obtain. The constants occurring in the solution of Eq. (12.2.38) can be determined by following the earlier procedure adopted in Neumann solution as the boundary conditions and interface conditions can be prescribed in terms of  $\eta$ . The solution involves lengthy calculations and big expressions. Refer [361] for complete solution. To get some idea about the type of solutions

obtained, the solution of  $T_S(\eta)$  in the case of temperature prescribed boundary condition is given below with parameters.

$$T_S(\eta) = \frac{-(-T_0 + \phi_1(+\infty))}{\operatorname{erfc}(k_L \lambda / k_S)} \frac{2}{\sqrt{\pi}} \int_{k_L \lambda / k_S}^{k_L \eta / k_S} \exp(-p^2) dp + \phi_1(\eta), \quad \eta > \lambda, \quad T_0 < 0, \quad (12.2.39)$$

$$\phi_1(\eta) = \frac{2l\sqrt{\pi}}{C_S} \int_{k_L \lambda / k_S}^{k_L \eta / k_S} \beta_S(p) \exp(p^2) [\operatorname{erf}(p) - \operatorname{erf}(k_L \eta / k_S)] dp, \quad \eta > \lambda. \quad (12.2.40)$$

The parameters  $k, l, C$  follow the notations followed in the earlier chapters of this book. The analytical solution for the case of flux prescribed condition given in Eq. (12.2.35) has also been obtained. As mentioned earlier, the unknown constant  $\lambda$  in Eq. (12.2.37) is obtained generally as a real positive root of a complicated transcendental equation. The authors in [361] have taken pains to prove the existence and uniqueness of  $\lambda$  by taking into account different parameters and behaviour of complicated functions occurring in the solutions. Proofs are lengthy and cannot be presented here. When  $\beta_{S,L}(\eta)$  has a form as given below then integrals in Eqs (12.2.39), (12.2.40) and in other solutions (not reported here) can be integrated and closed-form solutions have been obtained.

$$\beta_{S,L}(\eta) = \exp\left(-(\eta_{S,L} + d_{S,L}^2)\right), \quad d_{S,L} \text{ are constants, } \eta_{S,L} = x/(2\sqrt{k_{S,L}t}). \quad (12.2.41)$$

In a one-dimensional two-phase planar melting problem, variable latent heat has been considered in [362]. At time  $t = 0$ , a solid occupies the region  $0 \leq x < \infty$ . The heat equations considered in the two regions are of the same type as in Eq. (1.3.1) so they are not being given here. The other prescribed conditions are as follows.

$$K_L \frac{\partial T_L(0, t)}{\partial x} = -q_2, \quad q_2 > 0, \quad t > 0; \quad K_S \frac{\partial T_S}{\partial x} \Big|_{x \rightarrow \infty} = -q_1, \quad t > 0. \quad (12.2.42)$$

$$T_S(x, 0) = -q_1 x / K_S, \quad x > 0, \quad q_1 > 0; \quad T_S|_S = T_L|_S = 0, \quad t \geq 0. \quad (12.2.43)$$

$$K_S \frac{\partial T_S}{\partial x} - K_L \frac{\partial T_L}{\partial x} = r S(t) \frac{dS}{dt}, \quad \text{on } x = S(t); \quad S(0) = 0. \quad (12.2.44)$$

Latent heat  $l$  in Stefan condition given in Eq. (12.2.44) has been taken as proportional to  $x$  which is of the form  $l = rx$ , where  $r$  is a known constant and accordingly we have Stefan condition as in Eq. (12.2.44). For both the phases, density is taken the same but thermal conductivities and specific heats are different in different phases. Let  $T(x, t)$  be the solution of the heat equation  $T_t = kT_{xx}$ . To obtain analytical solution of this present problem a slightly different approach is followed. The temperature  $T(x, t)$  is expressed as

$$T(x, t) = 2\sqrt{l\eta}(\xi), \quad \xi = x/(2\sqrt{t}); \quad T_t = kT_{xx}, \quad t > 0, \quad 0 < x < \infty, \quad (12.2.45)$$

where  $T_t = kT_{xx}$  is the heat equation considered in this problem for both phases and  $k_S$  and  $k_L$  are the diffusivities in the solid and liquid regions, respectively. The following ordinary differential equation can be easily obtained when  $T(x, t)$  given in Eq. (12.2.45) is substituted in the heat equations.

$$(k/2)\eta''(\xi) + \xi\eta'(\xi) - \eta(\xi) = 0, \quad k = k_S, k_L \text{ for solid and liquid.} \quad (12.2.46)$$

The authors first prove a result that  $T(x, t)$  given in Eq. (12.2.45) is a solution of heat equation if and only if  $\eta = \eta(\xi)$  is a solution of Eq. (12.2.46) and this solution of Eq. (12.2.46) is given by

$$\eta(\xi) = -d_1 \left[ e^{-\xi^2/k} + \sqrt{\frac{\pi}{k}} \xi \operatorname{erf} \frac{\xi}{\sqrt{k}} \right] + d_2 \xi, \quad (12.2.47)$$

where  $d_1$  and  $d_2$  are some arbitrary real constants. Four arbitrary constants will appear in the solutions of heat equations for solid and liquid regions and fifth unknown constant is  $\delta$  in  $S(t) = 2\delta\sqrt{k_L t}$ . Five conditions are available to determine them. The procedure for determining these constants will not be repeated here as it has been explained earlier.

Another result which has been proved in [362] is that if  $0 < q_1 < q_2$  then a unique solution of the two-phase problem in which latent heat is proportional to  $x$  is obtained by the above procedure. The proof involves the behaviour of functions occurring in the equation obtained to determine  $\delta$ . The unique solution of the above problem, when  $q_1 = 0$  is taken in Eq. (12.2.43), can also be obtained by the above procedure. In this case  $T_S(x, t) = 0$ ,  $x > S(t)$ ,  $t > 0$  and we have a one-phase problem.

Explicit solutions of one-dimensional two-phase planar Stefan problem with either shrinkage or expansion have been obtained in [363]. Consider an infinite region  $0 \leq x \leq \infty$  which initially at time  $t = 0$  is occupied by a liquid. The solidification of liquid starts at  $x = 0$  by imposing appropriate heat flux at  $x = 0$ . Because of the difference in the densities of solid and liquid phases, the original region may expand or shrink. In [363] it has been assumed that liquid phase remains at rest and the solid phase expands or shrinks. The problem formulation is following.

$$k_S \frac{\partial^2 T_S}{\partial x^2} = \frac{\partial T_S}{\partial t} + r_0 \frac{dS}{dt} \frac{\partial T_S}{\partial x}, \quad r_0 S(t) < x < S(t), \quad r_0 = (1 - \rho_L/\rho_S), \quad t > 0, \quad (12.2.48)$$

subscript  $S$  stands for solid and  $L$  for liquid,  $x = S(t)$  is the moving boundary.

$$k_L \frac{\partial^2 T_L}{\partial x^2} = \frac{\partial T_L}{\partial t}, \quad x > S(t), \quad t > 0; \quad k \text{ is thermal diffusivity.} \quad (12.2.49)$$

$$T_L(\infty, t) = T_L(x, 0) = B > T_m, \quad T_m \text{ is melting temperature.} \quad (12.2.50)$$

$$K_S \frac{\partial T_S}{\partial x}(r_0 S(t)) = q_0/\sqrt{t}, \quad t > 0, \quad q_0 > 0. \quad (12.2.51)$$

$$T_S(r_0 S(t), t) = A, \quad t > 0, \quad A < T_m < B. \quad (12.2.52)$$

$$K_S \frac{\partial T_S}{\partial x}(r_0 S(t)) = \frac{h_0}{\sqrt{t}} (T_S(r_0 S(t)) - U_0), \quad U_0 < T_m. \quad (12.2.53)$$

$$T_S(S(t), t) = T_L(S(t), t) = T_m; \quad K_S \frac{\partial T_S}{\partial x}(S(t), t) = K_L \frac{\partial T_L}{\partial x}(S(t), t) + \rho_S l \frac{dS}{dt}. \quad (12.2.54)$$

$U_0$  is the ambience temperature in Eq. (12.2.53). Any of the boundary conditions given in Eqs (12.2.51)–(12.2.53) can be prescribed at  $x = r_0 S(t)$ . These conditions at  $x = r_0 S(t)$  can be prescribed independently of each other but if the constants  $q_0$ ,  $A$ ,  $B$  and  $U_0$  are suitably chosen then the solutions of these three different problems in which only the conditions at  $x = r_0 S(t)$  are different, are related to each other, refer [363].

We first take up the solution of problem in which only Eq. (12.2.52) is prescribed together with Eqs (12.2.48)–(12.2.54) leaving Eqs (12.2.51), (12.2.53). Define a new variable  $y$  and express  $S(t)$  as given below and then substitute them in Eqs (12.2.48)–(12.2.54).

$$y = x/2\delta\sqrt{k_L t}; \quad S(t) = 2\delta\sqrt{k_L t}, \quad t > 0, \quad 0 < y < \infty, \quad \delta > 0. \quad (12.2.55)$$

If  $T_S(x, t) = \theta_S(y)$ ,  $r_0 < y < 1$  and  $T_L(x, t) = \theta_L(y)$ ,  $y > 1$ , then  $\theta_S(y)$  and  $\theta_L(y)$  satisfy the following ordinary differential equations.

$$k_S/(2\delta^2 k_L)\theta_S'' + (y - r_0)\theta_S' = 0, \quad r_0 < y < 1; \quad \theta_L'' + 2\delta^2 y\theta_L' = 0, \quad y > 1, \quad (12.2.56)$$

dash denotes differentiation with respect to  $y$ . The equations in Eq. (12.2.56) are of the same type as Eq. (12.2.25) and their analytical solutions can be easily obtained. All the prescribed conditions can be expressed in terms of values of  $y$  at  $y = 0$  or at  $y = 1$  and so similarity solutions for all the three problems with one of the three conditions given in Eqs (12.2.51)–(12.2.53) can be obtained independently of each other. We shall see later that if we choose  $A$ ,  $q_0$  and  $U_0$  appropriately in these equations then the three solutions are related to each other. The steps for determining the unknown constants will not be presented here as it is a known procedure but just to give some idea about the type of solutions,  $\theta_S(y)$ , ( $r_0 < y < 1$ ) is given below for the condition in Eq. (12.2.52).

$$\theta_S(y) = A + (T_m - A)\text{erf}(\sqrt{k_L/k_S}(y - r_0)\delta)/\text{erf}\left(\sqrt{k_L/k_S}(1 - r_0)\delta\right). \quad (12.2.57)$$

The authors have established the existence and uniqueness of the solutions of three problems corresponding to three different boundary conditions in Eqs (12.2.51)–(12.2.53) by proving that  $\delta$  in each case can be obtained as a unique positive real root of the transcendental equation obtained by satisfying the Stefan condition. The method of proof depends on the behaviour of functions which occur in the solutions for  $x > 0$ . In the case of flux and convective conditions (12.2.51), (12.2.53), respectively, for uniqueness of solutions it is required that conditions given below are satisfied.

$$q_0 > K_L(B - T_m)/\sqrt{\pi k_L}, \quad \text{in flux case.} \quad (12.2.58)$$

$$h_0 > K_L(B - T_m)/\left(\sqrt{\pi k_L}(T_m - U_0)\right), \quad \text{in convective case.} \quad (12.2.59)$$

From the solution of flux prescribed case, the temperature at  $x = r_0 S(t)$  can be calculated which turns out to be a constant and not a function of time. The constant is in the form of an expression. If this expression is taken as  $A$  in Eq. (12.2.52) then we get the solution which is the same as we get from the flux condition given in Eq. (12.2.51). This establishes the equivalence of these two problems for a particular value of  $A$ . Under certain conditions, the equivalence of all the three solutions obtained under three different conditions given in Eqs (12.2.51)–(12.2.53) can be established and for this refer [363] as the proof involves several functions.

In the solutions obtained earlier we have seen that in the presence of source terms in the formulation, exact analytical solutions are possible provided source terms are of some special type so that a similarity solution is possible. If the source terms are arbitrarily prescribed then similarity solution may not be possible. For example in the oxygen-diffusion problem (ODP) given in Eqs (3.3.34)–(3.3.38), oxygen is absorbed at a constant rate and the problem is a

simple one-phase problem but no exact analytical solution could be obtained. The implicit type of Stefan condition is not the cause of this nonexistence of exact solution in ODP as the implicit-type Stefan condition can be converted to an explicit type of boundary condition.

Consider the following *Baiocchi transformation* [74] which converts implicit condition in Eq. (3.3.38) to an explicit Stefan condition.

$$\bar{c}(x, t) = \bar{c}_0(x) + \int_0^t T(x, t') dt', \quad T(x, t) = \bar{c}_t. \quad (12.2.60)$$

$c(x, t)$  in ODP and  $\bar{c}(x, t)$  here both denote concentration. Using Eq. (12.2.60) in ODP formulation and identifying  $T(x, t)$  with temperature, we get

$$T_t - T_{xx} = 0, \quad 0 < x < S(t), \quad t > 0; \quad T_x = 0, \quad \text{at } x = 0, \quad t > 0. \quad (12.2.61)$$

$$T = -\delta(0), \quad t = 0, \quad 0 \leq x \leq 1; \quad T = 0, \quad \frac{dS}{dt} = -T_x, \quad \text{on } x = S(t), \quad t > 0; \quad S(0) = 1, \quad (12.2.62)$$

$\delta(\cdot)$  is the Dirac-delta function. No exact analytical solution of Eqs (12.2.61), (12.2.62) exists although now Stefan condition is of explicit type. If the initial condition and boundary condition at  $x = 0$  are suitably changed then the following functions satisfy Eqs (12.2.61), (12.2.62).

$$T(x, t) = \exp(x + t - 1) - x - t, \quad S(t) = 1 - t, \quad x < S(t), \quad S(0) = 1. \quad (12.2.63)$$

The solution in Eq. (12.2.63) is given in [364], which has been used to check the results of a numerical procedure developed there. It may be noted that in obtaining Eq. (12.2.63) the actual prescribed conditions have to be changed to obtain the solution and no systematic method of solution can be assigned as in earlier problems. As mentioned in Section 3.3.1 the implicit boundary conditions at the moving interface can be converted to explicit boundary conditions under some assumptions with the help of some transformations which may help in obtaining analytical and numerical solutions. The problems with implicit boundary condition at the moving interface need detailed analysis as the solution may below up, refer Section 3.3.2 and [104] for it.

Some Stefan-like problems in Cartesian coordinates will be discussed in Section 12.2.2 and some quasisteady-state and steady-state problems will be discussed in Section 12.2.4.

It is difficult to obtain exact solutions in two- and three-dimensional cylindrical regions so the focus will be on Stefan problems in a single radially symmetric cylindrical coordinate system  $(r, 0, 0)$ . In cylindrical regions the analytical solutions are further restricted as the general solution of the heat equation (12.2.26), which is given in Eq. (12.2.27), is the only solution of its kind, which can be used for obtaining similarity solutions. If the cylindrical region is finite and enclosing  $r = 0$  then to obtain exact analytical similarity solution no other boundary condition can be prescribed at the finite boundary in terms of  $\eta = r/(2\sqrt{kt})$  except when  $r = \delta\sqrt{t}$ . This essentially means that we are looking either for solidification or a melting problem in which similarity solutions  $r = S(t) = \delta\sqrt{t}$  can be obtained. How will the freezing or melting start? Since at the finite boundary no boundary condition in terms of temperature or flux can be prescribed which starts phase change we have to consider line heat sources or sinks at  $r = 0$  and the finite boundary of the region is considered a moving boundary.

An exact analytical solution for a Stefan problem in a two-phase one-dimensional cylindrical region is given in [24]. The region  $0 \leq r < \infty$  is initially at  $t = 0$  occupied by a liquid at temperature  $V$ ,  $V > T_m$ . Solidification initiates due to a line heat sink at  $r = 0$ , along the axis of the cylinder which extracts heat at the constant rate  $Q$  per unit time for  $t > 0$ . Let  $r = R(t)$  be the phase-change boundary,  $T_m$  be isotherm temperature,  $0 \leq r < R(t)$  be the solid region and  $r > R(t)$  is occupied by the liquid for  $t > 0$ . The heat equations in both the regions are of the type of Eq. (12.2.26) with thermal diffusivities  $k_S$  and  $k_L$ , and the subscript  $S$  stands for solid and  $L$  for the liquid. At the boundary  $r = R(t)$ , we have

$$T_L = T_S = T_m; \quad K_S \frac{\partial T_S}{\partial r} - K_L \frac{\partial T_L}{\partial r} = l\rho_S \frac{dR}{dt}, \quad t > 0, \quad \text{on } r = R(t), \quad (12.2.64)$$

$K$  is thermal conductivity,  $l$  is latent heat and  $\rho$  is density. If  $R(t)$  is taken as  $r = R(t) = 2\delta\sqrt{k_S t}$ , then using the solution in Eq. (12.2.27), the temperature solutions in the solid and liquid region can be taken as given below:

$$T_S = T_m + \frac{Q}{4\pi K_S} \left\{ Ei\left(\frac{-r^2}{4k_S t}\right) - Ei(-\delta^2) \right\}, \quad 0 \leq r < R(t), \quad t > 0. \quad (12.2.65)$$

$$T_L = V - \frac{(V - T_m)}{Ei(-\delta^2 k_S/k_L)} Ei\left(\frac{-r^2}{4k_L t}\right), \quad r > R(t), \quad t > 0. \quad (12.2.66)$$

$\delta$  is a real positive root of the equation

$$\frac{Q}{4\pi} e^{-\delta^2} + \frac{K_L(V - T_m)}{Ei(-\delta^2 k_S/k_L)} e^{-\delta^2 k_S/k_L} = \delta^2 K_S l \rho_S. \quad (12.2.67)$$

It can be checked that the following conditions are satisfied.

$$\lim_{r \rightarrow 0} \left( 2\pi r K_S \frac{\partial T_S}{\partial r} \right) = Q, \quad Q < 0 \text{ is constant, } t > 0; \quad T_L|_{r \rightarrow \infty} = V. \quad (12.2.68)$$

It may be pointed out here that in the Neumann solution two independent source solutions were available and the solution could be written as a linear combination of two independent solutions. In the present case we have to think how to satisfy the boundary conditions knowing that the solution has to be of the form in Eq. (12.2.27).

If the liquid in the above problem is taken as supercooled with temperature  $V < T_m$  and occupying the region  $S(t) \leq r < \infty$ , then to start solidification, the region  $0 \leq r < S(t)$  can be taken as solid at temperature  $T_m$ . By following the above procedure, solution of this problem can be easily obtained in which  $T_S(r, t) \equiv T_m$ , for  $0 \leq x \leq S(t)$  and solidification is growing in the supercooled liquid. By following the above procedure solutions of several simple one-dimensional Stefan problems can be obtained which will not be discussed here. Some Stefan-like problems in cylindrical coordinates will be discussed in Section 12.2.2 and quasisteady-state and steady-state solutions in Section 12.2.4.

It is difficult to obtain exact solutions in two- and three-dimensional spherical polar coordinate  $(r, \theta, \phi)$ . Some exact analytical solutions of one-dimensional radially symmetrical spherical problems in which the heat equation is of the form of Eq. (12.2.20) are discussed below.



If a variable  $\eta$  is defined as

$$\eta = r/2\sqrt{kt}, \quad 0 \leq r < \infty, t > 0, \quad (12.2.69)$$

then Eq. (12.2.20) is transformed into the following equation in terms of  $\eta$

$$\frac{d^2 T}{d\eta^2} + 2\left(\eta + \frac{1}{\eta}\right) \frac{dT}{d\eta} = 0, \quad 0 < \eta < \infty. \quad (12.2.70)$$

The solution of Eq. (12.2.70) can be expressed as

$$T(\eta) = T(r, t) = A \int_{\eta}^{\infty} \frac{e^{-p^2}}{p^2} dp + B, \quad A \text{ and } B \text{ are unknowns} \quad (12.2.71)$$

$$= A \left\{ \frac{e^{-\eta^2}}{2\eta} - \frac{\sqrt{\pi}}{2} \operatorname{erfc} \eta \right\} + T(r, t)|_{r \rightarrow \infty}. \quad (12.2.72)$$

We now consider a two-phase freezing problem due to a continuous surface heat sink at  $r = 0$  for  $t > 0$  such that

$$\lim_{r \rightarrow 0} \left( 4\pi K_S r^2 \frac{\partial T_S}{\partial r} \right) = Q_S(t) = Q_0/\sqrt{t}, \quad Q_0 < 0 \text{ is a constant.} \quad (12.2.73)$$

The heat energy equations in the two phases, initial condition and interface condition, are as follows.

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T_S}{\partial r} \right) = \frac{1}{k_S} \frac{\partial T_S}{\partial t}, \quad 0 < r < S(t), t > 0. \quad (12.2.74)$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T_L}{\partial r} \right) = \frac{1}{k_L} \frac{\partial T_L}{\partial t}, \quad S(t) < r < \infty, t > 0. \quad (12.2.75)$$

$$T_S = T_L = T_m, \text{ on } r = S(t); \quad S(0) = 0, r = S(t) \text{ is the freezing front.} \quad (12.2.76)$$

$$-K_S \frac{\partial T_S}{\partial r} + K_L \frac{\partial T_L}{\partial r} = -\rho_S l \frac{dS}{dt}, \quad \text{on } r = S(t), \quad (12.2.77)$$

$$T_L(0, t) = T_0 > T_m; \quad \lim_{r \rightarrow \infty} T_L(r, t) = T_0. \quad (12.2.78)$$

To obtain a solution of the problem stated in Eqs (12.2.74)–(12.2.78), the solutions of Eqs (12.2.74), (12.2.75) can be written as in Eq. (12.2.71). This will involve four unknown constants. The freezing front can be taken as follows for obtaining a similarity solution.

$$r = S(t) = 2\delta\sqrt{k_S t}, \quad \delta > 0 \text{ is unknown, } t > 0. \quad (12.2.79)$$

One of the unknowns in the solution of liquid region temperature can be determined by using the condition as  $r \rightarrow \infty$  and three unknowns can be determined with the help of conditions given in Eqs (12.2.76), (12.2.77). So the condition (12.2.73) has to be used to determine an unknown (as  $\delta$  is unknown) and to obtain a similarity solution.  $Q_S(t)$  has to be of the form  $Q_S(t) = Q_0/\sqrt{t}$ , otherwise similarity transformation cannot be used. In the cylindrical problem also condition (12.2.68) was used but to obtain a similarity solution,  $Q$  in Eq. (12.2.68)

has to be a constant. For the spherical problem solution is given below. Since our interest here is in explaining the method, the thermo-physical parameters are not in dimensionless form in the above formulation and many other formulations discussed earlier.

$$T_S = T_m + \frac{Q_0}{4\pi K_S \sqrt{k_S}} [G(\eta_1) - G(\delta)], \quad \eta_1 = r/2\sqrt{k_S t}, \quad (12.2.80)$$

$$T_L = T_0 + \frac{(T_m - T_0)}{G(\delta\sqrt{k_S/k_L})} G(\eta_2), \quad \eta_2 = r/2\sqrt{k_L t}, \quad (12.2.81)$$

$$G(\eta) = \frac{1}{2\eta} e^{-\eta^2} - \frac{\sqrt{\pi}}{2} \operatorname{erfc}(\eta). \quad (12.2.82)$$

The unknown  $\delta$  can be determined as a real positive root of Eq. (12.2.83), if it exists.

$$\frac{Q_0 e^{-\delta^2}}{4\pi K_S \sqrt{k_S} (T_0 - T_m)} - \sqrt{\frac{k_L}{k_S}} \frac{K_L}{K_S} \frac{e^{-\delta^2 (k_S/k_L)}}{G(\delta\sqrt{k_S/k_L})} = \frac{4\delta^2 l}{C_S (T_0 - T_m)}. \quad (12.2.83)$$

The solution procedure stated earlier can be used to obtain some other analytical solutions of spherical problems. As in the cylindrical problems, exact analytical similarity solutions also cannot be obtained in finite spherical regions unless boundary conditions are of special types. The similarity transformation does not work for temperature or flux prescribed boundary conditions imposed at the finite radius  $r = r_0$ ,  $r_0 < \infty$ . Note that in Eqs (12.2.27)–(12.2.72) for  $t > 0$ , as  $r \rightarrow 0$ ,  $\frac{\partial T}{\partial r} \rightarrow \infty$ . In the absence of any heat source, no flux condition is to be satisfied at  $r = 0$  and  $\frac{\partial T}{\partial r} = 0$  for  $t > 0$ . This implies that the region under consideration should not include  $r = 0$  or if it includes it then conditions of the form Eqs (12.2.68), (12.2.73) should be prescribed in which  $r = 0$  is a sink or a source. Some solutions for Stefan-like problems in spherical coordinates  $(r, 0, 0)$  will be considered in Section 12.2.2. Quasisteady-state and steady-state solutions will be considered in Section 12.2.4.

Some exact solutions in ellipsoidal and paraboloidal coordinates will be considered now.

There are many physical problems of interest such as Hele-Shaw problems, crystal and dendrite growth problems arising in solidification of supercooled melt, Darcy flow problems in porous media, etc., in which the free boundary may take shapes different from planar, cylindrical and spherical. This happens due to the instability of the free boundary and the whole or part of the free boundary may take the shape of an ellipse, an ellipsoid, a paraboloid, a cusp (refer [176, 365–367]) or some other type of shapes. This deformation of the free boundary has to be analytically as well as experimentally studied to understand the instability and its causes.

Some exact analytical similarity solutions are discussed below when the free boundary is considered in ellipsoidal and hyperboloidal regions. Let  $(x, y, z)$  be the Cartesian coordinates of a point in  $R^3$  and  $(\xi_1, \xi_2, \xi_3)$  be the ellipsoidal coordinates of this point. If  $(X_1, X_2, X_3)$  coordinates are defined as

$$X = (X_1, X_2, X_3) = (x, y, z)/\sqrt{t}, \quad (12.2.84)$$

then  $X$  can be expressed in terms of  $(\xi_1, \xi_2, \xi_3) = (\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3)/\sqrt{t}$  as (cf. [368]),

$$X_1^2 = (\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)/a^2(a^2 - b^2), \quad (12.2.85)$$

$$X_2^2 = (\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)/b^2(b^2 - a^2), \quad (12.2.86)$$

$$X_3^2 = \xi_1^2 \xi_2^2 \xi_3^2 / a^2 b^2, \quad (12.2.87)$$

where  $a$  and  $b$  are arbitrary and at no stage they will be determined.  $\xi_1$ ,  $\xi_2$  and  $\xi_3$  satisfy the following relations.

$$0 \leq \xi_3 \leq b \leq \xi_2 \leq a \leq \xi_1 < \infty. \quad (12.2.88)$$

It is easy to check that

$$X_1^2/(\xi_1^2 - a^2) + X_2^2/(\xi_1^2 - b^2) + X_3^2/\xi_1^2 = 1. \quad (12.2.89)$$

For  $\xi_1 = \text{constant}$ , Eq. (12.2.89) represents an ellipsoid and in particular for  $\xi_1 = a$ , Eq. (12.2.89) reduces to an elliptic plate

$$X_1 = 0, \quad X_2^2/(a^2 - b^2) + X_3^2/a^2 \leq 1. \quad (12.2.90)$$

For  $\xi_3 = \text{constant}$ , we have a hyperboloid of two sheets which is given by

$$X_3^2/\xi_3^2 - X_1^2/(a^2 - \xi_3^2) - X_2^2/(b^2 - \xi_3^2) = 1. \quad (12.2.91)$$

For  $\xi_3 = 0$ , it is the  $(X_1, X_2)$  plane and for  $\xi_3 = b$ , we get a hyperbolic plate

$$X_2 = 0, \quad X_3^2/b^2 - X_1^2/(a^2 - b^2) \geq 1. \quad (12.2.92)$$

For  $\xi_2 = \text{constant}$ , we get a hyperboloid of one sheet which for  $\xi_2 \rightarrow a, b$ , respectively, degenerates into an elliptic plate and a hyperbolic plate. Physically realistic boundary conditions cannot be imposed in this case at  $a$  and  $b$ .

The formulation of the Stefan problem for which exact solution is sought was considered in [366, 369] and is given in Eqs (12.2.93), (12.2.94).

$$-\frac{1}{2}\vec{X} \cdot \vec{\nabla} T_{S,L} = k_{S,L} \nabla^2 T_{S,L}. \quad (12.2.93)$$

Eq. (12.2.93) is in  $(X_1, X_2, X_3)$  coordinates which are similarity variables,  $\vec{\nabla}$  is gradient vector and  $\nabla^2$  is the elliptic operator. Subscripts  $S$  and  $L$  stand for solid and liquid regions, respectively.

$$\left. \begin{aligned} T_S &= T_L = 0 \\ K_L \frac{\partial T_L}{\partial n} - K_S \frac{\partial T_S}{\partial n} &= -l \rho_S v_n \end{aligned} \right\} \text{ on the phase-change interface.} \quad (12.2.94)$$

In Eq. (12.2.94),  $n$  is the outward drawn normal to the respective regions under consideration and  $v_n$  is the normal component of velocity of the moving boundary. In Eqs (12.2.93), (12.2.94) all the thermo-physical parameters are taken as constant but different in different phases. Isotherm temperature is taken as 0 after sealing.

First, exact similarity solution for solidification will be discussed without natural convection in the liquid and afterwards natural convection will also be considered in the obtained

solution. Eq. (12.2.93) which is a steady-state equation in similarity coordinates ( $X_1, X_2, X_3$ ) can be obtained in terms of any one of the coordinates  $\xi_i$ ,  $i = 1, 2, 3$  and this results in the following equations.

$$k_{S,L} \frac{d}{d\xi_i} \left( f(\xi_i) \frac{d}{d\xi_i} (T_{S,L}) \right) + \frac{1}{2} \xi_i f(\xi_i) \frac{dT_{S,L}}{d\xi_i} = 0, \quad i = 1, 2, 3, \quad (12.2.95)$$

$$f(\xi_i) = \left\{ (\xi_i^2 - a^2)(\xi_i^2 - b^2) \right\}^{1/2}, \quad i = 1, 2, 3, \quad k \text{ is thermal diffusivity.} \quad (12.2.96)$$

The solution of the ordinary differential equation (12.2.95) can be easily obtained and we get on integration

$$T_{S,L} = A_1 \int_{A_2}^{\xi_i} \exp \left( -\frac{p^2}{4k_{S,L}} \right) \frac{dp}{f(p)}, \quad i = 1, 2, 3, \quad (12.2.97)$$

$A_1$  and  $A_2$  are unknowns. As we are looking for similarity solutions in terms of similarity variables  $\xi_1$  or  $\xi_2$  or  $\xi_3$ , the moving boundary has to be taken in the form  $\bar{\xi}_i = \xi_0^i \sqrt{t}$ ,  $i = 1, 2, 3$ , or if there is no confusion then  $\bar{\xi}_i = \xi_0 \sqrt{t}$ ,  $i = 1, 2, 3$ , where  $\xi_0$  is a constant to be determined as a part of the solution and  $\xi_0$  will be different for different  $\xi_i$ . The Stefan problem for  $T_{S,L}$  in terms of any of the  $\xi_i$  can be formulated as given below together with initial and boundary conditions.  $T_{S,L}$  satisfies Eq. (12.2.95) for

$$\left. \begin{aligned} &\xi_i, i = 1, 2, 3, \\ &T_S|_{\xi_i=\xi_0} = T_L|_{\xi_i=\xi_0} = 0; -\frac{1}{2} \rho_S \xi_0 = \left( K_L \frac{\partial T_L}{\partial \xi_i} - K_S \frac{\partial T_S}{\partial \xi_i} \right) \Big|_{\xi_i=\xi_0}. \end{aligned} \right\} \quad (12.2.98)$$

$\xi_0$  will be different for different  $\xi_i$ ,  $K$  is thermal conductivity. The region for each  $\xi_i$  is clearly specified in Eq. (12.2.88). What about the boundary conditions for  $\xi_1$  at  $\xi_1 = a$  and at infinity,  $\xi_2$  at  $a$  and  $b$  and  $\xi_3$  at  $b$  and  $0$ ? If the ellipsoid in  $(x, y, z)$  coordinates or in  $(\xi_1, \xi_2, \xi_3)$  has axes  $a$  and  $b$  which depend on time and which are proportional to  $\sqrt{t}$  then  $\xi_i = \text{constant}$  for  $i = 1, 2, 3$  at the boundaries/surfaces of the regions of different variables under consideration. However, if  $a$  and  $b$  are not a functions of time as indicated earlier then at  $a$  and  $b$ , temperature boundary conditions cannot be prescribed as in Eq. (12.2.98) and at them similarity variable does not have some constant value as in Eq. (12.2.98). This situation is similar to the one which arose in the solutions of Stefan problems in cylindrical and spherical finite or infinite regions,  $0 \leq r \leq r_0 < \infty$  or  $0 < r_0 \leq r < \infty$ . In the later case for solidification in the outer region, to obtain an exact similarity solution, the liquid has to be supercooled and  $T_S \equiv 0$  which is the isotherm temperature at the moving boundary.

The formulation and solution of Stefan problem for solidification of a supercooled liquid occupying the region  $a \leq \xi_1 < \infty$  will now be presented. The initial temperature of the liquid region  $a \leq \xi_1 < \infty$  is taken as  $T_L(\xi_1, t)_{t=0} = T_0$ ;  $T_L|_{\xi_1 \rightarrow \infty} = T_0$ ,  $T_0 < 0$ . For convenience the following shorter notations will be used.

$$F_1(\xi_1; k_{S,L}) = \int_a^{\xi_1} \exp \left( -\frac{p^2}{4k_{S,L}} \right) \frac{dp}{f(p)}; \quad F_{11}(\xi_1; k_{S,L}) = \int_{\xi_1}^{\infty} \exp \left( -\frac{p^2}{4k_{S,L}} \right) \frac{dp}{f(p)}. \quad (12.2.99)$$

$$F_3(\xi_3; k_{S,L}) = \int_0^{\xi_3} \exp\left(-\frac{p^2}{4k_{S,L}}\right) \frac{dp}{f(p)}; F_{31}(\xi_3; k_{S,L}) = \int_{\xi_3}^b \exp\left(-\frac{p^2}{4k_{S,L}}\right) \frac{dp}{f(p)}. \quad (12.2.100)$$

The function  $f(p)$  in Eqs (12.2.99), (12.2.100) is the same as given in Eq. (12.2.96). The constant  $A_1$  in Eq. (12.2.97) is determined from the condition  $T_L|_{\xi_1 \rightarrow \infty} = T_0$  and  $A_2$  can be easily guessed. Finally the solution of the solidification problem for the solid growing towards  $\xi_1 \rightarrow \infty$  in a supercooled liquid with initial temperature  $T_0 < 0$  and occupying the region  $a < \xi_1 < \infty$  is given as follows.

$$T_S \equiv 0, a \leq \xi_1 \leq \xi_0; \xi_1 = \xi_0 \text{ is the phase-change boundary.} \quad (12.2.101)$$

$$T_L = T_0 (1 - F_{11}(\xi_1; k_L) / F_{11}(\xi_0; k_L)), \quad \xi_0 \leq \xi_1 < \infty. \quad (12.2.102)$$

$\xi_0$  is the real positive root of the equation given in Eq. (12.2.103). This equation is obtained by using the Stefan condition given in Eq. (12.2.98).

$$\rho_S l \xi_0 = 2K_L T_0 F'_{11}(\xi_0; k_L) / F_{11}(\xi_0; k_L), \quad (12.2.103)$$

dash stands for differentiation with respect to  $\xi_1$  and evaluated at  $\xi_1 = \xi_0$ . It has been proved in [366] that Eq. (12.2.103) has a real positive unique root  $\xi_0$  provided

$$-\rho_S l k_L |K_L < T_0 \leq 0, \quad T_L|_{\xi_1 \rightarrow \infty} = T_0 < 0. \quad (12.2.104)$$

In the original coordinates  $(x, y, z)$ , the moving boundary is an ellipsoid which is given by

$$\frac{x^2}{(\xi_0^2 - a^2)} + \frac{y^2}{(\xi_0^2 - b^2)} + \frac{z^2}{\xi_0^2} = t, \quad t > 0, \quad \xi_0 > a. \quad (12.2.105)$$

For some other solutions in ellipsoidal region, see [366]. In  $\xi_2$  coordinate realistic physical problems cannot be formulated as when  $\xi_2 \rightarrow a$  or  $b$ , there are singularities.

When  $\xi_3 = \text{constant}$ , we get a hyperboloid of two sheets in which  $0 \leq \xi_3 \leq b$ . At  $\xi_3 = b$  there is a singularity and no boundary condition can be prescribed at  $\xi_3 = b$ . At  $\xi_3 = 0$  boundary condition can be prescribed. Both solidification and melting problems can be considered. If the temperature  $T_0$  prescribed at  $\xi_3 = 0$  is greater than zero then we have a melting problem in which liquid occupies the region  $0 < \xi_3 < \xi_0$ , where  $\xi_3 = \xi_0$  is the phase-change front. As no boundary condition can be prescribed at  $\xi_3 = b$ , the region  $\xi_0 \leq \xi_3 < b$  has to be necessarily occupied by a solid at the melting temperature taken as zero. Similarly if the prescribed temperature  $T_0$  at  $\xi_3 = 0$  is less than zero then we have a solidification problem in which the liquid occupying the region  $\xi_0 \leq \xi_3 < b$  has to be necessarily at melting temperature and the region  $0 < \xi_3 < \xi_0$  is solid. This problem in  $\xi_3$  coordinate has a typical feature which was not there in the solution in  $\xi_1$  coordinate. At  $t = 0$ , the surfaces represented by  $\xi_3 = \text{constant}$  are elliptic cones and for  $t > 0$ , when the temperature at  $\xi_3 = 0$  is changed to  $T_0$ , the moving boundary  $\xi_3 = \xi_0$  becomes a hyperboloid of two sheets as given following.

$$\left. \begin{aligned} z^2/\xi_3^2 - x^2/(a^2 - \xi_3^2) - y^2/(b^2 - \xi_3^2) &= 0, & t = 0, \\ z^2/\xi_0^2 - x^2/(a^2 - \xi_0^2) - y^2/(b^2 - \xi_0^2) &= t, & t > 0. \end{aligned} \right\}. \quad (12.2.106)$$

Solutions for both melting and solidification problems can be obtained following the procedure mentioned earlier for ellipsoidal case. The following solution has been obtained in [366] for the melting problem when temperature  $T_0 > 0$  is prescribed at  $\xi_3 = 0$  for  $t > 0$ .

$$T_S \equiv 0, \xi_0 \leq \xi_3 < b, \xi_3 = \xi_0 \text{ is the phase-change boundary.} \quad (12.2.107)$$

$$T_L(\xi_3; k_L) = T_0 (1 - F_3(\xi_3; k_L) / F_3(\xi_0; k_L)), \quad 0 \leq \xi_3 \leq \xi_0. \quad (12.2.108)$$

The initial data depend on  $\xi_0$ . This will be explained further after constructing the solution.

The temperature  $T_L(\xi_3; k_L)$  in Eq. (12.2.108) satisfies the isotherm condition and the prescribed condition at  $\xi_3 = 0$ . To obtain  $\xi_0$ , the Stefan condition at  $\xi_3 = \xi_0$  is to be satisfied which results in the equation

$$\rho_S l \xi_0 = 2K_L T_0 F'_3(\xi_0; k_L) / F_3(\xi_0; k_L). \quad (12.2.109)$$

If  $\alpha = \rho_S l k_L / K_L T_0$ ,  $\xi_0 = \sqrt{k_L} \xi_0^*$  and we rescale  $a$  and  $b$  with  $\sqrt{k_L}$  then Eq. (12.2.109) becomes

$$\alpha \xi_0^* = F'_3(\xi_0^*) / F_3(\xi_0^*). \quad (12.2.110)$$

It was proved in [369] that if  $\alpha$  is small then Eq. (12.2.110) has no real root, two roots if  $\alpha$  is large and a double root for a critical value between. Such a situation in similarity solutions is new. If both  $a$  and  $b$  are infinite in Eq. (12.2.110) then it is the same equation which is obtained in one-phase Neumann solution.

$$\alpha \xi_0 e^{\xi_0^2/4} \int_0^{\xi_0} e^{-p^2/4} dp = 1. \quad (12.2.111)$$

Eq. (12.2.111) has a real positive root  $\xi_0$  for all  $\alpha$ . The nonexistence of solution of Eq. (12.2.109) for small  $\alpha$  can be explained by arguing that the amount of latent heat released or absorbed at the moving front does not match the speed of the moving front. The nonuniqueness of the root  $\xi_0$  is possible as this Stefan problem in hyperboloidal region is of unusual type. The initial temperature at  $t = 0$  is prescribed on elliptic cones obtained for  $\xi_3 = \text{constant}$  and given by Eq. (12.2.106). This constant in  $\xi_3 = \text{constant}$  is  $\xi_0$ , which is determined by Eq. (12.2.109). Generally in Stefan problems considered earlier and in the ellipsoidal region considered earlier, the initial temperature is prescribed for the material present initially inside the region (inner problem) or outside the region (outer problem) which is fixed. But this is not so in the hyperboloidal case in which the initial temperature depends on the moving boundary and initial temperature is prescribed on a changing region.

A good number of experimental and mathematical studies have been reported concerning the dendrite growth in crystals growing in supercooled melts (see Sections 4.4 and 4.5). The dendrite growth near its tip has been analysed theoretically by many authors such as in [41, 365, 366, 370] by considering paraboloids of various cross-sections such as circular and elliptic. Several other references can be found in these above-mentioned references. The primary interest of investigation is to obtain results on the growth speed and radius of the tip of the dendrite and examine their effect on supercooling. Taking into account many different parameters, several theories have been proposed on functional relationships amongst them. The one functional relationship which will be discussed here is between the dimensionless Stefan number and Peclet number. Some exact analytical solutions have been obtained for the growth

speed of the tip of dendrite and its radius by formulating Stefan problems in paraboloidal coordinates involving both heat and mass transfer. First, only heat transfer will be considered by us in the formulation of Stefan problems. Solutions with fluid velocity and concentration gradients present in the supercooled melt will be presented in [Section 12.2.2](#).

Let the coordinate system be fixed at the dendrite tip's frame of reference so that with regard to the laboratory frame of reference in which the tip of the dendrite is at rest, the moving coordinates have a velocity  $V$  in the  $z$ -direction which is the direction of dendrite growth. If  $(x, y, z)$  are the Cartesian coordinates of a point and  $(\xi, \eta, \phi)$  are the parabolic coordinates of the same point then the two sets of coordinates are related as

$$x = \xi \eta \sin \phi, \quad y = \xi \eta \cos \phi, \quad z = (\xi^2 - \eta^2) / 2. \quad (12.2.112)$$

For an isothermal steady-state dendrite growth in the  $z$ -direction, the dendrite surface can be expressed as  $\xi = f(\eta, \phi)$  and this fixes the ranges of  $\xi$ ,  $\eta$  and  $\phi$ . In the solid region (i.e. within the dendrite),  $0 \leq \xi \leq f(\eta, \phi)$ ,  $0 \leq \eta < \infty$  and  $0 \leq \phi \leq 2\pi$ . In the supercooled melt,  $f(\eta, \phi) \leq \xi < \infty$ ,  $0 \leq \eta < \infty$  and  $0 \leq \phi \leq 2\pi$ . Dendrite is growing at isothermal temperature  $T_m$ . The length scale is taken as  $2k/V$ , where  $k$  is thermal diffusivity and the temperature is measured relative to  $T_m$  in units of  $T_m - T_\infty$ , where  $T_\infty < T_m$  and  $T_\infty$  is the far-field temperature as  $\xi \rightarrow \infty$ . There are no temperature gradients in the dendrite as it is growing at isotherm temperature. We now discuss the formulation of the Stefan problem as considered in [365].

The steady-state temperature  $T(x, y, z)$  in the liquid is given by the equation

$$\nabla^2 T + \frac{\partial T}{\partial z} = 0, \quad (12.2.113)$$

in which scaling has been done in [365] but not reported here. Eq. (12.2.113) when transformed into parabolic coordinates becomes

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} + \frac{1}{\xi} \frac{\partial T}{\partial \xi} + \frac{1}{\eta} \frac{\partial T}{\partial \eta} + \frac{(\xi^2 + \eta^2)}{\xi^2 \eta^2} \frac{\partial^2 T}{\partial \phi^2} + 2\xi \frac{\partial T}{\partial \xi} - 2\eta \frac{\partial T}{\partial \eta} = 0. \quad (12.2.114)$$

At the free boundary  $\xi = f(\eta, \phi)$ , the following conditions are to be satisfied

$$T = 0; \quad \frac{-V\rho}{2k} \left( f + \eta \frac{\partial f}{\partial \eta} \right) = \frac{d_S}{2} \left\{ \frac{\partial T}{\partial \xi} - \frac{\partial f}{\partial \eta} \frac{\partial T}{\partial \eta} - \frac{g^2 \frac{\partial f}{\partial \phi}}{\eta^2 f^2} \frac{\partial T}{\partial \phi} \right\}, \quad g^2 = \xi^2 + \eta^2, \quad (12.2.115)$$

$d_S$  in Eq. (12.2.115) is the dimensionless Stefan number or dimensionless supercooling and is defined as

$$d_S = C\Delta T/l; \quad \Delta T = T_m - T_\infty; \quad C \text{ is specific heat, } l \text{ is latent heat, } \rho \text{ is density.} \quad (12.2.116)$$

When Eq. (12.2.114) is considered in a single coordinate  $\xi$  (i.e. considering parabolic coordinates  $(\xi, 0, 0)$ ), the resulting equation obtained is somewhat similar to Eq. (12.2.26) and the solid-liquid interface corresponds to  $\xi = \xi_0$ . The surface of the dendrite can be obtained using Eq. (12.2.112) and is given by the equation

$$2z = \xi_0^2 - r^2/\xi_0^2, \quad r^2 = x^2 + y^2 = \xi_0^2, \quad \text{for } z = 0, \quad (12.2.117)$$

$r$  is the radius of the surface of the tip of dendrite at  $z = 0$ . The cross-section of the surface is circular of radius  $\xi_0$ . The solution of Eq. (12.2.114) in  $(\xi, 0, 0)$  coordinates is given as

$$T^0(\xi) = \left( Ei(\xi^2) - Ei(\xi_0^2) \right) / Ei(\xi_0^2), \quad Ei(\xi) = - \int_{-\xi}^{\infty} \frac{e^{-q}}{q} dq. \quad (12.2.118)$$

The temperature solution  $T^0(\xi)$  is known as *Ivantsov solution*, which was obtained in [370].  $Ei(\xi)$  is the *exponential integral* defined in Eq. (12.2.27) in connection with the solution of Eq. (12.2.26).  $T^0(\xi)$  satisfies the isotherm condition and the far-field condition  $T^0(\xi) \Big|_{\xi \rightarrow \infty} = -1$  in dimensionless form. To determine  $\xi_0$ , the Stefan condition given in Eq. (12.2.115) can be used after putting  $\phi = 0$  and  $\eta = 0$  in it.

The solution  $\xi_0$  of the equation obtained on using Stefan condition exists if

$$d_S(\text{Stefan number}) = Pe^P Ei(P), \quad P = V\rho/2k. \quad (12.2.119)$$

$P$  is called *Peclet number*. If the relationship between  $d_S$  and  $P$  holds as in Eq. (12.2.119) then we get  $\xi_0^2 = V\rho/2k = P$ . Eq. (12.2.119) tells us that the supercooling is not independent of the velocity of the tip of the dendrite. In Ivantsov solution, the cross-section of the surface of the tip of dendrite is circular and for this case the relationship between supercooling and Peclet number is given by Eq. (12.2.119). What would be this relationship if the cross-section is elliptic? This question was examined in [41] by taking the cross-section as elliptic.

The following equation of an isothermal dendrite whose cross-section is an ellipse was considered in [41].

$$\frac{x^2}{(w-b)} + \frac{y^2}{(w+b)} = w - 2z. \quad (12.2.120)$$

The isotherm corresponds to  $w = \text{constant}$  and the interface corresponds to  $w = P$ , where  $P$  is the Peclet number defined in Eq. (12.2.119). The constant  $b$  determines the aspect ratio of the elliptic cross-section of the dendrite tip and  $b = 0$  corresponds to a circular cross-section of the dendrite. A more general form of the ellipse was also considered in [41] in which the dimensionless radius of curvature is  $(P-b)$  in the  $(x, z)$  plane and it is  $(P+b)$  in the  $(y, z)$  plane. So the average curvature can be taken as  $P$ . In Eq. (12.2.120) the average curvature is  $P$  and so it is a symmetrized version of the general case. Appropriate changes can be made for the solution in elliptic case in the earlier solution given in Eq. (12.2.97). The temperature in the supercooled melt in which the isotherm dendrite tip is growing in the  $z$ -direction and whose cross-section is an ellipse is then given by

$$T(w) = \frac{G(w) - G(P)}{G(P)}, \quad G(w) = \int_w^{\infty} \frac{e^{-u}}{\sqrt{u^2 - b^2}} du, \quad (12.2.121)$$

$$T(w) \Big|_{w \rightarrow \infty} = -1, \quad T(w) \Big|_{w=P} = 0. \quad (12.2.122)$$

$P$  is the Peclet number defined in Eq. (12.2.119). If the relationship between Stefan number  $d_S$  and Peclet number  $P$  is given by

$$d_S = \sqrt{P^2 - b^2} e^P G(P), \quad (12.2.123)$$



then  $w = P$  is the moving boundary which is obtained as the solution of equation of Stefan condition given in Eq. (12.2.115). For  $b = 0$ , we get Ivantsov solution.

The twofold symmetry solution obtained in Eq. (12.2.121) has been further generalized first to fourfold symmetry solution and then to  $n$ -fold symmetry solution in [365]. The specific effort by authors in [365] is to investigate the effect of shape variation of the dendrite tip  $z = 0$  due to change in the relationship between the Stefan number  $d_S$  and Peclet number  $P$  and to obtain an approximate form of the tip of the dendrite in fourfold symmetry. This is carried out by perturbing the twofold symmetry solution obtained in [41] in parabolic coordinates. The Peclet number  $P$  is kept fixed so that the tip of interface is at  $z = P/2$ . This also fixes the average radius of curvature. As the ellipticity of the shape depends on the ratio  $(P-b)/(P+b)$ , to study the shape of fixed ellipticity, the ratio is made independent of  $P$  by writing  $b = P\epsilon$ , where  $|\epsilon| < 1$ . The twofold solution given in [370] in parabolic coordinates is then perturbed in powers of the parameter  $\epsilon \ll 1$ . For perturbed solutions and further discussion see [365]. It may be mentioned here briefly that perturbations are of the form

$$\left. \begin{aligned} T(\eta, \xi, \phi, \epsilon) &= T^0(\xi) + \epsilon T^{(1)}(\eta, \xi, \phi) + \frac{\epsilon^2}{2} T^{(2)}(\eta, \xi, \phi) + O(\epsilon^3), \\ \xi &= f(\eta, \phi, \epsilon) = \xi_0 + \epsilon f^{(1)}(\eta, \phi) + \frac{\epsilon^2}{2} f^{(2)}(\eta, \phi) + O(\epsilon^3), \\ d_S &= d_S^{(0)} + \epsilon d_S^{(1)} + \frac{\epsilon^2}{2} d_S^{(2)} + O(\epsilon^3). \end{aligned} \right\} \quad (12.2.124)$$

When the densities of solid and liquid in a solidification problem are not equal (i.e.  $\rho_S \neq \rho_L$ ), it gives rise to flow in the liquid which is often neglected in a simplified model. As mentioned earlier this phenomenon is known as natural convection and a simple solution with convective term appearing in the liquid region was presented in Section 1.3. Although natural convection can be modelled in the formulation of solidification problems in many different ways, we present below a simple model whose exact analytical solution is given in [366] in the context of solidification in ellipsoidal region.

The solidification of a supercooled liquid is considered in an infinite ellipsoidal region, and the liquid is occupying the region  $R^3$ ,  $-\infty < x < \infty$ ,  $-\infty < y < \infty$ ,  $-\infty < z < \infty$ , where  $(x, y, z)$  are the Cartesian coordinates. In the ellipsoidal coordinates  $(\xi_1, 0, 0)$ , an ellipsoid is given by  $\xi_1 = \text{constant}$ . Consider the following system of equations in the formulation of one-phase solidification problem in ellipsoidal region with natural convection.

$$T_S \equiv 0, \text{ i.e. melting temperature is taken zero, } 0 < \xi_1 \leq a. \quad (12.2.125)$$

$$\frac{\partial T_L}{\partial t} + \vec{\nabla} \phi \cdot \vec{\nabla} T_L = k_L \nabla^2 T_L, \quad a \leq \xi_1 < \infty. \quad (12.2.126)$$

$$\nabla^2 \phi = 0 \text{ in the liquid, } a \leq \xi_1 < \infty; \vec{\nabla} \phi \text{ is liquid velocity.} \quad (12.2.127)$$

It has been assumed that the variable  $\xi_1$  in Eqs (12.2.125)–(12.2.127) is a similarity variable and the ellipsoidal coordinates  $(\xi_1, \xi_2, \xi_3)$  are related to  $(x, y, z)$  coordinates through Eqs (12.2.85)–(12.2.87).

At the solid-liquid interface  $\xi_1 = \xi_0$ , we have the following conditions.

$$T_S = T_L = 0; \quad K_L \frac{\partial T_L}{\partial n} = -\rho_S l v_n; \quad \frac{\partial \phi}{\partial n} = -\frac{(\rho_S - \rho_L)}{\rho_S} v_n, \quad (12.2.128)$$

where  $n$  stands for outward normal to the region under consideration,  $v_n$  is the normal component of the velocity of the liquid and  $\partial(\cdot)/\partial n$  is the normal derivative. For the flow in the liquid to obtain an exact similarity solution a simple model has been considered in Eq. (12.2.127). For an infinite region, the far-field conditions are

$$\phi \sim O(1/r) \text{ and } T_L \rightarrow T_0 < 0 \text{ as } \xi \rightarrow \infty; \quad r^2 = x^2 + y^2 + z^2. \quad (12.2.129)$$

If we take  $k_{S,L} \rightarrow \infty$  in Eq. (12.2.95) and  $T_{S,L}$  is replaced by  $\phi$ , then an ordinary differential equation  $\nabla^2 \phi = 0$  is obtained in  $\xi_1$  coordinate whose solution is easy to determine. For the sake of using earlier notations adopted in Eq. (12.2.99), the solution of Eq. (12.2.127) is written as

$$\phi = B_1 F_{11}(\xi_1; \infty), \quad B_1 \text{ is unknown constant to be determined.} \quad (12.2.130)$$

It may be noted that  $F_{11}(\xi_1; \infty)$  is not obtained by taking  $k_{S,L} = \infty$  in Eq. (12.2.99) but it simply means that a similar type of solution can be obtained by taking  $k_{S,L}$  as  $\infty$ . The constant  $B_1$  is determined by using the third condition in Eq. (12.2.128). Finally the solution for  $\phi$  is given by

$$\phi = -\frac{1}{2} \xi_0 (1 - \rho_L / \rho_S) F_{11}(\xi_1; \infty) / F'_{11}(\xi_0; \infty) = -A_1 F_{11}(\xi_1; \infty). \quad (12.2.131)$$

On substituting  $\phi$  given in Eq. (12.2.131) into the differential equation (12.2.126), the following ordinary differential equation is obtained for the liquid region temperature.

$$\left. \begin{aligned} k_L \frac{d}{d\xi_1} \left( f(\xi_1) \frac{dT_L}{d\xi_1} \right) + \frac{1}{2} \xi_1 f(\xi_1) \frac{dT_L}{d\xi_1} + \frac{1}{2} A_1 \frac{dT_L}{d\xi_1} &= 0, \\ A_1 &= -1 / (2\rho_S F'_{11}(\xi_0; \infty)) (\rho_S - \rho_L) \xi_0. \end{aligned} \right\} \quad (12.2.132)$$

In view of the conditions to be satisfied by the solution in Eq. (12.2.132),  $T_L(\xi_1; k_L)$  can be taken as follows in which  $f(\xi_1)$  is given by Eq. (12.2.96).

$$T_L(\xi_1; k_L) = B_{11} \int_{\xi_1}^{\infty} \exp \left( \frac{-(u^2 - A_1 F_{11}(u; \infty))}{4k_L} \right) \frac{du}{f(u)} + T_0, \quad (12.2.133)$$

$$= B_{11} \bar{G}(\xi_1; k_L) + T_0. \quad (12.2.134)$$

$T_L(\xi_1; k_L)$  satisfies the required condition as  $\xi_1 \rightarrow \infty$ . To obtain the constant  $B_{11}$ , the isotherm condition at the moving boundary  $\xi_1 = \xi_0$  is used and to determine  $\xi_0$ , the second condition in Eq. (12.2.128) is used. Finally the solution of Eq. (12.2.132) is given as

$$T_L(\xi_1; k_L) = T_0 (1 - \bar{G}(\xi_1; k_L) / \bar{G}(\xi_0; k_L)), \quad T_0 < 0, \quad (12.2.135)$$

and  $\xi_0$  is the real positive root of equation

$$\frac{1}{2} \rho_S l \xi_0 = K_L T_0 \bar{G}'(\xi_0; k_L) / \bar{G}(\xi_0; k_L), \quad (12.2.136)$$

$K_L$  is thermal conductivity, dash denotes differentiation with respect to  $\xi_1$ , and evaluated at  $\xi_0$ . It has been proved in [366] that a real positive root  $\xi_0$  of Eq. (12.2.136) exists if and only if the following condition is satisfied

$$0 < \rho_S \quad lk_L/K_L < -T_0. \quad (12.2.137)$$

Some solutions of problems in which both temperature and concentration gradients are present have been discussed in [366].

### *Stefan Problems in Which Thermo-Physical Parameters Are Not Constant*

When thermo-physical parameters are not constant then they could depend on several quantities. In the literature this dependence has been considered of various forms and it is difficult to list all such forms but one may get some idea about it from Eq. (3.1.1). Further generalizations are also possible. Since our interest is in obtaining exact analytical solutions, the choice of the functional form of the dependence of thermo-physical parameters on other quantities becomes limited and it has to be of some specific form so that an exact solution can be obtained. The underlying idea in the analytical methods of solution discussed below is to convert the nonlinear heat equation into a linear equation by using some transformation in terms of a dependent variable other than the temperature. A similarity solution of the new formulation of original problem in which the heat equation is no more nonlinear in terms of the transformed system can then be attempted. However, it is not certain that a similarity solution of the transformed problem can be always obtained. To overcome this hurdle the original Stefan problem is often chosen selectively.

The solidification or thawing problem discussed below and whose solution has been obtained in [371] is a one-dimensional one-phase problem. A liquid at the freezing temperature  $T_m$  occupies initially a finite region  $0 \leq r \leq R < \infty$ . Here  $r$  could be a planar coordinate, a cylindrical polar coordinate or a spherical coordinate. Solidification starts at  $r = R$  and progresses inwardly. Weak formulation or enthalpy formulation does not require Stefan condition or the energy balance condition to be satisfied at the phase-change interface in the form of discontinuity of heat fluxes at the interface. Enthalpy formulation for the problem considered has been discussed in [371]. The latent heat released at the moving interface can be taken care by modifying the specific heat appropriately. Under certain conditions a weak solution is a classical solution and a classical solution is a weak solution (see Section 5.1).

In [371], thermal conductivity and specific heat have been taken as follows.

$$C(T) = \frac{C_0}{|T|}; \quad K(T) = K_0|T|^b, \quad (12.2.138)$$

where  $C_0$  is specific heat coefficient,  $K_0$  is thermal conductivity coefficient and  $b$  is some real exponent. Since our interest is in the method of solution it will be assumed without affecting the method of solution that all the quantities have been suitably made dimensionless. The one-phase heat equation is given by

$$\rho_0 \frac{C_0}{U_S} \frac{\partial U_S}{\partial t} = \text{div}(K_0(U_S)^b \text{grad } U_S), \quad S(t) \leq r < R; \quad t_0 \leq t < t_p, \quad (12.2.139)$$

where  $U_S$  is the dimensionless temperature of the solid,  $r = S(t)$  is the phase-change boundary,  $t_0$  is some reference time such as the time when solidification starts and  $t_p$  is the time when the solidification ends. At  $r = R$ , the imposed boundary condition is of convective type, and at  $r = 0$  no heat flux condition is required.

$$K_0 \frac{\partial U_S}{\partial r} \Big|_{r=R} = -h U_S(R, t), \quad h > 0; \quad K_0 \frac{\partial U_S}{\partial r} \Big|_{r=0} = 0. \quad (12.2.140)$$

The following transformation

$$z = \int_0^{U_S} |(v)^b| dv = \frac{1}{(1+b)} |U_S|^{b+1}, \quad (12.2.141)$$

transforms Eq. (12.2.139) into the following equation.

$$\frac{\partial z}{\partial t} = K_S z \nabla^2 z = K_S z \left( \frac{1}{r^m} \frac{\partial}{\partial r} \left( r^m \frac{\partial z}{\partial r} \right) \right), \quad K_S = (1+b)K_0/C_0 P_0, \quad (12.2.142)$$

where  $m = 0$  gives equation in planar case,  $m = 1$  is cylindrical case and  $m = 2$  is spherical case. Eq. (12.2.142) is a nonlinear partial differential equation. By using the invariance property of the nonlinear heat conduction equation under certain finite or infinite transformation groups, Eq. (12.2.142) can be transformed to an ordinary differential equation which involves a single independent variable.

The one-parameter finite transformation group used in [371] is given below in which  $a$  is the *group parameter*.

$$\bar{z} = a^p z; \quad \bar{r} = a^q r; \quad \bar{t} = a^n t. \quad (12.2.143)$$

As Eq. (12.2.142) is invariant under the above transformation, the real constants  $p, q, n$  should satisfy the relation  $p - 2q + n = 0$ . Take  $q = \lambda n$ , then two independent invariants of the group are obtained as follows.

$$J_1 = r(t/t_0)^{-\lambda}; \quad J_2 = z(t/t_0)^{(1-2\lambda)}, \quad z(r, t) = (t/t_0)^{(2\lambda-1)} J_2(J_1), \quad (12.2.144)$$

where  $t_0$  is some characteristic time and  $t > t_0$ ,  $J_2(J_1)$  is some arbitrary function of the new variable  $J_1 \cdot J_1$  and  $J_2$  or any of their functions are invariant. Define a new function  $y$  as given below and substitute  $y$  in Eq. (12.2.142). We obtain a nonlinear ordinary differential equation

$$y = K_S t_0 J_2; \quad y \frac{d^2 y}{dJ_1^2} + \left[ \frac{my}{J_1} + \lambda J_1 \right] \frac{dy}{dJ_1} - (2\lambda - 1) y = 0. \quad (12.2.145)$$

The free boundary is taken in the form  $r = S(t)$  which in this case can be re-written in the form [372, 373]

$$r = J_1(t/t_0)^\lambda, \quad (12.2.146)$$

by making use of the independent invariant  $J_1$ . As the free boundary should also be invariant under the group transformation, the parameter  $\lambda$  is taken as zero. The differential equation in Eq. (12.2.145) is then reduced to

$$y \frac{d^2 y}{dJ_1^2} + \frac{my}{J_1} \frac{dy}{dJ_1} + y = 0. \quad (12.2.147)$$

The nontrivial analytic solutions of Eq. (12.2.147) are given by

$$y(J_1) = \frac{-J_1^2}{2(1+m)} + \frac{A_1}{(1-m)J_1^{(m-1)}} + A_2, \quad m \neq 1, \quad (12.2.148)$$

$$y(J_1) = \frac{-J_1^2}{2(1+m)} + A_1 \log J_1 + A_2, \quad m = 1. \quad (12.2.149)$$

The constants  $A_1$  and  $A_2$  are determined by using the boundary conditions given in Eq. (12.2.140). Substituting  $J_1$  and  $J_2$  from Eq. (12.2.144) in Eqs (12.2.148), (12.2.149) and using the conditions given in Eq. (12.2.140), we get

$$U_S(r, t) = \left[ \frac{\rho_0 C_0}{K_0 t} \left( \frac{R^2 - r^2}{2(1+m)} + \frac{RK_0}{h(1+b)(1+m)} \right) \right]^{\frac{1}{(1+b)}}, \quad (12.2.150)$$

$$0 \leq m \leq 2, \quad b \neq -1, \quad -1 \leq U_S \leq 0. \quad (12.2.151)$$

The freezing temperature in dimensioned form is taken as  $-1$ . When  $b = -1$ , a linear equation in terms of the variable  $z$  defined in Eq. (12.2.141) can be easily obtained whose solution does not present any difficulty. The moving boundary  $r = S(t)$  can be obtained using the condition  $U_S(S(t), t) = -1$  and the total solidification time  $t_p$  can be obtained from the condition  $r = S(t_p) = 0$ . In the solid-liquid phase-change problems in which latent heat is evolved, the coefficient  $C_0$  in Eq. (12.2.138) can be identified with the latent heat. Some further references on the use of finite and infinite transformation groups in obtaining analytical solutions may be found in the cross-references given in [371]. The analytical solution was obtained and its numerical work was done in the context of freezing of foodstuffs.

The solution presented earlier with less details of the solution procedure has been presented in [374] in the context of soil freezing (thawing) with almost the same numerical work.

In the one-dimensional problem considered in [375], a liquid occupying the region  $0 \leq x < \infty$  is solidifying over a cooled solid substrate occupying the region  $-\infty < x \leq 0$ . The liquid is initially at the freezing temperature and the initial temperature of the base material is constant. Latent heat is released at the moving boundary and temperature gradients have been considered in both the solid phases, that is, in the base material denoted by the subscript  $i = 1$ , occupying  $-\infty < x \leq 0$ , and in the frozen solid  $0 \leq x \leq S(t)$  denoted by the subscript  $i = 2$ .  $x = S(t)$  is the solid-liquid interface. At  $x = 0$ , the continuity of temperatures and fluxes of the two solid regions have to be satisfied. In the heat equations considered for the two solid regions, specific heats and thermal conductivities are temperature dependent. The following transformation is used in the nonlinear heat equations in which both specific heat and thermal conductivity are temperature dependent.

$$u_i = \int_0^{T_i} C_i(\xi) d\xi, \quad i = 1, 2; \quad C \text{ is specific heat, } T \text{ is temperature.} \quad (12.2.152)$$

The heat equations are transformed into the following equations.

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x} \left( D_i(u_i) \frac{\partial u_i}{\partial x} \right), \quad i = 1, 2. \quad (12.2.153)$$

The temperature-dependent diffusivities  $D_i(u_i)$  in Eq. (12.2.153) have been taken in [375] in the form

$$D_i(u_i) = \frac{D_i(0)}{1 - 2\alpha_i u_i + \beta_i u_i^2}, \quad i = 1, 2. \quad (12.2.154)$$

$\alpha_i$  and  $\beta_i$ ,  $i = 1, 2$  are arbitrary constants except that  $\beta_i - \alpha_i^2 > 0$  so that  $D_i(u_i)$ ,  $i = 1, 2$  have no singularities. Each  $D_i$ ,  $i = 1, 2$  can have a single maximum at  $u_i = \alpha_i/\beta_i \in (0, 1)$ , or is monotonic over the range of temperature of interest. It has been mentioned that  $D_i(u_i)$  in Eq. (12.2.154) approximates the diffusivity of wide range of epoxy materials.

The method of similarity solution is based on the following *Boltzman scaling transformations*

$$\bar{x} = e^a x, \quad \bar{t} = e^{2a} t, \quad \bar{u}_i = u_i. \quad (12.2.155)$$

Under this transformation the nonlinear heat equations in Eq. (12.2.153) are invariant and if the free boundary is of the form

$$x = S(t) = \delta t^{1/2}, \text{ for some real positive constant } \delta,$$

then the whole moving boundary problem is scale-invariant under the transformation given in Eq. (12.2.155). The invariants of the scaling symmetry group in Eq. (12.2.155) must be functions of  $u_i$  and  $\sigma_i$  where  $\sigma_i$  are defined by

$$\sigma_i = \frac{(-1)^i x}{2\sqrt{D_i(0)t}}, \quad i = 1, 2. \quad (12.2.156)$$

Using the scale-invariant variables, Eq. (12.2.153) is transformed to an ordinary differential equation

$$\frac{1}{D_i(0)} \frac{d}{d\sigma_i} \left( D_i(u_i) \frac{du_i}{d\sigma_i} \right) = -2\sigma_i \frac{du_i}{d\sigma_i}, \quad i = 1, 2. \quad (12.2.157)$$

Eq. (12.2.157) is still nonlinear and further scalings and converting  $u_i$  and  $\sigma_i$  into some other variables are required to convert Eq. (12.2.157) into two first-order separable equations. This procedure and further procedures to obtain similarity solution are lengthy and involve many steps and equations. Due to space limitations this cannot be presented here. To obtain similarity solution, the boundary conditions at  $x = 0$  and at moving front also have to be transformed. This is a comparatively difficult problem. For further details refer [375]. Numerical results have been presented for temperature versus distance by taking different diffusivities. Several other research works have been mentioned in [375] where analytical solutions have been discussed using thermal diffusivities of the type given in Eq. (12.2.154) and its generalizations.

The exact solution of a one-dimensional one-phase problem in cylindrical radial coordinate  $r$  has been obtained in [376]. The liquid region initially at time  $t = 0$  occupies the region  $0 \leq r \leq R_0$  which is at the melting temperature taken as zero. The solidification is taking place inwardly. The solidification takes place due to prescribed temperature  $T_S(R_0, t) = T_1 < 0$ . If  $r = S_0(t)$  is the equation of freezing front then for inward solidification the problem formulation is given by the following equations.

$$\frac{\partial^2 T_S}{\partial r^2} + \frac{1}{r} \frac{\partial T_S}{\partial r} = \frac{\rho C(r)}{K} \frac{\partial T_S}{\partial t}, \quad (12.2.158)$$

$$\left. \frac{\partial T_S}{\partial r} \right|_{r=S_0(t)} = \frac{\rho l(S_0(t))}{K} \frac{dS_0(t)}{dt}; \quad T_S(S_0(t), t) = 0; \quad S_0(0) = R_0, \quad (12.2.159)$$

where  $C$  is specific heat and  $K$  is thermal conductivity. Latent heat  $l$  is considered as a function of  $S_0(t)$ .

If the following transformation of radial coordinate into the Cartesian planar coordinate  $x$  is made then the above problem can be reformulated in the semiinfinite region  $0 \leq x < \infty$ . Let the  $x$  coordinate be related to  $r$  coordinate as follows.

$$\frac{x}{x_0} = -\log\left(\frac{r}{R_0}\right); \quad \frac{S(t)}{x_0} = -\log\left(\frac{S_0(t)}{R_0}\right), \quad (12.2.160)$$

where  $x_0$  is some arbitrary length scale, and  $x = S(t)$  is free boundary in the planar case. If the above coordinate transformation is used then in terms of  $x$  coordinate, and we have  $u(x, t) = T_S(r, t)$  then for  $u(x, t)$  we can reformulate the Stefan problem in the region  $0 \leq x < \infty$ . Solidification will now start at  $x = 0$  and  $x = S(t)|_{t=0} = 0$ . The solid occupies the region  $0 \leq x \leq S(t), t > 0$  and the region  $x > S(t)$  is liquid at the freezing temperature taken as zero. The transformation of coordinates given in Eq. (12.2.160) has been used earlier also to study solutions of cylindrical problems [377]. Using  $r = R_0 \exp\left(-\frac{x}{x_0}\right)$ , the heat equation (12.2.158) is transformed into

$$K x_0^2 / r^2 \frac{\partial^2 u}{\partial x^2} = \rho C(r) \frac{\partial u}{\partial t}. \quad (12.2.161)$$

If  $S_0(t) = R_0 \exp\left(-\frac{S(t)}{x_0}\right)$  is used in the Stefan condition which is the first condition in Eq. (12.2.159) then we get

$$K \{X_0/S_0(t)\}^2 \left. \frac{\partial u}{\partial x} \right|_{x=S(t)} = \rho l(S_0(t)) \frac{dS(t)}{dt}. \quad (12.2.162)$$

In [376],  $C(r)$  and  $l(s_0(t))$  are taken as

$$C(r) = x_0^2 C_0 / r^2, \quad l(S_0(t)) = l_0 x_0^2 / (S_0(t))^2, \quad C_0, l_0 \text{ are constants.} \quad (12.2.163)$$

We use Eq. (12.2.163) in Eqs (12.2.161), (12.2.162), and take  $S(t) = \delta \sqrt{t}$ , where  $\delta$  is a real positive constant. The solution of the problem which is now transformed into  $(x, t)$  coordinates can be easily obtained as we are now looking for the solution of a one-phase Neumann problem whose solution is given in [24]. On using Eq. (12.2.160) in the solution of one-phase Neumann problem,  $T_S(r, t)$  and  $r = S_0(t)$  can be obtained. Some particular cases of the solution have also been discussed in [376].

Several types of transformations other than the finite and infinite group transformations have been used in the literature to transform the nonlinear heat equation into an invariant linear equation which could be of the type of a partial differential equation or an ordinary differential equation. In either case it should be amenable to an exact solution or to some analytical solution which may help in obtaining easy numerical results or provide some useful information about solutions in some particular cases. The functional form of the nonconstant

thermo-physical parameters depending on dependent and independent variables is chosen in a suitable manner so that an analytical solution can be obtained. Of course it is often supported by experimental data.

The Stefan problem discussed below and considered in [378] is an extension of the problem considered in [379]. This extension consists of prescribing an overspecified condition in addition to the formulation considered in [379]. The problem considered in [379] is related to the evaporation of moisture in soil mechanics and for the formulation of this problem refer [379]. Some other references on soil-freezing are also given in [379]. The following one-phase solidification problem has been considered in [378] in the semiinfinite region  $0 \leq x < \infty$ .

$$\rho C \frac{\partial T(x, t)}{\partial t} = \frac{\partial}{\partial x} \left( K(T, x) \frac{\partial T(x, t)}{\partial x} \right) - v(T) \frac{\partial T(x, t)}{\partial x}, \quad 0 < x \leq S(t), \quad t > 0. \quad (12.2.164)$$

$$T(0, t) = -T_0, \quad T_0 > 0; \quad K(T(0, t), 0) \frac{\partial T}{\partial x}(0, t) = \frac{q_0}{\sqrt{t}}, \quad q_0 > 0; \quad t > 0. \quad (12.2.165)$$

$$K(T(S(t), t), S(t)) \frac{\partial T}{\partial x}(S(t), t) = \rho l \frac{dS}{dt}, \quad t > 0; \quad T(S(t), t) = 0, \quad (12.2.166)$$

where  $x = S(t)$  is the phase-change boundary,  $S(0) = 0$ . The second condition in Eq. (12.2.165) is an additional overspecified condition which is not there in [379]. Because of this overspecified condition one of the parameters (to be identified later) can be taken as unknown. Let the formulation given in Eqs (12.2.164)–(12.2.166) be denoted by Problem ( $P_1$ ). The conductivity  $K(T, x)$  and velocity  $v(T)$  in Eq. (12.2.164) are taken as in Eq. (12.2.167) in which  $a, b, d$  are constant.

$$K(T, x) = \rho C \frac{(1 + dx)}{(a + bT)^2}, \quad v(T) = \rho C \frac{d}{2(a + bT)^2}, \quad (a, b, d) > 0. \quad (12.2.167)$$

The formulation given earlier need not be explained as this type of formulation has been explained earlier also. The following transformations are used in Problem ( $P_1$ ) to convert the formulation suitable for a similarity solution.

$$y = \frac{2}{d} \left[ (1 + dx)^{1/2} - 1 \right], \quad \bar{S}(t) = \frac{2}{d} \left[ (1 + dS(t))^{1/2} - 1 \right], \quad (12.2.168)$$

$$dy^* = (a + b\bar{T}) dy + \frac{b}{(a + b\bar{T})^2} \frac{\partial \bar{T}}{\partial y} dt, \quad \bar{T}(y, t) = T(x, t), \quad y^*(0, 0) = 0, \quad (12.2.169)$$

$$y^* = y^*(y, t) = \int_{\bar{S}(t)}^y (a + b\bar{T}(\sigma, t)) d\sigma + (\alpha b + a) \bar{S}(t), \quad \alpha = \frac{l}{C} > 0, \quad (12.2.170)$$

$$T^*(y^*, t) = \frac{1}{a + b\bar{T}(y, t)}; \quad S^*(t) = y^*|_{y=\bar{S}(t)} = (\alpha b + a) \bar{S}(t); \quad \bar{T}(y, t) \neq -\frac{a}{b}, \quad (12.2.171)$$

$$S^*(t^*) = \sqrt{2\delta} t^*, \quad \bar{S}(t) = \sqrt{2\delta} t, \quad \delta \text{ is a positive real constant, } t^* = t, \quad (12.2.172)$$

$$\xi^* = y^*/\sqrt{2\delta t^*}; \quad T^* = T_0^* = \frac{1}{a - bT_0}, \quad \text{at } \xi^* = bq_0^* \sqrt{\frac{2}{\delta}}, \quad T_0 \neq \frac{a}{b}. \quad (12.2.173)$$

The transformation used above is the *Bäcklund transformation* which is further discussed later.

In view of the overspecified flux condition at  $x = 0$  which is prescribed in Eq. (12.2.165) as the second condition, the authors in [378] have used the following transformation for  $y^*(y, t)$  in place of the earlier transformation given in Eq. (12.2.170) which was used in [379].



$$\left. \begin{aligned} y^*(y, t) = y^* &= \int_0^y (a + b\bar{T}(\sigma, t))d\sigma + V(t) - V(0), \\ V(t) &= bq_0^*/\sqrt{t}, \quad q_0^* = q_0/\rho C. \end{aligned} \right\} \quad (12.2.174)$$

By using the transformations given in Eqs (12.2.168)–(12.2.172) and using Eq. (12.2.174) in place of Eq. (12.2.170) the nonlinear heat equation (12.2.164) is transformed to a linear equation in terms of  $y^*$  and  $t$  but other conditions in Eqs (12.2.165), (12.2.166) still depend on time. So Eq. (12.2.173) is further used to obtain a similarity solution for  $T^*(y^*, t) = \theta^*(\xi^*)$  in terms of the similarity variable  $\xi^*$ . Finally the Problem ( $P_1$ ) is reduced to the following problem which we denote by Problem ( $P_2$ ). The formulation of the Problem ( $P_2$ ) is given below in Eqs (12.2.175)–(12.2.178) and reported in [378] but calculations should be checked. Procedure is alright.

$$\frac{d^2\theta^*}{d\xi^{*2}} + \delta\xi^* \frac{d\theta^*}{d\xi^*} = 0, \quad bq_0^*\sqrt{\frac{2}{\delta}} < \xi^* < \alpha b + a; \quad \theta^*(\xi^*) = T^*(y^*, t), \quad (12.2.175)$$

$$\theta^*(\xi^*) = T_0^*, \quad \text{at } \xi^* = bq_0^*\sqrt{\frac{2}{\delta}}, \quad (12.2.176)$$

$$\frac{d\theta^*}{d\xi^*} = -q_0^*b\sqrt{2\delta}T_0^*, \quad \text{at } \xi^* = bq_0^*\sqrt{\frac{2}{\delta}}, \quad (12.2.177)$$

$$\frac{d\theta^*}{d\xi^*} = -(\alpha b/a)\delta, \quad \text{at } \xi^* = \alpha b + a; \quad \theta^* = \frac{1}{a}, \quad \text{at } \xi^* = \alpha b + a. \quad (12.2.178)$$

The solution of Eq. (12.2.175) can be easily written and

$$\theta^*(\xi^*) = A \operatorname{erf}\left(\sqrt{\frac{\delta}{2}}\xi^*\right) + B. \quad (12.2.179)$$

$A$ ,  $B$  and  $\delta$  are unknown constants, which are to be determined along with one of the parameters taken as unknown from  $a$ ,  $b$ ,  $d$ ,  $l$ ,  $C$ ,  $\rho$ , as there is an overspecified condition. Once the temperature is known, the complete solution of the Problem ( $P_2$ ) can be obtained. For complete solution refer [378]. To give some idea about the solution, we present here only a portion of the solution. If

$$\xi = y/\sqrt{2\delta t} = \frac{2}{d} \left[ \frac{(1 + dx)^{1/2} - 1}{\sqrt{2\delta t}} \right], \quad (12.2.180)$$

then  $\xi$  and  $\xi^*$  are related as

$$\xi = \int_{bq_0^*\sqrt{2/\delta}}^{\xi^*} \left[ A \operatorname{erf}(\sqrt{\delta/2}\sigma) + Bt \right] d\sigma, \quad \xi^*|_{\xi=0} = bq_0^*\sqrt{2/\delta}. \quad (12.2.181)$$

$$T(\xi) = \frac{1}{b} \left[ \frac{1}{\left( A \operatorname{erf}\left(\sqrt{\frac{\delta}{2}}\xi^*\right) + B \right)} - a \right], \quad (12.2.182)$$

$$x = S(t) = \sqrt{2\delta t} + d\delta t/2. \quad (12.2.183)$$

Six pairs of unknowns  $(\delta, a)$ ,  $(\delta, \rho)$ ,  $(\delta, C)$ ,  $(\delta, l)$ ,  $(\delta, b)$  and  $(\delta, d)$  can be formed and existence and uniqueness of the solution of Problem ( $P_1$ ) has been investigated with respect to

each pair. The uniqueness proof of the solution of Problem  $(P_1)$  by taking any of the six pairs of the parameters listed earlier as unknown requires some conditions to be satisfied. These conditions involve functions occurring in the complete solution obtained for the problem. All of this cannot be reported here as the procedure is too lengthy. We refer the reader to the original paper [378] for these proofs which are in the form of six theorems.

The solutions obtained in [378, 379] are applications of Bäcklund reciprocal transformation which will be discussed later in this section. Obtaining the expression for  $\xi$  in Eq. (12.2.181) is straightforward, calculate  $d\xi/d\xi^*$  and integrate. On using Eqs (12.2.171)–(12.2.179),  $T(\xi)$  can be obtained as given in Eq. (12.2.182). Eq. (12.2.183) is obtained when  $x$  is written in terms of  $y$  and the argument that  $x = S(t)$  implies  $y = \bar{S}(t)$  is used.

In a later work reported in [380], some of the authors of [378] have reconsidered the same problem which was considered in [378] with some minor changes which do not alter the solution procedure and most of the equations presented in the formulation and in the solution obtained in [378] have been retained with the same notations. The changes to be made are as given below.

The flux condition prescribed at  $x = 0$  in Eq. (12.2.165) is now in [380] a prescribed condition and not an overprescribed condition and there is no overspecified condition. Therefore the first condition in Eq. (12.2.165) will not be there. Further, instead of a one-phase freezing problem it is now a one-phase melting problem and so a solid at melting temperature  $T_m > 0$  occupies the region  $0 \leq x < \infty$  at time  $t = 0$ . Take  $q_0$  as  $-q_0$ ,  $q_0 > 0$  in Eq. (12.2.165) and replace  $l$  by  $-l$  in Eq. (12.2.166). Isotherm temperature is taken as  $T_m > 0$  and not zero as in Eq. (12.2.166). In place of  $y^*$  given in Eq. (12.2.170), a simpler expression for  $y^*$  has been derived in terms of which the formulation of the Problem  $(P_2)$  becomes simple. The  $y^*$  given in Eq. (12.2.170) is differentiated with respect to time. After some manipulations, we get

$$\frac{\partial y^*}{\partial t} = b \int_0^y \frac{\partial \bar{T}}{\partial t}(\sigma, t) d\sigma + bq_0^*/\sqrt{t}. \quad (12.2.184)$$

Eq. (12.2.184) on integration and after some steps gives

$$y^*(y, t) = \int_0^y (a + b\bar{T}(\sigma, t)) d\sigma + 2bq_0^*\sqrt{t}. \quad (12.2.185)$$

Since the range of  $y$  is  $0 \leq y \leq \bar{S}(t)$ , the range of  $y^*$  will be  $2bq_0^*\sqrt{t} \leq y^* \leq S^*(t^*)$  and the range of  $\xi^*$  becomes  $bq_0^*\sqrt{\frac{2}{\delta}} < \xi^* < 1$ ,  $\xi^*$  is as defined in Eq. (12.2.173). Note that  $\xi^*$  is a similarity variable and the solution of Problem  $(P_2)$  is obtained in terms of  $\xi^*$ . The formulation of the Problem  $(P_2)$  can be written easily incorporating changes in Eqs (12.2.175)–(12.2.178). For the changed problem and changed range of  $\xi^*$ , the solution can be obtained. The solution procedure for this changed problem also remains the same as explained earlier and will not be reported here. The following result about the uniqueness of the solution has been proved in [380] after establishing the existence and uniqueness of the parameter  $\delta$ .

**Theorem 12.2.1.** *The FBP has a unique solution of the similarity type for all data  $q_0$ ,  $\rho$ ,  $C$ ,  $l$ ,  $T_m$ ,  $a$ ,  $b$  and  $d$ .*

The proof of the theorem cannot be presented here as it is too lengthy. The solution of temperature prescribed condition at  $x = 0$  has also been obtained in [380]. There is no overspecified condition. For this case also, the formulation of the problem is given

by Eqs (12.2.164)–(12.2.167) provided some changes are made. In the first condition in Eq. (12.2.165) replace  $-T_0$  by  $T_0$ ,  $T_0 > T_m > 0$  and this is the only condition prescribed at  $x = 0$ . At  $x = S(t)$ ,  $T(S(t), t) = T_m$ , where  $T_m$  is the melting temperature. By using  $y^*(y, t)$  given in Eq. (12.2.170) another expression for  $y^*$  has been obtained for temperature prescribed condition by first obtaining  $\partial y^*/\partial t$  and then integrating it. The expression of  $y^*$  on integration which has been used for further calculations is as follows.

$$y^*(y, t) = \int_0^y (a + b\bar{T}(\sigma, t))d\sigma + \frac{b}{(a + bT_0)^2} \int_0^t \frac{\partial \bar{T}(0, \tau)}{\partial y} d\tau. \quad (12.2.186)$$

Now the range of  $y^*$  becomes

$$\frac{b}{(a + bT_0)^2} \int_0^{t^*} \frac{\partial \bar{T}(0, \tau)}{\partial y} d\tau < y^* < S^*(t^*), \quad t^* > 0, \quad t^* = t. \quad (12.2.187)$$

In terms of  $\xi^*$  (see Eq. 12.2.173), the range of  $\xi^*$  is

$$\frac{b}{(a + bT_0)^2 \sqrt{2\delta t^*}} \int_0^{t^*} \frac{\partial \bar{T}(0, \tau)}{\partial y} d\tau < \xi^* < 1. \quad (12.2.188)$$

$\xi^*$  is a similarity variable and to obtain a similarity solution, all the equations should be in terms of only  $\xi^*$ . The boundary condition at  $x = 0$  is

$$T^* \left( \frac{b}{(a + bT_0)^2 \sqrt{2\delta t^*}} \int_0^{t^*} \frac{\partial \bar{T}(0, \tau)}{\partial y} d\tau \right) = T_0^*, \quad T_0^* = \frac{1}{(a - bT_0)}. \quad (12.2.189)$$

The expression of  $T_0^*$  in Eq. (12.2.189) is obtained by using the property of Bäcklund transformation.

Eq. (12.2.189) suggests that if a similarity solution exists then for some  $\xi_0^*$

$$\int_0^{t^*} \frac{\partial \bar{T}(0, \tau)}{\partial y} d\tau = \xi_0^* \frac{(a + bT_0)^2}{b} \sqrt{2\delta t^*}, \quad \xi_0^* > 0. \quad (12.2.190)$$

The condition in Eq. (12.2.190) is necessary as in obtaining the similarity solution in terms of  $\xi^*$ , the condition (12.2.189) should hold for some  $\xi^* = \xi_0^*$ . This fixes the range of  $\xi^*$  as  $\xi_0^* < \xi^* < 1$ . Now the formulation of the problem can be written in terms of  $\xi^*$  and the solution can be obtained. The solution procedure has already been explained earlier and will not be repeated here. The solution is lengthy and cannot be presented here, refer [380] for it. Note that  $S^*(t^*) = \sqrt{2\delta t^*}$  in Eq. (12.2.172),  $\bar{S}(t) = \sqrt{2\delta t^*}$  and  $\xi_0^*$  is given by Eq. (12.2.190). Using this information the following relations have been obtained in [380].

$$\delta_1 = \delta/(\alpha b + a)^2, \quad \xi_0^* = -bq_0^* T_0^{*2} \sqrt{2/\delta}, \quad (12.2.191)$$

$q_0^*$  is given by the relation  $\frac{\partial \bar{T}}{\partial y}(0, t) = -q_0^*/\sqrt{t}$ ,  $\forall t > 0$ . Proving this result requires the availability of the complete solution of the problem.

Uniqueness of the similarity solution of the above melting problem has been proved for the data provided for  $q_0$ ,  $\rho$ ,  $C$ ,  $l$ ,  $T_m$ ,  $a$ ,  $b$  and  $d$ .

A one-phase one-dimensional melting problem has been considered in [381] whose formulation is given below.

$$\rho(T)C(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}(K(T)T_x), \quad 0 < x < S(t), \quad t > 0. \quad (12.2.192)$$

$$T(0, t) = T_0; \quad T(S(t), t) = T_m; \quad \left(K\frac{\partial T}{\partial x}\right)\bigg|_{x=S(t)} = -\rho_0 l \frac{dS}{dt}, \quad (12.2.193)$$

$\rho_0$  is the density at the free boundary which is constant,  $S(0) = 0$ . Let

$$\bar{T}(x, t) = (T(x, t) - T_m)/(T_0 - T_m), \quad T_0 > T_m > 0, \quad (12.2.194)$$

$$N(T) = \rho(T)C(T)/\rho_0 C_0, \quad L(T) = K(T)/K_0, \quad (12.2.195)$$

$$\eta = x/(2\sqrt{\alpha_0 t}); \quad f(\eta) = \bar{T}(x, t); \quad \alpha_0 = K_0/(\rho_0 C_0), \quad S(t) = 2\eta_0\sqrt{\alpha_0 t}, \quad (12.2.196)$$

where  $\eta_0 > 0$  is to be determined and  $\rho_0, C_0, K_0$  are some reference values for respective parameters. Using the previous substitutions given in Eqs (12.2.194)–(12.2.196) in Eqs (12.2.189), (12.2.192), (12.2.193), we get

$$[L(f)f'(\eta)]' + 2\eta N(f)f'(\eta) = 0, \quad 0 < \eta < \eta_0, \quad (12.2.197)$$

$$f(0) = 1; \quad f(\eta_0) = 0; \quad f'(\eta_0) = -2\eta_0\alpha_0\rho_0 l/[K(T_m)(T_0 - T_m)]. \quad (12.2.198)$$

The problem formulated in Eqs (12.2.197), (12.2.198) is equivalent to the following nonlinear integral equation of the *Volterra type*.

$$f(\eta) = 1 - \phi[\eta, L(f), N(f)]/\phi[\eta_0, L(f), N(f)], \quad (12.2.199)$$

$$\phi[\eta, L(f), N(f)] = (2/\sqrt{\pi}) \int_0^\eta E(t, f)/L(f)(t) dt, \quad (12.2.200)$$

$$E(x, f) = \exp\left(-2 \int_0^x pN(f(p))/L(f(p)) dp\right). \quad (12.2.201)$$

The third condition in Eq. (12.2.198) after using Eqs (12.2.199)–(12.2.201) becomes

$$E(\eta_0, f)/\phi(\eta_0, L(f), N(f)) = \eta_0 l \sqrt{\pi}/(c_0(T_0 - T_m)). \quad (12.2.202)$$

The existence of the solution of Eqs (12.2.199)–(12.2.201) has been proved by showing that some  $\eta_0 > 0$  exists such that Eq. (12.2.202) (which is Stefan condition) has at least one solution, and then using fixed point theorem the uniqueness of the solution for any  $\eta > \eta_0 > 0$  has been established. If  $N, L, T(x, t), \eta$ , etc., satisfy some conditions (cf. [381]) then the integral equation (12.2.199) has a unique solution. For full details see [381].

Flux prescribed boundary condition at  $x = 0$  has also been considered in [381]. In this case rescale  $\bar{T}(x, t)$  by taking  $T_0 = 0$  in Eq. (12.2.194) and replace the first condition  $f(0) = 1$  in Eq. (12.2.198) by the condition given below in Eq. (12.2.203). This is possible provided the prescribed flux is of the form  $-q_0/\sqrt{t}$ .

$$K(T_m(\bar{T}(0, t) + 1))\bar{T}_x(0, t) = -q_0/(T_m\sqrt{t}), \quad q_0 > 0. \quad (12.2.203)$$

In terms of  $f(\eta)$  defined in Eq. (12.2.196), we have

$$K(T_m(f(0) + 1))f'(0) = -2\sqrt{\alpha_0} q_0/T_m. \quad (12.2.204)$$

In this case the Volterra integral equation is given by

$$f(\eta) = l\eta_0\sqrt{\pi} (\phi[\eta_0, L(f), N(f)] - \phi[\eta, L(f), N(f)]) / (C_0 T_m E(\eta_0, f)), \quad (12.2.205)$$

where  $\phi$  and  $E$  are the same as defined in Eqs (12.2.200), (12.2.201). In this case also existence and uniqueness of the solution have been proved. Although no exact analytical solution or numerical solution has been obtained for the Volterra integral equation, this problem has been included in the present section as it deals with similarity solution and it cannot be adjusted in any other section.

Applications of reciprocal Bäcklund transformation have been used in [378, 379] for the analytical solutions of Stefan problems in which thermo-physical parameters are not constant. Reciprocal Bäcklund transformation has been described in [382] and many of its applications in FBP's can be found in the references given in [383, 384]. The *Bäcklund transformation* can be briefly described as follows.

The conservation law

$$\frac{\partial}{\partial t} \left\{ T \left( \frac{\partial}{\partial x}; \frac{\partial}{\partial t}; u \right) \right\} + \frac{\partial}{\partial x} \left\{ F \left( \frac{\partial}{\partial x}; \frac{\partial}{\partial t}; u \right) \right\} = 0, \quad (12.2.206)$$

$$\text{where } T \left( \frac{\partial}{\partial x}; \frac{\partial}{\partial t}; u \right) = T \left( u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots; \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2}, \dots \right), \quad (12.2.207)$$

$$F \left( \frac{\partial}{\partial x}; \frac{\partial}{\partial t}; u \right) = F \left( u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots; \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2}, \dots \right), \quad (12.2.208)$$

is transformed to the reciprocally associated conservation law

$$\frac{\partial}{\partial t^*} \left\{ T^* \left( \frac{\partial}{\partial x^*}; \frac{\partial}{\partial t^*}; u \right) \right\} + \frac{\partial}{\partial x^*} \left\{ F^* \left( \frac{\partial}{\partial x^*}; \frac{\partial}{\partial t^*}; u \right) \right\} = 0, \quad (12.2.209)$$

by the following Bäcklund transformation defined in Eqs (12.2.210), (12.2.211).

$$\left. \begin{aligned} dx^* &= Tdx - Fdt, \quad t^* = t \\ T^* &= \frac{1}{T(D^*; \partial^*; u)}, \quad F^* = \frac{-F(D^*; \partial^*; u)}{T(D^*; \partial^*; u)} \end{aligned} \right\} R, \quad R^2 = I, \quad (12.2.210)$$

$$D^* = \frac{\partial}{\partial x} = \frac{1}{T^*} \frac{\partial}{\partial x^*}, \quad \partial^* = \frac{\partial}{\partial t} = \frac{F^*}{T^*} \frac{\partial}{\partial x^*} + \frac{\partial}{\partial t^*}, \quad I \text{ is identity.} \quad (12.2.211)$$

We now consider a simple one-phase one-dimensional problem in the region  $0 \leq x < \infty$ , in which the temperature gradients are present only in the region  $0 \leq x \leq S(t) < \infty$  which means that initially the region  $0 \leq x < \infty$  is at the melting temperature  $T_m$ . The formulation of the problem in [383] is as given below in which  $x = S(t)$  is the phase-change boundary.

$$\rho C(T) \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left[ K(T) \frac{\partial T}{\partial x} \right], \quad 0 < x < S(t); \quad K(T) \frac{\partial T}{\partial x} \Big|_{x=0} = Q(t), \quad t > 0. \quad (12.2.212)$$

$$K(T) \frac{\partial T}{\partial x} \Big|_{x=S(t)} = l\rho \frac{dS}{dt}; \quad T(x = S(t), t) = T_m; \quad S(0) = 0. \quad (12.2.213)$$

By changing the signs of  $Q(t)$  and of latent heat  $l$ , the problem can be treated as a solidification or a melting problem. We use the transformation

$$\phi(T) = \int_{T_0}^T U(p) dp, \quad U = \rho C(T), \quad (12.2.214)$$

to transform the first equation in Eq. (12.2.212) to the following form

$$\frac{\partial}{\partial t}(\phi(T)) - \frac{\partial}{\partial x} \left[ \frac{K(T)}{\phi'(T)} \frac{\partial \phi}{\partial x} \right] = 0. \quad (12.2.215)$$

If the reciprocal transformation is used ( $R^2 = I$ ) in which

$$dx^* = \phi(T)dx + K(T)T_x dt, \quad t^* = t; \quad T^* = 1/\phi(T), \quad (12.2.216)$$

then a linear canonical form for Eq. (12.2.215) can be obtained which is given by

$$\frac{\partial T^*}{\partial t^*} = \frac{1}{\bar{K}^*} \frac{\partial^2 T^*}{\partial x^{*2}}, \quad \bar{K}^* > 0, \text{ a constant}, \quad (12.2.217)$$

provided

$$\frac{\phi'}{\bar{K}^* \phi^2} = K(T). \quad (12.2.218)$$

On using Eq. (12.2.214) in Eq. (12.2.218), after some manipulations we get

$$\frac{1}{(KU)^{1/2}} \left\{ \ln \left( \frac{U}{\bar{K}} \right)^{1/2} \right\}' = (\bar{K}^*)^{1/2}, \quad \text{dash stands for } \frac{d(\cdot)}{dT}. \quad (12.2.219)$$

It was observed by Storm [385], who obtained this relation in Eq. (12.2.219) that for simple monatomic metals the variations of  $KU$  and  $(\ln(U/\bar{K}))^{1/2}$ ' over a wide range of temperature is not significant. Therefore for such metals the l.h.s. in Eq. (12.2.219) can be regarded as a constant. So for *Storm-type metals* the nonlinear equation (12.2.215) can be converted to a linear differential equation with the help of reciprocal transformation. The solution of linear heat equation (12.2.217) can be easily obtained as obtained in Eq. (1.3.11). The next question is to obtain prescribed conditions in Eqs (12.2.212), (12.2.213) in terms of the variables  $T^*$ ,  $x^*$  and  $t^*$ . First, we obtain  $x^*$  in terms of  $x$ . From Eq. (12.2.216), we have

$$\frac{\partial x^*}{\partial x} = \phi(T); \quad \frac{\partial x^*}{\partial t} = K(T) \frac{\partial T}{\partial x} = \int_0^x \frac{\partial}{\partial x} \left( K(T) \frac{\partial T}{\partial x} \right) dx + K(T) \frac{\partial T}{\partial x} \Big|_{x=0} \quad (12.2.220)$$

$$= \int_0^x \frac{\partial}{\partial t} (\phi(T)) dx + Q(t). \quad (12.2.221)$$

The relation between  $x^*$  and  $x$  is now given by

$$x^* = \int_0^x \phi(T) dx + \bar{Q}(t) - \bar{Q}(0), \quad \text{take } x^*(0, 0) = 0, \quad \frac{\partial \bar{Q}(t)}{\partial t} = Q(t). \quad (12.2.222)$$

By differentiating  $T^* = 1/\phi(T)$  with respect to  $x^*$ , using Eq. (12.2.218) and the flux condition in Eq. (12.2.212), we get

$$\frac{1}{\bar{K}^*} \frac{\partial T^*}{\partial x^*} = -Q(t)T^*, \text{ on } x^* = \bar{Q}(t) - \bar{Q}(0) \text{ or on } x = 0. \quad (12.2.223)$$

Next we have to find the relation between  $x = S(t)$  and  $x^* = S^*(t^*)$ .

$$\begin{aligned} \frac{\partial x^*}{\partial t} &= K(T) \frac{\partial T}{\partial x} = - \int_x^{S(t)} \frac{\partial}{\partial x} \left( K(T) \frac{\partial T}{\partial x} \right) dx + K(T) \frac{\partial T}{\partial x} \Big|_{x=S(t)} \\ &= - \int_x^{S(t)} \frac{\partial}{\partial t} (\phi(T)) dx + l\rho \frac{dS}{dt}. \end{aligned} \quad (12.2.224)$$

On integration, we get

$$x^*(x, t) = \int_{S(t)}^x \phi(T) dx + (\phi(T_m) + l\rho) S(t). \quad (12.2.225)$$

When  $x^* = S^*(t^*)$ , we have  $x = S(t)$  and we get

$$S^* = x^* \Big|_{x=S(t)} = (\phi(T_m) + l\rho) S(t), \quad S^*(0) = 0. \quad (12.2.226)$$

Isotherm condition in terms of  $T^*$  is easy to obtain from the second equation in Eq. (12.2.216).

$$T^* \Big|_{x^*=S^*(t^*)} = (1/\phi(T)) \Big|_{x=S(t)} = 1/\phi(T_m). \quad (12.2.227)$$

To obtain Stefan condition in terms of  $T^*$  and  $S^*$ , Eqs (12.2.214), (12.2.216) can be used to derive the equation

$$\frac{dS^*}{dt^*} = (1/T^*) \frac{dS}{dt} - \frac{\partial T^*}{\partial x^*} \Big/ \left( \bar{K}^* T^* \right). \quad (12.2.228)$$

Using Eqs (12.2.226)–(12.2.228), we get

$$\frac{1}{\bar{K}^*} \frac{\partial T^*}{\partial x^*} = \frac{-l\rho dS^*/dt^*}{\phi(T_m)(\phi(T_m) + l\rho)} \text{ on } x^* = S^*. \quad (12.2.229)$$

The Stefan problem formulated in Eqs (12.2.212), (12.2.213) has now been reformulated in terms of  $T^*(x^*, t^*)$  and  $S^*(t^*)$ . As Eq. (12.2.217) is linear its solution can be easily written as done in Eq. (1.3.11). The solution will not be presented here. The original problem is in terms of  $T(x, t)$ ,  $x$  and  $S(t)$ .  $T(x, t)$  and  $S(t)$  can be obtained from Eqs (12.2.214), (12.2.226), respectively, but  $x$  is yet to be obtained in terms of  $x^*$ .

$$\begin{aligned} \frac{\partial x}{\partial x^*} &= T^*, \quad \frac{\partial x^*}{\partial t^*} = K(T) \frac{\partial T}{\partial x^*} = \int_{x^{**}}^{x^*} \frac{\partial}{\partial x^*} \left( \frac{1}{\bar{K}^*} \frac{\partial T^*}{\partial x^*} \right) dx^* + \frac{1}{\bar{K}^*} \frac{\partial T^*}{\partial x^*} \Big|_{x^{**}}, \\ &= \int_{\bar{Q}(t)-Q(0)}^{x^*} \frac{\partial T^*}{\partial t} dx^* - Q(t)T^* \Big|_{x^*=\bar{Q}(t)-Q(0)}, \end{aligned} \quad (12.2.230)$$

$$x^* = \int_{\bar{Q}(t)-Q(0)}^{x^*} T^* dx^*, \quad x^{**} = x^* \text{ at } x = 0. \quad (12.2.231)$$

As far as method of solution of FBP with nonlinear heat equation as discussed earlier for Storm-type materials is concerned, it is complete. It may be pointed out here that the solution obtained with the help of the above procedure is in general not a similarity solution. However, in some particular type of boundary conditions similarity solution is also possible. For example, if  $Q(t) = q_0/\sqrt{t}$ ,  $q_0$  is constant, then a similarity solution in terms of the similarity variable  $x/\sqrt{t}$  is possible. Similarly similarity solution is also possible if a constant temperature is prescribed at  $x = 0$  for the problem discussed earlier.

By using reciprocal transformation discussed earlier, similarity solution of a two-phase one-dimensional Stefan problem will now be discussed as studied in [384]. The heat equation is nonlinear and is of the type of Eq. (12.2.212) in both the solid and liquid phases. The solution procedure is the same for solidification problem as well as for melting problem. We discuss below a solidification problem.

Let  $T_S(x, t)$  and  $T_L(x, t)$  be the temperatures in the solid and liquid regions, respectively, and similarly the thermo-physical parameters will also be identified by the subscripts  $S$  and  $L$  in the two solid and liquid regions. Initially at  $t = 0$ , liquid occupies the region  $0 \leq x < \infty$  at constant temperature  $T_1 > T_m$ , where  $T_m$  is the freezing temperature. Melting starts at  $x = 0$  when flux is prescribed at  $x = 0$ , for  $t > 0$ . Liquid occupies the region  $0 \leq x \leq S(t)$ ,  $t > 0$ . The mathematical formulation can be briefly described as follows in which  $x = S(t)$  is freezing front.

$$\rho c_{S,L}(T_{S,L}) \frac{\partial T_{S,L}}{\partial t} = \frac{\partial}{\partial x} \left( K_{S,L}(T_{S,L}) \frac{\partial T_{S,L}}{\partial x} \right); \quad T_{S,L}|_{x=S(t)} = T_m, \quad (12.2.232)$$

$$K_L(T_L) \left. \frac{\partial T_L}{\partial x} \right|_{x=0} = -q_0/\sqrt{t}; \quad T_S|_{t=0} = T_1 < T_m, \quad q_0 > 0, \quad (12.2.233)$$

$$K_S(T_S) \left. \frac{\partial T_S}{\partial x} \right|_{x=S(t)} - K_L(T_L) \left. \frac{\partial T_L}{\partial x} \right|_{x=S(t)} = -l\rho \frac{dS}{dt}. \quad (12.2.234)$$

The solution procedure using reciprocal transformation which has been discussed for the problem described in Eqs (12.2.212), (12.2.213) can be used for the present problem also. For example, nonlinear equations in Eq. (12.2.232) can be transformed into equations of the form Eq. (12.2.217) and similarity solution of these equations in the two phases can be obtained by defining appropriate similarity variables. However, a slightly different procedure has been adopted (for convenience) in [384] and the same procedure will be discussed here also so that the same results obtained in [384] can be discussed.

In the nonlinear heat equations (12.2.232), the transformation (12.2.214) is used and after that when Eq. (12.2.218) is used, we get

$$\frac{\partial \phi_{S,L}}{\partial t} = \bar{K}_{S,L}^* \frac{\partial}{\partial x} \left( \frac{1}{\phi_{S,L}^2} \frac{\partial \phi_{S,L}}{\partial x} \right), \quad (12.2.235)$$

$\bar{K}_{S,L}^*$  as defined in Eq. (12.2.219).  $\bar{K}_{S,L}^*$  are different for solid and liquid. Define as  $x = S(t)$ , the phase-change interface and a similarity variable  $\xi$  as follows.

$$x = S(t) = \sqrt{2\delta t}, \quad \delta > 0; \quad \xi = x/S(t) = x/\sqrt{2\delta t}, \quad \xi = 1 \text{ on } x = S(t). \quad (12.2.236)$$

We now seek solutions of Eq. (12.2.235) in the form of functions  $\Psi_{S,L}(\xi)$  which are defined below.



$$\Psi_{S,L}(\xi) = \phi_{S,L}(x, t). \quad (12.2.237)$$

Let the reciprocal transformation be defined as

$$\left. \begin{aligned} d\xi &= \Psi_{S,L}^*(\xi_{S,L}^*) d\xi_{S,L}^*, \\ \Psi_{S,L}^* &= 1/\Psi_{S,L}. \end{aligned} \right\} R, \quad R^2 = I, \quad (12.2.238)$$

$\xi_{S,L}^*$  are also similarity variables. On  $x = S(t)$ , temperature is constant and so  $\phi_{S,L}$  and  $\Psi_{S,L}$  are also constant on  $x = S(t)$  or on  $\xi = 1$  for both the subscripts  $S$  and  $L$ . Let

$$\xi_{S,L}^* = \lambda_{S,L}, \quad \text{for } \xi_{S,L} = 1, \lambda_{S,L} > 0 \text{ are unknown constants.} \quad (12.2.239)$$

The equations in Eq. (12.2.235) are first transformed in terms of  $\Psi_{S,L}(\xi)$  and then using the reciprocal transformation given in Eq. (12.2.238), these equations can be obtained in terms of linear canonical forms which are given below for  $\Psi_{S,L}^*$ .

$$\overline{K}_{S,L}^* \frac{d^2 \Psi_{S,L}^*}{d\xi_{S,L}^{*2}} + \delta \xi_{S,L}^* \frac{d \Psi_{S,L}^*}{d\xi_{S,L}^*} = 0. \quad (12.2.240)$$

In deriving equation (12.2.240), an important equation given below is also obtained which is useful in calculating  $\lambda_S$  and  $\lambda_L$ .

$$\overline{K}_{S,L}^* \frac{d \Psi_{S,L}^*}{d\xi_{S,L}^*} = -\delta \left( \xi_{S,L}^* \Psi_{S,L}^* - \xi \right). \quad (12.2.241)$$

The solutions of Eq. (12.2.240) have already been discussed earlier and they will be of the same type as that of Eq. (12.2.25). The boundary condition for  $T_L$  at  $x = 0$  becomes an appropriate condition for the derivative of  $\Psi_L^*$  at  $\xi_L^*|_{\xi=0}$  and initial temperature which is prescribed for  $T_S(x, t)$  at  $t = 0$  is now the condition for  $\Psi_S^*$  prescribed at  $\xi_S^*|_{\xi=\infty}$ . If Eq. (12.2.241) is used then the Stefan condition becomes

$$\phi_L(T_m) - \phi_S(T_m) + \lambda_S - \lambda_L = l\rho. \quad (12.2.242)$$

It may be noted that in deriving equation (12.2.242) if Eq. (12.2.241) and Stefan condition are used then unknown constant  $\delta$  will not appear in it. Therefore in the Stefan condition we have to use the solutions of Eq. (12.2.240) and calculate fluxes at the moving boundary by using these solutions. The equation so obtained will contain  $\delta$ . There are in all seven constants to be determined. Four constants appear in the solutions of Eq. (12.2.240), two constants for each region, and another three are  $\delta$ ,  $\lambda_S$  and  $\lambda_L$ . There are four conditions at the moving boundary including one in Eq. (12.2.242) and two given in Eq. (12.2.233). Therefore, one more condition is to be generated which is obtained by expressing  $\xi$  in terms of  $\xi_L^*$  by using reciprocal transformation and then using the condition  $\xi_L^* = \lambda_L$  when  $\xi = 1$ . The following equation is obtained.

$$2\xi = \int_{q_0(2/\delta)}^{\xi_L^*} \frac{1}{2} \left\{ A_L \operatorname{erf} \left[ \left( \delta/2 \overline{K}_L^* \right)^{1/2} \sigma \right] + B_L \right\} d\sigma, \quad (12.2.243)$$

where  $A_L$  and  $B_L$  are constants appearing in the solution of Eq. (12.2.240) for liquid.

The complete solution of the problem can be obtained without much difficulty except that it requires lengthy calculations which are not being reported here. The solution so obtained is a similarity solution in this particular problem.

The problem discussed in [384] was considered in [386] to investigate the existence and uniqueness of the solution of the problem. It was proved that a unique solution exists provided  $q_0$  (see Eq. 12.2.233) is large enough to satisfy the condition

$$q_0 > \sqrt{K_L^*} G^{-1} \left( \sqrt{\frac{K_L^*}{K_S^*}} \frac{1}{(\phi_S(T_m)/\phi_S(T_0) - 1)} \right), \quad T_0 < T_m, \quad (12.2.244)$$

$$G(x) = \operatorname{erf}(x) + \exp(-x^2)/(\sqrt{\pi}x), \quad x > 0, \quad G^{-1} \text{ is inverse of } G. \quad (12.2.245)$$

$T_0$  is the same as  $T_1$  in Eq. (12.2.233). The method of proof relies mostly on the behaviours of various functions occurring in the solution of the problem (refer [386]). If instead of flux prescribed condition, temperature is prescribed at  $x = 0$  then also uniqueness of the solution holds, in particular, provided the same temperature is prescribed at  $x = 0$  which is obtained from the solution of the problem with flux prescribed condition at  $x = 0$ .

### Fractional Diffusion Equations

The heat flux vector defined in Eq. (1.4.1) is called *Fickian flux vector* and such systems are known as *Fickian diffusion systems*. A typical property of Fickian diffusion systems is that initial scalar pulse spreads with the shape of a Gaussian distribution with a velocity of  $\sim t^{-1/2}$ , where  $t$  is time. In non-Fickian diffusion systems, the initial pulse can spread with time  $\sim t^\gamma$ ,  $1 > \gamma > \frac{1}{2}$  in *superdiffusion systems* and  $0 < \gamma < \frac{1}{2}$  in *subdiffusion systems* and the probability density function is no longer Gaussian. In the present section, the interest in non-Fickian diffusion systems has arisen as with a changed heat flux law also a Stefan problem can be formulated and a similarity solution can be constructed. The physical implications of *non-Fickian diffusion* will be discussed after obtaining the solutions.

Consider the following non-Fickian law for heat-flux  $q$  in one dimension.

$$q = -K_f \frac{\partial^\alpha T}{\partial x^\alpha}, \quad 0 < \alpha \leq 1, \quad K_f \text{ is the non-Fickian conductivity}, \quad (12.2.246)$$

and replace the Fourier heat conduction equation by the following *fractional diffusion equation* (FDE).

$$\rho C \frac{\partial^\beta T}{\partial t^\beta} = K_f \frac{\partial}{\partial x} \left( \frac{\partial^\alpha T}{\partial x^\alpha} \right), \quad 0 < \beta \leq 1, \quad 0 < \alpha \leq 1. \quad (12.2.247)$$

The changed thermal conductivity  $K_f$  has dimensions  $(JS^{-\beta}m^{\alpha-2}K^{-1})$ . The derivatives in Eq. (12.2.247) are called fractional derivatives. The *fractional derivative* of a function  $f(x)$  is defined in [387] as

$$\frac{\partial^\alpha f(x)}{\partial x^\alpha} \equiv D_x^\alpha f(x) = I^{n-\alpha} \frac{d^n(f(x))}{dx^n} = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\xi)^{n-1-\alpha} \frac{d^n f(\xi)}{d\xi^n} d\xi, \quad (12.2.248)$$

$\alpha > 0$ , can be any real number,  $n-1 < \alpha < n$ , for some integer  $n$ .

$D_x^\alpha f(x)$  as defined in Eq. (12.2.248) are known as *Caputo fractional derivatives*. The integral in Eq. (12.2.248) can be evaluated using Laplace transform. In particular

$$D_x^\alpha c_0 = 0, c_0 \text{ is constant}; D_x^\alpha x^{\alpha+1} = \Gamma(\alpha+2)x; D_x^\alpha x^\alpha = \Gamma(\alpha+1); \alpha > 0, \quad (12.2.249)$$

$$D_x^\alpha x^\eta = \frac{\Gamma(\eta+1)}{\Gamma(\eta+1-\alpha)} x^{\eta-\alpha}, \quad \eta \geq 1 \geq \alpha > 0 \text{ or } 0 < \eta \leq \alpha \leq 1, \quad (12.2.250)$$

$\Gamma(\alpha)$  is the *gamma function*. If  $n$  is an integer  $\geq 1$ ,  $\Gamma(n) = |n-1|$ .

A quasisteady-state solution of the one-phase one-dimensional Stefan problem with fractional derivative has been obtained in [387]. Take  $C \rightarrow 0$  in Eq. (12.2.247) and consider the following formulation.

$$\frac{\partial}{\partial x} (D_x^\alpha T) = 0, 0 \leq x < S(t), t > 0; T(x=0) = 1, \quad (12.2.251)$$

$$T(x=S(t)) = 0; -D_x^\alpha(T)|_{x=S(t)} = D_t^\beta S(t), \alpha > 0, \beta \leq 1, \quad (12.2.252)$$

$x = S(t)$  is the free boundary,  $S(0) = 0$ . We take  $T(x, t) = 1 - x^\alpha/S^\alpha$ , then  $D_x^\alpha(T) = -\Gamma(\alpha+1)/S^\alpha$  and diffusion equation (12.2.251) is satisfied. If in the Stefan condition (12.2.252) we take  $S(t) = At^\gamma$ , where  $0 < \gamma = \beta/(1+\alpha) < 1$ , then the constant  $A$  is obtained as

$$A = \{\Gamma(1+\gamma-\beta)\Gamma(\alpha+1)/\Gamma(1+\gamma)\} \frac{1}{1+\alpha}. \quad (12.2.253)$$

Without any further explicit discussion it may be mentioned that in non-Fickian diffusion processes there are ‘waiting times’. The flux is not a function of local temperature gradient. If  $\beta$  and  $\alpha$  are suitably chosen then we can have superdiffusion and subdiffusion effects in the movements of the free boundary in the above solution.

In the Stefan problem considered in [388], exact solutions for two different types of thermal conductivities have been obtained. The formulation of the problem is as follows in which fractional powers are there.

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( D(x, g(T)) \frac{\partial T}{\partial x} \right), 0 \leq x \leq S(t); T(0, t) = 1, t > 0, \quad (12.2.254)$$

$$T(x, 0) = 0; T(S(t), t) = 0, t > 0; -D(S, g(0)) \frac{\partial T}{\partial x} \Big|_{S(t)} = l \frac{dS}{dt}, \quad (12.2.255)$$

$$D(x, g(T)) = x^{\frac{2n-1}{n}}, 0 < n < 1. \quad (12.2.256)$$

If a similarity variable  $\xi$  is introduced and  $S(t)$  is taken as given below

$$\xi = x/t^n, 0 < n < 1; S(t) = a_1 t^n, a_1 > 0 \text{ is an unknown constant}, \quad (12.2.257)$$

then the nonlinear diffusion equation in Eq. (12.2.254) is converted to

$$\xi^{\frac{2n-1}{n}} \frac{d^2 T}{d\xi^2} + \left\{ \frac{(2n-1)}{n} \xi^{\frac{(n-1)}{n}} + n\xi \right\} \frac{dT}{d\xi} = 0, \quad 0 \leq \xi \leq a. \quad (12.2.258)$$

The boundary condition at  $x = 0$  and moving boundary conditions become

$$T(\xi = 0) = 1; T(\xi = a_1; t > 0) = 0; \quad -\frac{a_1^{\frac{(n-1)}{n}}}{n} \frac{\partial T}{\partial \xi} \bigg|_{\xi=a_1} = 1. \quad (12.2.259)$$

Eq. (12.2.258) has a general solution given by

$$T(\xi) = c_1 \int_{n^2 \xi^{\frac{1}{n}}}^{\infty} \alpha^{-n} e^{-\alpha} d\alpha + c_2 = c_1 \Gamma(1 - n, n^2 \xi^{1/n}) + c_2, \quad (12.2.260)$$

$\Gamma(1 - n, n^2 \xi^{1/n})$  is the *upper incomplete gamma function* (see [389]).

The unknown constants  $c_1$ ,  $c_2$  and  $a_1$  can be determined using the three conditions in Eq. (12.2.255). The remaining part of the solution requires knowledge of special functions and refer [388] for it. For determining constant  $a_1$ , a single transcendental equation is to be solved. Numerical work has been presented.

In another problem, the thermal conductivity is taken as  $D \sim \left(-\frac{\partial T}{\partial x}\right)^{\frac{(1-2n)}{n}}$ ,  $0 < n < 1$ , in the formulation given in Eqs (12.2.254)–(12.2.256). In terms of the similarity variable  $\xi = x/t^n$ ,  $0 < n < 1$ , the heat equation in Eq. (12.2.254) is transformed to

$$\frac{d^2 T}{d\xi^2} - \left(\frac{n^2}{1-n}\right) \xi \left(-\frac{dT}{d\xi}\right)^{\frac{(3n-1)}{n}} = 0. \quad (12.2.261)$$

$S(t)$  is again taken as  $S(t) = a_1 t^n$ ,  $0 < n < 1$ ,  $a_1$  is a constant to be determined. Only the third equation in Eq. (12.2.259) now changes to

$$-\frac{\partial T}{\partial \xi} \bigg|_{\xi=a_1} = (\ln a_1)^{\frac{n}{(1-n)}}. \quad (12.2.262)$$

The solution of Eq. (12.2.261) satisfying the condition at  $\xi = a_1$  is given by

$$T(\xi) = 1 - \xi \left(c_1 \frac{(1-2n)}{n}\right)^{\frac{n}{1-2n}} {}_2F_1\left(\frac{1}{2}, \frac{-n}{(1-2n)}; \frac{3}{2}; \frac{n^2 \xi^2}{2c_1(1-n)}\right), \quad (12.2.263)$$

where  ${}_2F_1(\dots)$  is the *hypergeometric function* (see [389]) and  $c_1$  is a constant to be determined. The two constants  $a_1$  and  $c_1$  can be obtained by satisfying the condition at  $\xi = 0$  and the Stefan condition (12.2.262). In this problem, to determine the constants  $a_1$  and  $c_1$ , two simultaneous transcendental equations are to be solved numerically.

In both the problems considered in [388], analytical and numerical solutions for both the temperature and the moving boundary have been obtained and presented. For the numerical solutions of the classical Stefan problem formulations considered above, enthalpy method has been used which requires some additional considerations such as integrated internode diffusivity treatment. In some ranges of  $n$  and Stefan number, both analytical and numerical solutions compare well.

In the presence of heterogeneity of the medium, fast transport paths and/or regions of hold-up can result in anomalous heat diffusion resulting in the breakdown of analogy between

diffusion and Brownian motion. This new heat diffusion is then considered as non-Fickian diffusion. By using non-Brownian random walk processes, the anomalous diffusion can be modelled in terms of fractional derivatives [390], which results in FDEs of the type (12.2.247).

A simple quasisteady-state one-dimensional one-phase problem formulation in Eqs (12.2.251), (12.2.252) and its solution was presented in [388] in which the free boundary in Stefan condition is taken as a sharp-interface with fractional time derivatives. In [391], the following sharp-interface model with fractional time derivatives in both transient heat equation and the Stefan condition has been considered.

$$C \frac{\partial^\beta T}{\partial t^\beta} = \frac{\partial}{\partial x} \left( \frac{\partial T}{\partial x} \right), t > 0, 0 \leq x \leq S(t); T(0, t) = 1, 0 < \beta \leq 1. \quad (12.2.264)$$

$$\frac{\partial^\beta S}{\partial t^\beta} = -\frac{\partial T}{\partial x}; T(S(t), t) = 0; S(0) = 0, t \geq 0, \quad (12.2.265)$$

where  $x = S(t)$  is the free boundary. The previous equations are suitably scaled and since the scalings do not affect the solution procedure, scalings are not being given here. The fractional derivative in time implies nonlocality. Essentially the memory of the operating flux at a point  $x$  in space is taken as weighted sum of the temperature gradient at that point back through time. This results in fractional derivatives or *Caputo derivatives* of order  $0 < \beta \leq 1$ . The fractional derivative in space coordinates implies the local balance of a nonlocal flux and the operating flux at a point  $x$  and is a weighted sum of the upstream local temperature gradients.

In [391], an exact similarity solution of the problem considered in Eqs (12.2.264), (12.2.265) has been obtained by considering a weak or enthalpy formulation of heat equation (12.2.264). The enthalpy  $H(T)$  which is the sum of the sensible heat and latent heat has been modelled in the following way in obtaining the analytical solution.

$$H(T) = CT + f(\epsilon, T), \quad T = 0 \text{ is the isotherm temperature}, \quad (12.2.266)$$

where  $C$  is specific heat and  $f(\cdot)$  is the liquid fraction in the temperature range of  $0 \leq T \leq \epsilon$ .  $f = 1$  when  $T = \epsilon$ , and  $f = 0$  when  $T = 0$ . We assume that  $H$  has been suitably dimensioned. The region in which  $0 \leq T \leq \epsilon$  is called mushy region in which both solid and liquid are present. If classical formulation such as in Neumann problem is considered, which is a sharp phase-change interface model, then as  $\epsilon \rightarrow 0$ , the sharp-interface model and enthalpy model are equivalent provided further conditions are imposed on enthalpy solution (see Section 5.1).

In the enthalpy formulation of a sharp-interface two-phase model, the function  $f(\epsilon, T)$  has been taken in the following form which is independent of  $\epsilon$ .

$$f(\epsilon, T) = 1, \text{ if } T \geq 0 \quad \text{and} \quad f(\epsilon, T) = 0, \text{ if } T < 0, \quad (12.2.267)$$

where  $T = 0$  is the isotherm phase-change temperature. When  $\epsilon \rightarrow 0$ , we get Eq. (12.2.267) in place of Eq. (12.2.266). The analytical solution which is being presented below is about the enthalpy formulation of Stefan problem in which enthalpy  $H(T)$  is given by Eq. (12.2.266). It is assumed that in the limit  $\epsilon \rightarrow 0$ , we get the solution of a sharp-interface problem. Consider the following sharp-interface model of a one-phase one-dimensional melting problem.

$$C \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\partial T}{\partial x} \right), 0 < x \leq S(t), t > 0; \quad T(0, t) = 1, T(S(t), t) = 0, S(0) = 0. \quad (12.2.268)$$

$$\frac{dS}{dt} = - \left( \frac{\partial T}{\partial x} \right) \Big|_{x=S}; \quad T(x, t) = 0, \text{ for } x \geq S(t), t \geq 0; \quad T(x, 0) = 0, \quad x \geq 0. \quad (12.2.269)$$

The following enthalpy formulation of the previous problem is considered in [391] in the region  $0 \leq x \leq 1$ .

$$\frac{\partial H}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad 0 \leq x \leq 1; \quad T(0, t) = 1; \quad T(x, 0) = 0; \quad \frac{\partial T}{\partial x} \Big|_{x=1} = 0. \quad (12.2.270)$$

$$H(T) = CT + f(\epsilon, T), \quad (12.2.271)$$

as in Eq. (12.2.266). If  $C \rightarrow 0$  in Eqs (12.2.268), (12.2.271) and we also consider fractional time derivatives in time, then we get

$$\frac{\partial^2 T}{\partial x^2} = 0; \quad \frac{\partial^\beta S}{\partial t^\beta} = - \left( \frac{\partial T}{\partial x} \right) \Big|_{x=S}, \text{ for sharp-interface model,} \quad (12.2.272)$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial^\beta f(\epsilon, T)}{\partial t^\beta}, \quad 0 \leq x \leq 1, \text{ for the enthalpy model,} \quad (12.2.273)$$

together with the boundary and initial conditions in the respective formulations. Without giving the procedure used for scalings, we assume that all the equations in Eqs (12.2.268)–(12.2.273) are suitably scaled. If  $\alpha = 1$  is taken in Eqs (12.2.251)–(12.2.253) then, we get

$$T(x, t) = 1 - x/S(t); \quad S(t) = \left( \frac{\Gamma(1 - \beta/2)}{\Gamma(1 + \beta/2)} \right)^{1/2} t^{\beta/2}, \quad t \geq 0, \quad 0 < \beta \leq 1. \quad (12.2.274)$$

To obtain the solution of Eq. (12.2.273), we scale  $x$  and  $t$ . Let  $X_1 = S/x$ ;  $\bar{t} = t^*/t$ , where  $t^*$  is the time at which the point  $x$  changes phase (melts). In Eq. (12.2.273), the r.h.s. after using the definition of Caputo fractional derivative given in Eq. (12.2.248) can be written as

$$\left( \frac{\partial^\beta f}{\partial t^\beta} \right)_x = \frac{1}{\Gamma(1 - \beta)} \int_0^t (t - t')^{-\beta} \left( \frac{\partial f}{\partial t'} \right)_x dt'; \quad \left( \frac{\partial f}{\partial t'} \right)_{(x, t^*)} = \delta(t' - t^*). \quad (12.2.275)$$

It may be noted that at  $t = t^*$ ,  $f(x, t^*)$  is discontinuous and so its derivative is Dirac delta function. On evaluating the integral in Eq. (12.2.275) and making use of Eq. (12.2.273) and scalings of  $x$  and  $t$ , we get the following equations.

$$\frac{\partial^2 T}{\partial x^2} = \frac{(t - t^*)^{-\beta}}{\Gamma(1 - \beta)}, \quad 0 \leq x \leq S(t); \quad \frac{\partial^2 T}{\partial X_1^2} = S^2 t^{-\beta} \frac{(1 - \bar{t})^{-\beta}}{\Gamma(1 - \beta)}, \quad 0 \leq X_1 \leq 1. \quad (12.2.276)$$

It is assumed that  $S = a_1 t^{\beta/2}$ ,  $a_1$  is unknown. To make the r.h.s. of second equation in Eq. (12.2.276), a function only of  $X_1$ , an inverse approach is followed and it is assumed that  $X_1 = \bar{t}^{\beta/2}$ . If in the end the constant  $a_1$  appearing in  $S(t) = a_1 t^{\beta/2}$  can be obtained as a real positive constant then the assumption about  $X_1 = \bar{t}^{\beta/2}$  is justified. Substituting  $\bar{t}$  in terms of  $X_1$  and  $t$  in terms of  $S$  in the second equation in Eq. (12.2.276), we get

$$\frac{d^2 T}{dX_1^2} = a_1^2 \frac{(1 - X_1^{2/\beta})^{-\beta}}{\Gamma(1 - \beta)}, \quad 0 \leq X_1 \leq 1; \quad T(X_1 = 0) = 1, \quad T(X_1) = 0, \quad \forall X_1 \geq 1. \quad (12.2.277)$$

To integrate Eq. (12.2.277),  $\frac{dT}{dX_1}$  at  $X_1 = 0$  is required which is obtained from  $\partial T/\partial x|_{x=0}$ . If Eq. (12.2.273) is integrated, then we get

$$-\frac{\partial T}{\partial x}\bigg|_{x=0} = \frac{d\beta}{dt\beta} \left( \int_0^1 f dx \right). \quad (12.2.278)$$

If it is assumed that in the limit as  $\epsilon \rightarrow 0$ , a sharp interface  $x = S(t)$  is recovered and the definition of Caputo fractional derivative is used then  $dT/dX_1$  is obtained as (see [391]) given below

$$\frac{dT}{dX_1} = \frac{a_1^2 X_1}{\Gamma(1-\beta)_2} F_1 \left( \frac{\beta}{2}, \beta; 1 + \frac{\beta}{2}; X_1^{2/\beta} \right) + c_1. \quad (12.2.279)$$

Further integration will give the solution of temperature involving two constants of integration which can be determined from the conditions  $T(X_1 = 0) = 1$  and  $T(X_1 = 1) = 0$ . The constant  $a_1$  appearing in the expression of  $S(t)$  can be obtained from Eq. (12.2.278) after calculating  $dT/dX_1|_{X_1=0}$ . Finally, leaving many intermediate calculations, we get

$$S(t) = a_1 t^{\beta/2} = \left( \frac{2}{\Gamma(1+\beta)} \right)^{1/2} t^{\beta/2}. \quad (12.2.280)$$

On comparing  $S(t)$  given in Eq. (12.2.280) with  $S(t)$  given in Eq. (12.2.274), we find that both will agree only when  $\beta = 1$ , i.e. when the problem is not formulated in fractional time derivatives of enthalpy as given in Eq. (12.2.273) in which enthalpy is given by Eq. (12.2.271). The author in [391] has used the term *diffuse interface models* for such enthalpy models. In the sharp-interface model as in Eqs (12.2.264), (12.2.265) fractional time derivatives are present but it is not a diffuse-interface model. The memory of the phase-change evolution is stored on the moving interface  $S(t)$ , whereas in the diffuse-interface model the memory of the phase-change evolution is stored in the melted fraction of the domain  $0 \leq x \leq S$ . Therefore the diffuse-interface model will not converge to the sharp-interface model as  $\epsilon \rightarrow 0$ .

In presenting the previous closed-form similarity solution another consideration was to highlight the above discrepancy in the solutions of two models and their causes. Before closing the discussion on the fractional Stefan problem it may be mentioned that according to the definition of Caputo fractional derivative given in Eq. (12.2.248).

$$\frac{\partial}{\partial z} \left( \frac{\partial^\beta \phi}{\partial z^\beta} \right) \neq \frac{\partial^{\beta+1} \phi}{\partial z^{\beta+1}}, \quad n-1 < \beta \leq n, \text{ for some integer } n, \beta > 0. \quad (12.2.281)$$

This means that the definition of heat flux in the heat equation cannot be the same as in Stefan condition. This difficulty can be overcome if another definition of fractional derivative known as *Riemann-Liouville fractional derivative* is used which is given below and which is the  $n$ th-order derivative of the  $(n - \beta)$  fractional derivative [392].

$$\left( \frac{\partial^\beta \phi}{\partial z^\beta} \right)_{RL} = \frac{1}{\Gamma(n-\beta)} \frac{\partial^n}{\partial z^n} \int_0^z (z-\xi)^{n-1-\beta} \phi(\xi) d\xi, \quad n-1 < \beta < n, \quad (12.2.282)$$

$$\frac{\partial}{\partial z} \left( \frac{\partial^\beta \phi}{\partial z^\beta} \right)_{RL} = \left( \frac{\partial^{\beta+1} \phi}{\partial z^{\beta+1}} \right)_{RL}. \quad (12.2.283)$$

The numerical solution requires special techniques to calculate the fractional time and space derivatives approximately and for all this reader is referred to [391]. The analytical and numerical solutions agree closely.

### *Problems With Kinetic Conditions at the Phase-Change Interface: Asymptotic Analysis of Small- and Large-Time Behaviours*

In Chapter 4, enough space has been devoted to the analysis of solidification of supercooled liquids. The analysis included formulations using phase-field model and Landau-Ginsburg free energy functional, finite-time blow-up, essential and nonessential blow-up, regularization of blow-up, analysis of solidification problems of supercooled melts with modified Gibbs-Thomson condition at the moving interface, etc. Brief discussions centred around the works presented in references [114–143]. Although occasionally some reporting of partial analytical solutions was done, no systematic effort in Chapter 4 was made to present quasianalytical solutions and their methods concerning Stefan problems with Gibbs-Thomson condition.

The problem of solidification of supercooled melts is not only interesting from analysis point of view, it is also a technologically important area. Supercooled liquids can solidify much more rapidly than a nonsupercooled liquid and when rapid solidification occurs, the liquid may not have time to form its usual crystalline structure. A material formed from a supercooled melt, which is usually called a glassy or amorphous solid, can present greater corrosion resistance, toughness strength, hardness and elasticity than common materials. Such materials are currently in use in medicine, defence and aerospace equipment, electronics and sports [393].

In the solidification of a supercooled liquid, the freezing temperature is not the temperature  $T_m$ , which is usually taken as the freezing or melting temperature of a pure liquid or solid. This new freezing temperature is considered to be a function of interface velocity and the velocity of the interface is given by Stefan condition. The freezing temperature of a supercooled liquid is unknown so a condition is required to obtain it. If we denote this unknown temperature by  $T_L(S(t)) = T_u(t)$  then ‘why and how  $T_u$  is related to the velocity of phase-change interface  $x = S(t)$ ’. Using statistical mechanics arguments it is shown in [394] that  $\frac{dS}{dt} = S_t$  can be expressed as a function of  $T_u$  and in linearized form  $T_u = T_m - \varepsilon S_t$ ,  $\varepsilon > 0$  is called kinetic undercooling coefficient. For the complete details of obtaining  $T_u$ , see [393].

We shall now analyse the behaviour of the solution of one-phase solidification problem of a supercooled liquid occupying the region  $0 \leq x < \infty$  as discussed in [393]. The scaling is done as follows.

$$\begin{aligned} \bar{\theta}_L &= (T_L - T_m)/\Delta T, \bar{t} = t/t_0, \bar{x} = x/a, \bar{S} = S/a, \bar{T}_u = (T_u - T_m)/\Delta T, \\ \Delta T &= T_m - T_\infty, t_0 = a^2/k_L \text{ is reference time, } a = \varepsilon k_L/\Delta T, T_\infty = T_L|_{x \rightarrow \infty}. \end{aligned} \quad (12.2.284)$$

Dropping the bar notation over different variables, the formulation of the problem is given by

$$\frac{\partial \theta_L}{\partial t} = \frac{\partial^2 \theta_L}{\partial x^2}, \quad S(t) < x < \infty; \theta_L|_{x=S(t)} = T_u(t); \beta S_t = -\frac{\partial \theta_L}{\partial x} \Big|_{x=S(t)}. \quad (12.2.285)$$

$$\theta_L \rightarrow -1 \text{ as } x \rightarrow \infty; \theta_L(x, 0) = -1; S(0) = 0; \beta = l/(C_L \Delta T), \quad (12.2.286)$$



$\beta$  is Stefan number. The scaled or dimensionless  $T_u$  is obtained by solving the following equation [394].

$$S_t = -T_u \exp\left(\frac{QT_u}{P + T_u}\right), \quad P = T_m/\Delta T, \quad (12.2.287)$$

for  $Q$  see [393]. Note that  $T_u(0) = T(0, 0) = -1$  and from Eq. (12.2.287),  $S_t(0) = 1$  provided linearized version of r.h.s. in Eq. (12.2.287) is taken. When  $T_u \equiv 0$  then  $T(S(t)) = 0$  and as  $S(0) = 0, T(0, 0) = 0$ . But  $T(x, 0) = -1$  implies  $T(0, 0) = -1$ . The discontinuity in the temperature at  $(0, 0)$  is the cause of infinite velocity of  $S(t)$  at  $t = 0$ .

Whilst dealing with the Stefan problems connected with supercooled melts several pertinent questions arise. Some of them are (1) existence and uniqueness of the analytical solutions and if the solutions exist then determination of solution; (2) Will the solution blow-up in finite-time? and How and at what time the blow-up occurs?; (3) behaviour of the solutions for small and large times and (4) effect of undercooling and/or of surface tension on the behaviour of solutions. Of course before all these questions are addressed preliminary information about the process of supercooling and the parameters affecting it should be discussed. We have made here till now and in the earlier chapters a very moderate beginning with the last enquiry. Because of the space limitations further discussion is not possible.

If the moving Boundary is immobilized with the help of the transformation

$$y = x - S(t), \quad T_L(x, t) = F(y, t), \quad (12.2.288)$$

then the problem stated in Eqs (12.2.285), (12.2.286) becomes as follows:

$$\frac{\partial^2 F}{\partial y^2} = \frac{\partial F}{\partial t} - S_t \frac{\partial F}{\partial y}, \quad 0 < y < \infty; \quad \beta S_t = \frac{-\partial F}{\partial y}, \quad \text{at } y = 0. \quad (12.2.289)$$

$$F|_{y=0} = T_u(t), \quad F = -1, \quad \text{as } y \rightarrow \infty; \quad F(y, 0) = -1; \quad S(0) = 0. \quad (12.2.290)$$

If a similarity transformation  $\eta = y/t^\alpha$ , for some positive real number  $\alpha$ , and  $F(y, t) = G(\eta)$  are used in the above equations then we get

$$G'' = -\left(\alpha \eta t^{2\alpha-1} + t^\alpha S_t\right) G', \quad 0 < \eta < \infty; \quad G(0) = T_u(t). \quad (12.2.291)$$

$$\beta t^\alpha S_t = -G'(0); \quad S(0) = 0; \quad G|_{\eta \rightarrow \infty} = -1. \quad (12.2.292)$$

Similarity solution is possible for  $\alpha = 0$  and  $\alpha = 1/2$  as this eliminates time  $t$  from the differential equation. Of course  $T_u(t)$  still remains to be managed.

For  $\alpha = 1/2$  and  $T_u(t) = 0$ , we can obtain a particular solution which is a particular case of Neumann solution. In this case if velocity of  $S(t)$  is written as  $S_t = 2\delta\sqrt{t}$  then  $\delta = -G'(0)/\beta$  and to determine  $\delta$  we have the equation

$$\beta\sqrt{\pi} \delta \operatorname{erfc}(\delta) e^{\delta^2} = 1. \quad (12.2.293)$$

A real positive  $\delta$  exists for  $\beta > 1$  and for  $\beta < 1$ , no solution exists. For  $\beta = 1, \delta \rightarrow \infty$ .

### Small-Time Solution

We now consider Stefan problem with linear kinetic undercooling which is formulated in Eqs (12.2.289), (12.2.290). In subsequent discussion, different forms of kinetic conditions and behaviours of  $T_L(x, 0)$  as  $x \rightarrow \infty$  will be considered. If  $T_u(0) = T(0, 0) = -1$  and linearized version of r.h.s. of Eq. (12.2.287) is taken then  $S_t(0) = 1$ . This indicates a different type of scaling of equations to be done. As  $F(y, 0) = -1$  in Eq. (12.2.290) a similarity solution close to  $F = -1$  should be attempted for short time. The variables  $\eta$  and  $F(y, t)$  are redefined as

$$\eta = y/t^{1/2}, \quad F(y, t) = -1 + t^{1/2}G(\eta). \quad (12.2.294)$$

From Eq. (12.2.292),  $\beta S_t = \frac{-\partial F}{\partial y}$  at  $y = 0$  or  $\beta S_t = -G'(0)$  which gives finite velocity for  $S(t)$ . To obtain small-time solution, transform equations (Eqs 12.2.289, 12.2.290) according to Eq. (12.2.294). Next define  $t \in \tau$  and  $S \in \hat{S}$  where  $\epsilon$  is an arbitrary small parameter. Expand  $G(\eta)$  and  $S(t)$  in powers of  $\epsilon^{1/2}$  as

$$G(\eta) = G_0(\eta) + O(\epsilon^{1/2}), \quad \hat{S}(\tau) = \hat{S}_0(\tau) + \epsilon^{1/2}\hat{S}_1(\tau) + O(\epsilon). \quad (12.2.295)$$

Analytical solutions of  $G_0(\eta)$ ,  $\hat{S}_0(\tau)$  and  $\hat{S}_1(\tau)$  can be obtained. Higher-order terms in temperature cannot be obtained analytically. In terms of the original variable  $t$

$$S(t) \approx t \left( 1 - \frac{4\beta\sqrt{t}}{3\sqrt{\pi}} \right), \quad S_t = 1 \text{ as } t \rightarrow 0; \quad G_0(\eta) = \beta \left[ \frac{2}{\sqrt{\pi}} e^{-\eta^2/4} - \eta \operatorname{erfc} \left( \frac{\eta}{2} \right) \right]. \quad (12.2.296)$$

### Large-Time Solution

Three cases arise.

**Case 1.**  $\beta < 1$ : We choose  $\alpha = 0$ . In this case from Eq. (12.2.292),  $S_t = -G'(0)/\beta$  and  $G'' = -\eta - S_t G'$ ,  $0 < \eta < \infty$ . If  $\delta = -G'(0)/\beta$  then  $G(\eta) = -1 + (1 - \delta)e^{-\delta\eta}$  and  $G'(0)/\beta = -\delta$  gives  $\delta = 1 - \beta$ . Finally we get a travelling wave solution

$$G = -1 + \beta e^{-\eta(1-\beta)}; \quad \delta = (1 - \beta), \quad S_t(0) \neq 1. \quad (12.2.297)$$

**Case 2.**  $\beta > 1$ : In this case take  $\alpha = 1/2$  and for large-time solution make the substitutions  $t = \tau/\epsilon$  and  $S(t) = \hat{S}(\tau)/\epsilon^{1/2}$  in Eqs (12.2.291), (12.2.292). The leading order temperature solution and moving interface solutions are

$$T_L \simeq -1 + \frac{\operatorname{erfc}(x/2\sqrt{t}) - C_1}{\operatorname{erfc}(\delta)}, \quad S(t) \simeq 2\delta\sqrt{t} + C_1, \quad (12.2.298)$$

where  $C_1$  is an integration constant which cannot be determined as no initial condition for  $S(t)$  at  $t = 0$  is available.  $\delta$  is obtained from Eq. (12.2.293). Note that this solution is not affected by undercooling. If  $C_1 = 0$  then the solution in Eq. (12.2.298) is a particular case of one-phase Neumann solution.

**Case 3.**  $\beta = 1$ : This case requires different type of arguments and the analysis is more delicate. See [393] for further details. The solution is given below

$$T_L \simeq -1 + \exp \left( \frac{x - S(t)}{(3t)^{1/3}} \right), \quad S \simeq \left( \frac{9}{8} \right)^{1/3} t^{2/3} + C_2. \quad (12.2.299)$$

The previous solution is the zeroth-order solution in which no similarity transformation is used and the power of the perturbation parameter  $\epsilon$  is not predecided but is found from the feasibility of the perturbation solution.

### Nonlinear Undercooling: Small- and Large-Time Solutions

We take  $F|_{y=0} = T_u(t)$  from Eq. (12.2.290) and substitute  $T_u = -1 + t^{1/2}G(0)$  in Eq. (12.2.287), the following expression for  $S_t$  is obtained.

$$S_t = (1 - t^{1/2} G(0)) \exp \left\{ \frac{Q(-1 + t^{1/2} G(0))}{P + (-1 + t^{1/2} G(0))} \right\}. \quad (12.2.300)$$

Using perturbation series expansions as in Eq. (12.2.295), a small-time solution is obtained which shows that  $S_t \simeq t^{1/2}$ .

For  $\beta < 1$ , in nonlinear supercooling as in Eq. (12.2.300), following the analysis used for obtaining the solution in Eq. (12.2.298), a travelling wave solution is obtained which can be considered a large-time solution as this solution does not satisfy the initial condition. The cases  $\beta > 1$  and  $\beta = 1$  were also discussed in [393]. For obtaining asymptotic behaviours in problems with linear and nonlinear undercooling, heat balance integral method (HBIM) was also employed. The HBIM solution will be discussed briefly at a later stage in the section dealing with approximate methods.

Asymptotic results obtained from perturbation analysis and results obtained through approximate HBIM were both compared with numerical solution. It was found that asymptotic results agree with numerical solution for only a small range of time whereas results obtained by HBIM agree very closely with numerical solution and found better than the asymptotic results. In the nonlinear case some of the results which could not be found by asymptotic analysis could be obtained by HBIM. Neumann solution in Eq. (12.2.293) was found to be highly inaccurate (not agreeing with numerical solution) as  $\beta \rightarrow 1+$  when  $T_u = -1$  is taken. For larger times velocities in linear and nonlinear models converged.

Gibbs-Thomson condition at the phase-change boundary given below in one-dimensional one-phase radially symmetric spherical solidification problem has been considered in [395]. It is assumed that radius of curvature of interface is large.

$$T_S = T_m \left( 1 + \frac{2\gamma}{l\hat{S}(t)} \right), \quad (12.2.301)$$

where  $\gamma$  is surface tension coefficient,  $l$  is latent heat,  $r$  is the radial coordinate,  $r = \hat{S}(t)$  is the phase-change boundary,  $\hat{S}(t) \leq r \leq r_0$ ,  $\hat{S}(0) = r_0$ ,  $t \geq 0$ . The problem formulation after suitable scalings of variables (see [395] for scalings) is given below.

$$\frac{\partial T_S}{\partial t} = \frac{2}{r} \frac{\partial T_S}{\partial r} + \frac{\partial^2 T_S}{\partial r^2}, S(t) \leq r \leq 1, t > 0; T_S = 1 \text{ at } r = 1; \frac{dS}{dt} = -\Delta \frac{dT_S}{dr} \Big|_{r=S}. \quad (12.2.302)$$

$$\Delta = C_S(T_m - T_S)/l \text{ is the Stefan number; } T_S = -\sigma/S(t) \text{ at } r = S(t). \quad (12.2.303)$$

If the following transformations are used

$$\eta = \frac{1-r}{1-S(t)} = \frac{1-r}{\tau(t)}; \quad \theta = rT_S = (1-\eta\tau)T, \quad (12.2.304)$$

then the freezing front is fixed at  $\eta = 1$  and the region under consideration becomes  $0 \leq \eta \leq 1$ .  $\tau(t) = 1 - S(t)$  is another variable describing the instantaneous position of  $r = S(t)$ ,  $S(0) = 1$ .

Finally the problem to be solved is as follows:

$$-\tau^2 \frac{dS}{dt} \frac{\partial \theta}{\partial \tau} + \eta \tau \frac{dS}{dt} \frac{\partial \theta}{\partial \eta} = \frac{\partial^2 \theta}{\partial \eta^2}, \quad \theta = 0, \text{ at outer surface } \eta = 0, \quad (12.2.305)$$

$$\frac{dS}{dt} = \frac{\Delta}{\tau} \left( \frac{1}{1-\tau} \frac{\partial \theta}{\partial \eta} \Big|_{\eta=1} + \frac{\tau}{(1-\tau)^2} \theta \Big|_{\eta=1} \right), \quad \theta = -\sigma, \text{ for } \eta = 1. \quad (12.2.306)$$

To obtain the solution of  $\theta(\eta, \tau)$ , it is expanded in a series as

$$\theta(\eta, \tau) = \sum_{m=0}^{\infty} \tau^m \theta_m(\eta). \quad (12.2.307)$$

$\frac{dS}{dt}$  obtained by using Eq. (12.2.306) plays an important part in using the series expansion given in Eq. (12.2.307) as Eq. (12.2.305) becomes an equation only in terms of  $\theta(\eta, t)$ . Solutions for  $\theta_0$  and  $\theta_1$  are reported in [395] but solutions have been obtained up to  $O(\tau^4)$  which have been used in the numerical work. Procedure may look simple but to obtain the solution lot of effort is required. Several other results such as freezing rates for  $\Delta \ll 1$ ,  $\tau \ll \Delta \ll 1$  and  $\Delta \ll 1 \simeq \tau$  have been obtained.

Numerical work using analytical solution was done and to improve the convergence rate of the series and for using the solution for wider time range nonlinear *Shank's transformation* [396] was used. It was found that surface tension increases the equilibrium temperature and so speeds up the freezing rate. The solution is not valid when the freezing front reaches close to the centre of the sphere as radius of curvature is no more large.

Using a solution procedure similar to the one adopted in [395], the authors in [397] have obtained a solution of outward solidification of supercooled melt in radially symmetric spherical geometry. Some changes are essential in the formulation of outward solidification. The one-dimensional region under consideration is  $0 < r_0 \leq r < \infty$ . Gibbs-Thomson condition at the phase-change interface  $r = \hat{S}(t)$  is taken as

$$T_L = T_m \left( 1 - \frac{2\gamma}{l\hat{S}(t)} \right), \quad \gamma \text{ is surface tension coefficient.} \quad (12.2.308)$$

The temperature of the liquid which is supercooled should be less than the temperature  $T_L$  given in Eq. (12.2.308) for crystal to grow. Therefore the size of the solid nucleus  $\hat{S}_0(t)$  should be greater than the minimum size for solidification to grow, i.e.

$$b = \hat{S}_0(t) > 2\gamma T_m / (l(T_m - T_\infty)), \quad T_\infty = T_L |_{r \rightarrow \infty}. \quad (12.2.309)$$

It is assumed that at  $t = 0$ ,  $b$  satisfies the condition (12.2.309). It seems appropriate to do the scaling as follows.

$$\bar{T}_L = (T_L - T_m)/(T_\infty - T_m), \bar{r} = r/b, \bar{S}(t) = \hat{S}(t)/b, \bar{t} = k_L t/b^2. \quad (12.2.310)$$

Dropping the bar over the variables defined in Eq. (12.2.310) and using the following transformations,

$$\eta = (r - 1)/(S(t) - 1) = (r - 1)/\tau(t); T_L = \theta_L/r = \theta_L/(1 + \eta\tau), \quad (12.2.311)$$

we get the transformed heat equation as in Eq. (12.2.312).

$$\tau^2 \frac{dS}{dt} \frac{\partial \theta_L}{\partial \tau} - \eta \tau \frac{dS}{dt} \frac{\partial \theta_L}{\partial \eta} = \frac{\partial^2 \theta_L}{\partial \eta^2}, \quad 1 \leq \eta < \infty; \theta_L \rightarrow 1 + \eta\tau \text{ as } \eta \rightarrow \infty. \quad (12.2.312)$$

$$\theta_L = \sigma \text{ at } \eta = 1, \sigma = \beta/\Delta, \beta = 2C_L \sqrt{T_m}/(bl^2), \Delta = C_L(T_m - T_\infty)/l. \quad (12.2.313)$$

In view of the condition (12.2.309), we have  $\sigma \leq 1$ . The Stefan condition transforms to

$$\frac{dS}{dt} = \frac{\Delta}{\tau} \left( \frac{1}{(1 + \tau)} \frac{\partial \theta_L}{\partial \eta} \Big|_{\eta=1} - \frac{\tau}{(1 + \tau)^2} \theta_L \Big|_{\eta=1} \right). \quad (12.2.314)$$

The solution for  $\theta_L$  is obtained by assuming a series expansion as in Eq. (12.2.307). The zeroth-order and first-order solutions are reported in [397] and solutions up to  $O(\tau^5)$  were obtained which were used in the numerical work. Convergence of results improved by using Shank's nonlinear transformation [396]. Many useful analytical results have been obtained and reported such as the growth rate of the nucleus, critical surface tension.

It was found that surface tension reduces the growth rate of the interface. When  $\gamma = 0$ , phase growth rate drops monotonically with time. The numerical results indicate the existence of a critical surface tension above which the decreasing of the phase growth rate becomes nonmonotonic.

The work which is being presented below does not deal with any analytical solution. It addresses an important question about the conservation of energy in the formulation of a one-phase one-dimensional solidification problem of a supercooled melt which initially occupies the region  $0 \leq x < \infty$ . The one-phase problem can be obtained as a limit of a two-phase problem in the region  $0 \leq x < \infty$  by taking thermal conductivity of solid  $K_S \rightarrow 0$ . It has been shown that in this case the energy of the system is not conserved. An appropriate formulation has been given in [398] which conserves energy and which is also valid for large ratios of the thermal conductivities of the solid and liquid phases and other physically realistic parameters. In the two-phase solid-liquid solidification problem, the Stefan condition at the phase-change interface  $x = S(t)$  is taken as given below and other equations are as in Neumann solution.

$$(\beta - (1 - C)S_t) S_t = \left( K \frac{\partial T_S}{\partial x} - \frac{\partial T_L}{\partial x} \right)_{x=S}, \quad S_t = \frac{dS}{dt}, T_L(x, 0) = -1. \quad (12.2.315)$$

$\beta = l/(C_L \Delta T)$ ,  $\Delta T = T_m - T_\infty$  which is the degree of supercooling,  $K$  is the ratio  $K_S/K_L$  and  $C$  is the ratio  $C_S/C_L$ . Eq. (12.2.315) is in dimensionless form in which scalings have been done in the following way.

$$\bar{T}_{S,L} = (T_{S,L} - T_m) / \Delta T, \bar{t} = t / \tau, \bar{x} = x / a, \bar{S} = S / a, \bar{T}_u = (T_u - T_m) / \Delta T, \quad (12.2.316)$$

$\tau = a^2 / k_L$  and  $a = k_L / \Delta T$ ,  $T_m$  is the freezing temperature in the absence of undercooling and  $T_u$  is the freezing temperature with undercooling,  $\bar{T}_u = -S_t < T_m$ ,  $T_\infty = T_L|_{x \rightarrow \infty} < T_m$ .

In most of the problems  $C = 1$  is taken but a general formulation of Stefan condition requires  $C \neq 1$  in Eq. (12.2.315) and it is so in Eq. (1.3.1). In further discussion bar over the different variables will be dropped. To get the formulation of one-phase solidification problem of a supercooled liquid in the region  $0 \leq x < \infty$ ,  $K \rightarrow 0$  is taken in the supercooled two-phase solidification problem which can be easily formulated and not given here. Taking the limit and after making appropriate changes in Eq. (12.2.315) we get the formulation presented in Eqs (12.2.285), (12.2.286). In the two-phase problem, the initial condition  $T(x, 0) = -1$  and boundary condition  $\frac{\partial T_S}{\partial x}|_{x=0} = 0$  are also to be prescribed for completing the formulation. In Eq. (12.2.317),  $\rho_S$  and  $\rho_L$  are taken almost equal so their ratio is taken equal to 1 and when  $K \rightarrow 0$

$$\left. \frac{\partial T_L}{\partial x} \right|_{x=S(t)} = -(\beta - (1 - C)S_t) S_t; T_L(S, t) = T_u(S, t) = -S_t, S(0) = 0. \quad (12.2.317)$$

The thermal energy  $E$  and  $\frac{dE}{dt}$  in the two-phase problem in the region  $0 \leq x < \infty$  are given by

$$E = \int_0^S CT_S(x, t) dx + \int_S^\infty T_L(x, t) dx. \quad (12.2.318)$$

$$\frac{dE}{dt} = \int_0^S C \frac{\partial T_S}{\partial t} dx + CT_S(S, t) \frac{dS}{dt} + \int_S^\infty \frac{\partial T_L}{\partial t} dx - T_L(S, t) \frac{dS}{dt}. \quad (12.2.319)$$

Using  $T_L(S, t) = T_S(S, t) = T_u(S, t) = -S_t$  and  $\left. \frac{\partial T_S}{\partial x} \right|_{x=0} = 0$ ,  $\left. \frac{\partial T_L}{\partial x} \right|_{x \rightarrow \infty} = 0$ , it is not difficult to obtain the energy balance  $\frac{dE}{dt} = \beta \frac{dS}{dt}$  from Eq. (12.2.319). As the boundaries are insulated, the rate at which thermal energy is changing is equal to the rate at which energy is produced by phase change. In essence this implies that the two-phase formulation is correct.

To obtain  $\frac{dE}{dt}$  for the one-phase problem as the limit of a two-phase problem removes terms of  $T_S$  from  $\frac{dE}{dt}$  in Eq. (12.2.319). Note that  $K$  will appear in  $\frac{dE}{dt}$  when  $\frac{\partial T_S}{\partial t}$  is replaced by  $\frac{\partial^2 T_S}{\partial x^2}$ . Replace  $\left. \frac{\partial T_L}{\partial x} \right|_{x=S(t)}$  by the expression given in Eq. (12.2.315) in  $\frac{dE}{dt}$  and we find that  $\frac{dE}{dt} \neq \beta \frac{dS}{dt}$ . This implies that one-phase supercooled problem formulation should not be obtained as the limit  $K \rightarrow 0$  of a two-phase supercooled problem if energy conservation is to be satisfied in the region  $0 \leq x < \infty$ . This inconsistency was realized by the authors of [399]. To remove this defect and to have energy-conserving formulation a boundary layer approach was suggested in [399]. A new coordinate  $\bar{x} = S(t) - Kx$  was introduced,  $K \ll 1$ , which transforms the equation  $\frac{\partial T_S}{\partial t} = \frac{K}{C} \frac{\partial^2 T_S}{\partial x^2}$  to the following equation.

$$\frac{dS}{dt} \frac{\partial T_S}{\partial \bar{x}} + K \frac{\partial T_S}{\partial t} = \frac{1}{C} \frac{\partial^2 T_S}{\partial \bar{x}^2}, \quad 0 \leq \bar{x} < \infty. \quad (12.2.320)$$

Neglecting the small-order term involving  $K$  and integrating Eq. (12.2.320), we get

$$\frac{1}{C} \frac{\partial T_S}{\partial \bar{x}} = S_t (T_S - T_S^0), \quad T_S \rightarrow T_S^0 \quad \text{as } \bar{x} \rightarrow \infty. \quad (12.2.321)$$

Replacing  $\frac{\partial T_S}{\partial \bar{x}}$  by  $-K \frac{\partial T_S}{\partial x}$  and using the condition  $T_S(S(t), t) = -\frac{dS}{dt}$ , we have

$$K \frac{\partial T_S}{\partial x} = -C S_t \left( \frac{dS}{dt} + T_S^0 \right), \quad \text{when } x = S(t). \quad (12.2.322)$$

Using Eq. (12.2.322) in Eq. (12.2.315), we get Stefan condition for one-phase problem as

$$-\frac{\partial T_L}{\partial x} = \left( \beta - S_t - C T_S^0 \right) \frac{dS}{dt}, \quad \text{at } x = S(t). \quad (12.2.323)$$

$$\frac{dE}{dt} = C \frac{dS}{dt} \left( \frac{dS}{dt} + T_S^0 \right) + \left( \beta - \frac{dS}{dt} - C T_S^0 \right) \frac{dS}{dt} + (1 - C) \left( \frac{dS}{dt} \right)^2 = \beta \frac{dS}{dt}. \quad (12.2.324)$$

The value of  $T_S^0$  has not been prescribed as yet. Although the value of  $T_S^0$  is not affecting the energy balance verification it cannot be left arbitrary and it will be discussed after discussing another approach. In conclusion, if Eq. (12.2.323) is used as the Stefan condition in one-phase problem then limit  $K \rightarrow 0$  can be taken in the two-phase problem to obtain one-phase formulation in which energy is conserved.

In [398], the authors have advanced the investigation further to obtain a suitable energy-conserving formulation for one-phase problem for the case  $K \rightarrow \infty$ . If  $K \rightarrow 0$ , the heat equation for the solid region reduces to  $\frac{\partial T_S}{\partial t} = 0$  but when  $K \rightarrow \infty$ , we have  $\frac{\partial^2 T_S}{\partial x^2} = 0$  or  $T_S = T_{S0}(t) + T_{S1}(t)x$ . Using the boundary conditions  $\frac{\partial T_S}{\partial x} = 0$  at  $x = 0$  and  $T_S = -S_t$  at  $x = S(t)$ , we get  $T_S = -S_t(t)$ . In the Stefan condition (12.2.315) the term  $K \frac{\partial T_S}{\partial x}$  appears which is 0 to leading order if  $T_S = -S_t(t)$ , therefore we take

$$K \frac{\partial T_S}{\partial x} = K \left( \frac{\partial T_{S0}}{\partial x} + \frac{1}{K} \frac{\partial T_{S1}}{\partial x} + O(1/K^2) \right), \quad (12.2.325)$$

$$\frac{\partial^2 T_{S0}}{\partial x^2} = 0; \quad C \frac{\partial T_{S0}}{\partial t} = \frac{\partial^2 T_{S1}}{\partial x^2}. \quad (12.2.326)$$

It is easy to check that

$$T_{S0} = -S_t, \quad T_{S1} = -C(S_t)_t (x^2 - S^2) / 2 \quad (12.2.327)$$

and

$$K \frac{\partial T_S}{\partial x} \Big|_{x=S} = \frac{\partial T_{S1}}{\partial x} \Big|_{x=S} = -CSS_{tt} \text{ for } K \gg 1, \quad S_{tt} = \frac{\partial^2 S}{\partial t^2}. \quad (12.2.328)$$

Using Eqs (12.2.315), (12.2.327), (12.2.328) in  $\frac{dE}{dt}$  (Eq. 12.2.319), as  $K \rightarrow \infty$  we get

$$\frac{dE}{dt} = -CSS_{tt} + (CSS_{tt} + \{\beta - (1 - C) S_t\} S_t) + (1 - C) (S_t)^2 = \beta \frac{dS}{dt}. \quad (12.2.329)$$

Numerical results were obtained using boundary immobilization method and *Keller box finite difference technique* [400]. For  $K \rightarrow 0$ , formulation in Eq. (12.2.322) is mathematically correct but was found highly inaccurate for practical problems and less accurate than the formulation for nonconserving energy form. In Eq. (12.2.322) the value of  $T_S^0$  is required. Two values  $T_S^0 = -1$  and  $T_S^0 = 0$  were tried and both gave similar results without improving the error. When Eq. (12.2.328) was used for  $K \gg 1$ , the one-phase reduction of the two-phase problem provided good agreement for a wide range of parameters. It has been suggested in [398] that if the one-phase Stefan problem formulation in one dimension given below for large  $K$  is used which is derived as limit of the two-phase one-dimensional Stefan problem then it provides a good accurate approximation for a wide range of physically realistic parameter values and supercooling. In dimensional form one-phase formulation is:

$$\frac{\partial T_L}{\partial t} = k_L \frac{\partial^2 T_L}{\partial x^2}, \quad S(t) < x < \infty; \quad T_L(S, t) = T_m - \varepsilon S_t, \quad (12.2.330)$$

$$S(0) = 0, S_t = 1; \quad T_L(x, 0) = T_\infty = T_L|_{x \rightarrow \infty}, \quad (12.2.331)$$

$$\rho_L C_S \varepsilon S_{St} + \rho_L (1 - (C_L - C_S) \varepsilon S_t) S_t = -K_L \left. \frac{\partial T_L}{\partial x} \right|_{x=S}, \quad (12.2.332)$$

where  $\varepsilon > 0$  is kinetic undercooling coefficient,  $k$  is diffusivity.

In the problems discussed earlier concerning undercooling, the initial temperature of the supercooled liquid was taken constant. In the next problem to be discussed now, the initial temperature is taken as a function  $\phi(x)$ ,  $0 \leq x < \infty$  and the behaviour of  $\phi(x)$  as  $x \rightarrow \infty$  decides the nature of solution. Several interesting results can be obtained depending on the behaviour of  $\phi(x)$  as  $x \rightarrow \infty$  and their comparison with the equilibrium temperature in the case of undercooling.

The kinetic condition imposed at the phase-change boundary can have significant effect on the behaviours of temperatures and moving boundary. The problem of solidification of a supercooled liquid in an one-dimensional semiinfinite region,  $0 \leq x < \infty$ , was considered in [116]. The temperature at the moving boundary  $x = S(t)$  is taken proportional to the velocity of the moving boundary. Further details are as follows.

$$\frac{\partial T_L}{\partial t} = \frac{\partial^2 T_L}{\partial x^2}, \quad S(t) < x < \infty, t > 0; \quad S(0) = 0. \quad (12.2.333)$$

$$T_L(x, 0) = \phi(x) \leq 0, \quad 0 \leq x < \infty; \quad \phi(x)|_{x \rightarrow \infty} \rightarrow -1 - \lambda. \quad (12.2.334)$$

$$T_L(S(t), t) = -\varepsilon \frac{dS}{dt}, \quad \varepsilon > 0, t > 0; \quad \frac{\partial T_L}{\partial x}(S(t), t) = -\frac{dS}{dt}, t > 0. \quad (12.2.335)$$

For convenience, all thermo-physical parameters are taken here unity. Freezing temperature is taken as 0 when  $\varepsilon = 0$  and at  $x = 0$ ,  $T_S(0, t) = 0, t > 0$ . As discussed in Section 5.3 for  $\varepsilon = 0$ , the solution of  $S_t$  which is obtained using Eqs (12.2.333)–(12.2.335) may blow up in finite time with  $\frac{dS}{dt} \rightarrow \infty$  if the initial data  $\phi(x)$  are not suitably prescribed. For  $\varepsilon = 0$ , similarity solution of the problem in Eqs (12.2.333)–(12.2.335) exists which can be easily obtained. We have

$$T_L(\xi, t) = \frac{\delta}{2} \exp\left(\delta^2/4\right) \int_{\xi}^{\delta} \exp\left(-y^2/4\right) dy, \quad \xi = x/\sqrt{t}, \quad S(t) = \delta \sqrt{t}, \quad (12.2.336)$$



$\delta$  is the root of the equation

$$\delta \exp\left(\delta^2/4\right) \int_{\beta/2}^{\infty} \exp(-y^2) dy = 1 + \lambda. \quad (12.2.337)$$

It is assumed that  $\phi(x) \in C^1(0, \infty)$  and is bounded together with its derivative. Eq. (12.2.337) has a positive real root only when  $-1 < \lambda < 0$  implying that the condition  $\phi(\infty) < -1$  should be satisfied to prevent the blow-up in finite-time. When  $\varepsilon = 0$ ,  $\lambda = 0$ , it is easy to obtain a travelling wave solution in the form

$$T_L(x, t) = \exp(-V(x - Vt)) - 1, \quad S(t) = Vt, \quad t > 0, \quad (12.2.338)$$

where  $V$  is any arbitrary positive constant representing wave speed. There is no condition available to determine  $V$ . Travelling wave solution does not exist when  $\varepsilon = 0$ ,  $\lambda < 0$ .

Next the effect of  $\varepsilon > 0$  is considered on the solution when  $\lambda < 0$ . We express the solution of Eqs (12.2.333)–(12.2.335) as

$$T_L(x, t) = T_L^{(1)}(\xi) + g(\xi)/\sqrt{t}, \quad \xi = x/\sqrt{t}, \quad S(t) = \delta\sqrt{t}, \quad t > 0, \quad (12.2.339)$$

where  $T_L^{(1)}(\xi)$  is the temperature solution of Eqs (12.2.336), (12.2.337) and  $g(\xi)$  satisfies differential equation  $g''(\xi) + (\xi g')/2 = 0$ , dash denotes differentiation with respect to  $\xi$ . If  $g(\delta) = -\varepsilon \delta/2$  and  $g'(\delta) = 0$ , then the solution of the differential equation for  $g(\xi)$  is

$$g(\xi) = \frac{\delta}{2} \left\{ \frac{\delta}{2} e^{-\xi^2/4} \int_{\xi}^{\delta} e^{y^2/4} dy - e^{(\delta^2 - \xi^2)/4} \right\}, \quad g(\xi) \rightarrow 0 \text{ as } \xi \rightarrow \infty, \quad (12.2.340)$$

where  $\delta$  is still given by Eq. (12.2.337) and  $\delta$  exists provided  $-1 < \lambda < 0$ .  $\varepsilon > 0$  has no effect on the moving front.

If  $\lambda > 0$  and  $\varepsilon > 0$ , then a travelling wave solution given below can be obtained.

$$T_L(x, t) = \exp(-V(x - Vt)) - (1 + \lambda), \quad S(t) = Vt = \lambda t/\varepsilon. \quad (12.2.341)$$

In this case  $V$  is not arbitrary. If  $\varepsilon > 0$  and  $\lambda < 0$  then no travelling wave solution exists.

To obtain asymptotic behaviour of  $S(t)$  as  $t \rightarrow \infty$ , modified Laplace transform (see Eq. 12.2.347) of Eqs (12.2.333)–(12.2.335) is taken and limit  $p \rightarrow 0$  is investigated. Large-time behaviour depends very much on the Laplace transform  $\hat{\Phi}(p)$  of  $\phi(x)$  and its limit as  $p \rightarrow 0$ .  $\hat{\Phi}(p)$  as  $p \rightarrow 0$  depends on  $\phi(x)$  as  $x \rightarrow \infty$ .

If  $\varepsilon > 0$ ,  $\lambda = 0$  then  $S(t) \sim A t^\alpha$  with  $\frac{1}{2} < \alpha < 1$  as  $t \rightarrow \infty$  and  $A$  depends on  $\hat{\phi}(p)$  as  $p \rightarrow 0$ . This is the large-time behaviour. If  $\varepsilon = 0$  and  $\lambda = 0$ , then several cases arise depending on the form of  $\phi(x)$  as  $x \rightarrow \infty$ . In particular no solution exists if the initial data are a unit step function.

For the derivation of results corresponding to the cases  $\varepsilon > 0$ ,  $\lambda = 0$  and  $\varepsilon = 0$ ,  $\lambda = 0$  and many more results associated with  $\varepsilon$  and  $\delta$  (see [116]) as their derivation is lengthy and depends on analysing the Laplace transform solution of the equations.

A problem similar to the above problem in one-dimensional radially symmetric spherical coordinates with Gibbs-Thomson condition at the solid-liquid interface has been studied in [116, 401]. The solidification of a supercooled liquid is considered and the formulation satisfies the following equations.

$$\frac{\partial T_L}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right), \quad S(t) < r < \infty, t > 0; \quad T_L(S(t), t) = -\varepsilon \dot{S} - 2\sigma/S. \quad (12.2.342)$$

$$\left. \begin{aligned} \frac{\partial T_L}{\partial r} &= -\frac{dS}{dt}, \text{ at } r = S(t), t > 0; T_L(r, 0) = \phi(r), S(0) \leq r < \infty, \\ \phi(r)|_{r \rightarrow \infty} &\rightarrow -1 - \lambda. \end{aligned} \right\} \quad (12.2.343)$$

$\sigma \geq 0$  is dimensionless surface tension. For convenience other thermo-physical parameters have been taken as unity. It is easier to tackle the spherical problem by using the transformation  $\theta_L(r, t) = rT_L(r, t)$  so that we have

$$\left. \begin{aligned} \frac{\partial \theta_L}{\partial t} &= \frac{\partial^2 \theta_L}{\partial r^2}, \quad S(t) < r < \infty; \quad \theta_L|_{r=S(t)} = -\varepsilon S \dot{S} - 2\sigma, \\ \frac{\partial \theta_L}{\partial r} \Big|_{r=S(t)} &= -(\varepsilon + S) \dot{S} - 2\sigma/S. \end{aligned} \right\} \quad (12.2.344)$$

The initial condition can be easily written. For  $\varepsilon = 0$ , a similarity solution exists which is obtained by taking  $S(t) = \delta\sqrt{t}$  and  $\xi = r/\sqrt{t}$ . The equation which determines  $\delta$  is given by

$$\frac{\delta^2}{2} \left( 1 - \delta e^{\delta^2/4} \int_{\delta/2}^{\infty} e^{-y^2} dy \right) = 1 + \lambda. \quad (12.2.345)$$

To obtain  $\theta_L(r, t)$  follow the procedure adopted in Eq. (12.2.339). It is easy to conclude that if  $\lambda < 0$  then the solution will blow up in finite time even in the presence of surface tension. If  $\varepsilon > 0$  and  $\lambda < 0$  then  $S(t) \sim \delta\sqrt{t}$  as  $t \rightarrow \infty$ . If  $\varepsilon > 0$  and  $\lambda > 0$  then a pseudo-travelling wave solution exists and

$$\begin{aligned} \theta_L(r, t) &= -(1 + \varepsilon V) r + 2(1/V - \sigma) + [2(Vt - 1/V) - r] e^{-V(r-Vt)} \\ &\quad - 2\sigma/(V\sqrt{t}) e^{-r^2/4t} \int_{-r/(2\sqrt{t})}^{(r-2Vt)/(2\sqrt{t})} e^{y^2} dy. \end{aligned} \quad (12.2.346)$$

$S(t) = Vt$ ,  $V = \lambda/\varepsilon$ . For  $\varepsilon = 0$ ,  $\lambda = 0$ ,  $S(t) = Vt$  is a solution for an arbitrary  $V > 0$ .

If  $\varepsilon > 0$ ,  $\lambda = 0$ , then as  $t \rightarrow \infty$ ,  $S(t) \sim bt^\alpha$ ,  $\frac{1}{2} < \alpha < 1$  and the constant  $b$  depends on  $\hat{\phi}(p)$  (Laplace transform of  $\phi(x)$ ) as  $p \rightarrow 0$ . Several cases arise here. The asymptotic or long-time behaviour of  $S(t)$  as  $t \rightarrow \infty$  is obtained by taking modified Laplace transform of the equations. The Laplace transforms of  $T_L(x, t)$  and  $\phi(x)$  are defined as

$$\hat{T}_L(p, t) = \int_{S(t)}^{\infty} e^{-px} T_L(x, t) dx; \quad \hat{\phi}(p) = \int_0^{\infty} e^{-px} \phi(x) dx. \quad (12.2.347)$$

For other detail refer [116, 401].

A self-similar solution has general relevance to a problem only if it can be shown that it is a limiting form to which a broad class of other solutions evolve. The asymptotic stability

of any planar solution with respect to shape perturbations reduces to studying the stability of similarity solution and travelling-wave solution and this has been shown in [402]. The asymptotic stability of the special type of similarity and travelling wave solutions have been established for some one-dimensional one-phase planar problems such as the one given below within the class of planar solutions:

$$\left. \begin{aligned} \frac{\partial \theta_L}{\partial t} &= \frac{\partial^2 \theta_L}{\partial x^2}, \quad S(t) \leq x < \infty, t > 0, \\ \frac{\partial \theta_L}{\partial x}(S(t), t) &= \theta_L(S(t), t); \quad S_t(t) = \theta_L(S(t), t). \end{aligned} \right\} \quad (12.2.348)$$

$$\lim_{x \rightarrow \infty} \theta_L(S(t), t) = \lambda; \quad \theta_L(x, 0) = \theta_L^0(x), \quad S(0) = 0, \quad x \geq 0. \quad (12.2.349)$$

To satisfy the kinetic condition at the moving boundary  $x = S(t)$ , the solutions can be sought in the following form.

$$\theta_L(r, t) = f(\xi) + g(\xi)/\sqrt{t}; \quad S(t) = \delta\sqrt{t}; \quad \xi = x/\sqrt{t}. \quad (12.2.350)$$

For  $\lambda \in (0, 1)$ , the following solution exists and is unique.

$$\begin{aligned} \theta_L(\xi, t) &= \frac{\delta}{2} e^{\delta^2/4} \int_{\delta}^{\xi} e^{-y^2/4} dy \\ &\quad + \frac{\delta}{2\sqrt{t}} \left\{ \frac{\delta}{2} e^{-\xi^2/4} \int_{\delta}^{\xi} e^{y^2/4} dy + e^{(\delta^2 - \xi^2)/4} \right\}, \end{aligned} \quad (12.2.351)$$

$$\lambda = \frac{\delta}{2} e^{\delta^2/4} \int_{\delta}^{\infty} e^{-y^2/4} dy. \quad (12.2.352)$$

For  $\lambda > 1$ , a travelling wave solution can be obtained which is given below.

$$\theta_L(x, t) = (V + 1) - e^{-V(x-Vt)}, \quad S(t) = Vt, \quad V = \lambda - 1. \quad (12.2.353)$$

Using Laplace transform method in which the Laplace transform is defined as in Eq. (12.2.347), the asymptotic behaviour of the moving boundary has been examined for  $t \rightarrow \infty$ . For convenience, thermo-physical parameters have been taken unity by us in Eqs (12.2.348)–(12.2.353).

Linear stability of the similarity and travelling wave solutions as solutions of the full two-dimensional one-phase problem with surface tension and Kinection condition has been discussed in [402] by taking time-dependent nonplanar small perturbations. For stability analysis see [402].

The results presented in this section on analytical solutions of Stefan problems with Gibbs-Thomson condition at the phase-change boundary are a mixed bag of analytical solutions, small- and large-time asymptotic solutions, series solutions, modelling, etc. We have seen earlier that exact solutions even without Gibbs-Thomson condition prescribed are possible in very special cases of one-dimensional problems with some special types of prescribed conditions. To think of obtaining exact solutions in this broader class of Stefan problems with Gibbs-Thomson condition at the interface seems to be an unrealistic proposition. Therefore the above material has been presented in this section of exact analytical solution.

### 12.2.2 Exact Analytical Solutions of Some Stefan-Like Problems

As mentioned in [Section 12.1](#), Stefan-like problems which are also called generalized-Stefan problems arise in many fields of science and engineering and vast literature already exists on such problems which is growing rapidly with time. In this chapter only such Stefan-like problems will be considered which resemble closely with the formulations of Stefan problems and in which the physics and formulation of the problem can be described with relative ease. Some Stefan-like problems have been included in supplementary references to have broader view of such problems arising in science and engineering.

This section includes some analytical solutions on the following topics.

1. Stefan-like problems in which only heat transport is taking place.
2. Stefan-like problems in which both heat and mass transports are taking place such as problems of dilute binary alloys.
3. Stefan-like problems in porous media.

#### *Some Stefan-Like Problems Which Have Only Heat Transport in the Formulation*

The sharp-interface model of solidification/melting problems is generally justified for pure metals. As discussed earlier, in the enthalpy model of sharp-interface formulation we have three regions, namely, solid, liquid and mush and in a simplified formulation there are no temperature gradients in the mush. In the solidification of dilute binary alloys even if concentration gradients are neglected, the sharp-interface formulation with only solid and liquid phases is not suitable to describe the process. This is because ahead of pure solid, a constitutionally supercooled mushy layer is frequently formed which is responsible for nucleation and crystal growth. Solid particles are formed in the mush, and pure solid region advances in solidification. Mush is a mixture of solid and liquid and there is no way to estimate or to find exactly the solid fraction present in the total volume of the mush at any given time. The phase change takes place over a temperature range. The topology of the mush, its temperature, thermo-physical properties, etc., are complicated and difficult subjects and largely nondeterministic in nature. Exact analytical solutions can be obtained only under some simplifying assumptions.

The one-dimensional solidification in the region  $0 \leq x < \infty$  considered in [403] has three distinct regions, viz., solid, mush and liquid. It is assumed that there are no temperature gradients in the mush and it is considered at the melting temperature  $T = 0$ .

In the solid region

$$\frac{\partial T_S}{\partial t} = k_S \frac{\partial^2 T_S}{\partial x^2}, \quad 0 \leq x \leq S(t), \quad t > 0; \quad T_S(0, t) = T_1 < 0, S(0) = 0. \quad (12.2.354)$$

In the liquid region

$$\frac{\partial T_L}{\partial t} = k_L \frac{\partial^2 T_L}{\partial x^2}, \quad x \geq r(t), \quad t > 0; \quad T_L(x, 0) = T_0 > 0, T_L(\infty, t) = T_0. \quad (12.2.355)$$

In the mush region

$$S(t) \leq x \leq r(t), \quad t > 0; \quad S(0) = r(0) = 0. \quad (12.2.356)$$

$$K_S \left. \frac{\partial T_S}{\partial x} \right|_{S(t)} - K_L \left. \frac{\partial T_L}{\partial x} \right|_{r(t)} = l\rho \left( \epsilon \frac{dS}{dt} + (1 - \epsilon) \frac{dr}{dt} \right), \quad t > 0, \quad 0 \leq \epsilon \leq 1. \quad (12.2.357)$$

$X = S(t)$  is called solidus and beyond this boundary pure solid region does not exist.  $x = r(t)$  is called liquidus and beyond this boundary only pure liquid exists.  $\epsilon l$  is the fixed fraction of the total latent heat  $l$  which is released at the solidus. The width of the mushy region is taken inversely proportional to the temperature gradient at  $S(t)$  in [404] and this is so in [403].

$$\left. \frac{\partial T_S}{\partial x} \right|_{S(t)} (r(t) - S(t)) = \gamma, \quad t > 0, \quad \gamma > 0 \text{ is a known constant.} \quad (12.2.358)$$

As the mush temperature is the freezing temperature which is taken as  $T = 0$ , we have

$$T_S(S(t), t) = 0, \quad T_L(r(t), t) = 0. \quad (12.2.359)$$

The formulation of the previous problem is well suited for a similarity solution and so we take

$$S(t) = \delta_1 t^{1/2}, \quad r(t) = \delta_2 t^{1/2}, \quad \delta_1, \delta_2 > 0, \quad \delta_1, \delta_2 \text{ are unknown constants.} \quad (12.2.360)$$

Such type of similarity solutions have been considered in Section 12.2.1 and the solution procedure will not be discussed further. The authors in [403] have considered also an overspecified condition at  $x = 0$  in the following form and obtained the solution.

$$K_S \left. \frac{\partial T_S}{\partial x} \right|_{(0,t)} = h_0 / \sqrt{t}, \quad t > 0, \quad h_0 > 0. \quad (12.2.361)$$

The similarity solution with this type of overspecified condition has already been considered earlier. Any one of the eight constants  $\epsilon, \gamma, l, \rho, C_S, C_L, K_S, K_L$  can be taken as unknown as there is an overspecified condition at  $x = 0$ . Note that this chosen unknown parameter is in addition to the unknowns occurring in the obtained solution of the problem formulated in Eqs (12.2.354)–(12.2.361).  $T_0, T_1$  and  $h_0$  are known quantities. First, theorems on necessary and sufficient conditions concerning uniqueness of the solution have been obtained in [403] by taking any one of the eight parameters as unknown. Then any two parameters are taken as unknown and necessary and sufficient conditions for the existence of at least one solution are given. This involves the behaviour of 58 functions to be examined and 48 such results are given. Three unknown parameters have also been considered and conditions for at least one solution to exist have been examined. Thirty-six results are given in this case. It is not possible to present even one result as it requires lot of space.

The next problem we consider is that of one-dimensional solidification in a radially symmetric cylindrical geometry. The freezing takes place over a temperature range. The formulation of the problem is as follows (cf. [67]).

In the solid region

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T_S}{\partial r} \right) = \frac{1}{k_S} \frac{\partial T_S}{\partial t}, \quad 0 < r < S_1(t), \quad t > 0, \quad S_1(0) = 0. \quad (12.2.362)$$

In the mushy region

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T_M}{\partial r} \right) + \frac{\rho l}{K_M} \frac{df_S}{dt} = \frac{1}{k_M} \frac{\partial T_M}{\partial t}, \quad S_1(t) \leq r \leq S_2(t), t > 0, \quad (12.2.363)$$

$$f_S = f_{Su} (1 - R_1), \quad R_1 = \frac{T_M - T_1}{T_2 - T_1}, \quad T_1 < T_M < T_2, \quad T_1 > 0, \quad (12.2.364)$$

where  $f_S$  is the solid fraction in the mush,  $f_{Su}$  is the solid fraction at  $r = S_1(t)$  in the mush.

In the liquid region

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T_L}{\partial r} \right) = \frac{1}{k_L} \frac{\partial T_L}{\partial t}, \quad r \geq S_2(t), t > 0; \quad T_L(r, 0) = T_0, \quad T_L(\infty, t) = T_0. \quad (12.2.365)$$

In the previous equations the subscripts  $S$ ,  $M$  and  $L$  stand for solid, mushy and liquid regions, respectively,  $K$  is thermal conductivity,  $k$  is thermal diffusivity,  $\rho_S = \rho_L = \rho$  is density.  $r = S_1(t)$ ,  $S_1(0) = 0$  is the solidus boundary, and  $r = S_2(t)$ ,  $S_2(0) = 0$  is the liquidus boundary. For scalings refer [67].

The freezing begins at  $x = S_1(t)$  at a constant temperature  $T_1$  at which  $T_M(r, t) = T_1 < T_0$  and ends at  $x = S_2(t)$  at temperature  $T_2$ ,  $T_M(S_2(t), t) = T_2 > T_1$ . The solid fraction  $f_S$  in the mush is assumed to be a linear function of the temperature in Eq. (12.2.364) and  $f_{Su}$  is the solid fraction at the *eutectic temperature* if the alloy has eutectic. The second term on the l.h.s. of Eq. (12.2.363) is the amount of latent heat liberated per unit volume/ $K_M$  and it is treated as a volumetric heat generation term in Eq. (12.2.363). The liquid is freezing due to the application of a heat sink at  $r = 0$  of strength  $Q$  which is given by the relation

$$Q = \lim_{r \rightarrow 0} \left( 2\pi r K_S \frac{\partial T_S}{\partial r} \right). \quad (12.2.366)$$

The boundary conditions at the solidus front are

$$T_S(r, t) = T_M(r, t) = T_1, \quad \text{at } r = S_1(t), \quad t > 0. \quad (12.2.367)$$

$$K_S \frac{\partial T_S}{\partial r} - K_M \frac{\partial T_M}{\partial r} = \rho l (1 - f_{Su}) \frac{dS_1}{dt}, \quad \text{at } r = S_1(t), \quad t > 0. \quad (12.2.368)$$

In the boundary condition (12.2.368),  $(1 - f_{Su})$  is the liquid fraction at  $r = S_1(t)$  which has become solid. The quantity  $f_{Su}$  may depend on the composition of the alloy if the alloy shows an eutectic.

The boundary conditions at the liquidus are

$$T_L(r, t) = T_M(r, t) = T_2, \quad r = S_2(t), \quad t > 0. \quad (12.2.369)$$

$$K_L \frac{\partial T_L}{\partial r} = K_M \frac{\partial T_M}{\partial r}, \quad r = S_2(t), \quad t > 0, \quad f_S = 0. \quad (12.2.370)$$

In Eq. (12.2.363), the derivative of  $f_S$  can be calculated using Eq. (12.2.364) and finally Eq. (12.2.363) can be written as

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T_M}{\partial r} \right) = \frac{1}{k_M^*} \frac{\partial T_M}{\partial t}, \quad 1/k_M^* = \rho l f_{Su} / (K_M (T_2 - T_1)) + 1/k_M. \quad (12.2.371)$$

The heat equations in all the three regions solid, mush and liquid are similar to Eq. (12.2.26) and their solutions in appropriate regions using Eq. (12.2.27) can be written as

$$T_i(r, t) = C_{1J} Ei \left( -r^2 / (4k_i t) \right) + C_{2J}, \quad i = S, M, L, \quad J = 1, 2, 3. \quad (12.2.372)$$

To obtain similarity solutions we take  $S_1(t)$  and  $S_2(t)$  as

$$S_1(t) = 2\delta_1 \sqrt{tk_S}, \quad S_2(t) = 2\delta_2 \sqrt{tk_M}; \quad \delta_1, \delta_2 > 0 \text{ are unknowns.} \quad (12.2.373)$$

Finally there are eight unknowns and eight conditions to determine them. To determine  $\delta_1$  and  $\delta_2$  two coupled equations are obtained which can be solved only numerically. The procedure for obtaining solution is straightforward but lengthy and therefore details of calculations are not being given here. Numerical results are generally obtained by assuming that  $\delta_1$  and  $\delta_2$  both are greater than zero as proving the existence and uniqueness of the solution requires special proofs. One more model for solid fraction  $f_S$  in the mush given below has been considered in [67] in which  $f_S$  is a function of position in the mush

$$f_S(r, t) = f_{Su} (1 - R_2), \quad \text{where } R_2 = (r - S_1(t)) / (S_2(t) - S_1(t)). \quad (12.2.374)$$

In this case only a series solution is possible which will be discussed in Section 12.3.

The formulation given in Eqs (12.2.362)–(12.2.371) has been largely followed in a planar solidification in [405] to obtain analytical solution of a one-dimensional solidification problem in the region  $0 \leq x < \infty$ . Some changes are mentioned below. In this problem cooling is done at  $x = 0$  by a prescribed temperature  $T_0 < T_m$ ,  $T_m$  is the freezing temperature. The densities of both solid and liquid are the same and taken constant and as a result mass fraction and volume fraction calculated in any phase are the same. Instead of  $l$  in Eq. (12.2.368) take  $l_{eq} = (C_L - C_S)T_m + l$  (see Eq. 1.3.17). An important difference is in the calculation of  $\frac{\partial f_S}{\partial t} = \frac{\partial f_S}{\partial T_M} \cdot \frac{\partial T_M}{\partial t}$ ,  $f_S$  is the solid fraction in the mush. In [67],  $\partial f_S / \partial T_M$  has been calculated using Eq. (12.2.364) or (12.2.374).

In [405] authors claim that without incorporating the *composition-solid fraction coupling* and phase diagram of temperature versus concentration in determining the temperature of the mush, the calculation of  $f_S$  and  $\partial f_S / \partial T_m$  may not give metallurgically consistent results. The details of derivation will be avoided here as it involves several other details and partial discussion of formulation will not be very useful. The final expression of  $f_S$  and  $\partial f_S / \partial T_M$  which have been obtained is as follows. For equilibrium solidification which is based on *lever rule*, we have

$$f_S = \frac{1}{(1 - 1/k_p)} \left( \frac{T_m - T_1}{T_m - T_{eq}} - \frac{1}{k_p} \right), \quad (1 - 1/k_p) (T_m - T_2) \frac{\partial f_S}{\partial T_M} = 1, \quad (12.2.375)$$

where  $k_p$  is the partition coefficient [42],  $T_1$  and  $T_2$  are temperatures as in Eqs (12.2.367)–(12.2.369), respectively,  $T_{eq}$  is the equivalent temperature of the mush which is calculated according to some procedure explained in [405] and  $f_S$  and  $\partial f_S / \partial T_M$  are also equivalent values of the respective quantities,  $T_m$  is the ideal equilibrium freezing/melting temperature. However, for determining similarity solution for temperature in the mush,  $f_S$  and  $\partial f_S / \partial T_M$  have to be taken as constant. In Eq. (12.2.375),  $f_S$  is not zero at the liquidus. Therefore, latent heat  $l_{eq}$  is to be accounted at the liquidus boundary as given below

$$K_M \frac{\partial T_M}{\partial x} - K_L \frac{\partial T_L}{\partial x} = \rho l_{eq} f_S \frac{dS_2}{dt}, \quad \text{at } x = S_2(t), \quad t > 0. \quad (12.2.376)$$

Analytical solution procedure for the above solidification problem is a simple extension of the solution procedure followed in obtaining Neumann solution and some other solutions obtained in Section 12.2.1. To obtain numerical results based on the analytical solution we require the calculation of  $f_S$  and  $\partial f_S / \partial T_M$ . For these calculations a procedure for interpolation of values has been outlined in seven steps in [405]. For numerical solution of the problem formulated, the problem domain is discretized using a fixed grid enthalpy method based on finite volume procedure.  $\text{NH}_4 \text{CL}-\text{H}_2\text{O}$  solution with 70% initial concentration was considered for values of thermo-physical properties.

### *Some Heat and Mass Transfer Problems of Dilute Binary Alloys*

As mentioned earlier vast literature exists on Stefan-like problems. Not every Stefan-like problem has an analytical solution but out of all those which have, only some can be discussed here. This is because the formulations of the Stefan-like problems require, in general, different types of mass, momentum and energy equations and their presentation needs lot of extra space. Without discussing the relevant physical situation, parameters involved and assumptions made, giving merely a set of differential equations, boundary conditions and solution procedure makes only little sense which may not be enough for understanding. However, in those Stefan-like problems whose formulations are similar to Stefan problems references can be made about formulations and solutions mentioning the material which already exists in the earlier pages. This saves some space and improves understandability.

The formulation of a one-dimensional two-phase solidification problem of a dilute binary alloy with coupled heat and mass transfer and natural convection in the liquid is presented below. Some of the equations of heat transfer are being repeated here to make the formulation complete and later on references in other problem formulations can be made mentioning this formulation. The following formulation has been considered in [406].

- (a) In the solid region,  $0 < x < S(t), t > 0$

$$\frac{\partial}{\partial x} \left( K_S \frac{\partial T_S}{\partial x} \right) = \rho_S C_S \frac{\partial T_S}{\partial t}; \quad \frac{\partial}{\partial x} \left( D_S \frac{\partial}{\partial x} (\rho_S \bar{c}_S) \right) = \frac{\partial}{\partial t} (\rho_S \bar{c}_S), \quad (12.2.377)$$

where  $C$  stands for specific heat,  $\bar{c}$  is the concentration of the solute or weight—fraction of the impurity in the liquid,  $D$  stands for mass diffusion coefficient and other notations are the notations which are being followed throughout this book.

- (b) In the liquid region,  $S(t) < x < h(t), t > 0$

$$\begin{aligned} \frac{\partial}{\partial x} \left( K_L \frac{\partial T_L}{\partial x} \right) - \frac{\partial}{\partial x} (\rho_L C_L T_L V) &= \rho_L C_L \frac{\partial T_L}{\partial t}, \\ \frac{\partial}{\partial x} \left( D_L \frac{\partial}{\partial x} (\rho_L \bar{c}_L) \right) - \frac{\partial}{\partial x} (\rho_L \bar{c}_L V) &= \frac{\partial}{\partial t} (\rho_L \bar{c}_L); \quad S(t) < x < h(t), t > 0, \end{aligned} \quad (12.2.378)$$

where  $V(t)$  is the velocity due to natural convection.

- (c) Initial conditions

$$T_L(x, 0) = T_0; \quad \bar{c}_L(x, 0) = \bar{c}_0; \quad S(0) = 0, \quad S(0) < x < h(0), \quad T_0 \geq T_m - m\bar{c}_0, \quad (12.2.379)$$

$T_0, T_m, \bar{c}_0, K_S, K_L, \rho_S, \rho_L, C_S, C_L, D_S, D_L$  and  $m$  are constant.



(d) Prescribed boundary conditions at  $x = 0$  and  $x = h(t)$ ,  $t > 0$

$$T_S(0, t) = T_1; T_L(x, t) \text{ or } \frac{\partial}{\partial x}(T_L(x, t)) \text{ is prescribed at } x = h(t), \quad (12.2.380)$$

$$\left. \frac{\partial}{\partial x}(\rho_S \bar{c}_S) \right|_{(0, t)} = 0; \frac{\partial}{\partial x}(\rho_L \bar{c}_L) \text{ is prescribed at } x = h(t); T_1 \leq T_m - m\bar{c}_0. \quad (12.2.381)$$

(e) Boundary conditions at  $x = S(t)$ ,  $t > 0$

$$K_S \frac{\partial}{\partial x}(T_S) - K_L \frac{\partial}{\partial x}(T_L) = l\rho_S \frac{dS}{dt}, \quad (12.2.382)$$

$$D_S \frac{\partial}{\partial x}(\rho_S \bar{c}_S) - D_L \frac{\partial}{\partial x}(\rho_L \bar{c}_L) = (1 - \bar{k})\rho_S \bar{c}_L \frac{dS}{dt}, \quad (12.2.383)$$

$$T_S = T_L = T_h(t); T_L = T_0 - m\bar{c}_L; \bar{c}_S = \bar{k}\bar{c}_L, \quad (12.2.384)$$

where  $m$  is the slope of the liquidus line,  $\bar{k}$  is distribution coefficient. If  $\bar{k} < 1$  then  $m > 0$ , whilst if  $\bar{k} > 1$ ,  $m < 0$ .  $T_h$  is the freezing temperature of the mixture and  $T_m$  is the freezing temperature of the pure solvent.

(f) In the case of natural convection  $V(t) = -(\rho_S/\rho_L - 1) \frac{dS}{dt}$ . Note that  $\frac{dS}{dt}$  is positive in the present case.

To obtain an exact analytical solution, the region is taken as  $0 \leq x < \infty$  (i.e.  $h(0) = \infty$ ). In the earlier chapters of this book, mass transfer was not given importance, therefore phase diagrams coupling the heat and mass transfer for thermodynamic equilibrium at the phase-change interface were not discussed. If the liquid mixture (alloy) consists of two components  $A$  and  $B$  then one of the component, say  $A$ , is called solvent and  $B$  is called solute or impurity. The mass fraction of  $B$  in the mixture is called concentration. Both the components may have different freezing/melting temperatures, say,  $T_A$  and  $T_B$  in pure state. During solidification or melting, the concentration of the component  $B$  (solute) in the mixture goes on changing and therefore the equilibrium temperatures and concentrations of the alloy go on changing. When temperature is plotted versus concentration then for a specified temperature the equilibrium concentrations of solid and liquid at which both phases may coexist in equilibrium in the mixture are given by a phase diagram. A simple two-component phase diagram shows: (i) the freezing temperature of a liquid of specified concentration over the concentration range, which is called liquidus curve extending from the freezing temperature  $T_A$  of the component  $A$  till temperature  $T_B$ , which is freezing temperature of component  $B$ ; (ii) the melting temperature of a solid (solute) of specified concentration, which is called solidus curve and it extends from  $T_A$  to  $T_B$  over the concentration range; (iii) the equilibrium concentrations of a mixture of solid and liquid in a mushy zone, which may coexist in equilibrium at a specified temperature.

The general phase diagrams for binary alloys solidification/melting including eutectic require lot of information for their understanding which is beyond the scope of this chapter. What is given below is some little discussion about a simple linear-phase diagram. In a *dilute binary alloy* the phase diagram which is given in Fig. 12.2.1 consists of two straight lines which are solidus and liquidus curves of different slopes and on the vertical temperature coordinate they start from  $T_A$  which is taken as  $T_m$  in Eq. (12.2.384). This is known as *linear-phase diagram* for dilute binary alloys. If the slope of liquidus is known, say  $m$ , then slope of solidus can be obtained in terms of concentrations of solid and liquid and vice versa and this is reflected in Eq. (12.2.384) through  $\bar{k}$ .

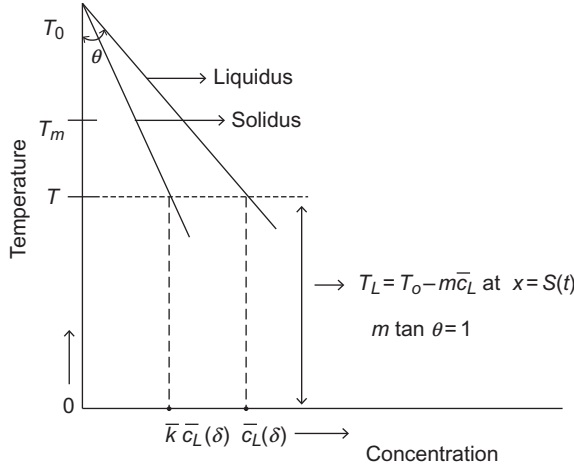


Fig. 12.2.1 Linear equilibrium phase diagram.

The exact solution of the formulation given in Eqs (12.2.377)–(12.2.384) with  $V(t)$  as given in condition (f) can be easily obtained in terms of a similarity variable  $\xi$  and,  $S(t)$  which are defined as

$$\xi = x / (2\sqrt{k_S t}) ; \quad S(t) = 2\delta\sqrt{k_S t}, \quad \delta > 0 \text{ is unknown.} \quad (12.2.385)$$

The solution procedure is similar to the procedure adopted in obtaining the solution given in Eqs (1.3.11)–(1.3.16). In the present problem there will arise eight unknown constants from the solutions of diffusion equations and one more unknown is  $\delta$ . There are nine conditions available to determine them. It is a matter of just lengthy calculations to determine the unknowns. We present here the complete solution as this solution and some of its particular cases will be referred in the subsequent discussion.

The dimensionless parameters in terms of which the solution is reported are as follows.

$$\left. \begin{aligned} P_1 &= \frac{m\bar{c}_0}{T_m - T_1}, \quad P_2 = \frac{T_0 - T_m}{T_m - T_1}, \quad P_3 = \frac{l}{C_S(T_m - T_1)}, \\ P_4 &= \sqrt{\frac{k_S}{D_L}}, \quad P_5 = \frac{K_L}{K_S}, \quad P_6 = \sqrt{\frac{k_S}{k_L}}, \quad P_7 = \frac{\rho_S}{\rho_L}, \\ P_8 &= 1 - (1 - \bar{k}) \sqrt{\pi} \delta P_4 P_7 \exp\left(\delta^2 P_4^2 P_7^2\right) \operatorname{erfc}(\delta P_4 P_7). \end{aligned} \right\} \quad (12.2.386)$$

$$T_S = T_1 + (T_m - T_1) \left(1 - \frac{P_1}{P_8}\right) \frac{\operatorname{erfc} \xi}{\operatorname{erf} \delta}. \quad (12.2.387)$$

$$T_L = T_0 - (T_m - T_1) \left(P_2 + \frac{P_1}{P_8}\right) \frac{\operatorname{erfc} \{[\xi + (P_7 - 1)\delta] P_6\}}{\operatorname{erfc}(\delta P_6 P_7)}. \quad (12.2.388)$$

$$\bar{c}_L = \bar{c}_0 \left[1 + \frac{(1 - P_8) \operatorname{erfc} \{[\xi + (P_7 - 1)\delta] P_4\}}{P_8 \operatorname{erfc}(\delta P_4 P_7)}\right], \quad \bar{c}_S = \bar{k} \bar{c}_L. \quad (12.2.389)$$

$$\begin{aligned} \sqrt{\pi}\delta P_3 = & \frac{\exp(-\delta^2)}{\operatorname{erf}(\delta)} - \frac{P_1}{P_8} \left[ \frac{\exp(-\delta^2)}{\operatorname{erf}(\delta)} + P_5 P_6 \frac{\exp(-\delta^2 P_6^2 P_7^2)}{\operatorname{erfc}(\delta P_6 P_7)} \right] \\ & - P_2 P_5 P_6 \frac{\exp(-\delta^2 P_6^2 P_7^2)}{\operatorname{erfc}(\delta P_6 P_7)}. \end{aligned} \quad (12.2.390)$$

$P_1$  and  $P_2$  can be positive or negative, whilst  $P_3, P_4, \dots, P_7$  are nonnegative. If  $P_1 = 0$  and  $P_7 = 1$ , then Neumann solution is obtained.  $P_1$  is the principal coupling parameter between heat and mass transfer. When  $P_1 = 0$ , the effect of  $P_7$  was found to be very small on  $\delta$ . In view of the second and third conditions in Eq. (12.2.384),  $T_h(t)$  is not required for obtaining solution but it can be determined from the solution.

Finite difference numerical solution using explicit scheme for the heat and mass transfer problem formulated in Eqs (12.2.377)–(12.2.384) was obtained in [407] in a finite slab  $0 \leq x \leq L$  with some changes in the formulation. Consider  $T_1 = T_1(t) \leq T_m$ ,  $\frac{\partial \bar{C}_S}{\partial x}|_{x=0} = 0$ ,  $T_0 = \bar{T}_0(x)$  and  $\bar{c}_0 = \bar{c}_0(x)$ ,  $0 \leq x \leq L$ .  $T_m$  is the melting temperature of the component A so the linear-phase diagram will be different and the slopes of the liquidus and solidus will be different from the linear-phase diagram considered in [406]. Natural convection is not considered in [407] and the boundary  $x = L$  is insulated for both temperature and concentration. Like enthalpy formulation for heat transfer phase-change problems, a function  $\phi(x, t)$  is defined in terms of temperatures  $T_L, T_S, T_m$ , concentrations  $\bar{c}_S, \bar{c}_L, m, \bar{k}$  and latent heat which is valid throughout  $0 \leq x \leq L$  but different in solid and liquid regions. For details refer [407]. Similarly a new concentration function is defined in terms of temperatures, concentrations,  $m, \bar{k}, T_m, D_S, D_L$ , which is valid throughout  $0 \leq x \leq L$ , but different in different phases. This new weak formulation has been developed by the author just for numerical scheme and no analytical solution has been presented.

The existence of the classical solution of a heat and mass transfer problem in terms of boundary and initial conditions which are taken as constant has been proved in [407] for semiinfinite region by using the classical formulation. The effect of concentration diffusion coefficient  $D_L$  of liquid on the solution has been investigated in several cases in conjunction with other parameter values. Linear-phase diagram has been considered. Numerical results have been discussed for comparable speeds of heat and mass transfer propagation, materials with high conductivity, and low diffusivity, etc.

The existence of the solution of a one-dimensional binary alloy problem in semiinfinite region has been examined in [408]. The equilibrium phase diagram consists of a liquidus curve  $T = f_L(\bar{c})$  for the component A in the mixture and solidus curve is given by  $T = f_S(\bar{c})$ , where  $\bar{c}, \bar{c}_0 < \bar{c} < 1$  is the concentration of the component A. It is assumed that  $f_L$  and  $f_S$  are monotonically increasing,  $f_L(\bar{c}) > f_S(\bar{c})$ ,  $f_L(0) = f_S(0) = T_m$ ,  $f_L(1) = f_S(1) = T_B$ . If  $T \leq f_S(\bar{c})$  then the solid is in equilibrium state and if  $T \geq f_S(\bar{c})$  then the liquid is in equilibrium state. If  $f_S(\bar{c}) < T < f_L(\bar{c})$  then the material is in unstable or heterogeneous or a mushy state. The isotherm temperature  $T_h(t)$  in Eq. (12.2.384) satisfies the following conditions.

$$T_h(t) = T_L(S(t), t) = T_S(S(t), t) = f_S(\bar{c}_S(S(t)^-, t)) = f_L(\bar{c}_L(S(t)^+, t)), \quad t > 0. \quad (12.2.391)$$

Other equations in the formulation of the problem considered in [408] are of the type similar to those considered in Eqs (12.2.377)–(12.2.384) but much simpler so that the solution of concentration in the solid is obtained as an unknown constant and not as a function of space

and time. The exact analytical similarity solution of the problem can be obtained following the procedure adopted for earlier problems. The following two results are the main findings.

*Result 1:* The similarity solution of the problem exists for all choices of initial and boundary conditions and thermal and diffusion parameters. Condition (12.2.391) is satisfied by the solution and  $T_h(t)$  is independent of time. If  $x = S(t) = 2\delta\sqrt{k_S t}$  then at least one value of  $\delta > 0$  exists which satisfies Eq. (12.2.391).  $\delta$  is bounded away from zero as  $k_S/D_L \rightarrow \infty$ .

*Result 2:* As  $D_L \rightarrow 0$  where  $D_L$  is the mass diffusivity of the liquid, the explicit analytical solution has a limit. The concentration tends to initial concentration except at  $x = S(t)$ , i.e. at the moving front.  $T_h \rightarrow f_S(\bar{c}_0)$ ,  $\bar{c}_0$  is initial concentration. When  $D_L$  is sufficiently small then it has been established that  $f_S(\bar{c}) < f_S(\bar{c}_0) = T_h \leq T_L < f_L(c)$ . This indicates that for  $x > S(t)$  but sufficiently close to  $S(t)$  the point  $(x, t)$  is in the mushy region. This contradicts the main assumption that  $x = S(t)$  is a sharp interface dividing the region into two distinct regions of solid and liquid.

This shortcoming or discrepancy in the above explicit solution which shows the existence of mushy region ahead of the liquid region was also pointed out in [409] with the help of numerical results. The sufficient condition for the appearance of mushy region ahead of the liquid region was obtained as

$$\frac{\partial}{\partial x} \{T_L(t, S^+(t)) - f_L(\bar{c}_L(t, S^+(t)))\} < 0. \quad (12.2.392)$$

It has been shown in [409] that if  $D_L$ , the mass diffusivity of liquid is very small then the condition in Eq. (12.2.392) is satisfied. A simple proof on the existence of explicit similarity solution of the problem similar to the one considered in [407] has also been given by showing that  $\delta > 0$  exists, when  $S(t) = 2\delta\sqrt{k_S t}$ . For physically realistic problems, the mass diffusivity  $D_L$  of the liquid is much smaller than  $k_L$ , the thermal diffusivity. It has been shown that for most of the realistic problems, the inequality (12.2.392) is satisfied and mushy region will occur.

A similarity solution of the solidification of a supercooled dilute binary alloy has been obtained in [410] by considering an one-dimensional two-phase problem in a semiinfinite region  $0 \leq x < \infty$ . The focus is on investigating the effect of *Lewis number*  $Le = k_L/D_L$ , which is the ratio of thermal diffusivity and the mass diffusivity of liquid. The method of obtaining analytical similarity solution does not offer anything new. The formulation of the problem is given below to show how Lewis number has been introduced into the equation. The scaling has been done as follows.

$$T_{S,L}^* = C_L(T_{S,L} - T_m - m\bar{c}_0)/l, \quad \bar{c}^* = \bar{c}/\bar{c}_0, \quad x^* = x/L, \quad S^* = S/L, \quad t^* = k_L t/L^2, \quad (12.2.393)$$

where  $L$  has dimension of length,  $\bar{c}_0$  is the initial concentration of the solute,  $m < 0$  is the slope of liquids line. Now we drop  $*$  over the transformed variables and write the formulation so that in the obtained similarity solution Lewis number ( $Le$ ) appears in the equations. Consider the following formulation:

$$\left. \begin{aligned} \frac{\partial T_S}{\partial t} &= \frac{k_S}{k_L} \frac{\partial^2 T_S}{\partial x^2}, \quad 0 \leq x \leq S(t); \quad \frac{\partial T_L}{\partial t} = \frac{\partial^2 T_L}{\partial x^2}, \quad x \geq S(t), \\ T_L(x, t)|_{x \rightarrow \infty} &= T_0 < 0, \quad \frac{\partial T_S}{\partial x} \Big|_{x=0} = 0. \end{aligned} \right\} \quad (12.2.394)$$

$$\left. \begin{aligned} \frac{\partial \bar{c}_S}{\partial t} &= \frac{1}{L_e} \frac{D_S}{D_L} \frac{\partial^2 \bar{c}_S}{\partial x^2}, \quad 0 \leq x \leq S(t); \quad \frac{\partial \bar{c}_L}{\partial t} = \frac{l}{L_e} \frac{\partial^2 \bar{c}_L}{\partial x^2}, \quad x \geq S(t), \\ \frac{\partial \bar{c}_S}{\partial x} \Big|_{x=0} &= 0; \quad c_L \rightarrow 1 \text{ as } x \rightarrow \infty. \end{aligned} \right\} \quad (12.2.395)$$

At the solid-liquid interface  $x = S(t)$

$$\bar{c}_L = \bar{c}_r(t), \quad \bar{c}_S = \bar{k} \bar{c}_r(t), \quad T_L = T_S = T_r(t) = M \bar{c}_0 (1 - \bar{c}_r), \quad M = -m \bar{c}_L / l, \quad (12.2.396)$$

$$\frac{K_S}{K_L} \frac{\partial T_S}{\partial x} \Big|_{x=S^-(t)} - \frac{\partial T_L}{\partial x} \Big|_{x=S^+(t)} = \frac{dS}{dt}; \quad \frac{D_S}{D_L L_e} \frac{\partial \bar{c}_S}{\partial x} - \frac{l}{L_e} \frac{\partial \bar{c}_L}{\partial x} = (1 - \bar{k}) \bar{c}_r \frac{dS}{dt}, \quad (12.2.397)$$

where  $\bar{k}$  is the partition coefficient,  $T_r(t)$  is the freezing temperature of the pure solvent which provides a solid seed for initiating solidification at the origin. The analytical solution exhibits the results as expected. When the Lewis number is large, the solute boundary layer is thinner than the thermal boundary layer and for small Lewis number the effect is opposite.

In a partial similarity solution or in other words analytical-numerical solution obtained in [411], in addition to solid and liquid phases, mushy region formulation has also been considered. The formulation of mushy region in the presence of both heat and mass transfer has been studied widely. Without making several assumptions it is not possible to present formulation and explicit solutions of such problems. We mention here very briefly, in particular some of the assumptions made in [411].

The solid region is fixed but liquid region can have motion due to natural convection. Temperature is uniform in the mush. At the solid-liquid interface  $\bar{c}_S = \bar{k} \bar{c}_L$ . In the solid solutal mass diffusion is negligible. Within a representative elementary volume (REV) of the mush, the temperature is uniform. After scaling, the freezing temperature of the solvent is taken zero and the liquidus line is given by  $\bar{c}_L = -m T_L$  where  $-m$  is the constant slope of the liquidus,  $\bar{c}$  is concentration. Thermo-physical properties are constant. The formulation is given below.

In the solid region,  $0 \leq x \leq S_1(t)$ ,  $S_1(0) = 0$ ,  $x = S_1(t)$  is solidus

$$\frac{\partial T_S}{\partial t} = (\rho_L / \rho_S) k_L \frac{\partial^2 T_S}{\partial x^2}; \quad T_S|_{x=0} = T_1 < T_{eu}, \quad T_{eu} \text{ is eutectic temperature.} \quad (12.2.398)$$

In the liquid region,  $S_2(t) \leq x < \infty$ ,  $S_2(0) = 0$ ,  $x = S_2(t)$  is liquidus

$$\frac{\partial T_L}{\partial t} = k_L \frac{\partial^2 T_L}{\partial x^2}, \quad T_L(x, 0) = T_I. \quad (12.2.399)$$

Initially at  $t = 0$  the liquid region has a uniform concentration  $\bar{c}_0$  and uniform temperature  $T_I$ . Therefore as  $x \rightarrow \infty$

$$T_L = T_I = T_A + \Delta T_{SH}, \quad \Delta T_{SH} \text{ is initial superheat in the liquid,} \quad (12.2.400)$$

and  $T_A$  is the equilibrium liquidus temperature.

$$T_A = -m \bar{c}_0, \quad \bar{c}_L|_{x \rightarrow \infty} = \bar{c}_0. \quad (12.2.401)$$

In the mushy region,  $S_1(t) \leq x \leq S_2(t)$ ,  $t \geq 0$ , equation for temperature is

$$U \frac{\partial T_M}{\partial x} + ((1 - \phi) + R\phi) \frac{\partial T_M}{\partial t} = K_L \frac{\partial^2 T_M}{\partial x^2} + (Rl/C) \frac{\partial \phi}{\partial t}, \quad (12.2.402)$$

where  $U$  is the volume flux of the interdendritic fluid in the mush, and  $\phi$  is the volume fraction of the solid in REV of the mush. Thermal conductivity  $K$  and specific heat  $C$  are taken the same for all the phases. Subscript  $M$  stands for the mushy region,  $R = \rho_L/\rho_S$  and  $k$  is diffusivity.

The equation for concentration is

$$(\beta \bar{k} R \phi + (1 - \phi)) \frac{\partial \bar{c}_L}{\partial t} + U \frac{\partial \bar{c}_L}{\partial x} = (1 - \bar{k}) R \bar{c}_L \frac{\partial \phi}{\partial t}, \quad (12.2.403)$$

where  $\bar{k}$  is the partition coefficient and  $\beta$  is microsegregation parameter.

$$\frac{\partial U}{\partial x} = (1 - R) \frac{\partial \phi}{\partial t}; \text{ equation for mass balance in the mush.} \quad (12.2.404)$$

Boundary conditions on  $x = S_1(t)$

$$\left. \begin{aligned} T_S &= T_M = T_{eu}, \\ k_L \frac{\partial T_M}{\partial x} \Big|_{x=S_1^+(t)} - \frac{\partial T_S}{\partial x} \Big|_{x=S_1^-(t)} &= -R(1 - \phi) \frac{l}{C} \frac{dS_1(t)}{dt}. \end{aligned} \right\} \quad (12.2.405)$$

$$U|_{S_1^+(t)} - U|_{S_1^-(t)} = (1 - R)(1 - \phi) \frac{dS_1}{dt}, \text{ on } x = S_1(t). \quad (12.2.406)$$

$x = S_1(t)$  is also the position of eutectic under the assumptions made in the formulation of the problem.

Boundary conditions on  $x = S_2(t)$

$$T_M = T_L = T_I; \quad \frac{\partial T_M}{\partial x} \Big|_{x=S_2^+(t)} = \frac{\partial T_L}{\partial x} \Big|_{x=S_2^-(t)}; \quad \phi = 0. \quad (12.2.407)$$

If a similarity variable  $\xi = x/(2\sqrt{\delta t})$  is defined and  $S_1(t)$  and  $S_2(t)$  are expressed as  $S_1(t) = 2\delta_1\sqrt{k_L t}$ ,  $S_2(t) = 2\delta_2\sqrt{k_L t}$ , where  $\delta_1$  and  $\delta_2$  are real positive constants to be determined, then similarity solutions for  $T_S(x, t)$  and  $T_L(x, t)$  can be obtained. However, similarity solutions for the complete formulation are not possible. The concentration  $\bar{c}_L$  can be eliminated from Eq. (12.2.403) by using the relation  $\bar{c}_L = -mT_L$ . The complete solution can be obtained only numerically. The numerical solution was obtained by adopting and extending the approach given in [412] and the procedure has been outlined in 11 steps in [411]. For further details see [411].

In [413], the author has addressed an important question about the boundary conditions at the mush-liquid boundary. We have seen in the earlier discussion that in the solidification of binary alloys, the sharp interface model of separating the solid from liquid by a sharp boundary is not appropriate as near the interface on the liquid side a mushy region develops. Therefore, the formulation of solidification of binary alloy should include mushy region formulation also. What should be the boundary conditions at the mush-liquid boundary? The continuity of variables and their fluxes leaves the problem under deterministic as the quantity of solid

fraction present at the mush-liquid boundary should be evaluated correctly by thermodynamic considerations. To make the problem deterministic some auxiliary conditions are prescribed at this interface such as  $\phi = 0$ ,  $\phi$  given in Eq. (12.2.407), which is the *marginal phase equilibrium condition*.

The focus in this interesting work in [413] is not on the formulation of mushy region but on showing that the under determinacy may be resolved by an application of an inequality assuring that the melt is not supercooled, together with the condition that mass fraction of solid has physically realistic value at the interface. The lack of supercooling is a natural corollary of the assumption of thermodynamic phase equilibrium in the mush. To derive the corollary, we proceed as follows.

Let the boundary between mush and melt (liquidus) be denoted by  $f(\vec{x}, t) = 0$  or  $\vec{x} = x_i(t)$ ,  $i = 1, 2, 3$  where  $\vec{x} = (x_1, x_2, x_3)$ . If  $|\nabla f| = 1$  on the interface then the normal vector on the interface can be expressed as

$$\vec{n}(x_i) = \nabla f(\vec{x}, t)|_{f=0}, \quad (12.2.408)$$

$\vec{n}$  points in the direction in which the interface is moving. Let  $n$  be the local normal coordinate (i.e. the distance along the normal from the interface). The relation between temperature and concentration at the liquidus is given by

$$T_M = G(\bar{c}_M, p_M), \quad \text{where } p_M \text{ is the pressure in the mush.} \quad (12.2.409)$$

If pressure is neglected then

$$\frac{\partial T_M}{\partial n} = \frac{\partial G}{\partial \bar{c}_M} \frac{\partial \bar{c}_M}{\partial n}. \quad (12.2.410)$$

The superheat  $H$  in the melt is given by

$$H = T_L - G(\bar{c}_L), \text{ and so } \frac{\partial H}{\partial n} = \frac{\partial T_L}{\partial n} - \frac{\partial G}{\partial \bar{c}_L} \cdot \frac{\partial \bar{c}_L}{\partial n}. \quad (12.2.411)$$

As temperature and concentration are continuous together with their fluxes across liquidus, we get the condition for *marginal equilibrium*, which is  $H = 0$  for  $n = 0$ . If there is no supercooling in the liquid just in front of the liquidus, we have  $\frac{\partial H}{\partial n} > 0$  for  $n > 0$ . Combining the two results, we get

$$\text{For } n = 0, \quad \frac{\partial H}{\partial n} \geq 0. \quad (12.2.412)$$

The condition  $\frac{\partial H}{\partial n} \geq 0$  at  $n = 0$  is a weaker condition than the condition of marginal equilibrium. Further the condition  $\frac{\partial H}{\partial n} \geq 0$  at  $n = 0$  is a corollary to the assumption of phase equilibrium and need not be imposed separately. It has been shown in [413] that this corollary helps in obtaining the correct boundary conditions at the mush-liquid boundary. First, melt boundary-layer equations are derived in [413] with flux conditions at  $n = 0$ . A combined flux condition in which the derivatives of the mush variables are eliminated from the two flux conditions of the temperatures and concentrations is obtained to form a single flux condition at  $n = 0$ . This single condition consists of two terms. Both the terms are nonnegative so that

each term should be zero separately. One of the conditions implies that the mass fraction of solid at the mush-liquid interface should be zero and another implies the marginal equilibrium condition. The lack of supercooling in the melt and the requirement that the mass fraction of the solid has a physically reasonable value, yields two informations rather than one and removes the indeterminacy. For further information see [413] as presenting the material only partially will not be of much help. The details are lengthy.

### Stefan-Like Problems in Porous Media

In the context of Stefan and Stefan-like problems considered in this chapter, to a large extent, information was available in this book itself which is about their modelling, parameters involved, assumptions, etc. It was easy to supplement some additional information. In problems of porous media the problem formulation is of different type and parameters involved are different. Due to space restrictions it is not possible to discuss all the aspects in detail. Attempts have been made here in a limited way to explain the presented material. For further information the reader has to see original works.

The problem discussed below is concerning melting in a frozen saturated porous medium occupying initially the space  $0 \leq x < \infty$ . A sharp interface  $x = S(t)$  divides the frozen and unfrozen part of the domain. The ice formed is due to water present in the soil of extent  $0 \leq x < \infty$ . An analytical solution has been obtained in [414] for the problem formulation given below.

$$\frac{\partial T_L}{\partial t} = k_L \frac{\partial^2 T_L}{\partial x^2} - b\rho \frac{dS}{dt} \frac{\partial T_L}{\partial x}, \quad 0 \leq x \leq S(t); \quad \frac{\partial T_S}{\partial t} = k_S \frac{\partial^2 T_S}{\partial x^2}, \quad x > S(t), t > 0. \quad (12.2.413)$$

Boundary conditions on  $x = S(t)$  and at  $x = 0$  and initial condition are as follows:

$$\left. \begin{aligned} T_L(S(t), t) = T_S(S(t), t) = d\rho S(t) \frac{dS}{dt}, \\ K_S \frac{\partial T_S}{\partial x} - K_L \frac{\partial T_L}{\partial x} = \alpha \frac{dS}{dt} + \beta \rho S(t) \left( \frac{dS}{dt} \right)^2. \end{aligned} \right\} \quad (12.2.414)$$

$$T_L(0, t) = B > 0; \quad T_S(x, 0) = T_S(\infty, t) = -A < 0, \quad S(0) = 0, \quad (12.2.415)$$

$$\left. \begin{aligned} b = \varepsilon \rho_{LC_L} / (\rho_{SC_S}), \quad d = \varepsilon \gamma \mu / K_0, \quad \rho = (\rho_L - \rho_S) / \rho_L, \quad \alpha = \varepsilon \rho_S l, \\ \beta = \varepsilon d \rho_S (C_L - C_S). \end{aligned} \right\} \quad (12.2.416)$$

In the previous equations  $\varepsilon > 0$  is porosity,  $\gamma$  is coefficient in Clausius-Clapeyron law,  $\mu > 0$  is viscosity of liquid,  $K_0$  is hydraulic permeability, the melting temperature is taken as zero,  $k$  is diffusivity and  $l$  is the latent heat.  $A$  and  $B$  are constants. There is motion in the liquid due to difference in densities but water does not percolate into the frozen soil. The previous equations are not scaled.

If a similarity variable  $\eta = x / (2\sqrt{k_L t})$  is defined and  $S(t)$  is taken as  $S(t) = 2\delta\sqrt{k_L t}$ ,  $\delta > 0$ , then it is easy to obtain the required quantities  $T_L(x, t)$ ,  $T_S(x, t)$  and  $\delta$  following the procedure as followed in Neumann solution. We get

$$T_L(x, t) = A_1 + A_2 \int_0^\eta \exp(-y^2 + 2b\rho\delta y) dy, \quad (12.2.417)$$

$$T_S(x, t) = A_3 + A_4 \operatorname{erf}(\eta\sqrt{k_L/k_S}), \quad (12.2.418)$$



$A_1, A_2, A_3$  and  $A_4$  are unknown constants which can be determined with the help of boundary and initial conditions. For the equation to determine  $\delta$  refer [414].

The existence of the solution so obtained can be validated by proving that  $\delta$  which is obtained by solving a transcendental equation exists and is positive. The necessary and sufficient conditions for the existence of a unique solution and also of multiple solutions have been carefully examined with the help of some parameters and many results concerning them have been reported in [414]. The analysis is interesting but because of its length considerations it cannot be presented here. Refer [414] for further details.

The problem considered in [415] is regarding absorption of water in the soil which forms saturated and unsaturated zones separated by a sharp interface denoted by  $x = S(t)$ . In the one-dimensional problem soil is considered as the region  $0 \leq x < \infty$ , and water is supplied at  $x = 0$  under pressure head  $\psi_0$  which percolates into the soil. The zone of water saturation progresses with time towards zone of unsaturation which is  $S(t) \leq x < \infty$ . If  $\psi(x)$  is the soil-water matrix potential and  $k(\psi)$  is the hydraulic conductivity then according to Darcy's law, the water flux  $F(x)$  is given by the rule

$$F = -K(\psi) \frac{\partial \psi}{\partial x}, \text{ after neglecting gravity.} \quad (12.2.419)$$

In the saturated zone

$$\psi(x, t) = \psi_0 - (\psi_0 - \psi_S)x/S(t), \quad 0 < x \leq S(t); \quad \psi_S = \psi|_{S(t)}. \quad (12.2.420)$$

If  $\theta$  is the volumetric water content in the soil then the problem formulation for the determination of  $\theta$  in the unsaturated zone is

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left( D(\theta) \frac{\partial \theta}{\partial x} \right), \quad x > S(t); \quad D(\theta) = a/(b - \theta)^2, \quad (12.2.421)$$

$$-D(\theta) \frac{\partial \theta}{\partial x} (S(t)^+, t) = K_S (\psi_0 - \psi_S) / S(t), \quad t > 0; \quad \theta(S(t)^+, t) = \theta_S, \quad t > 0, \quad (12.2.422)$$

$$\theta(x, 0) = \theta(+\infty, t) = \theta_0, \quad S(0) = 0, \quad (12.2.423)$$

where  $D$  is soil-water diffusivity,  $a, b, \theta_S, \theta_0, K_S$  are known constants,  $K_S$  is hydraulic conductivity at saturation. Eq. (12.2.421) is nonlinear therefore we have to linearize it with the help of some suitable transformations. The problem in Eqs (12.2.421)–(12.2.423) can be easily reformulated with the help of some simple transformations and some of them are

$$\bar{\theta} = (\theta - \theta_0) / (\theta_S - \theta_0); \quad R = (b - \theta_0) / (\theta_S - \theta_0); \quad \bar{x} = x - S(t). \quad (12.2.424)$$

We have not given here all the transformations given in [415] as it is a lengthy process which cannot be described here fully. Only main features are being described here. In the transformed formulation, we again make use of some transformations whose main features are given below.

$$g = R(R - 1) / (R - \bar{\theta}), \quad h = \frac{1}{\sqrt{R(R - 1)}} \int_0^{\bar{x}} (R - \bar{\theta}(y, t)) dy, \quad S(t) = \delta \sqrt{t}. \quad (12.2.425)$$

The problem is finally reduced to the solution of a nonlinear equation for  $g(S(t))$  and its solution is amenable to similarity solution. To obtain  $\delta$ , a transcendental equation is obtained.

To prove the existence of the solution of the problem it is necessary and sufficient to prove that a real positive  $\delta$  exists. This has been done in [415] and this type of proof is an interesting and important part of the research publications of Tarzia and coauthors. Whilst proving the existence result, the parameter  $R$  in Eq. (12.2.424) comes into the picture together with other parameters such as filtration rate, sorptivity.  $R$  is called *bifurcation parameter* and the solution has been discussed with respect to two ranges of  $R$ ,  $1 < R < R_1$  or  $R > R_1$ ,  $R_1$  depends on the feasibility of the solution. The equations presented earlier are not scaled but scaling has been done in the formulation in [415]. Presentation of anyone of the results requires many other lengthy details therefore a interested reader should go through [415] and the references mentioned in it.

The problem considered in [416] is about the solidification of a mixture of eutectic melt and vapour which initially at time  $t = 0$  occupies the porous space  $0 \leq x < \infty$ . We present below the heat and mass transfer equations in which for  $t > 0$ , the solid region is separated by a sharp interface  $x = S(t)$  from the mixture of liquid and vapour. In [416], the equations have been presented in dimensionless form. Their presentation here in dimensionless form giving scaling procedure will occupy too much space. Further the presentation of the similarity solution procedure does not require scaled variables. We present below the formulation of the problem in which the notations are the same as followed in this book. The following equations are not scaled.

$$k_S \frac{\partial^2 T_S}{\partial x^2} = \frac{\partial T_S}{\partial t}, \quad 0 < x \leq S(t); \quad \frac{\partial T_L}{\partial t} = k_L \frac{\partial^2 T_L}{\partial x^2} + \epsilon K_{0S} \frac{\partial \bar{c}_m}{\partial t}, \quad x \geq S(t), \quad t > 0. \quad (12.2.426)$$

$$\frac{\partial \bar{c}_m}{\partial t} = Lu \left( \frac{\partial^2 \bar{c}_m}{\partial x^2} + p_n \frac{\partial^2 T_L}{\partial x^2} \right), \quad x \geq S(t), \quad t > 0; \quad T_S(S(t), t) = T_L(S(t), t) = 1. \quad (12.2.427)$$

$$T_S(0, t) = T_0 < 1; \quad T_L(x, 0) = T_1 > 1. \quad (12.2.428)$$

$$\bar{c}_m(x, 0) = \bar{c}_0, \quad \bar{c}_m(\infty, t) = \bar{c}_0, \quad T_L(\infty, t) = T_1. \quad (12.2.429)$$

There is no concentration gradient in the solid. There are some minor corrections in the formulation which have been taken care here.  $\bar{c}_m(x, t)$  is the moisture content of the liquid-vapour region,  $T_L(x, t)$  is the temperature of the liquid-vapour region.  $\epsilon$  is phase transfer number and  $Lu$ ,  $p_n$  and  $k_{0S}$  are some known constants but if they are suitably scaled then they are called Luikov number, Posnov number and Kossovich number, respectively. For further details about these parameters refer [417]. At  $x = S(t)$ , we have the following conditions concerning heat and mass balance to be satisfied.

$$K_S \frac{\partial T_S}{\partial x} - K_L \frac{\partial T_L}{\partial x} = (1 + (\bar{c}_m|_{S(t)} - 1) \bar{c}_{mf}) l \frac{dS}{dt}, \quad \text{on } x = S(t), \quad S(0) = 0. \quad (12.2.430)$$

$$Lu \frac{\partial \bar{c}_m}{\partial x} \Big|_{x=S(t)} + \bar{c}_{mf} \frac{dS}{dt} = 0, \quad \bar{c}_{mf} \text{ is the frozen-moisture concentration,} \quad (12.2.431)$$

$$\bar{c}_{mf} \text{ is unknown. } \bar{c}_m(S(t), t) = \bar{c}_{mS}, \quad \bar{c}_{mS} \text{ is an unknown constant.} \quad (12.2.432)$$

The temperature in the liquid and concentration of liquid in the vapour are coupled. To uncouple them the following procedure can be adopted. Let

$$\theta_J = E_J T_L + F_J \bar{c}_m, \quad J = 1, 2 \text{ and } E_J \text{ and } F_J \text{ are unknown constants.} \quad (12.2.433)$$

If the constants  $E_J$  and  $F_J$  are chosen as given below then two uncoupled equations for  $\theta_1$  and  $\theta_2$  are obtained. Let

$$E_1 = 1, E_2 = P_2 p_n / (P_2 - 1), F_1 = (P_1 - 1) p_n / P_1, F_2 = 1, \quad (12.2.434)$$

where  $P_1$  and  $P_2$  are the roots of the equation

$$P_J - (1 + (1 + \epsilon k_{0SP_n}) Lu) p_J + Lu = 0, \quad J = 1, 2. \quad (12.2.435)$$

By using Eqs (12.2.433)–(12.2.435), temperature  $T_L$  and concentration  $\bar{c}_m$  can be uncoupled and the following equations are obtained for  $\theta_J$ ,  $J = 1, 2$ .

$$\frac{\partial \theta_1}{\partial t} = P_1 \frac{\partial^2 \theta_1}{\partial x^2}; \quad \frac{\partial^2 \theta_2}{\partial t} = P_2 \frac{\partial^2 \theta_2}{\partial x^2}; \quad x \geq S(t), \quad t > 0. \quad (12.2.436)$$

The equations for  $T_S$ ,  $\theta_1$  and  $\theta_2$  are linear parabolic equations whose solutions can be obtained as done in Neumann solution. To satisfy the boundary and initial conditions either change the boundary and initial conditions to suit  $\theta_1$  and  $\theta_2$  or with the help of  $\theta_1$  and  $\theta_2$  obtain  $T_L$  and  $\bar{c}_m$  using Eq. (12.2.433). To obtain complete solution unknown constants are to be determined for which equal number of boundary and initial conditions are available. Refer [416] for analytical solution. Numerical calculations were done for the obtained similarity solution using values of several parameters. In the numerical work,  $\bar{c}_{ms}$  in Eq. (12.2.432) was found less than the initial concentration  $\bar{c}_0$  and  $\bar{c}_{mf}$  in Eq. (12.2.430) was found having greater numerically calculated values than  $\bar{c}_{ms}$  which indicates discontinuity of  $\bar{c}_m$  at  $x = S(t)$ .

Nonlinear diffusion equations occur in many moving boundary problems of physical interest such as swelling of grains, polymers, wool grease in a wool scour. Such a swelling problem has been considered in [418]. We consider here swelling of grains. Let  $\phi(x, t)$  be the moisture content in the grain which changes as the grain is exposed to a pure liquid of moisture content  $\phi = 1$  on its surface. In an one-dimensional problem the surface is initially a static boundary  $x = 0, 0 \leq x < \infty$ . The grain absorbs the liquid and changes the initial moisture content  $\phi = \phi_0$  of the grain to the level  $\phi = \phi_1 < 1$ . Due to absorption of liquid and rising moisture level, swelling of grain takes place and the outer surface starts moving in the positive  $x$ -direction. As the liquid absorption continues in the grain, the depth up to which liquid has reached in the grain is moving in the negative  $x$ -direction due to swelling. Let  $x = S_1(t)$  and  $x = S_2(t)$  be the two moving boundaries in the positive and negative  $x$ -directions, respectively. Now the problem is to find the solution of the diffusion equation,  $S_1(t)$  and  $S_2(t)$  and we consider the following formulation.

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x} \left( D(\phi) \frac{\partial \phi}{\partial x} \right), \quad S_2(t) \leq x \leq S_1(t), \quad t > 0; \quad S_1(0) = S_2(0) = 0, \quad (12.2.437)$$

$D(\phi)$  is the nonlinear diffusivity. To obtain boundary condition at  $x = S_1(t)$ , we integrate Eq. (12.2.437) from  $x = -\infty$  to  $x = S_1(t)$  and apply boundary conditions presented in Eq. (12.2.439).

$$-\frac{dS_1(t)}{dt} \phi_1 + \frac{\partial}{\partial t} \left[ \int_{-\infty}^{S_1(t)} \phi dx \right] = \left[ D(\phi) \frac{\partial \phi}{\partial x} \right]_{-\infty}^{S_1(t)}, \quad (12.2.438)$$

$$\left. \frac{\partial \phi}{\partial x} \right|_{x \rightarrow -\infty} = 0, \quad \phi|_{x=S_1(t)} = \phi_1. \quad (12.2.439)$$

If there is no loss of solid material whose volume fraction is  $(1 - \phi)$  then

$$\int_{-\infty}^{S_1(t)} (1 - \phi) dx = \text{constant}, \quad (12.2.440)$$

$$\frac{dS_1}{dt} = \frac{D(\phi_1)}{(1 - \phi_1)} \left. \frac{\partial \phi}{\partial x} \right|_{x=S_1(t)}, \quad t > 0. \quad (12.2.441)$$

Eq. (12.2.441) is obtained when in the diffusion equation (12.2.438), boundary conditions are used. In a similar way the following equation can be derived.

$$\frac{dS_2}{dt} = -\frac{D(\phi_0)}{\phi_0} \left. \frac{\partial \phi}{\partial x} \right|_{x=S_2(t)}, \quad t > 0, \quad \phi|_{x=S_2(t)} = \phi_0. \quad (12.2.442)$$

The formulation of the problem now consists of Eqs (12.2.437), (12.2.440)–(12.2.442) and the two conditions given below.

$$\phi(x = S_1(t), t) = \phi_1; \quad \phi(x = S_2(t), t) = \phi_0, \quad t \geq 0. \quad (12.2.443)$$

If  $D(\phi) = \text{constant}$ , then similarity solution can be easily obtained otherwise some additional information is required about  $D(\phi)$  to obtain the solution. If the following transformations are used in the formulation then it is easier to obtain the similarity solution. However, without these transformations also solution can be obtained. Let

$$u^* = (\phi - \phi_0) / (\phi_1 - \phi_0), \quad x^* = x/x_0, \quad t^* = t/t_0, \quad S_1^*(t) = S_1(t)/x_0, \quad (12.2.444)$$

$$S_2^*(t) = S_2(t)/x_0, \quad t_0 = x_0^2/D_1, \quad D_1 = D(\phi_1), \quad x_0 \text{ is length scale.} \quad (12.2.445)$$

We now drop  $*$  over the various quantities. The formulation in terms of  $u(x, t)$  is given below for  $t > 0$ .

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( D(u) \frac{\partial u}{\partial x} \right), \quad S_2(t) \leq x \leq S_1(t); \quad u|_{S_1(t)} = 1, \quad u|_{S_2(t)} = 0, \quad (12.2.446)$$

$$\frac{dS_1}{dt} = a \left. \frac{\partial u}{\partial x} \right|_{x=S_1}; \quad \frac{dS_2}{dt} = -b \left. \frac{\partial u}{\partial x} \right|_{x=S_2}, \quad (12.2.447)$$

$$a = (\phi_1 - \phi_0) / (1 - \phi_1), \quad b = \frac{D(\phi_0)}{D(\phi_1)} \frac{(\phi_1 - \phi_0)}{\phi_0}. \quad (12.2.448)$$

In the quasisteady-state  $\partial u / \partial t = 0$ . Putting  $\frac{\partial u}{\partial t} = 0$  in Eq. (12.2.446) and integrating twice we get

$$\int D(u) \frac{\partial u}{\partial x} dx = c_1 x + c_2, \quad c_1 \text{ and } c_2 \text{ are constants.} \quad (12.2.449)$$

Define

$$\bar{D}(u) = \int_0^w D(w) dw = \left( \frac{x - S_2}{S_1 - S_2} \right) \bar{D}(1). \quad (12.2.450)$$

From Eqs (12.2.447), (12.2.448), we get

$$\frac{dS_1}{dt} = \frac{a\bar{D}(1)}{D(1)(S_1 - S_2)}; \quad \frac{dS_2}{dt} = \frac{b\bar{D}(1)}{D(0)(S_1 - S_2)}. \quad (12.2.451)$$

By taking  $S_1 = \delta_1\sqrt{t}$  and  $S_2 = -\delta_2\sqrt{t}$ ,  $\delta_1$  and  $\delta_2$  can be easily determined in terms of  $D(0)$ ,  $D(1)$ ,  $\bar{D}(1)$ ,  $a$  and  $b$  if these quantities are known.  $u(x, t)$  or  $\phi(x, t)$  can be determined only numerically. If  $D(u)$  is a constant or it is a suitable function such as  $D(u) = e^{\beta(u-1)}$  then  $u(x, t)$  can also be obtained analytically. The solution obtained earlier for general  $D(u)$  is neither a similarity solution nor an exact analytical solution but the procedure adopted is different from other solutions and in some special cases exact solution can be obtained.

In the transient case also analytical solution can be obtained in a limited way. Let

$$z = (x - S_2) / (S_1 - S_2), \quad z \in [0, 1], \quad \text{and } u(x, t) = \theta(z, t). \quad (12.2.452)$$

Now the new problem formulation in the domain  $0 \leq z \leq 1$  is

$$\frac{\partial \theta}{\partial t} = \frac{1}{(S_1 - S_2)^2} \frac{\partial}{\partial z} \left( D(\theta) \frac{\partial \theta}{\partial z} \right) - \frac{1}{(S_1 - S_2)} \left( \frac{dS_2}{dt} + \left( \frac{dS_1}{dt} - \frac{dS_2}{dt} \right) z \right) \frac{\partial \theta}{\partial z}, \quad (12.2.453)$$

$$\frac{dS_1}{dt} = \frac{a}{(S_1 - S_2)} \frac{\partial \theta}{\partial z} \Big|_{z=1}; \quad \frac{dS_2}{dt} = \frac{-b}{(S_1 - S_2)} \frac{\partial \theta}{\partial z} \Big|_{z=0}, \quad (12.2.454)$$

$a$  and  $b$  are as defined in Eq. (12.2.448).

$$\theta(z = 0, t) = 0, \quad \theta(z = 1, t) = 1, \quad S_1(0) = 0, \quad S_2(0) = 0. \quad (12.2.455)$$

Take  $S_1(t) = \delta_1\sqrt{t}$  and  $S_2(t) = -\delta_2\sqrt{t}$ . From Eq. (12.2.454),  $\delta_1$  and  $\delta_2$  are obtained as

$$\delta_1 = \sqrt{\frac{2a\theta_z(1)}{1 + \gamma}}, \quad \delta_2 = \sqrt{\frac{2b\gamma\theta_z(0)}{1 + \gamma}}, \quad \gamma = \delta_2/\delta_1, \quad \theta_z(1), \quad \theta_z(0) \text{ are unknown.} \quad (12.2.456)$$

Eq. (12.2.453) is highly nonlinear so to obtain some analytical solution it will be assumed that  $\partial\theta/\partial t \simeq 0$  and  $D(\theta) = 1$ ,  $a \neq b$ . With this simplification we get

$$\frac{d^2\theta}{dz^2} = \alpha(\beta - 2z) \frac{\partial \theta}{\partial z}; \quad \theta(0) = 0, \theta(1) = 1, \alpha = a(1 + \gamma)\theta_z(1)/2, \quad \beta = \frac{2\gamma}{(1 + \gamma)}. \quad (12.2.457)$$

The solution of Eq. (12.2.457) is

$$\theta = E \int_0^Z e^{\alpha(\beta y - y^2)} dy, \quad E = \theta(z)|_{z=1} = \left( \int_0^1 e^{\alpha(\beta y - y^2)} dy \right)^{-1}. \quad (12.2.458)$$

The quantities  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\theta_z(0)$ ,  $\theta_z(1)$  are all connected with each other and they can be determined only numerically. Numerical solution was also obtained for the transient problem using finite difference scheme and compared with analytical solution. For further information about problems concerning swelling see [418] and the references mentioned therein.

The problem discussed below is a mass transfer problem in which the mass flux takes place on account of the porosity in the sediment deposits. It has a characteristic feature of having two moving boundaries with one of them moving in opposite direction to the other. Formation of sedimentary ocean delta has been considered in [419] and analytical and numerical solutions have been obtained for the problem formulated. We present here very briefly the necessary details required in understanding problem formulation and for further information about physical, geometrical and modelling aspects together with assumptions made, the interested reader is referred to [419] and the references mentioned therein.

One moving boundary is the shoreline denoted by  $S_{sh}$  whose equation is  $x = S_{sh}(t)$  which is moving along  $x > 0$  in the one-dimensional problem considered in [419]. Another moving boundary is the landward movement of the alluvial-bedrock transition denoted by  $x = -S_{ab}(t)$ . The sediment unit flux  $q(x, t)$  in the sediment deposit is given by

$$q(x, t) = -v \frac{dh}{dx}, \quad h(x, t) \text{ is sediment height above a horizontal datum.} \quad (12.2.459)$$

The units of flux  $q$  are volumetric sediment bed transport per unit width and time; a definition that takes into account the porosity of the sediment deposit. The diffusivity  $v$  is taken as

$$v = v^* \left| \frac{\partial h}{\partial x} \right|^\theta, \quad v^* \text{ is constant with dimensions of diffusivity.} \quad (12.2.460)$$

The exponent  $\theta$  in Eq. (12.2.460) is  $\geq 0$ . Using the definition of flux given in Eq. (12.2.459), and mass balance in the sediment wedge, we get the following equation as modelled in [420].

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( v^* \left| \frac{\partial h}{\partial x} \right|^\theta \frac{\partial h}{\partial x} \right), \quad S_{ab}(t) \leq x \leq S_{sh}(t), \quad t > 0. \quad (12.2.461)$$

The boundary conditions are

$$h|_{S_{ab}(t)} = -\beta S_{ab}(t); \quad h|_{S_{sh}(t)} = 0, \quad t > 0. \quad (12.2.462)$$

$$v^* \left| \frac{\partial h}{\partial x} \right|^\theta \frac{\partial h}{\partial x} \Big|_{S_{ab}(t)} = -q_0; \quad v^* \left| \frac{\partial h}{\partial x} \right|^\theta \frac{\partial h}{\partial x} \Big|_{S_{sh}(t)} = -\gamma S_{sh} \frac{dS_{sh}}{dt}. \quad (12.2.463)$$

The boundary conditions need some explanation. The sediment delta is building on a nonsubsiding bedrock with constant slope  $-\beta > 0$  units into an ocean with a fixed relative sea level taken as  $z = 0$ .  $h(x)$  is the sediment height above  $z = 0$ . Initially a constant unit sediment flux  $q_0 > 0$  (area/time) is introduced upstream of the initial position of the shoreline  $x = 0$ . The bedrock slope is sufficient to transport this sediment flux to the ocean where it deposits to form a small sediment wedge. Over time this sediment wedge increases in size, consisting of a submarine component advancing into the deepening ocean and a subaerial fluvial component moving landward, up the bedrock slope. This deposition continues down the length of the fluvial surface. At the shoreline level the remaining flux is deposited offshore, contributing to the advancement of shoreline. This offshore transport deposition can be modelled as 'sediment wedge' with a constant angle of repose with slope  $\alpha$ .

The first condition in Eq. (12.2.462) gives the sediment height  $h(x, t)$  above sea level  $z = 0$  and the second condition tells that the fluvial surface meets the shoreline  $z = 0$  at the point up

to which the shoreline has progressed in the  $x+$  direction.  $q_0$  in Eq. (12.2.463) has already been explained and the first condition in Eq. (12.2.463) implies the continuity of flux at  $x = S_{ab}(t)$ . The second condition in Eq. (12.2.463) expresses the sediment balance at shoreline level in which  $\gamma = \alpha\beta/(\alpha - \beta)$  is referred to as the effective submarine slope and  $\gamma S_{sh}(t)$  gives the depth of the ocean at the delta toe. The initial conditions are

$$S_{sh}(t) = 0, \quad S_{ab}(t) = 0, \quad h(x, 0) = 0. \quad (12.2.464)$$

The previous equations are not scaled but after this stage of formulation scaling has been done in [419]. For any arbitrary exponent  $\theta \geq 0$ , analytical solution is not possible for this problem. For  $\theta = 0$ , a similarity solution can be obtained which can be derived without scaling of the equations. So to save space scaling is not being done here. We define a similarity variable and express moving boundaries as follows.

$$\xi = x/(2\sqrt{t}); \quad \eta = h(x, t)/(2\sqrt{t}); \quad S_{ab} = -2\delta_{ab}\sqrt{t}; \quad S_{sh} = 2\delta_{sh}\sqrt{t}. \quad (12.2.465)$$

If  $\xi$  and  $\eta$  defined in Eq. (12.2.465) are used in Eq. (12.2.461) with  $\theta = 0$ , then

$$\frac{d^2\eta}{d\xi^2} + 2R_{ab}\xi \frac{d\eta}{d\xi} - 2R_{ab}\eta = 0, \quad R_{ab} = -\frac{1}{\beta} \left. \frac{\partial h}{\partial x} \right|_{x=S_{ab}}; \quad -\delta_{ab} < \xi < \delta_{sh}. \quad (12.2.466)$$

The solution of Eq. (12.2.466) can be easily obtained as the differential equation is similar to that in Eq. (12.2.38). To determine the two constants which occur in the solution of Eq. (12.2.466), two boundary conditions in Eq. (12.2.462) are available. Finally, solution of Eq. (12.2.466) is obtained in terms of  $\delta_{sh}$  and  $\delta_{ab}$  which are unknowns and which are to be determined from two conditions in Eq. (12.2.463). Solution of the moving boundaries can be obtained only numerically. See [419] for details.

Another case in which similarity solution can be obtained is  $\theta \rightarrow \infty$ . In this case  $h(x, t)$  in Eq. (12.2.463) should be linear function of  $x$ . Therefore

$$\lim_{\theta \rightarrow \infty} \eta = R_{ab}(\delta_{sh} - \xi), \quad \text{as } h|_{x=S_{sh}} = 0. \quad (12.2.467)$$

The first condition in Eq. (12.2.462) implies  $\delta_{ab} = R_{ab}\delta_{sh}/(1 - R_{ab})$ .  $\delta_{ab}$  or  $\delta_{sh}$  can be obtained from Eq. (12.2.461) to complete the solution.

Numerical solution of the problem was also obtained using fixed grid enthalpy method by defining an enthalpy appropriate for this problem and formulating the differential equation and boundary conditions appropriate for enthalpy formulation in the region  $-\infty < x < \infty$ . The formulation in terms of enthalpy is interesting. The details of numerical scheme have been given. As reported in [419], the enthalpy method provides an accurate approximation of the similarity solution.

Experimental investigations suggest that freezing in a porous medium, such as wet soil, is accompanied by moisture migration towards the freezing front. The moisture migration effects the temperature distribution in the medium and the freezing front movement. Freezing in a radially symmetric one-dimensional cylindrical geometry in the region  $0 \leq r < \infty$  has been considered in [421]. At  $t = 0$ , the region  $0 \leq r < \infty$  is at the temperature  $T_0 > T_m$

and has moisture content concentration  $\bar{c}_0$ ,  $T_m$  is the freezing temperature. The freezing takes place due to a heat sink at  $r = 0$  of strength  $Q$  or in other words

$$\lim_{r \rightarrow 0} \left( 2\pi r \frac{\partial T_S}{\partial r} \right) = Q, \quad T_S \text{ is solid temperature in } 0 \leq r \leq S(t), \quad (12.2.468)$$

$r = S(t)$  is the position of the freezing front. In the problem formulation, the heat equation for the solid temperature  $T_S(r, t)$  in the frozen region  $0 \leq x \leq S(t)$ ,  $t > 0$  is similar to Eq. (12.2.26) with  $k$  replaced by  $k_S$  and heat equation for liquid temperature  $T_L(r, t)$  in the moisture region is also similar to Eq. (12.2.26) with  $k$  replaced by  $k_L$ .

The equation connecting the moisture concentration  $\bar{c}_m(r, t)$  and  $T_L(r, t)$  is given below.

$$\frac{\partial \bar{c}_m}{\partial t} = k_{mL} \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r \partial \bar{c}_m}{\partial r} \right) - k_{mL} \lambda \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r \partial T_L}{\partial r} \right), \quad S(t) \leq r < \infty, t > 0, \quad (12.2.469)$$

the subscript  $m$  in Eq. (12.2.469) stands for the moisture,  $\lambda$  is the thermal gradient coefficient and  $k_{mL} = k_m/k_L$ . The boundary conditions considered at  $x = S(t)$  are

$$K_S/K_L \frac{\partial T_S}{\partial r} - \frac{\partial T_L}{\partial r} = (1 - \bar{c}_m(S(t), t)) \frac{dS}{dt}; \quad T_S = T_L = T_m, \quad (12.2.470)$$

$$\frac{\partial \bar{c}_m}{\partial r} - \lambda \frac{\partial T_L}{\partial r} = 0, \quad \text{on } x = S(t); \quad \lambda \text{ connects the two gradients.} \quad (12.2.471)$$

The  $T_L$  and  $\bar{c}_m$  are coupled through Eq. (12.2.469). However, in this particular case, the two equations can be easily uncoupled by using the following procedure. Let

$$\theta(r, t) = T_L(r, t) + \gamma \bar{c}_m(r, t), \quad \gamma = (1 - k_{mL}) / (k_{mL} \lambda). \quad (12.2.472)$$

Now  $\theta(r, t)$  satisfies the equation

$$\frac{\partial \theta}{\partial t} = k_{mL} \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r \partial \theta}{\partial r} \right), \quad S(t) \leq r < \infty, t > 0. \quad (12.2.473)$$

The boundary and initial conditions for  $\theta$  can be derived using the conditions prescribed for  $T_L$  and  $\bar{c}_m$  or new diffusion equations for  $T_L$  and  $\bar{c}_m$  can be derived. We have not used dimensionless equations here which are given in [421] as our main interest is in the procedure for obtaining solution which can be explained without scaling. The complete solution can be obtained following the procedure explained earlier in the context of Eq. (12.2.26), and presented in Eqs (12.2.65), (12.2.66) and so it is not repeated here. For details of calculation see [421].

A problem somewhat similar to the above problem concerning vacuum sublimation of a frozen semiinfinite porous medium  $0 \leq x < \infty$  has been considered in [422]. Vacuum sublimation process is carried out for many heat sensitive products during the drying process. The vapour movement in porous media is due to the concentration and/or pressure gradients. We skip the details of assumptions made in [422] and also scalings of equations and discuss here directly the formulation of the problem. If  $T_S(x, t)$  is the temperature of the frozen moisture then  $T_S$  satisfies a linear parabolic equation such as Eq. (1.3.1) in the region  $S(t) \leq x < \infty, t > 0$ . The temperature in the drying material denoted by  $T_d(x, t)$  satisfies



an equation of the type Eq. (1.3.1) in the region  $0 < x < S(t)$ ,  $t > 0$ . If  $\bar{c}_V(x, t)$  is the concentration of the vapour then it is related to pressure, and concentration and pressure satisfy the equations

$$\frac{\partial \bar{c}_V}{\partial t} = k_m^* \frac{\partial^2 \bar{c}_V}{\partial x^2} + \beta \frac{\partial^2 P}{\partial x^2}; \quad \frac{\partial P}{\partial t} = k_p^* \frac{\partial^2 P}{\partial x^2}; \quad 0 < x < S(t), t > 0, \quad (12.2.474)$$

where  $k_m^*$  is the effective vapour diffusivity,  $\beta$  is permeability and  $k_p^*$  is the filtration motion diffusion coefficient of vapour.  $P$  is the pressure and  $x = S(t)$  is the position of the freezing front. The temperature, concentration and pressure at  $x = 0$  are constant and known. The three dependent variables which are prescribed at  $x = S(t)$  are unknowns but constant. The coupling between vapour concentration and pressure in Eq. (12.2.474) and the moisture balance at  $x = S(t)$  given below are the two things which are different in this formulation. Boundary and initial conditions together with energy balance condition at  $x = S(t)$  are of routine type and they are not being discussed here. Moisture and pressure balance at  $x = S(t)$  gives rise to the condition

$$\bar{k}_m^* \left. \frac{\partial \bar{c}_V}{\partial x} \right|_{x=S(t)} + \beta \left. \frac{\partial P}{\partial x} \right|_{x=S(t)} = \left( \bar{c}_{v0}^* - \bar{c}_V^*|_{x=S(t)} \right) \frac{dS}{dt}, \quad (12.2.475)$$

$\bar{c}_{v0}^*$  is the concentration of frozen bounded substance. Concentration and pressure can be easily uncoupled as suggested in Eq. (12.2.472) and for obtaining similarity solution the procedure adopted in Neumann solution can be followed. In the formulation some new parameters appear as it is a sublimation problem, refer [422] for details. As far as method is concerned, the parameters do not change the solution procedure. It is a simple matter to obtain similarity solution which is not being discussed here.

### 12.2.3 Explicit Solutions of $n$ -Phase Stefan Problems With $n > 3$

Some of the earliest references of explicit solutions of one-dimensional multiphase problems are available in [24, 423]. The basis of these solutions is the similarity solution procedure used in Neumann problem.

For  $n = 1, 2$ , and 3 phases, some explicit solutions have been discussed earlier in this chapter and some analysis aspects of such problems have been discussed in Section 3.2.3. We discuss below a Stefan problem in which  $n$ -phases exist in the region  $0 \leq x < \infty$  which are separated by the  $(n - 1)$  free boundaries  $S_{n-1}(t) < S_{n-2}(t) < S_{n-3}(t) < \dots < S_1(t)$ . The problem formulation reported in [424] is given below. There is not much new in the formulation except presenting various equations in  $n$ -different phases and to obtain recurrence relations for the solution some sequential numbering of solutions in different phases is to be done. Let

$$T(x, t) = T_i(x, t) \text{ if } S_i(t) < x < S_{i-1}(t), t > 0, \quad i = 1, \dots, n, \quad (12.2.476)$$

$$\frac{\partial T_i}{\partial t} = k_i \frac{\partial^2 T_i}{\partial x^2}, \quad S_i(t) < x < S_{i-1}(t), t > 0, \quad i = 1, \dots, n; \quad T_1(x, 0) = u_1, \quad x > 0. \quad (12.2.477)$$

$$\left. \begin{aligned} T_1(S_0(t)) &= u_1, \quad t > 0; \quad T_i(S_i^+(t), t) = T_{i+1}(S_i^-(t), t) = u_{i+1}, \\ i &= 1, \dots, n-1, \quad T_1(x, 0) = u_1, \quad S_0(t) = \infty. \end{aligned} \right\} \quad (12.2.478)$$

$$K_n \frac{\partial T_n}{\partial x}(0, t) = -q_0/\sqrt{t}, \quad q_0 > 0; \quad S_i(0) = 0, \quad i = 1, \dots, n-1, \quad S_n(t) = 0 \text{ for } t \geq 0. \quad (12.2.479)$$

$$K_i \frac{\partial T_i}{\partial x}(S_i^+(t)) - K_{i+1} \frac{\partial T_i}{\partial x}(S_i^-(t)) = l_i \frac{dS_i}{dt}; \quad u_{i+1} > u_i, \quad i = 1, \dots, n-1. \quad (12.2.480)$$

$l_i$  is latent heats evolved at  $S_i(t)$ . Densities of all the phases are taken the same. The solutions of  $T_i(x, t)$  are taken in the form  $T_i = A_i + B_i \operatorname{erf}(x/2\sqrt{k_i t})$  and  $S_i(t) = 2\delta_i \sqrt{t}$ ,  $\delta_i > 0$  in which  $A_i$ ,  $B_i$  and  $\delta_i$  which are unknowns can be easily obtained with the help of boundary conditions. Care should be taken to arrange these constants in a sequence to obtain recurrence relations. For further details see [424]. If this is done properly then necessary and sufficient conditions for the existence of the solution can be obtained in which  $q_0$  should satisfy the conditions  $q_0 > F_{n-1} > F_{n-2} > \dots > F_1$ .  $F_{n-1}, F_{n-2}$ , etc., are some functions involving various quantities. These functions cannot be presented here as their reporting requires lot of space and requires the complete solution of the problem.

If temperature  $u^*$  is prescribed at  $x = 0$  for  $T_n(x, t)$  instead of flux  $q_0$  then we write  $T_n(0, t) = u_{n+1}$ , where  $u_{n+1} > u_n$  and  $T(x, t) = T_i(x, t)$  if  $S_i(t) < x < S_{i-1}(t)$ ,  $i = 1, \dots, n$ ,  $t > 0$ . Let us now reconsider the earlier flux prescribed condition in Eq. (12.2.479).

For the temperature  $T_n(x, t)$ , the general solution is given by  $A_n + B_n \operatorname{erf}\left(\frac{x}{\sqrt{2k_n t}}\right)$  and suppose we have obtained  $A_n$  from recurrence relations (note that  $B_n = 0$ ) then  $A_n = u_n + P_n q_0$ , where  $P_n$  is some function involving some known quantities. When temperature is prescribed as  $T_n|_{x=0} = u_{n+1}$  then  $A_n = u_{n+1} = T_n(0, t)$ . So  $q_0$  which gives  $u_{n+1} = u_n + P_n q_0$  can be determined. Now the necessary and sufficient conditions for existence of solution which were obtained for the flux prescribed case can be used for the necessary and sufficient conditions in temperature prescribed case also with some minor modification. This proof is interesting as it saves lot of extra work. For obtaining solution of prescribed temperature case we have to proceed independently of the flux prescribed case. However, the necessary and sufficient conditions for the existence of the solution in temperature prescribed case can be obtained easily if they are known in flux prescribed case.

In [425] also, the one-dimensional region is  $0 \leq x < \infty$  and it is divided into  $n$ -phases with the help of  $(n-1)$  distinct phase-change boundaries  $x = S_i(t)$ ,  $i = 1, 2, \dots, (n-1)$ . In this problem  $S_0(t)$  is  $x = 0$  and the dependent variable  $T(x, t)$  at  $x = 0$  is prescribed as  $u_0$ . Let

$$T(x, t) = T_i(x, t), \quad S_{i-1}(t) < x < S_i(t) \text{ for } i = 1, \dots, n, \quad n \geq 2, \quad S_0 = 0, \quad (12.2.481)$$

$$\frac{\partial T_i}{\partial t} = k_i \frac{\partial^2 T_i}{\partial x^2}, \quad t > 0, \quad S_{i-1}(t) < x < S_i(t), \quad i = 1, \dots, n, \quad (12.2.482)$$

$$\left. \begin{aligned} T_i(S_i^-(t)) &= u_i + \Delta u_i, \quad i = 1, \dots, n; \quad T_i(S_i^+(t)) = T_i, \\ T(0, t) &= u_n, \quad x > 0; \quad T(t, x)|_{x \rightarrow \infty} \rightarrow u_n, \quad t > 0. \end{aligned} \right\} \quad (12.2.483)$$

$$l_i \frac{dS_i}{dt} = K_i \frac{\partial T_i}{\partial x}(S_i^-(t)) - K_{i+1} \frac{\partial T_{i+1}}{\partial x}(S_i^+(t)), \quad i = 1, \dots, n-1, \quad t > 0, \quad (12.2.484)$$

$$\left. \begin{aligned} S_n(t) &= \infty, \quad S_i(t)|_{t=0} = 0, \quad i = 1, \dots, n-1; \quad k_i > 0, \quad K_i > 0, \\ k_n \text{ and } K_n &> 0, \text{ or } k_n = 0 = K_n. \end{aligned} \right\} \quad (12.2.485)$$

Either  $u_{i-1} > u_i + \Delta u_i$  and  $l_i \leq 0$  for  $i = 1, \dots, n-1$  or  $u_{i-1} < u_i + \Delta u_i$  and  $l_i \geq 0$  for  $i = 1, 2, \dots, n-1$ . If  $k_n = K_n = 0$  then  $u_{n-1} = u_n$ . If  $u_{n-1} = u_n$  then  $l_{n-1} \neq 0$  is possible.

If the dependent variable is temperature then  $\Delta u_i = 0$  for  $i = 1, \dots, n-1$ . However, there are situations such as oxygen concentration in a diffusion-controlled oxidation of alloy in which the dependent variable could be discontinuous across phase-change boundary. We have not given here all the assumptions concerning the dependent variable and its derivatives, and the parameters occurring in the formulation. It can be easily verified that the following solutions satisfy the diffusion equations in different phases and also boundary and initial conditions when dependent variable is temperature.

$$T_i(x, t) = (u_{i-1} + \alpha_i) \operatorname{erfc}\left(x/2\sqrt{k_i t}\right) / \operatorname{erfc}\left(\delta_{i-1}\sqrt{k_{i-1}/k_i}\right) - \alpha_i. \quad (12.2.486)$$

In Eq. (12.2.486), we have  $S_{i-1} < x < S_i$ ,  $i = 1, \dots, n-1$ ;  $S_i(t) = 2\delta_i\sqrt{k_i t}$ ,  $i = 1, \dots, n-1$ ,  $\delta_0 = k_0 = 0$ ,  $\delta_i$  and  $\alpha_i$  are unknowns to be determined.

$$\begin{aligned} T_n(x, t) &= u_n \text{ if } k_n = 0, t > 0, x > S_{n-1}(t) \\ &= u_n + (u_{n-1} - u_n) \operatorname{erfc}\left(x/2\sqrt{k_n t}\right) / \operatorname{erfc}\left(\delta_{n-1}\sqrt{k_{n-1}/k_n}\right), \text{ if } k_n > 0. \end{aligned} \quad (12.2.487)$$

The condition of discontinuity of temperatures at the moving boundary implies that

$$\left. \begin{aligned} u_{i-1} + \alpha_i &= (u_{i-1} - u_i - \Delta u_i) \operatorname{erfc}\left(\delta_{i-1}\sqrt{k_{i-1}/k_i}\right) / A, \\ A &= \operatorname{erfc}\delta_i - \operatorname{erf}\left(\delta_{i-1}\sqrt{k_{i-1}/k_i}\right), \quad i = 1, \dots, n-1. \end{aligned} \right\} \quad (12.2.488)$$

From Eq. (12.2.488),  $\alpha_i$ ,  $i = 1, \dots, n-1$  can be expressed in terms of  $\delta_i$ . The  $\delta_i$ ,  $i = 1, \dots, n-1$  can be obtained from the Stefan condition (12.2.484). For further details see [425].

Existence and uniqueness of the solution have also been established which require some conditions to be imposed on the dependent variables and parameters. The existence proof requires proving that  $\delta_i > 0$  for  $i = 1, \dots, n-1$ . Note that  $\delta_i$  requires the knowledge of  $\delta_{i-1}$  and  $\delta_{i-2}$  for  $i = 1, \dots, n-1$ . To establish sufficient conditions for existence and uniqueness of the solution a sequence of functions  $\{f_i(x)\}$ ,  $i = 1, \dots, n$  is defined such that the desired coefficients  $\delta_i$ ,  $i = 1, \dots, n-1$ , satisfy  $f_i(x^*) = \delta_i$  for  $i = 1, \dots, n-1$  when  $f_n(x^*) = f_{n-2}(x^*)$ . Existence follows from  $f_n - f_{n-2}$  being initially negative and eventually positive in an interval where the functions are continuous. Uniqueness follows from  $f'_n - f'_{n-2}$  being positive when  $f_n = f_{n-2}$ . The appropriateness of assumptions made has been discussed.

A fairly difficult multiphase problem has been considered in [426]. Consider a solidifying material say material 1 which occupies the region  $x \geq 0$  abuts another material 2 occupying the region  $x \leq 0$ . At  $x = 0$ , continuity of temperatures and fluxes is maintained. Both the materials are originally homogeneous and maintained at uniform temperature. The material 1 is initially at temperature  $T_m$  and at temperature  $T_{-n}$  is the material 2,  $T_m > T_{-n}$ . Each material may exist in number of phases. Let  $m$  be the number of phases in which material 1 may exist and similarly  $n$  be the number of phases for material 2. Each phase may have distinct thermal properties which could be temperature dependent. At the commencement of solidification, for a time interval  $0 < t < t_f$ , the temperature  $T_0$  at  $x = 0$  will be within the temperature range for one phase of material 1 and for one phase of material 2. Only material 2 is solidifying on the base material.

Having discussed the physics and geometry of the problem, we present very briefly the formulation aspects of the problem skipping totally the scalings of variables, parameters and also mentioning equations except the nonlinear diffusion equations for temperatures  $u_i$  which

are given below. We have narrowed down our presentation to converting the nonlinear diffusion equations to forms which are amenable to similarity solution. It is just not possible to present other related material in a simple concise manner. The reader if interested in such a problem should refer [426] for further details. Boundary and initial conditions are of the same type as considered in earlier problems but special care should be taken in understanding notations used for identifying the materials 1 and 2. For  $0 < t < t_f$ , the temperatures  $u_i$  satisfy  $T_{i-1} < u_i < T_i$ ,  $i = 1, 2, \dots, m$  and  $u_J$  satisfy  $T_J < u_J < T_{J+1}$ ,  $J = -1, \dots, -n$ .  $T_i$  and  $T_J$  are explained below. The nonlinear diffusion equations are as follows.

$$C_i(u_i) \frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x} \left[ K_i(u_i) \frac{\partial u_i}{\partial x} \right], \quad i = -n, \dots, -1, 1, \dots, m. \quad (12.2.489)$$

Let the phase-change temperatures at the moving boundaries  $x = y_i(t)$ ,  $i = -n, \dots, -1, 1, \dots, m$  be denoted by  $T_i$ , for  $i = 1, \dots, m-1$  and  $T_J$ , for  $J = -1, \dots, -n+1$ , with boundary  $y_{-n}$  taken at the negative infinity and  $y_m$  at positive infinity. To obtain a similarity solution of the problem, the main objective is to transform Eq. (12.2.489) into a linear equation. This is done in several stages. We use Kirchoff transformation given in Eq. (1.4.29) and define

$$\mu_i = \int_{v_i}^{u_i^*} K_i^*(\bar{u}) d\bar{u} = \int_{w_i}^{\theta_i} D_i(\bar{\theta}) d\bar{\theta}, \quad i = -n, \dots, -1, 1, \dots, m, \quad (12.2.490)$$

where  $\theta_i = \int_0^{u_i^*} C_i^*(\bar{u}) d\bar{u}$  is the heat density,  $D_i = K_i^*/C_i^*$  is the heat diffusivity,  $v_i$  are arbitrary constants and  $w_i = \int_0^{v_i^*} C_i^*(\bar{u}) d\bar{u}$ . The superscript and subscript \* indicate that the quantities have been scaled suitably which is not explained here as the whole process is too lengthy. Assume that in each phase the diffusivity  $D_i$  is of the following form in which  $\alpha_i$  and  $\beta_i$  are known.

$$D_i = \alpha_i (\beta_i - \theta_i)^{-2}, \quad \alpha_i > 0, \quad i = -n, \dots, -1, 1, \dots, m. \quad (12.2.491)$$

Using this  $D_i$  given in Eq. (12.2.491) in Eq. (12.2.490), we get

$$\left. \begin{aligned} \mu_i &= \alpha_i / (\beta_i - \theta_i) - \alpha_i / (\beta_i - w_i) \text{ and } D_i = a_i^2 (b_i - \mu_i)^2, \\ a_i &= 1/\sqrt{\alpha_i}, \quad b_i = -\alpha_i / (\beta_i - w_i). \end{aligned} \right\} \quad (12.2.492)$$

Eq. (12.2.489) when expressed in terms of  $\mu_i$ , dimensionless time  $t_*$  and dimensionless space coordinates  $x_*$  becomes

$$\frac{\partial \mu_i}{\partial t_*} = a_i^2 (b_i - \mu_i)^2 \frac{\partial^2 \mu_i}{\partial x_*^2}, \quad y_{i-\text{Sgn}(i)}^* < |x_*| < |y_i^*|, \quad i = -n, \dots, -1, 1, \dots, m. \quad (12.2.493)$$

The previous equations are not linear and a modification using Storm transformation [385] is used to linearize them. Let

$$\chi_i = \int_{y_{i-\text{Sgn}(i)}^*}^{x_*} a_i^{-1} (b_i - \mu_i)^{-1} dx_*, \quad i = -n, \dots, -1, 1, \dots, m. \quad (12.2.494)$$

On using  $\chi_i$  in Eq. (12.2.493), linear diffusion equations are obtained with convective terms. The coefficients of convective terms are some unknown quantities and unless it is established

that they are constant, which requires their functional form, similarity solution cannot be obtained.

It has been shown in [427] that the whole FBP with nonlinear diffusion equations and boundary and initial conditions discussed earlier very briefly here is invariant under the following scaling group transformation.

$$\mu_i^* = \mu_i, x_1^* = \exp(\epsilon) x_*, t_1^* = \exp(2\epsilon) t_*, \quad (12.2.495)$$

where  $\epsilon$  is the real-valued *Lie group parameter*. The invariants of the group are functions of  $\mu_i$  and  $x_* t_*^{-1/2}$ . Therefore, self-similar solutions of the form

$$\mu_i = h_i(\xi), \quad \xi = x_* t_*^{-1/2}; \quad i = -n, \dots, -1, 1, \dots, m, \quad (12.2.496)$$

can be used.  $\xi = \text{constant}$  are planes of constant temperature and the interfaces between phases move as

$$y_i^*(t_*) = \delta_i t_*^{1/2}, \quad i = -n + 1, \dots, -1, 1, \dots, m - 1. \quad (12.2.497)$$

The invariance also implies that the coefficients of convective terms on using Eq. (12.2.494) in Eq. (12.2.493) are constant. Finally, the nonlinear diffusion equations take the following form.

$$\left. \begin{aligned} \frac{\partial \mu_i}{\partial \tau} &= \frac{\partial^2 \mu_i}{\partial \chi_i^2} + \lambda_i \tau^{-1/2} \frac{\partial \mu_i}{\partial \chi_i}, \quad \lambda_i = a_i \gamma_i + (2a_i b_i)^{-1} \delta_{i-\text{Sgn}(i)}, \quad \tau = t_*, \\ \gamma_i &= \left. \frac{d\mu_i}{d\xi} \right|_{\xi=\delta_{i-\text{Sgn}(i)}}; \quad i = -n, \dots, -1, 1, \dots, m. \end{aligned} \right\} \quad (12.2.498)$$

All the initial conditions and boundary condition are to be obtained in terms of the transformed variables. Similarity solution has been obtained in [426].

It is very easy to say that use this transformation and then this transformation but in practice it requires lot of hard work and experience to implement it. For further information about the above work and other works of this type refer [426]. In between details of obtaining solution are too lengthy to be presented here.

### 12.2.4 Analytical Solutions Obtained for Some Particular Cases of Stefan and Stefan-Like Problems, Quasisteady-State and Steady-State Solutions

In this section, the restriction that a analytical solution has to be exact has been relaxed. A particular solution has to be in closed form but not in terms of a series or other forms. The solutions presented now pertain to variety of cases such as quasisteady-state and steady-state solutions, solutions in which some condition and/or some equation is satisfied only approximately, solutions obtained only for some particular values of parameters. Both Stefan and Stefan-like problems have been considered here. We found the presentation of solutions of the particular cases somewhat difficult because if the problem related to any particular case has not been discussed in earlier sections then the presentation if done with clarity becomes fairly lengthy. The physical and mathematical background of the problem, the description of parameters and their scalings, etc., have to be done in a concise way to save space but with

clarity and understandability. This takes lot of effort. Further the authors of the articles whilst discussing the solutions refer to lot of previous works and give only incomplete information about the solutions. Because of all these difficulties, comprehensive treatment of the material presented here could be less satisfactory.

There are many physical problems of interest in which the effect of internal radiative heat transfer in melting/solidification of nonopaque or semitransparent materials cannot be neglected and it should be included in the formulation of problems with or without conduction and convection. Examples in which radiative heat transfer plays significant role are problems of low-temperature range such as solar heating and problems of high-temperature range such as problems of crystal growth, melting or solidification problems of uranium fuels, etc. In [428], solidification/melting problem with radiative internal heat transfer has been considered in a one-dimensional semiinfinite medium,  $0 \leq x < \infty$ . The material could be nonopaque or semitransparent. Solutions in three different cases of radiative heat transfer have been discussed in [428] by considering formulations of solid, mush and liquid regions. Mathematical formulations have been derived at the solidus and liquidus boundaries by taking into account temperature derivative and flux discontinuities. In [428], emphasis is on formation and formulation of mushy region, which for nonopaque or semitransparent liquid in solidification develops, and which also develops in the melting of solids in radiative heat transfer. Most of the studies deal with two-phase sharp-interface models in radiative heat transfer which seems more reasonable when one of the regions has very high grey absorption rate or the material is opaque grey material. For an opaque grey material, the radiative flux  $q_r$  is given by  $q_r = -\frac{4n^2\sigma}{3G_a} \frac{\partial T^4}{\partial x}$ , where  $G_a$  is the grey absorption coefficient,  $\sigma$  is *Stefan-Boltzman constant* and  $n$  is the index of refraction or mass flux. As  $G_a \rightarrow \infty$ ,  $q_r = 0$ . Three particular solutions have been discussed below.

**Problem 12.2.1.** Solidification of a liquid at melting point: radiation-controlled solidification.

The heat energy equation in the mushy region in the presence of conduction, convection and radiation together with latent heat generation as internal source term is given by

$$\rho C \frac{DT}{Dt} - \nabla \cdot (K \nabla T) + \nabla \cdot \vec{q}_r = lR, \quad 0 \leq x < \infty, \quad (12.2.499)$$

where  $K$  is thermal conductivity,  $l$  is latent heat per unit mass,  $R$  is mass solidification rate per unit volume of the mush and the subscript  $r$  indicates radiative heat transfer. If conduction and convection are not present and only internal radiative heat transfer is considered then in a one-dimensional case for the solid region, we have

$$\frac{\partial q_r}{\partial x} = lR; \quad \frac{\partial \rho_S}{\partial t} = -\frac{\partial}{\partial x}(\rho_S u) + R, \quad (12.2.500)$$

where  $u$  is the velocity of the solid. If  $u = 0$  and  $\alpha = \rho_S / \rho_S^*$ , where  $\rho_S^*$  is the true solid density and  $\rho_S$  is the superficial mass density then  $\alpha$  becomes the solidified volume fraction.

It is easy to obtain the following relations.

$$\frac{\partial \alpha}{\partial t} = R / \rho_S^*, \quad \text{or} \quad \frac{\partial \alpha}{\partial t} = 1 / (l \rho_S^*) \frac{\partial q_r}{\partial x}, \quad t > t_0, \alpha = 0 \text{ at } t = t_0, \quad (12.2.501)$$

$t_0$  is the time when the liquid temperature becomes freezing temperature at  $x = 0$ . For the two-phase mushy region some quantities of interest given in [428] are as follows and for their derivations see [429, 430]. The superscript \* is for a dimensionless quantity.

$$\frac{\partial q_r}{\partial x} = 2G_a n^2 \sigma (T_m^4 - T_w^4) E_2(\tau), \quad T_w \text{ is the wall temperature,} \quad (12.2.502)$$

$\tau$  is optical length taken as  $G_a x$ ,  $T_m$  is melting temperature,  $E_n$  is the exponential integral function of  $n$ th kind. Using Eq. (12.2.502) heat transfer  $q_r$  at the wall  $x = 0$  can be easily calculated and  $\alpha$  is obtained as

$$\alpha(x, t) = E_2(\tau) t^*, \quad t^* = 2G_a n^2 \sigma (T_m^4 - T_w^4) t / (\rho_S^* l), \quad t > t_0, \quad \alpha = 0, \quad \text{at } t = t_0. \quad (12.2.503)$$

Once  $\partial \alpha / \partial t$  is known from Eq. (12.2.501) the local solidification rate  $R$  can be easily calculated. Total solidification rate throughout the medium denoted by  $R_{\text{total}}$  is given by

$$R_{\text{total}} = \int_0^\infty R dx = n^2 \sigma (T_m^4 - T_w^4) / l. \quad (12.2.504)$$

Pure solidified thickness when all the solid fractions are packed together in the mush at the time  $t$  can be found from total solidification rate and the equivalent solidified thickness denoted by  $S_{eq}$  is obtained as follows.

$$S_{eq} = \frac{1}{\rho_S^*} \int_0^t R_{\text{total}} dt = \frac{n^2 \sigma}{l \rho_S^*} (T_m^4 - T_w^4) t, \quad t > t_0. \quad (12.2.505)$$

In the Neumann solution, the solidified thickness is proportional to  $t^{1/2}$  but in Eq. (12.2.505) it is proportional to  $t$ . From Eq. (12.2.503) it can be checked that  $\alpha \rightarrow 0$  when  $x \rightarrow \infty$ . If  $G_a$  is small, that is the medium is less opaque then it can be shown that the solidified thickness in the mush is a small fraction of the total volume of the mush. Therefore, if the medium is less opaque then mushy region is comparatively larger.

**Problem 12.2.2.** Solidification of a liquid above melting point: radiation-controlled solidification.

In this case of radiation-controlled solidification, the heat energy equation in the liquid is

$$\rho_L^* C_L \frac{\partial T}{\partial t} = -\frac{\partial q_r}{\partial x}, \quad t > 0, \quad 0 \leq x < \infty; \quad T = T_\infty \text{ at } t = 0; \quad T|_{S_2(t)} = T_m, \quad (12.2.506)$$

$x = S_2(t)$  is the liquidus. The expression of  $\frac{\partial q_r}{\partial x}$  in this case in terms of temperature is complicated in which a nonlinear integral term is involved and using which it is difficult to obtain temperature and other quantities of interest. If it is assumed that  $T^4 \gg T_w^4$ , where  $T_w$  is the wall temperature, then an approximate expression for  $\frac{\partial q_r}{\partial x}$  can be obtained and

$$\frac{\partial q_r}{\partial x} \simeq 2n^2 \sigma T^4(\tau) E_2(\tau), \quad \tau = G_a x. \quad (12.2.507)$$

Using Eq. (12.2.507) in Eq. (12.2.506), the solution for temperature can be obtained. Once the temperature is known other quantities of interest such as the time  $t = t_0$  when the mushy

region emerges,  $\alpha(\tau, t)$  which is the solidified volume fraction,  $q_{rw}(t)$  the radiative heat flux at the wall, etc., can be calculated. For further discussion see [428].

**Problem 12.2.3.** Solidification of a liquid at melting point: combined conduction and radiation.

By using an approximate method, viz. integral heat balance method, a analytical-numerical solution has been obtained.

In a one-dimensional finite slab  $0 \leq x < L < \infty$ , solidification of a liquid at temperature higher than the freezing temperature has been considered in [431]. Approximate solution has been obtained for the convective-type boundary condition at  $x = 0$  and the boundary  $x = L$  is insulated as in Eq. (12.2.508).

$$h_t(T_w - T_\infty) = K_S \frac{\partial T_S}{\partial x}, \quad x = 0, \quad t > 0; \quad \frac{\partial T_L}{\partial x} \Big|_{x=L} = 0; \quad T_L|_{t=0} = T_o > T_m. \quad (12.2.508)$$

In Eq. (12.2.508),  $T_w$  is the wall temperature and  $T_\infty$  is the ambience temperature,  $T_\infty \leq T_w < T_m$ , and  $h_t$  is the heat transfer coefficient. Heat equations (not given here) are the same as in Neumann solution (Eqs 1.3.1–1.3.6) with equal densities in the two phases. The exact analytical solution of this problem does not exist but approximate solutions have been obtained in various forms by different methods (refer [431]). Since the solution procedure in [431] involves a new idea of continuity of heat extracted at  $x = 0$  and it has been used for the approximate calculation of moving boundary, it will be discussed here. In this solution procedure temperatures obtained are discontinuous which should be continuous and they are not obtained. Without describing the scalings of parameters the procedure for solution cannot be explained so it is given below.

$$St_e = C_S(T_m - T_\infty)/l, \quad l > 0, \quad \eta = C_L(T_o - T_m)/(C_S(T_m - T_\infty)), \\ F_O = k_S t/L^2, \quad Bi = h_t L/K_S. \quad (12.2.509)$$

In Eq. (12.2.509),  $St_e$  is Stefan number,  $\eta$  is the ratio of sensible heats of liquid and solid,  $F_O$  is Fourier number,  $Bi$  is Biot number,  $k$  is diffusivity,  $K$  is conductivity and  $C$  is specific heat. The solution for convective boundary condition is obtained later as a suitable interpolation of two solutions which are obtained for  $Bi \rightarrow 0$  and  $Bi \rightarrow \infty$ . We consider the case  $Bi \rightarrow \infty$  first. In this case constant temperature is prescribed at  $x = 0$  as  $T_w = T_\infty$ . In this case Neumann solution which is valid for  $0 \leq x < \infty$  is used for the time, say  $F_O = F_{O1}$ , till sensible heat equivalent to the total sensible heat present in the finite slab  $0 \leq x \leq L$  at time  $t = 0$  is extracted from the region  $0 \leq x < \infty$  by cooling done at the boundary  $x = 0$  with  $T = T_w$ . This is only an approximation as the sensible heat of the remaining semiinfinite region in Neumann solution is also removed partly during the time  $F_{O1}$ . In this case solidification starts at  $t = 0$  itself and latent heat is also removed partly from the region  $0 \leq x/L \leq 1$ . The time  $F_{O1}$  can be easily obtained from the energy balance condition, that is heat extracted at  $x = 0$  is equal to heat given out by the system.

According to the notations followed in the present book the freezing front is denoted here by  $S(t)$  which is taken as

$$x/L = S(t)/L = 2\delta\sqrt{ik_S} = \xi/L = 2a_0(F_O)^{1/2}. \quad (12.2.510)$$



$\xi$ ,  $a_0$  and  $F_O$  have been used to match the notations used in [431],  $a_0$  comes in place of  $k$  as we have used  $k$  for diffusivity. From the Neumann solution, the extracted relative heat energy  $Q_r$  can be calculated at  $x = 0$  and

$$Q_r = \frac{2\sqrt{F_O}}{\sqrt{\pi}(1 + \eta + 1/St_e)\text{erf}(a_0)}, \quad (12.2.511)$$

which is exact for the Neumann problem but for the problem under consideration equation (12.2.511) can be used only as extracted heat energy approximation till  $F_O = F_{O1}$ . For  $F_O > F_{O1}$ , the authors in [431] take the liquid at freezing temperature and use the Neumann solution in Eq. (12.2.512) for  $\eta = 0$  and replace the constant  $a_0$  in Eq. (12.2.511) by  $a_1$ . This solution is continued till the moving front reaches  $x = L$ . Note that the boundary condition at  $x = L$  is not satisfied explicitly but the assumptions are made in such a way that it gets satisfied. For  $F_O > F_{O1}$ ,  $Q_r$  is given by modifying Neumann solution.

$$Q_r = \frac{2\sqrt{F_O}}{\sqrt{\pi}(1 + 1/St_e)\text{erf } a_1}, \quad F_O > F_{O1}. \quad (12.2.512)$$

$Q_r$  is discontinuous at  $F_O = F_{O1}$  and to make  $Q_r$  a continuous function of time, a time shift is made and for  $F_{O2} \geq F_{O1}$  and the time shift is  $F_{O2} - F_{O1}$ . First,  $Q_r$  in Eq. (12.2.512) is modified to account for the sensible heat present in the liquid for  $F_{O2} \geq F_{O1}$  and then Neumann solution is used to obtain  $Q_r$  in the following form.

$$Q_r(F_O \geq F_{O1}) = \frac{2}{\sqrt{\pi}} \left( \frac{(F_O - F_{O1} + F_{O2})^{1/2}}{\text{erf}(a_1)} + \eta \right) / (1 + \eta + 1/St_e). \quad (12.2.513)$$

The expression of  $Q_r$  in Eq. (12.2.513) is difficult to understand that how sensible heat has been accounted in Eq. (12.2.513).  $F_{O2}$  is determined by replacing  $F_O$  in Eq. (12.2.511) by  $F_O = F_{O1}$  and also taking  $F_O = F_{O1}$  in Eq. (12.2.513) and then equating these two  $Q_r$  so that  $Q_r$  is continuous. The freezing front is given by

$$\xi/L = 2a_1(F_O - F_{O1} + F_{O2})^{1/2}, \quad F_O \geq F_{O1}. \quad (12.2.514)$$

The temperature remains discontinuous.

### The Solution for $Bi \rightarrow 0$

In this case  $T_w$  is time dependent and the heat equation becomes

$$h_t(T_w - T_\infty) = -\rho l C_L \frac{dT_w}{dt}. \quad (12.2.515)$$

For  $T_w > T_m$ , the exact solution of Eq. (12.2.515) is given by

$$T_{wL} - T_\infty = (T_o - T_\infty) \exp(-Bi F_O C_S / C_L). \quad (12.2.516)$$

This solution is for liquid region till the time  $F_o = F_{o4}$ , where  $F_{o4}$  is the time when  $T_{wL} = T_m$  or in other words till the wall temperature  $T_{wL}$  does not become  $T_m$  or the entire region remains liquid.  $F_{o4}$  can be easily determined using Eq. (12.2.516). For  $F_o \geq F_{o4}$ , the solidification starts and temperature of the region can be considered as constant and equal to the transition

temperature. The extracted heat is the latent heat which is released during solidification. From heat balance condition, we have

$$h_t(T_m - T_\infty) = l\rho \frac{\partial \xi}{\partial t}. \quad (12.2.517)$$

The solution in Eq. (12.2.516) has not accounted for the temperature gradient in the solid. On solving Eq. (12.2.517) for  $\xi$ ,  $F_{O4}$  is obtained from the following equation.

$$BiSte(F_O - F_{O4}) = \xi/L, \quad \xi = 0 \text{ at } F_O = 0. \quad (12.2.518)$$

The relative extracted heat  $Q_r$  is given as follows:

$$Q_r = \eta(T_O - T_{wL})(T_O - T_L)/(1 + \eta + 1/Ste), \quad 0 \leq F_O \leq F_{O4}. \quad (12.2.519)$$

$$\left. \begin{aligned} Q_r &= \eta + (1/Ste)(F_O - F_{O4})/(F_{O5} - F_{O4})(1 + \eta + 1/Ste), \\ F_{O4} &\leq F_O \leq F_{O5}, \end{aligned} \right\} \quad (12.2.520)$$

where  $F_{O5}$  is the total solidification time. To calculate  $Q_r$  and  $F_O$  for any  $Bi$ , the authors construct a separate solution when  $C_S$  and  $C_L$  are negligible. In this case  $\partial T/\partial t = 0$  and temperature in the solid will be a linear function of  $x$ . Two equations are obtained on doing the heat balance at the wall  $x = 0$  and at the freezing front, which are

$$h_t(T_w - T_\infty) = K_S(T_m - T_w)/\xi, \quad \text{at } x = 0; \quad K_S(T_m - T_w)/\xi = l\rho \frac{d\xi}{dt}. \quad (12.2.521)$$

On eliminating  $T_w$  from both equations in Eq. (12.2.521), we get

$$BiSte dF_O = (1 + Bi\xi/L)d\xi/L. \quad (12.2.522)$$

On integrating Eq. (12.2.522), we get  $F_O$  and also the interpolation formula as follows.

$$F_O = Q_r/BiSte + Q_r^2/2Ste \quad \text{or} \quad F_O = (BiF_O)_{Bi \rightarrow 0}/Bi + (F_O)_{Bi \rightarrow \infty}. \quad (12.2.523)$$

For other details see [431].

A simplified mathematical model of solidifying melt in continuous casting is proposed and then exact solution of the mathematical model has been obtained in [432]. In a fairly simple sketch of solidification process in continuous casting given in Fig. 7.4.1 we further assume that the moving boundary is parabolic and the solidified ingot in being pulled with a uniform velocity  $V$  in the  $y$ -direction which is taken vertically upwards.  $x$ -coordinate is taken horizontally extending from 0 to  $\infty$ . Let  $(x', y')$  be the original coordinates and  $\bar{T}(x', y')$  be the temperature of the solid formed when liquid at melting temperature is poured in the mold. The formulation of this one-phase two-dimensional steady-state problem is considered in [432] as follows.

$$\bar{T}_{x'x'} + \bar{T}_{y'y'} = (V\rho C_S/K_S)\bar{T}_{y'}, \quad x' > 0, \quad y' > F(x'), \quad (12.2.524)$$

$y' = F(x')$  is the equation of solid-liquid phase-change boundary.

$$\bar{T} = T_1, \quad \text{at } x' = 0, \quad y' > 0; \quad \bar{T}_{x'} = 0, \quad F(0) < y' < 0; \quad \bar{T} = T_0 \quad \text{at } y' = F(x'). \quad (12.2.525)$$

$$\bar{T}_{y'}|_F = \frac{-l_1 \rho V}{K_S \left(1 + \left(\frac{dF}{dx'}\right)^2\right)}, \quad l_1 = l/(C_S(T_0 - T_1)), \quad l \text{ is latent heat.} \quad (12.2.526)$$

We introduce the following transformations.

$$\left. \begin{aligned} T &= (\bar{T} - T_1)/(T_0 - T_1), \quad x' = xK/\rho VC, \quad y' = yk/\rho VC_S \\ y &= f(x) \text{ is the equation of the phase-change boundary.} \end{aligned} \right\} \quad (12.2.527)$$

If  $u$  and  $v$  are parabolic coordinates given by  $x = 2uv$ ,  $y = u^2 - v^2$  and  $v = \phi(u)$  is the equation of the interface then  $T(u, v)$  and  $v = \phi(u)$  can be determined by solving the following system of equations.

$$T_{uuu} - 2uT_{uu} + T_{vv} + 2vT_v = 0; \quad u > 0, \quad 0 < v < \phi(u). \quad (12.2.528)$$

$$T = 0, \quad \text{at } v = 0, \quad u > 0; \quad T_u = 0, \quad u = 0, \quad 0 < v < \phi(0); \quad T = 1, \quad \text{for } v = \phi(u). \quad (12.2.529)$$

$$T_v = 2l_1(\phi + u\phi')/(1 + \phi^2), \quad (12.2.530)$$

when  $v = \phi(u)$ ,  $l_1$  as in Eq. (12.2.526). If in Eqs (12.2.528)–(12.2.530), it is assumed that  $T$  is independent of  $u$  and  $v = \phi(u) = \text{constant} = v_0$  then a similarity solution can be easily obtained which is given below.

$$T = \text{erf}v/\text{erf}(v_0); \quad l_1\sqrt{\pi}v_0 \text{erf}v_0 = e^{-v_0^2} \text{ on } v = v_0 \text{ or } y = x^2/4v_0^2 - v_0^2, \quad (12.2.531)$$

$v_0$  is given by the second equation in Eq. (12.2.531).

By using enthalpy formulation, a one-dimensional solidification/melting problem in a semiinfinite region  $0 \leq x < \infty$  has been considered in [433] with solid, liquid and mushy region formulations. The total enthalpy  $H(T)$  of the system is taken in the form

$$H(T) = (1 - \lambda(T))\rho_S C_S T + \lambda(T)\rho_L(C_L T + l), \quad (12.2.532)$$

$\lambda(T)$  is the temperature-dependent liquid fraction. The other notations are the same as used in this book. Due to  $\rho_S \neq \rho_L$ , liquid may have velocity  $v(t)$ . Solid is at rest. The energy equation is taken in the form which exists only in distributional sense.

$$\frac{\partial H}{\partial t} + v(t) \frac{\partial H}{\partial x} = \frac{\partial}{\partial x} \left( k(H) \frac{\partial H}{\partial x} \right), \quad 0 \leq x < \infty, \quad t > 0, \quad k \text{ is diffusivity.} \quad (12.2.533)$$

The motion of the liquid  $v(t)$  can be determined with the help of mass balance in the region  $0 \leq x \leq \xi(t)$ , where  $x = \xi(t)$  is a point in the liquid region away from liquidus  $x = S_2(t)$ . The total mass  $M_\xi$  per unit area of the transverse section area inside the interval  $0 \leq x \leq \xi(t)$  is given by

$$M_\xi(t) = \rho_S S_1(t) + \bar{\rho}_{SL}(S_2(t) - S_1(t)) + \rho_L(\xi(t) - S_2(t)), \quad (12.2.534)$$

where  $x = S_1(t)$  is the solidus boundary and  $\bar{\rho}_{SL}$  is the average density of the mush.

$$\bar{\rho}_{SL} = \rho_S + (\rho_L - \rho_S)\bar{\lambda} = \rho_S + \frac{(\rho_L - \rho_S)}{(T_L - T_S)} \int_{T_S}^{T_L} \lambda(T) dT, \quad (12.2.535)$$

$\bar{\lambda}$  is averaged liquid fraction. If  $v(t) = \frac{d\xi}{dt}$  is the velocity of the liquid and mass is conserved (i.e.  $\frac{dM\xi}{dt} = 0$ ), we get

$$v(t) = (1 - \rho_S/\rho_L) \left[ \bar{\lambda} \frac{dS_1(t)}{dt} - (1 - \bar{\lambda}) \frac{dS_2(t)}{dt} \right]. \quad (12.2.536)$$

Thermal conductivity  $K(T)$  in the mush is taken in the form

$$K(T) = (1 - \lambda(T))K_S + \lambda(T)K_L. \quad (12.2.537)$$

To make Eq. (12.2.533) in the mush linear, the following assumption is made for the thermal diffusivity in the mush.

$$k = K \frac{dH}{dT} = k_{SL} = \text{constant}, \quad K_S \text{ and } K_L \text{ are constant.} \quad (12.2.538)$$

To obtain  $\lambda(T)$  which is the liquid fraction in the mush, Eqs (12.2.532), (12.2.537), (12.2.538) are used and the following differential equation is obtained.

$$\left. \begin{aligned} (1 + pT) \frac{d\lambda(T)}{dT} + a\lambda(T) + b &= 0, \quad a = \{q(C_L - C_S/\mu) - (K_L - K_S)\}/(ql), \\ b &= (qC_S/\mu - K_S)/(ql), \quad p = (C_L - C_S/\mu)/l, \quad \mu = \rho_L/\rho_S, \quad q = k_{SL}\rho_L. \end{aligned} \right\} \quad (12.2.539)$$

At the liquidus  $x = S_2(t)$ ,  $\lambda(T) = 1$ . At  $x = S_1(t)$ ,  $\lambda(T) = 0$ , if  $T$  is not eutectic temperature, otherwise  $\lambda(T) = \lambda_0 \neq 0$  if  $T$  is eutectic at  $x = S_1(t)$ .  $k_{SL}$  can be determined if  $T$  is eutectic. Finally  $\lambda(T)$  and  $\bar{\lambda}(T)$  are obtained as

$$\lambda(T) = -\frac{b}{a} + \frac{a+b}{a} \left( \frac{1 + pT_{LL}}{1 + pT} \right)^{a/p}, \quad \bar{\lambda}(T) = \frac{G(T_{LL}) - G(T_{SS})}{T_{LL} - T_{SS}}, \quad (12.2.540)$$

$$G(X) = -\frac{a}{b}X - \frac{(a+b)(1+pX)}{a(a-p)} \left( \frac{1 + pT_{LL}}{1 + pX} \right)^{a/p}. \quad (12.2.541)$$

In obtaining  $\lambda(T)$  the condition for  $\lambda(T)$  at the liquidus as mentioned earlier has been used.  $T_{SS}$ ,  $T_{LL}$  are the temperatures at the solidus and liquidus, respectively. By taking  $k(H)$  as constant in each of the solid, mush and liquid regions Eq. (12.2.533) becomes linear but it has to be solved in all the three regions separately as it is valid at  $x = S_1(t)$  and  $x = S_2(t)$  in the distributional sense. If the following boundary conditions are imposed on  $H(T)$  then an exact similarity solution can be obtained on the lines of Neumann solution given in Eqs (1.3.11)–(1.3.16).

$$H(x > 0, t = 0) = H_{\text{init}} = H(T_{\text{init}}); H|_{x=0} = H_{\text{out}} = H(T_{\text{out}}), t \geq 0. \quad (12.2.542)$$

$$H|_{x \rightarrow \infty} = H_{\text{init}} = H(T_{\text{init}}). \quad (12.2.543)$$

In obtaining solutions for the mushy and liquid regions, appropriate shifting of coordinate  $x$  is required. The velocity  $v(t)$  of the liquid,  $S_1(t)$  and  $S_2(t)$  are taken as

$$v(t) = \gamma/(2\sqrt{t}), \quad \gamma = (\mu - 1)(\bar{\lambda}\delta_S + (1 - \bar{\lambda})\delta_L), \quad S_1(t) = \delta_S\sqrt{t}, \quad S_2(t) = \delta_L\sqrt{t}. \quad (12.2.544)$$

In terms of a new variable  $\xi$  which is defined as  $\xi = x^2/t$ , the enthalpy equation (12.2.533) can be written as

$$\frac{d^2 H}{d\xi^2} = -\frac{1}{4} \left\{ \frac{1}{k(H)} + \frac{2}{\xi} - \frac{\gamma}{k(H)\sqrt{\xi}} \right\} \frac{dH}{d\xi}. \quad (12.2.545)$$

Solutions of Eq. (12.2.545) on the three intervals  $(0, S_1(t))$ ,  $(S_1(t), S_2(t))$  and  $(S_2(t), \infty)$  can be obtained as obtained in Neumann solution.  $\delta_S$  and  $\delta_L$  are determined from the following conditions at solidus  $x = S_1(t)$  and liquidus  $x = S_2(t)$ , respectively.

$$k_S \left. \frac{\partial H}{\partial x} \right|_{S_1(t)-0} = k_{SL} \left. \frac{\partial H}{\partial x} \right|_{S_1+0} + \rho_S \lambda_0 l \frac{dS_1(t)}{dt}; k_{SL} \left. \frac{\partial H}{\partial x} \right|_{S_2-0} = K_L \left. \frac{\partial H}{\partial x} \right|_{S_2+0}. \quad (12.2.546)$$

For the exact solutions in the three regions, see [433].

Exact analytical solutions of Stefan problems in the presence of constant heat sources do not exist. This is because a systematic procedure for obtaining exact solutions is through similarity solution and a similarity variable in the presence of constant heat source required to obtain the solution is not available. However, in special type of heat sources such as in [361] a similarity solution was possible because a suitable similarity variable exists. A transient two-phase Stefan problem in a finite slab with constant heat sources of different intensity in both regions was considered in [144]. Initially at  $t = 0$ , a part of the slab is filled with solid or liquid and not whole of the slab is filled with one material and flux is prescribed at  $x = L$ . No exact analytical solution was obtained in [144] but integral representations of temperatures and their derivatives with the help of Green's functions were obtained. It was proved in [144] that near the moving boundary, regions of supercooling or superheating may appear in the presence of strong heat sources. Using linear stability analysis, several results concerning linear stability of free boundary were discussed in [144].

Quasisteady- and steady-state solutions of one-dimensional solidification and melting problems in the presence of constant heat source in a finite slab,  $0 \leq x \leq L$ , have been obtained in [434]. In addition to many analytical results obtained in [434] with quasisteady-state formulation for solidification, quasisteady-state solution for melting problem in the presence of heat source was also discussed. One key assumption is that in a melting problem, instead of a pure liquid region, a mushy region may appear.

The solidification problem considered in [434] in dimensionless form is as follows.

$$\frac{\partial^2 \theta_S}{\partial \xi^2} + \beta = Ste \frac{\partial \theta_S}{\partial \tau}, 0 \leq \xi \leq \xi_0, \xi = x/L, \beta = qL^2/(K(T_m - T_0)), \quad (12.2.547)$$

$T_0 < T_m$ ,  $q$  is volumetric heat generation rate,

$$T_S(x=0, t) = T_0; \theta_S = (T_S - T_0)/(T_m - T_0), \quad (12.2.548)$$

$$\frac{\partial^2 \theta_L}{\partial \xi^2} + \beta = Ste \frac{\partial \theta_L}{\partial \tau}, \xi_0 \leq \xi \leq 1; Ste = C(T_m - T_0)/l, \quad (12.2.549)$$

$$\theta_L = (T_L - T_0)/(T_m - T_0); \tau = Ste kt/L^2, \quad (12.2.550)$$

where  $l$  is latent heat,  $k$  is thermal diffusivity and  $K$  is conductivity.

Boundary and initial conditions are

$$\theta_S(0, \tau) = 0; \theta_S(\xi_0, \tau) = 1; \theta_L(\xi_0, \tau) = 1; \frac{\partial \theta_L}{\partial \xi}(1, \tau) = 0; \quad (12.2.551)$$

$$\frac{\partial \theta_S}{\partial \xi} \Big|_{\xi=\xi_0} - \frac{\partial \theta_L}{\partial \xi} \Big|_{\xi=\xi_0} = \frac{d\xi_0}{d\tau}, \quad \xi = \xi_0(\tau) \text{ is solid-liquid interface, } \xi_0(0) = 0. \quad (12.2.552)$$

If Stefan number (*Ste*) is small, say less than 0.1, then r.h.s. in Eqs (12.2.547), (12.2.549) can be ignored and the above formulation becomes a quasisteady-state solution provided free boundary is taken time dependent. The mathematical interpretation of quasisteady-state problem and quasisteady-state formulation of transient problems is to be understood this way. However, the effect of these assumptions on the behaviour of solutions of Stefan problems has not been discussed in the earlier sections and some discussion on this aspect seems to be in order.

A small Stefan number assumption on the basis of which the transient term in heat equation is neglected may mean two things. Either sensible heat in the system is small as compared to latent heat or latent heat is too large. In the former case, temperatures may change quickly or thermal effects propagate with infinite speed and so steady state is attained quickly. Since a small Stefan number also means that latent heat is large it will slow down the interface motion. However, neglecting sensible heat also means that the thermal resistance to heat flow is less and consequently interface velocity is over estimated in quasisteady-state model. When internal heat sources are present in both the solid and liquid phases in solidification, the interface velocity slows down. This complements the effect of small Stefan number. On the other hand, energy generation in melting can result in rapid changes, thus calling into question the quasisteady-state assumption. The strength of the heat source plays an important role. As the heat conduction term is of the order of one,  $\beta$  should also be of the same order or smaller to justify the quasisteady-state approximation in melting.

Taking  $Ste = 0$  in Eqs (12.2.547)–(12.2.549), we get

$$\frac{\partial^2 \theta_S}{\partial x^2} + \beta = 0, \quad \frac{\partial^2 \theta_L}{\partial x^2} + \beta = 0; \text{ for } \theta_S \text{ in } 0 \leq \xi \leq \xi_0; \text{ for } \theta_L \text{ in } \xi_0 \leq \xi \leq 1. \quad (12.2.553)$$

The solutions of  $\theta_S$  and  $\theta_L$  can be easily obtained and are not given here. The differential equation for  $\xi_0(\tau)$  can be obtained from Eq. (12.2.552) which on integration gives

$$\int_0^\tau d\tau = \int_0^{\xi_0} \frac{\xi d\xi}{(\beta/2)\xi^2 - \beta\xi + 1}, \quad \xi(\tau_0) = 1, \quad (12.2.554)$$

$\tau_0$  is the total solidification/melting time. The integral in Eq. (12.2.554) has to be solved for three different cases. Case (i):  $\beta > 2$ ,  $\tau \geq 0$ ; case (ii):  $\beta < 2$ ,  $0 \leq \tau \leq \tau_0$  and case (iii):  $\beta = 2$ ,  $\tau \geq 0$ .  $\xi = \xi_0(\tau)$  can be obtained implicitly as a function of  $\beta$  and  $\tau$  but it is not possible to extract any further information about  $\xi_0(\tau)$  in these three cases. The evaluation of integrals is elementary but expressions are complicated. However, in numerical computations these expressions can be used and they provide useful information. Some useful information can be extracted from steady-state solution which is a particular case of quasisteady solution in which the moving boundary  $\xi_0(\tau) = \text{constant}$ .

$$\xi_0|_{\text{steady}} = 1 - \sqrt{1 - 2/\beta}. \quad (12.2.555)$$

For further details see [434].

Melting problem considered in [434] is significantly different from solidification problem which was considered in Eqs (12.2.547)–(12.2.552). In this case at  $x = 0, t > 0$ , temperature  $T_0 > T_m$  (melting temperature) is prescribed. Because of low specific heat in small Stefan number together with heat generation in solid and the prescribed temperature at  $x = 0$ , the solid melts immediately and the temperature of the slab attains the melting temperature uniformly. The whole solid slab becomes a mixture of solid and liquid immediately at melting temperature in which the proportion of liquid rises continuously till the whole slab becomes liquid. Only liquid temperature changes in the mush which can be obtained by solving the following problem

$$\frac{\partial^2 \theta_L}{\partial x^2} + \beta = 0; \theta_L(0, \tau) = 0; \theta_L(\xi_0, \tau) = 1; \beta = qL^2/(K(T_0 - T_m)), \quad (12.2.556)$$

$Ste = C(T_0 - T_m)/l$ , and  $\theta_L$  defined

as in Eq. (12.2.550).

The energy balance equation at  $\xi = \xi_0$ , i.e. at the solid-liquid interface, should include the liquid mass fraction in the slab. If  $\gamma$  is the ratio of mass of liquid in mixture to the total mass of mixture then  $\gamma$  is determined from energy balance for the mixture and we get

$$\rho l \frac{d\gamma}{dt} = q \text{ and } \gamma = qt/\rho L, \quad \gamma(0) = 0, \quad (12.2.557)$$

$q$  is volumetric heat generation rate.  $\theta_L(\xi, \tau)$  ( $\xi = x/L$ ) is obtained from Eq. (12.2.556), whose solution is simple and not given here.  $\xi_0(\tau)$  can be obtained from the following energy balance condition at  $\xi = \xi_0(\tau)$ .

$$(1 - \beta\tau) \frac{d\xi_0}{d\tau} = \frac{\partial \theta_L}{\partial \xi} \Big|_{\xi_0}, \xi_0 = \sqrt{2\tau/(1 - \beta\tau)}, \xi_0(0) = 0, \tau_0 = 1/(2 + \beta). \quad (12.2.558)$$

Numerical work was done in [434] for transient solid and liquid temperatures for  $\beta$  values ranging from 0.5 to 4. For  $\beta > 2$ , steady state is reached without having total solidification. The effect of energy generation in steady-state case is as expected. In the melting problem solid-liquid mixture was considered for numerical work. One characteristic feature of melting with energy generation term is the absence of steady-state condition prior to total melting of the slab.

The effect of another parameter, Biot number  $Bi$ , has been discussed in [435] as  $Bi \rightarrow 0$  by considering a one-dimensional one-phase solidification problem in a semiinfinite region. The initial temperature of the solid is melting temperature  $T_m$  in the region  $0 \leq x < \infty$ , and any of the three types of boundary conditions given below can be prescribed at  $x = 0$  by taking  $q(t)$  differently.

$$\left. \begin{aligned} -K_L \frac{\partial T_L}{\partial x} \Big|_{x=0} &= q(t), q(t) \text{ may be prescribed as given below,} \\ \text{(i) } q(t) &= q_1(t); \text{ (ii) } q(t) = h[T_a(t) - T_w(t)]; \text{ (iii) } q(t) = \beta [T_a^4(t) - T_w^4(t)]. \end{aligned} \right\} \quad (12.2.559)$$

$q_1$  is the prescribed heat flux in condition (i), second condition is of convective type in which  $T_a$  is the ambience temperature and  $T_w$  is the wall temperature  $T_L(0, t)$ , the third condition is of radiative type. Various quantities are scaled as follows.

$$\theta_L = (T_L - T_m)/T^*, \quad \bar{q} = q/q^*, \quad \tau = t/t^*, \quad (\xi, \xi_0) = (x, S(t))/x^*, \quad (12.2.560)$$

$$Ste = C_L T^*/l, \quad Bi = q^* x^*/K_L T^*, \quad x^* = (t^* q^*)/(\rho l), \quad T^* = l/C_L. \quad (12.2.561)$$

$x = S(t)$ ,  $S(0) = 0$  or  $\xi = \xi_0(\tau)$  is the free boundary,  $T^*$ ,  $q^*$ ,  $x^*$ ,  $t^*$  are some reference quantities. The formulation of one-phase problem in  $0 \leq x < \infty$  can be easily written in terms of dimensionless quantities  $\theta(\xi, \tau)$ ,  $\bar{q}(\tau)$ ,  $\xi_0(\tau)$ . When  $Bi \rightarrow 0$ , we get a quasisteady-state problem.

$$L t \frac{\partial}{\partial \xi} \left( \frac{\bar{K}_L}{Bi} \frac{\partial \theta_L}{\partial \xi} \right) = Ste \frac{\partial \theta_L}{\partial t} \Rightarrow \frac{\partial^2 \theta_L}{\partial \xi^2} = 0, \quad (12.2.562)$$

$$- (\bar{K}_L/Bi) \left. \frac{\partial \theta_L}{\partial \xi} \right|_{\xi_0} = - (\bar{K}_L/Bi) \left. \frac{\partial \theta_L}{\partial \xi} \right|_{\xi=0} = \frac{d\xi_0}{d\tau}, \quad 0 \leq \xi < \xi_0; \quad \theta_L(\xi_0, t) = 0, \quad \tau > 0. \quad (12.2.563)$$

$\bar{K}_L$  is dimensionless thermal conductivity. The second equation in Eq. (12.2.562) is dimensionless heat equation which is obtained when  $Bi \rightarrow 0$  and on integration gives  $\partial \theta_L / \partial \xi = f(\tau)$ , a function of time. The first term equated to the third term in Eq. (12.2.563) is dimensionless Stefan condition and second term equated to  $\bar{q}(\tau)$  is the flux prescribed boundary condition. First and second terms are equal as heat transfer is in a steady state.

The scaling of flux is done as in Eq. (12.2.560) and when flux prescribed condition in Eq. (12.2.559) is made dimensionless then  $Bi$  will come in the denominator as in Eq. (12.2.563). The scaling of temperature in convective and radiative boundary conditions is done with reference temperature in a different way and  $T_a^*$  and  $T_w^*$  are the scales of  $T_a$  and  $T_w$ , respectively.  $q^*$  is the scale of  $hT^*$  and  $\beta(T^*)^4$ , and  $t^*$  is time scale of  $T_a(t)$ .

From first equation in Eq. (12.2.562), as  $Bi \rightarrow 0$ ,  $\frac{\partial \theta_L}{\partial \xi} = f(\tau)$  on integration. From boundary condition in Eq. (12.2.559),  $\left. \frac{\partial \theta_L}{\partial \xi} \right|_{\xi=0} = 0$ , and we have also  $\theta_L(\xi_0, \tau) = 0$ . This implies that as  $Bi \rightarrow 0$ ,  $\theta_L(\xi, \tau) \equiv 0$ ,  $\xi \geq 0$  and also  $T_w = T_a = 0$ . This is the discrepancy as Stefan condition cannot be used to determine  $\xi_0$  if  $Bi \rightarrow 0$  is taken in Stefan condition. If Eq. (12.2.563) is used and  $Bi$  is eliminated from the first two terms then second term is simply the prescribed flux at  $\xi = 0$  and  $\xi_0$  is determined as given below.

From the second term in Eq. (12.2.563) which is dimensionless flux, it is easy to obtain

$$\frac{d\xi_0}{d\tau} = \bar{q}(\tau), \quad \xi_0(\tau) = \int_0^\tau |\bar{q}(p)| dp, \quad \xi_0(0) = 0, \quad (12.2.564)$$

modulus sign in Eq. (12.2.564) takes care of both melting and solidification problems.

The author in [435] has called this problem a zero-phase Stefan problem as  $\xi_0$  can be obtained but temperatures are not to be tracked. Because of assumptions made, throughout the region temperature remains melting temperature and moving boundary can be obtained as in Eq. (12.2.564). Solutions of many other problems which are extensions of this problem and satisfy the earlier assumptions can also be obtained. If  $\xi = \xi_0(\tau)$ ,  $\tau \geq 0$  and  $\xi_0(\tau)|_{\tau=0} = b$ ,  $b \neq 0$ ,  $0 < b < 1$ , then  $\xi_0(\tau) = b + \int_0^\tau |\bar{q}(p)| dp$ ,  $\tau \geq 0$ . This is because under the assumptions made, the flux which is applied at the wall, i.e. at  $x = 0$  without disturbing the temperature of the region  $0 \leq \xi \leq b$  is acting at  $x = b$  instantaneously (this is interesting) and the free boundary starts moving. This is possible provided there is no changing of sign of prescribed flux at  $x = 0$ . If at any time the sign of flux is reversed at  $x = 0$  before free boundary



reaches  $x = 1$ , in a finite slab case, then reversal of the direction of the free boundary is not possible. This is because at no time the sensible heat of the system changes. Whatever energy the medium gets through the flux at  $x = 0$  is instantaneously consumed in releasing or absorbing latent heat and moving the phase-change boundary. Therefore, the new boundary starts afresh at  $x = 0$ . Conservation of energy is satisfied only by phase change, and not at all by the governing equation.

At  $t = 0$ , the region under consideration could be finite or infinite and number of phases present could be any finite number and of any extent such as solid for  $0 \leq x \leq b_1$ , then liquid for  $b_1 \leq x \leq b_2$ , again solid for  $b_2 \leq x \leq b_3$  and so on up to  $b_{n-1} \leq x \leq b_n$ ,  $\sum_{r=0}^n b_r = 1$ ,  $b_0 = 0$ . Starting could be from liquid and ending with liquid or solid. In a finite region problem, heat fluxes can be prescribed at the ends which could be of opposite or same signs. However, basic assumptions on which Eq. (12.2.564) has been derived for moving interface should be always satisfied. Thermal conductivities of solid and liquid could be different but sensible heats in both solid and liquid should remain the same for all time as at  $\tau = 0$ . If number of phases are more than one then to get physically meaningful results at least one phase change should take place.

One-dimensional radially symmetric cylindrical or radially symmetric spherical problems can also be considered in finite or infinite region occupied by a single phase or multiple phases. Multiple phases could be present in any random order. In the finite region, prescribed fluxes at both the ends could be of the same sign or opposite signs with same or different thermal conductivities. The assumptions under which the planar problem was studied should hold for these radially symmetric problems also.

A indetermination or inconsistency at the boundary  $x = 0$  in the planar case in the flux and temperature derivative should be resolved otherwise the solution obtained will be incorrect. The inconsistency in planar case was there but it did not affect the solution given in Eq. (12.2.564) as it got resolved when Stefan condition was written as in Eq. (12.2.563) and then we took the limit  $Bi \rightarrow 0$ . If the limit  $Bi \rightarrow 0$  is taken in the heat equation in hollow cylindrical problem, then we get the steady-state equation as given below.

$$\frac{\partial}{\partial \xi} \left( \frac{\bar{K} \xi}{Bi} \frac{\partial \theta}{\partial \xi} \right) = 0 \text{ or } \frac{\bar{K}}{Bi} \xi \frac{\partial \theta}{\partial \xi} = f(\tau); \theta(\xi, 0) = 0, \theta(\xi = \xi_0, \tau) = 0. \quad (12.2.565)$$

In Eq. (12.2.565) we are still continuing with the notations that  $\xi$  is dimensionless radial coordinate and  $\xi = \xi_0$  is the moving interface. At the inner hollow boundary  $\xi = r_0$ ,  $r_0 > 0$ ,  $\xi_0(0) = r_0$ , and if limit  $Bi \rightarrow 0$  is not taken in flux prescribed condition then we have

$$\bar{q}(\tau) = - \frac{\bar{K}}{Bi} \frac{\partial \theta}{\partial \xi} \bigg|_{\xi_0(0)}. \quad (12.2.566)$$

If  $Bi \rightarrow 0$  in Eqs (12.2.565), (12.2.566) is used then it can be easily shown that  $\theta(\xi, \tau) = 0$  throughout the region under consideration and so  $\partial \theta / \partial \xi = 0$ . But  $\partial \theta / \partial \xi$  in Eq. (12.2.565) is a function of time. However,  $\xi \frac{\partial \theta}{\partial \xi} = f(\tau)$  suggests that at the phase-change boundary, the equation to be solved for a single-phase hollow cylinder occupying  $r_0 \leq r < \infty$ ,  $r_0 \neq 0$  should be

$$\xi_0 \frac{d\xi_0}{d\tau} = \xi_0 \bar{q}(\tau) \text{ or } \xi_0^2(\tau) = \xi_0^2(0) + 2 \int_0^\tau \xi_0(0) |\bar{q}(p)| dp, \quad 0 \leq \tau \leq \tau_0, \xi_0(0) \neq 0. \quad (12.2.567)$$

The previous result is obtained if the Stefan condition is written in a form similar to Eq. (12.2.563) which is a modified Stefan condition. Modulus sign in Eq. (12.2.567) takes care of both melting and freezing problems of outward phase change.

In the spherical case, if the earlier notations for scaled radial coordinate and moving interface are assumed to hold then for a single-phase hollow sphere occupying the region  $0 < r_0 < r < \infty$ , the equation for phase-change interface will be

$$\xi_0^3(\tau) = \xi_0^3(0) + 3 \int_0^\tau \xi_0^2(0) |\bar{q}(p)| dp. \quad (12.2.568)$$

In both Eqs (12.2.567), (12.2.568) it has been assumed that the prescribed flux is such that phase change takes place at  $\tau = 0$  at the inner radius  $\xi_0(0)$  of the cylinder or sphere. Several other cases can be discussed for planar, cylindrical and spherical geometries. For further details refer [435]. Similar results when Stefan number  $\rightarrow 0$  were obtained in [21] see *Leibenzon's approximation*.

A quasisteady-state solution discussed in [436] pertains to the formation of rime ice layer and glaze ice layer when supercooled liquid droplets fall on a solid substrate which has the temperature below the freezing temperature  $T_m$  and occupies a region  $-L \leq x \leq 0$ . In mild temperatures or in the presence of a sufficiently thick ice layer, a water layer may subsequently appear and glaze ice will then form. Such problems arise in the accretion of ice and has applications in aircraft industry and other commercial icing models. Let  $T_S$ ,  $T_I$  and  $T_W$  be the temperatures of substrate, ice and water, respectively. The regions occupied by the temperatures are: for  $T_S(x, t)$ ,  $-L \leq x \leq 0$ ,  $t > 0$ , for  $T_I(x, t)$ ,  $0 \leq x \leq b(t)$ ,  $t > 0$  and for  $T_W$ ,  $b(t) \leq x \leq a(t)$ ,  $t > 0$ .  $b(t)$  is phase-change boundary between ice and water and  $a(t)$  is a moving boundary and no phase change is taking place at  $x = a(t)$ . Our interest here is more on boundary conditions as the boundary conditions are of a type not discussed earlier in this section.

At the substrate surfaces  $x = -L$  and  $x = 0$ , we have the prescribed conditions

$$K_S \frac{\partial T_S}{\partial x} = h_{as}(T_S - T_A), x = -L, t > 0; K_S \frac{\partial T_S}{\partial x} = K_I \frac{\partial T_I}{\partial x}, x = 0, t > 0. \quad (12.2.569)$$

At the phase-change boundary  $x = b(t)$ , if glaze ice is absent, we have

$$K_I \frac{\partial T_I}{\partial x} = Q_I + q_I(T_a - T_I), Q_I = \frac{1}{2} \dot{m} w^2 + \frac{1}{2} r h_{al} w^2 / C_a + \dot{m} l_f, \quad (12.2.570)$$

$$q_I = \dot{m} C_w + \lambda_1 e_o + h_{al}. \quad (12.2.571)$$

In Eqs (12.2.569)–(12.2.571), the subscripts  $S$  and  $I$  stand for substrate and ice, respectively. In Eq. (12.2.569),  $h_{as}$  is the heat transfer coefficient between substrate and the atmosphere below  $x = -L$  and  $T_A$  is the ambience temperature. In Eq. (12.2.570),  $Q_I$  is total energy gained by the rime ice in which  $\dot{m}$  is the incoming mass flux per unit area of the supercooled water,  $w$  is far-field air velocity. The first term in  $Q_I$  represents kinetic energy, the second term corresponds to aerodynamic heating in which  $r$  is the local recovery factor,  $h_{al}$  is the heat transfer coefficient between ice and air above ice,  $C_a$  is the specific heat of the air,  $C_w$  is the specific heat of water; in the third term  $l_f$  is the latent heat of freezing of droplets which form ice. Energy is also lost to surroundings at  $x = b(t)$  which is due to convective heat transfer and sublimation at  $x = b(t)$ . Energy lost is given by  $q_I(T_a - T_I)$ , in which  $T_a$  is the temperature of

the surrounding air above ice,  $T_a < T_I$ ,  $\lambda_1$  is sublimation coefficient,  $e(T) \simeq e_0 T$  is saturation vapour pressure.

If glaze ice is forming then energy equation for temperature  $T_W(x, t)$  should also be considered along with boundary conditions on  $x = a(t)$ . One boundary condition is of temperature continuity, i.e.  $T_W(x, t) = T_I(x, t)$ , at  $x = a(t)$ ,  $t > 0$ . Another condition is energy balance at  $x = a(t)$ . The energy gained by water at  $x = a(t)$  is  $Q_w$  and energy lost is  $q_w(T_a - T_W)$ , where

$$Q_w = \frac{1}{2} \dot{m} w^2 + r h_{aw} w^2 / (2C_a), \quad q_w = \dot{m} C_w + \lambda_2 e_0 + h_{aw}, \quad (12.2.572)$$

where subscript  $w$  is for water. In Eq. (12.2.572),  $h_{aw}$  is the heat transfer coefficient between water and air above water,  $\lambda_2$  is evaporation coefficient. Therefore, the energy balance condition at  $x = a(t)$  is

$$K_W \frac{\partial T_W}{\partial x} = Q_w + q_w(T_a - T_W), \quad T_a < T_W. \quad (12.2.573)$$

There is no phase change at  $x = a(t)$  but there is a phase change from ice to water at  $x = b(t)$ . Therefore if glaze ice is formed then we have the energy balance as

$$\rho_I l_f \frac{\partial b}{\partial t} = K_I \frac{\partial T_I}{\partial x} - K_W \frac{\partial T_W}{\partial x}, \quad \text{at } x = b(t). \quad (12.2.574)$$

In addition to conservation of energy we should have mass balance also at  $x = b(t)$

$$\rho_I \frac{\partial b}{\partial t} + \rho_w \frac{\partial h}{\partial t} = \frac{d\dot{m}}{dt}, \quad h(t) = \frac{\rho_I}{\rho_w} (t - b(t)), \quad t \geq t_w, \quad h(t_w) = 0, \quad (12.2.575)$$

where  $h(t)$  is the height of water above ice and  $t_w$  is the time at which water starts forming. It may be noted that Eqs (12.2.470)–(12.2.476) are not scaled. For scalings refer [436]. The formulations of two-phase system of substrate and ice and three-phase system of substrate, ice and water are now complete.

Quasisteady-state solutions have been obtained in [436] under some assumptions given below. The method of solution is not different from other methods discussed in this section. In brief, temperature solutions in three regions are considered as linear functions of one of the parameters  $\epsilon_S$ ,  $\epsilon_I$  and  $\epsilon_W$  with corresponding subscript. These parameters (see [436]) arise after scalings. The heat equations are transformed to equations in which transient terms get multiplied by a constant  $\epsilon$  assigned to the respective region and which is small. For example, the equation for ice becomes as given below in which we take  $T_I = T_I^0 + \epsilon_I T_I^1$ .

$$\frac{\partial^2 T_I}{\partial x^2} = \epsilon_I \frac{\partial T_I}{\partial t}, \quad \epsilon_I = C_I L \dot{m} / K_I, \quad \epsilon_I \ll 1. \quad (12.2.576)$$

In the first-order terms for  $T_I$ , transient terms can be neglected as  $\epsilon_I$  is very small. Similarly other two heat equations for  $T_S$ ,  $T_W$  and boundary conditions also become independent of time as  $\epsilon_S$  and  $\epsilon_W$  are even much smaller. In the solution of three-region problem, a differential equation for  $b(t)$  is obtained which can be solved only numerically. To save more space for formulation, scalings are not given here. Formulation using droplet energy as a source term has also been discussed.

Moving grid scheme as in [68] has been used to obtain the numerical solution of the coupled transient three-region problem and compared with the quasisteady-state solutions of three regions.

In [437], unidirectional solidification of binary solutions or melts with three phases, solid, mush and melt have been studied in terms of classical formulations. Analytical solutions have been obtained for transient temperature distributions in the solid and mush with some approximations so that they can serve approximate solutions to the transient problem. Solidus and liquidus are boundaries of the mushy region. Liquidus temperature is melt temperature which is taken as a known constant. Concentration is considered in the mush but no equation has been solved for concentration as it gets eliminated from equations and can be expressed in terms of temperature. Overall it is a good approximate model for analytical solution of mush and compares well with the models considered in this section. As an example for elucidation, the binary solution can be taken as saltish sea water. Above the sea is air and the upper layer of sea water has frozen. Below the frozen layer is deep sea water at constant temperature  $T_W$ .

Consider a semiinfinite region  $0 \leq x < \infty$ , in which we take  $x$ -axis is taken horizontally for convenience. For  $t \geq 0$ , it is filled with solid material occupying  $0 \leq x < S_1(t)$ , the mushy region is  $S_1(t) \leq x \leq S_2(t)$  and liquid occupies  $x > S_2(t)$ .  $S_1(t)$  and  $S_2(t)$  are solidus and liquidus boundaries, respectively,  $S_1(0) = S_2(0) = 0$ . Solid temperature denoted by  $T_S$  is taken in the form

$$T_S(x, t) = T_{at}(t) + xg(t), \quad 0 < x < S_1(t), \quad t > 0, \quad (12.2.577)$$

where  $T_{at}(t)$  is the atmospheric temperature which is known and  $g(t)$  is an unknown function. If  $T_M$  is the temperature of the mush then as in Eq. (12.2.577) for  $T_S$ ,  $T_M$  is also taken as

$$T_M(x, t) = T_1(t) + xT_2(t), \quad S_1(t) < x < S_2(t), \quad t > 0. \quad (12.2.578)$$

$T_M(x, t)$  in Eq. (12.2.578) satisfies approximately the classical heat transfer equation

$$\rho_M C_M \frac{\partial T_M}{\partial t} = \frac{\partial}{\partial x} \left( K_M(\phi) \frac{\partial T_M}{\partial x} \right) + l \frac{\partial \phi}{\partial t}, \quad (12.2.579)$$

where  $\phi$  is solid-phase fraction in mush,  $l$  is the latent heat of freezing. At  $x = S_1(t)$ ,  $\phi = \phi_{S_1}(t)$  and  $T_S = T_m$ ,  $T_m$  is freezing temperature of salty water.

The heat balance at  $x = S_1(t)$  results in the following equation.

$$l(1 - \phi_{S_1}) \frac{dS_1}{dt} = K_S \frac{\partial T_S}{\partial x} - [K_S \phi_{S_1} + K_W(1 - \phi_{S_1})] \frac{\partial T_M}{\partial x}, \quad \text{at } x = S_1(t). \quad (12.2.580)$$

In Eq. (12.2.580), the thermal conductivity  $K_m$  of mush is taken as volume-fraction-weighted average. The subscripts  $S$ ,  $W$  and  $M$  stand for solid, water and mush, respectively. If  $\bar{c}_M$  is the concentration of salt in sea water then according to linear-phase diagram of temperature and concentration within the mush has been taken approximately as

$$T_M = -m\bar{c}_M, \quad S_1(t) < x < S_2(t), \quad t > 0; \quad m \text{ is the liquidus slope.} \quad (12.2.581)$$

The transport of solute or salt takes place according to *Scheil's equation* [438] in which it is assumed that ice in mush is almost devoid of salt. This implies

$$\frac{\partial}{\partial t} ((1 - \phi)\bar{c}_M) = 0, \quad S_1(t) < x < S_2(t), \quad t > 0. \quad (12.2.582)$$

Mass balance at  $x = S_1(t)$  gives

$$\bar{c}_M(1 - \phi_{S_1}) \frac{dS_1}{dt} = -D_W(1 - \phi_{S_1}) \frac{\partial \bar{c}_M}{\partial x}, \quad x = S_1(t), \quad t > 0, \quad (12.2.583)$$

$D_W$  is the mass diffusion coefficient of pure water.

At the liquidus  $x = S_2(t)$ ,  $t > 0$

$$\phi = \phi_{S_2}, \quad T_M = T_W, \quad T_W = \text{constant}, \quad (12.2.584)$$

$$l\phi_{S_2} \frac{dS_2}{dt} = (K_S\phi_{S_2} + K_W(1 - \phi_{S_2})) \frac{\partial T_M}{\partial x}, \quad x = S_2(t), \quad t > 0. \quad (12.2.585)$$

If  $\phi_{S_1} = 1$  and  $\phi_{S_2} = 1$ , the mushy region is only a solid region. For  $\phi_{S_1} \neq 1$ ,  $\phi_{S_2} \neq 1$ , the unknowns,  $\phi(x, t)$ ,  $\phi_{S_1}$ ,  $\phi_{S_2}$ ,  $g(t)$ ,  $T_1(t)$ ,  $T_2(t)$ ,  $S_1(t)$ ,  $S_2(t)$  and  $\bar{c}_M$ , can be obtained by solving above equations.  $\bar{c}_M$  is unknown but can be eliminated from the equations. To determine  $S_1(t)$  and  $S_2(t)$ , two nonlinear coupled first-order differential equations are obtained which can be solved only numerically. To get any analytical solution lot of assumptions are to be made and finally it is possible to obtain some analytical results for  $S_1(t)$  and  $S_2(t)$  provided  $\phi_{S_2}$  is known. For further details of solution procedure and assumptions see [437] as they are lengthy.

A four-phase problem which arises in casting processes has been considered in [439] in which mold, gaseous gap between solid and mold, solid region and melt region are the four phases. Transient heat conduction without mushy region and concentration gradients have been considered in solid and liquid phases and by suitably choosing the gaseous gap exact analytical solutions of Neumann type have been obtained. The solution has been further extended to obtain a more generalized solution that incorporates an initial interfacial resistance to heat transfer. The resistance may arise from an initial gap, the use of mold coating, imperfect contact due to ruggedness of the mold surface, contamination of layers, etc.

Mold occupies the region  $-\infty < x < 0$  with its temperature denoted by  $T_{M_0}(x, t)$  and gaseous gap having temperature  $T_g(x, t)$  occupies the region  $0 \leq x \leq d(t)$  between mold and solid. As  $x \rightarrow -\infty$ ,  $T_{M_0} \rightarrow T_a < T_m$  where  $T_m$  is the constant freezing temperature. Solid and liquid temperatures are denoted by  $T_S(x, t)$  and  $T_L(x, t)$ , respectively. At  $x = 0$

$$T_{M_0} = T_g \text{ and } K_{M_0} \frac{\partial T_{M_0}}{\partial x} = K_g \frac{\partial T_g}{\partial x}, \quad t > 0. \quad (12.2.586)$$

At  $x = d(t)$  which is r.h.s. boundary of gaseous gap and l.h.s. boundary of solid region, we have

$$T_g = T_S \text{ and } K_g \frac{\partial T_g}{\partial x} = K_S \frac{\partial T_S}{\partial x}, \quad t > 0, \quad d(0) = 0. \quad (12.2.587)$$

Mold surface temperature and casting surface temperatures are constant but unknown. At  $x = S(t)$  which is r.h.s. boundary of solid region and l.h.s. boundary of liquid region and is also the phase-change interface, we have

$$T_S = T_L = T_m; \quad K_S \frac{\partial T_S}{\partial x} - K_L \frac{\partial T_L}{\partial x} = \rho l \frac{dS}{dt}, \quad S(0) = d(0) = 0. \quad (12.2.588)$$

The subscripts  $Mo$ ,  $g$ ,  $S$  and  $L$  refer to mold, gaseous gap, solid region and liquid region, respectively. Liquid region extends from  $x = S(t)$  to  $x = \infty$ . As  $x \rightarrow \infty$ ,  $T_L \rightarrow T_A$ , a constant. We have seen in earlier problems that exact analytical solutions are similarity solutions which can be obtained only in special type of formulations. Therefore, the gaseous gap  $d(t)$  is taken in the form

$$d(t) = \lambda(4k_g t)^{1/2}, \quad \lambda > 0 \text{ is unknown, } t > 0; \quad k \text{ is diffusivity.} \quad (12.2.589)$$

The temperatures in all four regions satisfy linear heat equations of the type given in Eq. (1.3.1) and  $S(t)$  is taken as  $\delta(4k_S t)^{1/2}$ ,  $\delta > 0$  is unknown. Neumann-type solutions can be easily obtained for temperatures. To determine  $\lambda$  in Eq. (12.2.589), a relationship between  $d(t)$  and  $S(t)$  is taken in the form

$$d(t) \approx \frac{\alpha_S S(t) \Delta T_{\text{avg}}}{1 + \alpha_S \Delta T_{\text{avg}}} \approx \alpha_S S(t) \Delta T_{\text{avg}}, \quad \Delta T_{\text{avg}} = (T_m - T_{CS})/2, \quad (12.2.590)$$

where  $\alpha_S$  is the coefficient of thermal expansion and  $T_{CS}$  is the unknown casting surface temperature. Note that Eqs (12.2.586)–(12.2.590) are not scaled.

The details of the coupled solution of the above problem of four phases will not be presented here as such solutions have been discussed earlier and the solution procedure is not new.  $\lambda$  in Eq. (12.2.590) depends on  $T_{CS}$  but  $T_{CS}$  requires the knowledge of  $\lambda$  so an iterative procedure is to be adopted. From the solution in Eq. (1.3.16) it can be seen that more is the superheat ( $T_A - T_M$ ), the less is the growth rate of solid-liquid interface. Therefore, the superheat of the liquid should be first removed in the mold before the solidification starts. For pure metals, in the solidification process, the melt has to be supercooled. A constant  $G$  should be added to the expression of  $S(t)$  and for practical purposes,  $S(t)$  is taken as  $S(t) = \delta(4k_S t)^{1/2} - G$ . For the determination of  $G$  see [439].

The initial resistance to heat transfer is present in all casting processes. To account for the resistance to heat transfer, four virtual components in the form of virtual gaps are introduced in the region  $0 \leq x < \infty$ . Suppose before the heat transfer exchange from mold starts, the melt is presolidified having a virtual thickness  $S_0$  which takes place in time  $t_0$ . Both  $S_0$  and  $t_0$  are known. The analytical solutions of all the five phases, four mentioned above and fifth is virtual gap equivalent to  $S_0$ , have to be obtained from the solutions of Eqs (12.2.586)–(12.2.590) by considering them coupled.

To include the effect of initial resistance due to various factors discussed above including the effect of  $S_0$ , the authors in [439] adopted a technique which was earlier followed in [440]. Let  $E_g$  be the virtual gap occupying the region  $0 \leq x \leq E_g$  to account for  $S_0$  and  $t_0$ .  $E_g$  is unknown. Another virtual gap  $d_g$  is introduced on the right of  $E_g$  occupying the region  $E_g \leq x \leq E_g + d_g$  which is to account for the resistance of heat transfer from the mold. After  $E_g + d_g$  is gaseous gap  $d(t)$  occupying the region  $E_g + d_g \leq x \leq E_g + d_g + d(t)$ .  $d(t)$  is not a virtual gap and after  $d(t)$  is another virtual gap  $d_g$  which accounts for the resistance from the casted solid and occupies the region  $E_g + d_g + d(t) \leq x \leq E_g + d_g + d(t) + d_g$ . After  $E_g + 2d_g + d(t)$  is region  $S_0$ , after  $E_g + 2d_g + d(t) + S_0$  is the solid region and after solid region is the liquid region. If the gap  $d_g$  is taken similar to the gaseous gap  $d(t)$ , then time  $t_0$  defined above can be expressed as  $t_0 = d_g^2/(\lambda^2 k_g)$  as the gap is  $2d_g$ . If the  $x$ -coordinate is changed to  $x'$  such that  $x' = x + E_g$  and time  $t' = t + t_0$ , then analytical similarity solutions for  $T_{Mo}$ ,  $T_g$ ,  $T_S$ ,  $T_L$  and  $S(t)$  can be obtained from the results obtained earlier when virtual gaps were not considered. But this new solution will be in  $x'$  and  $t'$  coordinates in which

$$\left. \begin{aligned} x' &= d'(t') = \lambda(4k_g t')^{1/2}, \quad S'(t') = \delta(4k_S t')^{1/2}, \quad S'(t_0) = D_0 = \delta(4k_S t_0)^{1/2}, \\ D_0 &= S_0 + 2d_g, \quad 2d_g = \lambda D_0 (k_g/k_S)^{1/2} / \delta. \end{aligned} \right\} \quad (12.2.591)$$

This new solution should be converted in  $(x, t)$  coordinates.  $E_g$ ,  $D_0$  and  $d_g$  are unknowns to be determined from the new solution. For the expressions of  $E_g$  and  $d_g$  and other solution details refer [439]. The numerical work indicates that the initial interfacial resistance has a profound effect on the gap and solidification growth rates.

After making some assumptions, one-dimensional one-phase Stefan problem in a body with variable cross-sectional area has been considered in [441]. If the body is insulated from the surroundings, the radius of curvature of the surface of body with variable cross-sections is taken as smooth, the body is very large and the change of cross-sectional area perpendicular to central line is relatively small then the temperature along the central line  $x$  can be taken as constant. For further simplification it is assumed that there is no natural convection and in the solidification problem the initial constant temperature is taken equal to the melting temperature  $T_m$ . The formulation of the problem is given below in which  $A(x)$  is the cross-sectional area.

The temperature  $T_S(x, t)$  of the solid satisfies the differential equation

$$\frac{\partial T_S}{\partial t} = k_S \left( \frac{\partial^2 T_S}{\partial x^2} + \frac{A'(x)}{A(x)} \frac{\partial T_S}{\partial x} \right), \quad x_0 < x < S(t), \quad t > 0, \quad (12.2.592)$$

$x = S(t)$  is the solid-liquid interface. At  $x = S(t)$ , we have

$$\frac{dS}{dt} = \frac{K_S}{\rho l} \left. \frac{\partial T_S}{\partial x} \right|_{x=S(t)}, \quad S(0) = x_0; \quad T_S(x, t) \Big|_{x=S(t)} = T_m. \quad (12.2.593)$$

Eq. (12.2.592) has not been derived in [441] but a 1942 reference is quoted.

At  $x = x_0$ , boundary condition could be of temperature prescribed or flux prescribed or of convective type. Using the following transformations

$$y = A(x_0) \int_{x_0}^x \frac{dp}{A(p)}; \quad \eta = A(x_0) \int_{x_0}^{S(t)} \frac{dp}{A(p)}, \quad \eta = \eta(t) \text{ or } t = t(\eta), \quad (12.2.594)$$

the formulation of the problem given in Eqs (12.2.592), (12.2.593) is transformed to

$$\frac{\partial T_S}{\partial t} = k_S \left( \frac{A(x_0)}{A(x)} \right)^2 \frac{\partial^2 T_S}{\partial y^2}, \quad 0 < y \leq y(S(t)); \quad T_S(y(S(t)), t) = T_m, \quad (12.2.595)$$

$$\frac{d\eta}{dt} = \frac{k_S}{\rho l} \left( \frac{A(x_0)}{A(S(t))} \right)^2 \frac{\partial T_S(\eta, t)}{\partial y}, \quad \text{at } x = S(t). \quad (12.2.596)$$

Boundary conditions at  $x = x_0$  or  $y = 0$  can also be written in terms of  $(y, \eta)$ . Now  $T_S$  is expressed in terms of  $(y, \eta)$  so that in the  $(y, \eta)$  plane  $x = S(t)$  becomes  $y = \eta$ . This is done as follows.

$$\frac{\partial T_S}{\partial t} = \frac{\partial T_S}{\partial \eta} \cdot \frac{\partial \eta}{\partial t}; \quad \frac{\partial \eta}{\partial t} = \frac{K_S}{\rho l} \left( \frac{A(x_0)}{A(S(t))} \right)^2 \frac{\partial T_S(\eta, \eta)}{\partial y}, \quad \text{at } y = \eta. \quad (12.2.597)$$

Substituting  $\partial\eta/\partial t$  from second equation of Eq. (12.2.597) in the first equation, we get

$$\frac{\partial^2 T_S}{\partial y^2} = \frac{K_S}{\rho k_S l} \left( \frac{A(x_0)}{A(S(t))} \right)^2 \frac{\partial T_S(\eta, \eta)}{\partial y} \frac{\partial T_S}{\partial \eta}. \quad (12.2.598)$$

In the neighbourhood of  $x = S(t)$ , we write,  $S(t) - \Delta x \leq x \leq S(t)$ ,  $y - \Delta y \leq y \leq \eta$ , so that

$$A(x) = A(S(t) - \epsilon) = A(S(t)) - \epsilon A'(S(t)) + \dots, \quad 0 \leq \epsilon \leq \Delta x. \quad (12.2.599)$$

$$\text{As } \Delta x \rightarrow 0, \text{ we get, } \lim_{\Delta x \rightarrow 0} \left( \frac{A(x)}{A(S(t))} \right)^2 = 1, \quad t \geq 0. \quad (12.2.600)$$

Now the system of equations is independent of the area of cross-section in the  $(y, \eta)$  plane near  $y = \eta$  or  $x = S(t)$ . If  $A(x) = \text{constant} = A_c$  then from Eq. (12.2.597)

$$\left( \frac{\partial \eta}{\partial t} \right)_{A=A_c} = \frac{K_S}{\rho l} \frac{\partial T_S(\eta, \eta)}{\partial y} = g(\eta); \quad \left( \frac{\partial \eta}{\partial t} \right)_{A(x)} = \left( \frac{A(x_0)}{A(S(t))} \right)^2 g(\eta). \quad (12.2.601)$$

It can be easily concluded that if  $dS/dt = g(\delta)$  when  $A(x)$  is constant then from Eqs (12.2.594), (12.2.601) for  $A(x)$  when it is function of  $x$ , we get

$$\left( \frac{dS}{dt} \right)_{A(x)} = \frac{A(x_0)}{A(S(t))} g \left( A(x_0) \int_{x_0}^{S(t)} \frac{dp}{A(p)} \right), \quad (12.2.602)$$

provided initial and boundary conditions of the two problems remain the same after transformation with the exception of interface velocity. However, this conclusion holds even if the initial temperature is different from the freezing temperature. As a particular case, Neumann solution can be obtained. In the cylindrical coordinates  $A(x)_{\text{cyl}} = 2\pi l x$  and spherical coordinates  $A(x)_{\text{sph.}} = 4\pi x^2$ . If  $\xi = S(t)$ ,  $S(0) = x_0$  then

$$\left( \frac{dS}{dt} \right)_{\text{cyl}} = (x_0/S(t)) g(x_0 \log(\xi/x_0)); \quad \left( \frac{dS}{dt} \right)_{\text{sph.}} = (x_0/\xi)^2 g(x_0 - x_0^2/\xi). \quad (12.2.603)$$

Eqs (12.2.592)–(12.2.603) are not scaled.

A one-dimensional one-phase melting problem in the region  $0 \leq x < \infty$  has been considered in [442] in which the solid is melting due to prescribed heat flux at  $x = 0$  and when the melting starts, the melted liquid is drained out from the system. Such problems are known as *ablation problems* in aerodynamics and arise due to aerodynamic heating of spacecrafts, rockets, etc. Such problems also arise in sublimating systems and are called sublimation problems. Without loss of generality it can be assumed that melting starts at  $t = 0$  and not at  $t = t_0$ ,  $t_0 > 0$  as before this time  $t_0$  only a heat conduction problem without phase change exists. If the melting interface is given by  $x = S(t)$ ,  $S(0) = 0$ , then at  $x = S(t)$ ,

$$x = S(t) = \int_0^t \frac{dS}{d\tau} d\tau = \int_0^t v(\tau) d\tau, \quad v(t) \text{ is the interface velocity.} \quad (12.2.604)$$

$$T_S = T_m \text{ and } q - \rho l v = -K_S \frac{\partial T_S}{\partial x}; \quad \text{at } x = S(t), t \geq 0, \quad (12.2.605)$$



where  $q$  is the applied heat flux at  $x = S(t)$ ,  $T_m$  is the sublimation temperature. If new coordinates  $(z, \tau)$  are defined as  $z = x - S(t)$ , and  $\tau = t$  then in terms of  $(z, \tau)$ ,  $z = 0$  at  $x = S(t)$ . The linear heat equation (not given here) is transformed to an equation with a convective term. Quasisteady-state solution of this problem can be easily obtained without resorting to Laplace transform solution which the author in [442] has obtained. In the transient case,  $v$  has been taken as constant, and so  $z = x - vt$ . Such similarity solutions have already been obtained in this section earlier and the solution will not be discussed here. From the solution,  $\frac{\partial T_s}{\partial z}$  can be obtained at  $z = 0$  which gives a relationship between  $q$  and  $v$  in the transient case and if  $v$  is known then  $q$  can be determined or vice versa.

The solution obtained in a particular case when the applied flux and velocity of the interface are proportional to  $t^{-1/2}$  has been used as an approximate solution to the solution of an ablating nose cone during reentry into earth's atmosphere.

The problem discussed earlier can be regarded as a one-phase problem. It can be generalized to a two-phase problem consisting of a sublimation phase and a solid phase. This is just a matter of simple extended formulation. In this very sublimation problem if we consider vaporization instead of sublimation and take vaporization temperature to be much higher than the melting temperature then we have vaporization only after melting takes place. If the temperature of the melted liquid which is not drained out rises to vaporization temperature then we shall have two moving boundary. The region under consideration could be semiinfinite or a finite slab. We can have more number of phases according to their melting temperatures. However, to obtain exact analytical solutions, the formulations have to be of special types. Having considered so many formulations of these problems, it does not seem necessary to present these formulations here. This will save some space.

A three-phase vaporization problem with appearing and disappearing phases in a one-dimensional finite slab was considered in [443] and studied only numerically with finite element method. For three-phase sublimation or vaporization problems, quasianalytical solutions or transient solutions under the assumptions of constant thermo-physical properties could be thought of in some cases but to obtain moving boundaries, coupled differential equations in terms of time derivatives of moving boundaries have to be solved which is possible only numerically. However, we are not aware of such analytical-numerical solutions. For some more references on such problems refer [78].

A heat source or a heat front in [444] is moving in an infinite region,  $-\infty < x < \infty$ , with constant velocity  $V$ . The heat source need not be situated at the phase-change boundary. If heat source or sink is not strong enough then phase change from solid to liquid or from liquid to solid will not take place. So it is assumed that the heat source or sink is strong enough for phase change to take place. The heat equation in which latent heat is also accounted and boundary conditions considered in the problem formulation in [444] are given below in Eqs (12.2.606), (12.2.607)–(12.2.609).

$$\frac{\partial^2 T}{\partial x^2} - m^2 T - \frac{\rho l}{k} \frac{dx}{dt} \delta(x - R_i) = \frac{1}{k} \frac{\partial T}{\partial t}, \quad -\infty < x < \infty, t > 0, i = a, b, \quad (12.2.606)$$

$R_a$  and  $R_b$  are explained below,  $m$  is the heat dissipation coefficient and  $\delta(\cdot)$  is the Dirac-delta function. Thermo-physical properties in both the solid and liquid phases are the same and  $T(x, t)$  is the temperature in the whole medium  $-\infty < x < \infty$ ,  $t \geq 0$ . The moving heat front divides the medium into two branches,  $-\infty < x < Vt$  and  $Vt < x < \infty$ . In the branch  $-\infty < x < Vt$ , behind the front, the Dirac-delta term models a source front that locates the

interface position in that branch. The interface ahead of the moving source front is designated  $R_a$ , whilst that behind the front is designated  $R_b$ . This is so if the coordinate system is shifted to  $x = Vt$  or in other words fixed at the moving heat front. In the fixed original coordinates,  $R_a$  and  $R_b$  are the same. In the branch  $Vt < x < \infty$ , the delta function models a sink front that locates the interface position in that branch.  $R_a$  and  $R_b$  are defined as earlier for the branch  $-\infty < x < Vt$ .

The Stefan condition has been incorporated in Eq. (12.2.606) at the phase-change interface and in addition to this condition isotherm conditions are to be satisfied.

$$T(R_i, t) = T_m, \quad i = a, b. \quad (12.2.607)$$

Eq. (12.2.606) is to be solved subjected to the conditions given below.

$$T(x, 0) = 0 < T_m; \quad T(-\infty, t) = \text{finite}, \quad t \geq 0; \quad T(\infty, t) = 0, \quad t > 0, \quad (12.2.608)$$

$$T(Vt, t) = T_0; \quad \text{or} \quad -K \left[ \frac{\partial T}{\partial x}(Vt^+, t) - \frac{\partial T(Vt^-, t)}{\partial x} \right] = q^+ + q^- = q. \quad (12.2.609)$$

Quasisteady-state solution of this problem has been obtained by fixing the coordinate system at  $x = Vt$ . To distinguish the interfaces in this case they are denoted by  $R_{0a}$  and  $R_{0b}$ . For the solution in the range  $Vt < x < \infty$ , the coordinate  $\xi = x - Vt$  is introduced and for the solution in the range  $-\infty < x < Vt$ , the coordinate  $\eta = Vt - x$  is introduced and finally the quasisteady solutions in the two branches are expressed in the same range 0 to  $\infty$ . In the  $(\xi, \eta)$  coordinates the problem formulation for quasisteady solution is given below.

$$\frac{d^2 T}{d\xi^2} + (V/k) \frac{\partial T}{\partial \xi} - m^2 T = \frac{\rho V l}{K} \delta(\xi - R_{0a}), \quad 0 < \xi < \infty. \quad (12.2.610)$$

$$\frac{d^2 T}{d\eta^2} + (V/k) \frac{\partial T}{\partial \eta} - m^2 T = \frac{\rho V l}{K} \delta(-\eta + R_{0b}), \quad 0 < \eta < \infty. \quad (12.2.611)$$

$$T(0) = T_0 \text{ if } T_0 \text{ is given or } \frac{dT(0)}{d\xi} = -\frac{q^+}{K}, \quad \frac{dT(0)}{d\eta} = -\frac{q^-}{K}, \text{ if } q \text{ is given,} \\ q = q^+ + q^-. \quad (12.2.612)$$

$$T(\xi = \infty) = 0; \quad T(\eta = \infty) = \text{finite}; \quad T(R_{0i}) = T_m, \quad i = a, b. \quad (12.2.613)$$

The notation in Eq. (12.2.612) means that derivatives of temperature are evaluated at  $\xi = 0$  and  $\eta = 0$ . For the solution, Laplace transform is used. Delta functions are expanded in terms of *Heaviside functions* and Laplace transform is applied [368]. The solution of  $T(\eta)$  in the region  $0 < \eta < \infty$  is changed from  $\eta$  to  $(-\xi)$ ,  $R_{0b}$  to  $(-R_{0b})$  and  $\frac{dT(0)}{d\eta}$  to  $-\frac{dT(0)}{d\xi}$ . This is done to obtain solutions of two branches in the region  $0 < x < \infty$ .  $T(\xi)$  is obtained in four regions,  $\xi < R_{0b}$ ,  $R_{0b} < \xi < 0$ ,  $\xi > R_{0a}$  and  $0 < \xi < R_{0a}$ . For further details of solution refer [444] as their presentation here will occupy lot of space. Twelve cases have been discussed that cover constant temperature or flux conditions imposed at the moving heat front and the position of the interface with respect to heat sources, etc. In eight cases explicit solutions have been derived.

An quasisteady-state analytical solution has been obtained in [445] for a problem in which phase change takes place due to internal heat sources in a long cylinder of radius  $r_0$ . The energy equations in transient state are

$$\left. \begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} \left( Kr \frac{\partial T_i}{\partial r} \right) + \bar{q} &= \rho c \frac{\partial T_i}{\partial t}, \quad i = S, L, \quad 0 \leq r \leq r_0, \quad t > 0, \\ T_S(r_0) &= T_0 < T_m, \end{aligned} \right\} \quad (12.2.614)$$

where  $r$  is the radial cylindrical coordinate,  $\bar{q}$  is the volumetric heat generation term and other notations are the same as defined in this book. It is easy to obtain  $T_L(r, t)$  and  $T_S(r, t)$  in the regions  $0 \leq r \leq S(t)$  and  $S(t) \leq r \leq r_0$ , respectively, in the quasisteady-state case. Calculate their derivatives as given below in which  $r = S(t)$  is the phase-change boundary,  $S(0) = r_0$ .

$$\left. \frac{dT_L}{dr} \right|_{r=S(t)} = -\frac{\bar{q}S(t)}{2K_L}; \quad \left. \frac{dT_S}{dr} \right|_{r=S(t)} = -\frac{\bar{q}S}{2K_S} + \frac{(S^2 - r_0^2)\bar{q} + 4K_S(T_m - T_0)}{4K_S(\ln S - \ln r_0)S}. \quad (12.2.615)$$

On substituting these temperature derivatives in the Stefan condition such as the one in Eq. (12.2.77), a differential equation for  $r = S(t)$  is obtained which can be solved only numerically. The above equations are not scaled.

Numerical solution of a two-dimensional two-phase cylindrical problem in  $(r, z)$  coordinates by considering coupled temperature and flow equations in  $r$ - and  $z$ -directions has also been obtained in [445]. If Stefan number  $C(T_m - T_0)/l$  is much smaller compared to unity then quasisteady-state solution and numerical solutions are in good agreement. The scale analysis of parameters shows the existence of four separate regimes in which four different parameters are dominating and different types of melt formations are observed. They provide information about the formation of convection cells in the liquid region. The scaled heat source was taken about 400 times of Stefan number. At the top of cylinder of finite height, a small recirculation zone is formed which becomes more pronounced for large Rayleigh number.

A solid rectangular block of height  $a$  and width  $b$  is melting due to heat applied on one of the vertical sides by prescribing constant heat flux. The formulation considered in [446] for melting problem is as follows.

Let  $x$  coordinate be along the width of the block which is taken horizontally,  $0 \leq x \leq b$ , and  $y$  coordinate be taken vertically,  $0 \leq y \leq a$ . Heat flux is applied at the vertical side  $x = 0$  and there is no heat loss from other boundaries. The solid block is initially at the melting temperature  $T_m$ . Volume change due to different densities is negligible but this gives rise to natural convection in the liquid as a result of which the width of melted portion towards the top of the block is more than at the bottom. The solid-liquid phase-change boundary is taken as slanted but straight. If  $\phi$  is the angle which this slanted boundary makes with the  $y$ -axis and  $x = S(y, t)$ ,  $S_0(0) = 0$ , is the moving boundary then we have

$$S(y, t) = S_0(t) + y \tan \phi, \quad t \geq 0, \quad 0 \leq y \leq a; \quad \text{on } y = 0, S = S_0(t). \quad (12.2.616)$$

The melting takes place in two stages, before and after the contact of the interface  $S(y, t)$  with the top right corner of the enclosure. If only conduction is the mode of heat transfer, which is so initially, then the heat balance results in the following equation provided there is no heat loss from other boundaries.

$$\int_0^a q dy = \int_0^a l \rho_S \frac{\partial S}{\partial t} dy, \quad l \text{ is latent heat of fusion, } \rho \text{ is density,} \quad (12.2.617)$$

$q$  is the heat flux applied at  $x = 0$ ,  $0 \leq y \leq a$ . The effect of natural convection is generally more at the top of the container than at the bottom. Therefore to determine  $S_0(t)$ , it can be

assumed that  $S_0(t) = 2\delta\sqrt{k_L t}$ ,  $\delta > 0$  is an unknown constant. For Stefan numbers higher than 0.25,  $\delta$  can be evaluated using the following correlation which has been obtained from experiments.

$$\delta = (St_e)^{0.53} / ((\rho_S/\rho_L)\sqrt{\pi}), \quad St_e \text{ is Stefan number.} \quad (12.2.618)$$

Stefan number is the ratio of latent heat and sensible heat. Using Eqs (12.2.617), (12.2.618), we get

$$\frac{qa}{l\rho_S} = \delta\sqrt{\frac{k_L}{t}}a + \frac{a^2}{2\cos^2\phi} \frac{d\phi}{dt}, \quad \phi = \tan^{-1} \left( \frac{2qt}{al\rho_S} - \frac{4\delta\sqrt{k_L t}}{a} \right). \quad (12.2.619)$$

As mentioned earlier, total melting takes place in two stages. First, the slanting interface reaches the point  $(x = b, y = a)$  with  $\phi$  depending on  $S_0(t)$  as given below. After that the melting continues with the top of the slanted side moving down towards  $(x = b, y = 0)$  and the other lower end sliding on side  $y = 0$  towards  $x = b$ . The melted fractions in these two stages denoted by  $F_{ab}$  and  $F_b$  are taken as given below.

$$F_{ab} = (2S_0 + a \tan \phi)/2b, \quad F_b = (1 - (b - S_0)^2)/(2ab \tan \phi). \quad (12.2.620)$$

In the first stage of melting in  $F_{ab}$ ,  $\cos \phi = a/(b - S_0)$  and in the second case in  $F_b$  also  $\cos \phi = a/(b - S_0)$ .

This analytical solution compares well with the numerical and experimental results obtained by other research workers till the natural convection does not affect the melting which is much less at the bottom of the slab in the initial stages but is more during the later stage of melting and is towards the top of the slab.

In the problem considered in [447], a supercooled liquid having temperature  $T_0 < T_m$ , where  $T_m$  is the freezing temperature, at time  $t = 0$  occupies a semiinfinite region  $0 \leq x < \infty$ ,  $-\infty < y < \infty$ . The  $x$ -axis is taken horizontally and  $y$ -axis is taken vertically. The frozen solid is maintained at  $T_m$  and in the liquid region the fluid flow is considered as *stagnation-point flow*. Thermal field and flow field in the liquid are partially decoupled but not completely. If *Prandtl number* which is the ratio of kinematic viscosity and diffusivity is small and flow is stagnation-point flow then potential flow field approximation can be used. The velocity components  $u$  and  $v$  in the  $x$ - and  $y$ -directions, respectively, can be taken as

$$u = -2A(x - S(t)), \quad v = 2Ay, \quad A \text{ is constant potential flow strain.} \quad (12.2.621)$$

If the solid-liquid interface  $x = S(t)$  is taken planar then  $v = 0$ . Further if  $\xi = x - S(t)$ , i.e. the moving coordinates are used then in the quasisteady-state case, temperature and moving boundary can be easily determined using the procedure developed in the solutions of problems considered earlier. To determine  $x = S(t)$ , the following differential equation results which is obtained after determining steady-state temperature and using Stefan condition in the one-phase problem

$$\dot{S} \exp(\dot{S}/2)^2 \operatorname{erfc} \dot{S}/2 = 2Ste/\sqrt{\pi}, \quad \dot{S} = \frac{dS}{dt}, \quad Ste \text{ is Stefan number.} \quad (12.2.622)$$

It is clear from Eq. (12.2.622) that  $S(t)$  should be proportional to  $t$  so that  $S$  is constant. This solution can be considered a long-time solution. Only Stefan number is scaled in the above discussion by the authors.

A short-time solution using similarity variable  $\eta = x/(2\sqrt{t})$ , taking  $S(t) = 2\delta\sqrt{t}$  and neglecting time derivative of temperature can be obtained easily and unknown  $\delta$  can be obtained by solving a transcendental equation obtained for it. The authors call this solution an instantaneous solution.

For numerical solution, only one-dimensional problem in  $x$ -coordinate was considered with transient heat conduction and  $u$  as given in Eq. (12.2.621). By using a trigonometric transformation the semiinfinite region is transformed to a finite closed interval  $[0.5, 1]$ . Long-time and short-time solutions have been compared with the numerical solution and the effect of Stefan number has been analysed.

In the following problem, a simple and very restricted analytical solution is obtained which is based on the suitable scalings of parameters. The formulation of a solidification problem in a rectangular enclosure has been considered in [448] with coupled transient heat flow and transient fluid flow. The  $x$ -axis is along horizontal direction,  $0 \leq x \leq W$  and  $y$ -axis is taken vertically,  $0 \leq y \leq H$ .  $W$  and  $H$  are the width and height of the container. On  $x = 0$ , a temperature  $T_0 < T_m$  is prescribed and on the wall  $x = W$ , temperature  $T_h > T_m$  is prescribed. The faces  $y = 0$  and  $y = H$  are insulated.

The two-dimensional formulation of solidification problem with coupled heat transfer and fluid flow equations in [448] can be obtained if they are considered as a particular case of general equations derived in Section 1.4.7 and using Eq. (2.4.16) together with appropriate boundary and initial conditions. Suitable scaling of equations results in six dimensionless parameters, viz. Rayleigh number  $R_a$  (product of Grashof number and Prandtl number), Stefan number  $St_e$ , dimensionless diffusivity  $k$ , Prandtl number  $P_r$ , ratio of density of solid and density of liquid  $\rho_S/\rho_L$  and a dimensionless coefficient  $\lambda$  which is  $(St_e \cdot (k_S/k_L) \cdot (C_L/C_S) \cdot (\rho_L/\rho_S))$ .  $C$  is specific heat at constant pressure. It is well known that when Stefan number  $St_e \ll 1$ , the transient terms can be dropped in both heat conduction equation and Navier-Stokes equations and time occurs only in the Stefan condition. If it is further assumed that  $\lambda \ll 1$  then  $\nabla\theta_L \cdot \vec{n} \sim (R_a)^{1/4}$  at the moving interface  $x = S(y, t)$  in the Stefan condition.  $S(y, t)$  separates solid region  $0 < x < S(y, t)$  from the liquid region  $S(y, t) < x < W$  and  $\vec{n}$  is the outward normal vector.  $\theta_L(x, y)$  is the scaled liquid temperature. Further simplification of the equations results by another scalings given below.

$$x = R_a^{-1/4}X, \quad S(y, t) = R_a^{-1/4}\bar{S}(y, \tau), \quad t = R_a^{-1/2}\tau. \quad (12.2.623)$$

On neglecting lower-order terms the heat equation for solid becomes one-dimensional in temperature (steady-state) and it can be easily solved and solution is given below.

$$\frac{\partial^2 \theta_S}{\partial x^2} = 0, \quad \theta_S = \frac{X}{\bar{S}(y, t)}, \quad S(y, 0) = \bar{S}(y, 0) = 0. \quad (12.2.624)$$

If it is further assumed that the width of the solid layer is much less as compared to the width of the container then boundary conditions can be effectively applied at  $X = 0$  or  $x = 0$ . On using Stefan condition at  $x = S(y, t)$  and not at  $X = \bar{S}(y, t)$ , we have

$$\frac{\partial S}{\partial t} = \frac{1}{S} - k_L \left( \frac{\partial \theta_L}{\partial x} \right)_{x=0}, \quad S(y, 0) = 0, \quad (12.2.625)$$

$\left( \frac{\partial \theta_L}{\partial x} \right)_{x=0}$  is the liquid temperature derivative calculated at  $x = 0$  and it is a function of  $y$  only. Under the above assumptions the liquid temperature  $\theta_L$  to the leading order can be treated

as that of steady-state problem with natural convection in the rectangular enclosure without solidification. The derivative of  $\theta_L$  is to be calculated from this solution of steady-state problem and substituted in Eq. (12.2.625).

For determining average solidified thickness refer [448]. Eq. (12.2.625) has to be modified suitably as given in Eq. (12.2.626) to consider time-dependent cooling at  $x = 0$ . In this case  $\lambda \sim 1$ . After suitable scaling of  $\theta_S$  and defining  $\theta_{\text{cold}}$  suitably (refer [448]),  $\theta_S$  is taken as

$$\theta_S = (1 - \theta_{\text{cold}}(y, t))x/S(y, t) + \theta_{\text{cold}}(t). \quad (12.2.626)$$

Suppose  $\theta_{\text{cold}}(y, t) = \epsilon \bar{\theta}_{\text{cold}}(y, t)$ , and for  $\epsilon \ll 1$ ,  $\bar{\theta}_{\text{cold}} \sim O(1)$ . Let  $S = S_0(y, t) + \epsilon S_1(y, t) + O(\epsilon^2)$ . If  $\bar{\theta}_{\text{cold}}$  is given then  $S_0$  and  $S_1$  can be determined from Eq. (12.2.627). By taking  $\bar{\theta}_{\text{cold}} = \sin wt$ ,  $S_0$  can be obtained from Eq. (12.2.627) and  $S_1$  can be obtained in terms of  $S_0$ .

$$\frac{\partial S}{\partial t} = \frac{(1 - \theta_{\text{cold}}(y, t))}{S} - k_L \left. \frac{\partial \theta_L}{\partial x} \right|_{x=0}. \quad (12.2.627)$$

Eq. (12.2.627) can be derived using Eq. (12.2.626) and Stefan condition given in Eq. (12.2.625) provided appropriate changes are made. In the initial formulation heat and mass transfer in Navier-Stokes equations are coupled (for complete formulation see [448]). By appropriate scalings of parameters and variables, finally the heat transfer is decoupled from fluid motion and in some cases analytical-numerical solution has been obtained.

For obtaining finite element numerical solution of the original transient problem *Cosmol Multiphysics Software* was used. Considerable numerical work has been reported.

The effect of solidification on surface tension-driven convection has been analysed in [449] by perturbing steady-state and quasisteady-state one-dimensional solidification problems. Linear stability analysis has been employed. Perturbation is imposed on the original mean state and boundary conditions and the equations are linearized resulting in a stationary solidification problem. Surface tension enters into the formulation of perturbation problem and in its solution when the boundary conditions imposed at the free surface of the liquid through the balancing of tangential forces are considered. Surface tension is taken as a linear function of liquid temperature. The conditions for the onset of convection in the liquid phase are determined through a linear stability analysis. The destabilizing effect of various parameters on the solid-liquid front has been analysed. For *dispersion relation* and discussion see [449] as stability analysis is not a part of this chapter. The analytical solutions of two steady-state problems on which stability analysis is done are discussed below.

Let the one-dimensional region  $0 \leq x \leq S + d$  in the steady-state case be occupied by solid and liquid with solid occupying the region  $0 \leq x \leq S$  and liquid occupying the region  $S \leq x \leq S + d$ .  $x = S(t)$  is phase-change interface which has acquired the steady state. At  $x = 0$ ,  $T_S(x) = T_1 < T_m$  and at  $x = S + d$ ,  $T_L(x) = T_2 > T_m$ . Out of  $S$  and  $d$  one of them is unknown. The solution of this problem is very easy and it is not presented here. Thermophysical parameters are taken constant. The parameter  $l/C_S(T_2 - T_1)$  was found to have a major effect on the *neutral stability curves*.

In the quasisteady-state solution one important assumption is made that the melt depth in solidification is always maintained constant at  $d$  by continuously pouring liquid into the container. Let the  $x$ -axis be taken vertically upwards and in the horizontal direction the extent be infinite. Originally the liquid is occupying the region  $0 \leq x \leq d$ . At  $x = d$ ,  $T_L = T_2 > T_m$

and at  $x = 0$ , flux  $q_0$  is prescribed which is cooling the liquid. The velocity of solid-liquid interface is taken to be a constant  $V_p$  but unknown. If the coordinate system is attached to the moving interface  $x = S(t)$  then in the new coordinate system  $\xi = x - V_p t$ ,  $x = S(t)$  implies  $\xi = 0$ . As the depth of the liquid is maintained at  $d$ , the upper boundary of liquid is  $\xi = d$ . The mass balance at  $x = S(t)$  gives  $u = V_p \left( \frac{1}{\lambda} - 1 \right)$ ,  $\lambda = \rho_L / \rho_S$ ,  $u$  is the basic flow velocity of the liquid which is taken in the direction of propagation of the interface. As both  $u$  and  $V_p$  are in the same direction, the velocity of the liquid in  $\xi$  coordinate is the sum of flow velocity  $u$  and the velocity  $V_p$  so that the liquid region has a flow velocity  $V_p / \lambda$  and the solid region acquires a velocity  $V_p$ . The solution of this problem for steady-state temperatures can be easily obtained and it will not be discussed here.  $V_p$  can be determined on satisfying Stefan condition. The solution given in [449] for liquid satisfies the boundary conditions at  $\xi = 0$  and at  $\xi = d$ . The given solid temperature satisfies the isotherm condition at  $\xi = 0$ , i.e.  $T_S = T_L = T_m$  and the solid temperature satisfies the flux condition at  $\xi = -d$ .

If it is assumed that the solid formed is being pulled in the negative  $x$ -direction then this problem has some similarity with continuous casting model. The parameter  $l/C_S(T_2 - T_m)$  was found to have a major effect on neutral stability curves.

Effect of inequality in the densities of solid and liquid phases in a one-dimensional solidification problem of a supercooled liquid on the stability of a planar phase-change interface has been analysed in [450] using the classical linearized theory of hydrodynamic stability. The initial amplitude of the perturbation is supposed to be infinitesimally small. The region occupied initially by solid is  $-\infty < x \leq 0$  and by liquid,  $0 \leq x < \infty$ . The volume change due to change in densities of solid and liquid on solidification results in a velocity  $V$  of the liquid region. The solid phase remains at rest.  $V$  is constant as  $\partial V / \partial x = 0$  from the continuity equation. Let the phase-change boundary  $x = S(t)$  move with a velocity  $v$ . If the original laboratory coordinate system is changed to moving coordinate system by shifting it to the interface with the help of the transformation  $\bar{x} = x - vt$ ,  $v$  is a constant, then the steady-state temperature equations become as follows.

$$k_L \frac{\partial^2 \theta_L}{\partial \bar{x}^2} + (v - V) \frac{\partial \theta_L}{\partial \bar{x}} = 0; \quad k_S \frac{\partial^2 \theta_S}{\partial \bar{x}^2} + v \frac{\partial \theta_S}{\partial \bar{x}} = 0; \quad \theta = \frac{(T - T_m)}{l} C_L. \quad (12.2.628)$$

$\theta_S$  is the scaled temperature of the solid occupying  $0 < x < S(t)$  and  $\theta_L$  is the scaled temperature of the liquid region  $S(t) < x < \infty$ . The moving interface is now  $\bar{x} = 0$ .

The mass, momentum and energy balance across the interface  $\bar{x} = 0$  result in the following equations.

$$\rho_L(v - V) = \rho_S v \quad \text{or} \quad V = (\rho_L - \rho_S)v / \rho_L. \quad (12.2.629)$$

$$p_S - p_L = \rho_L v(v - V) \quad \text{or} \quad p_L - p_S = \rho_S(\rho_L - \rho_S)v^2 / \rho_L, \quad (12.2.630)$$

$$-\rho_S E_S v - \frac{k_S l}{c_L} \left( \frac{\partial \theta_S}{\partial \bar{x}} \right)_{\bar{x}=0} = \rho_L E_L (V - v) + p_L V - \frac{k_L l}{C_L} \left( \frac{\partial \theta_L}{\partial \bar{x}} \right)_{\bar{x}=0}. \quad (12.2.631)$$

In Eqs (12.2.629)–(12.2.631),  $p$  is pressure,  $E_S$  and  $E_L$  are the specific total energies of the solid and liquid regions, respectively.  $E_S = e_S$  and  $E_L = e_L + \frac{1}{2}V^2$  where  $e_S$  and  $e_L$  are the specific internal energies.  $e_S = e_0 - l - p_S / \rho_S$  and  $e_L = e_0 - p_L / \rho_L$ , where  $e_0$  is some reference internal energy related to equilibrium phase-change temperature  $\theta = 0$ . The condition of local thermodynamic equilibrium at  $\bar{x} = 0$  requires  $e_S = e_L = 0$  as  $\theta = 0$  at  $\bar{x} = 0$ .

Finally, the solution of the following system of equations is to be obtained.

$$\frac{\partial^2 \theta_L}{\partial \bar{x}^2} + (v - V) \left/ k_L \frac{\partial \theta_L}{\partial \bar{x}} \right. = 0, \bar{x} \geq 0; \quad \frac{\partial^2 \theta_S}{\partial \bar{x}^2} + \frac{v}{k_S} \frac{\partial \theta_S}{\partial \bar{x}} = 0, \bar{x} \leq 0. \quad (12.2.632)$$

$$\left[ 1 - (\rho_L^2 - \rho_S^2) v^2 / (2 \rho_L l) \right] v = k_L \rho_L / \rho_S \left[ \frac{k_S}{k_L} \left( \frac{\partial \theta_S}{\partial \bar{x}} \right)_{\bar{x}=0} - \left( \frac{\partial \theta_L}{\partial \bar{x}} \right)_{\bar{x}=0} \right]. \quad (12.2.633)$$

The solution of Eqs (12.2.632), (12.2.633) is given below.

$$\theta_L = A \left( e^{-2\bar{x}/b_L} - 1 \right), \quad A = 1 - \left( \rho_L^2 - \rho_S^2 \right) v^2 / (2 \rho_L l), \quad (12.2.634)$$

$$b_L = 2k_L / (v - V), \quad \bar{x} \geq 0; \quad \theta_S = 0, \quad \bar{x} \leq 0. \quad (12.2.635)$$

For typical experimental situations  $A \geq 0$ . Therefore  $\theta_L(\infty) = -A$  and liquid is supercooled. The linear stability analysis showed that the effect of density difference between  $\rho_S$  and  $\rho_L$  on the critical wavelength is not negligible. In the perturbation analysis Gibbs-Thomson condition at the interface was taken into consideration.

An analytical procedure to obtain approximate solution of a phase-change problem with the help of an analytical solution of a corresponding pure heat conduction problem without phase change has been suggested in [451]. The method is different but has limited applications. Some simple test cases have been taken up such as Neumann solution in which latent heat is released at the phase-change boundary and the corresponding heat conduction problem with uniform properties throughout the region without phase change. From the Neumann solution without natural convection which can be obtained from Eqs (1.3.11)–(1.3.16), heat flux at  $x = 0$  can be calculated as temperature is prescribed at  $x = 0$ . The corresponding solution of a heat conduction problem without phase change and uniform properties throughout  $0 \leq x < \infty$  can be easily obtained and flux at  $x = 0$  can be calculated. On comparing the fluxes of these two problems at  $x = 0$  which are with and without phase change, respectively, an effective formula can be developed to obtain the heat flux to be applied in the Neumann problem (the solution of Neumann problem is for temperature prescribed condition) to obtain an approximate Neumann solution for only phase-change boundary but not temperatures. For example, in the solution of pure heat conduction problem, if  $K = K_S$  is taken and an effective value of thermal diffusivity is taken as

$$k_{\text{ef}} = k_S \left[ (T_0 - T_m) \text{erf}(\delta) / (T_m - T_b)^2 \right], \quad (12.2.636)$$

then flux at  $x = 0$  in the Neumann solution is obtained. Effective value of specific heat can also be similarly calculated. However, temperature field will be in error.

If the interest is in the accuracy of phase-change boundary then  $x = x_0$  can be obtained from the solution of pure heat conduction problem for which  $T = T_m$ . By comparing  $x_0$  and  $S(t)$  at the same value of time in Neumann solution an effective value of thermal diffusivity can be obtained. However, fluxes in the two cases of zero latent heat and nonzero latent heat will not match. Analytical solutions obtained with the help of effective values have been compared for a cylindrical problem and some other problems also. We have discussed this procedure here from the point of view of having a different approach to obtain approximate solutions.

The problem considered in [452] and discussed below also does not have general applicability but is presented here as it suggests a different procedure for approximate solution.



Solidification in a radially symmetric spherical shell is considered in which heat transport from liquid at the phase-change boundary is ignored. In this case energy balance at the interface  $r = S(t)$ ,  $r \leq S(0)$ , will result in an inequality. Following the notations used earlier in this book, we have

$$\left| 4\pi r^2 \left( -K_S \frac{\partial T_S}{\partial r} \right) \right| \geq \left| 4\pi S^2 \rho_S l \frac{dS}{dt} \right|, \text{ at } r = S(t), S(0) = S_0. \quad (12.2.637)$$

The l.h.s. in Eq. (12.2.637) is the heat conduction rate of solid at the radius  $r$  and if the l.h.s. is less than r.h.s. then solidification will not progress. If the liquid is at the freezing temperature then the inequality becomes an equality. Temperature of the solid at  $r = S(t) = T_m$ , is a constant. As an exact solution of the spherical shell inward solidification problem is not available, the authors have tried to obtain an approximate solution satisfying the heat equation, the condition Eq. (12.2.637) and isotherm condition at  $r = S(t)$ . The equation for  $S(t)$  is chosen in the form

$$S(t) = S(0) + S_1^2/S_2(1 - \sqrt{1 - 2S_2t/S_1}), \quad (12.2.638)$$

$S(0)$ ,  $S_1$  and  $S_2$  are constants and are the radius, velocity and acceleration of  $S(t)$ , respectively, at time  $t = 0$ . It is assumed that  $d^2S/dt^2 = A(dS/dt)^3$ ,  $A$  is constant,  $S_2 = AS_1^3 \neq 0$ . If  $S_2 = 0$  then  $S(t) = S_0 + S_1t$ , and for some special cases closed-form solution is possible. The temperature solution satisfies the energy equation, isotherm condition and inequality condition. Boundary condition cannot be arbitrarily prescribed at  $r = S_0$  if  $S_0 = S(t)$  is known at  $t = 0$  as there is no arbitrary constant in the temperature solution obtained in [452]. Temperature solution is of very special type and not presented here. There is no systematic method involved in determining it.

## 12.3 ANALYTICAL SOLUTIONS OF STEFAN AND STEFAN-LIKE PROBLEMS IN SERIES FORM

Series solutions have been considered in earlier sections and will be considered in the later sections also. What matters in their placing in different sections is the context in which they arise. Series solutions play an important role in the solutions of variety of problems of mathematical physics. If a series solution is obtained for a Stefan problem with kinetic condition then the context of kinetic condition is dominating and the solution is accommodated in that section which deals with the solutions of Stefan problems with kinetic condition. In the present section it is the method which is important and focus is on presenting solutions with variety in the method of solution, no matter whether the work is new or old.

### 12.3.1 Short-Time Analytical Solutions of Classes I, II and III Problems

Short-time solutions are also series solutions having some characteristic features different from many other series solutions. Several methods have been developed to obtain short-time solutions. Our interest here is in those methods which have wider applicability in terms of geometry, boundary and initial conditions. The method of solution and its characteristic features can be best explained with the help of solution procedure and the obtained solution and this is being done in what follows by discussing these methods one by one.

### Solutions Obtained by Using Embedding Techniques

Boley [244] proposed an analytical method known as 'Boley's embedding technique' and obtained a short-time solution for a one-dimensional two-phase solidification problem in the region  $0 \leq x < \infty$ . A one-phase ablation problem has also been considered in [244] but it seems more appropriate to demonstrate the method with the help of a two-phase problem. After discussing the formulation and the solution, the embedding technique will be better understood but at present a brief description of the technique will be sufficient to engage attention. In a solidification problem (melting problem is mathematically analogous), at any time after the solidification has started, both solid and liquid regions occupy separately only a portion of the region which is occupied by the liquid initially at time  $t = 0$ . However, in the method of solution by embedding technique both the solid and liquid regions are embedded in the original region such as  $0 \leq x < \infty$ . These are fictitious or imaginary extensions of solid and liquid regions once solidification has started. To satisfy the boundary conditions at the phase-change boundary, apply a fictitious flux at  $x = 0$  for the liquid region and prescribe fictitious initial temperature for the solid region. Flux prescribed at  $x = 0$  for liquid is fictitious as flux is prescribed for the solid and similarly initial temperature for the solid is fictitious as initial temperature is prescribed for liquid. Both the fictitious quantities are unknowns and are to be determined. Remaining information about determining unknown quantities will be discussed after formulation of the problem.

In the liquid region for  $0 \leq t \leq t_m$

$$\left. \begin{aligned} k_L \frac{\partial^2 T_L}{\partial x^2} &= \frac{\partial T_L}{\partial t}, \quad 0 < x < \infty, \quad 0 < t \leq t_m, \quad T_L(x, 0) = T_1 > T_m; \\ K_L \frac{\partial T_L}{\partial x} \Big|_{x=0} &= f_1(t). \end{aligned} \right\} \quad (12.3.1)$$

$t_m$  is the time at which  $T_L(0, t_m) = T_m$  and  $T_m$  is the melting temperature.

Up to time  $t = t_m$ , it is a problem of pure heat conduction whose solution is easily available in text books and also given below.

$$\left. \begin{aligned} T_m &= T_1 + \frac{1}{K_L} \sqrt{\frac{k_L}{\pi}} \int_0^{t_m} f_1(t_m - t_1) t_1^{-1/2} dt_1, \\ T_L(x, t) &= T_1 + \frac{1}{K_L} \sqrt{\frac{k_L}{\pi}} \int_0^t f_1(t - t_1) t_1^{-1/2} e^{-\frac{x^2}{(4k_L t_1)}} dt_1. \end{aligned} \right\} \quad (12.3.2)$$

After  $t \geq t_m$ , the following problem is to be solved which is obtained after embedding the liquid region  $S(t) \leq x < \infty$  in the region  $0 \leq x < \infty$  which is a fictitious extension of the liquid region  $S(t) \leq x < \infty$  after solidification has started. Similarly the solid region  $0 \leq x \leq S(t)$  is embedded in the region  $0 \leq x < \infty$  which is a fictitious extension of the solid region,  $0 \leq x \leq S(t)$ .  $x = S(t)$  is the moving boundary. For  $t \geq t_m$ , the formulation is given below.

$$k_L \frac{\partial^2 T_L}{\partial x^2} = \frac{\partial T_L}{\partial t}, \quad 0 < x < \infty, \quad t \geq t_m; \quad T_L(x, t_m) = \bar{T}_L(x); \quad \bar{T}_L(0, t_m) = T_m. \quad (12.3.3)$$

$$T_L(S(t), t) = T_m; \quad K_L \frac{\partial T_L}{\partial t}(S, t) = Q_1(t) - \rho_S l \frac{dS}{dt}; \quad t \geq t_m. \quad (12.3.4)$$

$$k_S \frac{\partial^2 T_S}{\partial x^2} = \frac{\partial T_S}{\partial t}, \quad 0 < x < \infty, \quad t \geq t_m; \quad K_S \frac{\partial T_S}{\partial x}(0, t) = f_1(t), \quad t \geq t_m. \quad (12.3.5)$$

$$T_S(S(t), t) = T_m; \quad K_S \frac{\partial T_S}{\partial x}(S, t) = Q_1(t); \quad t \geq t_m. \quad (12.3.6)$$

$x = S(t)$  is the phase-change boundary and  $S(t_m) = 0$ . If embedding technique is not used then Stefan condition is

$$K_S \frac{\partial T_S}{\partial x}(S, t) - K_L \frac{\partial T_L}{\partial x}(S, t) = \rho_S l \frac{dS}{dt}, \quad l \text{ is latent heat.} \quad (12.3.7)$$

Eq. (12.3.7) is divided into two parts. One part has been written as  $K_S \frac{\partial T_S}{\partial x}(S, t) = Q_1(t)$  as in Eq. (12.3.6) and another part  $K_L \frac{\partial T_L}{\partial x}(S, t) = Q_1(t) - \rho_S l \frac{dS}{dt}$  is adjusted in Eq. (12.3.4) so that ultimately Eq. (12.3.7) holds.  $Q_1(t)$  is unknown but we shall see later that it will not be determined as it can be eliminated.

The extended liquid region  $0 \leq x < \infty$  should have a boundary condition at  $x = 0$  to complete the formulation. We prescribe a fictitious flux for liquid at  $x = 0$ . So in addition to conditions in Eqs (12.3.3), (12.3.4) one more condition at  $x = 0$  as given below should be satisfied.

$$K_L \left. \frac{\partial T_L}{\partial x} \right|_{x=0} = f_1(t) + Q_2(t), \quad Q_2(t) \text{ is unknown.} \quad (12.3.8)$$

Why  $f_1(t) + Q_2(t)$ ? We can prescribe any flux, say  $Q_3(t)$  without including  $f_1(t)$  but flux should be continuous at  $(x = 0, t = t_m)$  so it is better to write it as  $f_1(t) + Q_2(t)$ . Why not prescribe the temperature? We require heat input as the boundary conditions at  $x = S(t)$  are to be satisfied. Similarly initial temperature for the solid region  $0 < x < \infty$  should be prescribed as at  $x = 0$  flux is already prescribed. Therefore in addition to conditions in Eqs (12.3.5), (12.3.6), the following fictitious initial temperature should be prescribed for the embedded solid region to complete the formulation.

$$T_S(x, t_m) = \bar{T}_S(x), \quad \bar{T}_S(x) \text{ is unknown, } \bar{T}_S(0) = T_m. \quad (12.3.9)$$

For liquid, the temperature is continuing from  $t \geq 0$  but solidification starts only at  $t = t_m$ . Therefore the formulation of liquid region should finally include Eqs (12.3.1)–(12.3.4), (12.3.8) and the formulation of solid region should include Eqs (12.3.5)–(12.3.7), (12.3.9). By using *Duhamel's theorem* [24], the expressions of temperatures of liquid and solid regions satisfying their respective formulations can be written as follows for  $t \geq 0$  for liquid and for  $t \geq t_m$  for solid.

$$T_L(x, t) = T_1 + \int_0^t f_1(t - t_1) \frac{\partial T_{0L}}{\partial t_1}(x, t_1) dt_1 + \int_0^{t-t_m} Q_2(t - t_1) \frac{\partial T_{0L}}{\partial t_1}(x, t_1) dt_1. \quad (12.3.10)$$

$$\left. \begin{aligned} T_S(x, t) = & \int_{t_m}^t f_1(t - t_1) \frac{\partial T_{0S}}{\partial t_1}(x, t_1 - t_m) dt_1 \\ & + \int_0^\infty \bar{T}_S(x_1) \{G(x, t; x_1, t_m) + G_1(x, t; x_1, t_m)\} dx_1 \end{aligned} \right\}. \quad (12.3.11)$$

$T_{0L}(x, t)$  is the temperature of liquid occupying the region  $0 \leq x < \infty$ , whose initial temperature is zero and unit flux is applied at  $x = 0$ . If  $f_1(t) = Q_0$ , a constant, then the first term on r.h.s. in Eq. (12.3.10) can be evaluated.  $T_{0S}$  is the temperature of a solid occupying the

region  $0 \leq x < \infty$ , whose temperature is zero and unit flux is applied at  $x = 0$ .  $G(x, t; x_1, t_m)$  is the Green's function and  $G_1$  is the mirror image of  $G$  in the plane  $x = 0$ .  $T_L(x, t)$  and  $T_S(x, t)$  have to satisfy the conditions given in Eqs (12.3.4), (12.3.6) at  $x = S(t)$ , respectively. To obtain a short-time solution, it is required to make use of nondimensional notations. Some of the dimensionless notations used in [244] are given below.

$$\left. \begin{aligned} y &= t/t_m - 1, \quad y_1 = t_1/t_m, \quad F_1(y) = Q_1(t)/Q_0, \quad F_2(y) = Q_2(t)/Q_0, \\ P &= \sqrt{\pi} K_L (T_1 - T_m), \quad X = Q_0 x_1 / P, \quad \theta(x) = \bar{T}_S(x_1)/T_m, \\ m &= \sqrt{\pi} C_L (T_1 - T_m) / (2\rho_S l), \quad D = (k_L/k_S), \quad \bar{K} = (K_L/K_S) (T_1 - T_m)/T_m, \\ \xi(y) &= Q_0 S(t)/P, \quad \xi(0) = 0, \quad f_1(t) = Q_0, \quad \theta(0) = 1. \end{aligned} \right\} \quad (12.3.12)$$

On satisfying the conditions prescribed in Eqs (12.3.4)–(12.3.6), we get the following equations

$$\int_0^y F_2(y - y_1) e^{-\xi^2/y_1} \frac{dy_1}{y_1^{1/2}} = 2 \left[ 1 - \sqrt{\pi(1+y)} \operatorname{ierfc} \frac{\xi}{\sqrt{1+y}} \right], \quad (12.3.13)$$

$$\xi(y) \int_0^y F_2(y - y_1) e^{-\xi^2/y_1} \frac{dy_1}{y_1^{1/2}} = \sqrt{\pi} \left[ \operatorname{erf} \frac{\xi}{\sqrt{1+y}} - \frac{2}{m} \frac{d\xi}{dy} + F_1(y) - 1 \right], \quad (12.3.14)$$

$$\int_0^\infty \theta(x) \left[ e^{-(\xi-X)^2 D/y} + e^{-(\xi+X)^2 D/y} \right] dx = \sqrt{\frac{\pi y}{D}} \left\{ 1 + \bar{K} \sqrt{\frac{\pi y}{D}} \operatorname{ierfc} \frac{\xi \sqrt{D}}{\sqrt{y}} \right\}, \quad (12.3.15)$$

$$\begin{aligned} & \int_0^\infty \theta(x) \left[ (\xi - x) e^{-(\xi-x)^2 D/y} + (\xi + x) e^{-(\xi+x)^2 D/y} \right] dx \\ &= \frac{\bar{K} \pi}{2} \left( \frac{y}{D} \right)^{3/2} \left\{ \operatorname{erfc} \frac{\xi \sqrt{D}}{\sqrt{y}} - F_1(y) \right\}. \end{aligned} \quad (12.3.16)$$

To obtain a short-time solution, we are guided by Neumann solution which indicates that  $\xi(y)$  could be written in a series of powers of  $y^{1/2}$  such as

$$\xi(y) = a_1 y^{1/2} + a_2 y + a_3 y^{3/2} + a_4 y^2 + \dots; \quad \xi(0) = 0; \quad y \ll 1. \quad (12.3.17)$$

The constants  $a_i$  for  $i = 1, 2, \dots$  are unknown constants to be determined. For  $y \ll 1$ , Eq. (12.3.13) suggests that the leading term in  $F_2(y)$  should be  $-(2/\pi)y^{1/2}$  and  $F_2(y)$  can be expanded in a series given below provided the unknown coefficients can be determined.

$$F_2(y) = -(2/\pi)y^{1/2} + b_1 y + b_2 y^{3/2} + \dots; \quad y \ll 1. \quad (12.3.18)$$

The leading term  $-(2/\pi)y^{1/2}$  is an a priori estimate but it can be verified from the solution that it is a correct guess.  $\theta(x)$  can be taken in the following series form as it is a function of space coordinate.

$$\theta(x) = 1 + p_1 x + p_2 x^2 + \dots; \quad \theta(0) = 1, \quad p_1, p_2, \dots \text{ are unknown.} \quad (12.3.19)$$

From Eq. (12.3.14) and r.h.s. of Eq. (12.3.16), for  $y \ll 1$ , we obtain, respectively, first and second equations in Eq. (12.3.20).

$$\frac{d\xi}{dy} = \frac{m}{\pi} y^{1/2} + \frac{m}{2} [F_1(y) - 1], \quad 1 - 2\xi \sqrt{\frac{D}{\pi y}} - F_1(y) = 0. \quad (12.3.20)$$

Note that the l.h.s. of Eq. (12.3.16) is zero for  $\theta(x) = 1$ . If from first equation in Eq. (12.3.20),  $F_1(y)$  is substituted in the second equation in Eq. (12.3.20) then after considering lowest-order terms in  $y$ , we get

$$\frac{d}{dy}\xi(y) = \frac{m}{\pi}y^{1/2}, y \ll 1 \text{ or } \xi(y) = \frac{2m}{3\pi}y^{3/2}, y \ll 1. \quad (12.3.21)$$

In a similar way comparing different powers of  $y^{1/2}$  on both sides of Eqs (12.3.13)–(12.3.16), other unknown coefficients in  $\xi(y)$ ,  $F_2(y)$  and  $\theta(x)$  can be obtained.  $F_1(y)$  can always be eliminated from equations used for determining unknown coefficients and no series expansion for  $F_1(y)$  is required.

Terms up to  $y^3$  in  $\xi(y)$ , up to  $y^2$  in  $F_2(y)$  and up to  $x^3$  in  $\theta(x)$  have been reported in [244]. It is clear from the above discussion regarding problem formulation and its solution that there are some essential requirements such as: (i) only flux boundary conditions can be prescribed; (ii) temperatures  $T_{0L}$ ,  $T_{0S}$  and Green's function should be available together with their short-time solutions; (iii) some a priori estimates of unknown functions could be required and their correctness should be established. Some plus points are (i) up to the level of Eqs (12.3.13)–(12.3.16), the formulation is exact; (ii) some one-phase radially symmetric problems in radial coordinate  $r$  and cylindrical coordinates  $(r, z)$  have been studied; (iii) class II and III problems as explained in Section 3.2.2 have been studied in the context of one-phase ablation problems; (iv) the idea of using fictitious flux is pioneering and has motivated further research which will be discussed next; (v) there is no restriction on the values of thermo-physical parameters but the solution obtained is valid for very short real time and is good mostly for starting numerical solutions and checking initial results; (vi) Neumann solution and some other similarity solutions can be reproduced using Boley's embedding technique.

It is not possible to give here the long list of publications of Boley and coworkers. Many of the publications are reported in the references of an article 'An applied overview of moving boundary problems' in the publication [453]. The problem considered above in Eqs (12.3.1)–(12.3.9) is a class I problem. A one-phase two-dimensional ablation problem of class I has been studied in [60] by considering a semiinfinite strip of finite width. Examples of problems of classes I, II and III have been demonstrated with the help of application of suitable flux in [60]. For the characteristics of class I, II and III problems see page 68.

In a class III problem studied in [454], melting is starting at a point  $(0, 0)$  in the two-dimensional region,  $0 \leq x < \infty$ ,  $-\infty < y < \infty$  due to a suitably prescribed flux. The melt spreads with time along the boundary  $x = 0$  and also progresses towards the interior  $x > 0$ . The problem considered is that of a one-phase ablation in which melt is removed as soon as it is formed. The method of solution is the same as described above but the solution requires extensive computations. This is because short-time behaviours of temperature solutions are required to obtain unknown quantities. The melt which spreads along  $x = 0$  is proportional to  $y^{1/2}$  and along  $x > 0$  it is proportional to  $y^{3/2}$ . A good knowledge of special functions is also required to obtain the leading terms in the growth along  $x > 0$  and spread along  $x = 0$ . However, there is great advantage in using fictitious flux and using Duhamel's theorem [24]. Although with much effort only the leading terms are obtained but the boundary conditions are taken care exactly in the application of Duhamel's theorem in this ablation problem.

A three-dimensional one-phase class III ablation problem was considered in [61] in the half space,  $x \geq 0$ ,  $-\infty < y < \infty$ ,  $-\infty < z < \infty$ . A one-phase ablation class III problem in a radially symmetric infinite cylindrical region was considered in [455]. Computations in

[60, 61, 455] which have been carried out are fairly difficult and extensive knowledge and behaviour of special functions are required.

In obtaining series expansions in time employed for obtaining phase-change interface along  $x = 0$  and along  $x > 0$ , and for fictitious flux at  $z = 0$ , closed-form analytical expressions were a priori assumed in [61] and their expansions were used. If  $Q(x, t)$  is the fictitious flux at  $z = 0$  and  $z = \xi(x, t)$  is the interface moving towards  $z > 0$ , then it was assumed in a planar two-dimensional problem that for  $t \ll 1$

$$Q(x, t) \propto t^{1/2} \left[ 1 - \left( \frac{x}{x_0(t)} \right)^2 \right]^{1/2}, \quad \text{and} \quad \xi(x, t) \propto t^{3/2} \left[ 1 - \left( \frac{x}{x_0(t)} \right)^2 \right]^{3/2}, \quad (12.3.22)$$

where  $x_0(t)$  is the growth along  $x = 0$ . For  $t \ll 1$ ,  $x_0(t)$  was found proportional to  $t^{1/2}$ .

Some sufficient conditions for the uniqueness of solution in the case of one-phase one-dimensional ablation problem were derived in [287]. Some results and references on upper and lower bounds on ablation rates can be found in [456].

Having obtained a short-time analytical solution of a two-phase one-dimensional radially symmetric cylindrical problem in [457] with the help of Boley's embedding technique, later on, Gupta developed a 'New embedding technique' and studied several class I, II and III problems with the help of a new embedding technique. To illustrate this new technique, we could have considered a simple one-dimensional problem but to emphasize that the new technique works with relative ease in obtaining solutions of problems in higher dimensions with multiple phases also, the following three-phase two-dimensional class I problem has been considered which is discussed in [458].

A superheated liquid occupying a radially symmetric cylindrical region  $0 \leq r \leq 1, |z| < \infty$  is in contact with a chilled mold at time  $t = 0$ . The mold and liquid have imperfect mold contact and the heat transfer coefficient is taken as a function of space and time.  $r = X(z, v)$  is phase-change boundary. The remaining information about the problem will be clear from the formulation given below.

In the liquid region,  $0 < r < X(z, v), |z| < \infty, v > 0$

$$\left. \begin{aligned} 2\alpha_1^2 \frac{\partial T_L}{\partial v} &= v \left( \frac{\partial^2 T_L}{\partial r^2} + \frac{1}{r} \frac{\partial T_L}{\partial r} + \frac{\partial^2 T_L}{\partial z^2} \right), \\ T_L(r, z, v)|_{v=0} &= \bar{f}_L^1(r, z), 0 \leq r \leq 1, |z| < \infty. \end{aligned} \right\} \quad (12.3.23)$$

In the solid region,  $X(z, v) < r \leq 1, |z| < \infty, v > 0$

$$2\alpha_2^2 \frac{\partial T_S}{\partial v} = v \left( \frac{\partial^2 T_S}{\partial r^2} + \frac{1}{r} \frac{\partial T_S}{\partial r} + \frac{\partial^2 T_S}{\partial z^2} \right). \quad (12.3.24)$$

In the mold,  $1 \leq r < \infty, |z| < \infty, v > 0$

$$\left. \begin{aligned} 2\alpha_3^2 \frac{\partial T_M}{\partial v} &= v \left( \frac{\partial^2 T_M}{\partial r^2} + \frac{1}{r} \frac{\partial T_M}{\partial r} + \frac{\partial^2 T_M}{\partial z^2} \right), \\ T_M(r, z, v)|_{v=0} &= \bar{f}_M^2(r, z), 1 \leq r < \infty, |z| < \infty. \end{aligned} \right\} \quad (12.3.25)$$

Boundary conditions at solid-liquid interface  $r = X(z, v)$

$$\left. \begin{aligned} & \left[ 1 + \left( \frac{\partial X}{\partial z} \right)^2 \right] \left( \frac{\partial T_S}{\partial r} - \beta_1 \frac{\partial T_L}{\partial r} \right) = \frac{2\lambda}{v} \frac{dX}{dv}, \\ & T_L(r, z, v) = 1; T_S(r, z, v) = 1; X(z, v)_{v=0} = 1, v \geq 0. \end{aligned} \right\} \quad (12.3.26)$$

Boundary conditions at  $r = 1, |z| < \infty, v > 0$

$$\frac{\partial T_S}{\partial r} = \beta_2 \frac{\partial T_M}{\partial r}; \quad \frac{\partial T_M}{\partial r} = H(z, v)(T_S - T_M). \quad (12.3.27)$$

In the above equations, subscripts  $S, L$  and  $M$  stand for solid, liquid and mold regions, respectively. Temperatures have been made dimensionless with the help of freezing temperature  $T_m$  which is taken as constant. The cylindrical coordinates  $(r, z)$  have been made dimensionless with the help of radius  $R$  of the cylinder.  $\bar{f}_L^1(r, z)$  and  $\bar{f}_M^2(r, z)$  are known initial temperatures. Heat transfer coefficient  $H(z, v)$  in Eq. (12.3.27) is a known quantity. Other dimensionless quantities are defined as follows.

$$\left. \begin{aligned} & \alpha_1^2 = k_S/k_L, \alpha_2^2 = 1, \alpha_3^2 = k_S/k_M, \beta_1 = K_L/K_S, \beta_2 = K_M/K_S, \lambda = l/C_S T_m, \\ & v = 2\sqrt{\left( \frac{k_S t}{R^2} \right)}, H(z, v) = \text{heat transfer coefficient, } R/K_M, t \text{ is real time,} \end{aligned} \right\} \quad (12.3.28)$$

$k$  is diffusivity,  $K$  is conductivity,  $l$  is latent heat and  $C$  is specific heat.

The solutions of Eqs (12.3.23)–(12.3.25) using Green's function [24] can be written as follows.

$$\begin{aligned} T_i = & \frac{2\alpha_J^3}{\sqrt{\pi}v^3} \left[ \int_{-\infty}^{\infty} \int_0^1 pe^{-\alpha_J^2(r^2+p^2+(z-q)^2)/v^2} I_0 \left( \frac{2\alpha_J^2 rp}{v^2} \right) \bar{f}_i^1(p, q) dp dq \right. \\ & \left. + \int_{-\infty}^{\infty} \int_1^{\infty} pe^{-\alpha_J^2(r^2+p^2+(z-q)^2)/v^2} I_0 \left( \frac{2\alpha_J^2 rp}{v^2} \right) \bar{f}_i^2(p, q) dp dq \right], \end{aligned} \quad (12.3.29)$$

where  $i = L, S$  and  $M$  stand for liquid, solid and mold, respectively,  $J = 1, 2, 3$ .  $I_0(x)$  is the modified Bessel function of the first kind of order zero.

We now describe the new embedding technique. It is important to note that  $\bar{f}_L^1(r, z)$  and  $\bar{f}_M^2(r, z)$  are the two known initial temperatures,  $\bar{f}_S^1(r, z)$ ,  $\bar{f}_S^2(r, z)$ ,  $\bar{f}_L^2(r, z)$  and  $\bar{f}_M^1(r, z)$  are unknown fictitious initial temperatures and they are prescribed in the fictitious extensions of the solid, liquid and mold regions, respectively. The solid region,  $X(z, v) < r < 1, v > 0$ , is first embedded in the region,  $0 \leq r \leq 1, |z| < \infty$  and for this region an initial temperature  $\bar{f}_S^1(r, z)$  is prescribed which is fictitious. This region is further extended fictitiously to the region,  $1 \leq r < \infty, |z| < \infty$  and a fictitious initial temperature is prescribed for it which is  $\bar{f}_S^2(r, z)$ . In a similar manner the liquid region,  $0 < r \leq X(z, v), |z| < \infty$  is first embedded into the region,  $0 \leq r \leq 1, |z| < \infty$  and then further extended fictitiously to the region,  $1 \leq r < \infty, |z| < \infty$  and a fictitious initial temperature  $\bar{f}_L^2(r, z)$  is prescribed for it. The fictitious temperature  $\bar{f}_M^1(r, z)$  arises due to fictitious extension of mold region to the region  $0 \leq r \leq 1$ . In principle, at the onset, this new embedding technique for obtaining analytical

solution seems alright as including  $X(z, v)$  there are five unknown functions and five boundary conditions to determine them. If a procedure to determine the unknowns can be developed then it can be claimed that it is possible to obtain some solution which could be a short-time solution or a closed-form solution. In fact both exact and short-time analytical solutions are obtained depending on the formulation of the problem and regions considered.

Is this new technique compatible with the physics of melting/freezing problems? Yes, it is. It is well known that the initial growth of the phase-change boundary depends on the initial temperature and boundary conditions. As the boundary and initial conditions are retained as they are, initial temperatures whether prescribed or fictitious should be able to give accurate solution at least for short time. It can be easily checked that this new embedding technique when applied to the solution of Neumann problem then Neumann solution is obtained exactly. A more complicated exact analytical solution presented in [406] for a heat and mass transfer problem can also be obtained exactly by the application of this new technique and similarly many other similarity solutions can be reproduced.

The procedure to obtain the solution will be discussed now. For small values of the argument of  $I_0(x)$ , an asymptotic expansion given in [24] can be used.

$$I_0(x) = \frac{e^x}{\sqrt{(2\pi x)}} \left\{ 1 + \frac{1}{8x} + \frac{9}{128x^2} + \cdots \right\}. \quad (12.3.30)$$

To obtain short-time solution only first two terms in the above expansion are retained and used in Eq. (12.3.29). More terms can also be retained but calculations become lengthy. Note that this asymptotic series does not converge and may give wrong results if several terms are retained. These temperatures are then substituted in the five boundary conditions given in Eqs (12.3.26), (12.3.27) and five equations are obtained. The derivation of these five equations is simple but the presentation of all of them here will be avoided to save space and only the energy balance condition at  $r = X(z, v)$  is given below as a sample equation. It is possible to describe the method of solution quite satisfactorily with the help of the following equation as similar operations of taking limits are used for other equations. The first condition in Eq. (12.3.26), i.e. Stefan condition, is transformed as follows.

$$\begin{aligned} 2\pi\lambda X^{5/2} \frac{\partial X}{\partial v} = & \left\{ 1 + \left( \frac{\partial X}{\partial z} \right)^2 \right\} \left[ \beta_1 \int_{\infty}^{-\infty} \int_{X\alpha_1/v}^{(X-1)/v} R_L f_L^1(p_1, q_1) e^{-(p^2+q^2)} dp dq \right. \\ & + \beta_1 \int_{\infty}^{-\infty} \int_{(X-1)\alpha_1/v}^{-\infty} \frac{R_L}{\sqrt{(X-vp/\alpha_1)}} f_L^2(p_1, q_1) e^{-(p^2+q^2)} dp dq - \\ & - \int_{\infty}^{-\infty} \int_{X/v}^{(X-1)/v} R_S f_S^1(p_2, q_2) x e^{-(p^2+q^2)} dp dq \\ & \left. - \int_{\infty}^{-\infty} \int_{(X-1)/v}^{-\infty} \left( R_S / \sqrt{(X-vp/\alpha_1)} \right) f_S^2(p_2, q_2) e^{-(p^2+q^2)} dp dq \right]. \end{aligned} \quad (12.3.31)$$

There is some printing mistake in Eq. (13) in [458] which has been corrected in Eq. (12.3.31). In Eq. (12.3.31), the following notations have been used.

$$p_1 = X - vp/\alpha_1, \quad p_2 = X - vp/\alpha_2, \quad q_1 = X - vq/\alpha_1, \quad q_2 = X - vq/\alpha_2, \quad (12.3.32)$$

$$R_L = 2p\alpha_1 \left\{ X^2 \left( X - \frac{vp}{\alpha_1} \right) + \frac{v^2 X}{16\alpha_1^2} \right\} + \frac{vX(X - vp/\alpha_1)}{2} + \frac{3v^3}{32\alpha_1^2}, \quad (12.3.33)$$



$$R_S = R_L|_{\alpha_1=1}, f_L^1(r, z) = \bar{f}_L^1(r, z)/\sqrt{r}, f_S^1(r, z) = \bar{f}_S^1(r, z)/\sqrt{r}. \quad (12.3.34)$$

It is assumed that the following series expansions hold for various known and unknown functions.

$$f_i^1(r, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\partial^{m+n}(f_i^1)}{\partial r^m \partial z^n} \bigg|_{r=1, z=0} (r-1)^m z^n, 0 \leq r \leq 1, |z| < \infty, \quad (12.3.35)$$

$$f_i^2(r, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\partial^{m+n}(f_i^2)}{\partial r^m \partial z^n} \bigg|_{r=1, z=0} (r-1)^m z^n, 1 \leq r \leq \infty, |z| < \infty, \quad (12.3.36)$$

$$X(z, v) = 1 + \sum_{n=1}^{\infty} A_n(z) v^n; H(z, v) = \sum_{n=0}^{\infty} H_n(z) v^{n-1}; v \geq 0, |z| < \infty, \quad (12.3.37)$$

the subscript  $i$  in Eq. (12.3.35), (12.3.36) stands for  $i = L, S$  and  $M$  depending on the region under consideration. In Eq. (12.3.30), there is a singularity at  $r = 0$ , therefore  $f_L^1(r, z)$  and  $f_S^1(r, z)$  as defined in Eq. (12.3.34) are used. Solidification is taking place at  $(r = 1, z = 0)$ , therefore for a short-time solution, expansions have been used about  $(r = 1, z = 0)$  in Eqs (12.3.35), (12.3.36). The term  $1/\sqrt{r}$  can be written as  $1/\sqrt{1 + (r-1)}$  and expanded in powers of  $(r-1)$  as  $(r-1)$  is small. The leading term in heat transfer coefficient in Eq. (12.3.36) is taken proportional to  $v^{-1}$  and this assumption is supported by many experimental observations refer [459].

To obtain unknown coefficients occurring in the series expansions given in Eqs (12.3.35), (12.3.36), the series are substituted in the five equations which are obtained on satisfying five boundary conditions as explained earlier. Limit  $v \rightarrow 0+$  of each of the five equations which are obtained on satisfying five boundary conditions is taken. Five equations in five unknowns  $A_1(z)$ ,  $f_S^1(1, z)$ ,  $f_S^2(1, z)$ ,  $f_L^1(1, z)$  and  $f_M^1(1, z)$  are obtained. To have some idea about the nature of these equations, they are given below.

$$(1 + \operatorname{erf} A_1 \alpha_1) [\lambda \sqrt{\pi} (1 + \operatorname{erf} A_1) A_1 + e^{-A_1^2} (f_S^1(1, z) - 1)] = \left. \begin{aligned} & \beta_1 \alpha_1 e^{-A_1^2 \alpha_1^2} (1 + \operatorname{erf} A_1) (f_L^1(1, z) - 1), \end{aligned} \right\} \quad (12.3.38)$$

$$f_S^1(1, z) = [\beta_2 H_0 \alpha_3 + 2\alpha_3 / \sqrt{\pi} + H_0 - \beta_2 H_0 \alpha_3 (1 + \operatorname{erf} A_1) f_M^1(1, z)] D_1, \quad (12.3.39)$$

$$D_1 = 2\alpha_3 \sqrt{\pi} + H_0 - \alpha_3 H_0 \beta_2 \operatorname{erf} A_1, \quad f_S^2(1, z) = \frac{2 - \operatorname{erfc}(A_1) f_S^1(1, z)}{1 + \operatorname{erf} A_1}, \quad (12.3.40)$$

$$f_L^2(1, z) = \frac{2 - \operatorname{erfc}(A_1 \alpha_1) f_L^1(1, z)}{1 + \operatorname{erf} A_1 \alpha_1}, \quad f_M^1(1, z) = \frac{f_M^2(1, z) - 2(1 - f_S^1(1, z))}{\beta_2 \alpha_3 (1 + \operatorname{erf} A_1)}. \quad (12.3.41)$$

Because of mold region and heat transfer coefficient considered, the equations are a bit lengthy. Except  $A_1(z)$  in Eq. (12.3.38), all other unknown coefficients in the functions  $f_S^1(1, z)$ ,  $f_M^1(1, z)$ , etc., are obtained by solving a system of linear algebraic equations in a simple and systematic manner. In the solutions of many Stefan problems with one or two phases in which flux prescribed condition is imposed,  $A_1$  comes out to be zero and the coefficient  $A_2$ ,  $A_3$ , etc., along with other unknowns are all obtained by solving linear system of algebraic equations in unknowns if the present method is used.

To determine  $A_2(z)$  and the first derivatives  $\frac{\partial f_L^2}{\partial r}(1, z)$ ,  $\frac{\partial f_S^1}{\partial r}(1, z)$ , etc., differentiate with respect to  $v$  all the five equations obtained on satisfying five boundary conditions and take the

limit  $v \rightarrow 0+$ . This procedure of calculating higher-order derivatives such as second, third, etc., is continued and limits  $v \rightarrow 0+$  are taken. The unknown coefficients are obtained in a systematic way. Temperatures in all the three regions can be obtained after substituting series expansions (12.3.35)–(12.3.37) in Eq. (12.3.29) and integrating term by term. The coefficients  $A_2(z)$ ,  $A_3(z)$  and other related coefficients are reported in [458] together with temperatures corresponding to first few terms.

Before proceeding further the process of taking limit  $v \rightarrow 0+$  should be justified. This is so because the two-dimensional limit  $r \rightarrow 1$ ,  $v \rightarrow 0$  does not exist. At no time in any of the integrals, the above two-dimensional limit is taken. The limit  $v \rightarrow 0+$  is taken along a particular path  $r = X(z, v)$  or along the path  $r = 1$  which is justified. It is possible that the limit  $v \rightarrow 0+$  may give some results which may not have anything to do with the actual solution of the given problem. In [458] as well as in many other works using this new technique sufficient analytical and numerical checks have been carried out. For all this discussion refer [458] and some other references discussed below.

Analytical and numerical solutions of radially symmetric one-dimensional two-phase inward solidification problem in spherical geometry were obtained in [246] by using this new embedding technique. In this relatively simple problem in which time-dependent flux is prescribed at the surface of the sphere,  $A_1 = 0$  is obtained in the solution and in this case coefficients  $A_2, A_3, A_4, A_5$  in the analytical solution could be easily obtained. Numerical solution was obtained with the help of finite difference *moving grid scheme* [460]. When the numerical solution is started (starting values) with analytical solution of moving boundary and temperatures obtained by the above method, there is complete agreement at later times (short time) between analytical and numerical values. If to start numerical scheme some minor error is introduced deliberately in the starting values of temperature and moving boundary in the starting values of numerical solution then there is large error occurring in the numerical solution for small solidified thickness which is reducing with time. Although analytical solution is a short-time solution, the comparison of analytical and numerical solutions compares well up to solidified thickness equal to 0.51. It may be clarified that for any set of given parameter values, comparison with numerical solution may not be good up to so much solidified thickness.

An important question regarding the time for which a short-time analytical solution is valid should be clarified. Calculate  $A_1 V, A_2 V^2, A_3 V^3, \dots$  If this sequence is decreasing sufficiently then till that value of  $V$  for which this sequence is decreasing sufficiently,  $V$  can be used in the analytical solution. This criterion has been checked with the help of numerical solutions obtained by numerical schemes and was found satisfactory. Rigorous justification of proofs of existence, uniqueness, convergence, etc., is difficult as it requires the existence proofs of fictitious initial temperatures and the convergence of series expansions which is difficult to obtain as general terms of solutions of temperatures and moving boundary cannot be obtained in explicit form.

This new technique is applicable to both inward and outward spherical and cylindrical problems. There is no necessity to obtain solution for the outward problem separately if the solution for the inward problem has been obtained and vice versa. It is just a matter of interpretation of initial regions and its fictitious extensions, and of initial temperature and fictitious initial temperatures.

Presented below is a two-phase class II solidification problem in a finite plate [63]. This problem is not only very difficult to study even its solution procedure presentation here in concise form to save space is difficult. The method of solution is based on the new embedding technique and some important modifications are required in it to obtain the short-time solution.

Consider a finite plate  $0 \leq X \leq 1$ ,  $-a \leq Y \leq a$  in which  $X$  and  $Y$  are dimensionless Cartesian coordinates. Initially at time  $t = 0$ , a superheated melt occupies the whole region of this plate. We study the following problem of solidification in which the solidification initiates over a limited portion of the face  $x = 0$  of the plate.

The various quantities have been made dimensionless as given in Eq. (3.2.58) in the context of the problem discussed in Section 3.2.4. As the method of solution is independent of the way parameters are made dimensionless, we have taken all the thermo-physical parameters equal to unity for the presentation of this problem here. For  $V > 0$  (time), the solid temperature  $T_S(X, Y, V)$  occupies the region  $(0 < X < R(Y, V), B_2(V) \leq Y \leq B_1(V))$  and liquid temperature  $T_L(X, Y, V)$  occupies the region  $(R(Y, V) < X < 1, (B_1(V) < Y < 1) \cup (-1 < Y < B_2(V)))$  (for  $B_1(V), B_2(V)$  see Eq. 12.3.43). The phase-change boundary is  $X = R(Y, V)$ . The temperatures  $T_S$  and  $T_L$  satisfy the linear heat equations (3.2.40), (3.2.48), respectively. The initial temperature of the melt is given below.

$$\left. \begin{aligned} T_L(X, Y, V)_{V=0} &= f_L^1(X, Y), \quad 0 \leq X \leq 1, -\infty \leq -a \leq Y \leq a < \infty, \\ V &= 2\sqrt{t}, t \text{ is dimensionless time, } V \geq 0. \end{aligned} \right\} \quad (12.3.42)$$

On  $X = 0$ ,  $-a \leq Y \leq a$ , there are two boundary conditions prescribed for  $V > 0$ .

$$\left. \begin{aligned} \frac{\partial T_L}{\partial X} \Big|_{X=0} &= \hat{d}_1 T_L \Big|_{X=0} - \hat{d}_2 \sum_{m=0}^{\infty} \hat{D}_m(Y) V^m, \quad Y \in \partial B_r^C; \\ \partial B_r^C &= \{Y | -a \leq Y \leq a\} - \partial B_r, \\ \partial B_r &= \{Y | -C_2 \leq -B_2(V) \leq Y \leq B_1(V) \leq C_1\}, \\ B_1(V) &\in [C_1, a], \quad B_2(V) \in [C_2, -a], \end{aligned} \right\} \quad (12.3.43)$$

$$B_1(0) = C_1, \quad B_2(0) = C_2, \quad C_1 \neq C_2, \quad 0 \leq C_1 \leq a, -a \leq C_2 \leq 0. \quad (12.3.44)$$

$$\frac{\partial T_S}{\partial X} \Big|_{X=0} = \sum_{m=0}^{\infty} Q_m(Y) V^{m-1}, \quad Y \in \partial B_r, \quad V > 0; \quad \partial B_r \cup \partial B_r^C = [-a, a]. \quad (12.3.45)$$

The boundary condition (12.3.45) implies that solidification initiates over a limited portion  $(0, C_1] \cup [C_2, 0)$  on the edge  $X = 0$  at time  $V = 0+$  with  $C_1$  and  $C_2$  as in Eq. (12.3.44). It is assumed that the prescribed flux at  $X = 0$ ,  $C_2 \leq Y \leq C_1$  satisfies a sufficient condition for solidification to start at  $V = 0+$ . This sufficient condition is reported in [63]. The boundary condition (12.3.43) states that on the remaining portion  $\partial B_r^C$  of the edge  $X = 0$ , flux is prescribed for the liquid as it is still liquid. In this problem, as the solidification is also spreading along  $X = 0$  the portion  $\partial B_r^C$  will be changing for  $V > 0$  and accordingly the boundary conditions are continuing to change on portions  $\partial B_r$  and  $\partial B_r^C$ . Spread along  $X = 0$  is  $B_1(V) \cup B_2(V)$ ,  $V \geq 0$ , where

$$B_1(V) = \{(X = 0, Y, V); Y \in [C_1, a]\}; \quad B_2(V) = \{(X = 0, Y, V); Y \in [C_2, -a]\}. \quad (12.3.46)$$

It may be noted that  $B_1(V) \neq B_2(V)$ .

In addition to the spread along  $X = 0$ , the solidification is growing towards the interior of the plate. This solidified growth towards  $X > 0$  is denoted by

$$\left. \begin{aligned} X &= R(Y, V), \quad Y \in [B_2(V), B_1(V)]; \quad R(Y, V)|_{Y=B_1(V)} = 0, \\ R(Y, V)|_{Y=B_2(V)} &= 0, \quad V \geq 0, \quad R(Y, 0) = 0. \end{aligned} \right\} \quad (12.3.47)$$

In essence, the phase-change boundary is the union of three boundaries and is nonsmooth. The derivatives do not exist at  $Y = B_1(V)$  and  $Y = B_2(V)$ ,  $V > 0$  but continuity of temperatures and phase-change boundary is there.

The boundary conditions for the liquid region on other three edges of the plate could be of any type but in [63] they are taken of convective type such as

$$\frac{\partial T_L}{\partial X} \Big|_{X=1} = \hat{e}_1 T_L \Big|_{X=1} - \hat{e}_2 \sum_{m=0}^{\infty} \hat{E}_m(Y) V^{m-1}, \quad -a < Y < a, V > 0. \quad (12.3.48)$$

Similarly, other boundary conditions can be written for other edges. These conditions should ensure on physical and mathematical grounds that  $X = R(Y, V)$  is the only moving boundary for  $X > 0$ . Other convective-type boundary conditions on the remaining edges are not given here. All the equations are scaled suitably and for scaling see [63]. The boundary conditions at  $X = R(Y, V)$  are similar to those given in Eq. (12.3.26) with the addition of Eq. (12.3.47).

To obtain a short-time solution, we first discuss determination of the phase-change front  $X = R(Y, V)$ . Later on this solution will be used in a novel way to obtain  $Y = B_1(V)$  and  $Y = B_2(V)$ . The new embedding technique explained earlier in connection with the problem considered in [458] is used here also. The description of fictitious extensions of original regions and prescribing initial temperatures for these regions requires much space and we refer [63] for this.

There are in all eight boundary conditions to be satisfied. One unknown function is the moving boundary  $X = R(Y, V)$  on which three boundary conditions are prescribed. Seven unknown functions according to the new technique have to be initial temperatures which are fictitious and prescribed in fictitious regions. The boundary condition (12.3.45) can be prescribed on  $X = 0, -\infty < Y < \infty$  without violating the boundary condition prescribed for solid. Following the discussion of the earlier problem in [458] mentioned earlier, the temperature solution for solid can be written using Green's function (cf. [24]) and two fictitious initial temperatures. Corresponding to two fictitious initial temperatures, the solid temperature which is used to satisfy boundary conditions will be the sum of two temperatures. The liquid temperature will be the sum of six temperatures or six integrals corresponding to six initial temperatures as the original region has been extended to  $-\infty < X < \infty, -\infty < Y < \infty$ . Five of the initial temperatures are fictitious and one is known as it is prescribed. The boundary condition (12.3.43) for liquid is extended to  $X = 0, -a \leq Y \leq a$  and fictitious initial temperature  $f_L^6(X, Y)$  has been assumed for the region  $-\infty < X \leq 0, -a \leq Y \leq a$ . Let the temperature corresponding to initial prescribed temperature be denoted by  $T_L^1(X, Y, V)$ . The liquid temperature which is the solution of Eq. (3.2.48) is the sum of many integrals corresponding to other fictitious initial temperatures in which  $f_L^2(X, Y)$  and  $f_L^6(X, Y)$  both are prescribed for the same region  $-\infty < X < 0, -a \leq Y \leq a$ . The sum of temperatures  $T_L^6(X, Y, V)$  corresponding to  $f_L^6(X, Y)$  and  $T_L^1$  is used exclusively to satisfy the condition (12.3.43) in which  $T_L^2(X, Y, V)$  is not participating. However, in the boundary conditions at the phase-change boundary, all the seven fictitious temperatures are participating and contributing in  $X(R, V)$ . For further discussion about the temperature solutions for solid and liquid regions refer [63]. The presentation and reasoning in choosing appropriate regions and the justification for fictitious initial temperatures is simple but lengthy and cannot be reported here. Except  $F_L^1(X, Y)$  all other initial temperature are fictitious.

The series expansions for fictitious initial temperatures are taken keeping the boundary conditions in mind. For example, if  $f_L^J(X, Y)$  are used to satisfy the boundary conditions at  $Y = \pm a, 0 \leq X < 1$  ( $J = 4$  and  $5$ ) then  $f_L^J(X, Y)$  are expanded as

$$f_L^J(X, Y) = \sum_{n=0}^{\infty} \frac{(Y \mp a)^n}{n!} \left. \frac{\partial^n f_L^J}{\partial Y^n} \right|_{(X, Y=\pm a)}, \quad J = 4, 5. \quad (12.3.49)$$

For the moving boundaries the following expansions are used.

$$\left. \begin{aligned} X = R(Y, V) &= \sum_{m=1}^{\infty} A_m(Y) V^m; \quad B_1(V) = C_1 + \sum_{n=1}^{\infty} b_n^{(1)} V^{n/2}, \\ B_2(V) &= C_2 + \sum_{n=1}^{\infty} b_n^{(2)} V^{n/2}; \quad -a \leq Y \leq a, V \geq 0. \end{aligned} \right\} \quad (12.3.50)$$

The temperatures in the solid and liquid regions and  $X = R(Y, V)$  can be determined by following the procedure suggested in [458]. We assume that it has been done and as the details require too much space they are not given here. The procedure to determine  $B_1(V)$  and  $B_2(V)$  is new so we present it rather explicitly. The conditions in Eq. (12.3.47) at  $Y = B_1(V)$  and  $Y = B_2(V)$  require that

$$\sum_{k=1}^{\infty} A_k \left( Y = C_J + \sum_{n=1}^{\infty} b_n^{(J)} V^{n/2} \right) V^k \equiv 0 \text{ for all } V \geq 0; J = 1, 2. \quad (12.3.51)$$

It is easy to obtain, the identity

$$\left\{ A_1(C_J) + \left( \sum_{n=1}^{\infty} b_n^{(J)} V^{n/2} \right) A_1'(C_J) + \left( \sum_{n=1}^{\infty} b_n^{(J)} V^{n/2} \right)^2 A_1''(C_J)/2 + \dots \right\} V \\ + \left\{ A_2(C_J) + \left( \sum_{n=1}^{\infty} b_n^{(J)} V^{n/2} \right) A_2'(C_J) + \dots \right\} V^2 + \dots \equiv 0; J = 1, 2. \quad (12.3.52)$$

$A_1(Y)$  and  $A_2(Y)$  are reported in [63]. Dash in Eq. (12.3.52) denotes differentiation with respect to  $Y$  and then evaluated at  $C_J$ . As reported in [63] except  $A_1(Y)$ , other coefficients  $A_2(Y), A_3(Y)$ , etc., are all determined by solving linear system of equations. Eq. (12.3.52) is an identity and on comparing different powers of  $V^{1/2}$  on both sides, the unknown coefficients  $b_n^{(J)}$  for  $n = 1, 2, 3, \dots; J = 1, 2$  can be obtained. Several cases arise in obtaining  $b_n^{(J)}$  which are reported in [63]. We take up only two cases here for illustration and for some more cases see [63].

**Case I.** If  $A_1'(C_J) = 0$  and  $A_1''(C_J) \neq 0, J = 1, 2$ , then

$$\left( b_1^{(J)} \right)^2 = -2A_2(C_J)/A_1''(C_J); \quad b_2^{(J)} = 0, J = 1, 2; A_2(C_J) \neq 0. \quad (12.3.53)$$

From the analytical solution obtained,  $A_2(C_J)$  and  $A_1''(C_J)$  have been obtained in [63] and with the help of rigorous physical and mathematical arguments it has been shown that  $b_1^{(J)}, J = 1, 2$  are real quantities. In [63], several checks have been proposed for the correctness of the obtained solutions of  $b_n^{(J)}, J = 1, 2$  and accordingly on asymptotic behaviour of  $B_1(V)$  and  $B_2(V)$  for  $V \ll 1$ .  $B_1(V)$  and  $B_2(V)$  are  $O(V^{1/2})$  or  $O(t^{1/4})$  for  $t \ll 1$  depending on the solution of  $A_1(Y), A_2(Y)$ , etc., and their derivatives in this Case I. This sort of behaviours has been reported earlier in the literature in connection with Fourier heat conduction-related Stefan problems [461].

**Case II.** In the second case, if  $A_1'(C_J) \neq 0$  and  $A_2(C_J) \neq 0$ , then we have

$$b_n^{(J)} = 0, \text{ if } n \text{ is odd, } J = 1, 2; \quad b_2^{(J)} = -A_2(C_J)/A_1'(C_J), J = 1, 2. \quad (12.3.54)$$

$A_1'(C_J)$  is always negative according to geometric considerations and if  $A_2(C_J) \neq 0$  then  $A_2(C_J)$  is positive from Eq. (12.3.52) and  $A_1(C_J)$  is always zero. Hence  $b_2^{(J)}, J = 1, 2$  are positive and  $B_1(V)$  and  $B_2(V)$  are of order  $O(V)$ ,  $V \ll 1$ . Similar results were obtained in [61, 455, 456] for the spread of moving boundary along the surface  $X = 0$ .

When  $C_1 = C_2 = 0$ , we have the results for a cold spot problem and  $A_1'(C_J) = 0$ . Class I problem in a finite strip for melting/solidification can be easily studied with some changes in the solution procedure by choosing suitably the fictitious initial temperatures and their corresponding fictitious regions in [63]. If  $a \rightarrow \infty$  we get results for a two-dimensional semiinfinite problem of two-phase solidification. Two-dimensional semiinfinite two-phase melting problem was considered in [462] but boundary condition for solidification is temperature prescribed so results cannot be compared with the solution obtained in [63]. A problem of coupled heat and mass transfer was considered in [463] in which temperature is prescribed as boundary condition in a class I problem. Using the above embedding method results in [463] cannot be compared with results in [461] as boundary conditions are different.

In [62], a two-phase class III cylindrical problem of two-dimensional solidification due to an axisymmetric cold spot has been studied in the region  $R_0 \leq r < \infty, |z| < \infty$ . The solidification initiates due to an initial temperature, begins at  $z = 0, r = R_0$ , continues along  $r = R_0, |z| < \infty$  and also towards  $r > R_0$ . Several type of growths along  $r = R_0$  and  $r > R_0$  are possible depending on initial temperature and boundary condition at  $r = R_0$ . In one case spreading of solidification along  $r = R_0$  was found to be  $O(t^{1/4})$ ,  $t \ll 1$ , but other types of growths are also possible and obtained in [62] as well as in [63]. All the cases cannot be discussed here, see [62, 63] for further details.

In the solutions of problems of Class II and III but not in Class I problems, there is a shortcoming of the application of new embedding technique. This is so for the problems considered in both [62, 63]. Two fictitious initial temperatures are prescribed for the same fictitious extension of the original region. To make this point clear we consider the problem in [63] which has been discussed above in details. The temperature  $T_L^6(X, Y, V)$  is used exclusively to satisfy the boundary condition prescribed on  $\partial B_r^C$ .  $T_L^2(X, Y, V)$  is also the fictitious temperature for the same fictitious region as  $T_L^6(X, Y, V)$  is. In calculating the moving boundary  $X = R(Y, V)$  both  $T_L^2$  and  $T_L^6$  are participating which they should. If we calculate  $\partial T_L^2(X, Y, V)/\partial X$  at  $X = 0$  then it may not be zero which it should be and this means that the boundary condition on  $\partial B_r^C$  is not satisfied exactly. To find out how significant this effect is, numerical work for the analytical solution was done.

Numerical values of both  $\left. \frac{\partial T_L^6}{\partial X} \right|_{X=0}$  and  $\left. \frac{\partial T_L^2}{\partial X} \right|_{X=0}$  were calculated for  $-1 \leq Y \leq 1$ , by taking  $C_1 = 0.5$ ,  $C_2 = -0.25$ ,  $f_L^1(x, y) = 1 + x/10 + x^2y^2/10$ ,  $Q_0 = (C_1 - Y)(C_2 + Y)$ ,  $Q_1 = 0.5$  and some parameter values. The maximum value of  $\left. \frac{\partial T_L^2}{\partial X} \right|_{X=0}$  is only 1.73% of the absolute value of  $\left. \frac{\partial T_L^6}{\partial X} \right|_{X=0}$  at  $V = 0.1$ , and  $-C_2 \leq Y \leq C_1$ . In [62], in a cold spot problem, two liquid temperatures for two fictitious initial temperatures in the same fictitious extension of the original region were compared with prescribed initial temperature and difference found was insignificant for short time. As the solidification at  $r = R_0, z = 0$  is initiated due to a suitably prescribed initial liquid temperature, initial temperature for liquid was found mainly responsible for the spread of solidification along  $r = R_0$ .

In the cold spot problem in [62], to obtain growth along  $z = 0$ ,  $r = R_0$  appropriate choice of series expansions of  $Z = B(V)$  is required. For example, if  $\left. \frac{\partial^2 f_L^1}{\partial z^2} \right|_{r=R_0, z=0} = 0$  and  $\left. \frac{\partial^4 f_L^1}{\partial z^4} \right|_{r=R_0, z=0} \neq 0$ , where  $f_L^1(r, z)$  is prescribed initial temperature, then  $B(V)$  has to be taken in the form  $B(V) = \sum_{n=1}^{\infty} b_n V^{n/4}$ . This type of series for  $B(V)$  is required to determine the unknown coefficients  $b_n, n = 1, 2, \dots$ , on using identity of the form in Eq. (12.3.52). In this case  $B(V)$  is  $O(V^{1/4})$  or  $O(t^{1/8})$  for  $t \ll 1$ . For further details see [62].

As far as we know except Boley and coworkers' works on one-phase ablation problems and Gupta's works on two-phase problems, no other work on analytical and/or numerical solution is available for Class II and III problems. It would be interesting to see some analytical solutions which take account of all the boundary conditions exactly and throw some light on the unusual behaviours of spread along the surface of the region on which the solidification/melting initiates.

### Solutions Obtained by Fixing the Phase-Change Boundary

The analytical solution considered in [464] uses the transformation which converts moving front to a fixed boundary. The method of solution in this method is entirely different from the methods of solutions discussed here earlier for short-time solutions. The method of solution in [464] and some other works which will be discussed here uses series expansions not in powers of  $t^{1/2}$  but in some other forms. The spherical radially symmetric Stefan problem in one-dimensional one-phase considered in [464] has the following formulation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2}, \quad R(t) < r < 1, \quad t > 0; \quad u(r, t) = 1 \text{ when } r = 1. \quad (12.3.55)$$

If  $\bar{c}(r, t)$  is the concentration in a shrinking core model and  $u(r, t) = r\bar{c}(r, t)$  then we get the differential equation in Eq. (12.3.55). The conditions at the moving front  $r = R(t)$  are

$$u(r, t)|_{R(t)} = 0; \quad \frac{\partial u}{\partial r}(R(t), t) = -\alpha R(t) \frac{dR}{dt}, \quad R(0) = 1. \quad (12.3.56)$$

In Eq. (12.3.56),  $\alpha = \alpha^*/\bar{c}_0$ , where  $\alpha^*$  is the nondiffusing reactant in the particle and  $\bar{c}_0$  is the maximum concentration outside the particle. Eqs (12.3.55), (12.3.56) have been suitably

scaled in [464]. The phase-change boundary is fixed at  $x = 1$  with the help of the following *Landau transformation*  $(r, t) \rightarrow (x, y)$  and new independent variables are defined as  $(x, y)$ .

$$x = (r - 1)/(R(t) - 1); y = R(t), y(0) = 1; u(r, t) = \varphi(x, y). \quad (12.3.57)$$

The new formulation of the above problem is as follows.

$$\alpha y \frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial \varphi}{\partial y}(1, y) \left[ x \frac{\partial \varphi}{\partial x} - (y - 1) \frac{\partial \varphi}{\partial y} \right], 0 \leq x \leq 1, 0 < y \leq 1. \quad (12.3.58)$$

$$\varphi(x, y) = 1 \text{ at } x = 0; \varphi(x, y) = 0 \text{ at } x = 1; \frac{\partial \varphi}{\partial x}(1, y) = -\alpha y(y - 1) \frac{dy}{dt}. \quad (12.3.59)$$

As reported earlier similarity solution of this problem does not exist. Exact analytical solution may not be possible and so a series solution is attempted as follows. Expand  $\phi(x, y)$  as given below

$$\varphi(x, y) = \sum_{n=0}^{\infty} A_n(x)(y - 1)^n. \quad (12.3.60)$$

On using Eq. (12.3.60) in Eq. (12.3.59) and equating different powers of  $(y - 1)^n, n \geq 0$  on both sides of Eq. (12.3.59), we get the following system of equations.

$$\left. \begin{aligned} A_0'' &= \alpha^{-1} x A_0'(1) A_0', \\ A_n'' + A_{n-1}'' &= \alpha^{-1} \sum_{j=0}^n A_j'(1) [x A_{n-j}' - (n - j) A_{n-j}], n \geq 1, \end{aligned} \right\} \quad (12.3.61)$$

where dash denotes differentiation with respect to  $x$  and all  $A_n$  for  $n \geq 0$  are functions of  $x$ . As far as method of solution is concerned, it ends here. It remains to obtain now the solution of the system of differential equations in Eq. (12.3.61). Note that in the methods of short-time solutions discussed earlier in the context of embedding techniques, it is not possible to obtain recursive relations for the unknown coefficients as in Eq. (12.3.61) and this is plus point. The complete procedure of solution of Eq. (12.3.61) cannot be described here but some outlines are given below. For  $n \geq 1$ , the differential equations in Eq. (12.3.61) can be written in the form

$$A_n'' + \gamma(x A_n' - n A_n) = f_n(x), \gamma = -A_0'(1)/\alpha, n \geq 1, \quad (12.3.62)$$

$$f_n(x) = -A_{n-1}'' + \frac{1}{\alpha} \sum_{j=1}^n A_j'(1) [x A_{n-j}' - (n - j) A_{n-j}]. \quad (12.3.63)$$

The solution of the homogeneous part of equations in Eq. (12.3.62) is easy to obtain if a substitution  $z = -\gamma x^2/2$  is made. For the particular integral, Wronskian  $W(x)$  is needed which is given by

$$W(x) = -e^{-\gamma x^2/2}. \quad (12.3.64)$$

Now we start determining coefficients from  $A_0(x)$  onwards by solving first equation in Eq. (12.3.61) whose solution can be obtained but it contains the unknown constant  $\gamma$ . For



determining  $\gamma$  the following equation is obtained by differentiating  $A_0(x)$  after  $A_0(x)$  is obtained.

$$\gamma \int_0^1 e^{-\gamma \lambda^2/2} d\lambda = \beta e^{-\gamma/2}, \quad \beta = \alpha^{-1}; \quad \gamma \approx \beta \text{ for small } \beta. \quad (12.3.65)$$

The coefficients  $A_0(x)$ ,  $A_1(x)$ ,  $A_2(x)$  are reported in [464]. To determine the moving boundary the third equation in Eq. (12.3.59) is to be used.  $y$  or  $R(t)$  can be determined only numerically.

Analytical solution can be used provided  $A'_0(1)$  is determined numerically or the approximation for small  $\beta$  is used. The convergence of the series expansion for moving boundary has not been discussed but the numerical work based on analytical solution with three terms in Eq. (12.3.61) indicates that complete solidification is possible.

### Total Freezing Time

If  $\delta_n = A'_n(1)$  and  $\varepsilon(t) = y - 1$  then from third equation in Eq. (12.3.59), we have

$$dt = -\frac{\alpha\varepsilon}{\delta_0} \left\{ 1 + \frac{(\delta_0 - \delta_1)\varepsilon}{\delta_0} + \frac{(\delta_1^2 - \delta_0\delta_1 - \delta_0\delta_1)\varepsilon^2}{\delta_0^2} + \dots \right\} d\varepsilon. \quad (12.3.66)$$

On integrating Eq. (12.3.66),  $t$  is obtained in a series of powers of  $(y - 1)$  and terms up to  $(y - 1)^4$  have been obtained. When freezing is complete  $y(t) = 0$  and total solidification time can be obtained. For  $\alpha \rightarrow \infty$ , the total solidification time  $t_c$  is given by

$$t_c = \frac{1}{6\gamma} + O(1). \quad (12.3.67)$$

$t_c$  in Eq. (12.3.67) is time for pseudo-steady-state solution of the above problem. Numerical results obtained from analytical solution have been compared with known results and the agreement in the results is good.

In an one-dimensional one-phase problem in [465], a saturated liquid inside a sphere is freezing. The boundary condition at  $r = a$ ,  $0 \leq r \leq a$ , is of convective type. The formulation and solution procedure is similar to one described above in [464]. In this case  $A_0(x) = 0$  and so the authors were able to determine  $A_1(x)$ ,  $A_2(x)$ ,  $\dots$ ,  $A_6(x)$ . Several results including the solidification time in the transient and quasisteady cases and upper and lower bounds for the total solidification time have been discussed. Analytical results are given but they have to be evaluated numerically. The numerical results agree with other known results.

Consider the following one-dimensional linear equation discussed in [466].

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{m}{r} \frac{\partial u}{\partial r}, \quad m = 0, 1, 2. \quad (12.3.68)$$

Here  $u(r, t)$  could be temperature in freezing/melting problems or concentration in a shrinking core model. For  $m = 0, 1, 2$ , we have one-dimensional problems in the planar, cylindrical and spherical geometry, respectively, and  $r$  is the corresponding coordinate in these geometries. The range of  $r$  can be specified according to region considered in the problem. For simplicity all the thermo-physical parameters are taken here to be unity in Eq. (12.3.68). Initial

and boundary conditions are similar to those given in Eqs (12.3.55), (12.3.56). If Landau transformation  $x = (1 - r)/(1 - R(t))$  is used in the planar problem where  $r = R(t)$  is the moving boundary in the formulation of the problem then we have a problem similar to Neumann problem in the region  $(-\infty, 1)$  in which the moving boundary is fixed at  $x = 1$ . The solution is given by  $R(t) = 1 - (2\gamma_1 t)^{1/2}$  and  $u(x, t) = A_0(x)$ , where

$$A_0(x) = \alpha\gamma_1 \int_x^1 \exp(\gamma_1(1 - \xi^2)/2) d\xi, \quad (12.3.69)$$

$\gamma_1$  is determined by solving a transcendental equation.

In the cylindrical problem, the following transformation can be used.

$$x = \log(r)/\log(R), \quad y = \log(R); \quad r = R(t) \text{ is moving boundary.} \quad (12.3.70)$$

$u(x, t)$  can be expressed in a series of the form

$$u(x, t) = A_0(x) + A_1(x)y + A_2(x)y^2 + \dots \quad (12.3.71)$$

$A_0(x)$  is given by Eq. (12.3.69). Other coefficients can be determined following the procedure discussed earlier but even the calculation of  $A_2(x)$  becomes complicated.

An alternate transformation for spherical problem can also be used as given below.

$$x = \frac{r^{-1} - 1}{R^{-1} - 1}, \quad y = R^{-1} - 1. \quad (12.3.72)$$

We look for a solution of the form

$$u(r, t) = A_0(x) + \bar{A}_1(x)y + \bar{A}_2(x)y^2 + \dots \quad (12.3.73)$$

$A_0(x)$  is the same as in Eq. (12.3.69) and for  $\bar{A}_1(x)$  and  $\bar{A}_2(x)$ , and  $R(t)$  see [466].

All the three transformations proposed in problems of plane, cylinder and sphere can be combined into a single transformation and this is being discussed below. Let

$$x = S_\lambda(1, r)/S_\lambda(1, R); \quad y = S_\lambda(1, R); \quad S_\lambda(x, y) = \int_y^x p^{-\lambda} dp. \quad (12.3.74)$$

If  $f(r) = S_\lambda(1, r)$  then it is easy to check that

$$\frac{d^2 f}{dr^2} + \frac{\lambda}{r} \frac{df}{dr} = 0; \quad f(1) = 0; \quad \frac{df}{dr}(1) = 1; \quad \lambda = 0, 1, 2. \quad (12.3.75)$$

The solution of  $f(r)$  can be easily obtained by solving Eq. (12.3.75). With the choice of  $x = f(r)/f(R)$ ,  $y = f(R)$  and  $u(r, t) = \varphi(x, y)$ , Eq. (12.3.68) is transformed into the following equation with  $\alpha$  as in Eq. (12.3.56).

$$\alpha \left\{ \frac{\partial^2 \varphi}{\partial x^2} \left( \frac{df}{dr} \right)^2 + y \frac{\partial \varphi}{\partial x} \left( \frac{d^2 f}{dr^2} + \frac{\lambda}{r} \frac{df}{dr} \right) \right\} = \left( \frac{df}{dr} \right)^2 \frac{\partial \varphi}{\partial x}(1, y) \left( x \frac{\partial \varphi}{\partial x} - y \frac{\partial \varphi}{\partial y} \right). \quad (12.3.76)$$

In view of first differential equation in Eq. (12.3.75), Eq. (12.3.76) is considerably reduced. When  $\varphi(x, y)$  is determined from Eq. (12.3.76),  $u(r, t)$  for all the three regions is obtained. In Eq. (12.3.70) as  $R \rightarrow 0$ , we have  $y \rightarrow \infty$  and so Euler's transformation  $y/(1+y)$  can be used. If the transformation  $x = (1-r)/(1-R)$ ,  $y = 1-R$  is used for the spherical problem,  $u(r, t)$  can also be expanded in a series of the form.

$$u(r, t) = \frac{1}{r} \{A_0(x) + A_1(x)y + A_2(x)y^2 + \dots\}. \quad (12.3.77)$$

Having described the method of solution for one-phase problems in detail, it is much easier to explain the method of solution for two-phase problems of plane, cylinder and sphere concerning solidification/melting in one dimension considered in [467]. Let  $T_S(r, t)$  and  $T_L(r, t)$  be the temperatures in the solid and liquid regions, respectively. Consider the heat equations for  $T_S$  and  $T_L$  similar to Eq. (12.3.68) in which  $T_S(r, t)$  occupies the region  $0 < r < R(t)$  and  $T_L(r, t)$  occupies the region  $R(t) < r < 1$ ,  $r = R(t)$  is the interface. The boundary and initial conditions are given below for a melting problem.

$$T_L(1, t) = 1; T_L(R(t), t) = T_S(R(t), t) = 0; \frac{\partial T_S}{\partial r}(0, t) = 0; T_S(r, 0) = -V \leq 0. \quad (12.3.78)$$

We assume that all the equations are in dimensionless form. For scalings see [467]. In addition to Eq. (12.3.78), there is Stefan condition at the phase-change boundary  $r = R(t)$ ,  $R(0) = 1$ , which is of similar type as that in the Neumann problem.

The method of solution is based on the following transformation which fixes  $R(t)$ , and the points  $r = 0$  as 1 and 0. Let

$$x = f(r)/f(R); y = f(R); T_L(r, t) = \varphi_L(x, y); T_S(r, t) = V[-1 + \varphi_S(x, y)]. \quad (12.3.79)$$

The function  $f(r)$  in Eq. (12.3.79) satisfies the differential equation in Eq. (12.3.75) with boundary conditions prescribed as  $f(1) = 0$ ,  $|f(0)| = \infty$ , so that the liquid region becomes  $[0, 1]$  and solid region is now  $[1, \infty]$  with moving boundary fixed at  $x = 1$ . For the cylindrical problem, take  $\lambda = 1$  in Eq. (12.3.75) and define  $x$  and  $y$  as in Eq. (12.3.70).

In terms of the functions  $\varphi_L(x, y)$  and  $\varphi_S(x, y)$ , the heat equations become

$$\left. \begin{aligned} \frac{\partial^2 \varphi_L}{\partial x^2} &= Q_1 W_1 \left[ x \frac{\partial \varphi_L}{\partial x} - y \frac{\partial \varphi_L}{\partial y} \right], Q_1 = \frac{e^{2(x-1)y}}{\alpha_1}, \\ W_1 &= \frac{\partial \varphi_L}{\partial x}(1, y) - V \frac{\partial \varphi_S}{\partial x}(1, y), \end{aligned} \right\} \quad (12.3.80)$$

$$k_L/k_S \frac{\partial^2 \varphi_S}{\partial x^2} = Q_1 W_1 \left[ x \frac{\partial \varphi_S}{\partial x} - y \frac{\partial \varphi_S}{\partial y} \right], \quad \alpha_1 = l/C_L(T_a - T_m), \quad (12.3.81)$$

$T_a$  is the temperature of liquid at the outer radius,  $T_m$  is the fusion temperature, and both are in nondimensionless form,  $\rho_S = \rho_L$ . The Stefan condition and boundary and initial conditions can be easily written in terms of  $\varphi_S(x, y)$ ,  $0 \leq x \leq 1$  and  $\varphi_L(x, y)$ ,  $1 \leq x < \infty$ . The following series expansions have been used to determine the solution.

$$\varphi_L(x, y) = \sum_{n=0}^{\infty} A_n(x)y^n, \quad \varphi_S(x, y) = \sum_{n=0}^{\infty} B_n(x)y^n. \quad (12.3.82)$$

For the cylindrical and spherical problems the transformations in Eqs (12.3.70), (12.3.72) have been used, respectively. In both the problems expressions of  $A_0, B_0, A_1$  and  $B_1$  have been obtained and only numerical evaluation is possible to determine them. Expressions of  $A_1(x)$  and  $B_1(x)$  are lengthy. Moving boundary can be obtained only numerically and for obtaining it solution procedure similar to the one discussed in the earlier references has to be used. In both cylindrical and spherical problems there is singularity at  $r = 0$ . In cylindrical case total solidification is not possible if transformation given in Eq. (12.3.70) is used. However, in the spherical case if  $x = (1 - r)/(1 - R), y = 1 - R$  is used then large-time solution is possible. Analytical solution obtained is in good agreement with the numerical solution obtained by enthalpy method. For other related works see references listed in [464–467].

### Use of Repeated Integrals of Error Functions

It was mentioned earlier that for each integer  $n, n \geq 0$  the functions given in Eq. (12.2.24) satisfy the one-dimensional linear heat equation. The function  $i^n \text{erfc}(x/V), V = 2\sqrt{t}$ , for  $n \geq 1$  is singular at  $x \neq 0, V \rightarrow 0$  but  $V^n i^n \text{erfc}(x/V)$  is not singular as  $V \rightarrow 0, x \neq 0$ . The two-dimensional limit  $(x, t) \rightarrow (0, 0)$  does not exist.  $V^n i^n \text{erfc}(-x/V)$  also satisfies the linear heat equation and is not singular as  $V \rightarrow 0$  for  $n \geq 0$ . By considering linear combinations of these iterated error functions, the solution of a two-phase one-dimensional planar solidification problem has been studied in [468]. The liquid region at time  $t = 0$  occupies the semiinfinite space  $0 \leq x \leq \infty$ . At  $t = 0+$ , the solidification starts at  $x = 0$  due to prescribed convective condition which is given below together with the initial condition for liquid.

$$\frac{\partial T_S}{\partial x}(0, t) = h [T_S(0, t) - U(t)], t > 0; T_L(x, 0) = \bar{V}(x), 0 \leq x < \infty, \quad (12.3.83)$$

$h$  is heat transfer coefficient. It is assumed that  $U(t)$  and  $\bar{V}(x)$  possess the following series expansions. Subscripts  $S$  and  $L$  stand for solid and liquid, respectively.

$$\left. \begin{aligned} U(t) &= \sum_{n=1}^{\infty} u_n t^n / n!, \quad u_n = (d^n U / dt^n)_{t=0}, \\ \bar{V}(x) &= \sum_{n=0}^{\infty} v_n (x/k_L^{1/2})^n / n!, \quad v_n = (k_L)^{n/2} (d^n \bar{V} / dx^n)_{x=0}. \end{aligned} \right\} \quad (12.3.84)$$

The linear heat equations for solid and liquid regions and boundary conditions at the phase-change boundary  $x = S(t)$  are of the same type as in Neumann problem and not given here to save space. At any time  $t > 0$ , the solid occupies the region  $0 < x \leq S(t)$  and the liquid region is  $S(t) \leq x < \infty$ . To express the temperatures  $T_S(x, t)$  and  $T_L(x, t)$  in series form the following notations for some of the functions are used.

$$E_n(\xi) = \frac{1}{2} [i^n \text{erfc}(-\xi) + i^n \text{erfc}\xi]; \quad F_n(\xi) = \frac{1}{2} [i^n \text{erfc}(-\xi) - i^n \text{erfc}\xi]. \quad (12.3.85)$$

$$G_n(\xi) = \frac{1}{2} [i^n \text{erfc}(-\xi) + (-1)^n i^n \text{erfc}(\xi)], \quad \xi = x/(4kt)^{1/2}, \quad (12.3.86)$$

$k$  is diffusivity.  $i^n \text{erfc}(\xi)$  is iterated error function or also called *repeated integral of error function* and defined in Eq. (12.2.24).  $G_n(\xi)$  is a polynomial of degree  $n$  and

$$G_n(\xi) = \sum_{m=0}^{[n/2]} \frac{\xi^{n-2m}}{2^{2m-1} m! (n-2m)!}, \quad [n/2] \text{ stands for largest integer in } n/2. \quad (12.3.87)$$

The temperatures  $T_S(x, t)$  and  $T_L(x, t)$  satisfying heat equations are constructed in the following way.

$$T_S = T_S^1 + T_S^2; \quad T_L = T_L^1 + T_L^2, \quad (12.3.88)$$

$$T_S^1(\xi_S) = \sum_1^\infty u_n(4t)^n G_{2n}(\xi_S), \quad T_S^2(\xi_S) = \sum_0 a_n(4t)^{n/2} \left[ E_n(\xi_S) + h(4k_S t)^{1/2} F_{n+1}(\xi_S) \right]. \quad (12.3.89)$$

$$T_L^1(\xi_L) = \sum_0 v_n(4t)^{n/2} G_n(\xi_L); \quad T_L^2 = \sum_0 b_n(4t)^{n/2} i^n \operatorname{erfc}(\xi_L), \quad (12.3.90)$$

$$\xi_S = x/(4k_S t)^{1/2}, \quad \xi_L = x/(4k_L t)^{1/2}; \quad a_n \text{ and } b_n, n = 0, 1, 2, \dots, \text{ are unknowns.} \quad (12.3.91)$$

$\xi_S$  and  $\xi_L$  are similarity variables and for linear heat equations similarity variables can be used to express the solution. Note that the similarity solution of the above problem does not exist because the boundary conditions at  $x = 0$  and at the phase-change interface cannot be expressed in terms of similarity variables only. Therefore the analytical solution can be obtained only in the form of a series and this is why the unknown constants  $a_n$  and  $b_n$  have been introduced and series expansions are considered.

Using Eq. (12.3.89), it can be checked that the boundary condition at  $x = 0$  is satisfied.

$$\begin{aligned} \left( T_S^1 - \frac{1}{h} \frac{\partial T_S^1}{\partial x} \right)_{x=0} &= \sum_1^\infty u_n(4t)^n G_{2n}(0) - \frac{1}{h(k_S)^{1/2}} \sum_0^\infty u_n(4t)^{n-1/2} G_{2n-1}(0) \\ &= \sum_1 u_n t^n / n! = U(t) \text{ as } G_{2n-1}(0) = F_n(0) = 0. \end{aligned} \quad (12.3.92)$$

$$\begin{aligned} \left( T_S^2 - \frac{1}{h} \frac{\partial T_S^2}{\partial x} \right)_{x=0} &= a_0 h F_1(0) + \sum_1^\infty a_n \left\{ (4t)^{n/2} \left[ E_n(0) + h(4k_S t)^{1/2} F_{n+1}(0) \right] \right. \\ &\quad \left. - \frac{1}{h(k_S)^{n/2}} (4t)^{(n-1)/2} \left[ F_{n-1}(0) + h(4k_S t)^{1/2} E_n(0) \right] \right\} = 0. \end{aligned} \quad (12.3.93)$$

If  $T_L^1$  and  $T_L^2$  given in Eq. (12.3.90) are used, it can be checked that  $T_L^1|_{t \rightarrow 0} = \bar{V}(x)$  and  $T_L^2|_{t \rightarrow 0} = 0$  because  $i^n \operatorname{erfc}(\infty) = 0$ . It seems appropriate to express the phase-change boundary in terms of the following series.

$$S(t) = (4k_S)^{1/2} \sum_0^\infty \lambda_n \tau^{n+1}, \quad \tau = t^{1/2}; \quad \lambda_n \text{ are unknowns.} \quad (12.3.94)$$

There are now three sets of unknowns  $a_n$ ,  $b_n$  and  $\lambda_n$  and three conditions to be satisfied at  $x = S(t)$ . In principle after substituting  $S(t)$ ,  $T_L$  and  $T_S$  and by comparing different integral powers of  $t^{1/2}$  on both sides of these conditions, these unknown coefficients can be determined as was done in Boley's embedding technique [244]. However, this procedure of comparing the coefficients is more difficult than the procedure of differentiating the equations with respect to  $\tau$  as followed in the new embedding technique in [458] and also followed in [468]. Successive differentiations of equations with respect to  $\tau$  do yield the unknown coefficients  $a_n, b_n, \lambda_n$  starting from  $n = 0$ . However, this procedure will not give system of equations in terms of

a general  $n$  or in other words recurrence relations for the unknowns. In both [244, 458] it is also not possible to develop a system of equations for a general  $n$  or in other words recurrence relations for the unknowns. To obtain a system of equations in [468] for a general  $n$  each  $E_n, F_n$  and  $G_n$  for  $n = 0, 1, 2, \dots$  is considered functions of  $x = x(t)$  and  $t$ . The derivatives with respect to time become material derivatives and we get

$$D^n f = \frac{d^n f}{dt^n} = \sum_{n=0}^N \sum_{m=0}^{N-n} \left( \frac{N!}{n!} \right) z_m^{N-n}(x) \frac{\partial^n}{\partial t} \frac{\partial^m}{\partial x} f(x, t), \quad (12.3.95)$$

$$z_m^n(x) = \sum_{\beta_k} \frac{1}{|\underline{\beta}_1| |\underline{\beta}_2| \cdots |\underline{\beta}_n|} (Dx)^{\beta_1} \left( D^2 x / 2! \right)^{\beta_2} \cdots (D^n x / n!)^{\beta_n}, \quad (12.3.96)$$

$$\sum_1^n \beta_r = m, \sum_1^n r \beta_r = n; Df = \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \cdot \frac{dx}{dt}, D^2 f = D \left( \frac{df}{dt} \right). \quad (12.3.97)$$

To show how difficult the calculations are, we give below only a small portion of them. Let  $w = (k_S/k_L)^{1/2}$ .

$$D_\tau^N \left[ (2\tau)^n G_n \left( \frac{S(t)}{(4k_L t)^{1/2}} \right) \right]_{\tau=0} = D_\tau^N \left[ (2\tau)^n G_n \left( w \sum_0^\infty \lambda_n \tau^n \right) \right]_{\tau=0} = N! A_n^N(w). \quad (12.3.98)$$

$$A_n^N(w) = 2^n \sum_{m=0}^p w^m G_{n-m}(w\lambda_o) \sum_{\beta_r} \left( \prod_{r=1}^p \frac{\lambda_r^{\beta_r}}{\beta_r!} \right), \sum_1^{N-n} \beta_r = m, \sum_1^{N-n} r \beta_r = N - n. \quad (12.3.99)$$

$D_\tau$  stands for derivative with respect to  $\tau$ .

When Stefan condition is differentiated with respect to  $\tau$ , the first derivative will involve negative power of  $\tau$  and so it should be first multiplied by  $\tau$  and then differentiated as it is a valid operation. Just a glance at Eqs (12.3.95)–(12.3.99) is enough to understand how much effort is required to obtain system of equations for unknown coefficients  $a_n, b_n$  and  $\lambda_n$  for any general  $n$ . The remaining part of the calculations and reporting of system of recurrence relations will not be presented here as it is too lengthy. The method of solution has been explained. For further details refer [468].

One important point to be noted in Eq. (12.3.99) is that the nonlinearity of  $\lambda_r$  is increasing with  $r$ . This implies that in a two-phase problem except the coefficient  $\lambda_0$  or perhaps  $\lambda_1$  other  $\lambda_2, \lambda_3, \dots$  are to be obtained by numerically solving coupled nonlinear equations. With only zeroth-order term or few terms taken in the series of  $S(t)$  in Eq. (12.3.94), which can be obtained analytically, the analytical solution can at best be regarded as a short-time solution. A simple proof can also be developed independently to show that temperature so obtained is not unique and this proof is given below.

We are not aware of analytical solutions of cylindrical and spherical problems of one or two phases in one or more dimensions obtained with the help of series of the form  $E_n, F_n$  and  $G_n$  given in Eqs (12.3.85), (12.3.86). A big list of publications of the author of [468] using these repeated integrals of error functions can be found in [800]. In the references only one-dimensional semiinfinite regions  $0 \leq x < \infty$  have been considered to obtain analytical solutions. However, for finite regions even in one-dimensional case, no analytical solution

even for a short time could be found. No numerical work regarding computation of analytical solution is available. It is not known how even for a very short time more than one-dimensional problems can be solved even for semiinfinite regions.

A proof for the convergence of phase-change boundary  $x = S(t)$  has been developed in [468] and similar type of proofs have been reported in all the publications in which solutions have been obtained using repeated integrals of error functions. It has been stated that the series of  $S(t)$  given in Eq. (12.3.94) is absolutely and uniformly convergent for all  $t < t_0$  for some  $t_0$ . Regarding the proof of convergence, the authors of [297, 298] have commented that the proofs are unconvincing. We do not want to go into the steps of the proofs and checking them but prove here with a simple argument given below that the convergence proof even if granted to be correct does not prove that the solution is a long-time solution as the temperature solutions constructed in Eqs (12.3.89), (12.3.90) are nonunique.

Consider the temperature solution  $T_L(x, t)$  given below in which  $T_L^1(x, t)$  and  $T_L^2(x, t)$  are as reported in Eq. (12.3.90) to which one more temperature is added.

$$T_L(x, t) = T_L^1 + T_L^2 + T_L^3, \text{ where } T_L^3(x, t, k_L) = \int_{(x+2)/(2\sqrt{k_L t})}^{(x+1)/(2\sqrt{k_L t})} e^{-p^2} dp. \quad (12.3.100)$$

At  $t = 0$ ,  $T_L^3(x, t)|_{t=0} = 0, x \geq 0$ . The derivatives of  $T_L^3(x = S(t), t)$  with respect to  $\sqrt{t}$  of all finite orders are zero as  $t \rightarrow 0+$ . Therefore  $T_L^3(x, t)$  does not contribute towards calculation of  $S(t), a_n$  and  $b_n$  but it is not zero for  $x > 0, t > 0$ . Similarly a third temperature solution  $T_S^3(x, t)$  can be added in Eq. (12.3.89) and  $T_S(x, t)$  can be taken as  $T_S(x, t) = T_S^1 + T_S^2 + T_S^3$ . If temperature is prescribed at  $x = 0$  then  $T_S^3(x, t)$  can be taken as

$$T_S^3(x, t) = T_L^3(x, t, k_S) - T_L^3(-x, t, k_S); T_S^3(0, t) = 0. \quad (12.3.101)$$

In the flux prescribed case at  $x = 0$ , the two terms on the right of Eq. (12.3.101) will be added so that  $\partial T_S^3(x, t)/\partial x = 0$  at  $x = 0$ . The nonuniqueness of  $T_L(x, t)$  is sufficient to prove that the solution of the problem considered in [468] is not unique. It is often extremely difficult to check the calculations and proofs presented in the works of a different research worker particularly when calculations are so heavy. The discrepancies cannot be easily pointed out. A convergence proof gives importance to the work and notwithstanding the contribution of such proofs, the works become noteworthy and cited generously.

A coupled heat and mass transfer problem similar to the problem considered in [406] with a difference that initial temperature and concentration are functions of  $x$  and prescribed temperature at  $x = 0$  is a function of time has been considered in [469]. At the solidus and liquidus curves, concentration is a function of temperature. The solution has been constructed with the help of series expansions of the form given in Eqs (12.3.85), (12.3.86). In this case five sets of unknowns and five sets of equations for the unknowns have been reported. The solution procedure is similar to the one explained in the context of [468]. Because of the presence of concentration gradients in the solid and liquid regions and the coupling between temperature and concentrations, the solution details are different. The motion in the liquid due to density jump at the interface has been studied in [470]. Except obtaining recurrence relations, no analytical or numerical solution has been reported in these works.

The nonlinearity of Stefan problem is not reflected in the linear heat equation if all the thermo-physical parameters are constant and the heat equation is homogeneous. The boundary

conditions at the phase-change interface make the problem highly nonlinear. In the context of a planar one-phase one-dimensional Stefan problem, it was suggested in [471] that if the transformation  $\xi = (a-x)/(a-S(t))$  is used in the formulation then the free boundary  $x = S(t)$  is fixed at  $\xi = 1$  and the region  $0 \leq x \leq a$  is transformed to  $0 \leq \xi \leq 1$ . This transformation as mentioned earlier is called Landau transformation. Similar type of transformations were later used in many other Stefan problems such as in Eqs (9.7.53), (12.3.56). The linear heat equation is transformed to a nonlinear equation if Landau transformation is used. Exact solution in [471] is reported for a heat conduction problem without phase change in a finite slab but several particular cases and approximate solutions such as in a steady-state case have been discussed for a Stefan problem in a semiinfinite region. This study is mainly concerned with determining melting rates in a finite and semiinfinite solid due to prescribed temperature. Transformations of Landau type are widely used in analytical and numerical studies of Stefan problems.

A short-time solution of a coupled heat and mass transfer planar problem of one-dimensional solidification has been considered in [472]. The formulation is similar to the one considered in [406] but in this case constant heat flux is prescribed and so the exact solution is not possible and series solutions using repeated integrals of error functions such as  $E_n$  and  $F_n$  given in Eqs (12.3.85), (12.3.86) have been used to construct solution. Although in the general formulation a finite one-dimensional slab  $0 \leq x \leq L < \infty$  has been considered, in view of short-time solution the effect of the boundary conditions at  $x = L$  has been neglected. Concentration flux is zero at  $x = 0$ . Initial concentration and temperature are constant. Three terms in the moving boundary and four terms in the concentrations have been obtained by comparing different integral powers of  $t^{1/2}$  on both sides of the equations of five prescribed conditions which are available to determine the five sets of unknowns.

By using a combination of Boley's embedding technique and iterated error functions, a planar one-dimensional two-phase solution of a melting problem for a short time has been obtained in [473]. The melting takes place due to internal heat generation term  $Q(x)$  in the solid which after scaling is taken in the form

$$Q = \sum_{n=0}^{\infty} H_{2n} x^{2n}, 0 \leq x \leq 1, t \geq 0; \left. \frac{\partial T_L}{\partial x} \right|_{x=0} = 0. \quad (12.3.102)$$

Heat source term  $Qk_S/k_L$  is present in the liquid also. This problem can be easily formulated as the formulation is on the same lines as in earlier formulations. For  $t \leq t_m$ , where  $t_m$  is the time till the temperature of plate at  $x = 0$  reaches the melting temperature, we have a heat conduction problem without phase change. After  $t \geq t_m$ , the temperature of the solid using Boley's embedding technique can be written as

$$T = T_S^1 + T_S^2, t \geq 0; T_S^1(x, t) = \sum_{n=0}^{\infty} \sum_{i=0}^n a_{ni} t^i x^{n-i}, \quad (12.3.103)$$

$$a_{ni} = \frac{1}{i!(n-i)!} \left( \frac{\partial^n T_S^1}{\partial t^i \partial x^{n-i}} \right)_{(x=0, t=0)}, a_{ni} \text{ are known.}$$

$T_S^1(x, t)$  is the temperature of pure heat conduction problem with no phase change and it is assumed to be known and so  $a_{ni}$  are known.



$$T_S^2(x, t) = 2 \sum_{i=1}^{\infty} q_{oi} t^{(i+1)/2} \frac{i^{i+1} \operatorname{erfc} \left\{ x/(2t^{1/2}) \right\}}{i^i \operatorname{erfc}(0)}, \quad (12.3.104)$$

where  $q = \sum_{i=1}^{\infty} q_{oi} t^{i/2}$  is the fictitious flux applied at  $x = 0$  after  $t \geq t_m$  as done in using the embedding technique. The solution of liquid temperature has been borrowed from some earlier work of a coworker of Boley. If  $x = S(t) = \sum_{i=1}^{\infty} S_{oi} t^{(i+2)/2}$  then with the help of interface conditions, the unknowns  $q_{oi}$  and  $S_{oi}$  and one more set of unknowns in the fictitious initial liquid temperature can be determined and first five  $S_{oi}$  are given in [473].

A two-phase two-dimensional melting problem with natural convection in the liquid and internal heat generation term has also been discussed in [473]. A perturbation solution for small  $\epsilon = (\rho_S - \rho_L)/\rho_L$  has been obtained. The authors themselves write that the solution is quite cumbersome but some unknowns in the moving boundary have been determined. Quasisteady conditions in the plate have also been discussed. All the equations have been suitably scaled in [473].

A two-phase solidification problem in a one-dimensional finite slab has been considered in [474] and the solution has been obtained by using iterated error functions. This work is an extension of the type of problem considered in [468]. In [474] a finite slab has been considered. The initial temperature of the liquid is some constant, say  $T_i$  which is also the prescribed temperature at the edge  $x = w$  of a slab which is of width  $w$ . The prescribed temperature at  $x = 0$  is also a constant. The formulation of the problem is similar to that of Neumann problem and is not being reported here. The transformation  $y = x/(x - w)$  has been used so that in terms of independent variables  $(y, t)$ , the domain becomes  $0 \leq y < \infty$  and  $x = S(t)$  becomes  $y = P(t)$ . The transformation  $(x, t) \rightarrow (y, t)$  looks simple in form but it changes the formulation considerably. For example, the homogeneous heat equations become nonhomogeneous. Using the invariance property of the linear parabolic equation, the similarity variables  $\eta_i = \frac{x}{2\sqrt{k_i t}}$ ,  $i = L, S$  can be used and solutions of heat equations can be written in terms of series expansions of the form in Eqs (12.3.85), (12.3.86). A procedure to obtain particular solutions of the transformed heat equations has been given. Interface conditions can be expressed in terms of variables  $(y, t)$ . The boundary  $x = S(t)$  is now  $y = P(t)$  and  $P(t)$  is expressed as

$$P(t) = 2\sqrt{k_S} \sum_{m=0}^{\infty} a_m \tau^{m+1}, \quad \tau = t^{1/2}, \quad a_m \text{ are unknowns.} \quad (12.3.105)$$

The boundary conditions at  $y = P(t)$  have been expressed in terms of double and triple power series expansions. Obtaining recursive relations are much more difficult now than in [468]. No attempt has been made to obtain analytical or numerical solution, not even at the zero level of indices. The analytical solution if obtained will be at best a very short-time solution.

Under Lie group theory, the heat equations remain invariant under the transformations used above and the solution of the original problem can be found from the transformed formulation including boundary and initial conditions (cf. [475]).

Bergman [476] proposed series expansions in terms of series of repeated integrations of error functions for elliptic problems in gasdynamics and elastostatics. The series expansions considered in Eqs (12.3.85), (12.3.86) are extensions of series expansions of Bergman type

and such series expansions could be called *Bergman-type series expansions* or expansions in terms of complementary error function integrals or repeated integrals of error functions. The shrinking core model of concentration considered in [464] has been considered in [477] also and the moving boundary has been fixed in the same way. By using series solution in terms of repeated integrals of error functions, a starting solution, i.e. the first term of the series, has been obtained to start a finite difference numerical solution.

### Power Series and Polynomials

In the early stages of progress of obtaining analytical solutions of one-dimensional nonlinear Stefan problems many series solutions using power series in time  $t^{1/2}$  for the moving boundary and double power series in  $(x, t)$  for temperature were constructed. Plenty of such solutions exist in the literature but here only some of them which have some characteristic features will be briefly mentioned. Power series solutions of two Stefan problems with temperature gradients in only one phase in the region  $0 \leq x < \infty$  have been given in [24, Section 11.3]. The boundary condition in one case is of constant prescribed flux and in another case it is  $\left. \frac{\partial T}{\partial x} \right|_{x=0} = hT$ . In both the problems a power series in time  $t$  for  $x = S(t)$  and power series of the form  $\sum_{n=0}^{\infty} \sum_{i=0}^n a_{ni} t^i x^{n-i}$  for temperature have been considered. However, if temperature solution is taken in a series of iterated error functions then series expansions in powers of  $t^{1/2}$  have to be considered. The method of solution, if valid, will itself force the unwanted coefficients to be zero. The method of solution is simple in this case. Substitute the power series expansions of various variables in the equations of problem formulation and by comparing different powers of  $t$  and  $x$  on both sides of the equations determine the unknowns.

In the one-dimensional Cauchy problem considered in [478] we determine temperature  $T(x, t)$  in the one-dimensional region,  $0 \leq x < \infty$ , when both temperature and heat flux are prescribed at some point  $x = x_0$ ,  $x_0$  could be zero or nonzero. Consider the heat transfer problem given below.

$$x^{-r} (x^r T_x)_x = T_t(x, t); \quad T(x_0, t) = f(t); \quad \frac{\partial T}{\partial x}(x_0, t) = q(t), \quad (12.3.106)$$

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, f^n(t) = \frac{d^n f}{dt^n}, f_n = f^n(t)|_{t=0}; \quad q(t) = \sum_{n=0}^{\infty} q_n \frac{t^n}{n!}. \quad (12.3.107)$$

For  $r = 0, 1, 2$ , we have planar, cylindrical and spherical problems, respectively, and  $x$  is Cartesian coordinate for exponent  $r = 0$  and radial coordinate for  $r = 1, 2$  in these problems. If  $x_0 \neq 0$  then the three temperature solutions in three different regions are given below.

$$T(x, t) = \sum_{n=0}^{\infty} \left[ \frac{f^n(t)(x - x_0)^{2n}}{(2n)!} - \frac{q^n(t)(x - x_0)^{2n+1}}{(2n+1)!} \right], \text{ for plane.} \quad (12.3.108)$$

$$T(x, t) = \sum_{n=0}^{\infty} \left[ f^n(t) \frac{(x - x_0)^{2n}}{(2n)!} \frac{(x + 2nx_0)}{(2n+1)!x} - q^n(t) \frac{(x - x_0)^{2n+1}}{(2n+1)!x_0} \right], \text{ for sphere.} \quad (12.3.109)$$

$$T(x, t) = \sum_{n=0}^{\infty} \left[ f^n(t) d^n(x^2/4) - \frac{1}{2} q^n(t) e^n(x^2/4) \right], \text{ for cylinder.} \quad (12.3.110)$$

Expressions of  $d^n$  and  $e^n$  which are functions of  $x^2$  in Eq. (12.3.110) have been derived and reported in [478] but they are too lengthy to be reported here. It can be easily checked that in Eqs (12.3.108), (12.3.109),  $T(x, t)|_{x \rightarrow x_0} = f(t)$  and  $(\partial T / \partial x)_{x \rightarrow x_0} = q(t)$ . Note that if  $x_0 = 0$

then for plane problem Eq. (12.3.108) is still valid but other two equations will change as follows.

$$T(x, t) = \sum_{n=0}^{\infty} \left[ \frac{f^n(t)x^{2n}}{(2n+1)!} + \frac{q^n(t)x^{2n-1}}{(2n)!} \right], \text{ for sphere.} \quad (12.3.111)$$

$$T(x, t) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{(n!)^2} \left[ f^n(t) - q^n(t) \left\{ \ln_e x - 1 - \frac{1}{2} - \dots - \frac{1}{n} \right\} \right]. \quad (12.3.112)$$

The temperature in Eq. (12.3.112) is for cylinder. It has been mentioned in reference [26] of this book that if  $f(t)$  and  $q(t)$  are analytic functions then  $T(x, t)$  is also analytic and nonanalytic solutions do not exist. The above series expansions for the case  $x_0 = 0$  can be obtained by taking

$$T(x, t) = \sum_{m,n} (a_{mn} + b_{mn} \cdot \ln_e x) x^m t^n. \quad (12.3.113)$$

If  $T(x, t)$  is substituted in heat equation then recursion relations for  $a_{mn}$  and  $b_{mn}$  can be obtained and  $T(x, t)$  can be decomposed into series of polynomial and quasipolynomial solutions of the heat equation. These series can be summed up to obtain solutions which are reported previously. For  $n = 0$  in Eq. (12.3.112), only  $\ln_e x$  term is to be considered in the expression mentioned in the curly brackets. Fourier series and Bessel series solutions of the heat equations can be obtained if  $f(t)$  and  $q(t)$  are chosen as

$$f(t) = \sum_{m=0}^{\infty} A_m \exp(-b_m^2 t) \text{ and } q(t) = \sum_{m=0}^{\infty} B_m \exp(-b_m^2 t). \quad (12.3.114)$$

If the series for  $f(t)$  and  $q(t)$  given in Eq. (12.3.107) is used then  $T(x, t)$  for  $x_0 = 0$  can be expressed in terms of series of polynomials.

$$T(x, t) = \sum_{n=0}^{\infty} \left[ f_n P^{2n}(x, t) - q_n P^{2n+1}(x, t) \right], \text{ for plane.} \quad (12.3.115)$$

$$T(x, t) = \sum_{n=0}^{\infty} \left[ f_n P^{2n+1}(x, t)/x + q_n P^{2n}(x, t)/x \right], \text{ for sphere.} \quad (12.3.116)$$

$$T(x, t) = \sum_{n=0}^{\infty} \left[ f_n Q^{2n}(x, t) - q_n R^{2n}(x, t) \right], \text{ for cylinder.} \quad (12.3.117)$$

The expressions of the polynomials  $P^{2n}$  and  $P^{2n+1}$  are given below and for  $Q^{2n}$  and  $R^{2n}$ , refer [478].

$$P^{2n}(x, t) = xW^{2n-1}(x, t) = \frac{t^n}{n!} + \frac{t^{n-1}x^2}{(n-1)!2!} + \frac{t^{n-2}x^4}{(n-2)!4!} + \dots + \frac{t^0x^{2n}}{0!2n!}, \quad (12.3.118)$$

$$P^{2n+1}(x, t) = xW^{2n+1}(x, t) = \frac{t^n x}{n!1!} + \frac{t^{n-1}x^3}{(n-1)!3!} + \dots + \frac{t^{n-2}x^5}{(n-2)!5!} + \dots + \frac{t^0x^{2n+1}}{(2n+1)!}. \quad (12.3.119)$$

The temperature solutions obtained in Eqs (12.3.108), (12.3.109) have been used to study inverse one-dimensional one-phase problems. If it is assumed that the phase-change boundary has constant velocity  $S_1$  then what should be the temperature or flux boundary condition to be prescribed at  $x = x_0, 0 \leq x < \infty, x_0 \neq 0$ . The solution of this inverse problem reported in [478] is given below. In the Cartesian case

$$f(t) = \exp[S_1(S - x_0)] - 1, \quad q(t) = S_1 \exp[S_1(S - x_0)], \quad (12.3.120)$$

$x = S(t) = S_0 + S_1 t$ ,  $S(t)$  is phase-change boundary.  $x = x_0 = S_0 = S(0)$ .

$$T(x, t) = \exp[S_1(S(t) - x)] - 1, \quad S(t) = S_0 + S_1 t. \quad (12.3.121)$$

For spherical and cylindrical problems also such solutions have been reported. In the spherical problem, the solution of inverse problem is given below. The cylindrical case is a bit complicated and see [478] for this solution. For spherical case

$$f(t) = (1 - 2/S_0 S_1)(\exp(S_1^2 t) - 1) + (2S_1 t/S_0) \exp(S_1^2 t), \quad (12.3.122)$$

$$q(t) = 2/S_1 + 2/S_1 [S_0^2 S_1^2/2 - 1 + (S_0 S_1 + 1)S_1^2 t] \exp(S_1^2 t), \quad (12.3.123)$$

$$T(x, t) = (1 - 2/(x S_1))(\exp(S_1(S - x)) - 1) - 2(1 - S/x) \exp[S_1(S - x)]. \quad (12.3.124)$$

These above results are useful in the context of inverse heat transfer problems with or without phase change.

A two-phase one-dimensional Stefan problem in the region  $0 \leq x < \infty$  was considered in [479] by using series expansions in conjunction with two-sided Laplace transform and similarity transformation. In one- and two-dimensional problems in plane, the new embedding technique [462] can also be interpreted as solutions obtained with the help of two-sided Laplace transforms in one and two dimensions, respectively. The difficulty comes in studying the spherical and cylindrical problems and also in class II and III problems. Without interpretation in terms of embedding and assigning fictitious initial temperatures in suitable fictitious extensions of the original regions, the solutions of these problems cannot be obtained. The two-sided Laplace transform solution obtained in [479] is as follows.

If the transformations  $\tau = t$  and  $x = yS(\tau)$  are used in the linear heat equations such as Eq. (1.3.1), they get transformed. Suppose  $T_L(x, t) = v(y, \tau)$  then in terms of  $v$ , we get

$$S^2(\tau) \frac{\partial v}{\partial \tau} - S(\tau) \frac{dS}{d\tau} y \frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial y^2}. \quad (12.3.125)$$

Let the two-sided Laplace transform of  $v(y, \tau)$  be denoted by  $\bar{v}(p, \tau)$ . Then the Laplace transform of Eq. (12.3.125) is

$$\bar{v}(p, \tau) = e^{p\tau}/S^2(\tau) \{F_1(p/S(\tau)) + F_2(p/S(\tau))\}, \text{ if } S(0)S'(0) \neq 0. \quad (12.3.126)$$

$F_1$  and  $F_2$  are the images of arbitrary one-sided functions  $\varphi_1$  and  $\varphi_2$ . If the problem is that of solidification then  $\varphi_1(y, \tau)$  in Eq. (12.3.127) is the known prescribed initial temperature of the liquid and  $\varphi_2(y, \tau)$  is some unknown function to be determined. Similarly the temperature solution in the solid region can be written with the help of two-sided Laplace transform. On inversion

$$v(y, t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \left\{ \exp \left[ \frac{-S^2(y-w)^2}{4t} \right] \varphi_1(Sw) + \int_{-\infty}^0 \exp \left[ \frac{-S^2(y-w)^2}{4t} \right] \varphi_2(Sw) \right\} \frac{S}{2\sqrt{t}} dw, \quad (12.3.127)$$

$\varphi_1(y, \tau)$  is the initial temperature in the region  $0 \leq x < \infty$  and  $\varphi_2(y, \tau)$  is the initial temperature in the region  $-\infty < x < 0$ . The initial temperature  $\varphi_2(y, \tau)$  is fictitious as the region  $-\infty < x < 0$  does not exist in the problem. It should be made clear that the interpretation of  $\varphi_1$  and  $\varphi_2$  as initial temperatures has been given by us and not reported in [479]. Depending on the problem under consideration, one of  $\phi_1$  and  $\phi_2$  could be known and the other unknown. For convenience, thermo-physical parameters have been taken as unity in Eq. (12.3.125) and the formulation of the problem is of routine type so it is not given here.

The following series expansions for known and unknown functions are assumed in which  $\phi_1$  and  $\phi_2$  are the initial temperatures for liquid as in Eq. (12.3.127) and  $\phi_3, \phi_4$  are the initial temperatures for the solid which will occur in the Laplace transform solution for solid. The problem could be of melting or solidification.

$$\phi_{i,j}(z) = \sum_{k=0}^{\infty} z^k \phi_{ik,j}, \quad i = 1, 2, \text{ for } j = L \text{ and } i = 3, 4 \text{ for } j = S; \quad S(\tau) = \sum_1^{\infty} S_k \tau^{k/2}. \quad (12.3.128)$$

The subscripts  $L$  and  $S$  stand for liquid and solid regions, respectively. These series expansions are substituted in temperatures and then in boundary and interface conditions which are to be satisfied. By following the procedure of repeated differentiations with respect to  $\tau^{1/2}$  as suggested in [462, 468], the unknown coefficients can be determined from recurrence relations which are in the form of integrals which cannot be evaluated analytically. Using this procedure of obtaining the solution, in [480] for a one-phase problem eight coefficients in the moving boundary  $x = S(t)$  have been determined. However, in a two-phase problem after first few terms the calculations become very lengthy.

If the prescribed functions are analytic in the whole complex plane then  $F_1$  and  $F_2$  in Eq. (12.3.126) are also regular and in the neighbourhood of infinity we have

$$F_i(\xi/p) = F_{i0} + F_{i1}(\xi/p) + F_{i2}(\xi/p)^2 + \dots, \quad i = 1, 2. \quad (12.3.129)$$

In both the phases temperatures will be of the following form.

$$T(y, t) = \frac{1}{2} \sum_{k=0}^{\infty} \left( 2t^{1/2} \right)^k F_{1k} i^k \operatorname{erfc} \left( -S(t)y/2t^{1/2} \right) + \frac{1}{2} \sum_{k=0}^{\infty} (-1)^{k+1} (2t^{1/2})^k F_{2k} i^k \operatorname{erfc} \left( S(t)y/2t^{1/2} \right), \quad (12.3.130)$$

$i^n \operatorname{erfc}(x)$  are iterated error functions, see Eq. (12.2.24).

This solution in Eq. (12.3.130) is more accurate for small values of  $S(t)/2t^{1/2}$  and is of the type obtained in [468]. The unknown constants in the series expansions in Eq. (12.3.129) can

be identified with coefficients in the series expansions of prescribed and unknown functions and with known and unknown coefficients which appear in Eqs (12.3.89), (12.3.90). The small-time solution can be obtained by comparing the powers of  $t^{1/2}$  on both sides of conditions to be satisfied or following the procedure of repeated differentiations.

A large-time asymptotic solution of a one-dimensional one-phase problem has been obtained in [481] by using Lagrange-Bürmann expansions. The solution is obtained in several steps which are all connected. Not a single step in the procedure can be left unreported otherwise the method does not make sense. This solution cannot be discussed here and the interested reader may refer [481] for this solution.

### 12.3.2 Semianalytical Solutions Using Series Expansions

It is difficult to obtain analytical solutions of Stefan problems in series form without the help of numerical computations. Invariably, the calculation of coefficients or various terms of the series expansions require their numerical solution and such solutions could be called analytical-numerical solutions or semianalytical solutions. To clarify this point further we consider an example. In the Neumann solution  $\delta$  is determined from Eq. (1.3.17). To calculate  $\delta$  we have to resort to some simple numerical scheme but still the solution in Eq. (1.3.17) is an analytical solution and except  $\delta$  all other quantities are known. Now we consider Eq. (12.3.66). To determine time  $t$  or  $y$ , first of all  $A'_0(1)$  is required which if  $\beta$  is not small is to be determined numerically.  $A'_1(1)$  is again determined numerically and so on for  $A'_2(1)$ , etc. These values are substituted in Eq. (12.3.66) and numerical integrations are done. This solution could be considered an analytical-numerical or semianalytical solution. Some authors consider this type of solutions also analytical solutions. We respect their view point also but would like to maintain a distinct identity for such solutions. The reason why this solution in Eq. (12.3.66) has not been considered in the present section is that in the problem in [464], the method of fixing the phase-change boundary and obtaining a solution is more important than obtaining semianalytical solutions in series form.

Variety of semianalytical solutions exist in the literature. For example on using Green's functions in some solutions, the problem is converted to that of an integral equation or an integro-differential equation and then suitable numerical schemes are used to obtain numerical solutions. Perturbation solutions and solutions by approximate methods are also semianalytical solutions. They have been put in separate sections because methods of solutions are very different in each of these methods. The variety in method decides the solution to be presented in a section or a subsection. However, one thing that is common in all the semianalytical solutions is to convert the formulation of the problem to another suitable analytical form so that instead of obtaining numerical solution of the problem formulation in its original form, it is comparatively easier with some advantage to obtain the numerical solution from the transformed analytical form. In some particular cases, exact solutions of transformed problem can be obtained.

The problem considered in [482] has a special feature. In this problem the temperature at the moving boundary can be prescribed arbitrarily. When all the boundary conditions are known, it is somewhat easier to choose the method of solution. A one-dimensional one-phase mass transfer problem has been considered in [482]. Let  $\bar{c}_L(x, t)$  be the mole fraction of impurity in the liquid which initially occupies the region  $0 \leq x < \infty$ . The region  $-\infty < x < 0$  is occupied by solid with mole fraction  $\bar{c}_S = \text{constant}$ . The diffusion equation in liquid in which the coordinate system is attached to the moving boundary is given by

$$\frac{\partial \bar{c}_L}{\partial t} = D \frac{\partial^2 \bar{c}_L}{\partial z^2} + f(\hat{c}_S/\hat{c}_L) \frac{\partial \bar{c}_L}{\partial z}, \quad (z, t) > 0, \quad (12.3.131)$$

where  $z = x - h$  and  $f \equiv h$  is the velocity of the moving boundary.  $\hat{c}_S$  is the total molar concentration of the solid and  $\hat{c}_L$  of that of liquid and  $D$  is the diffusion coefficient of the liquid. The factor  $\hat{c}_S/\hat{c}_L$  arises due to unequal densities of the solid and liquid. The mass balance at the moving interface  $z = 0$  and initial condition at  $t = 0$  are given below.

$$f(\hat{c}_S/\hat{c}_L)(\bar{c}_S - \bar{c}_L) = D \frac{\partial \bar{c}_L}{\partial z}, \text{ at } z = 0; \quad \bar{c}_L(z, 0) = \bar{c}_L(\infty, t) = \bar{c}_\infty. \quad (12.3.132)$$

The concentration  $\bar{c}_L(z, t)$  is not prescribed at  $z = 0$ .

The solution of Eq. (12.3.131) is attempted by introducing a similarity transformation and defining some new functions. Let

$$\xi = t^{1/2}, \eta = z/2(Dt)^{1/2}, u(\xi, \eta) = \frac{\bar{c}_\infty - \bar{c}_L(z, t)}{\bar{c}_S - \bar{c}_\infty}, \lambda(\xi) = (f(t)\hat{c}_S/\hat{c}_L)(t/D)^{1/2}. \quad (12.3.133)$$

The new formulation of the above problem can be written as follows.

$$2\xi \frac{\partial u}{\partial \xi} = \frac{\partial^2 u}{\partial \eta^2} + 2(\eta + \lambda) \frac{\partial u}{\partial \eta}, \quad (\xi, \eta) > 0; \quad \xi = 0, \text{ at } t = 0. \quad (12.3.134)$$

$$\eta = 0 \text{ at } z = 0; \quad u(\xi, \infty) = 0, \xi > 0; \quad -\partial u / \partial \eta = 2\lambda(1 + u) \text{ when } \eta = 0. \quad (12.3.135)$$

As the exact analytical solution of this problem does not exist, a series solution has been attempted in the following form.

$$u(\xi, \eta) = \sum_{n=0}^{\infty} u_n(\eta) \xi^n, \quad \lambda(\xi) = \sum_{n=0}^{\infty} \lambda_n \xi^n. \quad (12.3.136)$$

When these series solutions are substituted in Eqs (12.3.134), (12.3.135), we obtain

$$\begin{aligned} u_0'' + 2(\eta + \lambda_0)u_0' &= 0; \quad u_n'' + 2(\eta + \lambda_0)u_n' - 2nu_n = -2 \sum_{k=0}^{n-1} \lambda_{n-k} u_k' \\ &= -g_n, \quad n \geq 1. \end{aligned} \quad (12.3.137)$$

$$-u_n'(0) = 2\lambda_n + 2\lambda_0 v_n + 2 \sum_{k=0}^{n-1} \lambda_{n-k} v_k = \varphi_n, \quad n \geq 1. \quad (12.3.138)$$

$$\varphi_0 = 2\lambda_0(1 + v_0); \quad u_n(0) = v_n, \quad u_n(\infty) = 0; \quad n \geq 0. \quad (12.3.139)$$

Instead of  $u_n(0)$  in Eq. (12.3.139) it is appearing as  $u^n(0)$  in [482]. It is easy to obtain  $u_0$  and  $v$  as

$$\left. \begin{aligned} u_0 &= v_0 \operatorname{erfc}(\eta + \lambda_0) / \operatorname{erfc}(\lambda_0), \\ v_0 / (1 + v_0) &= \sqrt{\pi} \lambda_0 \exp(\lambda_0^2) \operatorname{erfc}(\lambda_0), \quad v_0 > -1. \end{aligned} \right\} \quad (12.3.140)$$

General solution of second equation in Eq. (12.3.137) for  $n \geq 1$  can be written in terms of repeated integrals of error function. For example, for  $n \geq 1$  if  $g_n = 0$  then  $u_n = i^n \operatorname{erfc}(\pm y)$  satisfies Eq. (12.3.137), where  $i^n \operatorname{erfc}(y)$  is iterated error function and  $y = \eta + \lambda_0$ . If  $F_n(\pm y) = i^n \operatorname{erfc}(\pm y)$  then Wronskian  $W$  has the form  $W\{F_n(y), F_n(-y)\} = W_n \exp(-y^2) \neq 0$ , and  $F_n(y)$  and  $F_n(-y)$  are linearly independent.

$$W_n(y=0) = \left(\sqrt{\pi} 2^{n-2} n!\right)^{-1}. \quad (12.3.141)$$

The solution of nonhomogeneous second equation in Eq. (12.3.137) has the following representation

$$u_n(y) = \varphi_n \frac{F_n(y)}{F_{n-1}(\lambda_0)} + \int_{\lambda_0}^{\infty} g_n(s) \exp(s^2) G_n(y/s) ds; \quad g_n = 2 \sum_{k=0}^{n-1} \lambda_{n-k} u'_k, \quad n \geq 1. \quad (12.3.142)$$

The Green's function  $G_n(y/s)$  is given by

$$G_n(y/s) = W_n^{-1} \begin{cases} F_n(y) \left[ F_n(-s) + \frac{F_{n-1}(-\lambda_0)}{F_{n-1}(\lambda_0)} F_n(s) \right], & \lambda_0 \leq s < y \\ F_n(s) \left[ F_n(-y) + \frac{F_{n-1}(-\lambda_0)}{F_{n-1}(\lambda_0)} F_n(y) \right], & s > y \geq \lambda_0. \end{cases} \quad (12.3.143)$$

$G_n$  can be evaluated at the surface  $y = \lambda_0$  and when it is used in Eq. (12.3.42) we get

$$v_n = u_n(0) = [F_{n-1}(\lambda_0)]^{-1} \left[ \varphi_n F_n(\lambda_0) + \exp(-\lambda_0^2) \int_{\lambda_0}^{\infty} g_n(s) \exp(s^2) F_n(s) ds \right]. \quad (12.3.144)$$

Even if the concentration is prescribed arbitrarily at the moving interface then  $v_n$  for  $n \geq 0$  are known. The problem is to determine  $\varphi_n$  and  $g_n$ . The quantities  $\varphi_0$  and  $\lambda_0$  are known from Eq. (12.3.140) as  $v_0$  is prescribed.  $\varphi_1$  and  $g_1$  involve only  $\lambda_1$ ,  $\lambda_0$  and  $v_1$ . Eq. (12.3.138) provides one equation to determine  $\varphi_1$  and  $\lambda_1$  and another equation is provided by Eq. (12.3.144).  $g_1$  is now known which can be used to obtain  $\varphi_2$  and  $\lambda_2$ . Generally numerical integration is required but in some special cases exact analytical solution can be obtained. For example if  $u(\xi, 0) = \text{constant}$ , i.e.  $v_0 = \text{constant}$  and  $v_n = 0$  for  $n \geq 1$  then an exact solution of the Neumann type is obtained as it can be shown that  $u_1, u_2, \dots, u_n$  are all zero along with  $\lambda_1, \lambda_2, \lambda_n, \dots$ . For some other types of special solutions see [482].

The problem considered in [483] is a three-dimensional ablation problem. As the formulation is new, it is briefly presented here.  $T$  is temperature in the solid,  $H(x, y, t)$  is heat input and  $z = S(x, y, t)$  is the moving boundary. The melting starts at  $t = 0$

$$k \nabla^2 T(x, y, z) = \frac{\partial T}{\partial t}, \quad z > S(x, y, t). \quad (12.3.145)$$

On  $z = S(x, y, t)$ , we have

$$\begin{aligned} T(x, y, z, t)|_{z=S} &= T_m, \quad \frac{H(x, y, t)}{K} + \left[ 1 + \left( \frac{\partial S}{\partial x} \right)^2 + \left( \frac{\partial S}{\partial y} \right)^2 \right] \left( \frac{\partial T}{\partial z} \right)_{z=S} \\ &= \frac{l \rho}{K} \frac{\partial S}{\partial t}, \quad t > 0. \end{aligned} \quad (12.3.146)$$



From the isotherm condition given in Eq. (12.3.146), we obtain

$$\frac{\partial T}{\partial t} = -\frac{\partial S}{\partial t} \frac{\partial T}{\partial z}; \quad \frac{\partial T}{\partial r} = -\frac{\partial S}{\partial r} \frac{\partial T}{\partial z}, \quad r = x \text{ or } y. \quad (12.3.147)$$

If it is assumed that heat equation holds on  $z = S$  also then using Eq. (12.3.147) in Eq. (12.3.145), we get

$$\begin{aligned} & \left( \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial x^2} \right) \left( \frac{\partial T}{\partial z} \right)_{z=S} - \left[ 1 + \left( \frac{\partial S}{\partial x} \right)^2 \right. \\ & \left. + \left( \frac{\partial S}{\partial y} \right)^2 \right] \left( \frac{\partial^2 T}{\partial z^2} \right)_{z=S} = \frac{1}{k} \frac{\partial S}{\partial t} \left( \frac{\partial T}{\partial z} \right)_{z=S}. \end{aligned} \quad (12.3.148)$$

Using Eq. (12.3.148) in second equation in Eq. (12.3.146), we get

$$\begin{aligned} & \frac{H(x, y, t)}{K} + \left( \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} \right. \\ & \left. - \frac{1}{k} \frac{\partial S}{\partial t} \right) \left( \left( \frac{\partial T}{\partial z} \right)^2 / \frac{\partial^2 T}{\partial z^2} \right)_{z=S} = \beta \frac{\partial S}{\partial t}, \quad \beta = \frac{\rho l}{K}. \end{aligned} \quad (12.3.149)$$

If it is assumed that  $\left( \frac{\partial T}{\partial z} \right)^2 / \frac{\partial^2 T}{\partial z^2}$  at  $z = S$  is constant or independent of time then it yields considerable simplification in results. An obvious choice for temperature distribution is to represent it in the following series.

$$T(x, y, z, t) = T_m + \sum_{k=1}^n \varphi_k(x, y, t) f_k(z - S). \quad (12.3.150)$$

Two methods have been proposed to obtain solution. We discuss here only one method in which thermal boundary layer approximation is used (refer Section 12.8.1 also). It is assumed that

$$T|_{z=\delta} = T_0 \text{ and } \left( \frac{\partial^k T}{\partial z^k} \right) = 0, \quad z = \delta, \quad \text{for } k = 1, \dots, n-1. \quad (12.3.151)$$

$z = \delta$  is the depth up to which heat has penetrated and changed the initial temperature. In this approach heat equation is not satisfied. In view of isotherm condition (see Eqs 12.3.146, 12.3.151), we take

$$T = T_m + \sum_{k=1}^n \varphi_k f_k \left( \frac{z-S}{\delta-S} \right), \quad S < z < \delta, \quad t > 0. \quad (12.3.152)$$

The isotherm condition should be satisfied at  $z = S$  which implies  $\sum_{k=1}^n \varphi_k f_k(0) = 0$ . Using Eq. (12.3.151), we get nonhomogeneous system of equations to determine  $\varphi_1, \varphi_2, \dots, \varphi_n$  which are constant, not all zero. This follows when Eq. (12.3.152) is used in the first and second conditions in Eq. (12.3.151). If we calculate  $n$  derivatives of  $T$  with respect to  $z$  for

$n = 1, 2, \dots, n-1$ , a system of  $n$  linear equation is obtained to determine  $\varphi_1, \varphi_2, \dots, \varphi_n$ . In view of the equation corresponding to isotherm condition for a nontrivial solution of  $\phi_1, \phi_2, \dots, f_k$  may be chosen suitably at  $z = \delta$ , such as  $f_k = 1$ , and the solution for temperature can be obtained. Let

$$f_k = \left( \frac{z-S}{\delta-S} \right)^k, k = 1, 2, \dots, n; \sum_1^\infty \varphi_k f_k(0) = 0, \text{ on } z = S. \quad (12.3.153)$$

$$\varphi_k = (-1)^k \frac{n!}{(n-k)!k!} (T_m - T_0), n \neq 1; T = (T_m - T_0) \left[ 1 - \frac{Z-S}{\delta-S} \right]^n + T_0. \quad (12.3.154)$$

A complete set of functions of the form  $\xi^k, k = 1, 2, \dots$  has been chosen for  $f_k$  in Eq. (12.3.153) to obtain a nontrivial solution for  $\phi_1, \phi_2$ , etc. From Eq. (12.3.149), a differential equation to determine  $S(x, y, t)$  can be obtained.

$$\left. \begin{aligned} \frac{1}{D} \frac{\partial S}{\partial t} - \left( \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} \right) &= Q(x, y, t); Q = \frac{H}{K(T_m - T_0)} \left( 1 - \frac{1}{n} \right), \\ \frac{1}{D} &= \frac{1}{k} \left[ 1 + \frac{k\beta}{(T_m - T_0)} \left( 1 - \frac{1}{n} \right) \right]. \end{aligned} \right\} \quad (12.3.155)$$

No numerical work has been done in both the approaches. Another solution approach is complicated and it is difficult to present it here in a concise way.

The problem of freezing of water in small diameter pipes considered in [484] is a mixed bag of experimental, numerical and analytical studies. Our main interest here is in analytical solution. In simple words without mentioning assumptions and scalings of equations, the problem is that of a one-phase one-dimensional solidification in a cylindrical region whose formulation is given below.

$$\left. \begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T_S}{\partial r} \right) &= \frac{1}{k_S} \frac{\partial T_S}{\partial t}, S(t) \leq r \leq R, t > 0, \\ -K_S \frac{\partial T_S}{\partial r}(R, t) &= U(T_S(R, t) - T_\infty), t > 0. \end{aligned} \right\} \quad (12.3.156)$$

$$T_S(S(t), t) = T_m; T_S(r, 0) = T_m; K_S \frac{\partial T_S}{\partial r} \Big|_{S(t)} = \rho_S \frac{dS(t)}{dt}, \quad (12.3.157)$$

$r = S(t)$ ,  $S(0) = R$  is the phase-change boundary and  $R$  is the radius of cylindrical pipe. Experimental set up was done to compare the obtained semianalytical solution with experiments.

The solution of  $T_S(r, t)$  is obtained as the sum of two functions as given below. Let

$$T_S(r, t) = \theta(r, t) + \phi(r). \quad (12.3.158)$$

It is easy to obtain governing equations in terms of  $\theta(r, t)$  and  $\phi(r)$  as given below.

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \theta}{\partial r} \right) = \frac{1}{k_S} \frac{\partial \theta}{\partial t}; \quad \frac{1}{r} \frac{d}{dr} \left( r \frac{\partial \phi}{\partial r} \right) = 0. \quad (12.3.159)$$

If  $\theta(r, t)$  is taken as product of two functions  $P(r)$  and  $Q(t)$  then it is easy to get

$$\frac{d^2 P_m}{dr^2} + \frac{1}{r} \frac{dP_m}{dr} + \lambda_m^2 P_m = 0; \quad \frac{dQ_m}{dt} + k_S \lambda_m^2 Q_m = 0. \quad (12.3.160)$$

$\lambda_m$ , for  $m = 1, 2, \dots$ , are called separation constants or *eigenvalues* which are unknowns. The general solutions of  $P_m$  and  $Q_m$  can be easily written (cf. [24]) as

$$P_m(r) = A_m J_0(\lambda_m r) + B_m Y_0(\lambda_m r); \quad Q_m(t) = D_m \exp(-k_S \lambda_m^2 t). \quad (12.3.161)$$

$J_0(x)$  and  $Y_0(x)$  are Bessel functions of order zero and are of first and second kinds, respectively.  $A_m$ ,  $B_m$  and  $D_m$  are constants to be determined. Let  $\theta(r, t)$  satisfy the conditions

$$\theta(S(t), t) = 0 \quad \text{and} \quad -K_S \frac{\partial \theta}{\partial r}(R, t) = U T_S(R, t). \quad (12.3.162)$$

The first condition in Eq. (12.3.162) gives a relation between  $A_m$  and  $B_m$  and  $\lambda_m$  are determined from the second condition. Finally  $\theta(r, t)$  has the following series solution.

$$\theta(r, t) = \sum_{m=1}^{\infty} E_m [G_m - F_m Y_0(\lambda_m r)] \exp(-k_S \lambda_m^2 t),$$

$$G_m = J_0(\lambda_m R), \quad F_m = \frac{J_0(\lambda_m r)}{Y_0(\lambda_m r)} \Big|_{r=S(t)}, \quad E_m = A_m D_m. \quad (12.3.163)$$

$$\lambda_m [J_1(\lambda_m R) - F_m Y_1(\lambda_m R)] = \frac{U}{K_S} [G_m|_{r=R} - F_m Y_0(\lambda_m R)]. \quad (12.3.164)$$

The solution of second equation in Eq. (12.3.159) is given by

$$\phi(r) = W_1 \ln r + W_2, \quad W_1 = \frac{U(T_m - T_{\infty})}{U \ln(S(t)/R) - (K_S/R)}, \quad W_2 = T_m - W_1 \ln(S(t)). \quad (12.3.165)$$

To determine  $W_1$  and  $W_2$ , the conditions  $\phi(S(t)) = T_m$  and  $-K_S \frac{d\phi}{dr} \Big|_{r=R} = U(\phi(r=R) - T_{\infty})$  have been used.

Finally,  $T_S(r, t)$  is given by

$$T_S(r, t) = T_m + W_1 \ln \left( \frac{r}{S(t)} \right) + \sum_{m=1}^{\infty} E_m [G_m - F_m Y_0(\lambda_m r)] \exp(-k_S \lambda_m^2 t). \quad (12.3.166)$$

The summation in Eq. (12.3.166) is from 1 to  $\infty$  as for  $m = 0$  it can be shown that  $\theta_0(r, t) = 0$ .

In a *Sturm-Liouville problem* such as the one concerned with the solution of first equation in Eq. (12.3.160), the constants  $E_m$ ,  $m = 1, 2, \dots$ , are determined by orthogonality of eigenfunctions or in the present case the orthogonality of functions  $G_m - F_m Y_0(\lambda_m r)$ . This orthogonality is difficult to prove. Resorting to numerical computations, it is claimed in [484] that  $E_m$  can be determined approximately by using the following equation.

$$W_1 \int_{S(t)}^R \ln \left( \frac{r}{S(t)} \right) dr + E_m \int_{S(t)}^R (G_m - F_m Y_0(\lambda_m r)) dr = 0. \quad (12.3.167)$$

$S(t)$  for different values of time  $t$  are required. Although the exact equation to determine exact value of  $S(t)$  is given by Stefan condition, an approximate value of  $S(t)$  was used.

$$\frac{dS(t)}{dt} \approx \frac{K_S W_1}{\rho_S l S(t)}, \text{ as other terms were found numerically very small.} \quad (12.3.168)$$

Six values of  $\lambda_m$ ,  $m = 1, 2, \dots, 6$ , were obtained numerically from Eq. (12.3.164) and then  $E_m$  can be determined numerically from Eq. (12.3.167). The above equations are not dimensionless.

For numerical work a MATLAB code was used to determine  $\lambda_m$  and  $E_m$ . Considerable numerical work was done to obtain results for different cooling rates, freezing times and material properties. Comparison with experimental and available solutions has been done.

Same problem of solidification as in [484] with same boundary conditions but in a different context was studied in [485]. In [485] it is the solidification of phase-change material (PCM) in a latent heat thermal storage device which is in the form a cylindrical shell and tube geometry with radial fins. Only one-dimensional problems in radial and axial coordinates have been considered with three distinct phases of solid, liquid and fin. The heat transfer in the solid is in radial direction with moving boundary or phase-change boundary. In fin it is only heat conduction in radial direction with linear heat equation and initial and boundary conditions of usual type. The heat transfer in solidified region is independent of fin but in fin it depends on solid region. If  $r = \Omega(t)$  is the phase-change boundary and the solid region is  $R_{in} < r < \Omega(t)$ .  $R_{in}$  forms inner boundary for the fin and at  $R_{in}$  a convective condition of the type in Eq. (12.3.156) is prescribed. The liquid at  $t = 0$  occupies the region  $R_{in} < r < R_{out}$ , where  $R_{out}$  is the outer boundary of the fin. The outer finite boundary of the fin is insulated. The energy balance governing heat transfer equation in the liquid is in terms of a first-order linear ordinary differential equation in the axial direction with laminar flow. Our interest here is in the solution of solid phase whose formulation is as given in Eqs (12.3.156), (12.3.157) with a simple change that the region under consideration is  $R_{in} \leq r \leq \Omega(t)$ , where  $R_{in}$  is the inner radius of the cylindrical shell.

Series solution has been used for the solution but there is no indication of the method of solution used. The separation constants or eigenvalues are the roots of an equation which is entirely different from Eq. (12.3.164). Although there is no mention of constants such as  $E_m$  in Eq. (12.3.167), we assume that they have been directly substituted in the solution presented in Eq. (6) in [485] which may have been determined by using orthogonality condition as the norm is given. In [484] the authors could not establish any orthogonality condition. In the absence of method of solution reported and discussed in [485], it is not possible for us to comment any further on the solution.

The analytical solution in [485] has been compared with the numerical solution obtained by a numerical method based on enthalpy formulation. The estimate of solid-liquid interface by analytical solution is in good agreement with the numerical solution. Numerical work was done for rectangular enclosure also and it was found that PCM solidifies in cylindrical shell storage faster than in rectangular enclosure.

An approximate analytical-numerical solution of a quasisteady-state two-dimensional problem in cylindrical coordinates  $(r, z)$  has been studied in [486]. The  $z$ -coordinate is taken

along the axis of the long cylinder taken vertically upwards and the radial coordinate  $r$  is in horizontal direction. In the vertical gradient freeze technique (VGF) with bottom seeding, the pulling velocity has been taken as constant and surface temperature has been prescribed. There is no natural convection in the melt. The formulation of the problem is as follows:

$$\nabla^2 T_J(r, z) = 0; \quad \left. \frac{\partial T_J}{\partial r} \right|_{r=1} = Bi(T_{\text{ref}} - T_J|_{r=1}), J = S, L. \quad (12.3.169)$$

$$\left. \frac{\partial T_S}{\partial n} \right|_{z=f(r)} = \sigma \left. \frac{\partial T_L}{\partial n} \right|_{z=f(r)} + (S_L)^{-1} P_{\text{ep}} n_z; \quad T_S|_{z=f(r)} = T_L|_{z=f(r)} = 0. \quad (12.3.170)$$

The above formulation is in dimensionless form and for scalings see [486].  $r = 1$  is radius of the cylinder,  $0 \leq r \leq 1$ ,  $Bi$  is effective Biot number,  $T_{\text{ref}}$  is some reference temperature,  $\sigma = K_L/K_S$ ,  $S_L$  is Stefan number which is small,  $P_{\text{ep}}$  is the Peclet number of pulling,  $n_z$  is the normal in the  $z$ -direction at  $z = f(r)$  where  $z = f(r)$  is the phase-change boundary. The solid occupies the region  $z < 0$  and liquid occupies the region  $z > 0$ . The thermal gradients in the both solid and liquid along side wall boundary are constant. At  $r = 1$ ,  $T_S = T_S^0 z$  for  $z < 0$  and  $T_L = T_L^0 z$  for  $z > 0$  and if curvature of the surface  $z = f(r)$  is small then  $n_z \approx 1$  and  $\sigma T_L^0 = T_S^0 - (S_L)^{-1} P_{\text{ep}}$ . In the steady-state case, the interface  $z = f(r)$  is planar which is  $z = 0$  and  $T = z$  can be taken as a solution which has continuity of temperature and fluxes at  $z = 0$ .

To obtain solution of the quasisteady-state problem in which  $z = f(r)$  is the interface with small curvature, a correction term is introduced. This correction term  $\theta$  can be taken as  $\theta = T - T_0$ , where  $T_0$  is the temperature in the steady-state case with  $z = 0$  being interface.  $T_0$  can be taken as reference temperature in Eq. (12.3.169). For  $Bi \rightarrow \infty$ ,  $T|_{r=1} = T_{\text{ref}} = T_0$  and for  $Bi = 0$ , we have an adiabatic wall. To obtain  $\theta$ , an iterative procedure can be adopted. Let  $f^{(n)}$  and  $\theta^{(n)}$  be the  $n$ th iterative approximations to  $f(r)$  and  $\theta$ , respectively, then

$$\nabla^2 \theta_{S,L}^{(n)} = 0; \quad \left\{ \frac{\partial \theta_S^{(n)}}{\partial z} - \sigma \frac{\partial \theta_L^{(n)}}{\partial z} \right\} \bigg|_{z=f^{(n)}(r)} = (S_L)^{-1} P_{\text{ep}} + \sigma - 1, n = 0, 1, 2, \dots \quad (12.3.171)$$

$$\theta_S^{(n)} = \theta_L^{(n)} = -f^{(n)}(r) \text{ on } z = f^{(n)}(r); \quad f^{(0)}(r) = 0. \quad (12.3.172)$$

The first approximation for  $f^{(1)}(r)$  can be taken as  $f^{(0)}(r)$  and then it can be updated by using the following procedure.

$$f^{(n+1)}(r) = -\theta_S^{(n)} \big|_{z=f^{(n)}(r)} = -\theta_L^{(n)} \big|_{z=f^{(n)}(r)}. \quad (12.3.173)$$

$$\theta_S^{(0)}(r, -z) = \theta_L^{(0)}(r, z) = A \sum_{n=0}^{\infty} \frac{2J_0(\lambda_n r) e^{-\lambda_n z}}{\lambda_n^2 J_1(\lambda_n)}, \quad A = \frac{(S_L)^{-1} P_{\text{ep}} + \sigma - 1}{1 + \sigma}, \quad (12.3.174)$$

$\lambda_n$ ,  $n = 0, 1, 2, \dots$  are the roots of the equation  $J_0(\lambda) = 0$ ,  $J_0(\cdot)$  is the Bessel function of zeroth order.  $f^{(1)}(r)$  can be updated analytically from Eq. (12.3.173) but for the determination of  $f^{(n)}(r)$  and  $\theta^{(n)}(r, z)$  for  $n > 1$ , only numerical solutions are possible. Although numerical solutions of  $\theta_{S,L}^{(0)}$  and  $f^{(1)}(r)$  from Eqs (12.3.169), (12.3.170), (12.3.174) have been obtained by numerical methods,  $\theta^{(n)}(r, z)$  and  $f^{(n)}(r, z)$  for  $n > 1$  have not been obtained even numerically. An analytical-numerical procedure has also been suggested for the case when the wall temperature is prescribed in the form  $T|_{r=1} = \sum_{m=1}^n C_m z^m$  and for this case refer [486].

Numerical results were obtained with the commercial fluid dynamics package FIDAP. Numerical results were obtained for some other cases also such as buoyancy-driven flows and it was found that buoyant melt convection has only a minor influence on the phase-boundary shape for the typical conditions of VGF growth.

A steady-state two-dimensional axisymmetric cylindrical problem in  $(r, z)$  coordinates was studied in [487]. The phase-change interface in an infinitely long cylinder with  $z$ -axis along the axis of the cylinder and  $r$ -axis along the horizontal line is known. The phase-change interface can have three type of shapes: (1)  $z = br^2$ , a paraboloid which is convex toward the melt side; (2)  $z = b(R^2 - r^2)$ , the concave interface in which  $R$  is the radius of the cylinder; (3)  $z = 0$ , the flat surface. The formulation is as given below.

$$\nabla^2 T(r, z) = 0, T(r, \infty) = 0; T(r, z) = T_m - T_a, \text{ at the interface.} \quad (12.3.175)$$

$$\left. \frac{\partial T}{\partial r} \right|_{r=0} = \frac{\varepsilon}{K} T \Big|_{r=R}, \quad \varepsilon \text{ is heat loss coefficient; } \left. \frac{\partial T}{\partial r} \right|_{r=R} = 0. \quad (12.3.176)$$

The solution of temperature satisfying the heat equation with conditions given in Eq. (12.3.176) can be expressed as

$$T(r, z) = \sum_{n=1}^{\infty} C_n \exp(-\lambda_n z) J_0(\lambda_n r); \quad \frac{\varepsilon}{K} J_0(\lambda R) - \lambda J_1(\lambda R) = 0. \quad (12.3.177)$$

$\lambda_n$ 's are the roots of the second equation in Eq. (12.3.177).  $C_n$  are unknown coefficients which can be determined from the interface condition. Fifteen coefficients of  $C_n$  were determined on a computer and analytical results were compared with experimental results.

## 12.4 ANALYTICAL-NUMERICAL SOLUTIONS OF STEFAN PROBLEMS

Although analytical-numerical solutions are also called semianalytical solutions, the term analytical-numerical solution seems to be more appropriate for the type of solutions we have in mind, i.e. first either transform the given formulation of the problem in totality to some other suitable formulation or obtain a partial solution of the full given formulation and then run numerical schemes to obtain numerical results. It may be further clarified that we are concerned with the problems in which the original formulation in its totality is transformed to an appropriate form such as integral equation formulation and then the problem is solved numerically in its totality. The discussion of any analytical-numerical solution does not mean 50% weightage for both type of solutions. The emphasis should be on analytical solution. Only outlines of numerical methods will be discussed here which may be just sufficient to indicate that the problem can be solved numerically.

### 12.4.1 Adomian Decomposition Method

Adomian proposed a method [488] to study a wide class of nonlinear operator equations by decomposing the nonlinear operator and writing it as a sum of three operators in terms of an

easily invertible linear highest-order operator  $L$ , a nonlinear operator  $N$  and a remaining linear operator  $R$ . Let us consider the following nonlinear operator equation.

$$P(\theta) = f. \quad (12.4.1)$$

Here  $P$  is a nonlinear operator,  $P:H \rightarrow G$ ,  $\theta$  is an element of some suitable Hilbert space  $H$  and  $f$  is some known element of another Hilbert space  $G$ . We are looking for some  $\theta$ , if possible a unique  $\theta$ , which satisfies Eq. (12.4.1) exactly. If  $\theta$  does not satisfy Eq. (12.4.1) exactly but only approximately then we ask ‘what is the procedure of obtaining this approximation’. As mentioned earlier we write  $P(\theta)$  as the sum of the following three operators.

$$P(\theta) = L(\theta) + N(\theta) + R(\theta), \quad (12.4.2)$$

where operators  $L$ ,  $N$  and  $R$  are as described earlier. The solution of Eq. (12.4.1) is sought in the form of an infinite series of functions of independent variables occurring in Eq. (12.4.2). Let

$$\theta = \sum_{i=0}^{\infty} g_i; \quad N(\theta) = \sum_{i=0}^{\infty} A_i, \quad A_0 = N(g_0), \quad g_0 = \bar{g} + L^{-1}(f), \quad (12.4.3)$$

$$g_n = -L^{-1}R(g_{n-1}) - L^{-1}(A_{n-1}), \quad n \geq 1;$$

$$A_n = \frac{1}{n!} \left[ \frac{d^n}{dw^n} N \left( \sum_{i=0}^n w^i g_i \right) \right]_{w=0}, \quad n \geq 1, \quad (12.4.4)$$

where  $\bar{g}$  is a function which depends on the initial and boundary conditions.  $A_i$  are called *Adomian polynomials*. To demonstrate the method of solution by using Adomian decomposition, the following problem studied in [489] will be first discussed. Although the following one-phase one-dimensional Stefan problem formulation has been discussed earlier, the formulation is being repeated here for better understanding of Adomian method.

$$k \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \quad \alpha < x < S(t^*), \quad t > 0; \quad T(x, 0) = \phi(x), \quad x \geq \alpha; \quad T(\alpha, t) = V(t), \quad (12.4.5)$$

$$T(S(t), t) = T_m; \quad K \left. \frac{\partial T}{\partial x} \right|_{x=S(t)} = \rho l \frac{dS}{dt}, \quad 0 \leq t \leq S(t^*), \quad S(0) = \alpha \geq 0, \quad t^* > 0. \quad (12.4.6)$$

The above problem is that of melting in which melting is starting at  $x = \alpha$  and accordingly the parameters and other quantities can be interpreted with appropriate signs and subscripts.  $T_m$  is the melting/freezing temperature,  $\phi(x) \leq 1$ ,  $x \geq \alpha$  and  $V(t) > 1$  for  $t > 0$ .  $t^*$  is some convenient reference time  $> 0$ . The operators  $L$ ,  $N$  and  $R$  as described in Eq. (12.4.2) for the above problem are

$$L(T) = \frac{\partial^2 T}{\partial x^2}, \quad N(T) = 0, \quad R(T) = -\frac{1}{k} \frac{\partial T}{\partial t}, \quad f = 0. \quad (12.4.7)$$

Define the inverse operator  $L^{-1}(T(x, t))$  as

$$L^{-1}(T(x, t)) = \int_{\alpha}^x \left[ \int_{\xi(t)}^q T(p, t) dp \right] dq. \quad (12.4.8)$$

It can be easily proved that

$$L^{-1}L(T(x, t)) = T(x, t) - V(t) - (\rho l/K) \frac{dS}{dt}(x - \alpha). \quad (12.4.9)$$

In Eq. (12.4.9), the boundary condition at  $x = \alpha$  and Stefan condition at  $x = S(t)$  have been incorporated and initial condition and temperature at  $x = S(t)$  are still to be satisfied. Note that  $L^{-1}L \neq I$ ,  $I$  is the identify operator. According to  $g_0$  in Eq. (12.4.3) and  $g_n$  in Eq. (12.4.4), we have

$$g_0 = \bar{g} = V(t) + (\rho l/K)(x - \alpha) \frac{dS}{dt}; \quad g_n = \frac{1}{k} \int_{\alpha}^x \left[ \int_{\xi(t)}^q \frac{\partial g_{n-1}}{\partial t} dp \right] dq, \quad n \geq 1. \quad (12.4.10)$$

The phase-change boundary  $S(t)$  is expressed in the following series of complete set of functions  $\psi_i(t)$ .

$$S(t) = \sum_{i=1}^{\infty} \delta_i \psi_i(t) \approx \sum_{i=1}^m \delta_i \psi_i(t), \quad m \text{ is some positive integer.} \quad (12.4.11)$$

In a similar way  $T(x, t)$  is approximated as  $T(x, t) \approx T_n(x, t)$  where

$$T_n(x, t) = \sum_0^n g_i(x, t). \quad (12.4.12)$$

To determine the unknowns  $\delta_i$ ,  $i = 1, 2, \dots, m$ , the functional  $J(\delta_1, \delta_2, \dots, \delta_m)$  is minimized, where

$$J(\delta_1, \delta_2, \dots, \delta_m) = \int_0^{t^*} \{T_n(S(t), t) - T_m\}^2 dt + \int_{\alpha}^{S(t^*)} \{T_n(x, 0) - \phi(x)\}^2 dx. \quad (12.4.13)$$

$\frac{\partial J}{\partial \delta_1}, \frac{\partial J}{\partial \delta_2}, \dots, \frac{\partial J}{\partial \delta_m}$  will give  $m$  equations to determine  $\delta_1, \delta_2, \dots, \delta_m$ .  $t^*$  could be the time for total solidification/melting or any convenient time required for data collection. Note that we did not satisfy in Eq. (12.4.9) the isotherm condition and initial condition. Therefore the functional is taken in the form as given in Eq. (12.4.13). Except in very simple cases, analytical solution is not possible in general for  $g_0, g_1, g_2$  and  $\delta_1, \delta_2, \dots, \delta_m$  as it involves solution of a system of coupled nonlinear equations. The absolute errors  $E_S$  and  $E_T$  in the moving boundary and temperature, respectively, can be defined as

$$E_T = \left\{ \iint_D (T(x, t) - T_n(x, t))^2 dx dt \right\}^{1/2},$$

$$E_S = \left\{ \int_0^{t^*} \left[ S(t) - \sum_{i=1}^m \delta_i \psi_i(t) \right]^2 dt \right\}^{1/2}. \quad (12.4.14)$$

$D$  is the  $(x, t)$  region under consideration in the problem. Percentage errors can be easily defined.  $T(x, t)$  and  $S(t)$  are exact analytical solutions in Eq. (12.4.14). The errors  $E_T$  and  $E_S$  can be found only numerically except in some simple cases in which the integrals in  $E_T$  and  $E_S$  can be evaluated analytically. If exact analytical solutions for  $T(x, t)$  and  $S(t)$  are not available then numerical solution is to be used. The domains desired in  $E_T$  and  $E_S$  have to be



discretized and numerical values of integrals are obtained by numerical methods. For  $T(x, t)$  and  $S(t)$  the original formulation is to be used.

It is clear that numerical solutions of  $T_n(x, t)$  in Eq. (12.4.12) and  $S(t)$  in Eq. (12.4.11) cannot be generally avoided. What are the numerical methods/procedures available? There are several of them which will be discussed one by one with examples.

An exact solution of the problem considered in [489] of the one-phase problem given in Eqs (12.4.5), (12.4.6) is taken as  $T(x, t) = e^{kt-x}$ ,  $V(t) = e^{kt}$ ,  $S(t) = kt$ ,  $\alpha = 0$ ,  $D = \{(x, t): 0 \leq x \leq S(t^*), 0 \leq t \leq t^*\}$ ,  $\phi(x) = \exp(-x)$ ,  $T_m = 1$ . The approximate analytical-numerical solution by ADM was obtained by taking different values of  $m$  such as,  $m = 2, 3, 4, 5$ , one by one and taking  $n = 1$  in Eqs (12.4.11), (12.4.12). The basis function  $\Psi_i(t)$  is to be taken as complete set of functions such as  $\Psi_i(t) = t^{i-1}$ ,  $i = 1, 2, 3, \dots, \infty$ . *Conjugate gradient method* [490] has been used to obtain coefficients  $\delta_i$ ,  $i = 1, 2, \dots, m$ , by taking  $m = 2, 3, 4$  and 5, one by one and minimizing the functional in Eq. (12.4.13). The absolute errors and percentage errors were found insignificant. Symbolic computation software such as MATLAB can be used for performing sequence of suboperations.

Minimization of the functional given in Eq. (12.4.13) can be avoided by following a different approach and this has been shown in [491]. The problem considered in  $0 \leq x \leq S(0)$  is similar in notations to one-dimensional problem as considered in Eqs (12.4.5), (12.4.6) with  $\alpha = 0$ .

The moving boundary is fixed at  $y = 1$  by using the transformation

$$y = x/S(t), \quad \tau = t; \quad S(0) = S_0. \quad (12.4.15)$$

The problem formulation is transformed as given below.

$$\frac{\partial T(y, \tau)}{\partial \tau} = \frac{y}{S(\tau)} \frac{dS(\tau)}{d\tau} \frac{\partial T(y, \tau)}{\partial y} + \frac{k}{(S(\tau))^2} \frac{\partial^2 T(y, \tau)}{\partial y^2}, \quad 0 < y \leq 1, 0 < \tau < \tau^*, \quad (12.4.16)$$

$$T(y, 0) = \phi(yS_0); T(0, \tau) = V(\tau); T(1, \tau) = T_m; (K/S(\tau)) \left. \frac{\partial T}{\partial y} \right|_{y=1} = \rho l \frac{dS(\tau)}{dt}. \quad (12.4.17)$$

The differential equations and boundary conditions are discretized for obtaining finite difference numerical solutions. Let  $y_i = i\Delta y$ ,  $i = 1, 2, \dots, n$ , and  $\Delta y = 1/n$ . The interval  $0 \leq y \leq 1$  is discretized into  $n$  intervals with  $y_0 = 0$ ,  $y_n = n\Delta y$  and  $T_n(\tau) = T(n\Delta y, \tau)$ . We have the following discretized equations.

$$\left. \begin{aligned} \frac{dT_i(\tau)}{d\tau} = \frac{1}{(S(\tau)\Delta y)^2} \left\{ \frac{Ky_i}{\rho l} (T_{n+1}(\tau) - T_n(\tau)) (T_i(\tau) - T_{i-1}(\tau)) \right. \\ \left. + k (T_{i-1}(\tau) - 2T_i(\tau) + T_{i+1}(\tau)) \right\}, i = 1, 2, \dots, n; T_{n+1}(\tau) = T_m. \end{aligned} \right\} \quad (12.4.18)$$

$$\begin{aligned} \frac{dS(\tau)}{d\tau} = \frac{K}{\rho l S(\tau)} \frac{(T_{n+1}(\tau) - T_n(\tau))}{\Delta y}; \quad T_i(0) = \phi(S_0 y_i), i = 1, 2, \dots, n, \\ T_0(\tau) = V(\tau). \end{aligned} \quad (12.4.19)$$

Eqs (12.4.18), (12.4.19) can be rewritten as a system of following equations.

$$T_i(t) = a_{i,0} + L^{-1} f_i(t, T_1, T_2, \dots, T_n), \quad i = 1, 2, \dots, n; L^{-1}(h(t)) = \int_0^t h(p) dp. \quad (12.4.20)$$

As done in Eq. (12.4.3), the solutions of  $T_i, i = 1, 2, \dots, n$ , are sought in terms of a series of the form

$$T_i = \sum_{r=0}^{\infty} g_{i,r} \approx \sum_{r=0}^m g_{i,r}, i = 1, 2, \dots, n; m \text{ is some finite positive integer.} \quad (12.4.21)$$

Using ADM

$$f_i(t, T_1, T_2, \dots, T_n) = \sum_{r=0}^{\infty} A_{i,r}(g_{i,0}, g_{i,1}, \dots, g_{i,r}), \quad i = 1, 2, \dots, n. \quad (12.4.22)$$

In Eq. (12.4.21), by taking some finite integer value of  $r$  the summation has been approximated.  $A_{i,r}$  are Adomian polynomials given below.

$$A_{i,r}(g_{1,0}, \dots, g_{n,r}) = \frac{1}{r!} \left\{ \frac{d^r}{d\lambda^r} f_i \left( t, \sum_{J=0}^r \lambda^J g_{1,J}, \dots, \sum_{J=0}^r \lambda^J g_{n,J} \right) \right\}_{\lambda=0}, \quad (12.4.23)$$

$$g_{i,0} = a_{i,0}; g_{i,n} = L^{-1}(A_{i,n-1}), \quad i = 1, 2, \dots, n, n \geq 1. \quad (12.4.24)$$

If  $\phi(x) = \exp(1-x)$ ,  $V(t) = \exp(kt+1)$ ,  $k = 0 \cdot 1$ ,  $T_m = 1$ ,  $S(0) = 1$ ,  $\rho l/K = -10$  then  $T(x, t) = \exp((kt-x)+1)$ , and  $S(t) = kt+1$  is the solution of the one-phase problem considered in Eqs (12.4.5), (12.4.6). If these data are accepted as given in [491] then domain of the problem is not  $0 \leq x \leq S(0)$  as mentioned in [491]. We are concerned here with the method of solution so these details do not matter.

For obtaining  $S(t)$  and  $T_i(t)$ ,  $n$  is taken as 6 in Eq. (12.4.22) and  $m = 4$  in Eq. (12.4.21). Whilst using Adomian decomposition method (ADM) time is not discretized. Both  $S(t)$  and  $T(yS(t), t)$  are obtained as continuous functions of time. Numerical solution was also obtained by using fourth-order Runge-Kutta method (RKM) in which time is discretized by considering  $t_i = i\Delta t$ ,  $i = 0, 1, 2, \dots, p$ ,  $p = 20$  and  $\Delta t = t^*/p$  where  $t^*$  could be some convenient time. All computations were done using Mathematica [492]. Absolute errors for the results obtained by RKM and ADM were compared and it was found that ADM is more accurate as well as less computer time consuming.

With some modifications, ADM method has been used in [493] for the analytical-numerical solution of a one-dimensional one-phase spherical problem of shrinking core model. For clear understanding of the solution procedure, the formulation is being given below although it has been discussed earlier.

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial r^2} + \frac{2}{r} \frac{\partial \theta}{\partial r}, S(t) < r < 1, 0 < t < t^*; \frac{\partial \theta}{\partial r}(1, t) = I^*. \quad (12.4.25)$$

$$\left. \begin{aligned} \theta(r, t) &= \bar{\theta}_0, 0 \leq r < S(t), 0 < S(0) = S_0 < 1, 0 < t < t^* \\ \theta(S(t), t) &= \lambda, (\bar{\theta}_0 - \lambda) \frac{dS}{dt} = \frac{\partial \theta}{\partial r} \Big|_{r=S(t)} \end{aligned} \right\} \quad (12.4.26)$$

$\theta(r, t)$  is the Lithium concentration in a Lithium iron-phosphate particle during the process of Lithium intercalation into the particle with shrinking core spherical model during the discharge process. The formulation given above is for the initial stage of discharge process.  $r = S(t)$  is the inner radius of the spherical shell,  $t^*$  is the time when the two-phase interface disappears,  $I^*$  is the current,  $\bar{\theta}_0$  is the initial concentration of Lithium in iron-phosphate,  $\lambda$  is

the concentration of Lithium at  $r = S(t)$ . We would like to clarify that the authors in [493] have considered initial concentration remaining unchanged for  $t > 0$  in the changing region. The only way a one-phase problem can be formulated as in Eqs (12.4.25), (12.4.26) is to assume that  $\theta(r, t)$  remains as  $\bar{\theta}_0$  in the region  $0 \leq r < S(t)$  for  $0 \leq t \leq t^*$ . We assume here that all the quantities are dimensionless.

In the operator form, Eq. (12.4.25) can be written as

$$L_t(\theta) = L_{rr}(\theta), \quad L_{rr}(\theta) = r^{-1} \frac{\partial^2}{\partial r^2} [r(\theta)]. \quad (12.4.27)$$

$$L_{rr}^{-1}(\theta) = r^{-1} \int_1^r \left[ \int_1^p \theta(q, p) dq \right] dp, \quad L_t^{-1}(\theta) = \int_0^t \theta(p) dp. \quad (12.4.28)$$

Applying  $L_{rr}^{-1}$  on both sides of the first equation in Eq. (12.4.27), we get

$$\theta = L_{rr}^{-1} L_t(\theta) + r^{-1} A(t) + B(t), \quad A(t) \text{ and } B(t) \text{ are arbitrary functions.} \quad (12.4.29)$$

If Eq. (12.4.29) is differentiated with respect to  $r$  and second equation in Eq. (12.4.25) is used then we get  $A(t) = -I^*$  but  $B(t)$  is not determined. Following ADM,  $\theta(t)$  and  $B(t)$  are taken in the form

$$\theta(r, t) = \sum_0^\infty \theta_n(r, t) \approx \sum_0^m \theta_n(r, t); \quad B(t) = \sum_0^\infty b_n t^n \approx \sum_0^m b_n t^n. \quad (12.4.30)$$

In both the series expansions in Eq. (12.4.30) the same  $m$  can be taken as some finite positive integer. When these series expansions are substituted in Eq. (12.4.29) and  $\theta_0(r, t)$ ,  $\theta_1(r, t)$ , etc., are determined with  $A(t)$  taken as  $-I^*$ , we get

$$\theta_0(r, t) = -I^*/r + \sum_{n=0}^8 b_n t^n; \quad \theta_1(r, t) = ((r+2)(r-1)^2/6r) \sum_{n=1}^8 n b_n t^{n-1}. \quad (12.4.31)$$

For computations  $m = 8$  was taken. For  $\theta_2(r, t), \dots, \theta_8(r, t)$  refer [493]. These functions were obtained performing symbolic operations using MATLAB. This solution in Eq. (12.4.31) is not valid near  $r = 0$  and another formulation without phase change has been considered for a very small region around  $r = 0$ .

The boundary conditions at  $r = S(t)$  given in Eq. (12.4.26) are yet to be satisfied. When the first condition at  $r = S(t)$  in Eq. (12.4.26) is differentiated with respect to  $t$  and substituted in the second condition at  $r = S(t)$  in Eq. (12.4.26) along with  $\theta(r, t) = \sum_{n=0}^8 \theta_n(r, t)$ , we get

$$\left[ \sum_{n=0}^8 \frac{\partial \theta_n}{\partial r}(S(t), t) \right]^2 + (\bar{\theta}_0 - \lambda) \sum_{n=0}^8 \frac{\partial \theta_n}{\partial t}(S(t), t) = 0. \quad (12.4.32)$$

We also have

$$\sum_{n=0}^m \theta_n(S(t), t) - \lambda = 0; \quad \sum_{n=0}^m \theta_n(S_0, 0) - \lambda = 0. \quad (12.4.33)$$

To determine  $S(t)$ , its positions  $S(t_1) = 0.1, S(t_2) = 0.2, \dots, S(t_9) = 0.9$  are fixed and times  $t_1, t_2, \dots, t_9$  are taken as unknowns. In all there are 18 unknowns,  $b_0, b_1, \dots, b_8$  and  $t_1, t_2, \dots, t_9$ , to be determined numerically. In a small neighbourhood of  $r = 0$  another problem of diffusion without moving boundary has been studied. Refer [493] for its solution.

Authors have given MATLAB code for generating the system of Eqs (12.4.32), (12.4.33) with appropriate details of obtaining solution which is useful. After generating the system of equations, solutions in a given range of time were sought using the optimization software 1st Opt to obtain high precision solution. Considerable numerical work has been reported graphically.

In references [489–493] several useful cross-references connected with ADM and solution are given.

### 12.4.2 Variational Iteration Method by JH He

In this section those solutions of Stefan problems will be discussed which have been obtained by the application of the variational iteration method (VIM) proposed by He (cf. [494]). To explain the application of the method, we consider a simple one-dimensional problem of heat conduction, which is governed by the following differential equation.

$$\frac{d^2 T}{dx^2} + T + x = 0, \quad 0 < x < 1; \quad T(0) = T(1) = 0. \quad (12.4.34)$$

The method is being explained first for a general nonlinear operator equation. According to the variational method a *correctional functional* is to be constructed as explained below. Consider a nonlinear operator equation in the following form.

$$L_t u(t, x, y, z) + L_x u + L_y u + L_z u + Nu = g(t, x, y, z), \quad (12.4.35)$$

where  $L_t, L_x, L_y$  and  $L_z$  are linear operators.  $L_x$  means that it contains only partial derivatives with respect to  $x$  and so on for other independent variables  $t, y$ , etc.  $N$  is a nonlinear operator and  $g$  is inhomogeneous or a forcing term. The following four correction functionals in  $t, x, y, z$  directions can be defined.  $u_0, u_1, u_2, \dots, u_n, n \geq 0$ , are successive iterations of  $u(t, x, y, z)$  in Eqs (12.4.36)–(12.4.39).

$$u_{n+1}(t, x, y, z) = u_n(t, x, y, z) + \int_0^t \lambda_1 \{L_\tau u_n + (L_x + L_y + L_z + N)\bar{u}_n - g\} d\tau. \quad (12.4.36)$$

$$u_{n+1}(t, x, y, z) = u_n(t, x, y, z) + \int_0^x \lambda_2 \{L_\xi u_n + (L_t + L_y + L_z + N)\bar{u}_n - g\} d\xi. \quad (12.4.37)$$

$$u_{n+1}(t, x, y, z) = u_n(t, x, y, z) + \int_0^y \lambda_3 \{L_p u_n + (L_t + L_x + L_z + N)\bar{u}_n - g\} dp. \quad (12.4.38)$$

$$u_{n+1}(t, x, y, z) = u_n(t, x, y, z) + \int_0^z \lambda_4 \{L_q u_n + (L_t + L_x + L_z + N)\bar{u}_n - g\} dq. \quad (12.4.39)$$

In any given problem any of the correctional functional can be used subjected to its suitability in terms of linearity of the operator.  $\lambda_i, i = 1, 2, 3, 4$  are general Lagrange multipliers and they are to be determined according to the variational principles in an optimal way.  $u_0$  is

an initial approximation or trial function and  $\bar{u}_n$  is considered as *restricted variations* [495] (i.e.  $\delta\bar{u}_n = 0$ ).

The correction functional for the problem considered in Eq. (12.4.34) is as follows [496].

$$T_{n+1}(x) = T_n(x) + \int_0^x \lambda(p) \left\{ \frac{\partial^2 T_n(p)}{\partial p^2} + T_n(p) + p \right\} dp, n \geq 0, \quad (12.4.40)$$

$T_0, T_1, T_2, \dots, T_n$  are successive iterations.

Taking variation with respect to the independent variable  $T_n$ , we get

$$\begin{aligned} \delta T_{n+1}(x) &= \delta T_n(x) + \delta \left( \int_0^x \lambda \left\{ \frac{\partial^2 T_n(p)}{\partial p^2} + T_n(p) + p \right\} dp \right) \\ &= \delta T_n(x) + (\lambda \delta T'_n(p))_{p=x} - (\lambda' \delta T_n(p))_{p=x} \\ &\quad + \int_0^x \left( \frac{\partial^2 \lambda}{\partial p^2} + \lambda \right) \delta T_n dp, \delta T_n(0) = 0. \end{aligned} \quad (12.4.41)$$

For stationary conditions all the variations  $\delta T_n$  ( $T$  is prescribed) and  $\delta T'_n$  ( $T'$  is not occurring) in the r.h.s. of Eq. (12.4.41) should be zero. This implies that

$$\lambda(p)|_{p=x} = 0; 1 - \lambda'(p)|_{p=x} = 0; \left( \frac{\partial^2 \lambda}{\partial p^2} + \lambda \right)_{p=x} = 0; \lambda'(p) = \frac{d\lambda}{dp}. \quad (12.4.42)$$

The solution of equations in Eq. (12.4.42) determines the Lagrange multiplier which in this problem is

$$\lambda = \sin(p - x). \quad (12.4.43)$$

We take  $T_0(x) = A \cos x + B \sin x$  and then  $T_1(x)$  is given by the relation

$$\begin{aligned} T_1(x) &= T_0(x) + \int_0^x \sin(p - x) \left\{ \frac{\partial^2 T_0(p)}{\partial p^2} + T_0(p) + p \right\} dp \\ &= T_0(x) + \int_0^x p \sin(p - x) dp. \end{aligned} \quad (12.4.44)$$

$$T_1(x) = \frac{\sin x}{\sin(1)} - x, \text{ which is also the exact solution.} \quad (12.4.45)$$

The constants  $A$  and  $B$  have been determined in Eq. (12.4.45) by taking  $T_1(0) = 0$  and  $T_1(1) = 0$ .  $T_0(x)$  is to be suitably taken and it is not prescribed.

In Eq. (12.4.34) there is no nonlinear operator and so no restricted variation has been introduced. Now we introduce a restricted variation for the problem in Eq. (12.4.34) and consider the correction functional as

$$T_{n+1}(x) = T_n(x) + \int_0^x \lambda(p) \left\{ \frac{\partial^2 T_n(p)}{\partial p^2} + \bar{T}_n(p) + p \right\} dp, \quad (12.4.46)$$

$\bar{T}_n$  is restricted variation and  $\delta \bar{T}_n = 0$ . The Lagrange multiplier can be obtained by solving the following system of equations.

$$\lambda(p)|_{p=x} = 0; \quad 1 - \lambda'(p)|_{p=x} = 0; \quad \left. \frac{d^2 \lambda}{dp^2} \right|_{p=x} = 0. \quad (12.4.47)$$

It can be checked that  $\lambda = p - x$  is the Lagrange multiplier.  $T_0(x)$  is taken as  $A \cos x + B \sin x$  and arbitrary constants  $A$  and  $B$  are determined by forcing  $T_1(x)$  to satisfy  $T_1(0) = T_1(1) = 0$  as  $T_0(x) \equiv 0$  if boundary conditions are applied.  $T_0''(p) + T_0(p) = 0$  and this simplifies the calculation of  $T_1(x)$  if  $T_0(x) = A \cos x + B \sin x$ .  $T_1, T_2, \dots, T_n$  have to satisfy prescribed boundary conditions.

$$\begin{aligned} T_1(x) &= T_0(x) + \int_0^x (p-x) \left\{ \frac{\partial^2 T_0(p)}{\partial p^2} + T_0(p) + p \right\} dp \\ &= \int_0^x p(p-x) dp + T_0(x). \end{aligned} \quad (12.4.48)$$

It is clear from the above example that restricted variation may not be very useful in linear operator equations and it may require several iterations to get a good approximate solution as  $(p-x)$  is only the first term in the series expansion of  $\sin(p-x)$ .

Considerable literature exists on the applications and solutions of variational iterative method (VIM) in different fields of partial differential equations, system of differential equations and problems of mathematical physics. Our interest here is to give a brief introduction to this iterative method so that some solutions of Stefan problems can be presented. In brief, the observations mentioned below provide some useful information about the applications of this method.

- (1) The application of restricted variations in correction functionals makes it easier to obtain Lagrange multiplier and approximate solutions of nonlinear problems of mathematical physics can be obtained easily.
- (2) The initial approximation can be selected with unknown constants, which can be determined via various methods without much difficulty.
- (3) There is no restriction on the range of values of parameters which is there in perturbation methods. First-order approximations are generally fairly accurate if initial approximation is good.
- (4) Nonlinear problems with fractional derivative can also be studied with the help of this method.
- (5) This variational method can be used in conjunction with other iterative techniques.
- (6) Comparison with Adomian method shows that the approximations obtained by VIM converge faster to the exact solution.

Many references of works related to this variational method and its use in nonlinear problems with fractional derivatives, modifications of this method, some convergence aspects, comparison with Adomian method and applications to several problems of mathematics, physics and engineering are reported in [497].

Having introduced VIM, its application to a Stefan problem will be discussed now. Consider the one-phase problem formulated in Eqs (12.4.5), (12.4.6) with some changes to consider the problem discussed in [498]. Take  $\alpha = 0$ , and consider a solid at melting temperature  $T_m$  which occupies the region  $S^0 \leq x < \infty$  and the region  $0 \leq x \leq S^0$  is occupied by a superheated liquid. This clarification about the changes in problem formulation considered in [491] comes by seeing Fig. 2 presented in [498].  $S(t) = at + 1$  is meaningful now if  $S^0 = 1$ . Note that in [491], temperature is prescribed at  $x = 0$  whereas in [498] flux is prescribed at  $x = 0$ . Flux can be calculated from the analytical solution prescribed in [491]. Melting starts at  $x = S^0$  where  $S(0) = S^0$ . At  $x = S(t)$  temperature is melting temperature and at  $x = 0$  flux is prescribed. The analytical solution is taken as  $T(x, t) = \exp(at - x + 1)$ ,  $S(t) = at + 1$ ,  $a = 0.1$ ,  $T_m = 1$ ,  $\rho l/k = -10$ . With the help of the transformation  $y = x/S(t)$ , the moving boundary is fixed at  $y = 1$ . The transformed equation, and boundary and initial conditions will be similar to Eqs (12.4.16), (12.4.17) except that flux prescribed boundary condition at  $y = 0$  will be  $\partial T(0, t)/\partial y = V(t) S(t)$  which before transformation was  $\partial T(x = 0, t)/\partial x = V(t)$ .

Following the procedure described for solution by VIM, the  $(n + 1)$ th iterative solution is given by

$$T_{n+1}(y, t) = T_n(y, t) + \int_0^t \lambda_1(p) \left( \frac{\partial T_n(y, p)}{\partial p} - N_1(\bar{T}_n, \bar{S}_n; y, p) \right) dp, \quad (12.4.49)$$

$$S_{n+1}(t) = S_n(t) + \int_0^t \lambda_2(p) \left( \frac{\partial S_n(p)}{\partial p} - N_2(\bar{T}_n, \bar{S}_n; p) \right) dp, \quad (12.4.50)$$

$$N_1(T, S; y, t) = \frac{y}{S(t)} \frac{dS(t)}{dt} \frac{\partial T(y, t)}{\partial y} + \frac{k}{(S(t))^2} \frac{\partial^2 T(y, t)}{\partial y^2}, \quad (12.4.51)$$

$$N_2(T, S(t); t) = \frac{\rho l}{K S(t)} \frac{\partial T(y, t)}{\partial y} \Big|_{y=1}. \quad (12.4.52)$$

The stationary conditions of the correctional functionals in Eqs (12.4.49), (12.4.50) give  $\lambda_1(p) = -1$  and  $\lambda_2(p) = -1$ . Note that  $\delta \bar{T}_n = 0$  and  $\delta \bar{S}_n = 0$ . Also  $\delta T_n(0) = 0$  and  $\delta^n S_n(0) = 0$ . To obtain successive iterations,  $\bar{T}_n$  and  $\bar{S}_n$  are replaced by  $T_n$  and  $S_n$  in the expressions of  $N_1$  and  $N_2$  given in Eqs (12.4.51), (12.4.52). Initial approximation for  $T_0(y, S(t))$  is required. For  $S(t)$ ,  $S_0(t) = S^0$  is given. The approximation of  $T_0(y, t)$  in [498] is taken as

$$T_0(y, t) = a_1 y + b_1 t + c_1 y t + d_1. \quad (12.4.53)$$

To obtain the unknown constants  $a_1$ ,  $b_1$ ,  $c_1$  and  $d_1$ , the four conditions  $T(0, 0)$ ,  $T(0, t^*)$ ,  $T(1, 0)$  and  $T(y, 0)$  are used. Take  $t^*$  as some desired time chosen for computations.

$$S_0(t) = S^0; T_0(y, t) = S^0 (V(t^*) - V(0))t/(t^*)(y - 1/2) + \phi(S^0 y). \quad (12.4.54)$$

Approximations  $S_0$ ,  $S_1$  and  $S_2$  for  $S(t)$  and  $T_0$ ,  $T_1$  and  $T_2$  for  $T(y, t)$  have been calculated and considered in the numerical work. The exact analytical solution is the same as considered in [491]. Absolute errors defined in Eq. (12.4.14) and similarly percentage errors have been calculated and error distributions in satisfying boundary condition at  $x = 0$  and interface temperature have been presented.

The numerical method for numerical solution presented in [499] using VIM for the problem considered in [491] is different. The VIM solution procedure is the same as considered above in [498]. The only difference between the two problems in [498, 499] is that in [498] flux is prescribed at  $x = 0$  and in [499] temperature is prescribed at  $x = 0$ . The exact analytical solution considered for temperature and for  $x = S(t)$  is similar with some changes in both [498, 499], and  $t^* = 0.5$ .

Comparison of numerical solutions obtained by ADM and VIM has been done in [500]. The exact analytical solution considered in [500] is  $T(x, t) = e^{at-x}$ ,  $S(t) = at$ ,  $a = 0.1$ . This solution gives the initial temperature  $\phi(x) = \exp(-x)$  and flux  $q(t)$  at  $x = 0$  as  $q(t) = \lambda \exp(at)$ ,  $\lambda = 1$ ,  $T_m = 1.0$ ,  $t^* = 0.5$ . From this data, the problem formulation of the one-phase Stefan problem can be easily constructed and it will not be given here. For ADM, the solution procedure is the same as discussed in [489] and for numerical solution by VIM, an algorithm presented in [501] has been used. VIM method was found only slightly more effective.

### 12.4.3 Use of Green's Functions and Integral Equation Approach in Analytical-Numerical Solutions

The advantage of using integral equation formulation is that the calculation of temperature distribution or dependent variable at points within the domain by solving the differential equations becomes unnecessary to find the solution of the problem. The unknown quantities can be obtained at all points by calculating their values at boundary points and this saves lot of computational effort.

#### *Problems in Cartesian Coordinates*

A one-phase one-dimensional problem and a one-phase two-dimensional problem have been considered in [502]. Initially at  $t = 0$ , in a one-dimensional problem, a liquid occupies the region  $0 \leq x \leq b$ ,  $b > 0$ , having temperature  $T(x, 0) = \phi(x) \geq 0$ ,  $\phi(b) = 0$ . Melting temperature is taken zero and the phase-change boundary  $x = S(t)$  at  $t = 0$  is given by  $S(0) = b$ . At  $x = 0$ ,  $T(0, t) = f(t)$ ,  $f(t) \geq 0$ ,  $t > 0$ . Other equations are given below.

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < S(t), t > 0; \quad T(S(t), t) = 0; \quad T_x(S(t), t) = -\frac{dS(t)}{dt}, \quad t > 0. \quad (12.4.55)$$

For convenience all the thermo-physical parameters have been taken as unity by us.

In the following discussion, we first try to express temperature  $T(x, t)$  in terms of an integral over the boundary of the region  $\{(x, \tau); 0 \leq x \leq S(\tau), 0 \leq \tau \leq t\}$  such that  $T(x, t)$  satisfies heat equation, isotherm condition, initial and boundary conditions.

The Green's function is chosen in such a way that at  $x = 0$  it gives the temperature  $T(0, t) = 0$ . We define

$$\bar{G}(x, t; \xi, \tau) = K_N(x, t; \xi, \tau) - K_N(-x, t; \xi, \tau), \quad \bar{G}|_{\xi=0} = 0, \quad (12.4.56)$$

$$K_N(x, t; \xi, \tau) = 1 / \left( 2\sqrt{\pi(t-\tau)} \right) \exp \left[ -(x-\xi)^2 / (4(t-\tau)) \right], \quad t > \tau. \quad (12.4.57)$$



The Green's function  $K_N$  in Eq. (12.4.57) can be obtained if in Eq. (12.2.14),  $t$  is replaced by  $(t - \tau)$ ,  $t > \tau$ . It can be easily checked that for  $t > \tau$

$$\overline{G}_\tau = -\overline{G}_{\xi\xi}; \quad \overline{G}(x, t; 0, \tau) = 0; \quad (\overline{G}T_\xi - T\overline{G}_\xi)_\xi - (\overline{G}T)_\tau = 0. \quad (12.4.58)$$

The third equation in Eq. (12.4.58) is called *Green's identity* and if Green's identity is integrated with respect to  $\xi$  from 0 to  $S(\tau)$  and for  $\tau$  from  $\tau = 0$  to  $\tau = t - \varepsilon$ ,  $0 < \varepsilon < \tau < t - \varepsilon$ , we get (refer [368])

$$\begin{aligned} 0 = & \int_0^{t-\varepsilon} \left\{ (\overline{G}T_\xi - T\overline{G}_\xi)_{\xi=S(\tau)} - (\overline{G}T_\xi - T\overline{G}_\xi)_{\xi=0} \right\} d\tau \\ & - \int_0^{t-\varepsilon} \left\{ \int_0^b (\overline{G}T)_\tau + \int_b^{S(\tau)} (\overline{G}T)_\tau \right\} d\xi d\tau. \end{aligned} \quad (12.4.59)$$

To obtain integral representation of  $T(x, t)$ , we proceed as discussed below.

In the first integral in the r.h.s. of Eq. (12.4.59) we use  $\overline{G}|_{\xi=0} = 0$ ,  $T(S(t), t) = 0$  and take the limit  $\varepsilon \rightarrow 0$ . In the second term of second integral, the order of integration is changed and the limits of  $\tau$  will be from  $S^{-1}(\xi)$  to  $t - \varepsilon$  and for  $\xi$  they will be from  $b$  to  $S(t - \varepsilon)$ . In the first term of the second integral, limit  $\varepsilon \rightarrow 0$  can be taken but limit  $\varepsilon \rightarrow 0$  in the second term requires careful handling (cf. [19, 503]).

After carrying out suggested operations and some steps in calculations the following integral equation is obtained using Eq. (12.4.59)

$$\begin{aligned} T(x, t) = & \int_0^t \overline{G}(x, t; S(\tau), \tau) T_\xi(S(\tau), \tau) d\tau + \int_0^t \overline{G}_\xi(x, t; 0, \tau) f(\tau) d\tau \\ & + \int_0^b \overline{G}(x, t; \xi; 0) T(\xi, 0) d\xi. \end{aligned} \quad (12.4.60)$$

Integral equation for Stefan condition can be obtained by differentiating  $T(x, t)$  given in Eq. (12.4.60) with respect to  $x$  and taking limit  $x \rightarrow S(t)$ . However, this derivation is not simple. The complete derivation is given in [203].

$$\left. \begin{aligned} V(t) = \frac{\partial T}{\partial x} \Big|_{x=S(t)} = & \frac{1}{2} V(t) + \int_0^t \overline{G}_x(S(t), t; S(\tau), \tau) V(\tau) d\tau \\ & + \int_0^t \overline{G}_{x\xi}(S(t), t; 0, \tau) f(\tau) d\tau + \int_0^b \overline{G}_x(S(t), t; \xi, 0) \phi(\xi) d\xi. \end{aligned} \right\} \quad (12.4.61)$$

$$S(t) = b - K \int_0^t V(\tau) d\tau. \quad (12.4.62)$$

In [502], the numerical solution of  $S(t)$  has been carried out using Eq. (12.4.62) in which  $V(t)$  is given by Eq. (12.4.61). A more convenient form of  $V(t)$  for numerical computations could be used as given in [203] which is as follows.

$$\left. \begin{aligned} V(t) = & 2[\phi(0) - f(0)]N(S(t), t; 0, 0) + 2 \int_0^b \phi_\xi(\xi) N(S(t), t; \xi, 0) d\xi \\ & - 2 \int_0^t \dot{f}(\tau) N(S(t), t; 0, \tau) d\tau + 2 \int_0^t V(\tau) \overline{G}_x(S(t), t; S(\tau), \tau) d\tau, \end{aligned} \right\} \quad (12.4.63)$$

$$N(x, t; \xi, \tau) = \bar{G}(x, t; \xi, \tau) + \bar{G}(-x, t; \xi, \tau). \quad (12.4.64)$$

Algorithm for numerical computations runs as follows.

- (1) Guess  $S_0(t)$  and  $V_0(t)$  initially for  $S(t)$  and  $V(t)$ .
- (2) Substitute  $S_0(t)$  and  $V_0(t)$  in Eqs (12.4.61), (12.4.62) to obtain  $V_1(t)$  and  $S_1(t)$ , which are the next estimates and iterate till convergence. Then use  $V_1(t)$  and  $S_1(t)$  to obtain  $V_2(t)$  and  $S_2(t)$  and so on till  $S_n(t)$  and  $V_n(t)$  are obtained for some suitable  $n$ . Temperature can be obtained from Eq. (12.4.60) by using  $S_n(t)$  and  $V_n(t)$  in place of  $S(t)$  and  $V(t)$ . For further details of finite element solution refer [502].

In [21], a more general one-dimensional Stefan problem of a single phase with strong nonlinearity has been considered which is mentioned below. The integral representation for  $T(x, t)$  and Stefan-condition have been derived. However, no numerical work has been done so it is not being discussed further.

$$\begin{aligned} \frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial x^2} + F\left(x, t, T, \frac{\partial T}{\partial x}, S(t), \dot{S}(t)\right) &= 0 \text{ in } \{0 < x < S(t); 0 < t \leq t^*\}, \\ \frac{\partial T}{\partial x} &= f(t, T), \quad x = 0, t > 0; T = \phi(x), \quad t = 0, \quad 0 \leq x \leq S(0); T = \psi(x) \text{ on } x = S(t), \\ \dot{S}(t) &= Z\left(t, T, \frac{\partial T}{\partial x}, x\right), \text{ for } x = S(t); t > 0; S(0) = L > 0. \end{aligned} \quad (12.4.65)$$

A two-dimensional one-phase problem has also been studied in [502] by obtaining integral representations of temperature  $T(x, y, t)$  and the normal derivative of temperature at the phase-change boundary. Consider a finite region  $\Omega(0) \subset R^2$  in  $(x, y)$  plane whose only boundary which is the outer boundary  $\Gamma_{\text{out}}$  is smooth.  $\Omega(0)$  is expanding with time which is denoted by  $\Omega(t)$ , for  $t > 0$ . The region outside  $\Omega(0)$  is occupied at  $t = 0$  by a solid having melting temperature  $T(x, y, t) = 0$ . The region  $\Omega(t) - \Omega(0)$  is denoted by  $\Omega^*(t)$ , i.e.  $\Omega^*(t) = \Omega(t) - \Omega(0)$ ,  $t > 0$ . The region  $\Omega^*(t)$  is bounded by an outer boundary  $\Gamma(t)$  and an inner boundary denoted by  $\Gamma_{\text{out}}$ .  $\Gamma(t)$  is changing with time as it is the phase-change boundary.  $\Omega^*(0)$  and  $\Omega(0)$  have a common boundary  $\Gamma_{\text{out}}$ . We assume that it is a melting problem and melting starts at time  $t = 0$ . The region occupied by the liquid at time  $t > 0$  is denoted by  $\Omega^*(t)$  and the phase-change boundary is given by  $S(x, y, t) = 0$  which is denoted by  $\Gamma(t)$ . The formulation of the problem is given below.

$$T_t(x, y, t) = k(T_{xx} + T_{yy}), \quad (x, y) \in \Omega(t), \quad 0 < t < t^*, \quad (12.4.66)$$

$$T(x, y, t) = 0, \quad (x, y) \in \Gamma(t); \quad T(x, y, 0) = \phi(x, y), \quad (x, y) \in \Omega(0), \quad (12.4.67)$$

$$\frac{dx}{dt} = -K_1 T_x(x, y, t), \quad (x, y) \in \Gamma(t); \quad \frac{dy}{dt} = -K_2 T_y(x, y, t), \quad (x, y) \in \Gamma(t). \quad (12.4.68)$$

Let  $G(x, y, t; \xi, \eta, \tau)$  be the two-dimensional Green's function then for  $t > \tau$

$$G(x, y, t; \xi, \eta, \tau) = 1/(4\pi k(t - \tau)) \exp\{-(x - \xi)^2 - (y - \eta)^2\}/(4k(t - \tau))\}. \quad (12.4.69)$$

$$\frac{\partial G}{\partial \tau} + k\left(\frac{\partial^2 G}{\partial \xi^2} + \frac{\partial^2 G}{\partial \eta^2}\right) = 0, \quad k[(T_\xi G)_\xi + (T_\eta G)_\eta - (TG_\xi)_\xi - (TG_\eta)_\eta] - (TG)_\tau \equiv 0. \quad (12.4.70)$$

If the identity in Eq. (12.4.70) is integrated over  $\tau$  from 0 to  $t - \epsilon$  and with respect to  $\xi$  and  $\eta$  over  $\Omega(\tau)$  (triple integration) then finally after several steps of calculations, we get

$$T(x, y, t) = \int_{\Omega(0)} G|_{\tau=0} \phi(\xi, \eta) d\xi d\eta + \int_0^t \int_{\Gamma(\tau)} kG \frac{\partial T}{\partial n_{\xi n}} d\sigma d\tau. \quad (12.4.71)$$

$$\left. \begin{aligned} V(x, y, t)|_{\Gamma(t)} &= \frac{\partial T}{\partial n} \Big|_{\Gamma(t)} = \int_{\Omega(0)} \frac{\partial(G|_{\tau=0})}{\partial n_{xy}} \phi(\xi, \eta) d\xi d\eta \\ &+ \int_0^t \int_{\Gamma(\tau)} k \frac{\partial G}{\partial n_{xy}} V d\sigma d\tau + \frac{1}{2} V(x, y, t) |_{\Gamma(t)}. \end{aligned} \right\} \quad (12.4.72)$$

$$\left. \begin{aligned} x(\tau, \sigma)|_{\Gamma} &= x(0, \sigma) - K_L \int_0^\tau T_x(x, y, p)|_{\Gamma} dp; \\ y(\tau, \sigma)|_{\Gamma} &= y(0, \sigma) - K_S \int_0^\tau T_y(x, y, p)|_{\Gamma} dp. \end{aligned} \right\} \quad (12.4.73)$$

The outer boundary of  $\Omega(t)$  for  $t > 0$  is  $\Gamma(t)$  which is appearing in Eqs (12.4.71)–(12.4.73).  $\frac{\partial T}{\partial n} \Big|_{\xi\eta}$  and  $\frac{\partial T}{\partial n} \Big|_{xy}$  are the normal derivatives along the normal in the outward direction to the region in  $(\xi, \eta)$  and  $(x, y)$  coordinates, respectively, and  $n$  stands for normal.

$$\frac{\partial T}{\partial n} \Big|_{\xi\eta} = T_\xi n_\xi + T_\eta n_\eta \text{ and } \frac{\partial T}{\partial n} \Big|_{xy} = T_x n_x + T_y n_y. \quad (12.4.74)$$

$$\frac{\partial T}{\partial n} \Big|_{\Gamma(t)} \text{ is the limit of } \frac{\partial T}{\partial n} \text{ as } (x, y) \rightarrow \Gamma(t). \quad (12.4.75)$$

For derivation of Eq. (12.4.71) refer [502] and for the derivation of Eq. (12.4.72) refer [19, 503]. Algorithm used for numerical work is as follows.

Step 1. Guess initial values  $V_0(x, y, t)|_{\Gamma}$ ,  $x_0(x, y, t)|_{\Gamma}$ ,  $y_0(x, y, t)|_{\Gamma}$ .

Step 2. Calculate  $V_1(x, y, t)|_{\Gamma}$  by using  $V_0$  in the r.h.s. of Eq. (12.4.72) then use  $V_1|_{\Gamma}$  in Eq. (12.4.71) and calculate  $T_x|_{\Gamma}$  and  $T_y|_{\Gamma}$ . Using these values calculate  $x_1|_{\Gamma}$  and  $y_1|_{\Gamma}$ .

Step 3. Iterate till convergence. Use converged values to calculate  $T(x, y, t)$  from Eq. (12.4.71). The subscripts 1, 2, ..., in  $V_1, x_1$ , etc., indicate successive iterations.

For numerical work,  $\Omega(0)$  was taken as  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ . For finite element solution of Eq. (12.4.72), 20 finite elements for  $\Gamma(t)$  and 25 elements for  $\Omega(0)$  were considered. Twelve iterations were done.  $\Gamma(t)$  and  $T(x, y, t)$  have been presented graphically.

The integral representations for two-phase Stefan problems in one-dimensional finite slab using Green's functions have been given in [21] for temperatures and Stefan-condition. Existence and stability of the solution have been proved for large time. No analytical or numerical solution has been obtained. Boundary conditions of the first and third kind can be prescribed.

A two-phase two-dimensional inverse problem in the region  $-\infty < x < \infty$ ,  $0 \leq y \leq a < \infty$  has been formulated in [257] in terms of integral representations of temperatures and phase-change velocity. The moving boundary is known and has been taken in the form  $y = S(x, t)$ ,  $S(x, 0) = b(x)$ ,  $x \in R$ . Initial temperatures in the two phases are taken different. At  $y = 0$  temperature is unknown and the face  $y = a$  is insulated. Numerical work has been done for the regularization of this inverse problem which is concerned with determining temperature at  $y = 0$  when phase-change boundary is known. This is an ill-posed problem.

For the formulation of the problem refer [257] which can be used for direct problems also with appropriate changes.

The boundary integral formulation considered in [504] has been further extended in [505] to a more general problem and so only the formulation considered in [505] will be discussed. In [502] the region  $\Omega(t)$  is expanding with time and  $\Omega(0) \subset \Omega(t)$ , for  $t > 0$ , and the outer boundary of  $\Omega(t)$  is a moving boundary. In [505], a two-dimensional heat equation is considered in a region  $B(t)$  whose outer boundary is a fixed boundary denoted by  $\partial B^*$  and the inner boundary  $\partial B^+(t)$  is a moving boundary or a phase-change boundary. Over the fixed boundary  $\partial B^*$ , either temperature is prescribed or heat flux is prescribed. In the region  $B(t)$ , the temperature  $T(x, t)$  satisfies the following equation together with the isotherm and Stefan condition.

$$\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial y^2} = 0, \quad \rho_C/K = 1 \text{ after scaling, } t > 0, (x, y) \in B(t). \quad (12.4.76)$$

$$T|_{t=f} = T_M(t); \quad \left. \frac{\partial T}{\partial n} \right|_{t=f} = \lambda(t) V_n; \quad t = f(x, y) \text{ is the interface,} \quad (12.4.77)$$

$\lambda(t)$  is the Stefan number.

The moving boundary  $\partial B^+(t)$  is a surface  $S^+$  in space-time region and is taken as  $t = f(x, y)$ .  $\partial B^*$  may be regarded as a cylindrical surface  $S^*$  in space-time region and is taken to coincide with  $\partial B^+$  at time  $t = 0$ .  $V_n$  is the normal component of the velocity of the moving boundary in Eq. (12.4.77). Let  $G(\vec{r}, \vec{r}'; t, t')$  be the Green's function defined as

$$\left. \begin{aligned} G(\vec{r}, \vec{r}', t, t') &= H(t - t') \exp[-|\vec{r} - \vec{r}'|^2 / (4(t - t'))] / (4\pi(t - t')), \\ H(t - t') &= 1, t > t' \text{ and } = 0 \text{ for } t < t'. \end{aligned} \right\} \quad (12.4.78)$$

$$\frac{\partial G}{\partial t'} + \frac{\partial^2 G}{\partial x'^2} + \frac{\partial^2 G}{\partial y'^2} = -\delta(t - t')\delta(\vec{r} - \vec{r}'); \quad G(\vec{r}, \vec{r}', t, t') = G(x, y, t; \xi, \eta, t'). \quad (12.4.79)$$

In Eq. (12.4.79) shorter notations have been used otherwise  $G(\vec{r}, \vec{r}', t, \tau)$  is the usual two-dimensional Green's function. Let  $T' = T(\vec{r}', t')$ ,  $\nabla' = \left( \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'} \right)$  and  $\vec{r}$  and  $\vec{r}'$  be the position vectors of the points  $(x, y)$  and  $(x', y')$ , respectively. The following identity holds. Note that dashes do not stand for derivatives.

$$\frac{\partial}{\partial t'} (T' G) + \nabla' \cdot (T' \nabla' G - G \nabla' T') = -T' \delta(t - t') \delta(\vec{r} - \vec{r}'). \quad (12.4.80)$$

Integrating Eq. (12.4.78) over the space-time region between  $S^*$  and  $S^+$ , taking unit outward normal  $\vec{n}$  vector as  $(\nabla f, -1)/(1 + |\nabla f|^2)^{1/2}$  and using divergence theorem in space-time region, we get

$$\left. \begin{aligned} &\int_0^{t+0} \int_{\partial B^*} \left( \frac{T' \partial G}{\partial n'} - \frac{G \partial T'}{\partial n'} \right) ds' dt' \\ &- \int_{S^t} \frac{[\nabla' T' \cdot \nabla' f' + T_M(t') G - T_M(t') \nabla' G \cdot \nabla' f]}{(1 + |\nabla' f|^2)^{1/2}} ds' = -\alpha T(\vec{r}, t), \end{aligned} \right\} \quad (12.4.81)$$

where  $S^t$  is the portion of  $S^+$  for  $0 \leq t' \leq t$  and  $\partial T'/\partial n'$  is the outward normal derivative on  $\partial B^*$  in  $(x', y')$  coordinates.  $\alpha = 1/2$  if  $(x', y')$  is a regular boundary point in boundary element

scheme and  $\alpha = 1$  if  $(x', y')$  is away from the boundary. Stefan condition can be obtained in a simplified form as  $\nabla T' \cdot \nabla f' = \lambda(t')$ . If the projection of  $S^t$  is taken on  $(x', y')$  plane then the second integral on the l.h.s. in Eq. (12.4.81) can be written as

$$- \int_B \{[\lambda(t') + T_M(t')]G - T_M(t')\nabla' G \cdot \nabla' f'\}_{t'=f'} dx' dy'. \quad (12.4.82)$$

$T(x, y, t)$  and  $f(x, y)$  both are unknowns.  $f(x, y)$  should satisfy  $T|_{t=f} = T_M(t)$  so together with Eq. (12.4.81) we have two coupled integral equations.

The effect of concentration, if present, has also been accounted in [505] in a limited way in  $\lambda(t)$ . If  $\bar{c}_L$  and  $\bar{c}_S$  are the concentrations of impurity in the liquid and solid, respectively, and linear-phase diagram of the type in Fig. 12.2.1 is taken then we have

$$\gamma_S \bar{c}_S = T_M - T_A, \quad \gamma_L \bar{c}_L = T_M - T_A. \quad (12.4.83)$$

If  $A_0$  is the initial cross-sectional area of the cylindrical region in space time,  $A$  is the cross-sectional area at any time  $t$  and  $\beta = \gamma_L |\gamma_S - 1|$ , then it has been shown in [505] that by considering concentration of the impurity, the Stefan number  $\lambda(t)$  can be estimated without taking into account diffusion equations and

$$\lambda(t) = \frac{\rho_L l}{K} + \gamma_L \frac{\bar{c}_L \rho_L}{K} \beta \left( \frac{A}{A_0} \right)^\beta \bar{c}_{0L}, \quad \bar{c}_{0L} = \bar{c}_L|_{t=0}. \quad (12.4.84)$$

For numerical work boundary element method (BEM) has been employed. An infinite rod of square cross-section with a circular hole at the centre has been considered as geometry. Some other geometries have also been considered. Numerical work is more clearly explained in [504]. Considerable numerical work has been reported.

Lightfoot's source and sink method (SSM) [506] discussed earlier (refer Eq. 12.2.29) has been employed in [507] to study a one-phase one-dimensional problem in semiinfinite medium  $x > 0$ . A cyclic temperature is prescribed at  $x = 0$  which gives rise to alternating half cycles of melting and cooling. Suppose in the first stage, heating is done by prescribed temperature at  $x = 0$  in the region  $0 \leq x < \infty$  which initially at  $t = 0$  was at the melting temperature. Let  $x = S_1(t)$  be the phase-change boundary in the first stage. Till the time heating stops and cooling begins the moving boundary has moved up to  $x = S_1(t_1) > 0$ . If the solutions of temperature and  $S_1(t)$  can be derived then  $S_1(t_1) > 0$  can be calculated as  $t_1$  is known and  $t_1 = P/2$  where  $P$  is the period of the cycle. For  $t \geq t_1$ , if the temperature less than melting temperature is prescribed at  $x = 0$  then solidification starts at  $t = t_1$ . Let  $x = S_2(t)$  be the solidification front for  $t > t_1$ . If cooling continues till time  $t = t_2$  then  $S_2(t_2)$  can be calculated by solving another phase-change problem as  $t_2 = P$  is known. This is the second stage of the problem. In the second cycle other phase-change boundaries may occur. Let the region on the right of  $x = S_1(t)$  be called region  $a$ , the region between  $S_2(t) \leq x \leq S_1(t)$  be called region  $b$  and the region to the left of  $x = S_2(t)$  be the region  $c$ . Thus we have a three-region problem.

If in some melting problem initially flux is prescribed at  $x = 0$  then the solidification does not start at  $t = t_1$ , and  $t_1$  may not be  $P/2$ . In this case also we have a three-region formulation but solution will be different. There could be several such cycles of heating and cooling taking place such as in solar energy devices. In this case numbers of phases and phase-boundaries may go on increasing. For an initially subcooled medium exposed to a large sinusoidal temperature

cycle, the medium may melt or freeze simultaneously and the problem could be a four-phase problem.

As discussed earlier, SSM requires that thermo-physical properties of all the phases remain the same as that of the original medium. The diffusion equations of all the three phases in one cycle in a three-phase problem could be combined into a single equation as given below.

$$\frac{\partial^2 T}{\partial x^2} - \frac{\rho l}{K} \frac{dS_1}{dt} \delta_1(x - S_1(t)) + \frac{\rho l}{K} \frac{dS_2(t)}{dt} \delta_2(x - S_2(t)) = \frac{1}{k} \frac{\partial T}{\partial t}, \quad (12.4.85)$$

$\delta_1$  and  $\delta_2$  are Dirac-delta functions and so Eq. (12.4.85) is an equation in distributional sense. Other conditions in the problem could be as follows.

Boundary conditions at  $x = 0$

$$T(0, t) = F(t) \text{ or } -K \frac{\partial T(0, t)}{\partial x} = Q(t). \quad (12.4.86)$$

Initial conditions in temperature and flux prescribed cases could be as given below.

$$T(x, 0) = T_m = 0, \quad 0 \leq x < \infty; \quad T(x, t_1) = T_1(x, t). \quad (12.4.87)$$

Isotherm conditions

$$T(S_1(t), t) = T_m = 0; \quad T(S_2(t), t) = T_m = 0. \quad (12.4.88)$$

This three-phase problem in the classical sense has already been formulated in earlier sections and its formulation will not be presented here. If  $\delta_2$  term in Eq. (12.4.85) is deleted and  $T(S_2(t), t) = 0$  is not there in Eq. (12.4.88) then we call it a phase I problem. By suitably changing signs of latent heat  $l$  and heat flux, changes due to heating and cooling conditions at  $x = 0$  can be accounted. Letting  $x \neq S_1(t)$  and  $x \neq S_2(t)$  and by suitably choosing the ranges of  $x$  in the three phases, heat equations in the three phases can be obtained (cf. Section 5.2.1). If Eq. (12.4.85) is integrated from  $S_2(t) - \epsilon$  to  $S_2(t) + \epsilon$ , limit  $\epsilon \rightarrow 0$  is taken and correct limiting process is followed then the energy balance condition at  $x = S_2(t)$  can be obtained. Similar procedure can be applied for  $S_1(t)$ . Note that temperatures are continuous across  $x = S_2(t)$  but not fluxes.

**Temperature for phase I:** Using SSM the temperature  $T_1(x, t)$  can be obtained by solving Eq. (12.4.85).

$$T_1(x, t) = T_0(x, t) - \frac{l}{C} \int_{\tau=0}^{\tau=t} \frac{dS_1}{d\tau} \bar{G}(x, t; S_1(\tau), \tau) d\tau. \quad (12.4.89)$$

$$T_0(x, t) = \sqrt{k/\pi} \int_{\tau=0}^t \frac{E(\tau)}{(t-\tau)^{1/2}} \exp \left[ -\frac{x^2}{k(t-\tau)} \right] d\tau. \quad (12.4.90)$$

$E(\tau) = \frac{1}{2k} \frac{\bar{F}(\tau)}{t-\tau}$ . If  $F(t)$  is prescribed at  $x = 0$  then  $\bar{F}(t) = xF(t)$  and if flux is prescribed at  $x = 0$  then  $\bar{F}(t) = Q(t)/K$ . If  $F(t)$  is prescribed then  $\bar{G}(x, t; x', \tau)$  is given by Eq. (12.4.56) with  $(t - \tau)$  replaced by  $k(t - \tau)$ ,  $k$  is thermal diffusivity, and if flux is prescribed then there will be plus sign between the two terms in Eq. (12.4.56).  $S_1(t)$  is determined by satisfying  $T_1(S_1(t), t) = 0$ .  $S_1(t)$  is to be obtained by solving an integral equation. If temperature is

prescribed at  $x = 0$  then  $t_1 = P/2$  and for  $t \geq t_1 + 0$  solidification starts at  $x = 0$ . We call it a phase II problem and the phase II problem is for  $t > t_1$ . However, if flux is prescribed solidification may not start at  $t = P/2$  and then  $t_1 \neq P/2$ . In the phase II for the flux prescribed case, although cooling has started at  $x = 0$ , till the temperature at  $x = 0$  does not become zero (freezing temperature) solidification will not start. In this case  $t = t_1$  is obtained by solving the following equation.

$$T_1(0, t_1) = 0 = T_0(0, t_1) - \frac{l}{C} \int_{\tau=0}^{t_1} \frac{dS_1(\tau)}{d\tau} \bar{G}(0, t_1; S_1(\tau), \tau) d\tau. \quad (12.4.91)$$

If  $T_2(x, t)$  is the temperature in second phase then using the procedure followed earlier in connection with Lightfoot method, we get

$$\left. \begin{aligned} T_2(x, t) = & T_0(x, t - t_1) + \int_{x'=0}^{\infty} T_1(x', t_1) \bar{G}(x, t - t_1; x', 0) dx' \\ & + \frac{l}{C} \int_{\tau=0}^{t-t_1} \left[ -\frac{dS_1(\tau + t_1)}{d\tau} \bar{G}(x, t - t_1; S_1(\tau + t_1), \tau) \right. \\ & \left. + \frac{dS_2(\tau + t_1)}{d\tau} \bar{G}(x, t - t_1; S_2(\tau + t_1), \tau) \right] d\tau. \end{aligned} \right\} \quad (12.4.92)$$

The second term in the r.h.s. of Eq. (12.4.92) involves implicitly a double integral which is not suitable for numerical purpose. Following a lengthy procedure two identities have been derived (not given here) in [507] and with their help finally  $T_2(x, t)$  can be written as

$$\left. \begin{aligned} T_2(x, t) = & T_0(x, t) - \frac{l}{C} \int_{\tau=0}^t \frac{dS_1(\tau)}{d\tau} \bar{G}(x, t; S_1(\tau), \tau) d\tau \\ & + \frac{l}{C} \int_{\tau=t_1}^t \frac{dS_2(\tau)}{d\tau} \bar{G}(x, t; S_2(\tau), \tau) d\tau. \end{aligned} \right\} \quad (12.4.93)$$

Eq. (12.4.93) is valid for one cycle of heating and then cooling or for one cycle of cooling and then heating. If there are  $n$ -phases,  $n > 1$  then the temperature  $T(x, t)$  is given by

$$\begin{aligned} T(x, t) = & \frac{l}{C} \sum_{i=1}^n \hat{H} \left( t - \frac{(i-1)P}{2} \right) \int_{\tau=(i-1)P/2}^t (-1)^{m+i-1} \frac{dS_i(\tau)}{d\tau} G(x, t; S_i(\tau), \tau) d\tau \\ & + T_0(x, t), \end{aligned} \quad (12.4.94)$$

$m = 1$  or  $2$  as heat is supplied first or withdrawn first.  $P$  is time for one cycle when temperature is prescribed,

$$\hat{H}(t - (i-1)P/2) = 1 \text{ for } t > (i-1)P/2 \text{ and } = 0 \text{ if } t < (i-1)P/2. \quad (12.4.95)$$

In the flux prescribed case  $(i-1)P/2 = t_1$ . Considerable numerical work was done and its discussion is reported in [507]. In the finite difference method which was used for discretization, the local linearization of velocities of moving boundaries was done so that terms, like  $dS_1/dt$ ,  $dS_2/dt$ , can be taken out of the integral. By using initial values of moving boundaries and temperatures, the solution for next time step can be started. Both cyclic temperature and cyclic flux boundary condition were used in numerical work for one cycle. If there are  $n$ -phases in one cycle then  $n$  integral equations which are coupled have to be solved to determine the phase-change boundaries and temperatures.

In [508], the emphasis is on the numerical solutions obtained by the SSM and in showing that SSM and BEM are two methods in one. SSM is simpler to use and hybrid methods are also possible combining SSM with complex variable method and other methods. In SSM the problem is solved in a fixed domain so that the boundary motion terms are absent. They are incorporated in the form of moving heat sources. First, a general expression for temperature is presented in [508] in terms of an integral equation in a region  $\Omega(t)$  whose boundary is  $\Gamma$ . Multiple phases and moving boundaries are present in  $\Omega(t)$  in the integral equation formulation. Afterwards numerical solutions have been obtained for one-dimensional problems having two boundaries.

The temperature in the general case can be expressed as

$$T(\vec{r}, t) = \int_{\Omega'} G(\vec{r}, t; \vec{r}', 0) T_i(\vec{r}') d\vec{r}' + \frac{1}{C} \sum_{J=1}^n l_J \int_{t_{J-1}}^{t_J} \int_{\Omega'} G(\vec{r}, t; \vec{r}', s) \times \frac{\partial S_J(t)}{\partial s} \delta_J[\vec{r}' - S_J(\vec{r}', s)] d\vec{r}' ds + \sum_i \quad (12.4.96)$$

Any point in the region  $\Omega(t)$  is represented by a position vector  $\vec{r}$ ,  $ds$  is an element on the boundary  $\Gamma$  of  $\Omega(t)$ ,  $\vec{r}'$  is a position vector of the point at which point heat source is acting in Green's function  $G(\vec{r}, t; \vec{r}', t')$ .  $l_J$ ,  $\delta_J(\cdot)$  and  $S_J(\vec{r}', t) = 0$  are, latent heats, Dirac-delta functions and moving boundaries, respectively, and  $J = 1, 2, \dots, n$ .  $T_i(r)$  is the initial temperature in the  $i$ th region. The last term in brackets in Eq. (12.4.96) accounts for the multiple boundary conditions. An equation similar to Eq. (12.4.85) can be easily written in terms of  $(\vec{r}, t)$  with summation sign. Also equation similar to Eq. (12.4.94) can be written in terms of  $(\vec{r}, t)$ . Numerical procedure is similar to the one described in [507] and local linearization of the velocities of moving boundaries can be done in finite difference discretization of integral equations in time. Using SSM, numerical results have been presented graphically for both single and two moving boundaries.

A procedure has been suggested in [509] for two-dimensional one-phase problems in rectangular region using SSM. It is assumed that only one phase-change boundary is present which can be written as  $y = S(x, t)$ . The two-dimensional heat equation in a plane similar to Eq. (12.4.85) in two dimensions can be discretized in the  $x$ -direction using finite differences as follows.

$$\left. \begin{aligned} & \frac{\partial^2 T_i(y, t)}{\partial y^2} + \frac{T_{i+1}(y, t) - 2T_i(y, t) + T_{i-1}(y, t)}{(\Delta x)^2} - \frac{\rho l}{K} \frac{dS_i(t)}{dt} \delta(y - S_i(t)) \\ & = \frac{1}{k} \frac{\partial T_i(y, t)}{\partial t}, \quad 0 \leq y \leq d, 0 \leq x \leq b, \end{aligned} \right\} \quad (12.4.97)$$

where  $T_i(y, t) = T(y, x_i, t)$  and  $S_i(t) = S(x_i, t)$ .  $x_i, i = 1, 2, \dots, n$ , are grid points in the  $x$ -direction.

Using Green's function,  $T_i(y, t)$  can be written in integral form as

$$T_i(y, t) = -\frac{l}{C} \int_{t_0}^t \frac{dS_i}{d\tau} G_i(y, t; S_i(t), \tau) d\tau + K \int_{t_0}^t \int_0^d g_i(y', t) G_i(y, t; y', \tau) dy' d\tau, \quad (12.4.98)$$

$$g_i(y', \tau) = \{T_{i+1}(y', \tau) - 2T_i(y', \tau) + T_{i-1}(y', \tau)\} / (\Delta x)^2.$$



The initial temperature  $T(x, y, t)$  has been taken to be zero in Eq. (12.4.98). Depending on boundary conditions, Green's function satisfying boundary conditions has to be constructed. Numerical solution procedure using SSM for two-dimensional problem has been discussed in great detail in [509].

In [507–509] several references concerned with SSM and hybrid methods used in some other geometries are also given. For further details refer these references.

### Problems in Cylindrical and Spherical Geometries

Green's function has been used in [510] to obtain integral representation of temperature  $T(r, t)$  in a one-dimensional radially symmetric problem in cylindrical coordinates and then numerical solution has been obtained. An equation similar to third equation in Eq. (12.4.58) (Green's identity) can be easily derived in cylindrical coordinates  $(r, 0, 0)$  and temperature can be expressed as

$$T(r, t) = 2\pi \int_0^{R_0} T_0 \xi G|_{\tau=0} d\xi + 2\pi \int_{R_0}^{S(t)} T_m(t) \xi G|_{S(t)} d\xi \left. \vphantom{\int_0^{R_0}} \right\} + 2\pi k \int_0^t \xi \left[ G \frac{\partial T}{\partial \xi} \Big|_{S(\tau)} - T_m(t) \frac{\partial G}{\partial \xi} \Big|_{S(\tau)} \right] d\tau, \quad (12.4.99)$$

The derivation of Eq. (12.4.99) involves some steps. First, the third equation in Eq. (12.4.58) (Green's identity in  $(r, t)$ ) is doubly integrated with respect to  $\xi$  from  $\xi = R_0$  to  $\xi = S(\tau)$  and with respect  $\tau$  from 0 to  $t - \epsilon$  and then using Green's theorem the double integral is converted to line integral. The line integral consists of lines or curves joining the points: (i)  $(\xi = 0, \tau = 0)$  to  $(\xi = R_0, \tau = 0)$ ; (ii)  $(\xi = R_0, \tau = 0)$  to  $(\xi = S(\tau = t - \epsilon), \tau = t - \epsilon)$ ; (iii)  $(\xi = S(\tau = t - \epsilon), \tau = t - \epsilon)$  to  $(\xi = 0, \tau = t - \epsilon)$ ; (iv)  $(\xi = 0, \tau = t - \epsilon)$  to  $(\xi = 0, \tau = 0)$ . On taking limit  $\epsilon \rightarrow 0$ , we get Eq. (12.4.99). Taking limits requires some expertise.

In Eq. (12.4.99),  $r = S(t)$  is the phase-change boundary,  $S(0) = R_0$ ,  $T_0$  is the initial temperature in the region  $0 \leq r \leq R_0 < \infty$ ,  $T_m(t)$  is the temperature at  $r = S(t)$ . The temperature  $T(r, t)$  satisfies the linear heat equation (12.2.26) and energy balance condition at  $r = S(t)$  has been taken in [510] as

$$K \frac{\partial T}{\partial r} \Big|_{r=S(t)} = h[T_B(t) - T_m(t)] + \rho l \frac{dS}{dt}. \quad (12.4.100)$$

The temperature  $T_B(t)$  is known and so is  $T_m(t)$ .  $h$  is heat transfer coefficient. The Green's function  $G(r, t; \xi, t)$  is as given in Eq. (12.2.23) with  $t$  replaced by  $(t - \tau)$ . If  $T_m(t)$  is known then  $r = S(t)$  is the only unknown. To obtain analytically  $\partial T / \partial r|_{r=S(t)}$  and  $T(S(t), t) = T_m(t)$ , which are required to satisfy boundary conditions at the interface, the limit  $\tau \rightarrow t$  and  $\xi \rightarrow S(t)$  has to be taken simultaneously (two-dimensional limit) for each of them. This requires evaluation of contour integration which will result only in *Cauchy-principle value* which will not give accurate results. As an approximation in the numerical solution using BEM or some other methods, when time interval  $[0, t]$  is discretized to obtain numerical results in the small intervals  $(t_k, t_{k+1})$ ,  $k = 0, 1, 2, \dots, n$ ,  $T(r, t)$  can be approximated by  $T_m(t)$  under the assumptions that  $T_m(t)$  and  $T_B$  do not change significantly from their actual values. Both  $T(r, t)$  and its derivative at  $r = S(t)$  can be taken out of integral sign. With this approximation, numerical solution of Eq. (12.4.99) or in other words solution of temperature and  $S(t)$  can be

obtained by any of the standard methods and at present several software codes are available for this.

A radially symmetric one-dimensional spherical problem in a spherical shell has been considered in [511]. The problem is concerned with the diffusion of concentration and is related to the study of particle precipitation from a supersaturated solution. To save space, the scalings of equations and the original problem formulation will be skipped. In any case method of solution has no bearing on scalings. If  $\bar{c}(r, t)$  is the concentration in which  $r$  is the spherical coordinate then under the transformation  $u(r, t) = r\bar{c}(r, t)$ , the formulation is transformed as follows.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2}, \quad S(t) < r < R_S; \quad u = r\bar{c}_0, \quad \text{at } t = 0; \quad u = r\bar{c}_S, \quad \text{at } r = S(t). \quad (12.4.101)$$

$$\frac{\partial(u/r)}{\partial r} = 0, \quad \text{at } r = R_S; \quad S(0) = R_0 < R_S; \quad \left. \frac{\partial(u/r)}{\partial r} \right|_{r=S(t)} = \left( Q - \frac{u}{S(t)} \right) \frac{dS}{dt}. \quad (12.4.102)$$

Eqs (12.4.101), (12.4.102) are dimensionless (cf. [511]) obtained from the nondimensional formulation in terms of  $\bar{c}(r, t)$ .

In Eq. (12.4.102),  $r = R_S$  is the outer boundary,  $r = S(t)$  is the moving boundary,  $Q$  is the dimensionless concentration of a particle with solute concentration  $\bar{c}_I$  which is diffusing towards the interface. The formulation in terms of  $\bar{c}(r, t)$  can be easily obtained from Eqs (12.4.101), (12.4.102). Now the Green's function for the one-dimensional Cartesian case can be used with  $x$  replaced by  $r$  without making any further transformation to spherical coordinates and let it be denoted by  $G_1(r, t; \xi, \tau)$ .  $G_1$  is not Green's function in spherical coordinate but  $G_1/4\pi r\xi$  is. The third equation in Eq. (12.4.58) (Green's identity for  $G_1$ ) if doubly integrated with respect to  $\xi$  from  $R_0$  to  $S(\tau)$  and with respect to  $\tau$  from 0 to  $t- \in$  results in the double integral over the region  $D$ ,  $D = \{R_0 < \xi < S(\tau); 0 < \tau < t- \in\}$ .

$$\iint_D \{G_1 u_{\xi\xi} - u(G_1)_{\xi\xi}\} d\xi d\tau - \iint_D \frac{\partial(uG_1)}{\partial \tau} d\xi d\tau = 0. \quad (12.4.103)$$

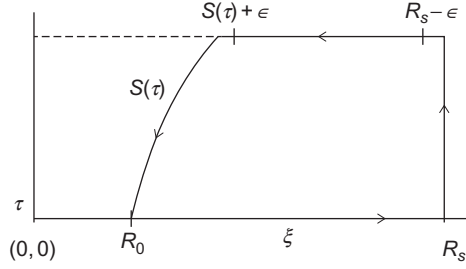
The Green's function  $G_1$  will be specified later keeping in mind the condition of no flux at  $r = 0$ .

Using Green's theorem the area integral can be converted to line integral and we have

$$\oint \{G_1 u_{\xi} - u(G_1)_{\xi}\} d\tau + \oint u G_1 d\xi = 0. \quad (12.4.104)$$

The line integration path is depicted in Fig. 12.4.1. Note that  $\tau < t$  and  $\tau = t$  are taken only in the limit  $\in \rightarrow 0$  of  $\tau = t- \in$ . This is important as there is a singularity at  $\tau = t$  if also  $r = \xi$  is taken simultaneously. Taking the limit as  $\in \rightarrow 0$  and substituting  $u(r, t) = r\bar{c}(r, t)$  in the evaluated line integral, we get the following equation after replacing  $G_1$  by  $G$ .  $G$  is given below after Eq. (12.4.105).

$$\begin{aligned} \bar{c}(r, t) = & \int_{R_0}^{R_S} (\bar{c}_0 \cdot 4\pi \xi^2 G)|_{\tau=0} d\xi + \int_{S(t)}^{R_0} (\bar{c} \cdot 4\pi \xi^2 G)|_{S(\tau)} d\xi \\ & + \int_0^t 4\pi \xi^2 \left[ G \frac{\partial \bar{c}}{\partial \xi} - \bar{c} \frac{\partial G}{\partial \xi} \right]_{R_S} d\tau - \int_0^t 4\pi \xi^2 \left[ G \frac{\partial \bar{c}}{\partial \xi} - \bar{c} \frac{\partial G}{\partial \xi} \right]_{S(\tau)} d\tau. \end{aligned} \quad (12.4.105)$$



**Fig. 12.4.1** Integration path in Eq. (12.4.104).

If the no flux boundary condition at  $R_S$  is to be satisfied in Eq. (12.4.105) and  $r$  is spherical polar coordinate then  $G_1$  should be modified and taken as  $G$  which is given below.

$$G(r, t; \xi, \tau) = (1/A) \left[ \exp \left\{ -(r - \xi)^2 / (4(t - \tau)) \right\} - \exp \left\{ - \left[ (r + \xi)^2 / 4(t - \tau) \right] \right\} \right], \quad (12.4.106)$$

$$A = 8\pi r \xi \sqrt{\pi(t - \tau)}, \quad \frac{\partial G}{\partial r} \Big|_{r=0} = 0; \quad G_1(r, t; \xi, \tau) (\text{modified}) = 4\pi r \xi G.$$

On satisfying the prescribed boundary conditions at  $x = S(t)$ , Eq. (12.4.105) gets further simplified and we have

$$\begin{aligned} \bar{c}(r, t) = & \int_{R_0}^{R_S} \left( \bar{c}_0 \cdot 4\pi \xi^2 G \right) \Big|_{\tau=0} d\xi - \int_0^t \left( \bar{c} \cdot 4\pi \xi^2 \frac{\partial G}{\partial \xi} \right) \Big|_{R_S} d\tau \\ & - Q \int_0^t \frac{dS(\tau)}{d\tau} \cdot 4\pi \xi^2 G \Big|_{S(\tau)} d\tau + \int_0^t \left( \bar{c} \cdot 4\pi \xi^2 \frac{\partial G}{\partial \xi} \right) \Big|_{S(\tau)} d\tau. \end{aligned} \quad (12.4.107)$$

In Eq. (12.4.107),  $\bar{c}(S(t), t) = \bar{c}_S$  is known but  $\bar{c}(R_S, t)$  is not known as flux is prescribed at the outer boundary  $r = R_S$ . In principle, two coupled equations are to be solved to determine  $\bar{c}(R_S, t)$  and  $S(t)$ . The two equations are provided by the limits of Eq. (12.4.105) as  $r \rightarrow R_S$  and  $r \rightarrow S(t)$ . Since  $\partial G / \partial \xi$  is discontinuous across both  $r = S(t)$  and  $r = R_S$ , the r.h.s. of Eq. (12.4.107) in the limit will give  $\frac{1}{2} \bar{c}(S(t), t)$  and  $\frac{1}{2} \bar{c}(R_S, t)$ , respectively. These values if used in numerical scheme will give wrong results in numerical computations. These limits can be obtained analytically only as Cauchy-principle values and so the factor of half comes. An alternate approach is to replace these two equations by directly using Eq. (12.4.107) for two positions  $r = R_S$  and  $r = S(t)$  infinitesimally close to the respective boundaries but inside the region. Both the intervals  $[R_0, R_S]$  and  $[0, t]$  are to be discretized in numerical computations and if it is assumed that  $\bar{c}(S(t_m), t_m)$  remains approximately constant in any time interval  $(t_k, t_{k+1})$ ,  $k = 0, 1, 2, \dots, N$ ,  $t_k \leq t_m \leq t_{k+1}$  then  $\bar{c}(S(t_m), t_m)$  can be taken out of the integral sign, for example, in the fourth term in the r.h.s. of Eq. (12.4.107). Similar type of suitable approximations can be made for  $\bar{c}_0, \bar{c}(R_S, t)$  and  $dS(t)/dt$ . This helps in evaluating the intervals in the r.h.s. of Eq. (12.4.107) as  $\bar{c}_0, \bar{c}(R_S, t), dS/dt$  and  $\bar{c}(S(t), t)$  are no more within the integral sign.

In [511], for numerical solutions, integrals of Green's function and its derivative have been evaluated analytically after discretization. Starting from the initial time  $t = t_0 = 0$ ,  $S(t)$  and

$\bar{c}(R_S, t)$  are solved for next time step  $t = t_1$  by requiring that Eq. (12.4.107) is satisfied for points very close to  $S(t_1)$  and  $\bar{c}(R_S, t_1)$ . This procedure continues for other points also. The numerical solution has been first checked with an exact analytical solution given in [512] and then after confirmation further numerical work was carried out. By prescribing  $\bar{c}(S(t), t) = \bar{c}_S$  suitably, surface tension effects can also be taken care in the numerical solution. One such expression of  $\bar{c}_S$  including surface tension is given in [511]. Considerable numerical works have been done and for details see [511].

A porous metal sphere of radius  $R_0$  and of temperature less than the temperature of a liquid metal pool of infinite extent is put in the liquid metal pool. A spherical solid dense shell grows over the porous sphere for some time, say up to  $t = t_a$ . At  $t = t_a$ , the solid dense shell remelts without lapse of time and the porous sphere regains its original radius  $r = R_0$ . At  $t = t_a$ , the temperature of porous sphere can be obtained by solving a two-phase Stefan problem in which the material properties of dense sphere and porous sphere are taken different and the phase-change boundary is growing into the liquid. After  $t > t_a$ , the porous metal sphere starts melting. For  $t < t_a$ , we have a two-phase problem and the temperature of porous metal sphere is changing with time. For  $t > t_a$ , only a one-phase problem has been considered as it has been assumed that the temperature of infinite pool does not change for both  $t < t_a$  and  $t > t_a$ . For  $t > t_a$ , temperature of the porous sphere is required at  $t = t_a$ . This problem has been considered in [513] and its numerical solution has been obtained by considering its formulation in terms of integral equations. The problem has been first formulated in terms of equations without scalings and then presented after doing scalings. Since the method of solution is independent of scalings, we present below the formulation after scaling without giving it in details and also take all the thermo-physical parameters to be unity for convenience.

**Porous region I.**  $t < t_a, 0 \leq r \leq 1$ .

$$\left. \begin{aligned} \frac{\partial T_1}{\partial t} - \left( \frac{\partial^2 T_1}{\partial r^2} + \frac{2}{r} \frac{\partial T_1}{\partial r} \right) &= 0, 0 < r < 1; \frac{\partial T_1}{\partial r} \Big|_{r=0} = 0, \\ T_1(r, 0) &= 0, r = 1; T_1(r, 0) = -1, r < 1. \end{aligned} \right\} \quad (12.4.108)$$

**Dense region.**  $t < t_a, 1 \leq r \leq S_1(t) < \infty, S_1(0) = 1$ .

$$\frac{\partial T_d}{\partial t} - \left( \frac{\partial^2 T_d}{\partial r^2} + \frac{2}{r} \frac{\partial T_d}{\partial r} \right) = 0, \quad 1 < r < S_1(t); T_d(S_1(t), t) = 0. \quad (12.4.109)$$

$$\frac{\partial T_d}{\partial r} - T_{0m} = \frac{dS_1(t)}{dt}; (T_1)_{r=1} = (T_d)_{r=1}; \frac{\partial T_1}{\partial r} \Big|_{r=1} = \frac{\partial T_d}{\partial r} \Big|_{r=1}. \quad (12.4.110)$$

**Porous region II.**  $t > t_a, 0 \leq r \leq S_2(t) \leq 1, S_2(0) = 1, T_2(r, t), 0 \leq r < 1, t > t_a$ , satisfies the same heat equation as in Eq. (12.4.108).

$$\left. \begin{aligned} \frac{\partial T_2}{\partial r} \Big|_{r=0} &= 0; \frac{\partial T_2}{\partial r} \Big|_{r=S_2(t)} - T_{0m} = \frac{dS_2}{dt}; T_2(r, t_a) = T_1(r, T_a); T_2(S_2(t), t) = 0. \end{aligned} \right\} \quad (12.4.111)$$

In Eqs (12.4.110), (12.4.111),  $T_{0m} > 0$  is constant temperature of the melt which is not changing with time and space. The subscript  $d$  stands for the dense region. Other notations are self-explanatory. In [513], thermo-physical parameters in the porous solid region and dense solid are taken different and all the parameters are taking part in the formulation.

To obtain the integral representations, we proceed as in [511]. Green's function  $G(r, t; \xi, \tau)$  for the present problem is the same as in Eq. (12.4.106). Let the heat equation in Eq. (12.4.108) be written for temperature  $T(\xi, \tau)$ . Multiply by  $T(\xi, \tau)$  the equation which  $G$  given in Eq. (12.4.106) satisfies and subtract the product from the product of Eq. (12.4.108) by  $G$  as mentioned earlier. Green's function  $G$  should have zero gradient at  $r = 0$  because of spherical geometry. We get the identity mentioned in the third equation in Eq. (12.4.58). Let this identity be integrated over the domain  $(\xi, \tau)$  with upper limits of  $\xi$  and  $\tau$  being fixed according to the regions under consideration. The area integral using Green's theorem can be converted to line integral and this results in line integral of the type Eq. (12.4.104). The contour integration depends on the region under consideration and is discussed below for each region.

**Porous region I.**  $t \leq t_a$ . Integral equation for temperature.

The line integral in Eq. (12.4.104) for the present problem consists of the path joining the four points: (i)  $(\xi = 0, \tau = 0)$  to  $(\xi = 1, \tau = 0)$ ; (ii)  $(\xi = 1, \tau = 0)$  to  $(\xi = 1, \tau = t - \epsilon)$ ; (iii)  $(\xi = 1, \tau = t - \epsilon)$  to  $(\xi = 0, \tau = t - \epsilon)$ ; (iv)  $(\xi = 0, \tau = t - \epsilon)$  to  $(\xi = 0, \tau = 0)$ . All the contours are traversed in anticlockwise direction. When  $\epsilon \rightarrow 0$  in the contour integration and boundary and initial conditions given in Eq. (12.4.108) are incorporated then the solution of temperature  $T_1(r, t)$  for porous region I can be written as

$$T_1(r, t) = \int_0^t 4\pi\xi^2 \left( G \frac{\partial T_1}{\partial \xi} \right) \Big|_{\xi=1} d\tau - \int_0^t 4\pi\xi^2 \left( T_1 \frac{\partial G}{\partial \xi} \right) \Big|_{\xi=1} d\tau - \int_0^1 G \Big|_{\tau=0} 4\pi\xi^2 d\xi. \quad (12.4.112)$$

**Dense region.**  $t \leq t_a$ : Integral equation for temperature.

The contour integration for dense region consists of the path: (i) from  $(\xi = 1, \tau = 0)$  to  $(\xi = S_1(\tau), \tau = t - \epsilon)$ ; (ii) from  $(\xi = S_1(\tau), \tau = t - \epsilon)$  to  $(\xi = 1, \tau = t - \epsilon)$ ; (iii) from  $(\xi = S_1(t - \epsilon), \tau = t - \epsilon)$  to  $(\xi = 1, \tau = 0)$ . The temperature  $T_d(r, t)$  in the dense region, when  $\epsilon \rightarrow 0$  is taken, can be written as given below after incorporating conditions given in Eq. (12.4.110).

$$T_d(r, t) = \int_0^t 4\pi\xi^2 \left[ G_d \left( \frac{dS_1}{d\tau} + Bi \right) \right] \Big|_{\xi=S_1(\tau)} d\tau - \int_0^t 4\pi\xi^2 \left( G_d \left( \frac{\partial T}{\partial \xi} \right) \right) \Big|_{\xi=1} d\tau + \int_0^t 4\pi\xi^2 \left( T_d \frac{\partial G_d}{\partial \xi} \right) \Big|_{\xi=1} d\tau. \quad (12.4.113)$$

In Eq. (12.4.113),  $G_d$  is different from  $G$  used in Eq. (12.4.112) in the sense that in  $G_d$  thermo-physical parameters of dense region are used. If in actuality thermo-physical parameters of both porous and dense regions are the same then there are no two separate regions. We have taken parameters equal to unity just for the sake of convenience. In the first term in Eq. (12.4.113),  $Bi$  is Biot number which has not been taken equal to one so that there is no confusion.

**Porous region** for  $t > t_a$ . Integral equation for temperature.

The time  $t_a$  can be obtained by putting  $dS_1/dt = 0$  in the first condition in Eq. (12.4.110) and solving for  $t$ . The integration path is traversed in the anticlockwise direction and consists of lines/curves joining the points: (i)  $(\xi = 0, \tau = t_a)$  to  $(\xi = 1, \tau = t_a)$ ; (ii)  $(\xi = 1, \tau = t_a)$  to  $(\xi = S_2(t - \epsilon), \tau = t - \epsilon)$ ; (iii)  $(\xi = S_2(t - \epsilon), \tau = t - \epsilon)$  to  $(\xi = 0, \tau = t - \epsilon)$  to

( $\xi = 0, \tau = t_a$ ). Let  $T_2(r, t)$  be the temperature of porous region for  $t > t_a$ .

$$\lim_{\epsilon \rightarrow 0} 4\pi \int_0^{S_2(\tau=t-\epsilon)} (T_2 G)_{\tau=t-\epsilon} \xi^2 d\xi = T_2(r, t). \quad (12.4.114)$$

If the conditions mentioned in Eq. (12.4.111) are incorporated then we have

$$T_2(r, t) = \int_{t_a}^t 4\pi \xi^2 \left\{ G \left( \frac{dS_2}{dt} + Bt^* \right) \right\}_{\xi=S_2(\tau)} d\tau + \int_0^1 4\pi \xi^2 (GT_2)_{\tau=t_a} d\xi. \quad (12.4.115)$$

$Bt^*$  is Biot number. In all the contour integrations take  $\xi$ -axis horizontally and  $\tau$ -axis vertically. The integration paths in obtaining Eqs (12.4.113), (12.4.115) by taking the limits  $\epsilon \rightarrow 0$  are similar to that in Fig. 12.4.1 with appropriate changes.

In Eqs (12.4.112)–(12.4.114),  $dS_1/dt$ ,  $dS_2/dt$ ,  $t_a$  together with temperature and temperature gradients at  $r = 1$  are unknown quantities. In the numerical solution, to avoid singularities which arise when  $\tau \rightarrow t$  and  $r \rightarrow S_1(t)$  or  $S_2(t)$ , the points are considered inside the domains under consideration but infinitesimally close to the boundaries as in [511]. In the numerical solution when time is discretized, quantities such as temperature, temperature derivatives,  $dS_1/dt$  and  $dS_2/dt$  can be taken out of the integral. The numerical solution using this procedure has been compared with an exact solution in a limiting case. For further information refer [513].

The references [510–513], although not recent, contain useful information about the theoretical aspects of method of analytical solutions using integral formulation and its numerical solution. Several cross-references given in them are also useful.

The analytical-numerical solutions of one-dimensional radially symmetric spherical problems have been further extended in [514]. A two-phase Stefan problem with source terms in heat equations has been considered. The integral representations of temperatures in the solid and liquid regions are given in terms of operators involving single- and double-layer potentials. The numerical method of solution in [514] is altogether different from the method discussed in [511–513]. The singularity arising in single-layer potential at the boundaries has been removed by adding a term of singularity subtraction in the corresponding operator. However, in the operator of double-layer potential, singularity still remains at the boundaries.

The heat equations in solid and liquid regions are similar to Eq. (12.4.108) with source terms occurring in the r.h.s. Initial conditions could be function of space coordinate and temperature at the phase-change front could be functions of time so that kinetic conditions at the interface can be taken into account. The temperature  $T(r, t)$  which satisfies the heat equation with source term and initial temperature can be represented in an integral operator form as given below provided  $0 \leq r \leq S(\tau) < \infty$ .

$$\lambda T(r, t) = -kPT(r, t) + kM \left( \frac{\partial T}{\partial r} + \frac{T}{k} \frac{dS}{dt} \right) (r, t) + AT_0(r, t) + Nq(r, t), \quad (12.4.116)$$

$M$  and  $P$  are single- and double-layer potentials, respectively,  $A$  is the initial potential and  $N$  is the Newton potential. These potentials are defined in [514] as follows.

$$Pg(r, t) = \int_0^t \left\{ \frac{\partial G}{\partial \xi} (r, t; \xi, \tau) g(\xi, \tau) \right\}_{\xi=S(\tau)} (S(\tau))^2 d\tau. \quad (12.4.117)$$

$$Mg(r, t) = \int_0^t \{G(r, t; \xi, \tau)g(\xi, \tau)\}_{\xi=S(\tau)} (S(\tau))^2 d\tau. \quad (12.4.118)$$

$$Ag(r, t) = \int_0^{S(0)} G(r, t; \xi, 0)T_0(\xi)\xi^2 d\xi. \quad (12.4.119)$$

$$Nq(r, t) = \int_0^t \int_0^{S(\tau)} G(r, t; \xi, \tau)q(\xi, \tau)\xi^2 d\xi d\tau. \quad (12.4.120)$$

In Eqs (12.4.116)–(12.4.120),  $k$  is the thermal diffusivity,  $r = S(t)$  is the phase-change boundary and  $G(r, t; \xi, \tau)$  is the Green's function in spherical coordinates  $(r, 0, 0)$ .  $G$  in the above equations can be obtained from  $G$  given in Eq. (12.4.106) if  $G$  given there is multiplied by  $4\pi$  and  $(t - \tau)$  is replaced by  $k(t - \tau)$ . Except Newton potential all other potentials are similar to the potentials we have discussed earlier.  $G$  solves the heat equations with  $q(\xi, \tau) = \xi^2 \delta((\xi - r), (t - \tau))$  and homogeneous initial conditions.  $\lambda$  occurring in Eq. (12.4.116) is equal to 1,  $\frac{1}{2}$  or 0 depending on whether the evaluation point  $r$  is inside, on or outside the boundary  $r = S(t)$ .

The solution of the exterior problem in the region  $S(t) \leq r < \infty$  can be written if in Eq. (12.4.116) the sign of the first term is changed to positive and the positive sign in the second term changed to negative. When  $r \rightarrow S(t)$  and  $\tau \rightarrow t$ ,  $G(r, t; S(\tau), \tau)$  and  $\partial G / \partial \xi|_{\xi=S(\tau)}$  have singularities and this is why in the earlier works presented here, some approximations were made in numerical solutions. For numerical work considered in [514], the singularity in Eq. (12.4.118) has been taken care by an analytical singularity subtraction method which is described below. Let

$$Pg(t) = \int_0^t \frac{w^p(t, \tau)}{\sqrt{t - \tau}} g(\tau) d\tau; \quad Mg(t) = \int_0^t \frac{w^d(t, \tau)}{\sqrt{t - \tau}} g(\tau) d\tau. \quad (12.4.121)$$

The kernels  $w^p$  and  $w^d$  can be calculated by taking  $r = S(t)$  and  $\xi = S(\tau)$  in Eqs (12.4.117), (12.4.118). If it is assumed that  $S(t)$  is smooth then the  $w^p$  and  $w^d$  are smooth in a closed rectangle  $\Delta t_* = \{(\tau, t); \Delta \leq \tau \leq t \leq t_*\}$ ,

$$w^d(t, t) = \left(1/\sqrt{4\pi\alpha}\right), \quad w^p = \frac{1}{\sqrt{4\pi k}} \left(\frac{1}{2k} \frac{dS}{dt} - \frac{1}{S(t)}\right). \quad (12.4.122)$$

A simple proof for the expressions of  $w^d$  and  $w^p$  in Eq. (12.4.122) has been developed in [514].

The integrand in Eq. (12.4.118) as  $\tau \rightarrow t$  can be written in the form

$$Mg(t) = \int_0^t \frac{w^d(t, \tau)g(\tau) - w^d(t, t)g(t)}{\sqrt{t - \tau}} d\tau + 2\sqrt{t} w^d(t, t)g(t). \quad (12.4.123)$$

The new integrand is now  $O((t - \tau)^{1/2})$  and hence the trapezoidal rule will converge with convergence rate 3/2. The derivatives of singularity corrected integrand are still singular. The description of the quadrature rule used to obtain numerical solution requires lot of space and is not of much use if described here partially. However, briefly it may be mentioned that *Nyström discretization method* in time given in [515] has been used. The integral operators are discretized by quadrature rule and the resulting equations are evaluated at quadrature nodes. Newton potential is difficult to evaluate and so analytical solution, if possible, is suggested. The Stefan condition is the key equation for initialization of the numerical solution in which

first moving boundary  $r = S(t)$  is calculated at the next time step and then temperatures are calculated. Two analytical solutions, one of them about solidification problem in a supercooled liquid and another about laser melting, have been considered for checking the numerical solution.

The existence, uniqueness and stability of the solutions of Stefan problems using integral equation formulations with the help of Green's functions of a more general one-phase problem have been discussed in Chapter I, part two of [21] and for two-phase problems in Chapter II in [21]. No numerical work has been carried out.

### 12.4.4 Solutions Using Iterative Method and Approximate Analytical Method

#### *Picard's Iterative Method*

Analytical and numerical solutions of a one-phase problem by Picard's iterative method have been considered in [516] and the same procedure has been followed for a two-phase one-dimensional problem in [517]. The two-phase problem will be discussed here. Consider a one-dimensional slab occupying the region  $0 \leq x \leq d$ . At time  $t = 0$  the portion  $0 \leq x \leq a < d$  is occupied by liquid and the remaining region is solid. The initial and boundary conditions are

$$\left. \begin{aligned} T_L(x, 0) = \phi_L(x), \quad 0 \leq x \leq a; \quad T_S(x, 0) = \phi_S(x), \quad a \leq x \leq d, \\ T_L(0, t) = f_L(t); \quad T_S(d, t) = f_S(t). \end{aligned} \right\} \quad (12.4.124)$$

The subscript  $L$  stands for liquid and  $S$  for solid. The heat equations for  $T_L$  and  $T_S$ , isotherm conditions and Stefan condition at  $x = S(t)$  are of the type similar to those in the Neumann problem (Section 1.3).

If heat equations are integrated with respect to time, for  $0 < t \leq t^*$ , we get

$$T_L(x, t) = T_L(x, 0) + k_L \int_0^t \frac{\partial^2 T_L}{\partial x^2}(x, \tau) d\tau, \quad 0 < x < S(t), \quad (12.4.125)$$

$$T_S(x, t) = T_S(x, 0) + k_S \int_0^t \frac{\partial^2 T_S}{\partial x^2}(x, \tau) d\tau, \quad S(t) < x < d. \quad (12.4.126)$$

In Picard's iterative method, the  $m$ th iteration is obtained as follows.

$$T_{L,m}(x, t) = \phi_L(x) + k_L \int_0^t \frac{\partial^2 T_{L,m-1}(x, \tau)}{\partial x^2} d\tau, \quad m = 0, 1, 2, \dots \quad (12.4.127)$$

$$T_{S,m}(x, t) = \phi_S(x) + k_S \int_0^t \frac{\partial^2 T_{S,m-1}(x, \tau)}{\partial x^2} d\tau, \quad m = 0, 1, 2, \dots \quad (12.4.128)$$

The initial approximations  $T_{L,0}(x, t)$  and  $T_{S,0}(x, t)$  are required which are chosen in such a way that if both boundary and initial conditions are satisfied well and good, otherwise at least one condition is satisfied by both the temperatures. The sequences  $\{T_{L,m}\}_{m=0}^\infty$  and  $\{T_{S,m}\}_{m=0}^\infty$  have been assumed to be convergent. Sufficient conditions for convergence of these sequences are available in [21, 518] to some extent. It is not simple to apply those sufficient conditions as the boundary and initial conditions may not match for which sufficient conditions have been given. Further if the sufficient conditions involve some estimates of some functions involved, it may not be easy to calculate the estimates.



The initial approximations have been taken in the following form in [517].

$$\left. \begin{aligned} T_{L,0}(x, t) &= e^x[f_L(t) - f_L(0)] + \varphi_L(x), \\ T_{S,0}(x, t) &= e^{d-x}[f_S(t) - f_S(0)] + \varphi_S(x). \end{aligned} \right\} \quad (12.4.129)$$

$S(t)$  is taken as

$$S(t) = \sum_{n=1}^m p_i \psi_i(t), \quad p_1, p_2, \dots, p_m, \text{ are unknown constants.} \quad (12.4.130)$$

$\psi_i(t)$ ,  $i = 1, 2, \dots$ , form an infinite set of linearly independent functions out of which only finite number of terms have been considered in Eq. (12.4.130). In [517],  $\psi_i(t) = t^{i-1}$ ,  $i = 1, 2, \dots, m$ , have been taken and  $p_i$ ,  $i = 1, 2, \dots, m$ , are obtained by minimizing the functional

$$\begin{aligned} J(p_1, p_2, \dots, p_m) &= \int_0^{t^*} (T_{L,n}(S(t), t) - T_m)^2 dt + \int_0^{t^*} (T_{S,n}(S(t)) - T_m)^2 dt \\ &+ \int_0^{t^*} \left\{ K_S \frac{\partial T_{S,n}}{\partial x} \Big|_{x=S(t)} - K_L \frac{\partial T_{L,n}}{\partial x} \Big|_{x=S(t)} - l \frac{dS}{dt} \right\}^2 dt, \end{aligned} \quad (12.4.131)$$

$t^*$  is the time up to which solution is required. The subscript  $n$  in Eq. (12.4.131) for  $T_L$  and  $T_S$  stand for  $n$ th iteration,  $T_m$  is the isotherm temperature,  $K$  stands for thermal conductivity and  $l$  for latent heat. The functional in Eq. (12.4.131) is guided by the reasoning that in Eqs (12.4.125)–(12.4.127), the boundary conditions at the interface have not been taken into account. Gradient method can be used to determine  $p_1, p_2, \dots, p_m$  by minimizing  $J(p_1, p_2, \dots, p_m)$ .  $S(t)$  is calculated from Eq. (12.4.130). Absolute errors in  $S(t)$  and  $T(x, t)$  can be calculated as in Eq. (12.4.14) provided exact analytical solutions are known. An exact analytical solution was constructed in [517] and absolute errors were found insignificant. Percentage errors can also be calculated. Higher iterates have to be generally calculated numerically.

An approximate solution of one-phase one-dimensional solidification/melting problem has been constructed in [519] by using an approach which is simple to explain. Consider the linear heat equation in the region  $\{S(t) \leq x \leq d < \infty, 0 \leq t \leq t_1\}$  in which the moving boundary  $x = S(t)$  at  $t = 0$  is given by  $S(0) = d$ . Although other type of boundary conditions can also be considered; first, temperature prescribed boundary condition is considered at  $x = d$  to explain the method. To save space complete problem formulation will not be given here as it is of the same type as considered earlier in many other problems. Let  $\xi(t) = d - S(t)$ . Temperature  $T(x, t)$  is expressed as

$$T(x, t) = \sum_{i=0}^{\infty} A_i(t) \frac{(x - d + \xi(t))^i}{i!}, \quad A_i(t), \quad i = 0, 1, 2, \dots, \text{ are unknowns.} \quad (12.4.132)$$

When Eq. (12.4.132) is substituted in the linear heat equation and it is assumed that  $(x - d + \xi(t))^i$ ,  $i = 0, 1, 2, \dots$ , are a set of independent functions with respect to coefficient  $A_i(t)$ , it can be concluded that

$$\sum_{i=0}^{\infty} \left[ \frac{A'_i(t) y^i}{i!} + \xi'(t) A_{i+1}(t) \frac{y^i}{i!} \right] = k \sum_{i=0}^{\infty} A_{i+2}(t) \frac{y^i}{i!}, \quad y = x - d + \xi(t). \quad (12.4.133)$$

Dash denotes differentiation with respect to time. On comparing the coefficients of  $y^i$ ,  $i = 0, 1, 2, \dots$ , on both sides of Eq. (12.4.133), we obtain a system of equations

$$A_i'(t) + \xi'(t)A_{i+1}(t) = kA_{i+2}(t), \quad i = 0, 1, 2, \dots; \quad k \text{ is diffusivity.} \quad (12.4.134)$$

If isotherm condition and Stefan condition are used in Eq. (12.4.134) then  $A_i(t)$ ,  $i = 0, 1, 2, \dots$ , can be determined systematically. However,  $A_i(t)$  will contain  $\xi(t)$ , derivatives of  $\xi(t)$  and integral powers of  $\xi(t)$ .  $A_0(t) = T_m$ ,  $A_1(t) = -\frac{l\rho}{K}\xi'(t)$ ,  $A_2(t) = -\frac{l\rho}{kK}(\xi'(t))^2$  and so on.  $T_m$  is the constant isotherm temperature,  $l$  is the latent heat,  $\rho$  is the density,  $k$  is the diffusivity and  $K$  is the conductivity which are the notations followed in this book. On satisfying the temperature prescribed boundary condition  $T(d, t) = \psi(t)$ , we get

$$\sum_{i=0}^{\infty} A_i(t) \frac{\xi^i(t)}{i!} = \psi(t) \simeq \sum_{i=0}^n A_i(t) \frac{\xi^i(t)}{i!}. \quad (12.4.135)$$

By taking summation in Eq. (12.4.135) for different values of  $n = 1, 2, 3, \dots$ , one after another and using  $A_i(t)$  determined from Eq. (12.4.134), differential equations to determine  $\xi(t)$  can be obtained.  $\xi(t)$  in general is determined by numerical methods. Only in some simple cases first few equations can be solved to determine  $\xi(t)$  analytically which gives only some approximation. In general only numerical solutions of  $A_i(t)$  and then of  $T(x, t)$  from Eq. (12.4.132) are possible.

If  $\xi(t)$  is approximated by broken lines in the discretized time intervals  $(t_J, t_{J+1})$ ,  $J = 0, 1, 2, \dots$ , with discretized space intervals  $(x_J, x_{J+1})$ ,  $J = 0, 1, 2, \dots$ , then  $\xi(t)$  can be approximated as

$$\left. \begin{aligned} \xi(t) &= x_J + \lambda_J(t - t_J), t \in (t_J, t_{J+1}), x_0 = 0, t_0 = 0, t_{J+1} > t_J \\ x_J &= x_{J-1} + \lambda_{J-1}(t_J - t_{J-1}); J = 1, 2, \dots, x_{J+1} > x_J. \end{aligned} \right\} \quad (12.4.136)$$

$\lambda_J$ ,  $J = 0, 1, 2, \dots$ , can be determined by using the appropriate boundary conditions. In this way calculation of  $A_i(t)$  becomes simple. Using Stefan condition and Eq. (12.4.134), we have

$$A_0(t) = T_m; A_i(t) = -\frac{\rho l \lambda_J^i}{Kk}, \quad t \in (t_J, t_{J+1}), \quad J = 0, 1, 2, \dots, i = 1, 2, \dots \quad (12.4.137)$$

It should be remembered that in Eq. (12.4.136),  $\xi(t)$  will be obtained as a step function and on the basis of this it is difficult to say that the numerical solution will be correct after few steps.

The discretization in Eq. (12.4.136) can be used if *moving space grids* are considered or variable time step numerical methods are used to determine  $x_J$  or  $t_J$ . However, if exact analytical solution is available and used for comparison then both  $x_J$  and  $t_J$  can be calculated exactly. To match the time  $(t_{k+1} - t_k)$  in which  $\xi(t)$  has moved from  $x_k$  to  $x_{k+1}$ , the Stefan condition can also be used and  $t_{k+1}$  can be iterated till Stefan condition is satisfied by the improved corrected initial guess of  $t_{k+1}$ ,  $k = 0, 1, 2, \dots$ .

This above numerical scheme of satisfying Stefan condition for obtaining  $t_{k+1}$  suggested in [519] seems to be more appropriate in our opinion than the one suggested in [520] by the authors for the same problem considered in [519]. If  $\xi(t)$  and  $A_i(t)$  given in Eqs (12.4.136), (12.4.137) are substituted in Eq. (12.4.132) then the infinite series in summation sign can be summed up as an exponential function in which  $\lambda_J$  are unknowns,  $J = 0, 1, 2, \dots$ . To determine

$\lambda_J$  the boundary condition at  $x = d$  can be used which could be of first, second or third kind. For each  $\lambda_J$ ,  $J = 0, 1, 2, \dots$  a transcendental equation is to be solved. If exact location of  $x_k$  and exact time  $t_k$  are not known then in our opinion it is difficult to comment about the accuracy.

A quasisteady-state two-dimensional one-phase problem has been considered in [521]. The method of analytical-numerical solution is similar to that used in [519, 520]. In view of a continuous casting problem considered in [521], the ingot has a constant velocity  $V$  in the  $z$ -direction which is taken vertically upwards and  $x$ -axis is taken horizontally. The problem formulation does not present any difficulty and it will be skipped here. The phase-change boundary is  $x = S(z)$ ,  $S(z) \leq x \leq d$ ,  $0 < z \leq \bar{z}$ . If  $\xi(z) = d - S(z)$  then temperature  $T(x, z)$  can be expressed in a series of the form Eq. (12.4.132) with  $t$  replaced by  $z$ . We follow the method as explained earlier and obtain recurrence relations for  $A_i(z)$ ,  $i = 0, 1, 2, \dots$  which involve  $\xi'(z)$  and  $A'_i(z)$ . Dash denotes differentiation with respect to  $z$ . Any one of the boundary conditions which could be of first, second or third kind can be prescribed at  $x = d$ . The Stefan condition at  $x = \xi(z)$  will be different in this problem and is given below

$$K \left( \frac{\partial T}{\partial x} \Big|_{x=\xi(z)} - \xi'(z) \frac{\partial T}{\partial z} \Big|_{x=\xi(z)} \right) = \rho l V \xi'(z), \quad V \text{ is ingot velocity.} \quad (12.4.138)$$

For determining the approximate analytical solution for this problem, the procedure adopted in [519, 520] is followed. For numerical solution, the solution procedure is similar to the one adopted in [520]. In [519–521] exact analytical solutions have been suitably constructed or chosen for comparison with the numerical solutions.

A short-time analytical solution of a one-dimensional one-phase solidification problem has been constructed in [360] using Lightfoot's SSM. The boundary conditions considered at  $x = 0$  and  $x = S(t)$  are given below.

$$K_S \frac{\partial T_S}{\partial x} \Big|_{x=0} = h[T_S(0, t) - T_0]; \quad T_S(S(t), t) = T_m - T_0. \quad (12.4.139)$$

$T_0$  is ambience temperature,  $T_m$  is the constant melting temperature and  $x = S(t)$  is moving boundary. Heat equation and Stefan condition are as in Neumann solution (Section 1.3) which are not given here. In the solution considered in Eq. (12.2.29) which is obtained for the region  $0 \leq x < \infty$ , at  $x = 0$  temperature is prescribed. In [360] as flux is prescribed we would like flux to be zero at  $x = 0$  and so a plus sign is taken between the two exponential terms in Eq. (12.2.29). To incorporate the first boundary condition in Eq. (12.4.139), the solution given in Eq. (12.2.29) is to be modified and another solution given below is to be considered.

$$T_S^*(x, t) = T_S^{**}(x, t) - \frac{lh}{CK} \int_0^t \left[ S(t') \exp \left\{ \frac{h^2 a}{K^2} (t - t') \right\} + \frac{h}{K} [x + S(t')] \right] \times \operatorname{erfc} \left\{ \frac{x + S(t')}{2\sqrt{k(t - t')}} + \frac{h}{K} \sqrt{k(t - t')} \right\} dt'. \quad (12.4.140)$$

$T_S^{**}(x, t)$  is the modified solution in Eq. (12.2.29) as suggested earlier. As explained in Eq. (12.2.31), another solution is to be added to this  $T_S^*(x, t)$  which would satisfy the condition of initial temperature  $T_m$  and the radiation condition at  $x = 0$ . This solution in Eq. (12.4.140)

which we call  $W_S^*(x, t)$  is guided by the solution given in [24, Section 2.7]. Therefore, the temperature  $T_S(x, t) = T_S^*(x, t) + W_S^*(x, t)$  and  $x = S(t)$  is determined from the condition

$$\{T_S^*(x, t) + W_S^*(x, t)\}_{x=S(t)} = T_m - T_0. \quad (12.4.141)$$

If  $t \ll K^2/(h^2k)$  then the following approximation can be used in Eq. (12.4.141) for a short-time solution of  $S(t)$  and then of  $T_S(x, t)$ .

$$\left. \begin{aligned} S(t') &= S(t) + S(t)(t' - t) + S(t)(t' - t)^2/2 + \dots, \\ S(t) &= S(0) + S(0)t + S(0)t^2/2 + \dots \end{aligned} \right\} \quad (12.4.142)$$

The above solution cannot be continued for longer-time and for long-time solution heat balance technique has been employed which is described very briefly here as it will be discussed later in detail. If the linear heat equation is integrated with respect to  $x$  from 0 to  $S(t)$  and boundary conditions in Eq. (12.4.139) are used, we get

$$\dot{S}(t)\rho l/K = T_S(0, t)h/k + \frac{1}{k} \int_0^{S(t)} \frac{\partial T_S}{\partial t}(x, t) dx. \quad (12.4.143)$$

If iterations are used then on using Eq. (12.4.143),  $S(t)$  and  $T_S(x, t)$  can be determined analytically and numerically but only approximately. The integral in Eq. (12.4.143) is to be evaluated using approximate expressions of  $S(t)$  and  $T_S(x, t)$  for short time. For large-time solution at least three terms are required in  $S(t')$ .

### 12.4.5 Approximate Solutions in Series Form

For a two-phase one-dimensional solidification problem in a finite slab, a finite sine transform has been used in [522] to obtain the solution. We assume for presentation here that the formulation of this solidification problem is in dimensionless form (see [522] for scalings).  $x = S(t)$  is the freezing front,  $S(0) = 0, 0 \leq x \leq 1$ , and liquid occupies the region  $S(t) < x \leq 1, t > 0$ . At  $t = 0, T_L(x, 0) = 2$  and  $\partial T_L/\partial x|_{x=1} = 0$ . Solid occupies the region  $0 \leq x < S(t)$  and  $T_S(0, t) = 0$ . One is the freezing temperature in which our notation is  $T_m$  without scaling. Heat equations and Stefan condition are as is Neumann solution (Section 1.3). Finite sine transform  $\bar{T}_S(n, t)$  of  $T(x, t)$  and its inversion are defined as given below.

$$\bar{T}_S(n, t) = \frac{2}{S(t)} \int_0^{S(t)} T_S(x, t) \sin(\alpha_n x) dx, \quad \alpha_n = n\pi/S(t). \quad (12.4.144)$$

$$T_S(x, t) = \sum_{n=1}^{\infty} \bar{T}_S(n, t) \sin(\alpha_n x), \quad 0 < x < S(t). \quad (12.4.145)$$

If finite sine transform of heat equation for solid is taken and boundary condition at  $x = 0$  and isotherm condition are incorporated then

$$\bar{T}_S(n, t) = (-1)^{n+1} 2n\pi \exp\left(-\int_0^t \alpha_n^2 dt'\right) \left(\int_0^t \exp\left(\int_0^{t'} \alpha_n^2 dt''\right) / (S(t')^2) dt'\right). \quad (12.4.146)$$

For temperature  $T_L(x, t)$  another type of sine transform defined below is used.

$$\bar{T}_L(n, t) = \frac{2}{(1 - S(t))} \int_{S(t)}^1 [T_L(x, t) - 1] \sin(\beta_n(x - S(t))) dx, \quad \beta_n = \frac{(2n - 1)\pi}{2(1 - S(t))}, \quad (12.4.147)$$

$$T_L(x, t) = 1 + \sum_{n=1}^{\infty} \bar{T}_L(n, t) \sin[\beta_n(x - S(t))], \quad S(t) < x < 1. \quad (12.4.148)$$

If sine transform as defined in Eq. (12.4.147) is taken of the heat equation for liquid and isotherm and initial conditions are taken into consideration then we get a following differential equation whose solution can be constructed.

$$\left. \begin{aligned} \frac{d\bar{T}_L(n, t)}{dt} &= -\beta_n^2 \frac{k_L}{k_S} \bar{T}_L(n, t); \quad \bar{T}_L(n, t) = \bar{T}_L(n, 0) \exp(-\lambda(t)) \\ &= \frac{4}{\pi(2n - 1)} \exp(-\lambda(t)), \quad \lambda(t) = - \int_0^t \beta_n^2 k_L / k_S dt' \end{aligned} \right\} \quad (12.4.149)$$

To determine  $S(t)$ , Stefan condition is to be used.

$$S(t) = \int_0^t \left[ \gamma_S \sum_{n=1}^{\infty} (-1)^n \alpha_n \bar{T}_S(n, t') - \frac{k_L}{k_S} \gamma_L \sum_{n=1}^{\infty} \beta_n \bar{T}_L(n, t') \right] dt', \quad (12.4.150)$$

$\gamma_S$  and  $\gamma_L$  are Stefan numbers. Solutions of some particular cases have been discussed. Numerical work has been presented graphically for both one-phase and two-phase problems. Some earlier published research papers concerning application of sine transform can also be referred such as [523] for further information.

The one-phase one-dimensional problem considered in Eqs (12.2.212)–(12.2.214) and two-phase one-dimensional problem considered in Eqs (12.2.232)–(12.2.234) have been considered in [524] with a little difference that instead of flux prescribed condition at  $x = 0$ , temperature is prescribed in [524]. The method of approximate analytical solution in [524] does not use reciprocal transformation, instead parameterization of Storm condition (cf. Eq. 12.2.218) is done. We revisit Storm condition given below.

$$\frac{d}{dT} ((V(T)/K(T))^{1/2}) V(T) = \text{Constant} = \lambda; \quad V(T) = \rho C(T) \quad (12.4.151)$$

$$\text{Let } \xi = x/S(t), \quad S(t) = (2\delta t)^{1/2} \text{ and } T(x, t) = \varphi(\xi). \quad (12.4.152)$$

The nonlinear heat equation (12.2.212) can now be written as

$$K(\varphi)\varphi'' + \varphi'^2 \frac{dK}{d\varphi} + \delta\xi V(\varphi)\varphi' = 0. \quad (12.4.153)$$

If another variable  $y$  is defined as  $y = (K(\varphi)/V(\varphi))^{1/2}$ , then a parameterization of Storm condition in Eq. (12.4.151) is possible.

$$V(\varphi) = -\frac{1}{\lambda y^2} \frac{dy}{d\varphi}, \quad K(\varphi) = -\frac{1}{\lambda} \frac{dy}{d\varphi}. \quad (12.4.154)$$

By using Eq. (12.4.154) in Eq. (12.4.153), we get

$$\frac{d^2 y}{d\xi^2} + \frac{\delta \xi}{y^2} \frac{dy}{d\xi} = 0. \quad (12.4.155)$$

The solution of Eq. (12.4.155) has been obtained in terms of two coupled transcendental equations. The coupling variable is  $u$  which is given below.

$$u = \delta^{1/2} \xi / y - \delta^{-1/2} \frac{dy}{d\xi}. \quad (12.4.156)$$

The steps leading to the coupled equations are not easy and involve many steps and cannot be explained partially as it will not make any sense. Approximate solutions have been obtained for both one- and two-phase problems. Numerical work has also been reported. For further details refer [524].

An exact analytical solution has been presented in [525] for a one-dimensional two-phase problem in a finite slab. In another problem in [525] with the help of coordinates fitted to the phase-change boundary, numerical solution of a two-dimensional two-phase transient problem in a radially symmetric cylindrical mold of finite length has been obtained. We first discuss one-dimensional problem in a slab  $0 \leq x \leq h$ . The linear heat equations in solid and liquid phases and Stefan condition are similar to those in Neumann solution (Section 1.3) and are not being given here. The other conditions are given below.

$$\left. \begin{aligned} T(x, 0) = T_0(x) = m_S(x - S(t)), \quad 0 \leq x \leq S(t) \\ = m_L(x - S(t)), \quad S(t) \leq x \leq h. \end{aligned} \right\} \quad (12.4.157)$$

$$\left. \begin{aligned} T_S(0, t) = -T_1 - \lambda_S t = m_S S(t) - \lambda_S t, \\ T_L(h, t) = T_2 - \lambda_L t = m_L(h - S(t)) - \lambda_L t. \end{aligned} \right\} \quad (12.4.158)$$

$x = S(t)$  is the phase-change boundary and  $T = 0$  is the isotherm temperature. There is no convection in the liquid,  $m_S$  and  $m_L$  are temperature gradients, and  $\lambda_L, \lambda_S$  are cooling rates.

If Laplace transform  $\bar{T}_S(x, p)$  of linear heat equation is taken over the interval  $0 \leq x \leq S(t)$ , we get

$$\frac{d^2 \bar{T}_S}{dx^2} = (p \bar{T}_S - T_0(x)) / k_S. \quad (12.4.159)$$

If another Laplace transform denoted by  $T^*(z, p)$  of Eq. (12.4.159) is taken, we get

$$T_S^*(z, p) = - \left( \frac{T_1}{p} + \frac{\lambda_S}{p^2} \right) \frac{z}{(z^2 - p/k_S)} + (\bar{T}_S)_x(0, p) - \frac{(T_S^*)_0(z)}{k_S(z^2 - p/k_S)}. \quad (12.4.160)$$

In arriving at Eq. (12.4.160), initial and boundary conditions for the solid region have been used.  $T_S^*(z, p)$  can be inverted to get  $\bar{T}_S(x, p)$  and then  $\bar{T}_S(x, p)$  can be inverted. Using residue theorem  $T_S(x, t)$  can be obtained which is given in [525]. As the expression of  $T_S(x, t)$  is lengthy it is not being given here. A similar procedure can be adopted to obtain  $T_L(x, t)$ . Stefan condition gives a differential equation to determine  $S(t)$ . The solution has not been compared further with any analytical solution. No numerical work has been done.

It is not possible to check the calculations but if they are correct it is a good solution as it is an exact analytical solution. Surprisingly this solution has not been referred as much as some solutions such as in [468] which are just short-time solutions.

For the transient two-dimensional problem only numerical solution is presented in [525]. The problem is related to continuous casting problem without any type of convection in the liquid. Heat equation is linear and Stefan condition is as considered earlier. The moving boundary is taken as  $z = S(r, t)$ ,  $z$  is taken vertically upward and  $r$  is the radial coordinate. If  $(\xi, \eta)$ ,  $z = z(\xi, \eta)$ ,  $r = r(\xi, \eta)$  are moving boundary-fitted coordinates then the transformation of all the equations occurring in the formulation in terms of  $(\xi, \eta)$  coordinates is required and it has been explicitly described in [525] with procedure also outlined for general three-dimensional heat equation. These are useful derivations given in [525]. In the liquid region above the bottom of moving boundary, temperature gradient is prescribed at the wall  $r = w/2$  and below  $z = S(r, t)$ , temperature is prescribed at  $r = w/2$  in the form  $T = T_b + m_S z - m_S V_0 t$ .  $m_S$  is temperature gradient,  $T_b$  depends on  $T_m$  and  $m_S$ , and  $V_0$  is equal to the crystal growth at  $r = w/2$ .  $w$  is the diameter of the cylinder. Finite difference discretized formulation for the numerical solution in  $(\xi, \eta)$  coordinates is reported and numerical scheme has been explained.

A steady-state two-dimensional Stefan problem has been studied in [526] in the context of continuous casting. The region  $\Omega = \{(x, y): -1 < x < 1, y < S(x)\}$  is mapped to a finite region  $\Omega'$  by the conformal map  $W = u + iv = \exp(-\pi i(x + y)/2)$ .  $\Omega' = \{(u, v): -1 < v < 1, 0 < u < F(v)\}$ ,  $u = F(v)$  is the image of free boundary  $y = S(x)$  in the  $W$  plane. The problem formulation in  $(u, v)$  coordinates is given below.

$$T_{uu} + T_{vv} = 0 \text{ in } \Omega'; -\gamma_1 \pi/2 \nu T_u = 1 + T, u = 0, 0 < v < 1. \quad (12.4.161)$$

$$\frac{\gamma_2 \pi}{2} \nu T_u = 1 + T, u = 0, -1 < v < 0; T = 0 \text{ on } u = F(v). \quad (12.4.162)$$

$$\frac{\partial T}{\partial n} = \frac{2}{\pi} (F(v) - \nu F'(v)) (v^2 + F(v)^2)^{-1} (1 + F'(v)^2)^{-1/2}, \text{ on } u = F(v). \quad (12.4.163)$$

$\gamma_1$  and  $\gamma_2$  are dimensionless constants which involve casting velocity along with other parameters.

Using the Green's function  $G(u, v; u', v')$ , where

$$G(u, v; u', v') = \frac{1}{4\pi} \left\{ \log \left[ (u - u')^2 + (v - v')^2 \right] + \log \left[ (u + u')^2 + (v - v')^2 \right] \right\}, \quad (12.4.164)$$

the temperature can be expressed as

$$T(u, v) = -\frac{1}{\lambda} \int_{\partial\Omega'} G(u, v; u', v') \frac{\partial T}{\partial n'}(u'(s), v'(s)) ds. \quad (12.4.165)$$

In Eq. (12.4.165),  $\partial\Omega'$  is the boundary of  $\Omega'$ ,  $ds$  is the arc-length along the boundary  $\partial\Omega'$  and  $n'$  is the outward drawn normal.  $\lambda = 1$  if  $(u, v) \in \Omega'$ ,  $\lambda = 1/2$  if  $(u, v) \in \partial\Omega'$  (except at  $(0, \pm 1)$ ).

Now the problem is to solve the equation

$$\int_{\partial\Omega'} G(u, v; u', v') \frac{\partial T}{\partial n'}(u', v') ds = \begin{cases} 0, & \text{if } u = F(v) \\ 2 - \pi \gamma_1 \nu \frac{\partial T}{\partial n}, & \text{if } u = 0, 0 < v < 1 \\ 2 - \pi \gamma_2 \nu \frac{\partial T}{\partial n}, & \text{if } u = 0, -1 < v < 0. \end{cases} \quad (12.4.166)$$

Note that  $u = F(v)$  is not known so some approximations are done in obtaining numerical solution. Method of lines in the  $u$  direction has been employed for numerical solution. For further details of numerical solution see [526].

### 12.4.6 Regularization of Dirac Delta Function and a Meshless Method for Numerical Solution

In this new method discussed in [527] discretization of transient heat equation in one dimension is done and mesh points are used but only in time. The motivation comes from the following discretization in time (cf. [78]) of linear transient heat equation in one-dimension planar case in which  $T^{n+1}$  is the temperature at the  $(n + 1)$ th time step.

$$\frac{d^2 T^{n+1}}{dx^2} - \frac{2}{\Delta t} T^{n+1} = -\frac{d^2 T^n}{dx^2} - \frac{2}{\Delta t} T^n, \quad n = 0, 1, 2, \dots \quad (12.4.167)$$

Starting from  $T^0(x)$  we obtain  $T^1(x)$  and so on. We write

$$T^{n+1}(x) = T_p^{n+1}(x) + q_1^{n+1} \psi(x, \xi_1) + q_2^{n+1} \psi(x, \xi_2),$$

where  $T_p^{n+1}$  is a particular solution,  $q_1^{n+1}$  and  $q_2^{n+1}$  are real constants and  $\psi(x, \xi)$  is an *approximate fundamental solution* (AFS) which is a modification of fundamental solution  $\psi(X - Y)$  which satisfies homogeneous equation. The motivation for AFS comes if solution of an operator equation  $L(T(X)) = 0$ ,  $X \in \Omega \subset R^n$  with  $T(X) = g(X)$  on the boundary  $\partial\Omega$  of  $\Omega$  is derived in terms of source solution such as Green's function which is singular and then by approximating the solution by a continuous solution. Instead of considering the solution of this equation in infinite series, we consider an approximate solution by considering

$$T_N(X) = \sum_{J=1}^N q_J \psi(X - y_J) = g(X), \quad \psi(X - Y) \text{ satisfies operator equation.} \quad (12.4.168)$$

$\psi(X - y_J)$  are singular at  $X = y_J$  and so these singular points should be situated outside the solution domain. Of course, an exact fundamental solution of the problem under consideration should be available in analytical form.  $q_J, J = 1, 2, \dots, N$ , are unknowns which can be determined from boundary data which is given in the problem.

To obtain appropriate  $\psi(X - y_J)$ , a method of AFS can be developed by using the solution of the operator equation in the following way. Let  $L\psi^*(x, \xi) = I_{M,\theta}(x, \xi)$ ,  $I_{M,\theta}(x, \xi)$  is the delta-shaped function which differs from zero only outside some neighbourhood of the source point and which is similar in some sense to Dirac-delta function  $\delta(x - \xi)$ . Unlike Dirac-delta function it is regular at  $x = 0$ . Let us consider a complete set of orthonormal functions which are solutions of  $\varphi''(x) = -\lambda^2 \varphi$ ,  $\varphi(-1) = \varphi(1) = 0$  and take

$$\varphi_m(x) = \sin(\lambda_m(x + 1)), \quad \lambda_m = m\pi/2, \quad m = 1, 2, \dots; \quad \int_{-1}^1 \varphi_n(x) \varphi_m(x) dx = \delta_{n,m}. \quad (12.4.169)$$

Because of the orthonormality condition, we can write

$$\delta(x - \xi) = \sum_{m=1}^{\infty} \varphi_m(x) \varphi_m(\xi). \quad (12.4.170)$$



The above series diverges at every point in the interval  $[-1, 1]$  which is the solution domain. We have to regularize this fundamental solution so that it becomes a smooth function having the shape of a delta-shaped function as  $\delta(x - \xi)$  is infinite at  $x = \xi$  and 0 elsewhere. Regularization and regularization techniques have been discussed extensively in Chapter 9. These regularization techniques are different for different classes of problems. The one used in [527, 528] is known as *Lanczos regularization technique* and if this is used then

$$I_{M,\theta}(x, \xi) = \sum_{m=1}^M r_m(M, \theta) \varphi_m(\xi) \varphi_m(x), \quad (12.4.171)$$

$$r_m(M, \theta) = (\sigma_m(M))^\theta, \quad \sigma_m(M) = \frac{\sin[\alpha(m, M)]}{\alpha(m, M)}, \quad \alpha(m, M) = \frac{m\pi}{M+1}. \quad (12.4.172)$$

With the above regularization, the AFS  $\psi^*(x, \xi)$  satisfies the equation

$$\frac{d^2 \psi^*}{dx^2} - p \psi^* = I(x, \xi), \quad p = 2/\Delta t. \quad (12.4.173)$$

The subscripts  $M$  and  $\theta$  have been dropped in Eq. (12.4.173). The subscript  $\theta$  and  $M$  should be taken in coupling. For further details regarding  $\theta$  and  $M$  refer references given in [527]. It is not necessary that in every problem the solution interval is  $[-1, 1]$ . The solution of Eq. (12.4.173) can be written as

$$\psi^*(x, \xi) = - \sum_{m=1}^M \frac{r_m(M, \theta) \varphi_m(\xi)}{(\lambda_m^2 + p)} \varphi_m(x) = \sum_{m=1}^M d_m(\xi) \varphi_m(x). \quad (12.4.174)$$

$\lambda_m$  are as in Eq. (12.4.169). Now the solution of heat equation at each time level can be written as

$$T(x, q_1, q_2) = T_p(x) + q_1 \psi^*(x, \xi_1) + q_2 \psi^*(x, \xi_2). \quad (12.4.175)$$

If the boundary conditions are  $T(a) = T_a$  and  $T(b) = T_b$  then

$$T_p(a) + q_1 \psi^*(a, \xi_1) + q_2 \psi^*(a, \xi_2) = T_a; \quad T_p(b) + q_1 \psi^*(b, \xi_1) + q_2 \psi^*(b, \xi_2) = T_b. \quad (12.4.176)$$

Eq. (12.4.167) can be rewritten as

$$\left. \begin{aligned} \frac{d^2 T^{n+1}}{dx^2} - p T^{n+1} &= f^{n+1}, \quad f^{n+1} = -\frac{d^2 T^n}{dx^2} - p T^n, \\ T^n &= T(x, t^n), \quad t^n = n\Delta t, \quad p = 2/\Delta t. \end{aligned} \right\} \quad (12.4.177)$$

$T^n(x)$  can be expressed in terms of a finite series as

$$\left. \begin{aligned} T^n(x) &= \sum_{m=1}^M \bar{T}_m^n \varphi_m(x); \quad f^{n+1} = \sum_{m=1}^M (\lambda_m^2 - p) \bar{T}_m^n \varphi_m(x), \\ \bar{T}_m &= F_m + q_1 d_m(\xi_1) + q_2 d_m(\xi_2). \end{aligned} \right\} \quad (12.4.178)$$

To obtain  $q_1$  and  $q_2$ , we should know  $T_p$  whose exact analytical solution is generally not possible. Approximation of the particular integral  $T_p$  has to be obtained which has been discussed in [529] and  $T_p$  and  $T(x, q_1, q_2)$  using Eq. (12.4.175) have the forms

$$\left. \begin{aligned} T_p(x) &= -\sum_{m=1}^M \frac{f_m}{(\lambda_m^2 + p)} \varphi_m(x) = \sum_{m=1}^M F_m \varphi_m(x), F_m \text{ is given below.} \\ T(x, q_1, q_2) &= \sum_{m=1}^M (F_m + q_1 d_m(\xi_1) + q_2 d_m(\xi_2)) \phi_m(x). \end{aligned} \right\} \quad (12.4.179)$$

$F_m, m = 1, \dots, M$ , are still unknown. To obtain them in the context of *Sturm-Liouville problem* for  $\varphi(x)$  in the interval  $[-1, 1]$  which was mentioned in Eq. (12.4.169), a special technique called CICE scheme is developed which is explained in [529]. On the basis of CICE scheme

$$T_p^{n+1} = -\sum_{m=1}^M \frac{(\lambda_m^2 - p)}{(\lambda_m^2 + p)} T_m^n \varphi_m(x), F_m = \frac{(\lambda_m^2 - p)}{(\lambda_m^2 + p)} T_m^n. \quad (12.4.180)$$

This whole procedure is for the meshless solution of transient linear heat equation without phase change.

If  $T(x, 0) = T^0(x)$ , then to start numerical scheme,  $T^0(x)$  is required which can be obtained by using CICE scheme in Eq. (12.4.180) and the solution can be continued for next time step. The numerical procedure for one-phase one-dimensional Stefan problem is also given in [527] and it is on the same lines as discussed above for heat equation with appropriate modifications. Once the computer programme for temperature is ready, moving boundary can be determined from Stefan condition.

In the ablation problem the melted solid or evaporated liquid, which results on applying heat load to the solid, is absent so it is a one-phase problem. It has been discussed in [530] that under some assumptions this one-phase problem can be studied both analytically and numerically as a two-phase problem. To explain this two-phase approach we consider a simple one-dimensional ablation problem. The complete formulation is given earlier whilst discussing the problem in [244]. If  $x = S(t)$ ,  $S(0) = 0$  is the moving boundary then at  $x = S(t)$  the boundary conditions for ablation problem in the region  $x \geq S(t)$  for prescribed flux  $\bar{q}(t)$  are

$$\bar{q}(t) = l\dot{S} - K_S \frac{\partial T_S}{\partial x}, x = S(t); T(S(t), t) = T_m; \bar{q}(t) = -K_S \frac{\partial T_S}{\partial x} \Big|_{x=0}. \quad (12.4.181)$$

Next we consider a two-phase problem in the region  $x \geq 0$ . In the region  $0 \leq x < S(t)$ , fictitious liquid is present and if  $0 \leq x < S(t)$  does not have sensible heat, i.e.  $C_L = 0$  then

$$-K_L \frac{\partial T_L}{\partial x} \Big|_{S(t)-0} = -K_L \frac{\partial T_L}{\partial x} \Big|_{x=0} = \bar{q}(t); \frac{\partial^2 T_L}{\partial x^2} = 0, 0 \leq x < S(t). \quad (12.4.182)$$

The Stefan condition for the two-phase problem is to be satisfied at  $x = S(t)$ .

The two-phase enthalpy formulation used in [431] is as given below.

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x} \left( K(T) \frac{\partial T}{\partial x} \right) + Q(x, t), \text{ in } x > 0; -K \frac{\partial T}{\partial x} = \begin{cases} \bar{q}(t), & \text{if } T(x=0, t) < T_m, \\ 0, & \text{if } T(x=0, t) \geq T_m, \end{cases} \quad (12.4.183)$$

together with the initial condition which is prescribed.  $H(x, t)$  is enthalpy and  $H = \rho C_S (T - T_m)$ , for  $T < T_m$ ,  $H = \rho l$  if  $T \geq T_m$ . The solution of steady-state heat equation is  $T = T_m$ . Let  $Q(x, t)$  be the total amount of heat delivered per unit time which is given by the thermal load  $\bar{q}(t)$  applied at  $x = S(t)$  and it is no longer applied at  $x = 0$ . Further

$$\int_0^\infty Q(x, t) \varphi(x) dx = \bar{q}(t) \varphi(S(t)). \quad (12.4.184)$$

Eq. (12.4.184) holds for every arbitrary but smooth test function  $\varphi(x)$ . The introduction of  $Q(x, t)$  becomes necessary as in the one-dimensional case  $\bar{q}(t)$  applied at  $x = 0$  is immediately transferred to  $x = S(t)$ . However, in the multidimensional problem this is not true as the applied heat is redistributed in the fictitious region.  $Q(x, t)$  is proportional to Dirac-delta function. It can be shown that the enthalpy formulation in Eq. (12.4.183) gives the Stefan condition in the classical sense also for one-dimensional as well as multidimensional case provided the weak solution satisfies some conditions (refer Chapter 11 and [431]). Two-phase enthalpy formulation for the one-phase ablation problem has been reported for the multidimensional problem also.

Numerical results have been presented for one-dimensional as well as multidimensional problems. For numerical solution refer [530].

A one-dimensional three-phase problem has been considered in [531] in the region  $0 \leq x \leq L$ . This region is originally occupied by solid. When heating is done at  $x = 0$ , solid starts melting and if it is assumed that only part of the liquid formed evaporates or is drained out then three regions may coexist. Suppose liquid occupies the region  $S_2(t) \leq x < S_1(t)$ , there is a mushy region occupying  $S_1(t) < x \leq S(t)$  and a solid region occupying  $S(t) < x \leq L$ .  $x = S_1(t)$ ,  $S_1(0) = 0$  is the liquid boundary,  $x = S_2(t)$  is the liquid-mush boundary and  $x = S(t)$  is the mush-solid boundary. Constant temperature is prescribed at  $x = S_2(t)$ . For the mushy region the model given in Eqs (12.2.357), (12.2.358) has been taken. Temperatures in the solid and liquid have been obtained in the integral form using Green's function. Similarity solution has been assumed for  $S(t)$  and  $S_1(t)$ . The three constants of proportionality in similarity solutions of  $S_1(t)$ ,  $S_2(t)$  and  $S(t)$  are unknowns and the constants  $\varepsilon$  and  $\gamma$  in Eqs (12.2.357), (12.2.358) together count five unknowns. The prescribed temperature at  $x = S_2(t)$  and  $S(t)$ , and the conditions in Eqs (12.2.357)–(12.2.359) provide five conditions to determine five unknowns.

## 12.5 ANALYTICAL-NUMERICAL SOLUTIONS OF INVERSE STEFAN PROBLEM

### 12.5.1 Introduction

Inverse Stefan problems which are essentially control problems have been studied from many points of view. Some of the aspects related to the study of inverse problems were discussed in Chapter 9. Few important points which emerged out of the earlier discussion can be summarized as: (i) the direct Stefan problem formulation requires the 'input' data exactly or in other words prescribes the 'cause' clearly and the problem is to find out the 'output' data or the effect; (ii) in the inverse problem, invariably, the effect is known but what causes it is not known and as the causes could be many inverse problems are inherently ill-posed. In Chapter 9 the main effort was to regularize the ill-posed problem and several regularization techniques were discussed. To give some practical touch to regularization techniques some numerical

procedures were discussed briefly. Numerical procedures were discussed only theoretically without numerical results and it was done to emphasize that the regularization techniques can be applied effectively to get a reasonably accurate solution.

In what follows in this section is the discussion about quasianalytical solutions of some inverse problems with or without regularization.

### 12.5.2 Solutions Obtained Without Regularization

Variety of inverse problems exist depending on the type of control and the problem formulation and several analytical-numerical methods exist in the literature. As in direct Stefan problems, exact solutions of inverse Stefan problems are also not possible except in some special type of formulations. Two exact analytical solutions were presented in Eqs (12.3.120)–(12.3.124). In the problems considered in [478], the phase-change boundary is known and the boundary conditions are to be prescribed so that they would give this known moving boundary as a part of solution of the formulation. Many other this type of problems will be discussed now. Another type of problems which have been discussed in an earlier chapter are related to determination of unknown parameters when an additional or over specified boundary condition is prescribed [239, 241, 243, 403]. The problem formulation is considered in such a form that an exact similarity solution is obtained. No regularization is required. The existence and uniqueness of the solution of problems have been discussed with the help of the similarity solution obtained in several problems with overspecified conditions.

Regularization of inverse problems is not necessitated if suitable smoothness conditions and restrictions are satisfied by prescribing suitable data. Continuous dependence of the solution of dependent variables on the imposed data can be established and stability estimates for dependent variables are satisfied. These conditions have been briefly described in Chapter 9. Often these conditions are ignored as it is difficult to satisfy them and analytical-numerical solutions are obtained and compared with some known exact solutions, if available. This does provide some justification for the validity of the solution obtained.

The problems discussed in Chapter 9 in which regularization was discussed and numerical procedures were outlined will not be discussed here for further additional details about them will be taken care.

### 12.5.3 Solutions Using Adomian Decomposition Method and Homotopy Analysis Method

The inverse problem considered in [532] is similar to the problem formulated in Eqs (12.4.5), (12.4.6) with a difference that the temperature  $V(t)$  prescribed at  $x = \alpha$  is unknown and  $x = S(t)$  is known. Here we are concerned with the procedure and not whether it is a solidification problem or a melting problem. The problem is to determine  $V(t)$  When  $S(t)$  is prescribed together with initial temperature and boundary conditions at  $x = S(t)$ . As far as application of Adomian method is concerned, in the present case also we start from Eq. (12.4.1) and repeat the steps till Eq. (12.4.10). As  $S(t)$  is known and  $V(t)$  is not known, Eq. (12.4.11) will now change to

$$V(t) = \sum_{i=1}^{\infty} \delta_i \psi_i(t) \approx \sum_{i=1}^m \delta_i \psi_i(t), \quad \delta_i \text{ are unknowns.} \quad (12.5.1)$$

$T(x, t)$  is approximated by a finite series as in Eq. (12.4.12).

To find unknown  $\delta_i$ , the following functional is minimized as the solution should satisfy initial and isotherm conditions.

$$J(\delta_1, \delta_2, \dots, \delta_m) = \int_0^{t^*} \{T_n(S(t), t) - T_m\}^2 dt + \int_{\alpha}^{S(0)} \{T_n(x, 0) - \varphi(x)\}^2 dx. \quad (12.5.2)$$

The remaining procedure for numerical solution is similar to the one discussed for the direct Stefan problem. To obtain dimensionless form of  $E_T$  and  $E_S$  given in Eq. (12.4.14), divide  $E_T$  by  $(1/D)^{1/2}$  where  $D = \iint dx dt$  and  $E_S$  by  $(1/t^*)^{1/2}$ .

The numerical solution has been compared with two exact analytical solutions. These exact solutions have been constructed by first fixing  $S(t)$  and values of parameters and other conditions and by some trials appropriate boundary temperature and temperature  $T(x, t)$  can be obtained. We give below one of the exact solutions used for comparison. Take  $T_m = 0$ ,  $\alpha = 0$  and all other parameters equal to unity and  $\varphi(x)$  and  $S(t)$  as in Eq. (12.5.3).

$$\varphi(x) = \exp\left(1 - (1+x)/\sqrt{2}\right) - 1; S(t) = \left(t + 2 - \sqrt{2}\right)/\sqrt{2}. \quad (12.5.3)$$

$$T(x, t) = \exp\left(1 - (1+x)/\sqrt{2} + t/2\right) - 1; V(t) = \exp\left(1 - 1/\sqrt{2} + t/2\right) - 1. \quad (12.5.4)$$

$\psi_i(t) = e^{-it}$ ,  $i = 1, 2, \dots$ , have been taken as the set of linearly independent functions.

If in a problem similar to the one described previously it is assumed that  $S(t)$  is known and unknown flux  $q(t)$  is prescribed at  $x = \alpha$  then in this case  $g_0$  given in Eq. (12.4.10) is to be modified.  $g_0 = \bar{g}$  and  $q(t)$  can be taken in the following form. Note that  $g_0$  is not arbitrarily chosen.

$$g_0 = \frac{1}{K} q(t) (S(t) - x) + T_m; q(t) = \sum_{i=1}^m p_i \psi_i(t), \quad p_i \geq 0, i = 1, 2, \dots, m. \quad (12.5.5)$$

Accordingly minimization functional will change and it will now be

$$J(p_1, p_2, \dots, p_m) = \int_{\alpha}^{S(0)} (T_n(x, 0) - \varphi(x))^2 dx + \int_0^{t^*} \left( K \frac{\partial T_n}{\partial x}(S(t), t) \pm l \rho \frac{dS}{dt} \right)^2 dt. \quad (12.5.6)$$

For  $\psi_i(t)$  any set of linearly independent functions can be taken such as  $\psi_i = t^{i-1}$ ,  $i \geq 1$ . Plus and minus signs occurring in Eq. (12.5.6) means that any one of the solidification or melting problems can be considered. Accordingly sign of  $q(t)$  will change.

As discussed earlier, inverse problems are ill-posed and so the existence and uniqueness of the solution should be addressed. Some existence and uniqueness conditions concerning the above problems have been discussed in [533] and Chapter 9.

### 12.5.4 Application of VIM for Inverse Stefan Problems

VIM described in Section 12.4.2 can be used for solutions of inverse problems also which is done in [534]. Consider the one-phase problem described in Eqs (12.4.5), (12.4.6) with  $\alpha = 0$ . The correction functional for the heat equation can be written as

$$T_n(x, t) = T_{n-1}(x, t) + \int_0^x \lambda(p) \left( \frac{\partial^2 T_{n-1}(p, t)}{\partial p^2} - \frac{1}{k} \frac{\partial \bar{T}_{n-1}(p, t)}{\partial t} \right) dp, \quad (12.5.7)$$

$\bar{T}_{n-1}(x, t)$ ,  $n = 1, 2, \dots$ , are the restricted variations and  $\lambda$  is the Lagrange multiplier.  $\lambda$  can be obtained by considering the stationary conditions for the functional. Following the procedure discussed earlier such as in obtaining Eq. (12.4.47),  $\lambda(p)$  is obtained as  $\lambda = p - x$ . The initial approximation  $T_0(x, t)$  is required to obtain successive iterations in Eq. (12.5.7). We take  $T_0(x) = A + Bx$ , where  $A$  and  $B$  are unknown parameters which could be functions of time. The inverse problem considered in [534] is to determine the boundary temperature  $V(t)$  at  $x = 0$  which is compatible with the known phase-change boundary  $x = S(t)$ , and to determine  $T(x, t)$  which is the solution of the problem. The constants  $A$  and  $B$  are determined by satisfying the condition  $T_0(0) = V(t)$  and satisfying the Stefan condition which gives  $B = -(l\rho/K) \frac{dS}{dt}$ . Using

$$T_0(x, t) = V(t) - (l\rho/K) \frac{dS}{dt} x, \quad (12.5.8)$$

and  $\lambda(\rho) = p - x$  in Eq. (12.5.7), successive approximations of  $T(x, t)$  can be obtained provided  $V(t)$  is known from Eq. (12.5.9).  $V(t)$  can be expressed in an infinite series which can be used for computations and is taken as.

$$V(t) = \sum_{i=1}^m p_i \psi_i(t), \quad (12.5.9)$$

$\psi_i$ ,  $i = 1, 2, \dots, m$ , are a set of linearly independent functions and  $p_i$ ,  $i = 1, 2, \dots, m$ , are constants which are unknowns and determined by minimizing the functional following least squares approach.

$$J(P_1, P_2, \dots, P_m) = \int_0^{t^*} \{T_n(S(t), t) - T_m\}^2 dt + \int_0^{S(0)} \{T_n(x, 0) - \varphi(x)\}^2 dx. \quad (12.5.10)$$

Note that the isotherm condition and initial temperature have not been satisfied by  $T_n(x, t)$  and they must be satisfied in any approximate solution and so they are appearing in Eq. (12.5.10). This dictates the functional form in Eq. (12.5.10). The obtained approximate analytical solution has to be computed numerically as  $p_1, p_2, \dots, p_m$  can be obtained only numerically and so is the case in evaluating the integrals in Eq. (12.5.10). The numerical solution has been compared with an exact analytical solution.

If in a problem similar to the one considered earlier, heat flux is prescribed at  $x = 0$  such as  $-K \frac{\partial T}{\partial n}(0, t) = q(t)$ , in which  $q(t)$  is unknown and  $x = S(t)$  is known then also VIM can be applied. In this case  $q(t)$  is expressed in a series of the form given in Eq. (12.5.9).  $T_0(x, t)$  has the following form.

$$T_0(x, t) = T_m + \frac{1}{K} q(t)(S(t) - x). \quad (12.5.11)$$

The minimization functional is similar to Eq. (12.5.6).  $\alpha$  could be zero or nonzero.

Homotopy analysis method (HAM) is based on homotopy which is a basic concept in topology. Some authors call this method as homotopy perturbation method (HPM). This is because the dependent variable is expressed as a power series in terms of an embedding parameter which is so in perturbation methods. This power series whose coefficients are functions is substituted in the equations governing the formulation of the problem and on comparing the different integral powers of this parameter on both sides of equations, solution is obtained by solving a set of equations. This process is similar to the one employed in perturbation techniques. However, in other perturbation techniques, the parameter chosen is small or large whereas there is no restriction in HPM on the embedding parameter except that it belongs to  $[0, 1]$ . The use of homotopy in numerical techniques such as homotopy continuation method was done much earlier (cf. [535]) but its use in the form of a perturbation technique was perhaps done earliest in the reference [536].

The application of HPM will now be discussed in the context of nonlinear operator equations and inverse problems. Let  $A(u) = L(u) + N(u)$  be a nonlinear operator and  $A(u) = L(u) + N(u) = f(z)$ ,  $z \in \Omega$  be an operator equation in which  $L$  is a linear operator and  $N$  is nonlinear operator,  $f$  is a known function and the problem is to find  $u$  which satisfies  $A(u) = f$ . At present we do not want to discuss the topological spaces to which  $u$  and  $f$  belong. HPM will be further discussed in Chapter 9 with a brief discussion concerning analysis. In this section HPM is being discussed with regard to its application in inverse Stefan problems.

We define a homotopy map  $H(w, p)$  as

$$H(w, p) = (1 - p) (L(w) - L(w_0)) + p (L(w) + N(w) - f(z)), \quad p \in [0, 1]. \quad (12.5.12)$$

$w_0$  is the initial approximation of the operator equation  $A(w) = f(z)$ ,  $(w, p) \in \Omega \times [0, 1]$ . For  $p = 0$ ,  $H(w, 0) = L(w) - L(w_0)$  and  $H(w, 0) = 0$  implies  $w_0$  is a solution of the trivial problem  $L(w) - L(w_0) = 0$ . For  $p = 1$ ,  $H(w, 1) = L(w) + N(w) - f(z)$  and  $H(w, 1) = 0$  implies  $w$  is the solution of  $A(w) = f(z)$ . It will be assumed without proof that as  $p$  varies from 0 to 1, the solution of the operator equation  $H(w, p) = 0$  varies continuously from initial approximation  $u = u_0$  to the exact solution of  $A(u) - f(z) = 0$ . We are trying to determine the exact solution of the operator equation  $H(w, p) = 0$  or  $w = f(u, p)$ . On expanding  $f(u, p)$  in a Taylor series in powers of  $p$  and denoting by  $u_m$  the  $m$ th derivative of  $f(u, p)$  at  $p = 0$ , we have

$$w = \sum_{i=0}^{\infty} p^i u_i \text{ and } u = \sum_{i=0}^{\infty} u_i, \quad \text{for } p = 1. \quad (12.5.13)$$

Some remarks about the convergence of the above series can be found in [536].

Using HPM, quasianalytical solution of one-phase one-dimensional Stefan problem considered in Eqs (12.4.5), (12.4.6) has been obtained in [537]. The temperature at  $x = 0$  is prescribed but is considered unknown. Similarly if Neumann boundary condition is prescribed it could be considered unknown.  $x = S(t)$  is known together with all other parameters involved in the formulation. We are seeking solution of  $H(w, p) = 0$  in which  $w = \sum_{i=0}^{\infty} p^i T_i$ ,  $L(T_i) = \frac{\partial^2 T_i}{\partial x^2} - \frac{1}{k} \frac{\partial T_i}{\partial t}$ ,  $N(w) = 0$ . Using Eqs (12.5.12), (12.5.13) in  $H(w, p) = 0$  it is easy to obtain

$$\sum_{i=0}^{\infty} p^i \frac{\partial^2 T_i}{\partial x^2} = \frac{\partial^2 T_0}{\partial x^2} - p \frac{\partial^2 T_0}{\partial x^2} + \frac{1}{k} \sum_{i=1}^{\infty} p^i \frac{\partial T_{i-1}}{\partial t}. \quad (12.5.14)$$

On comparing different integral powers of parameter  $p$  on both sides of Eq. (12.5.14), a system of partial differential equations is generated as given below.

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{k} \frac{\partial T_0}{\partial t} - \frac{\partial^2 T_0}{\partial x^2}; \quad \frac{\partial^2 T_i}{\partial x^2} = \frac{1}{k} \frac{\partial T_{i-1}}{\partial t}, \quad \text{for } i \geq 2. \quad (12.5.15)$$

When boundary conditions at  $x = S(t)$  are satisfied, we get

$$T_0(S(t), t) + T_1(S(t), t) = T_m; \quad -K \left( \frac{\partial T_0}{\partial x}(S(t), t) + \frac{\partial T_1}{\partial x}(S(t), t) \right) = \rho l \frac{dS}{dt}. \quad (12.5.16)$$

$$T_i(S(t), t) = 0, i \geq 2; \quad -K \frac{\partial T_i}{\partial x}(S(t), t) = 0, i \geq 2. \quad (12.5.17)$$

If the initial approximation  $T_0(x, t)$  is known then  $T_1(x, t)$  can be determined by solving the first equation in Eq. (12.5.15) subjected to the boundary conditions given in Eq. (12.5.16) as  $S(t)$  is known.  $T_i(x, t)$  for  $i \geq 2$  can be determined by solving the set of equations in Eq. (12.5.15) subjected to boundary conditions given in Eq. (12.5.17). As initial condition has not been used till now, we take  $T_0(x, 0) = \phi(x)$  and determine  $T_1(x, t)$  and subsequently all other  $T_i(x, t)$ ,  $i \geq 2$ . As an approximation of an infinite convergent series, the summation can be taken as some finite sum. If  $T_i(x, t)$ ,  $i \geq 1$  can be determined analytically well and good otherwise they have to be obtained numerically and the boundary temperature at  $x = 0$  or flux at  $x = 0$  can be calculated accordingly.

Two analytical solutions have been constructed in [537] by assuming suitable expressions of  $S(t)$ , parameter values and initial and boundary temperature. Analytical-numerical solutions have been compared with exact analytical solutions so constructed.

A two-phase one-dimensional inverse Stefan problem in the finite domain  $0 \leq x \leq d$  has been considered in [538]. The moving boundary  $x = S(t)$  is known together with other parameters occurring in the formulation,  $S(0)$  lies between 0 and  $d$ . Initial temperatures are prescribed in the two phases differently. Boundary temperature at  $x = 0$  is prescribed and the only unknown function is the boundary temperature or flux at  $x = d$ . As the method of solution is independent of solidification or melting problem any one of them can be considered and appropriate signs for fluxes can be assigned together with that of latent heat. Equations of the form as in Eq. (12.5.15) have to be obtained for both the phases using homotopy maps as in Eq. (12.5.12). All the boundary, initial and interface conditions are to be satisfied by temperatures. Some simplifications such as in Eq. (12.5.17) have been made in [538]. The equations in Eq. (12.5.16) and the equations in Eq. (12.5.17) will accordingly change together with Stefan condition for the two-phase problem. An exact analytical solution has been constructed for testing purpose and analytical-numerical solution has been compared with it.

HPM has been applied to a heat conduction problem without phase change in [539]. It is a good application of homotopy method. The authors call this method as HAM. The heat equation is nonlinear and the region is a finite spherical radially symmetric region. The boundary condition at the fixed boundary is of convective-radiative type. For further discussion of HAM or HPM refer [540].



### 12.5.5 Solutions Using Integral Heat-Balance Method

A separate Section 12.9 is devoted to the discussion of method and solutions by integral heat balance method. Some solutions are being considered here as they are more relevant in the context of inverse problems.

A one-dimensional inverse Stefan problem with one- and two-phase formulations has been considered in [541]. The phase-change boundary is known and the inverse problem is to determine the temperature or its gradient at the fixed boundary of the region. The region under consideration could be planar or spherical. Results are given for inward solidification but the results for outward solidification/melting problems can also be obtained. It is well known that both planar and spherical one-dimensional heat equations can be written as a single equation. If the notation  $R$  is used for coordinates in both coordinate systems then the one-dimensional heat equations can be written as

$$\frac{1}{R^i} \frac{\partial^2}{\partial R^2} (R^i T) = \frac{1}{k} \frac{\partial T}{\partial t}, i = 0 \text{ for plane and } i = 1 \text{ for sphere.} \quad (12.5.18)$$

Further if  $\theta = R^i T$ ,  $i = 0, 1$ , then we have the following formulation for a one-phase problem considered in [541].

$$\left. \begin{aligned} \frac{\partial^2 \theta}{\partial R^2} &= \frac{\partial \theta}{\partial t}, S(t) < R \leq 1, t > 0; \frac{\partial \theta}{\partial R} = \frac{S^i}{St_e} \frac{dS}{dt} \text{ on } R = S(t), \\ \theta &= 0, \text{ on } R = S(t), S(t) \text{ is known.} \end{aligned} \right\} \quad (12.5.19)$$

$St_e$  is Stefan number,  $i = 0, 1$ . Without going into the details of scalings, we assume that Eq. (12.5.19) is dimensionless and  $R = S(t)$  is the phase-change boundary,  $S(0) = 1$ . Let the problem be a solidification problem so we write  $\theta_S$  in place of  $\theta$ . If the heat equation in Eq. (12.5.19) is integrated from  $R = S(t)$  to 1 then we get

$$\left. \frac{\partial \theta_S}{\partial R} \right|_{R=S} - \left. \frac{\partial \theta_S}{\partial R} \right|_{R=1} = \frac{d\Phi_S}{dt}, \quad \Phi_S = \int_1^{S(t)} \theta_S dR. \quad (12.5.20)$$

The previous equation is called heat balance integral equation [542] and overall heat balance in the entire region is satisfied by this equation. Solution obtained by using Eq. (12.5.20) will always be an approximate solution as this equation is not accurate pointwise. To solve Eq. (12.5.20),  $\theta_S$  is expressed as

$$\theta_S = a + b \left( \frac{R-S}{1-S} \right) + c_1 \left( \frac{R-S}{1-S} \right)^2 + d \left( \frac{R-S}{1-S} \right)^3, \quad (12.5.21)$$

the coefficients  $a$ ,  $b$ ,  $c_1$  and  $d$  are unknowns and are in general functions of  $S(t)$ . Four conditions are required to determine the four constants. Two conditions are available at  $R = S(t)$  and two more are developed further. If isotherm condition  $\theta = 0$  is differentiated with respect to time and then the Stefan condition and heat equations are used, we get two more conditions given below.

$$\frac{\partial \theta_S}{\partial R} \frac{dS}{dt} + \frac{\partial \theta_S}{\partial t} = 0; \quad \frac{\partial^2 \theta_S}{\partial R^2} = -\frac{S^i}{St_e} \left( \frac{dS}{dt} \right)^2, \quad i = 0, 1. \quad (12.5.22)$$

Finally after determining the unknowns  $a, b, c_1, d$ , and on using Eq. (12.5.20) and  $\phi_S$  as given in Eq. (12.5.20), we get

$$\frac{d\phi_S}{dS} = \frac{12\phi_S}{(1-S)^2} \left( \frac{dS}{dt} \right)^{-1} + \frac{6}{St_e} S^i - \frac{(1-S) S^i}{St_e} \frac{dS}{dt}, \quad i = 0, 1. \quad (12.5.23)$$

On using Eq. (12.5.23)  $\phi_S$  can be obtained and then  $\theta_S(1, t)$  and  $\partial\theta_S(1, t)/\partial R$  can be obtained in simple analytical forms for  $i = 0, 1$ . For the expressions of  $a, b, c_1$  and  $d$ , and  $\theta_S(1, t)$ ,  $\partial\theta_S(1, t)/\partial R$  refer [541].

Numerical solutions for the planar cases have been compared with three known solutions, two of them are exact solutions and are obtained when: (1)  $S(t) = 1 - \beta t$ ,  $\beta$  is a constant; (2)  $S(t) = 1 - (\beta t)^{1/2}$ , this is Neumann solution; (3)  $S(t) = 1 - (\beta t)^{1/3}$  for which no exact solution is available but numerical results obtained by using a different method are available for comparison. For the spherical case when  $S(t) = 1 - \beta t$ , simple analytical solutions for temperature and its derivative at  $R=1$  can be obtained. For the two exact analytical solutions in planar case and one for spherical case refer [478, 541]. It may be noted that the solutions obtained by heat balance method are only approximate solutions even if the unknown quantities are obtained in closed form.

For a two-phase inverse solidification problem in a finite region  $0 \leq R \leq 1$ , thermal-layer approximation has been used in [542]. The formulation of the two-phase problem is similar to the formulation of Neumann problem and so it is not presented here. Let  $R$  stand for both Cartesian and radial coordinates. The method of approximate solution does not depend on scalings therefore let us assume that the equations given below are scaled. Suppose at any time  $t$ , the boundary conditions prescribed at  $R = 1$  have effect on the liquid temperature up to a depth  $\lambda(t)$ . Therefore in the region  $0 \leq R \leq \lambda(t)$ , the initial temperature of the liquid which is a prescribed function in a solidification problem, remains unaffected up to the time  $t$  for which  $\lambda(t)$  holds. The region to be considered for solution is  $\lambda(t) \leq S(t) \leq R \leq 1$  in which  $R = S(t)$  is the phase-change boundary,  $S(t) < R \leq 1$  is the solid phase and  $\lambda(t) < R < S(t)$  is the liquid phase. Integrate the heat equation for the solid phase from  $R = 1$  to  $R = S(t)$  and heat equation for the liquid phase from  $R = S(t)$  to  $R = \lambda(t)$ . Using isotherm conditions  $\theta_S = \theta_L = 0$  at  $R = S(t)$ , we get

$$\left. \frac{\partial\theta_L}{\partial R} \right|_{R=\lambda(t)} - \left. \frac{\partial\theta_L}{\partial R} \right|_{R=S(t)} = \frac{k_S}{k_L} \left[ \frac{d\varphi_L}{dt} - \theta_L(\lambda, t) \frac{d\lambda}{dt} \right], \quad (12.5.24)$$

$$\left. \frac{\partial\theta_S}{\partial R} \right|_{R=S(t)} - \left. \frac{\partial\theta_S}{\partial R} \right|_{R=1} = \frac{d\varphi_S}{dt}; \quad \varphi_L = \int_{S(t)}^{\lambda(t)} \theta_L dR, \quad \varphi_S = \int_1^{S(t)} \theta_S dR. \quad (12.5.25)$$

To determine  $\theta_S$  a cubic polynomial of the form in Eq. (12.5.21) is considered but the constants which are functions of time are different in this case. For  $\theta_L$ , the polynomial considered is as follows

$$\theta_L = A_L + B_L \left( \frac{R - \lambda}{S - \lambda} \right) + C_L \left( \frac{R - \lambda}{S - \lambda} \right)^2 + D_L \left( \frac{R - \lambda}{S - \lambda} \right)^3. \quad (12.5.26)$$

The initial temperature of liquid is constant which is higher than the isotherm temperature and in dimensionless form initial temperature is taken as unity. To obtain the unknowns, we use the conditions given below.

$$\left. \frac{\partial \theta_L}{\partial R} = i, \theta_L = (\lambda(t))^i, \frac{\partial^2 \theta_L}{\partial R^2} = 0 \right\} \text{ on } R = \lambda(t), i = 0, 1; \theta_L = 0 \text{ on } R = S(t), \quad (12.5.27)$$

$i = 0$  for Cartesian and  $i = 1$  for spherical geometry. Finally after determining the constants in Eq. (12.5.26), we get  $\theta_L$  in Cartesian (the superscript 0 is for Cartesian) and spherical geometry ( $i = 1$ ) as given below.

$$\theta_L = R^i - S^i \left( \frac{R - \lambda}{S - \lambda} \right)^3, \quad i = 0, 1. \quad (12.5.28)$$

For the constants  $A_L$ ,  $B_L$ , etc., see [542].

For solid region, we have the isotherm condition, the Stefan condition and an additional condition, which can be derived by differentiating isotherm condition and also using some other conditions.

$$\frac{\partial^2 \theta_S}{\partial R^2} - \frac{K_L}{K_S} \frac{k_L}{k_S} \frac{\partial^2 \theta_L}{\partial R^2} = - \frac{S^i(t)}{St_e} \left( \frac{dS^i}{dt} \right)^2, \text{ on } R = S(t), i = 0, 1. \quad (12.5.29)$$

To determine the coefficients occurring in the cubic expression of  $\theta_S$ , the definition of  $\phi_S$  given in Eq. (12.5.25) and expression of  $\theta_S$  (similar to given in Eq. 12.5.21) is used. When  $\theta_S$  is substituted in Eq. (12.5.25),  $\partial \phi_S / \partial S$  is obtained in an explicit form in terms of  $S(t)$  and  $\lambda(t)$  which are still unknown. Once  $\partial \phi_S / \partial S$  is obtained by using the definition of  $\phi_S$ , then  $\theta_S(1, t)$  and  $(\partial \theta_S / \partial R)_{R=1}$  can be calculated and also  $T_S(1, t)$  and  $(\partial T_S / \partial R)|_{R=1}$  which are the required quantities. If heat balance equation (12.5.24) for liquid phase is used then

$$z(t) \frac{dz}{dS} - \left( 4 - i \frac{z}{S} \right) z - 12 \frac{k_L}{k_S} \left( \frac{dS}{dt} \right)^{-1} = 0, z = S - \lambda; i = 0, 1. \quad (12.5.30)$$

If  $S(0) = 1$ , then  $\lambda(0)$  can be obtained from a heat conduction problem without phase change and  $z(0)$  is known.  $z(t)$  is obtained in terms of  $S(t)$ . In general, exact analytical solution of Eq. (12.5.30) is not possible. However, in some special cases such as when  $i = 0$  (Cartesian geometry), and  $S(t) = 1 - \beta t$  or  $S(t) = 1 - (\beta t)^{1/2}$ ,  $S(t)$  can be obtained in terms of  $\lambda(t)$ . Expressions of  $T_S(1, t)$  and gradient of temperature at  $R = 1$  can be obtained as functions of  $S(t)$  in such cases. Considerable numerical work has been done for both one- and two-phase problems and compared with the earlier results obtained by other authors.

In an one-dimensional three-phase inverse Stefan problem considered in [543] in the planar region  $a \leq x \leq b$ ,  $S_1(t)$  is the phase-change boundary separating solid and mush and  $S_2(t)$  is the phase-change boundary separating mush and liquid. In this solidification problem, classical formulations for all the three regions have been considered. Mushy region is sandwiched between solid and liquid regions. The initial temperature of the liquid, which at  $t = 0$  occupies the region  $a \leq x \leq b$ , is higher than the phase-change temperature  $T_2$  at  $x = S_2(t)$ . Heat is extracted at  $x = b$  and a convective-type boundary condition is prescribed at  $x = b$  in which heat transfer coefficient  $\alpha(t)$  is unknown. The boundary  $x = a$  is insulated. Both  $S_1(t)$  and  $S_2(t)$  are known.

When at  $t = t^*$ , the temperature at  $x = b$  becomes  $T_2$  then  $S_2(t)$  starts growing. When at  $t = t^{**}$ , the temperature at  $x = b$  reaches  $T_1$ , which is the constant temperature at  $x = S_1(t)$ , then  $S_1(t)$  starts growing. For  $0 \leq t < t^*$ , we have purely a heat conduction problem in the

region  $a \leq x \leq b$ . At  $x = a$ , the boundary condition is known. In the direct Stefan problem the boundary condition at  $x = b$  should be prescribed but in the present inverse problem, the boundary condition at  $x = b$  is unknown. The time  $t = t^*$  at which the temperature at  $x = b$  becomes  $T_2$  is known as  $S_2(t^*) = b$  and  $S_2(t)$  is known. We have to determine the unknown temperature for  $a \leq x \leq b$  at  $t = t^*$  as it is required to obtain solution of another problem for  $t > t^*$ . In a direct Stefan problem the temperature and time  $t^*$  when  $x = b$  attains temperature  $T_2$  can be calculated. In the inverse problem since  $S_2(t)$  is known,  $t^*$  can be calculated but not temperature in  $a \leq x \leq b$  because boundary condition at  $x = b$  is not known. Between  $t^* < t < t^{**}$ , no solid region exists and for  $t > t^{**}$ , we have a classical three-region formulation. The formulation reported in [543] will not be given here to save space but on the basis of the formulations considered earlier for three-phase regions it can be written as it is of similar type. Some equations which are essential for understanding the formulation are being reported in following paragraphs.

Since  $x = S_1(t)$  is known  $t = t^{**}$  can be calculated. Whether it is the direct problem or an inverse problem; at  $t = t^{**}$ , we require temperatures in the liquid and mush. In the inverse problem as  $x = S_1(t)$  is known,  $t = t^{**}$  becomes known but  $\alpha(t)$ , the heat transfer coefficient is not known (convective boundary condition is prescribed) so temperature will involve an unknown  $\alpha(t)$ . Although no analytical solution of the three-phase problem has been obtained in [543], the analytical procedure for obtaining the numerical solution is new as far as this chapter is concerned. Therefore, it is briefly presented in following paragraphs.

*In the liquid.* For  $t > 0$ , solve the linear transient heat equation in the region  $a \leq x \leq S_2(t)$  with prescribed initial temperature  $\varphi_0(x) > T_2$ , and boundary conditions given below.

$$-K_L \frac{\partial T_L}{\partial x}(a, t) = 0; T_L(S_2(t), t) = T_2; t \geq 0, S_2(t^*) = b. \quad (12.5.31)$$

*In the mushy region.* For  $t > t^*$ , solve the linear transient classical heat equation in the region  $(S_2(t), \min(b, S_1(t)))$  with the conditions

$$\left. \begin{aligned} T_M(x, t^*) &= T_L(x, t^*); T_M(S_2(t), t) = T_2; \beta_1 = K_L/l_L\rho_M, \beta_2 = K_M/l_L\rho_M \\ T_M(S_1(t), t) &= T_1 \text{ if } S_1(t) = \min(b_1, S_1(t)); \frac{\partial S_2(t)}{\partial t} = \left(-\beta_1 \frac{\partial T_L}{\partial x} + \beta_2 \frac{\partial T_M}{\partial x}\right) \Big|_{S_2(t)} \\ -K_M \frac{\partial T_M}{\partial x}(b, t) &= \alpha(t)(T_M(b, t) - T_\infty) \text{ if } b < \min(b, S_1(t)). \end{aligned} \right\} \quad (12.5.32)$$

*In the solid phase.* For  $t > t^{**}$ , solve linear transient heat equation in the region  $(S_1(t), b)$ . The other conditions to be satisfied are given below.

$$\left. \begin{aligned} T_S(x, t^{**}) &= T_M(x, t^{**}); T_S(S_1(t), t) = T_M(S_1(t), t), \\ -K_S \frac{\partial T_S}{\partial x}(b, t) &= \alpha(t)(T_S(b, t) - T_\infty). \end{aligned} \right\} \quad (12.5.33)$$

$$l_S \rho_S \frac{\partial S_1(t)}{\partial t} = -K_M \frac{\partial T_M}{\partial x}(S_1(t), t) + K_S \frac{\partial T}{\partial x}(S_1(t), t). \quad (12.5.34)$$

In Eqs (12.5.30)–(12.5.33), the subscripts  $L$ ,  $M$  and  $S$  stand for liquid, mush and solid regions, respectively. Eq. (12.5.31) is to be first solved by backward Euler's method. For solving Eq. (12.5.32), an approximate formula for  $\alpha(t)$  has been derived in [543] which is

updated with time and used in Eq. (12.5.33) also. Limited numerical work has been presented. Numerical scheme could have been presented more explicitly with details.

### 12.5.6 Use of Tikhonov Regularization and Some Other Approaches

The work presented in [544] is concerning experimental, analytical and numerical studies related to PCM undergoing solid-liquid transformations. When an inert matrix during phase change is filled with PCM such as polymer, shape is stabilized. When the phase change is taking place over a temperature range  $T_2 \leq T \leq T_1$ , enthalpy-temperature relationship is required. The work which will be presented now is about the inverse problem of obtaining enthalpy-temperature relationship when the data or thermal response  $T(r, t)$ ,  $0 \leq r \leq R$ , of PCM at  $r = 0$  is recorded by a thermocouple in an experiment. As discussed earlier, the relation between enthalpy  $H(r, t)$  and temperature  $T(r, t)$  in which  $r$  is the radial coordinate in the cylinder is given by

$$H = [\rho C_S(1 - f) + \rho C_L f] T + \rho l f, \quad f(r, t) \text{ is liquid fraction.} \quad (12.5.35)$$

The densities of solid and liquid are taken the same,  $l$  is the latent heat,  $f = 1$ , if  $T > T_m$  and  $f = 0$  if  $T < T_m$ , and  $T_m$  is the melting temperature. The energy conservation equations are given below.

$$\left. \begin{aligned} \frac{\partial H(r, t)}{\partial t} &= K \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T(r, t)}{\partial r} \right), \quad 0 < r < R, \quad t > 0, \\ \frac{\partial T}{\partial t} &= k_S \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \psi(r, t; T), \end{aligned} \right\} \quad (12.5.36)$$

$$\psi(r, t; T) = -(C_L/C_S - 1) \frac{\partial [f(r, t; T) T(r, t)]}{\partial t} - \frac{l}{C_S} \frac{\partial f(r, t; T)}{\partial t}. \quad (12.5.37)$$

It is easy to obtain the following equation

$$\frac{\partial H(r, t)}{\partial t} = \rho C_S \left[ \frac{\partial T(r, t)}{\partial t} - \psi(r, t; T) \right], \quad 0 < r < R, \quad t > 0. \quad (12.5.38)$$

The boundary and initial conditions are

$$\left. \frac{\partial T(r, t)}{\partial r} \right|_{r=0} = 0; \quad -K \left. \frac{\partial T(r, t)}{\partial r} \right|_{r=R} = h(T(R, t) - T_\infty(t)); \quad T(r, 0) = T_0 \text{ for } \forall r. \quad (12.5.39)$$

Thermal conductivity  $K$  is the same for both phases, heat transfer coefficients  $h$  and  $T_0$  are known and  $T_\infty$  is the temperature at the interface of steel and PCM. All other parameters  $\rho C_S$ ,  $\rho C_L$  and  $\rho l$  are known once the relation between  $H$  and  $T$  is known for example  $\frac{\partial H}{\partial T} = \rho C_S$  if  $f = 0$ . On integrating Eq. (12.5.38), we get

$$H(r, t) = H(r, 0) + \rho C_S \left[ T(r, t) - T(r, 0) - \int_{\tau=0}^t \psi(r, \tau; T) d\tau \right]. \quad (12.5.40)$$

Now the problem is to obtain  $\psi(r, t; T)$  with the help of observed data  $T(0, t) = y_m(t)$  which is recorded by taking finite number of observations and also temperature  $T(r, t)$  which

is the solution of second equation in Eq. (12.5.36). For this inverse problem Tikhonov regularization technique [271] was used in [544] which is also described briefly in Chapter 9. Following Tikhonov regularization method, the following functional is to be minimized.

$$J = \frac{1}{2} \int_{t=0}^t [y_m(\tau) - T(0, \tau)]^2 d\tau + \frac{\mu}{2} \int_{t=0}^t \int_{r=0}^R 2\pi \bar{r} w(\bar{r}) \psi^2(\bar{r}, \tau) d\bar{r} d\tau, \quad (12.5.41)$$

$\mu > 0$  is the Tikhonov regularization parameter and  $w(r)$  is the weighting function.

*Finite volume method* has been used in the numerical solution of heat equation for temperature in Eq. (12.5.36) and other equations and this results in matrix equations. The minimization is achieved by using Lagrange theory and minimum is attained when three conditions are satisfied in matrix form. One condition pertains to the solution of a direct problem in which the time derivative of temperature in matrix form satisfies an equation involving  $\psi(t)$ . Second condition is related to an adjoint problem and the third is the stationary condition. All the conditions are in terms of matrix equations. For further information about the numerical procedure refer [544, 545].

For testing the numerical solution, a test problem was considered in which

$$f = \frac{1}{2} \left[ 1 - \tanh \left( \frac{T_m - T}{T_1 - T_2} \right) \right], \quad \bar{H}(\theta) = (H(T) - H(T_m)) / (\rho C_S (T_1 - T_2)), \\ \theta = (T - T_m) / (T_1 - T_2). \quad (12.5.42)$$

$\bar{H}(\theta)$  is dimensionless form of  $H(r, t)$ ,  $T_1 > T_2$ . For scalings of equations, refer [544]. Considerable numerical work has been reported and numerical results are in good agreement with the experiments. For conducting experiment and mathematical modelling of the inverse problem, several assumptions have been made for which refer [544].

Both direct and inverse Stefan problems have been considered in [546]. The formulation of the inverse problem depends on the solution of the direct problem. We first consider a direct problem. If the prescribed temperature in a planar two-dimensional one-phase problem at  $y = 0$  is constant then the temperature will be independent of  $x$  and the solution will be only in  $(y, t)$  coordinates. The initial temperature is taken equal to melting temperature in the region  $0 \leq x < \infty, 0 \leq y < \infty$ .

The prescribed temperature  $T(x, 0, t) = F(x, t) = T_0^*$  is perturbed by sinusoidal perturbation as given below

$$T(x, 0, t) = F_1(x, t) = T_0^* + T_1^*(t) \cos mx, \quad T_1^*(t) \ll T_m - T_0^*. \quad (12.5.43)$$

The resulting two-dimensional one-phase problem is given below.

$$\left. \begin{aligned} \nabla^2 T(x, y, t) &= \frac{1}{k} \frac{\partial T}{\partial t}; \quad T(x, y = S(x, t); t) = T_m, \\ K \frac{\partial T}{\partial y}(x, S(x, t), t) &= l \rho \frac{\partial S}{\partial t}(x, t). \end{aligned} \right\} \quad (12.5.44)$$

The boundary temperature is prescribed as in Eq. (12.5.43). The Stefan condition is only approximate as the term  $(\partial T / \partial x)^2$  has been neglected at the phase-change boundary  $y = S(x, t)$ ,  $S(x, 0) = 0, x \geq 0$ . It is assumed that due to perturbation given in Eq. (12.5.43), corresponding temperature field and  $S(x, t)$  will change and so they can be written as

$$T(x, y, t) = T_0(y, t) + T_1(y, t) \cos mx; \quad S(x, t) = S_0(t) + S_1(t) \cos mx, \quad (12.5.45)$$

$T_0(y, t)$  and  $S_0(t)$  are the solutions of one-phase one-dimensional Stefan problem with  $T_0(0, t) = T_0^*$ . This solution is one-phase Neumann solution which is known and this is a similarity solution. On substituting  $T(x, y, t)$  given in Eq. (12.5.45) in the formulation given in Eq. (12.5.44), it is easy to obtain partial differential equations for  $T_0(y, t)$  and  $T_1(y, t)$ . If it is assumed that  $T_1(y, t) \ll T_0(y, t) - T_0^*$  and  $S_0(t) \gg S_1(t)$  then  $T(x, y, t)$  and  $S(x, t)$  can be expanded in Taylor series around  $T_0(y, t)$  and  $S_0(t)$ , respectively. After substituting them in the interface conditions and boundary conditions, the square and higher powers and product of small quantities can be neglected. Finally on comparing sinusoidal terms on both sides of the equations, we get equations to determine the pairs  $(T_0(y, t), S_0(t))$  and  $(T_1(y, t), S_1(t))$ . The solutions of  $T_0(y, t)$  and  $S_0(t)$  can be identified with the Neumann solution, and  $T_1(y, t)$  and  $S_1(t)$  must satisfy the following equations.

$$\frac{\partial^2 T_1}{\partial y^2} - m^2 T_1 = \frac{1}{k} \frac{\partial T_1}{\partial t}; \quad S_1(t) \frac{\partial T_0}{\partial y}(S_0, t) + T_1(S_0, t) = 0; \quad T_1(0, t) = T_1^*(t). \quad (12.5.46)$$

$$K \left[ \frac{\partial T_1}{\partial y} + S_1(t) \frac{\partial^2 T_0}{\partial y^2} \right] \bigg|_{(y=S_0)} = l\rho \frac{dS_1}{dt}. \quad (12.5.47)$$

The solution of first equation in Eq. (12.5.46) can be attempted in a separable variable form such as  $T_1(y^*, \tau) = f(y^*)g(\tau)$ ,  $\tau = m^2 kt$  and  $y^* = y/\sqrt{4kt}$ .  $S_1(t)$  can be eliminated in the interface conditions to obtain a single first-order partial differential equation. Finally the solution of  $T_1$  is obtained as

$$T_1(y^*, t) = e^{-\tau y^{*2}} \sum_{n=0}^{\infty} \tau^n \left[ A_n \varphi(1/2 + n, 1/2; y^{*2}) + B_n y^* \varphi(1 + n, 3/2; y^{*2}) \right]. \quad (12.5.48)$$

$\phi$  in Eq. (12.5.48) is *confluent hypergeometric function* [389] and the separation constant  $\alpha$  which occurs whilst obtaining separable variable solution has been taken as  $\alpha = 4n$ ,  $n = 0, 1, 2, \dots, A_n$  and  $B_n$  are unknown constants to be determined from the condition  $T_1(0, \tau) = T_1^*(\tau)$  and the interface condition obtained as a single equation after eliminating  $S_1(t)$  between two interface conditions in Eq. (12.5.46).  $S_1(t)$  is given below.

$$S_1(t) = -\frac{\sqrt{\tau} e^{-(\tau + \delta^2)}}{\delta(l/c)} \sum_{n=0}^{\infty} \tau^n \left[ A_n \varphi(1/2 + n, 1/2; \delta^2) + B_n \delta \varphi(1 + n, 3/2; \delta^2) \right]. \quad (12.5.49)$$

$\delta$  in Eq. (12.5.49) occurs when the moving boundary in one-phase Neumann solution is expressed in the form  $S(t) = \delta\sqrt{4kt}$ . Obtaining the coefficients  $A_n$  and  $B_n$  is in general not easy but if  $T_1^*(\tau) = E_1 e^{p\tau}$ ,  $E_1$  and  $p$  are constants, then it is possible to obtain  $A_n$  and  $B_n$  in closed form.

The inverse problem consists of obtaining the perturbation  $T_1^*(\tau)$  if  $S_1(\tau)$  is known, which it is, and is as given by Eq. (12.5.49).  $T_1^*(t)$  can be expanded in powers of  $\tau$  then  $S_1(\tau)$  can be expanded in a series of odd powers of  $\tau^{1/2}$  such as

$$S_1(\tau) = \sum_{n=0}^{\infty} \lambda_n \tau^{n+1/2}. \quad (12.5.50)$$

Temperature field is taken in the form of Eq. (12.5.48). To determine the constants  $A_n$  and  $B_n$ , substitute  $S_1(\tau)$  from Eq. (12.5.50) in Eq. (12.5.49). This provides one equation to determine  $A_n$  and  $B_n$ . To obtain another equation for obtaining  $A_n$  and  $B_n$ , substitute  $T_1(y^*, \tau)$  and  $S_1(\tau)$  from Eq. (12.5.50) in the dimensionless Stefan condition. For further details refer [546].

The inverse problem considered in [547] is different from the inverse problems discussed earlier. This problem is concerned with obtaining the solution of a liquid or a solid phase in a two-phase one-dimensional problem when the solution of solid or a liquid phase is known. The solution of the inverse problem in this method requires that thermo-physical properties of both solid and liquid phases are the same and they are constant. To explain the method, we consider a direct one-phase one-dimensional Stefan problem in the solid region  $S(t) \leq x \leq H, S(0) = 0$ . The equations which will be used to determine  $T_S(x, t)$  are part of the formulation of the two-phase solidification problem. This formulation is not being given here as we are familiar with it. In determining  $T_L(x, t)$  the initial constant temperature is taken  $T_I$ , boundary temperature at  $x = 0$  is  $T_w$  and  $T(H, t) = T_H$ . The isotherm temperature at the phase-change boundary  $x = S(t)$ , i.e.  $T_S(x = S(t), t) = T_m$  and  $K_S(\partial T_S / \partial x)_{S(t)} = q_{FS}$ ,  $q_{FS}$  is just a notation. Stefan condition for two phases is satisfied after determining the liquid temperature in the region  $0 \leq x \leq S(t)$ . It will now be assumed that  $T_S(x, t)$  is known but not completely as  $S(t)$  has been determined only from  $T_S(x = S(t), t) = T_m$ . For example, if  $x = S(t) = 2\delta\sqrt{k_S t}$  then  $\delta$  is still unknown. If  $\delta$  is determined using Stefan condition for one-phase problem then it will not be valid for a two-phase problem.

Next we consider the problem of determining liquid temperature satisfying the linear heat equation in the region  $0 \leq x \leq S(t)$  and satisfying the condition  $T_L(x = S(t), t) = T_m$ . If  $y = S(t) - x$  then  $T_L(0, t) = T_m$  and  $(\partial T_L / \partial t)_{y=0} = 0$ . Let  $q(S(t)^-, t) = q_{FL} = K \frac{\partial T_L}{\partial y}(0^+, t)$ . If it is assumed that  $T_L(y, t)$  can be expressed in a Taylor series in powers of  $y$  around  $y = 0$  which also implies that partial derivatives of  $T_L(y, t)$  of all orders at  $y = 0$  exist then using the heat equation and isotherm conditions, we obtain

$$T_L(y, t) = T_m + \frac{y}{K} q_{FL} + \frac{y^2}{2!kK} q_{FL} \frac{dS(t)}{dt} + \frac{y^3}{3!kK} \left\{ \frac{dq_{FL}}{dt} + \frac{1}{k} q_{FL} \left( \frac{dS(t)}{dt} \right)^2 \right\} + \dots \quad (12.5.51)$$

Let  $d_1 = q_{FL}/K$ ,  $d_2 = q_{FL} \dot{S}(t)/kK$  and  $d_n = \frac{d_{n-2}}{k} + \frac{d_{n-1}}{k} \dot{S}(t)$ , for  $n \geq 3$ . Dot denotes differentiation with respect to time. We write

$$T_L(x, t) = T_m + \sum_{n=1}^{\infty} \frac{d_n (S(t) - x)^n}{n!} = T_m + \sum_{n=1}^{\infty} \frac{(S(t) - x)^n}{n!} \left( \bar{d}_n + \frac{A_n l}{C} \right), \quad 0 \leq x \leq S(t). \quad (12.5.52)$$

In obtaining the second series in Eq. (12.5.52), Stefan condition for the two-phase problem has been used and  $q_{FL}$  is replaced by  $\bar{q}_{FL}$  and  $\bar{q}_{FL} = q_{FS} + \rho l \dot{S}(t)$ . Expressions for  $A_n$  can be easily obtained from Eq. (12.5.51). To obtain  $\bar{d}_n$ , replace  $q_{FL}$  by  $q_{FS}$  in  $d_n$ ,  $n \geq 1$ . Finally we have

$$T_L(x, t) = T_S(x, t) + \frac{l}{C} \sum_{n=1}^{\infty} G_n \left( 1 - \frac{x}{S(t)} \right)^n, \quad 0 \leq x \leq S(t). \quad (12.5.53)$$



It may be noted that if in Eq. (12.5.52),  $l$ , which is latent heat, is taken as zero then  $T_L(x, t)$  can be replaced by  $T_S(x, t)$  on the l.h.s. of Eq. (12.5.52).  $G_n$  in Eq. (12.5.53) is given below.

$$\left. \begin{aligned} G_n &= (S(t))^n A_n/n!; A_1 = \dot{S}(t)/k, A_2 = (\dot{S}(t)/k)^2, \\ A_n &= \dot{A}_{n-2}/k + A_{n-1}\dot{S}(t)/k \text{ for } n \geq 3. \end{aligned} \right\} \quad (12.5.54)$$

To obtain  $S(t)$ , we use  $T_S(S(t)) = T_m$ . The above procedure to obtain the solution of the inverse planar problem can also be extended to cylindrical and spherical geometries. Calculations for  $G_n$  in Eq. (12.5.53) have been reported in [547] when  $S(t) = bt^m$  ( $b$  is constant) in Cartesian geometry and for  $S(t) = \lambda\sqrt{t}$  in cylindrical and spherical geometries. More general heat equations such as

$$k \frac{1}{r^n} \frac{\partial}{\partial r} \left( r^n \frac{\partial T}{\partial r} \right) + \frac{P}{r^n} \frac{\partial T}{\partial r} + Q = \frac{\partial T}{\partial t}, \quad n = 0, 1, 2, \dots, \quad (12.5.55)$$

can also be considered provided  $Q$  can be represented in a power series in  $(S(t) - x)$  and  $P$  is constant. Convergence and error bounds have also been discussed.

## 12.6 QUASIANALYTICAL SOLUTIONS OF HYPERBOLIC STEFAN PROBLEMS

### 12.6.1 Introduction

In the context of non-Fickian or non-Fourier heat flux law, the heat energy equation is formulated as a hyperbolic equation. This hyperbolic equation is commonly known as Telegrapher's equation. Non-Fickian heat flux law in its generality requires knowledge of many areas such as irreversible thermodynamics, stress-strain rates. Using a simple Maxwell-Cattaneo model, Telegrapher's equation was derived in Chapter 8 (refer Eq. 8.1.3). The Fick's law assumes that the heat diffusion flux at a point  $x$  in space at time  $t$  depends on the temperature gradient  $\nabla T$  at the same space-time point. In other words heat flux is a local phenomenon. However, heat transport is inherently a nonlocal phenomenon as heat is transported from one point in space to another and so both distance and time are involved. The effect which comes from distance can be called nonlocal space effect and the effect which occurs due to the time lapse between flux and the temperature gradient generated at  $x$  is the nonlocal time effect. Because of nonlocal effects there is no local thermodynamical equilibrium whereas in Fick's law there is local thermodynamic equilibrium.

These nonlocal effects give rise to relaxation time  $\tau$ . A particle which at time  $t$  was at the point  $x - d$  or  $x + d$ ,  $d$  is some characteristic length, comes to the point  $x$  at time  $t + \tau$ . This implies that there is a time lag between the diffusion flux and gradient of temperature at  $x$ . The relationship between heat flux and temperature gradient is the starting point of building different models or in other words deriving energy equations for non-Fickian laws. Some of the models which include (1) extended irreversible thermodynamics, (2) fluctuation theory, (3) two concentration model, (4) discrete model, (5) diffusion-stress coupling model and application to rapid solidification have been discussed in [548] and references given therein. The discussion provides useful information.

Many mathematical formulations or models for deriving energy equations which include nonlocal effects are based on the response of heat flux to the temperature gradient. In a simple

one-dimensional case, heat flux  $q(x, t)$  is proportional to temperature difference and its gradient which can be expressed as

$$q(t + \tau, x) \propto T\{(x - d, t) - T(x + d, t)\}. \quad (12.6.1)$$

This temperature difference arises because a particle comes to the point  $x$  at time  $t + \tau$  from a point  $x - d$  or  $x + d$ , where it was at the time  $t$ .  $d$  is the mean-free path. The constant of proportionality depends on the mean velocity  $v$  of the particles.

If both  $\tau$  and  $d$  are small then expanding in Taylor series both sides of Eq. (12.6.1), we get

$$\sum_{n=0}^{\infty} \tau^n \frac{\partial^n q}{\partial t^n} \propto \sum_{m=0}^{\infty} \frac{d^{2m+1}}{(2m+1)!} \frac{\partial^{2m+1} T(x, t)}{\partial x^{2m+1}}. \quad (12.6.2)$$

By the principle of conservation of energy, we have  $\rho CT_t + q_x = 0$ . Without going into the elementary details of obtaining the constant of proportionality in Eq. (12.6.2), if only zeroth-order terms in Eq. (12.6.2) are considered then we get  $q = -KT_x$  in which  $K$  is thermal conductivity. Combining the energy equation and flux equation, we get a linear parabolic equation. If first two terms on l.h.s. are retained in Eq. (12.6.2) and only zeroth-order term is retained on r.h.s., then combining this equation with energy equation, we get the hyperbolic equation given in Eq. (8.1.3). Higher-order terms can also be retained provided it is justified. Amongst many other criteria it should also depend on of what order  $\tau$  and  $l$  are with respect to each other and with respect to other parameters of the physical system under consideration. However, there are models such as free energy models and models with diffusion-stress coupling in which the approach is different.

In Chapter 8, our main concern was to formulate hyperbolic Stefan problem for one-phase and two-phase problems in the context of hyperbolic equation (8.1.3). The questions of existence and uniqueness were addressed and the convergence of solutions of hyperbolic Stefan problems as the relaxation parameter  $\tau \rightarrow 0$  was discussed in some particular cases as no general result exists. Temperature could be discontinuous across the phase-change boundary and there could also be delay in the response of energy to the latent and sensible heats in some formulations. The discussion presented in Chapter 8 and here forms together only a small fraction of the material existing in the literature regarding formulation of constitutive equations for heat as well as heat and mass transfer flux vectors. Some references provided in Chapter 8 and here should be referred for further information.

In this section, quasianalytical solution of classical formulations of hyperbolic Stefan problem as given in Sections 8.2.1 and 8.2.2 will be discussed.

## 12.6.2 Solutions With Temperature Continuity Across Phase-Change Boundary

The solution of Eq. (8.1.3) has been obtained in [549] by using Laplace transform method. Consider the following equations in the one-dimensional semiinfinite region  $0 \leq x < \infty$ .

$$\frac{\partial q}{\partial x} + \rho c \frac{\partial T}{\partial t} = 0; \quad \tau \frac{\partial q}{\partial t} + q + K \frac{\partial T}{\partial x} = 0; \quad \tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}. \quad (12.6.3)$$

The first two equations are arising by retaining terms up to first-order terms in Eq. (12.6.2) and this process can be called *first-order relaxation process*. When first two equations in Eq. (12.6.3) are combined then we get the third equation which is hyperbolic heat conduction equation in which  $k$  is diffusivity.

If  $\bar{q}(p, x)$  and  $\bar{T}(p, x)$  are the Laplace transforms of  $q(x, t)$  and  $T(x, t)$  then taking Laplace transform of first two equations in Eq. (12.6.3) (after making them dimensionless (cf. [549])), we get

$$\frac{d\bar{q}}{dx} + p\bar{T} = 0 \text{ and } (d^2p + 1)\bar{q} + \frac{d\bar{T}}{dx} = 0, \quad d^2 = \tau k/L^2, \quad (12.6.4)$$

$L$  is some characteristic length. If  $\bar{q}$  is eliminated from the two equations in Eq. (12.6.4), then we have

$$\left. \begin{aligned} \frac{d^2\bar{T}}{dx^2} - p(d^2p + 1)\bar{T} &= 0 \text{ or } \bar{T}(x, p) = A(p)e^{+bx} + B(p)e^{-bx}, \\ b &= d\sqrt{p(p + 1/d^2)}. \end{aligned} \right\} \quad (12.6.5)$$

The problem considered in [549] is only a hyperbolic heat conduction problem without phase change. In phase-change problems also we require solutions of type of the system of equations given in Eq. (12.6.3). Consider the boundary and initial conditions as given below.

$$T[0, t] = 1, \quad t > 0; \quad T(x, t) = 0, \quad t > 0, x \rightarrow \infty; \text{ or } q(0, t) = 1, \quad t > 0. \quad (12.6.6)$$

$$q(x, t) = 0, \quad t > 0, x \rightarrow \infty; \quad q(x, 0) = T(x, 0) = 0, \quad 0 \leq x \leq \infty. \quad (12.6.7)$$

Either the temperature or the flux is prescribed at  $x = 0$  as given in Eq. (12.6.6). The problem is to obtain temperature at  $x = 0$  if flux is prescribed at  $x = 0$  and to obtain flux at  $x = 0$  when temperature is prescribed at  $x = 0$ . If flux is prescribed as in Eqs (12.6.6), (12.6.7) then it is not difficult to obtain the following solution after satisfying the conditions and obtaining inverse Laplace transform of  $\bar{T}(0, p)$ . It is not easy to obtain inverse of  $\bar{T}(x, p)$  which has the form given in Eq. (12.6.5) and this will be discussed later.

$$\bar{T}(0, p) = b/p^2; \quad T(0, t) = H(t)d \exp(-t/2d^2)[(1+t)I_0(t/2d^2) + tI_1(t/2d^2)], \quad (12.6.8)$$

where  $I_0$  and  $I_1$  are modified Bessel functions of the first kind of zeroth and first orders and  $H(t)$  is unit *Heaviside step function*. As  $t \rightarrow 0+$ ,  $T(0, 0+) = dq(0, 0+)$  as compatibility condition is to be satisfied on the characteristic curves. Similarly if temperature is prescribed then

$$\bar{q}(0, p) = 1/b \text{ and } q(0, t) = H(t)d^{-1} \exp(-t/2d^2)I_0(t/2d^2); \quad q(0, 0) = 1/d. \quad (12.6.9)$$

Closed-form analytical solutions as in Eqs (12.6.8), (12.6.9) are possible only at  $x = 0$ .

Non-Fourier melting of a semiinfinite slab has been considered in [550]. Both solid and liquid phases have been considered in this one-dimensional problem with liquid occupying the region  $0 \leq x \leq S(t)$ ,  $S(0) = 0$  and solid region extends from  $x = S(t)$  to infinity. The initial temperature of the solid is  $T_0 < T_m$ ,  $T_m$  is the melting temperature.  $T_L(0, t) = T_w H(t)$ ,  $H(t)$  is the unit Heaviside step function,  $T_w > T_m$  for  $t > 0$ . It is not difficult to obtain the following interface condition from Eq. (8.2.10) by differentiating it with respect to time provided,  $\rho_S = \rho_L = \rho$  and  $\tau = \tau_L = \tau_S$ .

$$l\rho \left( \tau \frac{d^2 S}{dt^2} + \frac{dS}{dt} \right) = K_S \left. \frac{\partial T_S}{\partial x} \right|_{x=S(t)} - K_L \left. \frac{\partial T_L}{\partial x} \right|_{x=S(t)}. \quad (12.6.10)$$

Other equations in the formulation of this problem can be easily written by referring to [Section 8.2.1](#). In terms of the dimensionless variables, we define the distances in liquid and solid regions as

$$\lambda_{L,S} = x/(2\sqrt{k_{L,S}\tau}) \text{ and } \beta = t/2\tau. \quad (12.6.11)$$

The solution of hyperbolic equation for solid satisfying the initial condition and solution for liquid satisfying the hyperbolic equation and boundary condition can be written as

$$T_S(\lambda_S, \beta) = T_0 + B F_S(\lambda_S, \beta); \quad T_L(\lambda_L, \beta) = T_w - A F_L(\lambda_L, \beta), \quad (12.6.12)$$

$$F_{S,L}(\lambda_{S,L}, \beta) = H(\beta - \lambda_{S,L}) \left( e^{-\lambda_{S,L}} + \lambda_{S,L} \int_{\lambda_{S,L}}^{\beta} \frac{e^{-p} I_1(\sqrt{p^2 - \lambda_{S,L}^2})}{\sqrt{p^2 - \lambda_{S,L}^2}} dp \right). \quad (12.6.13)$$

$H(\cdot)$  is the Heaviside function and  $I_1(\cdot)$  is the modified first kind Bessel function. The solutions given in Eq. (12.6.13) have been obtained in [551]. The constants  $A$  and  $B$  in Eq. (12.6.12) can be determined by satisfying isotherm conditions and  $S(t)$  is determined from Eq. (12.6.10). Analytical solution is not possible and only numerical solution of  $S(t)$  can be obtained which has been carried out in [550] by using an approximate empirical equation based on the assumption that  $\lambda_{S,L} = \beta = \bar{S}(\beta)$  and there exists a time  $\beta = \beta^*$ ,  $0 < \beta < \beta^*$  up to which the non-Fourier thermal wave is supposed to carry solid-liquid interface  $\bar{S}(\beta)$ . Beyond  $\beta^*$ , the wave discontinuity is less than  $T_m$  and therefore wave travels in the solid ahead of the phase-change boundary  $\bar{S}(\beta)$ . Non-Fourier effects in temperature were found significant only for a short time. On the wave front we have,  $x = \sqrt{k/\tau}t$  and hence  $\lambda_{S,L} = \beta$  and  $F_{S,L} = e^{-\beta}$ .

A particular solution of one-phase hyperbolic Stefan problem was constructed in [213] in which the phase-change boundary is taken as  $x = S(t) = Mt$ ,  $M > 0$ ,  $t > 0$ . Temperature solution is given in [213] without any details of procedure to obtain it. It appears that the solution is motivated by the expression of  $F_{S,L} = e^{-\beta}$  and modifying it suitably. Three cases  $M < (k/\tau)^{1/2}$ ,  $M > (k/\tau)^{1/2}$  and  $M = (k/\tau)^{1/2}$  were discussed in [213] for one-phase problem which have been reported in [Section 8.2.1](#) and will not be discussed here. Enthalpy formulation for two-phase problem in non-Fourier case has also been given in [213].

It was shown in [213] that  $M > (k/\tau)^{1/2}$  is not acceptable as in this case the velocity of phase-change front exceeds the velocity of the wave front. This is because condition of positive entropy production is not satisfied. In particular, the condition  $-(\partial T/\partial x)q(x, t) \geq 0$  is violated which should be satisfied at each point  $(x, t)$  in the interval  $0 \leq x \leq S(t)$ .  $M = (k/\tau)^{1/2}$  is not suitable as interface condition is not satisfied.

In [552] the assumption of continuity of temperature has been examined. Consider the melting of a semiinfinite one-dimensional solid which is melting due to prescribed heat input at  $x = 0$ . Consider the non-Fourier heat flux law  $q(x, t) = -KT_x(x, t - \tau)$ . This means that temperature gradient has built up at  $x = S(t)$  at time  $t - \tau$  which implies that  $T(S(t)^-, t) > T_m$ .  $T_m$  is the melting temperature. But heat has not reached  $x = S(t)$  as it takes some time  $\tau$  to reach  $x = S(t)$  from neighbourhood.  $T(S(t)^-, t) > T_m$  is possible if in some neighbourhood of  $x = S(t)^+$  solid is superheated which is not possible as the solid region was considered below

the temperature  $T_m$ . To rectify this contradiction in the formulation, it was proposed in [552] that temperature has a jump discontinuity across  $x = S(t)$ .

The new proposed formulation is for one-phase melting and includes equations in Eq. (12.6.3) together with the following equations.

$$\rho[l + C(T[t] - T_m)] \frac{dS}{dt} = q[t]; \quad \frac{\tau}{k} q[t] \frac{dS}{dt} = T[t] - T_m; \quad t > 0, S(0) = 0, \quad (12.6.14)$$

$$T(0, t) = T_1 > T_m \text{ or } q(0, t) = q_1 \text{ or } q(0, t) = h(f(t) - T(0, t)); \quad T(x, 0) = T_m. \quad (12.6.15)$$

$T[t]$  and  $q[t]$  stand for jump in  $T(x, t)$  and  $q(x, t)$  across  $x = S(t)$ . Any one of the boundary conditions in Eq. (12.6.15) can be prescribed at  $x = 0$ . To test the correctness of the above mathematical formulation in the sense that it is thermodynamically consistent and no superheating in the solid arises, a simple exact analytical solution similar to the one obtained in [213] was developed which is given below.

Assume that  $x = S(t)$  is given by  $x = mt, m > 0$  and  $T[t] = T_L^0$  at  $x = S(t)$ . It can be checked that the following expressions of  $T(x, t)$  and  $q(x, t)$  given below satisfy equations in Eq. (12.6.3).

$$\left. \begin{aligned} T(x, t) &= A + (T_L^0 - A) \exp(g(t)); \quad q(x, t) = B + D \exp(g(t)), \\ g(t) &= m(x - mt)/(\tau m^2 - k). \end{aligned} \right\} \quad (12.6.16)$$

The constants  $A, B, D$  and  $T_L^0$  can be determined using the boundary conditions and initial condition. The solution so obtained is thermodynamically consistent in the sense that  $q(x, t)T_x(x, t) > 0$  in the interval  $0 \leq x \leq S(t)$ . If  $m < \sqrt{k/\tau}$  or in other words if the speed of phase-change front is less than the wave speed then  $T_1 > T_m$  and  $q_1 > 0$ . As  $\tau \rightarrow 0$ ,  $T_L^0 = T_m + \tau m^2 l / (C(k - \tau m^2)) \rightarrow T_m$  and we get the solution of a corresponding parabolic Stefan problem.

A quasisteady-state analytical solution has also been obtained in [552] without making any assumption about the form of  $S(t)$ . Temperature and  $q(x, t)$  are taken as

$$T = T_1 + x(T_L(t) - T_1)/S(t); \quad q = -k(T_L[t] - T_1) S(t); \quad T = T_1 \text{ at } x = 0. \quad (12.6.17)$$

Equations in Eq. (12.6.3) for steady-state heat transfer are satisfied.  $T_L[t]$  and  $S(t)$  can be obtained on satisfying interface conditions.  $T_L[t]$  has an explicit solution but  $S(t)$  can be obtained only numerically. From the obtained exact analytical solution several observations about the qualitative behaviour of solution can be made. One such observation is that ‘hypercooling’ phenomenon must coincide with the interface curve being space like.  $|dS(t)/dt| > \sqrt{k/\tau}$  iff  $T_L[t] < T_m - l/C$ .

In [213] by taking a particularly simple form of phase-change boundary, some observations were made about the solution. In the formulations temperature is continuous across phase-change boundary. However, no existence and uniqueness theorem was proved. In [216] by considering the same one-phase problem as in [213] with temperature continuity at the interface, it has been proved that there exists a time interval in which the classical solution exists and is unique. For the conditions imposed on the class of functions involved in the formulation and proof refer [216] and Section 8.2.2. No analytical and numerical solutions have been presented in [216].

In [217] the author does not impose continuity of temperature at the phase-change interface but insists that both the equations in Eq. (12.6.3) should be interpreted as conservation equations and accordingly both temperature and flux should follow conservation laws which must hold across  $x = S(t)$ . This in turn implies that the following Rankine-Hugoniot relations should hold both for flux and temperature across discontinuities or shocks such as given below. In dimensionless form if  $e$  is internal energy, we should have

$$\frac{dS}{dt}(e_- - e_+) = q_- - q_+; \quad \frac{dS}{dt}(q_- - q_+) = T_- - T_+; \quad (12.6.18)$$

the subscripts  $-$  and  $+$  indicate the limits of  $e$  and  $q$  as  $x \rightarrow S(t)^-$  and  $x \rightarrow S(t)^+$ , respectively. The formulation for one-phase one-dimensional problem is presented and it has been shown that the formulation is consistent thermodynamically. Existence issues have been discussed. Asymptotic solutions for small time and large times have been obtained and a special numerical method has been suggested to obtain an approximate solution. For further details see [217] and Section 8.3.1.

In the framework of rapid solidification, high crystal growth velocities can be achieved with the help of high-speed pulsed lasers. It seems appropriate to pose rapid solidification problem within the framework of hyperbolic heat conduction model. The rapid solidification of materials with low entropy transformation between liquid and solid states, in most metals, occurs via the growth of dendrites. As discussed in earlier references [365–370], the tip of the dendrite crystal can be modelled as a parabolic plate for two-dimensional solidification or a paraboloid of revolution in three dimensions.

In [553] quasisteady-state rapid solidification of a melt at constant melting temperature has been obtained numerically by considering heat conduction equation as a hyperbolic equation. The emphasis is on supercooling which is necessary in the nucleation of crystal growth and therefore temperature is to be determined. As far as velocity of the advancement of the tip of the dendrite is concerned, with each solidification in nature a unique solidification velocity is associated with a given undercooling which results from the condition  $VR = \text{constant}$  in which  $V$  is the growth velocity of an isolated dendrite and  $R$  is the radius of curvature at the dendrite tip. If the Cartesian coordinate system is shifted to the tip of the dendrite which is moving with velocity  $V$  in the  $z$ -direction then it is not difficult to obtain a transformed quasistatic hyperbolic equation as given below.

$$\nabla^2 T + \frac{V}{k} \frac{\partial T}{\partial z} - \frac{\tau V^2}{k} \frac{\partial^2 T}{\partial z^2} = 0. \quad (12.6.19)$$

$z = 0$  is now tip of the dendrite. Note that  $\tau$  enters in the quasisteady-state heat equation also. If  $(\xi, \eta)$  are comoving parabolic coordinates in two dimensions and  $R$  is radius of curvature of the dendrite tip then

$$\xi^2 = \left( \sqrt{x^2 + z^2} + z \right) / R, \quad \eta^2 = \left( \sqrt{x^2 + z^2} - z \right) / R. \quad (12.6.20)$$

In a paraboloid of rotational symmetry in three dimensions, we have

$$\xi^2 = \left( \sqrt{r^2 + z^2} + z \right) / R, \quad \eta^2 = \left( \sqrt{r^2 + z^2} - z \right) / R, \quad (12.6.21)$$

$(r, \varphi, z)$  are cylindrical coordinates,  $\xi = \text{constant}$  are concentric paraboli and  $\eta$  is a running coordinate along the parabola.

The hyperbolic heat equation in Eq. (12.6.19) after transformation in  $\xi$  and  $\eta$  coordinates considered in two- and three-dimensional regions mentioned earlier is given below in the form of equations  $J_{2,3}(\xi, \eta) = 0$ , respectively,

$$J_{(2,3)}(\xi, \eta) = \frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} + 2P \left( \xi \frac{\partial T}{\partial \xi} - \eta \frac{\partial T}{\partial \eta} \right) + \lambda \left( \frac{\partial T}{\partial \xi} + \frac{\partial T}{\partial \eta} \right) - \frac{\tau V^2}{k} \frac{1}{(\xi^2 + \eta^2)} \left[ \xi^2 \frac{\partial^2 T}{\partial \xi^2} - 2\xi \eta \frac{\partial^2 T}{\partial \xi \partial \eta} + \eta^2 \frac{\partial^2 T}{\partial \eta^2} + \xi \left( \frac{3\eta^2 - \xi^2}{\xi^2 + \eta^2} \right) \frac{\partial T}{\partial \xi} + \eta \left( \frac{3\xi^2 - \eta^2}{\xi^2 + \eta^2} \right) \frac{\partial T}{\partial \eta} \right], \quad (12.6.22)$$

The subscripts 2 and 3 hold for two- and three-dimensional problems, respectively.  $T(\xi, \eta)$  is dimensionless temperature,  $\lambda = 0, 1$  in two- and three-dimensional problems, respectively.  $P$  is Peclet number and  $P = VR/2k$ ,  $\bar{T} = C(T - T_\infty)/l$ , bar over  $T$  has been dropped in Eq. (12.6.22), and  $T_\infty$  is faraway temperature. In two dimensions when  $\tau \rightarrow 0$ , the equation  $J_{2,3}(\xi, \eta) = 0$  is separable in  $\xi$  and  $\eta$  coordinates (parabolic case). In this case the solutions of equations in  $\xi$  coordinate are obtained as

$$\bar{T}_{(2,3)}(\xi) = \left( \sqrt{\pi} P \exp(P) \operatorname{erfc}(\sqrt{P}\xi), -P \exp(P) Ei(-P\xi^2) \right). \quad (12.6.23)$$

Here  $\bar{T}(\xi, \eta) = \bar{T}(\xi)\bar{T}(\eta)$ . The subscripts 2 and 3 in Eq. (12.6.23) hold for two and three dimensions, respectively;  $Ei$  is exponential integral defined earlier (cf. Eq. 12.2.27). Stefan condition if flux is time independent is taken as in Eq. (12.2.128).

Numerical solution of Eq. (12.6.19) has been obtained in [553] using finite element method. To start the numerical solution for the kinetic condition, which enters through  $R$ , empirical values have been taken first which are updated by iteration along with the temperature in the numerical scheme. The total supercooling is taken as the sum of three types of supercooling, and general expressions of radius of curvature  $R$  and Gibbs-Thomson parameter have been considered empirically for initial calculations. For further details of calculating supercooling, refer [553]. It is not clear whether Stefan condition has been used in [553] or not as there are other ways of calculating the growth velocity and updating it with temperature derivative. It was found in [553] that stable dendrite growth is only possible if the growth velocity is less than half the thermal wave velocity. The effect of the diffusivity anisotropy on dendrite growth introduced by the hyperbolic conductivity model has also been analysed numerically. If  $\bar{k}$  is new diffusivity then  $\bar{k} = k(1 - V^2/w^2)$ ,  $w$  is wave velocity.

The discussion presented in [554] is a shorter version of the work presented in [553] for the one-dimensional planar problem in semiinfinite medium. The solution for temperature  $T(x, t)$  satisfying hyperbolic equation (12.6.3) and boundary conditions prescribed for  $T(x, t)$  as in Eq. (12.6.6) has not been given in [550]. However, for the inverse problem, solution of  $T(0, t)$  is given in Eq. (12.6.8). The solution of  $T(x, t)$  for the problem considered in [550] has been reported in [554] which is given below. Let

$$\left. \begin{aligned} T(x, 0) &= 0; \quad \frac{\partial T}{\partial t}(x, 0) = 0 \text{ or } q(x, 0) = 0, \\ q(0, t) &= -K \frac{\partial T}{\partial x}(0, t) - \tau \frac{\partial q}{\partial x}(0, t) = F_0. \end{aligned} \right\} \quad (12.6.24)$$

$T(x, t)$  and  $q(x, t)$  both are zero for  $t < 0$ . The thermal conductivity  $K$  is subjected to a step change in the surface flux  $F_0$ . Then

$$T(x, t) = F_0 \sqrt{k\tau} / K H(t - x/A) \exp(-t/2\tau) \left\{ I_0 \sqrt{\frac{t^2}{4\tau^2} - \frac{x^2}{4k\tau}} + \frac{1}{\tau} \int_0^t \exp(-u/2\tau) I_0 \left( \sqrt{\frac{u^2}{4\tau^2} - \frac{x^2}{4k\tau}} \right) du \right\}, A^2 = k/\tau. \quad (12.6.25)$$

The above solution is for heat conduction problem without phase change.  $H(t)$  is the Heaviside unit step function;  $H(t) = 1$  for  $t \geq 0$  and  $H(t) = 0$  for  $t < 0$ .

A one-dimensional one-phase planar solidification problem in the region  $0 \leq x < \infty$  has been studied in [555] using energy-integral method also known as integral heat balance method which was proposed in [542]. The formulation consists of hyperbolic equation as in Eq. (12.6.3) with temperature prescribed on phase-change boundary  $x = S(t)$  as constant melting temperature. Energy balance condition at  $x = S(t)$  and boundary condition at  $x = 0$  are as given below (see Eqs 8.2.16–8.2.18).

$$T_x(S(t), t)[\gamma^2 - \dot{S}^2(t)] = -A[\ddot{S}(t) + 1/\tau \dot{S}(t)], \quad t > 0; T(0, t) = T_0(t), \quad t > 0, \quad (12.6.26)$$

$\gamma^2 = k/\tau$  and  $A = l/C$ ,  $k$  is diffusivity,  $C$  is specific heat and  $l$  is latent heat. Dot above  $S(t)$  denotes differentiation with respect to time.

If hyperbolic heat equation is integrated with respect to  $x$  from 0 to  $S(t)$ , we get

$$\left( \tau \frac{d^2}{dt^2} + \frac{d}{dt} \right) [\theta(t) + A S(t)] = -k T_x(0, t); \quad \theta(t) = \int_0^{S(t)} T(x, t) dx. \quad (12.6.27)$$

For convenience, the isotherm temperature has been taken zero in Eq. (12.6.27). An approximate solution of Eq. (12.6.27) is considered as

$$T(x, t) = a(t)(x - S(t) + b(t)(x - S(t))^2, \quad a(t) \text{ and } b(t) \text{ are unknowns.} \quad (12.6.28)$$

To determine  $a(t)$  and  $b(t)$ , the boundary condition at  $x = 0$  can be used and another condition to be satisfied is Eq. (12.6.27) which is not convenient in the present form. Therefore another condition is derived. If the isotherm condition is differentiated with respect to  $t$ , we get  $T_t(S(t), t)$  which if further differentiated with respect to  $x$  then we get  $T_{tx}(S(t), t)$ . Second differentiation of isotherm condition with respect to time yields

$$T_{tt}(S(t), t) = T_{xx}(S(t), t)\dot{S}^2(t) - T_x(S(t), t)\ddot{S}(t). \quad (12.6.29)$$

If in hyperbolic equation (12.6.3), first equation in Eq. (12.6.26), Eq. (12.6.29) and  $T_{tx}(S(t), t)$  are all used then we get

$$A T_{xx}(S(t), t) = [T_x(S(t), t)]^2. \quad (12.6.30)$$

Boundary condition at  $x = 0$  and Eq. (12.6.30) constitute a complete set of independent relations for the determination of the two unknowns  $a(t)$  and  $b(t)$ .  $S(t)$  is obtained from Eq. (12.6.27) in which  $T(x, t)$  is used which is obtained from Eq. (12.6.28). Finally



$$\left. \begin{aligned} T(x, t) &= T_m - A[\lambda(x - S) + \lambda^2(x - S)^2/2], \\ \lambda &= 1 - (\sqrt{1 + \mu})/S, \mu = 2(T_m - T_0(t))/A. \end{aligned} \right\} \quad (12.6.31)$$

$$\left( \tau \frac{d^2}{dt^2} + \frac{d}{dt} \right) (S(t)h(t)) = r(t)/S(t), \quad h(t) = 5 + \mu(t) + \sqrt{1 + \mu(t)},$$

$$r(t) = 6k \left[ 1 + \mu(t) - \sqrt{1 + \mu(t)} \right]. \quad (12.6.32)$$

$S(0) = 0, \dot{S}(0) = b$ . Isotherm temperature in Eq. (12.6.31) is now  $T_m$  and not zero.

In a particular case when  $T(0, t) = T_0(t) = \text{constant}$  and  $\tau \rightarrow 0$ , an exact analytical solution can be obtained which is also reported in [542]. Analytical solution of  $S(t)$  which is obtained using Eq. (12.6.32) is in general not possible. However, using singular perturbation method (discussed in Section 12.9), an approximate solution can be obtained. The relaxation time  $\tau$  is generally small, i.e.  $0 < \tau \ll 1$ . To obtain a uniformly valid solution for  $S(t)$  ( $S(0) = 0$  in Eq. 12.6.32) using singular perturbation method, the following procedure has been adopted in [555].

Introduce two new functions  $v(t)$  and  $f(t)$  as given below and substitute them in Eq. (12.6.32). We get an equation for  $v(t)$  involving  $f(t)$ . Let

$$v(t) = S(t)h(t), \quad f(t) = r(t) h(t), \quad v(0) = 0, v'(0) = b h(0). \quad (12.6.33)$$

Eq. (12.6.32) gives

$$v(t) \left[ \tau \frac{d^2 v}{dt^2} + \frac{dv}{dt} \right] = f(t). \quad (12.6.34)$$

To obtain uniformly valid approximation of  $v(t)$  another function  $\bar{v}(t)$  is defined as

$$\bar{v}(t) = \bar{S}(t, \tau) h(t) = \phi(t) + p(\xi) - CP, \quad \xi = t/\tau. \quad (12.6.35)$$

$\phi(t)$  and  $p(\xi)$  are to be determined and  $CP$  is the common part. To determine  $\phi(t)$  and  $p(\xi)$  the following equations given in Eq. (12.6.36) are to be solved which result when  $\bar{v}(t)$  is substituted in Eq. (12.6.34).

$$\frac{d^2 p}{d\xi^2} + \frac{dp}{d\xi} = 0; \quad p(0) = 0, p'(0) = b \tau h(0); \quad \varphi \frac{d\varphi}{dt} = f(t). \quad (12.6.36)$$

The solutions of equations in Eq. (12.6.36) are given below.

$$p(\xi) = b \tau h(0) \{1 - \exp(-\xi)\}; \quad \varphi = \sqrt{(2F(t) + M)}, \quad \frac{dF}{dt}(t) = f(t). \quad (12.6.37)$$

The constant  $M$  in the solution of  $\varphi$  in Eq. (12.6.37) is determined by the matching requirement  $\varphi(0) = p(\infty)$  which gives  $M = b^2 \tau^2 h^2(0) = (CP)^2$ . Finally

$$\bar{v}(t) = \bar{S}(t, \tau) h(t) = \sqrt{(2F(t) + b^2 \tau^2 h^2(0))} - b \tau h(0) e^{-\xi}. \quad (12.6.38)$$

If  $T_0(t) = \text{constant}$  then  $\bar{v}(t)$  gets considerably simplified. Some particular cases have also been discussed. The flux prescribed boundary condition at  $x = 0$  can also be considered and some results have been obtained in [555] for this case also. However, in the flux prescribed case, a series solution for  $S(t)$  has been obtained. The procedure in this case is different. For further discussion and many other particular cases, refer [555].

### 12.6.3 Theoretical Background of Green's Functions for Parabolic and Hyperbolic Stefan Problems: Solutions of Hyperbolic Stefan Problem Using Green's Functions

In Section 12.2 Green's functions in Cartesian, radially symmetric spherical and cylindrical regions, have been discussed for parabolic heat equations. The approach to obtain them was not rigorous. We relied on the solutions of parabolic heat equations, which can be obtained by Laplace transform or Fourier transform methods or just with the help of a simple source solution. Next step was to look for a singular function with singularity of a typical type. This singular function should satisfy the heat equation everywhere except at one point. Such a singular function was obtained by taking a nonhomogeneous term in the r.h.s. of parabolic equation in the form of a Dirac-delta function. In an unbounded region by using the properties of Dirac-delta function we could claim that Green's function has been obtained having required properties. Rigorous approach was not followed initially to built up the subject gradually. We do not claim that the discussion being presented now for parabolic case is fully rigorous but it is not heuristic.

Consider the operators  $P_2$  and  $P_3$  as follows.

$$P_2 = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}, P_3 = \frac{\partial^2}{\partial x'^2} + \frac{\partial}{\partial t}; P'_2(T) = \frac{\partial^2 T}{\partial x'^2} - \frac{\partial T}{\partial t'}, P'_3(T) = \frac{\partial^2 T}{\partial x'^2} + \frac{\partial T}{\partial t'}. \quad (12.6.39)$$

It can be checked that

$$vP'_2u - uP'_3v = ((vu_{x'} - uv_{x'})_{x'} - (uv)_{t'}); \quad u = u(x', t'), v = v(x', t'). \quad (12.6.40)$$

In Eq. (12.6.40),  $(vu_{x'})_{x'}$  stands for  $vu_{x'x'} + u'_{x'}v'_{x'}$  and  $v_{x'}$  implies differentiation of  $v$  with respect to  $x'$ .  $P_3$  is the *adjoint operator* of  $P_2$  as the r.h.s. of Eq. (12.6.40) can be written as  $E_{x'} + F_{t'}$  for some functions  $E$  and  $F$ .

The problem is to find  $u(x, t)$  which is the solution of  $P_2u = g(x, t)$  when  $u$  or  $u_x$  is prescribed on  $x = \text{constant}$  and  $u(x, t)$  is prescribed on the line  $t = \text{constant}$ . Consider now the region  $D = \{(x', t'): -b < x' < a; b, a > 0, 0 \leq t' \leq R, R > t\}$ , and integrate Eq. (12.6.40) over  $D$ . We get Eq. (12.6.41) when divergence theorem is used.

$$\begin{aligned} \int_D (vP'_2u - uP'_3v) dx' dt' &= \int_0^R (vu_{x'} - uv_{x'})_{x'=a} dt' - \int_0^R (vu_{x'} - uv_{x'})_{x'=-b} dt' \\ &\quad + \int_{-b}^a (uv)_{t'=0} dx' - \int_{-b}^a (uv)_{t'=R} dx'. \end{aligned} \quad (12.6.41)$$

In Eq. (12.6.41) take  $v$  as the solution  $G(x, t; x', t')$  of the following equation.

$$P'_3G = \delta(x' - x)\delta(t' - t). \quad (12.6.42)$$

The last term in the r.h.s. of Eq. (12.6.41) is not required in the initial value problem as  $u$  is not prescribed at  $t = R$  and this condition can be obtained only from the solution. As there is no boundary condition prescribed at  $t' = R$  the contribution of first and second integrals should be made zero for  $t < t' < R$ . We take  $G \equiv 0$  for  $t' > t$  and replace  $R$  in Eq. (12.6.41) by  $t$  in the upper limit of integrals in which the integration is with respect to  $t'$ .

Heat equation could be homogeneous or nonhomogeneous and accordingly we have  $P_2u = 0$  or  $P_2u = g(x, y)$ . When  $G(x, t; x', t')$  which is the solution of Eq. (12.6.42) is substituted in place of  $v$  in l.h.s. of Eq. (12.6.41), we get

$$u(x, t) = \int_D G g dx' dt' - \int_0^t (Gu_{x'} - uG_{x'})_{x'=a} dt' + \int_0^t (Gu_{x'} - uG_{x'})_{x'=-b} dt' - \int_{-b}^a (uG)_{t'=0} dx', \quad G \equiv 0 \text{ for } t' > t. \quad (12.6.43)$$

Different types of boundary conditions could be prescribed separately at  $x = -b$  and  $x = a$  and accordingly different  $G(x, t; x', t')$  are to be obtained in accordance with boundary conditions at  $x = -b$  and  $x = a$ . For example, if  $g(x, t) = 0$  and  $u(-b, t)$  and  $u(a, t)$  are prescribed then in this case for the solution of Eq. (12.6.42),  $G(x, t; x', t')$  should satisfy the conditions

$$G(x, t; -b, t') = 0 \text{ and } G(x, t; a, t') = 0. \quad (12.6.44)$$

It seems appropriate to call a function  $G_1(x, y; x', t')$  satisfying  $P_2G_1 = \delta(x - x')\delta(t - t')$  as a source solution for heat equation in an unbounded region and a source solution as a Green's function when  $G_1(x, y; x', t')$  also satisfies the conditions prescribed on a given curve. We are not concerned at present with obtaining Green's functions satisfying different types of boundary conditions and for this books on Mathematical methods and Partial Differential equations should be referred ([12, 228, 359, 368]).

Our interest at present lies in obtaining source solution or Green's function for the parabolic heat equation in one-dimensional case through rigorous justification as much as possible in which  $|x'| < \infty, 0 < t' < R, R > t$ . As  $u(x, t)$  remains bounded at infinity, the integrals in Eq. (12.6.43) at  $x' = \pm\infty$  should vanish. Therefore, it is necessary that  $G(x, t; x', t') = G'_x = 0$  as  $|x'| \rightarrow \infty$ . In this case, Eq. (12.2.43) when both  $b$  and  $a$  are infinite, reduces to

$$u(x, t) = \int_D G g dx' dt' - \int_{-\infty}^{\infty} (uG)_{t'=0} dx', \quad G = G'_x = 0, |x| \rightarrow \infty, G \equiv 0, t' > t. \quad (12.6.45)$$

To obtain  $G$  (just forget the earlier expression of  $G$ ) we have to find the solution of Eq. (12.6.42) under the conditions  $G \equiv 0, t' > t$  and satisfying the conditions at  $|x'| \rightarrow \infty$  as mentioned above together with some other properties (cf. [228]). This solution can be found by Laplace transform method but in taking Laplace transform of Eq. (12.6.42), we require  $G \equiv 0$  for  $t' < t$  and not for  $t' > t$ . If in place of Eq. (12.6.42), we consider the adjoint Green's function  $G^*$  which is the solution of the equation.

$$P'_2G^* = \delta(x' - x)\delta(t' - t) \text{ and } G^* \equiv 0 \text{ for } t' < t, \quad (12.6.46)$$

then  $G^*$  can be used in place of  $G$  in Eq. (12.6.45) provided

$$G(x, t; x', t') = G^*(x', t'; x, t). \quad (12.6.47)$$

This result in Eq. (12.6.47) has been proved in [12]. Note that the solution of Eq. (12.6.42) is the adjoint Green's function of the solution of Eq. (12.6.46). Similarly the solution of Eq. (12.6.46) is the adjoint Green's function of the solution of Eq. (12.6.42). If Green's function is associated with any differential equation then adjoint Green's function is associated with adjoint differential equation.

Let  $\bar{G}^*(x, t; x', p)$  be the Laplace transform of  $G^*$  so that

$$\bar{G}^*(x, t; x', p) = \int_0^\infty e^{-pt'} G^* dt'; \quad \frac{d^2}{dx'^2} \bar{G}^* - p \bar{G}^* = e^{-pt} \delta(x' - x). \quad (12.6.48)$$

In Eq. (12.6.48),  $G^* = 0$  at  $t' = 0$ . The solution  $\bar{G}^*$  of second equation in Eq. (12.6.48) has to be obtained such that  $G^*$  is continuous at  $x' = x$  which is so by definition of  $G^*$  and should be such that  $G^* \rightarrow 0$  as  $|x'| \rightarrow \infty$ . A suitable expression of  $\bar{G}^*$  is given by

$$\bar{G}^* = A e^{\sqrt{p}(x-x')}, \text{ for } x < x' \text{ and } \bar{G}^* = A e^{-\sqrt{p}(x-x')}, \text{ for } x > x', \quad (12.6.49)$$

where  $A$  is to be determined and  $\sqrt{p}$  is that branch of the square root whose real part is positive. It can be shown that (cf. [12])

$$\left( \frac{d\bar{G}^*}{dx'} \right)_{x'=x+0} - \left( \frac{d\bar{G}^*}{dx'} \right)_{x'=x-0} = e^{-pt} \text{ and } A = -e^{-pt} / (2\sqrt{p}). \quad (12.6.50)$$

The required  $\bar{G}^*$  is then given by

$$\bar{G}^* = -e^{-pt} e^{-\sqrt{p}|x-x'|} / (2\sqrt{p}). \quad (12.6.51)$$

By taking the inverse Laplace transform of  $\bar{G}^*$  and using the *Heaviside shift theorem*,  $G^*$  is obtained (cf. [12]) as

$$G^*(x, t; x', t') = \frac{1}{2\sqrt{\pi(t'-t)}} H(t' - t) \exp \left\{ -\frac{(x-x')^2}{4(t'-t)} \right\}. \quad (12.6.52)$$

The inverse of  $(1/\sqrt{p}) \exp(-\sqrt{p}|x-x'|)$  is  $\exp(-(x-x')^2/4t)$  and the Laplace transform of  $f(y-a)H(y-a)$  is  $e^{-ap}\bar{f}(p)$  where  $H(z) = 1$  if  $z \geq 0$  and  $H(z) = 0$  if  $z < 0$ . The final form of inverse of  $\bar{G}^*$  in Eq. (12.6.51) is obtained as  $G^*$  which is given in Eq. (12.6.52). The solution  $G(x, t; x', t')$  of Eq. (12.6.42) for an unbounded domain  $-\infty < x' < \infty$  is then given by

$$G(x, t; x', t') = G^*(x', t'; x, t). \quad (12.6.53)$$

When  $u(x, 0)$  is prescribed,  $u(x, t)$  can be calculated from Eq. (12.6.45) by putting  $t' = 0$  in  $G$  and also in  $u(x, t')$ .

$$u(x, t) = 1/(2\sqrt{\pi t}) \int_{-\infty}^{\infty} u(x', 0) \exp\{-(x-x')^2/4t\} dx'. \quad (12.6.54)$$

The Green's function appropriate for solving a particular differential equation is that which is associated with the adjoint operator. For the explicit construction of Green's function for

Telegrapher's equation refer [12, 368]. However, it may be mentioned that the procedure for obtaining Green's functions in an unbounded medium for elliptic and hyperbolic equations is on the same lines as discussed earlier. Rigorous justification is required for many of the steps in the earlier discussion presented here.

Some semianalytical solutions are presented below in which integral representation of temperature has been considered for the hyperbolic Stefan problem. Using the Lighthfoot's moving heat source method [506], in the solution of hyperbolic Stefan problem a heat source can also be introduced as done below in the one-dimensional parabolic energy equation and so we have

$$C\rho \frac{\partial T}{\partial t} + \frac{\partial q}{\partial x} = \rho l v \delta(x - S(t)), \quad x = S(t) \text{ is the interface,} \quad (12.6.55)$$

and  $v = \dot{S}(t)$ . The hyperbolic heat equation is now given by

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + T_A v \delta(x - S(t)) + T_A \tau \frac{\partial}{\partial t} v \delta(x - S(t)), \quad (12.6.56)$$

where  $T_A = l/C$  is the adiabatic temperature,  $l$  is latent heat,  $C$  is heat capacity and  $\tau$  is relaxation time. All the parameters are constant.

The formulation of two-phase one-dimensional melting problem considered in the region  $0 \leq x < \infty$  in [214] is as follows:

$$\frac{\partial^2 T_{L,S}}{\partial t^2} + \frac{\partial T_{L,S}}{\partial t} = \frac{\partial^2 T_{L,S}}{\partial x^2} - T_A(v + \dot{v})\delta(x - S(t)), \quad 0 < x < \infty, t > 0, \quad (12.6.57)$$

$$T_{L,S} = \frac{\partial T_{L,S}}{\partial t} = 0, x > 0, t = 0; \quad \frac{\partial T_L}{\partial x} = -[1 + \delta(t)], x = 0, \quad (12.6.58)$$

$$T_L = T_S = T_m > 0 \text{ at } x = S(t); \quad \frac{\partial}{\partial x}(T_S - T_L) = B(v + \dot{v}), \text{ at } x = S(t), \\ T_S = 0, x \rightarrow \infty, t \geq 0, \quad (12.6.59)$$

$B$  is a dimensionless constant and is inversely proportional to the Stefan number. Eqs (12.6.57)–(12.6.59) are dimensionless. For scalings see [214]. As mentioned earlier that in the application of Lighthfoot's method, the thermo-physical properties of both solid and liquid phases have to be considered the same.

The solution of Eq. (12.6.57) has been obtained as

$$T(x, t) = T_0(x, t) + B T_1(x, t), \quad (12.6.60)$$

where  $T_0(x, t)$  is the solution of Eq. (12.6.57) without the source term and  $T_1(x, t)$  is the solution of Eq. (12.6.57) with source term satisfying the energy balance condition at  $x = S(t)$ . The solution  $T_0(x, t)$  has been obtained in [556] and reported in [214] which is as given below.

$$T_0(x, t) = H(t - x) \left\{ \exp(-t/2) (I_0(t^2 - x^2)^{1/2}/2) + \int_0^t \exp(-p/2) (I_0(p - x)^{1/2}/2) H(p - x) dp \right\}. \quad (12.6.61)$$

The procedure to obtain the above solution has not been given but appears to be the application of Laplace transform to obtain solution of hyperbolic equation and whilst obtaining inverse, the Heaviside shift theorem has been used. The solution of  $T_1(x, t)$  is

$$T_1(x, t) = -\left(1/2\right) \int_0^t [(\dot{v}(t') + v(t'))(H(t - t' - |x - S(t')|)F(t, x; t')) + H(t - t' - (x + S(t'))E(t, x; t'))] dt', \quad (12.6.62)$$

$$F(t, x; t') = \bar{z}(t, t')I_0 \left\{ \frac{1}{2}[(t - t')^2 - (x - S(t'))^2]^{1/2} \right\}; \bar{z}(t, t') = \exp(-(t - t')/2),$$

$$E(t, x, t') = \bar{z}(t, t')I_0 \left\{ \frac{1}{2}[(t - t')^2 - (x + S(t'))^2]^{1/2} \right\}. \quad (12.6.63)$$

$T_1(x, t)$  satisfies energy balance condition at  $x = S(t)$ . All the initial and boundary conditions are satisfied in the above solution except the isotherm condition which has been used to obtain  $S(t)$ .

$H(t - t' - |x' - x|)/2$  is the Green's function for the wave equation  $F(w) = w_{xx} - w_{tt} = 0$  and satisfies  $F(w) = \delta(x - x')\delta(t - t')$ . Green's function is discontinuous across half-lines  $x' = x \pm (t - t')$ ,  $t' < t$ . No procedure has been given in [214] to obtain  $T_1(x, t)$  given in Eq. (12.6.62). If prescribed heat flux at  $x = 0$  is very strong then it is reasonable to assume that if  $v_0$  is some reference velocity and  $v$  is made dimensionless by dividing it by  $v_0$  then this dimensionless  $v$  will be very small as compared to the wave velocity  $(k/\tau)^{1/2}$ . So the constant  $B$  in Eq. (12.6.59) is also small as it is directly proportional to  $v_0$ .  $T_1(x, t)$  can be neglected compared to  $T_0(x, t)$  and  $S(t)$  can be evaluated approximately with the help of the relation  $T_0(S(t), t) = T_m$ . For a short-time solution, taking  $t \geq x$ ,  $x$  small and expanding  $S(t)$  in series of powers of  $t$ ,  $S(t)$  can be obtained approximately as  $S_0(t)$ .

$$S(t) \approx S_0(t) = (8 - \sqrt{a(t)})/3, \quad a(t) = 64 - 48 \left( 1 - T_m + \frac{1}{2}t - t^2/16 \right). \quad (12.6.64)$$

To obtain an improved estimate, take  $S(t) = S_0(t) + BS_1(t)$  and determine  $S_1(t)$  by the relation

$$S_1(t) = -T_1(t, S_0(t)) \left/ \left( \frac{\partial T_0}{\partial x}(x, t) \right) \right|_{x=S_0(t)}. \quad (12.6.65)$$

Lightfoot's [506] moving heat source method has been used in [557] for the rapid solidification of a supercooled liquid. The formulation is different from that given in [214]. The hyperbolic equation considered in [557] is given by Eq. (12.6.56). The temperature  $T(x, t)$ ,  $|x| < \infty$ ,  $t_0 \leq t < \infty$  is discontinuous at the phase-change boundary  $x = S(t)$ . The complete formulation consists of Eq. (12.6.56) in which the relaxation time  $\tau$  is denoted by  $\tau_h$  as the symbol  $\tau$  is used in the integrals. Other equations are given below.

$$T(x, t_0) = T_0 < T_m; \quad \frac{\partial T}{\partial t} \Big|_{t=t_0} = 0, -\infty < x < \infty, t \geq t_0; \quad \frac{\partial T}{\partial x} = 0, |x| \rightarrow \infty. \quad (12.6.66)$$

$$S(t) = \int_{t_0}^t S(\tau') d\tau' + S_0, \quad S_0 = S(t_0); [T] = T_+ - T_- = T(S(t) + 0, t) - T(S(t) - 0, t). \quad (12.6.67)$$

$$\ddot{S}(t) = S_v \left\{ 1 - \exp \left[ -\frac{l^*}{R} \left( \frac{1}{T_-} - \frac{1}{T_m} \right) \right] \exp \left[ -\frac{E}{R} \left( \frac{1}{T_-} - \frac{1}{T_+} \right) \right] \right\}. \quad (12.6.68)$$

Eq. (12.6.68) has been derived in [558].  $l^*$  is latent heat of solidification per mole,  $R$  is gas constant,  $T_m$  is equilibrium solidification temperature,  $S_v$  and  $E$  are some thermodynamic parameters. The problem in [557] is concerned with the rapid solidification of a supercooled liquid. It has been assumed that  $\dot{S}(t)$  is of the order of heat wave velocity  $V_h$ .

The solution of hyperbolic equation (12.6.56) without source term for a heat conduction problem without phase change in an unbounded medium has been obtained with the help of Green's function  $G$  which is given below.

$$\left. \begin{aligned} G(x - \xi, t - \tau) &= (V_h/2k)P_1P_2P_4, \quad V_h \text{ is speed of heat wave,} \\ P_1 &= \exp(-(t - \tau)V_h^2/2k), \quad P_2 = H[(t - \tau)V_h - |x - \xi|], \\ P_3 &= (V_h/2k)\sqrt{(t - \tau)^2V_h^2 - (x - \xi)^2}, \quad P_4 = I_0(P_3), \quad V_h = k/\tau_h. \end{aligned} \right\} \quad (12.6.69)$$

The above expression of  $G$  has been obtained in [368].  $I_0$  is the Bessel function of first kind of zeroth order. In deriving Eq. (12.6.69) it has been assumed that  $\dot{S}(t)$  is of the order of speed of heat wave propagation. Using the above Green's function,  $T(x, t)$  can be obtained as given below by following the procedure followed for the parabolic case.

$$\begin{aligned} T(x, t) &= \int_{t_0}^t d\tau \int_{-\infty}^{\infty} \varphi(\xi, \tau) G d\xi + \int_{-\infty}^{\infty} T(\xi, t_0) \left( G - \tau_h \frac{\partial G}{\partial \tau} \right)_{\tau=t_0} d\xi \\ &\quad + \int_{-\infty}^{\infty} \left[ \tau_h G \frac{\partial T}{\partial \tau} \right]_{\tau=t_0} d\xi, \end{aligned} \quad (12.6.70)$$

$$\varphi(\xi, \tau) = (l/C) \left( \dot{S}(\tau) \delta(\xi - S(\tau)) + \tau_h \frac{\partial}{\partial \tau} (\dot{S}(\tau) \delta(\xi - S(\tau))) \right).$$

To do scaling, time is made dimensionless by dividing it by  $\tau_h$ ;  $x, S(t)$ ,  $S_0$  are divided by  $(k\tau_h)^{1/2}$ , velocity is divided by  $V_h$  and  $C(T - T_0)/l = T$ . In integrals in Eq. (12.6.70), integration with respect to  $\xi$  can be evaluated. Using  $\phi(\xi, \tau)$  and conditions at  $x = \pm\infty$ , we get

$$\begin{aligned} T(x, t) &= \frac{1}{4} \int_{t_0}^t d\tau w(\tau) \bar{P}_2 \bar{P}_1 \bar{P}_4 + \frac{1}{4} \int_{t_0}^t d\tau w(\tau) \bar{P}_2 \bar{P}_1 I_1(\bar{P}_3)(t - \tau)/(2\bar{P}_3) \\ &\quad + \frac{1}{2} \int_{t_0}^t d\tau w(\tau) \delta((t - \tau) - |x - S(\tau)|) \bar{P}_1 \bar{P}_4. \\ &\quad + \frac{1}{2} [w(\tau) \bar{P}_2 \bar{P}_1 \bar{P}_4 \bar{z}(t, \tau)]|_{\tau=t_0}^t, \end{aligned} \quad (12.6.71)$$

where  $\bar{P}_1 = P_1|_{V_h^2/k=1}$ ,  $\bar{P}_2 = P_2|_{(\xi=S(\tau), V_h=1)}$ ,  $\bar{P}_3 = P_3|_{(V_h=1, k=1, \xi=S(\tau))}$ ,  $\bar{P}_4 = P_4(\bar{P}_3)$ ,  $w = \dot{S}(t)/V_h$ ,  $w_0 = S_0/V_h$ ,  $I_1$  is the Bessel function of first kind of order one.

The function  $\varphi(\xi, \tau)$  involves the derivative of  $\dot{S}(t)\delta(x - S(t))$ . In the first term in Eq. (12.6.70), the time derivative is to be integrated first and  $T(x, t_0) = T_0$  or 0 in

dimensionless form will be used. The integrals with respect to  $\xi$  are evaluated from  $-\infty$  to  $S(t)$  and  $S(t)$  to  $\infty$  after time integration. Along the heat wave, temperature remains  $T_0$  for all  $t$ .

Temperature discontinuity has been calculated for both steady-state and transient heat transfer. In the steady-state case  $S(t) = wt$ . Shifting  $(x, t)$  coordinates to moving coordinates  $(z, t)$  where  $z = x - wt$ ,  $T(z)$  has been obtained as follows.

For  $w < 1$  and  $z > 0$ :

$$T(z) = \exp(-wz/f(w))/f(w), \quad f(w) = 1 - w^2, \quad (12.6.72)$$

$z > 0$  is the supercooled liquid region and  $z \leq 0$  is the solid crystal phase. In the crystal phase  $T(z) = 1$ ,  $w < 1$ ,  $z \leq 0$ .

For  $w > 1$ ,  $z > 0$ :  $T(z) = 0$  for supercooled liquid phase and

$$T(z) = 1 - \exp(-wz/f(z))/f(z), \quad \text{for } z \leq 0 \text{ (solid crystal phase)}. \quad (12.6.73)$$

The temperature discontinuity at phase-change boundary  $z = 0$  for  $|w| < 1$  is given by

$$[T] = T_+ - T_- = w^2/f(z). \quad (12.6.74)$$

Many interesting conclusions can be derived from the results given in Eqs (12.6.72)–(12.6.74) and for them refer [557].

Transient heat transfer has also been considered to obtain temperature discontinuity. In this case consider a infinitesimally small neighbourhood  $\Delta$  around  $x = S(t)$  such that

$$\left. \begin{aligned} S(t-) = S(t) - \Delta \leq x \leq S(t) + \Delta = S(t+), \\ x = S(t\pm) = \int_{t_0}^t w(\tau') d\tau' + y_0 \pm \Delta, \quad S(0) = y_0. \end{aligned} \right\} \quad (12.6.75)$$

The integrals in Eq. (12.6.71) have been evaluated for  $w < 1$  and  $w > 1$ .  $\Delta$  is an arbitrarily small parameter related to crystal lattice in the solid phase. We skip the derivations and report the temperature discontinuity. For  $w < 1$

$$[T] = T_+ - T_- = w^2/f(w) - (w/f(w))_{t=t_0} \bar{P}_1 \bar{P}_4|_{(\tau=t_0, S(t)=S(t_0))}. \quad (12.6.76)$$

$$[T] = w^2/f(w), \quad \text{for } w > 1. \quad (12.6.77)$$

$\bar{P}_1$  and  $\bar{P}_4$  are as defined in Eq. (12.6.71). Discontinuity at  $x = S(t)$  when  $\Delta = 0$  in Eq. (12.6.75) has also been calculated approximately and in this case, for  $w < 1$

$$T_+ = 1/f(w) + \frac{dw}{dt} / w^3, \quad T_- = 1 + \frac{dw}{dt} / w^3. \quad (12.6.78)$$

$$\text{For } w > 1, [T] = w^2/f(z) \text{ as in Eq. (12.6.74).} \quad (12.6.79)$$

In the constant velocity case,  $\dot{S}(t) = w$ , if the initial supercooling  $\Delta T$  is higher than the adiabatic temperature  $T_Q = l/C$  of solidification, i.e.  $\Delta T = (T_m - T_0)/T_Q > 1$ , then region of steady-state solidification exists. In nonsteady heat transfers, the decreasing-in-time velocity



$\dot{S}(t) \sim t^{-1/2}$  occurs for  $\Delta T < 1$  and a case of decreasing-in-time velocity  $\dot{S}(t) \sim t^{-1/3}$  occurs in nonsteady case when  $\Delta T = 1$ . Solutions of self-consistent regimes in which the heat shock wave is excluded can be obtained by considering kinetic conditions at the interface of the form in Eq. (12.6.68). Both steady-state and transient regimes have been studied for this case and refer [557] for discussion.

Heat transfer has been studied in [559] in a rapidly solidifying supercooled pure melt during final transient phase when the phase-change boundary is approaching the fixed surface. Towards the last stage of rapid solidification of a supercooled melt in a one-dimensional planar problem, the interface approaches the fixed surface or a specimen surface with a constant velocity  $V$ , i.e. if  $x = S(t)$  is the moving interface then  $\dot{S}(t) = V$ . Let the temperatures and their derivatives or fluxes be known in some region near the fixed surface in which the assumption that the ratio  $V/V_h$  of interface velocity and wave velocity  $V_h$  tends to  $\infty$  as  $\tau \rightarrow 0$  holds. Then in such a small region a parabolic problem can be studied and solved. This is so because, not in general, but in many physical situations it has been proved rigorously that as  $\tau \rightarrow 0$  the solution of a hyperbolic Stefan problem tends to the solution of a corresponding parabolic Stefan problem (refer Chapter 8).

From thermodynamical point of view, in general, the velocity  $V$  of the phase-change interface is determined by the undercooling  $\Delta T = T_m - T_I$ , at the interface where  $T_m$  is the equilibrium solidification temperature and  $T_I$  is the actual interface temperature in solidification. Let  $V = f(\Delta T)$ . It has been shown in [40, 560, 561] that at the beginning, the growth velocity  $V$  increases with increasing undercooling  $\Delta T$ , reaching a maximum value and then some critical undercooling  $\Delta T^*$  is attained.  $V$  is practically kept constant in some range  $\Delta T$  of undercooling (cf. [561]). When the undercooling is large enough the amount of latent heat released on the solidification front may not be sufficient to heat the interface to the level of equilibrium temperature  $T_m$ . This discussion suggests that the undercooling at the interface will be different from zero and it will be so during the whole solidification process. When the interface moves near the fixed surface of the region, the undercooling  $\Delta T$  in general will change which also changes the velocity of interface. Let  $V_f$  be this velocity. If  $V_f$  is high due to high undercooling [560] and  $V_f$  depends only slightly on  $\Delta T$  then changes in  $\Delta T$  will not affect  $V_f$  much and it can be assumed that the interface approaches the fixed surface with a constant velocity  $V_f$ .

It is clear that there is lot of change in the interface velocity which is initially small and becomes near constant value in steady-state case. It is of interest to investigate the temperature field from the initial stages to the temperature towards the end of the solidification. The temperature in a rapidly solidifying melt can be considered within the scope of a Stefan problem of parabolic type. However, to obtain the solution of interest, the authors initially consider a corresponding auxiliary hyperbolic Stefan problem and interestingly they found that the boundary conditions at the interface can be satisfied more easily in a hyperbolic Stefan problem.

Although the assumptions under which solution of hyperbolic Stefan problem has been obtained have been discussed above, we repeat them all together as they are important in this problem: (i) The interface approaches the specimen surface with a constant velocity; (ii) the latent heat of solidification depends linearly on the interface temperature; (iii) all the physical quantities involved at the phase-change boundary can be expressed as linear combinations of exponential functions of the interface position, see Eq. (12.6.83). In the reference [562] which was published earlier than [559], terminal solute distribution of the formed solid in

solidification of binary melt was considered but the temperature changes during the interface motion near the fixed surface were not considered.

The temperature  $T(x, t)$  and flux  $q(x, t)$  are made dimensionless as follows:

$$\bar{T}_{L,S} = \rho C(T_{L,S} - T_m)/l_m; \quad \bar{q}_{L,S} = q_{L,S}/l_m V_h, \quad l_m = bT_m, \quad (12.6.80)$$

where  $T_m$  is equilibrium solidification temperature,  $l$  stands for latent heat,  $l = bT_l = l_m - b\Delta T$ ,  $b$  is a constant,  $V_h$  is heat wave velocity,  $\Delta T = T_m - T_l$ ,  $\alpha = V/V_h$ ,  $\tau = k/V_h^2$ ,  $k$  is diffusivity,  $\bar{l} = l/l_m$ . Dimensionless time and  $x$  are  $t/\tau$  and  $x/V_h$ , respectively.

Now considering dimensionless quantities, the formulation of the problem is given below.

$$\frac{\partial^2 \bar{T}_{L,S}}{\partial t^2} + \frac{\partial \bar{T}_{L,S}}{\partial t} = \frac{\partial^2 \bar{T}_{L,S}}{\partial x^2}; \quad \frac{\partial^2 \bar{q}_{L,S}}{\partial t^2} + \frac{\partial \bar{q}_{L,S}}{\partial t} = \frac{\partial^2 \bar{q}_{L,S}}{\partial x^2}; \quad t \leq 0, x \geq 0. \quad (12.6.81)$$

The liquid with subscript  $L$  occupies the region  $x + \alpha t < 0$ .  $x + \alpha t = 0$  is the equation of the phase-change boundary and  $x = 0$  is the fixed boundary. Time  $t$  has been taken negative just for convenience.

$$\left. \begin{aligned} \bar{q}_L|_{t=-x/\alpha} &= q_0(x); \quad \frac{\partial \bar{q}_L}{\partial t} \Big|_{t=-x/\alpha} = q_1(x); \quad q_0(x) \text{ and } q_1(x) \text{ are known,} \\ (\bar{q}_L - \bar{q}_S)|_{x+\alpha t=0} &= -\alpha \bar{l}|_{x+\alpha t=0}; \quad \bar{q}_L(x, t)|_{x=0} = 0, t \leq 0. \end{aligned} \right\} \quad (12.6.82)$$

The solid occupies the region between  $x + \alpha t = 0$  and  $x + t = 0$  in which  $V = V_f$  is considered for the velocity of moving boundary and the solution of the liquid region is sought in the region  $x + \alpha t < 0$ . At  $x + \alpha t = 0$ ,  $\bar{T}_L = \bar{T}_S$ . Any physical quantity  $F$  such as temperature, or flux, at the phase-change boundary  $t = t(x)$  or any other curve on which boundary conditions are prescribed can be expressed as given below.

$$F(x, t)|_{t=t(x)} = A_0^{(F)} + A_1^{(F)} e^{-\gamma_1 x/2} + A_2^{(F)} e^{-\gamma_2 x/2} + \dots, \quad (12.6.83)$$

the coefficients  $A_n^{(F)}$  and the powers of the exponents  $\gamma_n, n = 1, 2, \dots$ , are to be defined from the interface conditions and conditions at the fixed surface. Such expansions were used in [562] also. The superscript  $(F)$  stands for the physical quantity under consideration.

To solve the above hyperbolic Stefan problem, *Riemann's method* or *Riemann's function* was employed which is similar to the method of Green's function. The starting point of both the methods is the identity Eq. (12.6.40), and its integration over a suitable region  $D$  and then using divergence theorem. For a general hyperbolic equation also, Green's function exists which is not singular but is discontinuous. Both Green's function and Riemann's function have been discussed in [12].

If the substitution  $\bar{q} = ue^{-t/2}$  is made for any of the two fluxes in the second equation in Eq. (12.6.81) then it is reduced to a simpler form.

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} + \frac{1}{4}u = 0; \quad u|_{t=g(x)} = q_0 e^{g(x)/2} = \varphi_1(x), \quad (12.6.84)$$

$$\frac{\partial u}{\partial t} \Big|_{t=g(x)} = \left( \frac{1}{2}q_0 + q_1 \right) e^{g(x)/2} = \psi_1(x); \quad \frac{\partial \bar{q}}{\partial t} \Big|_{t=g(x)} = q_1(x), \quad (12.6.85)$$

$t = g(x)$  is some curve  $\Gamma$  on which  $\bar{q}$  and  $\partial\bar{q}/\partial t$  are prescribed.  $\Gamma$  could be the phase-change boundary. If  $\varphi_1(x)$  and  $\psi_1(x)$  are prescribed on  $t = g(x)$  then they are initial conditions for  $u(x, t)$  and  $\partial u/\partial t$ , respectively, and using them we get

$$\left. \frac{\partial u}{\partial x} \right|_{t=g(x)} = \varphi_1'(x) - \psi_1(x)g'(x). \quad (12.6.86)$$

The characteristics of the differential equation for  $u$  given in Eq. (12.6.84) are  $x \pm t = \text{constant}$ . According to Riemann's method if the characteristics go through the point  $M$  and intersect some curve  $\Gamma$  at the points  $P$  and  $Q$  as shown in Fig. 12.6.1, then  $u(x, t)$  at any point  $M$  in the region  $D$  is as given below provided  $\Gamma$  is not a characteristic.

$$u(M) = -\frac{1}{2} \int_{PQ} \left[ v \left( \frac{\partial u}{\partial x_1} dt_1 + \frac{\partial u}{\partial t_1} dx_1 \right) - u \left( \frac{\partial v}{\partial x_1} dt_1 + \frac{\partial v}{\partial t_1} dx_1 \right) \right] + \frac{1}{2} (u_P + u_Q). \quad (12.6.87)$$

All the points of region  $L$  or  $S$  should lie on only one side of  $\Gamma$ .  $v$  is Riemann's function (see Eq. 12.6.88).

The integral in Eq. (12.6.87) is taken along the curve  $\Gamma$  from  $P$  to  $Q$  and  $u_P$  and  $u_Q$  are values of  $u$ , at the points  $P$  and  $Q$ . Riemann's function  $v(x, t; x_1, t_1)$  for Eq. (12.6.84) has the form

$$v(M, M_1) = J_0 \left( \frac{1}{2} \sqrt{(x - x_1)^2 - (t - t_1)^2} \right), \quad M = (x, y), \quad M_1 = (x_1, y_1), \quad (12.6.88)$$

$J_0(x)$  is the Bessel function of order zero. In the Riemann's method if we know the characteristic of the governing hyperbolic equation and the points  $P$  and  $Q$  at which the characteristics intersect the curve  $\Gamma$ , on which data is prescribed, then integration is required only along the arc  $PQ$  on the curve  $\Gamma$ . The initial data is prescribed on  $t = -x/\alpha$  which is the phase-change boundary and it is the curve  $t = g(x)$  in the present problem. In Fig. 12.6.1 the arc  $PQ$  is shown for any general curve  $t = g(x)$ . When  $t = -x/\alpha$  is the curve on which initial

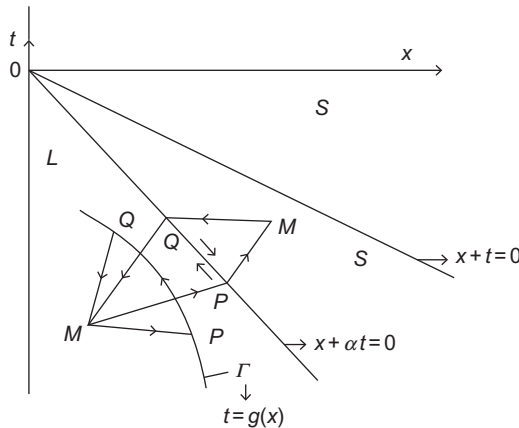


Fig. 12.6.1 Graphical presentation of Eq. (12.6.87).

data is prescribed then the points  $P$  and  $Q$  will be on the line  $x + \alpha t = 0$  as shown in figure. For determining solid temperature, the point  $M$  will be taken in the region between the lines  $x + \alpha t = 0$  and  $x + t = 0$  as our interest is only in this limited solid region.

The presentation of calculation details of obtaining liquid temperature using Eq. (12.6.87) and then evaluation of integrals are not possible here as it involves very lengthy processes. However, some steps through which the authors in [559] attempted solutions are being indicated here briefly. We discuss these steps firstly for  $\bar{q}_L$ .

Using Eqs (12.6.82), (12.6.83) and the integral representation of  $\bar{q}_L(x, t)$  using Riemann's method (the solution not given here) which contain  $\varphi(x)$  and  $\psi(x)$ ,  $\varphi(x)$  and  $\psi(x)$  can be written as

$$\varphi(x) = q_0(x) = A_0 + A_1 e^{-\gamma_1 x/2} + A_2 e^{-\gamma_2 x/2} + \dots, \quad (12.6.89)$$

$$\psi(x) = \frac{1}{2} q_0(x) - \frac{1}{\alpha} q'_0(x) - \frac{1 - \alpha^2}{\alpha^2} q_1(x) = B_0 + B_1 e^{-\gamma_1 x/2} + B_2 e^{-\gamma_2 x/2} + \dots, \quad (12.6.90)$$

where constants  $\gamma_n \geq 0, n = 1, 2, \dots$ , and  $A_n$  and  $B_n, n = 0, 1, 2, \dots$ , are to be determined. After obtaining suitable integral representation of  $u(x, y)$  as in Eq. (12.6.87) with  $P = -\alpha(t + x)/(1 - \alpha)$ ,  $Q = \alpha(x - t)/(1 + \alpha)$ , and substituting  $\varphi(x)$  and  $\psi(x)$  in Eq. (12.6.87) the recurrence relations for unknown coefficients are obtained with the help of the condition at the fixed surface given in Eq. (12.6.82).  $\bar{T}_L$  is also determined in the same way. The unknown coefficient in  $\bar{T}_L$  is defined in such a way that heat equation in the parabolic case is also satisfied. The authors in [559] have provided sufficient details about the procedure in three appendices.

For the flux  $\bar{q}_S$  and temperature  $\bar{T}_S$  also the above procedure is followed. For determining the unknowns  $\gamma_n$  and the coefficients in Eqs (12.6.89), (12.6.90) which for solid region will be different from liquid, the continuity of temperature at the interface and Stefan condition are used. Note that for the complete determination of the temperature field in the liquid, the interface boundary condition of continuity of  $\bar{T}_S$  and  $\bar{T}_L$  and flux discontinuity at the interface  $x + \alpha t = 0$  must be used. It is sufficient to obtain solutions of  $\bar{q}_S$  and  $\bar{T}_S$  in the region  $x + \alpha t = 0$  and  $x + t < 0$  and not for the whole solid region  $x + \alpha t > 0$ .

To obtain the solution of the liquid region in the parabolic model, limit  $\alpha = V/V_h \rightarrow 0$  (or  $V_h \rightarrow \infty$ ) is taken in the solution of the liquid region. From the final expression of  $\bar{T}_L(x, t)$  obtained in the solution (which is not given here) it can be shown that the disturbances in the liquid temperature ahead of the solidification front propagate only over distances proportional to  $\tau V_h$  or proportional to  $V^{-1}$ . Numerical work has been presented for  $\bar{T}_L$  and  $\bar{T}_S$  during the final stages of solidification, i.e. by considering parabolic model.

## 12.7 SOLUTIONS OF STEFAN PROBLEMS USING CONFORMAL TRANSFORMATION METHOD

### 12.7.1 Introduction

The nomenclature *Cauchy integral method* is also used for obtaining solutions of Stefan problems using *conformal mapping method*. The description of the methods and procedures for obtaining analytical-numerical solutions using conformal transformations are different from the methods of solutions considered earlier. In the earlier sections of this chapter, at some

places, we had to mention that partial description of obtaining solution will be of no help and reader was referred to the original works to get better insight. This becomes necessary when one after another several substitutions and/or transformations are required in obtaining solution and some assumptions are also made repeatedly in the transformed equations. Even with all the difficulties, in such problems, we have tried as far as possible that the reader gets some insight into the method of solution within the constraints on the space available. For example, in the presentation of method of solution employed in [559], about four pages have been devoted but we could report only the integral in Eq. (12.6.87) and not its integration to obtain the unknown coefficients in Eqs (12.6.89), (12.6.90). No temperature or flux solution could be given without the knowledge of unknown coefficients. Our interest was that the reader understands broadly the method of solution and the difficulties which are to be overcome if any solution is to be attempted in the similar direction.

Our difficulties in presenting solutions obtained using conformal transformations are much more serious. The original problem and its geometry is transformed repeatedly into different complex planes with different geometry. Each part of the figure in each plane is to be explained completely as it corresponds to some boundary condition or some useful information. In addition, contour integrations are required and appropriate branches are to be selected. Each figure is to be drawn with complete details and the transformed variables are to be reported with description of parameters. Finally the physical quantities of interest are obtained numerically with appropriate interpretation of parameters with the help of figures. Analytical-numerical solutions of only interface shapes have been obtained by authors but the similar type of solutions are not available for temperature. Such a presentation is possible in a monograph or a report which is exclusively devoted to such type of methods but not in a book of the present type.

Instead of leaving the presentation of conformal mapping and Cauchy integral method and the solutions obtained using them, we have tried to make a suitable compromise by describing the solution of a particular problem in somewhat detail with limited figures and provide as much information as possible with the help of its partial description. In several other works which are discussed further in this section, only problem considered and highlights of procedures will be mentioned.

### 12.7.2 Solutions of Steady-State Two-Dimensional Stefan Problems Using Conformal Transformation Method

The work which is being discussed below and available in [563] has been widely referred and application of conformal mapping method has been explicitly described in it in the context of the following problem.

A plate of width  $2a$  is insulated on its two sides. The length  $L$  of the plate which is taken along the plate and perpendicular to the width is large so that a two-dimensional formulation can be considered. The temperature of the plate  $T_w$  is lower than the solidification temperature  $T_m$ . A warm fluid of temperature  $T_L > T_m$  is flowing over the plate. The flow of liquid provides a constant heat transfer coefficient  $h$  between the warm liquid temperature  $T_L$  and the surface of the frozen layer  $S(x, y) = 0$  which is at temperature  $T_m$ . The Cartesian coordinate  $x$  is taken horizontally along the width of the plate and  $y$  coordinate is taken perpendicular to the plate vertically so that  $(x, y)$  are coordinates in a vertical plane. A two-dimensional problem in  $(x, y)$  coordinates is considered so it is assumed that temperature in steady-state does not vary along the length of the plate. It is assumed that the shape of the frozen layer of the liquid will get

adjusted when the heat transferred to it by the flowing fluid is exactly balanced by the heat conducted through the frozen layer to the plate.  $T_L$ ,  $T_m$ ,  $T_w$  and heat transfer coefficient  $h$  have been taken constant together with other thermo-physical properties. Under the assumptions considered in the formulation, a steady-state formulation of the solidification problem has been considered.

Various physical quantities can be made dimensionless as follows.

$$\left. \begin{aligned} T &= (T - T_w)/(T_m - T_w); \quad \frac{\partial T}{\partial n} \Big|_{S(x,y)=0} = (h/K)(T_L - T_m)/(T_m - T_w), \\ (X, Y) &= (x, y)/\gamma; \quad \gamma = K(T_m - T_w)/h(T_L - T_M); \quad A = a/\gamma, \end{aligned} \right\} \quad (12.7.1)$$

$\gamma$  is the length scale parameter and  $T_w$  = surface temperature. The problem is to obtain the steady-state dimensionless temperature  $T(X, Y)$  in the solid and the phase-change boundary  $S(X, Y) = 0$  satisfying the equations given below. It has been assumed that temperature gradients in the liquid are negligible. The formulation is:

$$\nabla^2 T = \frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} = 0; \quad q|_{S=0} = K \frac{\partial T}{\partial (n/\gamma)} \Big|_{S=0} = \frac{\partial T}{\partial N} \Big|_{S=0} = 1; \quad T|_{S=0} = 1, \quad (12.7.2)$$

$$T = 0, \quad (-A \leq X \leq A; Y = 0); \quad T(x, y) = T_w, \quad -a \leq x \leq a, \quad y = 0, \quad (12.7.3)$$

$N = n/\gamma$  is the outward normal at the interface  $S(X, Y) = 0$ .

The configuration corresponding to above formulation in the physical complex-plane  $Z = X + iY$  in dimensionless form is given in Fig. 12.7.1.

In the present problem, the solidified region is simply connected and temperature is the solution of a Laplace equation. Therefore, if we define a function  $W = -T + i\psi$ , then  $T$  will be the real part of  $W$ ,  $W$  is an analytic function of  $Z$ .  $-T$  is taken as heat flows from higher temperature to a lower temperature. From analyticity of  $W$  and *Cauchy-Riemann conditions*, we have

$$\frac{dW}{dZ} = -\frac{\partial T}{\partial X} + i\frac{\partial \psi}{\partial X} \quad \text{and} \quad \frac{\partial T}{\partial Y} = \frac{\partial \psi}{\partial X}; \quad \frac{dW}{dZ} = -\frac{\partial T}{\partial X} + i\frac{\partial T}{\partial Y}. \quad (12.7.4)$$

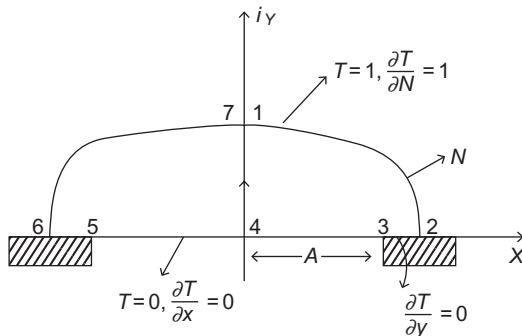


Fig. 12.7.1 Graphical presentation in  $Z$ -plane.

One of the property of an analytic function is that it can be completely determined by its boundary values. Our goal is becoming clear now. Find an analytic function  $W = f(Z)$  or use further transformation  $u$  which satisfies the boundary conditions prescribed in the  $Z$ -plane and then by integrating  $dZ/dW$  or  $dZ/du$ , obtain  $Z$ . The boundary values will give required interface. As it turns out temperature determination is difficult but in the process shape of the interface is determined. Fig. 12.7.1 in  $Z$ -plane after transformation will look like Fig. 12.7.2 in the  $W$ -plane. The lines 2–6 are the boundary  $S(X, Y) = 0$  in  $W$ -plane but interface conditions are not satisfied on it as yet.

The lines  $T = \text{constant}$  and  $\psi = \text{constant}$  form an orthogonal net in the  $W$ -plane. The lines  $\psi = \text{constant}$  are normal to lines  $T = \text{constant}$  and along  $\psi = \text{constant}$ ,  $\partial T / \partial Y = 0$ . If

$$\frac{dW}{dZ} = G = -\frac{\partial T}{\partial X} + i\frac{\partial T}{\partial Y} \text{ then } Z = \int \frac{dW}{G} + D_1, D_1 \text{ is a constant.} \quad (12.7.5)$$

By taking  $-\frac{\partial T}{\partial X}$  horizontally and  $i\frac{\partial T}{\partial Y}$  vertically, Fig. 12.7.2 in  $W$ -plane is transformed to another Fig. 12.7.3 in  $G$ -plane. Now the phase change boundary in  $G$ -plane is semicircular arc 2–1–7–6 in Fig. 12.7.3 on which the prescribed boundary conditions are satisfied. To make the

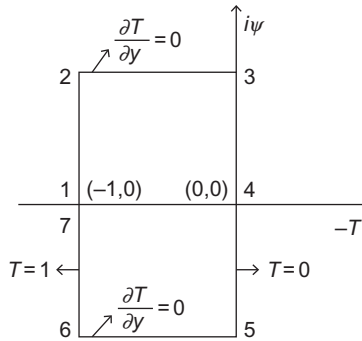


Fig. 12.7.2 Plane  $W = -T + i\psi$ .

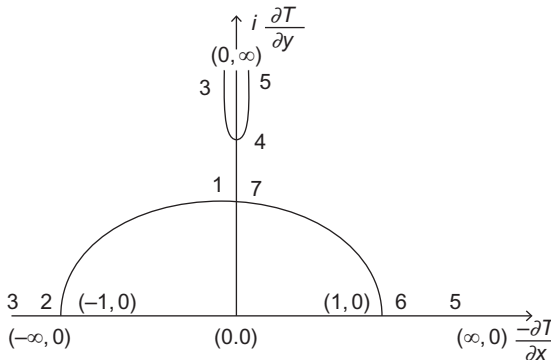


Fig. 12.7.3  $G$ -plane,  $G = -\frac{\partial T}{\partial X} + i\frac{\partial T}{\partial Y}$ .

interface a straight boundary, again a mapping  $F = \log |G| + i\theta$  is used in which the boundary 2-1-7-6 in Fig. 12.7.3 is mapped into the line 2-1-7-6 along  $i\theta$  in Fig. 12.7.4. The region in Fig. 12.7.4 is a generalized rectangle which can be mapped into half-plane by using *Schwarz-Christoffel transformation* [564] discussed later. As this transformation has sufficient degrees of freedom, the points 3 and 5 in Fig. 12.7.4 can be mapped to  $-1$  and  $+1$ , respectively, on the real axis in  $u$ -plane in Fig. 12.7.5. The points 2 and 6 are positioned at  $-b$  and  $+b$  in Fig. 12.7.5 with  $|b| < 1$ ,  $b$  is unknown. An appropriate Schwarz-Christoffel transformation is given below.

$$\frac{dF}{du} = \frac{D_2}{\sqrt{u-b}\sqrt{u+b}\sqrt{u-1}\sqrt{u+1}}; \quad F = \log |G| + i\theta. \quad (12.7.6)$$

Fig. 12.7.4 transformed from  $F$ -plane to  $u$ -plane is given in Fig. 12.7.5 in which  $u = \xi + i\eta$ .

Eq. (12.7.6) can be integrated to obtain  $F$  as a function of  $u$  which will have two unknown constants  $D_2$  and  $D_3$ . The point 1 in  $F$ -plane is at  $i\pi/2$  which is the point  $u = 0$  in  $u$ -plane. Also as  $F$  approaches 0 and  $i\pi$ ,  $u$  approaches  $b$  and  $-b$ . Using this information constants  $D_2$  and  $D_3$  can be determined.

The upper half  $\eta \geq 0$  in the  $u$ -plane can be mapped into a rectangle in the  $V$ -plane as shown in Fig. 12.7.6. The mapping for the present problem can be obtained using the following Schwarz-Christoffel transformation whose general expression is discussed in Eq. (12.7.17).

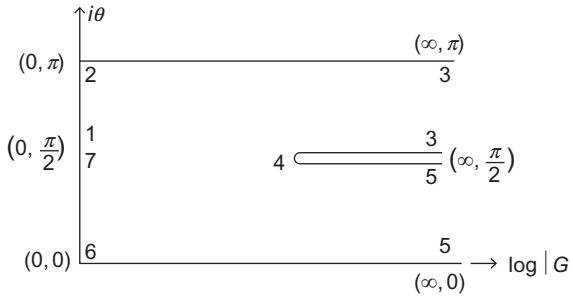


Fig. 12.7.4  $F$ -plane,  $F = \log |G| + i\theta$ .

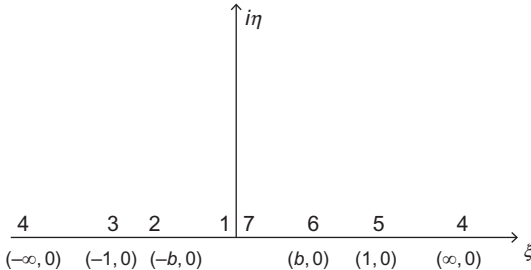
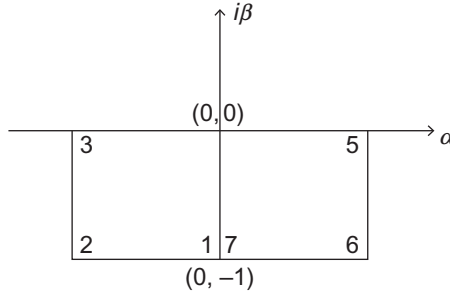


Fig. 12.7.5  $u$ -plane,  $u = \xi + i\eta$ .





**Fig. 12.7.6**  $V$ -plane,  $V = \alpha + i\beta$ .

$$\frac{dV}{du} = \frac{D_4}{\sqrt{u-b}\sqrt{u+b}\sqrt{u-1}\sqrt{u+1}}; \quad W = -iV. \quad (12.7.7)$$

To determine  $Z$ , the second equation in Eq. (12.7.5) is used.

$$Z = \int \frac{1}{G(u)} \frac{dW}{dV} \frac{dV}{du} du + D_1, \quad G = e^F. \quad (12.7.8)$$

$F$  is obtained by integrating Eq. (12.7.6) as discussed above in which arbitrary constants have been determined as suggested earlier. Skipping the calculation details, we give below the equation for  $Z$ .

$$Z = \frac{D_4}{2b} \left( -\log \frac{u-1}{u+1} + \log \frac{\sqrt{u^2-b^2} - \sqrt{1-b^2}}{\sqrt{u^2-b^2} + \sqrt{1-b^2}} \right) + D_1. \quad (12.7.9)$$

Note that in Fig. 12.7.4 there is a branch cut between  $-b$  and  $b$  along the real axis of the  $u$ -plane. When  $Z = 0$ ,  $u = \pm\infty$  and  $Z = A$  and  $-A$  are mapped on  $u = -1$  and  $u = 1$ , respectively. This gives

$$D_1 = 0, \quad D_4 = Ab / \log \sqrt{1-b^2}. \quad (12.7.10)$$

As  $z = x + iy$ ,  $z/a = Z/A = (X + iY)/A$ , we get

$$\left. \begin{aligned} \frac{x}{a} \Big|_{s=0} &= \frac{1}{2 \log \sqrt{1-b^2}} \log \left( \frac{1+\xi}{1-\xi} \right), \\ \frac{y}{a} \Big|_{s=0} &= -\frac{1}{\log \sqrt{1-b^2}} \tan^{-1} \sqrt{\frac{b^2-\xi^2}{1-b^2}}, \end{aligned} \right\} \quad (12.7.11)$$

$|\xi| < b$ .

Note that  $b$  is still undetermined and to determine  $b$  it should be related to a physical quantity. We note that  $b$  or  $A$  is a function of a single parameter  $a/\gamma$ . Heat flow through the frozen layer to the plate is also related to  $\gamma/a$  so we calculate the total flow rate of heat  $Q$  in the following way.

From Fig. 12.7.2,  $\psi(3) = \psi(2)$  and  $\psi(4) = \psi(1)$ . The total flow rate of heat  $Q$  is

$$\left. \begin{aligned} Q &= 2L \int_4^3 K \frac{\partial T}{\partial y} dx = 2KL(T_m - T_w) \\ \int_4^3 \frac{\partial \psi}{\partial X} dX &= 2KL(T_m - T_w)(\psi(2) - \psi(1)), \end{aligned} \right\} \quad (12.7.12)$$

$$\psi(2) - \psi(1) = \text{Im}[W(2) - W(1)] = \text{Im} \int_{u=0}^{-b} \frac{dW}{dV} \frac{dV}{du} du. \quad (12.7.13)$$

Im stands for the imaginary part of a complex function. From Eqs (12.7.12), (12.7.13), we have

$$\frac{Q}{2KL(T_m - T_w)} = \int_{\xi=0}^{-b} \frac{D_4}{\sqrt{b^2 - \xi^2} \sqrt{1 - \xi^2}} d\xi, \quad D_4 = \frac{(T_L - T_m)hab}{(T_m - T_w)K \log \sqrt{1 - b^2}}. \quad (12.7.14)$$

$D_4$  has been determined earlier and is given in Eq. (12.7.10). The length  $b$  is still unknown. The constant  $D_4$  can be obtained in a different way also. In Fig. 12.7.2, the size of the rectangle should be such that  $T(2) - T(3) = 1$ . Using the same procedure as in Eq. (12.7.13), we have

$$1 = -[W(2) - W(3)] = - \int_{-1}^{-b} \frac{dW}{dV} \frac{dV}{du} du = - \int_{\xi=-1}^{-b} \frac{D_4}{\sqrt{\xi^2 - b^2} \sqrt{1 - \xi^2}} d\xi. \quad (12.7.15)$$

The values of  $D_4$  in Eq. (12.7.14) and in Eq. (12.7.15) should be the same and this relation is used to determine  $b$  and finally  $b$  is determined from Eq. (12.7.16).

$$\frac{K(T_m - T_w)}{ha(T_L - T_M)} = \frac{-b}{\log \sqrt{1 - b^2}} \int_{-1}^{-b} \frac{d\xi}{\sqrt{\xi^2 - b^2} \sqrt{1 - \xi^2}}. \quad (12.7.16)$$

Numerical work has been done to determine  $S(x, y) = 0$  using Eq. (12.7.11) and the total heat flow rate  $Q$  using Eq. (12.7.14) but no analytical and/or numerical procedure has been indicated to determine temperature. It may be difficult but temperature can be obtained numerically if Laplace equation is solved in a region whose boundary is now known (as obtained earlier) and boundary conditions are already satisfied on the boundary.

### Schwarz-Christoffel Transformation

A general Schwarz-Christoffel transformation (cf. [564]) is given by the function given in Eq. (12.7.17).

$$S(z) = \int_0^z \frac{d\xi}{(\xi - A_1)^{\beta_1} \dots (\xi - A_n)^{\beta_n}}, \quad (12.7.17)$$

where  $A_1, A_2, \dots, A_n$  are  $n$ -distinct points on the real axis of the complex  $z$ -plane. The points are arranged in an increasing order. The exponents  $\beta_k$  will be assumed to satisfy conditions  $\beta_k < 1$  for each  $k$  and  $1 < \sum_{k=1}^n \beta_k$ . The case  $\sum_{k=1}^n \beta_k \leq 1$  can also be considered but then  $S(z)$  will no longer be bounded in the upper half plane.

The integrand in Eq. (12.7.17) is such that  $(z - A_k)^{\beta_k}$  is that branch (defined in the complex-plane slit along the infinite ray  $\{A_k + iy : y \leq 0\}$ ) which is positive when  $z = x$  is real and  $x > A_k$ . As a result

$$\left. \begin{aligned} (z - A_k)^{\beta_k} &= (x - A_k)^{\beta_k}, \text{ if } x \text{ is real and } x > A_k, \\ &= |x - A_k|^{\beta_k} e^{i\pi\beta_k}, \text{ if } x \text{ is real and } x < A_k. \end{aligned} \right\} \quad (12.7.18)$$

The complex plane slit along the union of the rays  $\bigcup_{k=1}^n \{A_k + iy : y \leq 0\}$  is simply connected, so the integral that defines  $S(z)$  is holomorphic in this open set.

**Theorem 12.7.1.** *There exist complex numbers  $\alpha_1$  and  $\alpha_2$  such that conformal map  $F$  of the upper half of  $z$ -plane denoted by  $H$ , onto the interior of a closed polygon  $P$  in the complex plane can be expressed as*

$$F(z) = \alpha_1 S(z) + \alpha_2, \quad (12.7.19)$$

$S(z)$  is the same as in Eq. (12.7.17). The points  $-\infty < A_1 < A_2 < \dots < A_n < \infty$  be the list of points which are mapped onto the  $n$ -vertices of the polygon  $P$ ,  $n \geq 3$ .

In the above theorem, none of the vertices of polygon  $P$  corresponds to the point at infinity. There are real numbers  $a_1, a_2, \dots, a_n$ , such that  $F(A_k) = a_k$  for all  $k$ .

If the geometry and boundary conditions in some problem are different from the one considered in [563] then the transformations including Schwarz-Christoffel transformation could be different but the essential features of conformal mapping method remain the same. Mappings from one complex plane to another plane are to be selected carefully so that boundary conditions are satisfied at last stage of mappings and the quantities of interest can be evaluated easily. It is not possible for us to draw figures or discuss the transformations in other works which will be discussed now and give their calculation details because of space restrictions. In this conformal mapping method partial description of the solution procedure does not serve any better purpose than the elaborate statement of the problem studied and so the later approach will be followed.

A steady-state continuous-casting two-dimensional problem has been considered in [565]. Let  $Y$  coordinate be taken vertically upwards along the height of the mold with  $Y = 0$  taken at the bottom of the mold and  $Y = b > 0$ , be the tip of the solid-liquid interface,  $X$ -direction be taken horizontally and width of the mold be  $2a$ . The partially solidified ingot occupies the portion  $b \leq Y \leq b + d$  and is withdrawn with constant velocity  $u$  in the negative  $Y$ -direction. The total height of the partially solidified ingot is  $d$ . A portion of partially solidified ingot has its cylindrical surface is in contact with the mold surface. On a portion  $b + d - H \leq Y \leq b + d$  the boundary condition on the mold surface is  $\frac{\partial T}{\partial X} = 0$ ,  $X = \pm a$ . For  $0 < Y < b + d - H$ ,  $T = 0$  and  $\frac{\partial T}{\partial Y} = 0$ ,  $X = \pm a$ , are the two boundary conditions.  $T(X, Y)$  is the temperature of the ingot. The boundary conditions on the interface  $S(X, Y) = 0$ ,  $-a < X < a$ ,  $b \leq Y \leq b + d$  are given below.

$$\left. \frac{\partial T}{\partial X} \right|_{S(X,Y)=0} = \frac{\rho l}{K} u \sin \theta \cos \theta; \quad \left. \frac{\partial T}{\partial Y} \right|_S = \frac{\rho l}{K} u \cos^2 \theta; \quad T|_{S=0} = T_m, \quad (12.7.20)$$

$\theta$  is the angle which the normal at any point on the interface makes with the vertically upwards  $Y$ -axis. Note that on partially solidified portion which is  $(b + d - H < Y < b + d, X = \pm a)$ ,

boundary conditions are different as in this portion direct cooling is done. Temperature satisfies the Laplace equation. Phase-change boundary has been determined and numerical work has also been presented. Temperature has not been obtained. The solution of interface has been obtained in terms of elliptic and theta functions (cf. [566]). For shape determination during directional solidification by continuous casting also refer [567].

The problem considered in [568] is about the flow of coolant into a porous reservoir through an opening along its surface. The coolant exits from the reservoir surface through some opening  $S$  on the surface whose shape is to be determined. The remaining boundary is impervious to flow and is insulated. When the coolant exits through the opening it receives a heat flux distribution. If it is desired to maintain the exit surface at a uniform design temperature then it is required to know the shape which the exit porous region should have to meet the imposed conditions. Conservation of mass, Darcy's law and heat energy equations have all been considered. Both temperature and pressure have been taken in some specific form so that under suitable assumptions a steady-state two-dimensional problem can be considered. The flux or the normal derivative at the interface is a function of interface and not a constant. The problem formulation and its configuration in  $(X, Y)$  coordinates and subsequently in the  $Z$ -plane is similar to the one considered in [563, 565]. Temperature has not been determined. The shape of the opening  $S$  has been obtained using conformal mapping method.

Solidification of a warm fluid flowing axially along  $Z$ -direction through a long pipe of rectangular cross-section has been considered in [569]. In this steady-state problem, the cross-section of pipe along its length has been taken as  $(-a \leq X \leq a, -b \leq Y \leq b)$ ,  $Z = 0$ , with  $X$ -axis taken horizontally and  $Y$ -axis taken vertically upwards. Constant temperature is prescribed on all the boundaries so that because of symmetry it is sufficient to consider only one quadrant  $(0 \leq X \leq a, 0 \leq Y \leq b)$ . Temperature on the solidification interface is taken as constant and so is the normal derivative. This is so because assumptions similar to those in [563] have been made in [569] also.

Some of the initially applied transformations are similar to those used in [563]. But in choosing Schwarz-Christoffel transformation from  $\Omega$ -plane, which is  $F$ -plane in Fig. 12.7.4, to  $t$ -plane, which is upper half plane in [569], three of the  $t$ -values can be assigned arbitrarily. One of the points in  $\Omega$ -plane is mapped to  $+\infty$  and  $-\infty$ . This has been supported by citing reference [570]. In the second Schwarz-Christoffel transformation two unknowns  $m$  and  $n$ ,  $m < 0 < 1 < n$ , are related to the width and breadth of the rectangle. Integral equations are to be solved to obtain the interface and the procedure to obtain these integral equations is different in [569]. Total heat flow from solid to liquid has been calculated. Numerical work has been done for plate of square cross-section.

The shape of the phase-change boundary has been obtained in [571] in a two-dimensional steady-state problem by considering the following formulation in dimensionless form.

$$\nabla^2 T(X, Y) = 0, X \geq 0, 0 \leq Y \leq B; T = 0 \text{ on } X = 0, 0 \leq Y \leq B, \quad (12.7.21)$$

$$\frac{\partial T}{\partial Y} = 0, Y \in (0, B), T = 1 \text{ on } S(X, Y) = 0; q_S = q|_{S=0} = \left. \frac{\partial T}{\partial N} \right|_{S=0}. \quad (12.7.22)$$

The container is partially filled with warm fluid and remaining portion is solid.  $T$  is the temperature in the solidified region.  $Y$ -axis is taken vertically upwards,  $X$ -axis is taken horizontally and  $S(X, Y) = 0$  is the phase-change boundary between solid and liquid which is unknown. The essential difference between this problem and other problems considered above is that flux  $q_S$  at the interface is a function of the interface position and is not a constant.

However,  $q_S = q|_{S=0}$  can be obtained from some physical considerations and therefore  $G(S)$  in Eq. (12.7.23) is assumed to be known. The arguments following which heat balance is reached and steady-state results are similar to those proposed in [563]. We skip the details of calculating unknown constants occurring in the Schwarz-Christoffel transformations and some other procedural changes and discuss here only the handling of normal derivative at the interface. Let

$$\frac{q_S}{q_r} = G(S(\xi)) = G(\xi), \quad q_r \text{ is some reference flux,} \quad (12.7.23)$$

$\xi$  is real axis as in Fig. 12.7.5. Using the boundary condition for normal derivative at the interface, the following equations can be obtained.

$$\left. \frac{\partial T}{\partial X} \right|_S = \frac{q_r}{\bar{q}} G(S) \cos \theta(S); \quad \left. \frac{\partial T}{\partial Y} \right|_S = \frac{q_r}{\bar{q}} G(S) \sin \theta(S), \quad (12.7.24)$$

$\bar{q}$  = average heat flux through the solid region.  $G(S)$  is assumed to be known so  $\theta(S)$  or  $\theta(\xi)$  is to be obtained. A Cauchy integral equation results for  $-1 < \xi \leq \theta$  with limits of integration from  $\xi = -1$  to  $\xi = 0$  as interface lies in this region on the real axis in the upper half  $u$ -plane. For further details of the solution of Cauchy integral equation which is singular but solved by writing it in a nonsingular form refer [571, 572]. For some related references refer [573, 574].

In the earlier problems concerning continuous casting, natural convection in the liquid region was neglected which has been considered in [575]. It has been mentioned by the authors in [575] that even for very strong natural convection the effect of flow field due to natural convection on the temperature is not much. Its significant effect is to enhance the formation of an almost isothermal region at the bottom of the pool. In the formulation of the continuous casting problem considered in [575],  $X$ -axis has been taken vertically downward and  $Y$ -axis taken horizontally. Formulation of two-dimensional steady-state quasiincompressible Newtonian fluid with constant material properties has been considered which can be obtained as a particular case of equations given in Section 1.4.1 and so it is not being given here. Boundary layer approximation has not been used.

Because of symmetry only half of the pool has been considered for applying boundary conditions and instead of boundary conditions at  $Y = \pm Y_0$ , they have been considered at  $Y = 0$  and at  $Y = Y_0$ . The length  $0 < X < \infty$  has been divided into two regions  $0 < X \leq X_p$  and  $X > X_p$ , where  $X = X_p$  is the tip of the interface. On the first portion there is indirect primary cooling and on second it is direct secondary cooling. On the interface  $Y = \bar{Y}(X)$  the boundary conditions are

$$U = U_w, V = 0; \quad T = T_m; \quad -K_S \left. \frac{\partial T_S}{\partial n} \right|_{\bar{Y}} + K_L \left. \frac{\partial T_L}{\partial n} \right|_{\bar{Y}} = \rho_S l U_w \cdot \bar{n}, \quad (12.7.25)$$

$U_w$  and  $V$  are constant velocity components in  $X$  and  $Y$  directions, respectively, and  $\bar{n}$  is outward normal at the interface.  $U_w$  is the withdrawal velocity. Finally the formulation is considered in terms of stream function of vorticity. For other boundary conditions, assumptions, scalings of quantities involved and the final formulation refer [575].

The work presented in [575] is essentially a numerical work based on finite-difference schemes. It is well known that the boundaries of solid and liquid regions do not fall on coordinate lines in the numerical discretization when flat interface is not present. Fixing the interface makes the equations highly nonlinear in which partial derivatives occur. To avoid

noneven and nonorthogonal nodal arrangements at the interface, conformal transformations can be used as the orthogonality of coordinate system is retained, resulting in relatively simple transformed equations and single-term boundary conditions. However, in this case additional difficulties in choosing the transformations arise and for an interface of arbitrary shape this will require a numerical iterative procedure. Initial guess of all the quantities of interest is to be made and after solving the transformed equations in the transformed plane, we have to transform the solution and in particular the interface temperature gradients into the physical plane.

Both solid and liquid regions were separately mapped in [575] by a sequence of transformations into parts of the unit circle. *Theodorsen's integral equation* [576] was used to obtain numerically the conformal mapping of the general interface  $Y = \bar{Y}(X)$  into the unit circle. The mapping from  $z = x + iy$  plane to  $W = u' + iv'$  plane is taken as

$$W = \sin h(\pi z/2), \quad W = u' + iv' = re^{i\theta}, \quad (r, \theta) \text{ are polar coordinates.} \quad (12.7.26)$$

The interface which is  $S$  in  $z$ -plane is transformed to  $S'$  in  $W$  plane and is further transformed to a unit circle by a sequence of mappings one of which is through stream function defining natural convection.

The procedures for obtaining analytical and numerical solutions have been explicitly described in [575]. Considerable numerical work has been done to obtain the interface and temperatures in solid and liquid regions for different Reynolds and Grashof numbers. Streamlines have been plotted for natural and forced convection. The complete details of mapping procedure and the convergence of the numerical iterative procedure are reported in [576].

A three-dimensional one-phase solidification problem has been studied in [577]. There is no conformal transformation used in obtaining the solution. The  $Z$ -axis is taken vertically upwards and  $(x, y)$  plane is perpendicular to  $Z = 0$  at which is cooling is done. The boundaries  $y = \pm b$  are insulated and a warm fluid is flowing through the opening in the  $x$ -direction which is solidifying till a steady-state is reached. The thickness of the solidified region  $Z = S(x, y)$  varies periodically in both  $x$  and  $y$  directions which is due to a spatially periodic variation in convective heating  $q_S(x, y)$  over the upper boundary of the solidified liquid. In the Stefan condition normal derivative at  $Z = S(x, y)$  is a function of  $S(x, y)$  and temperature at the free boundary is taken as constant. The dimensionless temperature  $\varphi(X, Y, Z)$  is expressed in the following form in terms of two arbitrary real functions  $f$  and  $g$ .

$$\varphi(X, Y, Z) = DZ + \frac{1}{2i} \int_{X-iZ}^{X+iZ} f(\cos \pi \xi) d\xi + \frac{L}{2\pi} \int_{(1/b)(Y-iZ)}^{(1/L)(Y+iZ)} g(\cos \pi \beta) d\beta, \quad (12.7.27)$$

where  $L = b/w$ ,  $w$  is half-wavelength of heating variation in  $X$ -direction and  $b$  is half-wavelength in  $Y$ -direction,  $D$  is some constant. The functions  $f$  and  $g$  are not determined but have been prescribed in [577] in terms of some arbitrary parameter so that they will give the desired shape of the interface. There does not seem to be a systematic method in handling such a problem. When  $\partial\varphi/\partial Z|_{Z=0}$  is calculated then the unknown functions  $f$  and  $g$  directly determine heat flux distribution through the base  $Z = 0$ . Using this flux, the flux  $q_S(x, y)$  is determined at  $Z = S(x, y)$  as done in an inverse problem. To handle such an inverse problem lot of previous experience is required in introducing arbitrary parameters in arbitrary functions  $f$  and  $g$  and determining them. For further details refer [577] which are not easy to understand. The method has limited applications.

Two-phase problem of obtaining shape of solidification interface has been considered in [578]. There is no velocity in the liquid. In [579], a one-phase solidification problem is formulated in the complex plane and perturbation solution has been obtained. The independent variables  $x$  and  $y$  which are the Cartesian coordinates are treated as dependent variables of temperature potential function  $\Phi$  and heat flow function  $\Psi$ . The interface shape is determined with the help of a perturbation solution in terms of a parameter  $K/hy_m$ , where  $K$  is thermal conductivity,  $h$  is heat transfer coefficient and  $y_m$  is the solidified thickness. Perturbation solutions will be discussed later in detail in Section 12.9. The problem formulation considered in [579] is similar to that in [563] with mainly two differences. Convective boundary condition is prescribed at the cooled boundary and Stefan condition is different.

$$K \frac{\partial T}{\partial y} \Big|_{y=0} = h(T_w(x) - T_c); \quad K \frac{\partial T}{\partial n} = \alpha q_r(x) \cos \theta, \quad \text{on interface,} \quad (12.7.28)$$

where  $\alpha$  is the radiant absorptivity of surface,  $q_r$  is the incident radiant flux and  $\theta$  is the angle between outward normal  $n$  on free boundary and  $y$ -axis.  $y = 0$  is the cooled boundary,  $T_c$  is coolant temperature,  $T_w$  is wall temperature and  $h$  is heat transfer coefficient.

Complex-variable method using some conformal transformations has been used in [580] to study the solution of uncoupled quasisteady-state heat transfer solidification/melting problems in the presence of potential incompressible fluid flow according to Darcy's law. The solid-liquid interface is time-dependent and time-dependent complex-variable transformations have been used. The main thrust in this study is on stability and blow-up of dendrite which is discussed with the help of an explicit solution. This problem will be discussed later along with Hele-Shaw problems which is our next topic of discussion. The solution procedure in [580] is different from the solution procedure of [563].

### 12.7.3 Use of Conformal Transformation in Solutions of Hele-Shaw Problems

Hele-Shaw problem has been briefly introduced in Sections 3.3.2 and 6.2.1. The context in the earlier sections concerning Hele-Shaw injection problem was to study its relation with quasistatic Stefan problem. The concern in the earlier sections was that of degeneracy or the singularity development in the free boundary movement in solutions of Hele-Shaw problems. Our interest in this section is only in those analytical-numerical solutions in which conformal mapping has been used.

We begin with the one-phase two-dimensional Stefan problem of solidification having formulation as given below.

$$\left. \begin{aligned} C\rho \frac{\partial T}{\partial t} &= K \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right), \quad (x, y) \in \Omega(t); \quad T|_S = 0, \\ K(\vec{n} \cdot \text{grad } T)|_S &= \rho l V \cdot \vec{n}|_S. \end{aligned} \right\} \quad (12.7.29)$$

$S(x, y, t) = 0$  is the phase-change boundary,  $l > 0$  is latent heat,  $T$  is the temperature of the solid phase which is adjacent to the liquid phase in which  $T(x, y) = 0+$ . The boundary of solidified region  $\Omega(t)$  is  $\partial\Omega(t)$  whose equation is  $S(x, y, t) = 0$ . When specific heat  $C = 0$  and  $T = -u$ ,  $u$  is fluid pressure, we call the above problem Hele-Shaw problem. Physically the formulation represents the flow of a fluid between two vertical plates, separated by a small distance  $2d$  between them. We take  $y$ -axis vertically,  $x$ -axis horizontally and  $z$ -axis

perpendicular to  $x$ - $y$  plane. The fluid flows in a narrow channel and a two-dimensional flow can be assumed.  $u$  is the pressure in the Hele-Shaw problem and when  $u > 0$  and  $T < 0$ , which it should be for the solid,  $\Omega(t)$  is increasing with time and we have a solidification problem. When  $u < 0$  and  $T > 0$ , the region is superheated and  $\Omega(t)$  is shrinking as time increases and it is interesting to discuss behaviour of solutions in this case. In the Hele-Shaw problem for  $u < 0$ , when fluid flow is considered, we have a withdrawal of fluid (suction) from the Hele-Shaw cell and even if  $\partial\Omega(0)$  and initial pressure  $u_0(x, y)$  are taken smooth enough for the solution to exist for  $t > 0$ , singularities can develop in infinite time even in one dimension.

In Eq. (12.7.29), replace  $T(x, y)$  by  $-u(x, y)$ ,  $u > 0$ , take  $C = 0$  and  $\rho l = K$ . The moving boundary conditions become [176]

$$\Delta u \cdot \Delta u - \frac{\partial u}{\partial t} = 0, \text{ or } \operatorname{Re} w = \operatorname{Re} \frac{\partial w}{\partial t} + \left| \frac{dw}{dz} \right|^2 = 0, z = x + iy, w = w(z). \quad (12.7.30)$$

To obtain Eq. (12.7.30), we use the interface conditions given in Eq. (12.7.29). As  $u(x, y)$  satisfies Laplace equation, it is the real part of a complex valued function  $w(z)$ . The explicit solution to Hele-Shaw problem with suction will be discussed now in the context of a single sink at  $z = 0$ . Let  $\partial\Omega(0)$  be bounded which is shrinking due to a sink of the following form as suggested in [581]

$$w \sim (Q/2\pi) \log z, \text{ as } z \rightarrow 0, Q < 0. \quad (12.7.31)$$

With the help of conformal transformation  $z = f(\alpha, t)$ ,  $\Omega(t)$  the region occupied by fluid at any time  $t$  with boundary  $\partial\Omega(t)$  is mapped onto the interior of a unit disk  $|\alpha| = 1$  in the  $\alpha$ -complex plane. This mapping exists and is unique if it is specified that  $f(0, t) = 0$  and  $f'(0, t)$  is real and positive [582].  $f$  is also analytic in a neighbourhood of  $\partial\Omega(t)$  if the later is analytic. If  $W = (Q/2\pi) \log \alpha$  then the condition  $(-u) = 0$  on  $|\alpha| = 1$  is satisfied and as  $z \rightarrow 0$ ,  $W$  has the behaviour as in Eq. (12.7.31). The energy balance condition at  $|\alpha| = 1$  is also satisfied if

$$\operatorname{Re} \left( \alpha \frac{df}{d\alpha} \frac{\overline{\partial f}}{\partial t} \right) = \frac{Q}{2\pi}, \text{ on } |\alpha| = 1, \text{ bar is for complex conjugate.} \quad (12.7.32)$$

To obtain the mapping  $f(\alpha, t)$ ,  $f$  is considered as a polynomial of degree  $N$  in  $\alpha$ . The determination of the coefficients of this polynomial is a complicated process but when  $N = 2$ ,  $f = a_1(t)\alpha + a_2(t)\alpha^2$  in which if we take  $a_1$  and  $a_2$  to be real then they satisfy the following equations.

$$a_1 \dot{a}_1 + 2a_2 \dot{a}_2 = 2Q/\pi; \quad \dot{a}_1 \dot{a}_2 + 2a_1 \dot{a}_2 = 0. \quad (12.7.33)$$

When  $\partial\Omega(0)$  is a suitable limaçon,  $\dot{a}_i$  becomes infinite at a finite time  $t = t^*$  and  $\partial\Omega(t^*)$  is a cardioid. The singularity occurs at  $(z = z_0, t = t^*)$  whenever  $z_0$  is a zero of  $dz/d\alpha$  which reaches  $|\alpha| = 1$  from the region  $|\alpha| > 1$  at  $t = t^*$ . If this happens at  $\alpha = \alpha_0$  then in general

$$\alpha - \alpha_0 \sim \text{constant} \times (z - z_0)^{1/2} \text{ as } z \rightarrow z_0. \quad (12.7.34)$$

If  $\partial\Omega(0)$  is a unit circle then suction at  $z = 0$  does not give rise to any singularity.

Suction at infinity, i.e. when  $x \rightarrow \infty$  in a long channel flow with  $|y| \leq \pi$ , was also discussed in [176] by using conformal transformation and it was found that singularity



develops after a time  $t = t^*$  and the singularity in  $\partial\Omega(t^*)$  is of the type mentioned in Eq. (12.7.34). Suction from infinity is less likely to cause singularity or blow-up than suction at a finite distance. Cuspidal blow-up seems to be generic for Hele Shaw-problem in  $R^2$  and will not be affected by introduction of nonzero-specific heat. Time  $t = t^*$  for blow-up has not been estimated. For many other results refer [176].

Given below are two explicit solutions of Hele-Shaw problems in cylindrical polar coordinates which could be useful in checking numerical solutions. The formulation considered in [583] to obtain explicit solutions is given below.

$$\left. \begin{aligned} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} &= 0, \quad (r, \theta) \in D(t), \quad t > 0, \\ u &= f(\theta, t), \quad (r, \theta) \in \partial D_1(t), \quad t > 0. \end{aligned} \right\} \quad (12.7.35)$$

$$u = 0 \text{ and } \text{grad } u = -L \frac{dR}{dt} = -L \left( \frac{dr}{dt}, r \frac{d\theta}{dt} \right); \quad (r, \theta) \in \partial D_2(t), t > 0. \quad (12.7.36)$$

Here  $L$  is some parameter,  $u$  is the fluid pressure in an incompressible fluid contained in a reservoir  $D(t)$ . The fluid is contained between the inner well boundary  $\partial D_1(t)$ , usually taken as the circle of radius  $r = r_0$ , outer boundary is  $\partial D_2(t)$ .  $R(t)$  is the position vector from  $r = 0$  to any point on the moving free boundary which is  $r = S(t)$  or  $\partial D_2(t)$  at which  $u = 0$  and  $dS/dt$  is prescribed.  $D(0)$  is given. The second condition in Eq. (12.7.36) can also be expressed as given below by using isotherm condition  $u = 0$  at the interface.

$$\text{grad } u \cdot \text{grad } u = Lu_t, \quad (r, \theta) \in \partial D_2(t). \quad (12.7.37)$$

In the above model the well  $\partial D_1(t)$  is usually interpreted as an approximation to a point source or sink and

$$u(r_0, \theta, t) = \hat{K} \log r_0, \quad r_0 \ll 1. \quad (12.7.38)$$

If  $\hat{K} < 0$  then the fluid is injected and this problem is not interesting as there is no blow-up as discussed earlier. For  $\hat{K} > 0$  it is a suction problem. Whether  $\hat{K} < 0$  or  $> 0$ , when  $u$  is independent of  $\theta$ , the following solution is obtained

$$u(r, t) = -\hat{K} \log r_0 \left( \frac{\log r - \log r_0}{\log S - \log r_0} \right) + \hat{K} \log r_0; \quad -L \frac{dS}{dt} = \frac{\hat{K} \log r_0}{S(\log S - \log r_0)}. \quad (12.7.39)$$

As the moving boundary  $S$  approaches  $r_0$ ,  $dS/dt \rightarrow \infty$ . However, if instead of  $u$  prescribed on  $\partial D_1(t)$ ,  $\partial u/\partial r = \text{constant}$  is prescribed at  $r = r_0$  then  $|dS/dt| < \infty$  for all  $t$ . After integration,  $r = S(t)$  can be obtained numerically for a given  $t$  by a simple numerical method as  $S$  is the root of a transcendental equation.

In the two-dimensional case consider  $D(t)$  as some region in the  $z$ -plane,  $z = x + iy$ ,  $x = r \cos \theta$  and  $y = r \sin \theta$ . A conformal transformation  $z = a_1(t)w + a_2(t)w^2$  maps  $D(t)$  onto the interior of a unit circle  $|w| = 1$  in the  $w$ -plane. The solution  $u(x, y, t)$  of Laplace equation in  $D(t)$  satisfying Eq. (12.7.37) and vanishing on  $\partial D_2(t)$  can be expressed as

$$u(x, y, t) = \text{Re } \varphi(z, t) = \text{Re } \hat{K} \log w; \quad \varphi(z, t) \text{ is analytic.} \quad (12.7.40)$$

As  $\varphi(z, t)$  is analytic,  $\text{grad } u \cdot \text{grad } u = |\partial\varphi/\partial z|^2$  and  $\partial\varphi/\partial z = \partial\varphi/\partial w \partial w/\partial z$ .

If follows that

$$\frac{\partial\varphi}{\partial z} = \hat{K}/(\omega(a_1 + 2a_2\omega)). \quad (12.7.41)$$

The Stefan condition can also be written as

$$\text{grad } u \cdot \text{grad } u - \partial u/\partial t = 0; \quad u_t = \text{Re}(\partial\varphi/\partial w \partial w/\partial t). \quad (12.7.42)$$

It is not difficult to obtain  $\hat{K}$  as

$$\hat{K} = L \left\{ \frac{1}{2} \frac{d}{dt} (|a_1|^2) + \frac{d}{dt} (|a_2|^2) + \text{Re}(w(\bar{a}_1 \dot{a}_2 + 2\bar{a}_1 a_2)) \right\}. \quad (12.7.43)$$

If this condition is to hold on  $|w| = 1$  then the last term in Eq. (12.7.43) should be zero which is satisfied if  $a_1^2 a_2 = 1$ . Finally

$$|a_1|^2 + 2|a_2|^2 = -2(\hat{K}/L)t + A, \quad A \text{ is an arbitrary constant.} \quad (12.7.44)$$

For calculation details of  $a_1$  and  $a_2$ , refer [583]. For numerical solution by finite difference method, method of lines in time has been used. In the initial stages, the free boundary is a limaçon with smooth boundary and encloses a convex domain. These are the observations from the numerical work done in [583].

The occurrence of singularity in the Hele-Shaw problem has been investigated in [79] in the presence of surface tension. In many studies (cf. [584]) by obtaining approximate analytical solutions using complex-variable theory and other methods, it was found that in the absence of surface tension singularity is in the form of 3/2-power cusp in the free boundary at some finite time  $t = t^*$  and for  $t > t^*$  and it is not possible to continue the solution. Most of the questions which have been raised in [79, 176, 367, 581, 584, 585], and dozens of references mentioned in them, are related to existence and/or nonexistence of singularities in Hele-Shaw flows in the presence of suction in finite regions and suction at infinity in infinite regions. By assuming suitable forms of mapping function in the neighbourhood of the cusp or the behaviour of pressure at infinity some approximate analytical solutions have been obtained and the effect of surface tension has been examined. In the absence of surface tension Hele-Shaw problem with suction is an ill-posed problem. Depending on finite or infinite regions considered and the position of suction, several questions have been examined. Some of the questions examined are: (1) Can all the fluid be removed from finite and infinite regions? (2) Are there flows in which some residual fluid remains? (3) Can the solution be continued after  $t > t^*$  if the power of cusp is different such as  $(4n+1)/2$ ? (4) Can the surface tension regularize the solutions assuming that the surface tension is small so that Hele-Shaw model is still valid? In addition to this discussion some description of solutions which do not develop singularities are also reported. For example 'Ivantsov parabola' [370] is one family of curves in Hele-Shaw flow in which fluid is totally removed from infinite region in the presence of suction pressure  $\sim (-\text{Re}(\sqrt{z}))$  at infinity.

By considering the problem of determining pressure  $u$  as formulated in [176] and taking  $u = -\sigma K_c$  at the free boundary in place of  $u = 0$ , the surface tension effects have been analysed in [79] when suction is at infinity.  $\sigma > 0$  is surface tension and  $K_c$  is the curvature

of the free boundary convex towards the fluid. The purpose is to propose a model for the free boundary for times after which cusp would form in the absence of surface tension. If surface tension is introduced after  $t \geq t^*$ ,  $t^*$  is blow-up time, and even if smooth solution is obtained, it is not necessary that for  $t < t^*$  this solution will resemble that of solution without surface tension. The only type of solutions which are smooth in the presence of suction at infinity and without surface tension are of ‘travelling-wave finger type’ [79]. ‘Ivantsov parabola’ [370] are the only solutions of travelling-wave finger type which are compatible with the suction mechanism with pressure,  $p \sim -A_0 \text{Re} \sqrt{z}$ ,  $|z| \rightarrow \infty$ .

It was suggested in [367] that the local ‘crack-tip’ model could match the travelling-wave solutions in the channel flow in which the pressure of the type given above is prescribed far upstream in the channel. This is the only morphology which can evolve when surface tension is small and nonzero for  $t = t^* + O(\sigma^{1/3})$ ,  $p = O(1)$  and  $f = 0$  is a crack of width  $O(\sigma^{2/3})$  with tip  $O(\sigma^{1/3})$ . The thin long ‘crack model’ seems to be the simplest nonsmooth possibility for continuing the solution beyond  $t > t^*$  which was proposed and pursued a bit but remained unresolved.

The asymptotic solutions of crack-model under certain assumptions were carried out in [367]. The two crucial assumptions made are about the analyticity of initial data and that of the evolving solution until it reaches a blow-up point. The crack proceeds along a path determined by its centre-line until the tip reaches a blow-up point which is determined by the analytic continuation of the initial thickness and path. For motion of the crack tip refer [367].

The only roles postulated for surface tension are that it should: (a) permit a local solution near the crack tip which matches with a certain parabola and moves with the certain speed (cf. [367]), and (b) control the evolution of a ‘Cuspidal crack’. On the basis of this speculative theory some solution was obtained which is not well-posed as the thickness of crack depends on time. Cracks with singularities of various types were also considered to gain information about cracks which blow up in finite time by considering them as time reversals of well-posed blowing problems. By considering several cases of blow-up and the role of surface tension in these blow-up, the discussion presented in [367] gives a good overview of singularity development in Hele-Shaw problems.

Another speculative approach to continue analytically the solution of Hele-Shaw problem considered earlier in references [367, 584] was suggested in [585]. The objective is to continue analytically the solution beyond  $t = t^*$ ,  $t^*$  is the time for blow-up, when the fluid is driven by suction at infinity and there is no surface tension. Based on the physical evidence and so-called ‘crack-model’ discussed in [367], the model proposed in [585] relies on allowing the conformal map from the flow domain to the interior of the unit disc to have singularities or zeros of its derivative on the unit disc, thereby generating slits in the physical-free boundary. The authors call it ‘slit method’ or continuation of solution by slit evolution. In the authors’ own words ‘we have only been able to admit this generalization at the expense of allowing so much freedom in the prescription of path and speed of the slit tip that the problem is seriously under determined. When we tried to remove this indeterminacy by proceeding to second order in the slit thickness, we found that only “crack model” is retrieved and even the slit model also becomes ill-posed’.

The starting point to obtain motion of fluid for  $t > t^*$  is Eq. (12.7.32) which is also known as *Polubarinova-Galin equation*, in which if  $t - t^* = \epsilon \tau$ ,  $\epsilon \ll 1$ ,  $\tau > 0$ , is taken then to the lowest order in  $\epsilon$ , we have  $\text{Re} \left( \frac{\partial f_S}{\partial \tau} / \alpha \frac{\partial f_S}{\partial \alpha} \right) = 0$ , on  $|\alpha| = 1$ ,  $f(\alpha, t^* + \epsilon \tau) = f_S(\alpha, \tau)$ . The next step is to find  $f_S(\alpha, \tau)$  keeping in mind that  $\partial f_S / \partial \alpha$  has a zero at  $\alpha = -1$  whilst moving

along the axis of symmetry. The determination of  $f_S(\alpha, \tau)$  is a long process which cannot be presented here concisely.

In the references [79, 367, 581–585] a good number of cross-references are mentioned which are useful for better understanding of the intricacies of methods used in dealing with singularities of these suction problems and their analytical solutions.

Some explicit solutions of Stefan-like problems involving fluid flow and diffusion have been discussed in [586] using complex-variable methods. The physics of the problems and mathematical formulations discussed in [586] are different from the problems we have considered and so they cannot be pursued here.

In the above Hele-Shaw problems only incompressible fluid flow with Darcy's law [8] was considered in which the fluid flow can be represented by a complex-potential  $W = \varphi + i\psi$ , where  $\varphi$  is the real part of complex potential and  $\psi$  is the streamfunction. Heat transfer was not considered. In the two-dimensional freezing or melting problem considered in [580], the frozen part of the domain occupies a semiinfinite two-dimensional region  $x > 0$  and an oncoming extensional-type stagnation-point fluid flow  $(u, v) \sim (-\gamma x, \gamma y)$  as  $|z| \rightarrow \infty$  is present from left to right. We consider solidification problem in the context of dendrite growth in which the oncoming liquid is supercooled with temperature  $T_\infty < T_m$ , where  $T_m$  is the freezing temperature.  $T_\infty$  is constant but the far-field velocity  $\vec{U}_\infty$  need not be constant.

To save space, the underlying assumptions and scalings done in the formulation in [580] will not be discussed here. A quasisteady-state model has been considered for the heat flow in liquid. The velocity vector  $\vec{U} = (u, v) = \text{grad } \varphi$ ,  $P_e \vec{U} \cdot \text{grad } \theta = \nabla^2 \theta$ ,  $P_e$  is Peclet number,  $P_e = U_\infty L/D$ ,  $L$  is length scale,  $D$  is diffusivity,  $\theta$  is temperature,  $z \in \Omega(t)$ ,  $\Omega(0) = \Omega_0$ .  $\partial\Omega(t)$  is the phase-change boundary. Other conditions are

$$\left. \begin{aligned} \theta &= 0, \quad \frac{\partial \theta}{\partial n} = -U_n, \quad \text{and} \quad \frac{\partial \varphi}{\partial n} = 0; \quad \text{on } z \in \partial\Omega(t), \\ \vec{U} &\rightarrow U_\infty \text{ as } |z| \rightarrow \infty; \quad \lim_{x \rightarrow -\infty} \theta = \theta_\infty = -1; \quad \lim_{y \rightarrow \pm\infty} \frac{\partial \theta}{\partial y} = 0. \end{aligned} \right\} \quad (12.7.45)$$

$n$  is normal derivative outwards to  $\partial\Omega(t)$ . Surface energy effects have been neglected. By time-reversal the melting problem can also be considered. Continuing the analogy with the zero-surface tension in Hele-Shaw model, the melting problem in [580] corresponds to the stable well-posed problem in which pressure is everywhere positive in the fluid and the freezing problem corresponds to the unstable ill-posed problem in which pressure is everywhere negative in the fluid. Hence in general it is expected that the solutions to melting problems will evolve smoothly but solutions to freezing problems may undergo finite-time blow-up.  $\partial\Omega(t)$  is assumed to be analytic.

The first step in obtaining solution is to exploit the relationship between the complex-potential  $W(z)$  of the Darcy flow and Schwarz function of the free boundary  $\partial\Omega(t)$  separating the frozen and unfrozen regions. Such a Schwarz function  $g(z, t)$  of the free boundary  $\partial\Omega(t)$  exists (cf. [587]) which is unique and analytic in some neighbourhood of the boundary such that  $\bar{z} = g(z, t)$  describes  $\partial\Omega(t)$ . The Schwarz function exists only for analytic curves in  $(x, y)$  plane. For the solidification problem the basic relationship between  $W(z, t)$  and  $g(z, t)$  is derived in [588] (see also [589]) which holds throughout the fluid domain and is as follows.

$$\left(1/\sqrt{W}\right) \frac{dW}{dz} = \frac{i}{2\sigma} \frac{\partial g}{\partial t}, \quad \sigma = \sqrt{P_e/\pi}. \quad (12.7.46)$$

In some previous references such as in [585], using Polubarinova-Kochina [590] and Galin [591] complex-variable method, an analytic function  $f(z, t)$  in Eq. (12.7.32) was introduced which is analytic in the unit disc  $|z| < 1$  and maps  $|z| \leq 1$  into  $\bar{\Omega}(t)$ .  $f(z, t)$  is unique under some conditions. Alternatively the problem in [585] can be posed in terms of the Schwarz function  $g(z, t) = \bar{z}$ .

The heat transfer and fluid flow in the problem in [580] can be decoupled via the Boussinesq transformation  $(x, y) \rightarrow (\varphi, \psi)$  in terms of the complex variable transformation  $z \rightarrow W$ . Under this transformation the frozen region is transformed into a slit along the positive  $\varphi$  axis and, more importantly, the equation for temperature  $\theta$  remains invariant. If further transformation  $W = Q^2$  is made, where  $Q = \xi + i\eta$ , then there exists an explicit similarity solution for  $\theta$  as a function of  $\eta$  only.  $\theta$  and  $V_n$  for the freezing problem are given below.  $\theta$  is a function of only  $\eta$ .  $V_n$  is the normal component of the velocity of dendrite growth in which time has been made dimensionless suitably.

$$\theta = -2\sigma \int_0^\eta \exp(-P_e q^2) dq; \quad -V_n = \frac{\partial \theta}{\partial n} = -\frac{\sigma}{\sqrt{W}} \frac{\partial \psi}{\partial n} \Big|_{\partial \Omega(t)}. \quad (12.7.47)$$

Using properties of Schwarz function (cf. [588]) and because the dendrite boundary is a streamline of the flow, we get

$$\left. \begin{aligned} \frac{dz}{ds} &= \left( \frac{\partial g}{\partial z} \right)^{-1/2}; \quad V_n = \frac{i}{z} \frac{\partial g}{\partial t} \left( \frac{\partial g}{\partial z} \right)^{-1/2}, \\ \frac{dW}{dz} \Big|_{\partial \Omega(t)} &= \left( \frac{dw}{ds} \Big/ \frac{dz}{ds} \right)_{\partial \Omega(t)} = \left( \frac{\partial g}{\partial z} \right)^{1/2} \frac{\partial \psi}{\partial n} \Big|_{\partial \Omega}. \end{aligned} \right\} \quad (12.7.48)$$

From Eq. (12.7.48) and second equation of Eq. (12.7.47), it is easy to check that Eq. (12.7.46) holds. Both sides of the identity in Eq. (12.7.46) are analytic on the boundary  $\partial \Omega(t)$ . Hence by analytic continuation this equality holds wherever both the functions exist.

To obtain solution or in other words the evolution of free boundary  $\partial \Omega(t)$ , a univalent map  $z = f(\zeta, t)$ , is required, which is one-to-one and analytic, and which maps a unit disc in complex  $\zeta$ -plane,  $\zeta = \alpha + i\beta$ , onto the fluid domain  $\Omega(t)$ . As  $\partial \Omega(t)$  is assumed to be analytic this is always possible by *Riemann's mapping theorem* and by imposing the conditions  $f(0, t) = 0$  and  $f'(0, t) < 0$ , it is fixed uniquely. Schwarz function for  $\partial \Omega(t)$  can now be expressed as

$$g(z, t) = \bar{z} = \overline{f(\zeta, t)} = \overline{f(1/\bar{\zeta}, t)} = \bar{f}(1/\bar{\zeta}, t), \quad |\zeta| = 1. \quad (12.7.49)$$

Both the functions  $g(z, t)$  and  $\bar{f}(1/\bar{\zeta}, t)$  are analytic in some neighbourhood of  $\partial \Omega(t)$  and so by analytic continuation Eq. (12.7.49) holds globally. The mapping  $f$  is analytic in the unit disc, hence all the singularities of  $g$  lie within the unit disc in the  $\zeta$ -plane.

A suitable expression for  $f(\zeta, t)$  in [580] is taken in the form

$$z = f(\zeta, t) = a(t) \left( \zeta^2 + b(t)\zeta + d(t) \frac{\zeta}{1 - \zeta} \right), \quad a, b, d \text{ are real}, \quad (12.7.50)$$

which amounts to symmetry about  $x$ -axis of the dendrite. The condition  $a(b + d) < 0$  assures that  $f(0, t) = 0$  and  $f'(0, t) < 0$ .

$$g(f(\zeta, t), t) = a \left( \frac{1}{\zeta^2} + \frac{b}{\zeta} - \frac{d}{1-\zeta} \right), \quad \zeta = 0 \Rightarrow z = 0 \text{ and } \zeta = 1 \Rightarrow z = \infty. \quad (12.7.51)$$

The local behaviour of  $g(z, t)$  near  $z = 0$  is given by

$$g(z) = af'(0)^2/z^2 + a(f''(0) + bf'(0))/z + O(1). \quad (12.7.52)$$

The derivatives  $f'(0)$  and  $f''(0)$  can be calculated and substituted in Eq. (12.7.52). As the complex potential  $W(z)$  corresponds to flow domain it is regular at  $z = 0$  and we have from Eq. (12.7.46)

$$a^3(b+c)^2 = e_1; \quad a^2(2(1+d) + b(b+d)) = e_2, \quad (12.7.53)$$

where the constants  $e_1$  and  $e_2$  can be determined from initial conditions. Third equation is generated by the condition when  $|z| \rightarrow \infty$ .

$$\left. \begin{aligned} g(z) &= -z + a(2(1+b) - d) + O(1/z), \text{ as } z \rightarrow \infty \\ \frac{d}{dt} a(2(1+b) - d) &= -2\sigma\sqrt{2\gamma} \end{aligned} \right\} \quad (12.7.54)$$

The far-field behaviour of  $W$  has already been specified as  $W \sim \gamma z^2/2$  as  $|z| \rightarrow \infty$ . Using this condition and  $g(z)$  in Eq. (12.7.54) and using both of them in Eq. (12.7.46), the second condition in Eq. (12.7.54) is obtained as the behaviour on both sides should match.

The unknowns  $a(t)$ ,  $b(t)$  and  $d(t)$  can be determined from Eqs (12.7.53), (12.7.54). Behaviour of these coefficients near blow-up time has been discussed in [580]. Also discussed are the heat flux and flow speeds as blow-up is approached. Considerable numerical work has been done to determine the free boundary at different times and for different values of parameters. Numerical work includes the  $5/2$ —power cusp solution. For all this information, refer [580].

On the same lines as in [580], a Hele-Shaw problem without heat transfer has been considered in [592]. A suction problem has been considered in which Eq. (12.7.31) holds. If we follow the notations used by us in discussing the problem considered in [580] (note that at some places our notations are different from those in [580]), then

$$\left. \begin{aligned} \frac{1}{\sqrt{W}} \frac{dW}{dz} &= \frac{i}{2} \frac{\partial g}{\partial t}; \quad z = f(\zeta, t) = a(t)\zeta + b(t)\zeta^2 + d(t)\zeta^3; \\ g(z, t) &= \frac{a(t)}{\zeta} + \frac{b(t)}{\zeta^2} + \frac{d(t)}{\zeta^3}. \end{aligned} \right\} \quad (12.7.55)$$

If there is a sink in the physical plane  $z = x + iy$  of strength  $Q$  at  $z = 0$ , then  $f(\zeta, t)$  given in second equation of Eq. (12.7.55) could be considered a solution of Eq. (12.7.32) and  $a(t)$ ,  $b(t)$  and  $d(t)$ , which have to be real as discussed earlier, may be determined. Note that in [592],  $z = f(\zeta, t)$  and  $\bar{z} = g(z, t)$ , where  $g(z, t)$  is the Schwarz function which has to be different from what is given in Eq. (12.7.50). As  $z \rightarrow 0$ , we have

$$\left. \begin{aligned} \frac{\partial W}{\partial z} &\sim Q/(2\pi z), \quad z \rightarrow 0; \quad g(z, t) = \frac{da^3}{z^3} + \frac{a^2b+3abd}{z^2} + \frac{a^2+2b^2+3d^2}{z} \\ &\quad + O(1), \quad z \rightarrow 0. \end{aligned} \right\} \quad (12.7.56)$$

Using Eq. (12.7.56) and the first equation in Eq. (12.7.55),  $b(t)$  and  $d(t)$  can be determined which are independent of  $a(t)$ , and  $a(t)$  turns out to be only a multiplicative factor. Some numerical work has been reported.

In [593], complex-variable method in conjunction with ‘least action principle’ has been used to obtain qualitative properties of the growth of fingers and bubbles on the interface between two fluids of different densities under zero gravitational field and no surface tension. We have only marginal interest in this problem and some other problems discussed in [594–596] which do not deal with analytical-numerical solutions and the scope of this chapter cannot be enlarged much. In [593], an asymptotic solution has been obtained for the type of *Rayleigh-Taylor instability* in the movement of the interface by using a suitable time-dependent two-to-one conformal map with at least two singularities in the fluid domain. The heavy fluid is mapped on the outside of a unit circle and the light one is mapped on the inside of a unit circle. Total kinetic energy of both the fluids has been calculated and variational principle is used in conjunction with conformal mapping to obtain the dynamical equations for the parameters of the time-dependent conformal mapping.

The reference [594] is a review article which describes the contributions of P. Ya. Kochina [590] concerning FBPs for harmonic functions. The applications of Kochina’s investigations in the areas of science, engineering and mathematics have been highlighted by devoting some sections to Hele-Shaw problem, seepage models, electrochemical machining, flows of immiscible fluids, ill-posedness and well-posedness, etc. There is a good list of useful references.

We have discussed earlier that Hele-Shaw problem in the presence of suction at infinity is unstable. The approach followed in [595] is to regularize this ill-posed problem by incorporating an extra term in the dynamic condition at the free boundary to penalize the large normal derivative. The idea is to include a kinetic undercooling term proportional to the normal velocity at the free boundary such as given below.

$$p + F \left( \frac{\partial p}{\partial n} \right) = 0, \text{ on the free boundary; } \frac{\partial p}{\partial n} \geq 0, \quad (12.7.57)$$

$p$  is pressure.  $F \left( \frac{\partial p}{\partial n} \right)$  could be of the form such as,  $F \left( \frac{\partial p}{\partial n} \right) = \left( \frac{\partial p}{\partial n} \right)^\beta$ ,  $\beta > 1$  and  $F$  satisfies some conditions (cf. [595]). Uniqueness of the classical solution has been proved in [595] for a general kinetic undercooling such as given above only for a short time.

Some analytical solutions of Hele-Shaw problem with suction at infinity have been obtained in [580, 581, 583, 592]. The author in [596] has remarked that the method used earlier is not complete—by itself as it does not enable one to assert categorically whether the parameterized form of the mapping function is a solution of the Hele-Shaw problem or not. As we have seen earlier that all the solutions obtained are special representatives of a class of solutions in which the partial derivatives of the mapping function are rational functions in the auxiliary plane. However, it has been shown in [597] that there are solutions which do not fall into this class. In [596], an exact solution of the most general form of solutions of this class of Hele-Shaw problem has been constructed with fluid flow in the channel. The discussion is rigorous and mostly theoretical except some simple numerical results. We cannot discuss this interesting work any further as this would require lot of space in formulation, explaining the general form of the mapping function and then the results.

## 12.8 APPROXIMATE METHODS OF SOLUTIONS OF STEFAN AND STEFAN-LIKE PROBLEMS

### 12.8.1 Introduction

The term ‘approximate method’ has been used here for some special class of methods. To illustrate our point, the Neumann solution is again considered. To obtain Neumann solution, the formulation of the problem was considered in the same differential form in which it is formulated. The solution holds pointwise, i.e. for each  $x$  and  $t$ . We have seen that if flux prescribed boundary condition at  $x = 0$  is considered in Neumann problem then an analytical solution is obtained which will not be an exact solution and will give only approximate values of the unknown physical quantities for some limited values of  $x$  and  $t$ . In this way, we have obtained an approximate solution. In this approximate solution, the formulation has been used in its original or prescribed form. Suppose the differential equation for heat energy is still the same but instead of finding its solution pointwise over the whole region under consideration it is satisfied in an integrated average sense over the region with or without a weight function. Using some method, a solution is obtained of this new formulation which gives reasonably accurate results for some physical quantities, not only for one or two problems but for a class or classes of problems, then this method of solution is different. This solution does not hold pointwise but it can be used pointwise approximately. We would like to call such or similar methods as *approximate methods* in which we need not work with the exact formulation of the problem.

The solution obtained by an approximate method is also an approximate solution and such solutions could be included in the class of approximate solutions. But definitely approximate methods have different identity. Variety of approximate methods have been used to obtain solutions of heat and mass transfer problems as well as many other problems in branches of science and engineering. Because of its simplicity in application, HBIM has found maximum applications. HBIM and some of the refinement of HBIM and some applications will be discussed here. Galerkin, collocation and variational methods will be explained with the help of one or two sample problems related to phase change. Although Stefan and Stefan-like problems have not been discussed in [495], for applications of several approximate methods, reference [495] is a very good source of information and understanding available in a single book.

### 12.8.2 Solutions Using Goodman’s HBIM: Standard HBIM

HBIM has been extensively used in literature because of its simplicity and reasonably accurate approximate solutions. Because of its utility and easyness in application, several variations, refinements and use of HBIM in conjunction with other methods have been proposed from time to time and efforts in these directions are continuing. A voluminous literature exists concerning HBIM. We have tried to accommodate as much variations of HBIM as possible within the constraints on space availability.

HBIM has already been introduced earlier in the solutions of some problems, for example, in [542]. We describe it here more explicitly. Consider a solution of heat equation, for example, a source solution. The temperature satisfies the heat equation at every point in space except at which heat source is applied and at every point of the time interval for which solution is required. In HBIM, the heat equation is satisfied only in an integrated average sense over



the region which is under consideration or only over a portion of the region with some assumptions.

The method proposed in [542] by Goodman is analogous to the momentum and energy integral methods of Von Kármán and Pohlhausen (cf. [598]) which were used for boundary layer approximations in fluid mechanic problems. The concept in HBIM is the ‘temperature penetration depth’ and ‘overall heat balance’ instead of heat balance over an elementary infinitesimal volume. To explain the method explicitly, a one-phase problem in a semiinfinite region  $0 \leq x < \infty$  is considered here in which the region is initially at the melting temperature  $T_m$  and the problem considered could be a solidification or a melting problem. Let  $T(x, t)$  be the temperature and  $x = S(t)$  be the phase-change interface. When linear heat equation is integrated with respect to  $x$  between the limits  $x = 0$  to  $x = S(t)$ , we get

$$k \left( \frac{\partial T}{\partial x} \Big|_{S(t)} - \frac{\partial T}{\partial x} \Big|_{x=0} \right) = \int_0^{S(t)} \frac{\partial T}{\partial t} dx = \frac{d}{dt} \left( \int_0^{S(t)} T dx \right) - T|_S \frac{dS}{dt}. \quad (12.8.1)$$

Eq. (12.8.1) can be written as follows if interface conditions which are of type considered in earlier one-phase problems without kinetic condition are used.

$$k \frac{\partial T}{\partial x} \Big|_{x=0} = \frac{\rho l}{k} \frac{dS}{dt} - \frac{d\theta}{dt} + T_m \frac{dS}{dt}; \quad \theta = \int_0^{S(t)} T dx, \quad (12.8.2)$$

where  $k$  is thermal diffusivity,  $l$  is latent heat and  $\rho$  is density.

To demonstrate how temperature and  $S(t)$  are obtained, the solution obtained in [359] will now be discussed. The boundary condition considered at  $x = 0$  is of the form

$$-\frac{\partial T}{\partial x} \Big|_{x=0} = f(T), \quad 0 \leq x < \infty, \quad t > 0; \quad f(T) \text{ is known.} \quad (12.8.3)$$

Temperature is approximated in the form of a second degree polynomial in  $(x - S)$ .

$$T(x, t) = A_1 + A_2(x - S) + A_3(x - S)^2, \quad A_1, A_2, A_3 \text{ are unknowns.} \quad (12.8.4)$$

The unknown coefficients such as  $A_1, A_2, \dots$  in Eq. (12.8.4) could be constant or functions of time. In some of the works which will be discussed later, these unknowns are also considered as functions of time to satisfy the heat balance at the phase-change boundary uniformly in time. If they are functions of time, a differential equation results to determine each of them. No set of rules can be prescribed for choosing the approximating polynomials or functions for temperature such as in Eq. (12.8.4). However, the usual strategy is that the approximate temperature is chosen in such a way that some of the unknown coefficients are easily determined with the help of the boundary, interface and initial conditions. Further, the determination of remaining unknowns could also be achieved easily. The process of determining the unknown coefficients occurring in the approximation for temperatures guides us to consider them as functions of time or not, even if we start the procedure by taking them as constant.

It is clear from Eq. (12.8.1) that the heat equation which is a partial differential equation is reduced to an ordinary differential equation which is a great advantage in HBIM.  $A_1$  and  $A_2$  can be easily determined if isotherm and Stefan condition are used. Constant  $A_3$  and  $S(t)$  are yet to be determined. We use Eq. (12.8.1) to determine  $S(t)$  and generate an extra boundary

condition to determine  $A_3$  by differentiating isotherm condition with respect to  $t$  and using the linear heat equation. The following extra boundary condition in a melting problem is obtained when we try to satisfy heat equation at  $x = S(t)$ .

$$\frac{k}{\rho l} \left( \frac{\partial T}{\partial x} \right)^2 \bigg|_{S(t)} = \frac{\partial^2 T}{\partial x^2} \bigg|_{S(t)}. \quad (12.8.5)$$

A quadratic equation for  $A_3$  results when Eq. (12.8.4) is used in Eq. (12.8.5). Which root is to be used is decided by the condition that when  $t = 0$ , it is required that  $T = T_m$  and  $S(t) = 0$ . To determine  $S(t)$ , Eq. (12.8.1) can be used.  $S(t)$  can be obtained only numerically. For further calculations which are lengthy, refer [359].

At this stage few questions arise. What should be the degree of the polynomial in Eq. (12.8.4)? Do we get a better approximation if degree of the polynomial is increased? In this case definitely, the determination of the unknown coefficients becomes much more difficult without any certainty that a better approximation of solution will be obtained. How to generate extra boundary conditions to determine unknown coefficients? There is no fixed rule and it depends on the problem under consideration and easiness in obtaining the solution. All these factors become important and they can be answered in the context of individual problems. Considerable literature exists on the solutions obtained by using HBIM alone and on solutions obtained by different methods in conjunction with HBIM method.

Our interest in this section is not to discuss all the solutions obtained by HBIM. However, in some solutions there is variety in application and this interests us. In many problems refinements of the above HBIM method have been considered or the approximation of temperature is considered in some different form or the region is divided into different zones, etc. Then there are multiphase problems as well as multidimensional problems. Such problems will be briefly discussed here. In general, calculations are lengthy and difficult also and they cannot be reported here.

In a heat conduction problem without phase change, the HBIM can be thought of as a *penetration depth method*, i.e. the boundary condition will affect the temperature of the liquid or solid up to a certain depth  $\delta(t)$  at the time  $t$ .  $\delta(t)$  is unknown. If the initial temperature of the solid is a constant, say,  $T_0$  then  $T(x = \delta(t), t) = T_0$  and for  $x > \delta(t)$ , the temperature remains  $T_0$ . No other condition is required at  $x = \delta(t)$  in a problem with no phase change. A suitable polynomial approximation can be assumed for temperature and unknowns are determined by satisfying heat equation and using prescribed boundary conditions and if necessary extra boundary conditions are to be generated.

The HBIM can be used for two-phase, solidification and melting problems also. To demonstrate its application, we use penetration depth assumption to determine liquid temperature in a solidification problem discussed below. Two coupled integro-differential equations are obtained which will involve two unknown functions  $S(t)$  and  $\delta(t)$ .  $S(t)$  is the phase-change interface and  $\delta(t)$  is the penetration depth, i.e. the depth up to which temperature of the liquid is affected. Consider the formulation of a two-phase problem as considered in Neumann problem, refer Eqs (1.3.1)–(1.3.7), without natural convection. Heat equation (1.3.1) for the solid when integrated from  $x = 0$  to  $x = S(t)$  results in the following equation.

$$\frac{d\theta_S}{dt} - T_m \frac{dS}{dt} = K_S \left[ \frac{\partial T_S}{\partial x} \bigg|_{x=S(t)} - \frac{\partial T_S}{\partial x} \bigg|_{x=0} \right], \theta_S = \int_0^{S(t)} T_S dx. \quad (12.8.6)$$

In the liquid region it is assumed that temperature of the liquid has changed up to the distance  $x = \delta(t)$ ,  $\delta(t) > S(t)$  and beyond  $x > \delta(t)$  the liquid temperature remains unchanged, i.e.  $T_L(\delta(t), t) = T_0$ ,  $x \geq \delta(t)$ , and  $\frac{\partial T_L}{\partial x}(\delta(t), t) = 0$ . In this case Eq. (1.3.3) (with  $u_x = 0$ ) is integrated from  $x = S(t)$  to  $x = \delta(t)$  and we get

$$\frac{d\theta_L}{dt} - T_0 \frac{d\delta(t)}{dt} + T_m \frac{dS}{dt} = -k_L \left. \frac{\partial T_L}{\partial x} \right|_{x=S(t)}, \quad \theta_L = \int_{S(t)}^{\delta(t)} T_L dx, \quad (12.8.7)$$

$k_S$  and  $k_L$  are diffusion coefficients. As discussed in the one-phase Stefan problems, suitable expressions for  $T_S$  and  $T_L$  are assumed and unknown coefficients are determined by satisfying all the prescribed boundary conditions and if necessary extra boundary conditions are generated.

The method of solution by HBIM in the case of two moving boundaries can be satisfactorily explained with less computational details with the help of the solution of an ablation problem which also has two moving boundaries. The ablation problem presented in [599] is about the melting of a semiinfinite solid  $0 \leq x < \infty$ . We take  $t = 0$  to be the time after scaling at which the temperature of the solid at  $x = 0$  attains the melting temperature taken as  $T = 0$ . Let the temperature of the solid at  $t = 0$  beyond  $x \geq \delta(t)$  be  $-T_0$ ,  $T_0 > 0$ .  $\delta(t)$  is the depth up to which the temperature of the solid has changed till the time melting started at  $x = 0$ . Note that the solid before  $t = 0$  at some time  $t = -t^*$  had the temperature  $-T_0$  and heat flux was applied to raise its temperature at  $t = -t^*$ . At  $t = 0$  the melting starts. The melting takes place due to prescribed heat flux  $f(t)$  and the melted solid or liquid formed is instantaneously drained out. The formulation of the problem is as given below:

$$\left. \begin{aligned} \frac{\partial T_S}{\partial t} &= K \frac{\partial^2 T}{\partial x^2}, x > S(t), \quad S(0) = 0, \quad t > 0, \\ f(t) + K \frac{\partial T_S}{\partial x} &= \rho l \frac{dS(t)}{dt}, \quad x = S(t), \quad t \geq 0, \\ T|_{S(t)} &= 0; \quad T|_{\delta(t)} = -T_0; \quad \left. \frac{\partial T}{\partial x} \right|_{\delta(t)} = 0. \end{aligned} \right\} \quad (12.8.8)$$

As there is no liquid region, the heat equation in Eq. (12.8.8) will be integrated from  $x = S(t)$  to  $x = \delta(t)$ . It is easy to obtain the heat balance integral which is given below.

$$\frac{d}{dt} [\theta + T_0 \delta(t) + k \rho l / K S(t)] = k f(t) / K; \quad \theta = \int_{S(t)}^{\delta(t)} T_S dx. \quad (12.8.9)$$

To obtain the solution, solid temperature  $T_S(x, t)$  is assumed in the following form (cf. [599])

$$T_S = A_1 + A_2 \frac{(x - S)}{\delta - S} + A_3 \left( \frac{x - S}{\delta - S} \right)^2 + A_4 \left( \frac{x - S}{\delta - S} \right)^3 + A_5 \left( \frac{x - S}{\delta - S} \right)^4. \quad (12.8.10)$$

The five constants  $A_1, \dots, A_5$  require five independent conditions to determine them. Three conditions are given in Eq. (12.8.8). Two more can be easily generated and they are given below.

$$\left. \frac{\partial^2 T_S}{\partial x^2} \right|_{x=\delta} = 0 \quad \text{and} \quad \left. \frac{\partial^3 T_S}{\partial x^3} \right|_{x=\delta} = 0. \quad (12.8.11)$$

Finally we get  $A_1 = 0$ ,  $A_2 = -4$ ,  $A_3 = 6$ ,  $A_4 = -4$  and  $A_5 = 1$ .

To determine  $S(t)$  and  $\delta(t)$ , one differential equation for  $S(t)$  is obtained from the Stefan condition in Eq. (12.8.8) in which derivative of  $T_S$  given in Eq. (12.8.10) can be substituted. Another differential equation in terms of  $(\delta - S)$  is obtained from Eq. (12.8.9) when  $T_S$  from Eq. (12.8.10) is substituted in Eq. (12.8.9). Thus we get two coupled differential equations. If we put  $y = \delta - S$  then the heat balance equation (12.8.9) can be written as

$$\frac{dy}{dt} - \frac{A}{y} = Bf(t), \quad A = (2_0KT_0/\rho l)(1 + k\rho l/KT_0), \quad B = 5/\rho l. \quad (12.8.12)$$

Initial condition for  $y$  is

$$y(t = 0) = \delta(0) - S(0), \quad S(0) = 0. \quad (12.8.13)$$

$\delta(0)$  in Eq. (12.8.13) is a prescribed quantity in the sense that when melting started at  $(x = 0, t = 0)$  the temperature of the solid at  $t = 0$  can be calculated by solving a heat conduction problem using HBIM or some other method and distance from  $x = 0$  to  $x = \delta(t)$  up to which temperature of the solid has changed can be obtained. Note that  $T_S = -T_0, x \geq \delta(t)$  before the melting started and we have taken  $t = 0$  as the time when melting started. Once  $y(t)$  is obtained from Eq. (12.8.12),  $\delta(t)$  is known in terms of  $S(t)$  and  $S(t)$  can be obtained as discussed above. Some particular solutions have been obtained in [599] which will not be discussed here.

What are the main considerations in obtaining this above solution or for that matter in obtaining any other solution by HBIM? The first is suitable representation of the approximate temperature such as in Eq. (12.8.10) or in some other form. Generating independent extra boundary conditions, if necessary. Sometimes introducing more unknown constants spoils the solution. Rearranging the obtained coupled integral equations so that if possible some analytical-numerical solution can be obtained and/or obtaining numerical solution becomes easy. As mentioned earlier, fairly large literature exists on solutions using HBIM. Each obtained solution has some interesting features. Our main interest here is on variations, refinements or extensions of the *standard HBIM* presented above and discuss some sample problems. The calculation details will be avoided unless they are required to explain the method of solution and are essential.

The basic features of HBIM have already been explained with the help of solutions of two sample problems and some discussion reported thereafter about some main considerations. Some of the works which are available with us are being discussed here but discussion will be limited to those features which suggest some variation, improvements or extensions of HBIM. We would like to clarify here that the diffusion equations, initial conditions, boundary and interface conditions if they are of routine type will not be presented in the discussion. The various quantities are dimensionless or have dimensions do not change method of solution.

In [600], one-phase one-dimensional problems of planar, cylindrical and spherical geometries have been considered with temperature prescribed boundary condition. All the three problems can be described in terms of one formulation. In addition to the integration of the heat equation as done in Eq. (12.8.1), heat equation is also multiplied by  $(1 - X)^\lambda \frac{\partial \theta}{\partial X}$ ,  $\lambda = 0, 1, 2$  and then integrated to obtain another integral equation.  $X$  and  $\theta(X, t)$  are scaled variables.

This new dimensionless independent variable  $X$  can be easily defined so that for  $\lambda = 0$  we have planar case and for  $\lambda = 1, 2$  we have cylindrical and spherical problems, respectively. To determine the temperature and phase-change boundary  $x = S(t)$ , two procedures are suggested. In the first method, temperature  $\theta(X, t)$  is approximated by  $X/S(t)$  and in the second method dimensionless temperature is approximated in the following way.

$$\theta = X/S(t) + g(t)(X/S(t) - X^2/(S(t))^2), \quad g(t) \text{ is unknown.} \quad (12.8.14)$$

$g(t)$  and  $S(t)$  are determined from the two obtained integral equations. The first integral equation is obtained by integrating heat equation and the second one is obtained by multiplying the heat equation by  $(1 - X)^\lambda \frac{\partial \theta}{\partial X}$  and then integrating with respect to  $x$  from  $x = 0$  to  $x = S(t)$ . Substitute Eq. (12.8.14) for temperature in them. The initial condition for  $S(t)$  is  $S(0) = 0$  and for  $g(t)$  it is  $S(t)g(t) = 0$  as  $S(t) \rightarrow 0$ . This later condition has been derived using total thermal energy of solidified thickness (cf. [600]).

A problem of fluidized-bed coating of thin plates has been studied in [601]. When an hot object is dipped in a bed of fluidized plastic powder, a film of fused plastic coating is formed on the object. The temperature of the object at its boundary has been considered as a function of time but the temperature of the plastic pool is taken as constant. Convective heat transfer  $h(t)$  has been considered at the wall surface (plate) and also at coated film boundary  $x = S(t)$ . Heat balance equations formulated at both the wall surface and coated film boundary involves  $S(t)$  together with heat conduction equation in coated film thickness make the problem complicated. To obtain the solution the moving boundary  $x = S(t)$  is fixed at  $\xi = 1$ ,  $\xi = x/S(t)$  and then the transformed heat equation is integrated twice with respect to  $\xi$  from 0 to  $\xi$  and  $\xi$  to 1 as double integral which after using boundary conditions gives an equation in terms of  $S(t)$  and temperature. The formulation is considered in dimensionless form. The intermediate steps require some nontrivial operations. A quadratic profile in terms of  $(1 - \xi)$  is assumed for temperature. Two nonlinear coupled equations are obtained to be solved numerically. This method of double integration with respect to the space variable has been applied in [602] and later on used in [601, 603].

The problem considered in [603] is that of one-phase radially symmetric shrinking core model in spherical geometry similar to the problem considered in [511]. By double integration of diffusion equation (in dimensionless form) with respect to spacial coordinate, an equation is obtained for concentration in terms of an integral involving phase-change boundary  $\delta(t)$  and derivative of concentration with respect to  $\delta(t)$ . First approximation of concentration is taken as  $\varphi(x, \delta) = 1 - x/\delta(t)$  which can be improved when obtained solution of  $\varphi(x, \delta)$  is used in the r.h.s. of the equation for  $\varphi(x, \delta)$  obtained in terms of  $\delta$  and integrals of  $\partial\varphi/\partial\delta$ .

Stefan problem with radiative heat transfer has been considered in [429] and the problem considered in [604] is that of recrystallization by radiative heat transfer of a melt which is maintained at constant temperature. The surface  $x = 0$  is considered at another constant temperature and solidification is taking place due to conduction and radiation. In the heat equation (energy balance equation), radiative heat flux  $q_r$  has been included.  $q_r$  is a function of time. The approximate temperature is taken in the form of a quadratic polynomial in  $(x - S(t))$ ,  $S(t)$  is the phase-change boundary. The application of HBIM is similar to one described in [359]. The function  $q_r(t)$  should be known at  $x = 0$  and at  $x = S(t)$  in the application of HBIM. The exact expression of  $q_r(t)$  is fairly complicated but some approximate values at  $x = 0$  and  $x = S(t)$  have been taken into consideration which are required, and temperature and  $S(t)$  have been obtained.

The solutions obtained by HBIM are in general analytical-numerical solutions and moving boundary can be obtained only numerically. In rare cases it has been obtained analytically. A radially symmetric one-dimensional spherical problem has been considered in [605] in which in the energy balance at the moving interface a convective term is also present in addition to the latent heat generation term. The problem could be that of solidification or melting. The method of solution is as discussed earlier. The dimensionless approximate temperature has been taken in the following series form.

$$\theta = -\frac{2}{\pi} \frac{S(t)}{r} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(n\pi \frac{r}{S(t)}\right) \exp(-n^2 \pi^2 \alpha t / (S(t)^2)), \quad (12.8.15)$$

$r = S(t)$  is the phase-change boundary and  $\alpha$  is thermal diffusivity. The solution for  $S(t)$  has been obtained by taking  $\theta$  in a very simple known form and not as in Eq. (12.8.15).

Two substrates of semiinfinite extent in Cartesian coordinate  $x$  are placed one above the other which have common boundary  $x = 0$ . The upper substrate has a lower melting point than the lower one and initially both are at the same melting temperature as that of the upper one. Suddenly on the plane  $x = 0$ , a heat source per unit area  $F$  is applied. The upper substrate starts melting and the lower substrate receives heat. The flux  $F$  is distributed as  $F_u$  and  $F_L$  on the upper and lower substrates, respectively,  $F = F_u + F_L$ . Using penetration depth method, the depth  $\delta(t)$  and temperature in the lower one can be calculated from which the temperature  $\theta(t)$  at  $x = 0$  can be calculated. Quadratic polynomial has been considered for temperature in the solid region (cf. [606]).

The temperature at  $x = 0$  calculated from solid temperature contains unknown  $\delta(t)$  and will also contain unknown flux  $F_u$ . Note that if  $F_u$  is unknown then  $F_L$  is known only in terms of  $F_u$ . The upper substrate problem is a melting problem in which the phase-change boundary  $S(t)$  can be calculated by taking a suitable approximate quadratic polynomial temperature profile which will have  $F_L$  and  $S(t)$  as unknowns. Since temperature is continuous at  $x = 0$ , this condition can be used to determine  $F_u$  or  $F_L$ . Finally two coupled integral equations have to be solved numerically which arise when boundary conditions are satisfied. The solution in [606] for the above problem has been obtained only in a particular case but the equations to determine the unknowns such as  $S(t)$ ,  $\delta(t)$ ,  $F_u$  are reported for the problem considered.

A finite slab  $0 \leq x \leq d$  has been considered in [607] which is initially occupied by a superheated liquid having temperature  $T_L > T_m$ . In the two-phase problem the face  $x = d$  is insulated and the temperature at  $x = 0$  is suddenly dropped to  $T_W$  which is below the freezing temperature  $T_m$ . In the liquid, penetration depth method is used and an appropriate quadratic profile is considered for temperature. In solid phase the quadratic polynomial profile for temperature has to satisfy the prescribed temperature condition at  $x = 0$ . Only one unknown constant is taken in quadratic temperature profile of solid which is a function of time. The authors argue that this way it is not possible to satisfy the heat balance at the phase-change interface uniformly in time till the interface attains the end of solidification. A procedure to extend the length of time for which solution is valid has been proposed. Suppose at  $t = \bar{t}$  the thermal penetration depth  $\delta(t)(x = \delta(t))$  in the liquid becomes equal to the thickness of the slab, i.e.  $\delta(\bar{t}) = d$ . The next stage of solution now begins. The earlier temperature approximation for solid is multiplied by a function  $f(t)$ , such that  $f(\bar{t}) = 1$ . Let the new solidified thickness be denoted by  $\bar{S}(t)$ ,  $\bar{S}(t) = S(\bar{t})$  (here  $S(t)$  is the thickness of solid obtained by earlier procedure). For  $t \geq \bar{t}$ , the heat balance at the interface and heat balance integral equation are then rewritten for  $\bar{S}(t)$  and  $f(t)$  which result in coupled differential equation for

$\frac{d\bar{S}(t)}{dt}$  and  $\frac{df}{dt}$  with initial conditions,  $\bar{S}(t) = S(\bar{t})$  and  $f(\bar{t}) = 1$  for  $t \geq \bar{t}$ . Moving boundaries are taken proportional to  $\sqrt{t}$ . According to authors this solution procedure is different from the one proposed in [608]. For further discussion of a particular solution, refer [607].

In [609], temperature and flow fields develop near the vertical interface (y-axis) between a solid and its own liquid, when the two are placed in thermal contact at  $x = 0$ . Initially the solid and liquid are at temperatures  $T_{Si}$  and  $T_{Li}$ , respectively,  $T_{Si} < T_m < T_{Li}$ , where  $T_m$  is the melting temperature. The problem is considered in  $(x, y)$  coordinates in which the coordinate system is attached to the two-phase interface. In this system the rate at which the interface advances by melting into the solid appears as a 'blowing' velocity  $u_0$ , that is, the velocity of the liquid that is generated at the interface and pushed to the right into the liquid boundary that descends along the interface. The liquid is considered Newtonian and Boussinesq incompressible. The mass, momentum and energy equations and approximations to temperatures and velocity are given below.

$$\left. \begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0; \quad v \frac{\partial^2 v}{\partial x^2} + g\beta(T_L - T_{Li}) = 0, \\ \frac{\partial T_L}{\partial t} + \frac{d}{dx}(uT_L) + \frac{\partial}{\partial y}(vT_L) &= \alpha_L \frac{\partial^2 T_L}{\partial x^2}. \end{aligned} \right\} \quad (12.8.16)$$

$$T_S = T_{Si} + (T_m - T_{Si})(1 + x/S(t))^2; \quad \frac{\partial T_S}{\partial t} + u_0 \frac{\partial T_S}{\partial x} = \alpha_S \frac{\partial^2 T_S}{\partial x^2}, \quad (12.8.17)$$

$$T_L = T_{Li} - (T_{Li} - T_m)(1 - x/\delta(t))^2; \quad v = V(y, t)((x/\delta(t))(1 - x/\delta(t)))^2. \quad (12.8.18)$$

In the above equations,  $\nu$  is kinematic viscosity,  $\alpha$  is thermal diffusivity and  $V(y, t)$  is velocity coefficient. The  $x$ -axis is taken horizontally and  $y$ -axis vertically. On the right of  $x = 0$  is the boundary layer  $x = \delta(y, t)$  arising due to natural convection and on the left of  $x = 0$  is the solid-liquid interface  $x = -S(y, t)$ .  $S(y, t)$  is the thickness of the solid which has melted.  $u$  and  $v$  are the velocity components in  $x$  and  $y$  directions, respectively. For the assumptions, scalings of equations and identification of parameters, refer [609].

To obtain solution of the natural convection boundary layer, first, second and third equations in Eq. (12.8.16) are integrated from  $x = 0$  to  $\delta(y, t)$ . The integrals will involve four unknowns  $V(y, t)$ ,  $u_0(y, t)$ ,  $u_\delta(y, t)$  and  $\delta(y, t)$ .  $u_0$  is the blowing velocity at the interface and  $u_\delta$  is velocity at  $\delta(y, t)$ .  $u_\delta$  and  $V$  can be eliminated and a single equation is obtained in  $u_0$  and  $\delta$ . For the solid phase, variation with respect to  $y$  has been neglected and integration of second equation in Eq. (12.8.17) from  $x = 0$  to  $x = -S(t)$ , results in an equation in  $u_0$  and  $S(t)$ .

Heat balance equation at  $x = 0$  which is given below is used as the third equation to determine three unknowns  $u_0$ ,  $S(y, t)$  and  $\delta(y, t)$ .

$$\rho l u_0 = K_L \left( \frac{\partial T_L}{\partial x} \right)_{x=0} - K_S \left( \frac{\partial T_S}{\partial x} \right)_{x=0}. \quad (12.8.19)$$

For other details refer [609].

A heat and mass transfer problem has been considered in [610] in a one-dimensional slab. Temperature gradients in solid and liquid phases are present and concentration gradient is present only in the liquid phase.  $x = S(t)$  is the solid-liquid phase-change boundary and  $\delta_1(t)$  and  $\delta_2(t)$  are the thermal and diffusion boundary layers in liquid, respectively. Natural convection has been considered in the liquid. For assumptions, scalings and formulation of the

problem refer [610]. All the equations of the formulation have been discussed in the earlier sections of this chapter and so not reported here. Approximate temperature in the solid is taken of the form  $x/S(t)$  and in the liquid it is taken as a cubic polynomial in  $(x - S(t))/\delta_1(t)$  in which only one coefficient which is a function of time remains undetermined after satisfying boundary conditions. A differential equation can be obtained for this coefficient. Cubic polynomial for concentration is of the form  $(x - S(t))/\delta_2(t)$ . A coupled system of three first-order differential equations in time is to be solved numerically which is obtained on satisfying heat and mass balance conditions at the phase-change interface. The obtained short-time analytical solution is extended for longer time by a finite difference numerical scheme. Experimental verification of results was also done.

Having discussed some solutions using HBIM, the following comments about the solution procedure can be easily understood.

1. The solution is invariably obtained as an analytical-numerical solution. As seen in the Neumann solution, the initial velocity of interface is infinite when temperature and/or concentration is prescribed at  $x = 0$ . The integral equations obtained in the solution by HBIM contain  $\frac{dS}{dt}$  and/or  $\frac{d\delta}{dt}$ . Therefore starting values in numerical solutions are required which can be obtained either from some available analytical solutions or short-time solutions are to be developed.
2. It has been observed by some authors such as in [611] that sometimes cubic profile for temperature gives less accurate results than a quadratic profile. Therefore solutions by HBIM should be compared with other known solutions. The optimization of the exponent in temperature concentration profiles will be discussed later.

### 12.8.3 Refinement and Variations of HBIM

The solution procedure used above and in particular in [359, 598] could be called standard solution procedure for HBIM and these methods as *standard HBIM*. The refined HBIM will be henceforth called *RHBM*. As far as possible, the discussion on the refinement and variations of HBIM will be restricted to the analytical solution part of the solution. The RHBM suggested in [612] is in the context of a melting problem in the region  $0 \leq x \leq \infty$ . The solid is initially at the melting temperature  $T_m$  and temperature  $T_0 > T_m$  is prescribed at  $x = 0$  which starts melting. Let the temperature range  $T_0$  to  $T_m$  be divided into  $n$  equal intervals so that the temperature  $T_i$  is given by

$$T_i = T_m + \frac{i}{n}(T_0 - T_m), \quad i = 0, 1, 2, \dots, n; \quad T_n = T_0 \text{ at } x = 0. \quad (12.8.20)$$

We consider  $T_0, T_1, \dots, T_m$  as isotherms and let  $X_i(t)$  be the penetration depth or melted thickness when  $T_i$  is the isotherm temperature. When linear heat equation is integrated between  $X_i(t) \leq x \leq X_{i+1}(t)$ , we get an equation similar to Eq. (12.8.1) and the integral is given by

$$\frac{d}{dt} \left[ \int_{X_{i+1}(t)}^{X_i(t)} T(\xi, t) d\xi + T_{i+1}X_{i+1} - T_iX_i \right] = -k \left( \frac{\partial T}{\partial x} \right)_{X_{i+1}} + k \left( \frac{\partial T}{\partial x} \right)_{X_i}, \quad (12.8.21)$$

$i = 0, 1, \dots, n$ ,  $X_n(t) = 0$ . For integration from  $X_1$  to  $X_0$  and 0 to  $X_{n-1}$ , the r.h.s. of Eq. (12.8.21) will be different. Next step is to approximate temperature in the  $n$ -subdivisions  $(X_{i+1} - X_i)$  and obtain recurrence relation for the  $n$ -differential equations to obtain solutions of  $dX_i/dt$ ,  $i = 0, 1, \dots, n - 1$ . If a linear temperature profile is chosen for the interval



$X_{i+1} \leq x \leq X_i$ , flux at the end points  $X_i$  and  $X_{i+1}$  will be discontinuous. In a quadratic temperature profile, flux at the end points can be made continuous by using continuity of fluxes. At  $X_0$  condition (12.8.5) is to be used as  $x = x_0$  corresponds to  $x = S(t)$ . Note that latent heat is evolved only at phase-change boundary. The process of obtaining recurrence relation involves lot of manipulations and it will not be discussed here. The time-dependent first-order  $n$ -differential equations are initially singular and short-time solutions are required to start numerical solution for each of them. Comparison of numerical solution with other known solutions has been done.

Instead of subdividing the temperature range as in [612], in [613] the penetration depth or the portion of the melt formed in  $0 \leq x \leq S(t)$  in a melting problem has been subdivided into  $n$  equal parts. The problem considered is a simple one-phase problem in which  $T(0, t) = 1$ ,  $T(x, t) = 0$  on  $x = S(t)$ ,  $t > 0$ ,  $T(x, 0) = 0$ ,  $x \geq 0$ ,  $\partial T / \partial x = -\beta dS/dt$  on  $x = S(t)$ ,  $t > 0$ ,  $\beta = l/C(T_0 - T_m)$ , which is dimensionless Stefan number ( $T_0 = 1$  and  $T_m = 0$ ). Let  $v_i$  be the approximation of  $T(x_i, t)$  in the  $i$ th cell, where  $v_0 = 1$  at  $x = 0$  and  $v_n = 0$  at  $x = S(t)$ . If a linear piecewise approximate temperature profile is chosen, we have

$$v = v_i + n(x - x_i)(v_{i+1} - v_i)/S(t), \quad x_i \leq x \leq x_{i+1}, \quad i = 0, 1, \dots, n-1. \quad (12.8.22)$$

The piecewise constant temperature gradient is defined by

$$\frac{\partial v}{\partial x} = n(v_{i+1} - v_i)/S, \quad x_i \leq x \leq x_{i+1}, \quad i = 0, 1, \dots, n-1; \quad \left. \frac{\partial v}{\partial x} \right|_{S(t)} = -\beta \frac{dS}{dt}. \quad (12.8.23)$$

The heat balance integral equation for each cell results in the following equations.

$$\int_{x_i}^{x_{i+1}} \frac{\partial v}{\partial t} = \left. \frac{\partial v}{\partial x} \right|_{x=x_{i+1}} - \left. \frac{\partial v}{\partial x} \right|_{x=x_i}, \quad i = 0, \dots, n-1. \quad (12.8.24)$$

Evaluating l.h.s. of Eq. (12.8.24) and using Eqs (12.8.22), (12.8.24), we get

$$S \frac{dS}{dt} = (2n^2/(2i+1))(v_{i+2} - 2v_{i+1} + v_i)/(v_i - v_{i+1}), \quad i = 0, 1, \dots, n-2, \quad (12.8.25)$$

$$S \frac{dS}{dt} = 2n^2 v_{n-1}/((2n-1)v_{n-1} + 2n\beta). \quad (12.8.26)$$

As  $S(t)$  is the same in Eqs (12.8.25), (12.8.26), this can be used to obtain a recurrence relation for  $v_i$  and we get after some calculations

$$v_i = v_{n-1} \sum_{r=0}^{n-i-1} \frac{(n+\alpha+1/2)^r}{(\alpha+r)!} \alpha!, \quad \alpha = n\beta/v_{n-1}, \quad i = 1, 2, \dots, n. \quad (12.8.27)$$

$$v_{n-1} \sum_{r=0}^{n-1} \frac{(n+\alpha-1/2)^r}{(\alpha+r)!} \alpha! = 1 \text{ as } v_0 = 1. \quad (12.8.28)$$

Eliminating  $v_{n-1}$  from Eqs (12.8.27), (12.8.28),  $v_i$ ,  $i = 0, 1, \dots, n-1$  are determined. When  $v_{n-1}$  is substituted in terms of  $\alpha$  in Eq. (12.8.28),  $\alpha$  can be obtained from the solution of a nonlinear equation in  $\alpha$ .  $S(t)$  can be obtained from Eq. (12.8.26) and

$$S(t) = 2n\sqrt{t}/\sqrt{2n+2\alpha-1} = 2\alpha^*\sqrt{t}, \quad \alpha^* = n/\sqrt{2n+2\alpha-1}. \quad (12.8.29)$$

It may be noted that the solution is valid for all values of  $\beta$  from 0 to  $\infty$ . Convergence of the solution as  $n \rightarrow \infty$  has also been analysed. Firstly it has been shown that as  $n \rightarrow \infty$ ,  $\left. \frac{\partial T}{\partial x} \right|_{x=0} = -1/(\sqrt{\pi t} \operatorname{erf} \alpha)$ ,  $\alpha = S/(2\sqrt{t})$ , which corresponds to the exact solution. For  $\beta$  (Stefan number)  $> 0$  including  $\beta \approx 1$  and  $\beta$  large, the rate of convergence is  $O(n^{-1})$  and for  $\beta = 0$  it is  $O(n^{-1/2})$ . For  $n \rightarrow \infty$ ,  $v_i = 1 - \operatorname{erf}(x/2\sqrt{t})/\operatorname{erf}(\alpha)$ , which is the exact solution. The rate of convergence is  $O(n^{-1})$ . There is a scope of further refinement by considering mesh refinement and higher-order piecewise approximations. The convergence proof makes it a good and useful work.

When the temperature gradient at  $x = 0$  is not prescribed and approximate temperature distribution is substituted to calculate it, which is required in Eq. (12.8.1), it becomes the main source of error. This point has been addressed in [614] and by double integration of heat equation with respect to  $x$ , the temperature derivative could be removed at  $x = 0$  in Eq. (12.8.1). A one-phase one-dimensional problem of solidification has been considered. The dimensionless formulation is similar to one considered in [613] except  $T(0, t) = F(t)$ ,  $t > 0$ . The double integration of heat equation with respect to  $x$  from 0 to  $x$  and then from 0 to  $S(t)$  results in the following equation (for scalings refer [614]).

$$\int_0^{S(t)} \left( \int_0^x \frac{\partial T}{\partial t} d\xi \right) dx - T(S(t), t) + T(0, t) + S(t) \left. \frac{\partial T}{\partial x} \right|_{x=0} = 0. \quad (12.8.30)$$

Eliminating  $\left. \frac{\partial T}{\partial x} \right|_{x=0}$  between Eqs (12.8.2), (12.8.30) and using the boundary conditions, we get

$$\frac{d}{dt} \left( \int_0^{S(t)} x T(x, t) dx \right) = F(t) - \frac{1}{2S_{te}} \frac{dS^2}{dt}, \quad S_{te} = C(T_m - T_{\text{ref}})/l. \quad (12.8.31)$$

Let the temperature be approximated by a quadratic polynomial of the form

$$T(x, t) = A_1 + A_2(1 - x/S) + A_3(1 - x/S)^2. \quad (12.8.32)$$

$A_1, A_2$  and  $A_3$  are shape functions. It is easy to check that on satisfying the boundary conditions, we get

$$T(x, t) = A(1 - x/S) + (F - A)(1 - x/S)^2; \quad 2S_{te}A = \frac{dS^2}{dt}. \quad (12.8.33)$$

The interface temperature condition determines  $A$  if  $T(x, t)$  from Eq. (12.8.33) is used in it. Using Eqs (12.8.31), (12.8.33), another equation is obtained as

$$(F + A + 6/S_{te}) \frac{dS^2}{dt} + S^2(t) \left( \frac{dA}{dt} + \frac{dF}{dt} \right) = 12F(t). \quad (12.8.34)$$

If  $A$  is used from Eq. (12.8.33) in Eq. (12.8.34) and Eq. (12.8.34) is integrated once, we get

$$\frac{dS^2}{dt} - \frac{24S_{te}}{S^2} \int_0^t F(t) dt = -2(S_{te}F + 6). \quad (12.8.35)$$

Eq. (12.8.35) is singular and for its numerical solution short-time analytical solution is required to start the numerical scheme. When  $F(t) = 1$  then exact analytical solution of Eq. (12.8.35) is available and  $S(t) = 2\lambda\sqrt{t}$  where  $\sqrt{\pi}\lambda \operatorname{erf}(\lambda)e^{\lambda^2} = S_{te}$ .  $A(t)$  can be determined from second equation in Eq. (12.8.33) and  $S(t)$  from Eq. (12.8.35). The obtained solution of the original problem has been compared with other solutions and seems to be a better solution.

In the ablation problem considered in [615], the flux is time dependent and instead of using the standard HBIM solution procedure as in [599], a different solution procedure has been followed. Consider the linear heat equation in  $S(t) \leq x < \infty, t > 0, T(x, 0) = T_0$ , as  $x \rightarrow \infty, T(x = S(t), t) = 1$  and

$$-K(T_m - T_a)\frac{\partial T}{\partial x} + \rho l \frac{dS}{dt} = q_0(t), \quad \text{at } x = S(t), \quad t > 0, \quad (12.8.36)$$

temperature is scaled,  $q_0(t)$  is the flux and  $T_a$  is the ambience temperature. We ignore here the preablation formulation and its solution. A variation of HBIM has been considered for the solution which the authors call *T-Moment Integral Method* which was suggested in [616]. In addition to the conventional heat balance integral equation, one more integral equation has also been obtained as follows:

$$\frac{d}{dt} \int_{S(t)}^{\infty} T^2 dx + \frac{dS}{dt} = -2k \left. \frac{\partial T}{\partial x} \right|_{x=S} - 2k \int_S^{\infty} \left( \frac{\partial T}{\partial x} \right)^2 dx, \quad (12.8.37)$$

$S(t)$  is ablation thickness. The heat equation is multiplied by  $T(x, t)$  and the product is integrated with respect to  $x$  from  $S(t)$  to  $\infty$ .  $T(x, t)$  can be regarded as weight. Approximate temperature is taken in the form

$$T(x, t) = \exp(-(x - S)\delta(t)), \quad \delta(t) \text{ is the thermal penetration depth.} \quad (12.8.38)$$

For  $\delta(t)$  an independent equation is obtained. For calculation of  $S(t)$  a differential equation is obtained which involves  $\delta(t)$ .  $\delta(0)$  is to be obtained from the solution of a preablation problem. Coupled equations are to be solved numerically to obtain  $S(t)$  and  $\delta(t)$ .

In [617], the authors suggested that to determine the solidified crust  $S(t)$  deposited on a cold wall in a transient freezing problem, the average value of lower and upper bounds for  $S^2(t)$  should be used. The most of the formulation of this one-phase one-dimensional problem is as discussed in [612] except that at the interface  $x = S(t)$  an arbitrary heat flux has also been considered.  $S(t)$  is approximated as  $S(t) \approx 2\lambda(t)t^{1/2}$  where  $\lambda(t)$  is the average of the upper and lower bounds of  $S(t)$ . Heat equation is integrated twice with respect to spacial variable as in Eq. (12.8.30) and a quadratic polynomial for temperature has been considered. The wall temperature does change with time in the formulation but it is considered constant in calculating bounds and numerical work.

As in [606], a two-stage solution procedure has been adopted in [618]. A warm fluid is flowing over a cold wall of finite thickness. The liquid is deposited as a solid of thickness  $S(t)$  measured from  $x = 0$  to the right from the wall surface. Another surface of the wall is  $x = -d$ . The temperature of the wall at  $x = 0$  denoted by  $T_{ws}(t)$  is a function of time which is unknown. In the first stage of solution procedure, the penetration depths in solid and the wall are obtained as done in a standard HBIM which will involve  $T_{ws}$ . When the penetration depth in the wall reaches  $x = -d$ , the prescribed boundary temperature at  $x = -d$  is to be

used for the temperature approximation in the wall and so earlier approximation will change. Once again penetration depths are to be obtained in both solid crust and wall by using suitable polynomials.  $T_{ws}(t)$  for the first and second stage of solutions will be different. If the first stage is for  $0 \leq t \leq t^*$ , then for  $t > t^*$ , the expression of temperature approximation in the solid crust does not change as the boundary conditions remain the same but solution obtained will be different. Dimensionless equations have been considered. Coupled system of differential equations is to be solved numerically to obtain  $S(t)$ .

The solidification problem considered in [619] has the following formulation in dimensionless form.

$$\left. \begin{aligned} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, 0 < x < S(t), t > 0; \quad -\frac{\partial S}{\partial t} = N_1 \frac{\partial T}{\partial x} \Big|_{x=S}, t \geq 0, \\ T(S(t), t) = 0; \quad S(0) = 0. \end{aligned} \right\} \quad (12.8.39)$$

$$\frac{\partial T}{\partial x} \Big|_{x=0} = N_2 (T|_{x=0} - 1), N_1 \text{ is Stefan number and } N_2 \text{ is Biot number.} \quad (12.8.40)$$

Instead of approximating temperature as a polynomial, it is taken as

$$T(x, t) = a_1(t)f_1(x, t) + a_2(t)f_2(x, t); f_1(S(t)) = f_2(S(t)) = 0, \quad (12.8.41)$$

$$\frac{\partial f_1}{\partial x} \Big|_{x=0} = \frac{\partial f_1}{\partial x} \Big|_{x=S}; \quad \frac{\partial f_2}{\partial x} \Big|_{x=S} = 0; a_1, a_2 \text{ are unknowns.} \quad (12.8.42)$$

The heat balance integral of the heat equation gives

$$\dot{y} = a_2 h_2, \quad h_2 = -\frac{\partial f_2}{\partial x} \Big|_{x=0}; \quad y = \int_0^{S(t)} T dx. \quad (12.8.43)$$

If Eq. (12.8.40) is taken into consideration then

$$a_1 = (\dot{y}g_2 - N_2 h_2)/h_2 g_1, g_1 = \frac{\partial f_1}{\partial x} \Big|_{x=0} - N_2 f_1|_{x=0}, g_2 = N_2 f_2|_{x=0} - \frac{\partial f_2}{\partial x} \Big|_{x=0}. \quad (12.8.44)$$

The pseudo-steady-state solution  $PSS(S(t))$  can be easily obtained for the above problem. The velocity of  $S(t)$  is a function of  $S(t)$  which follows from interface condition. This implies that the asymptotic pseudo-steady-state solution of transient problem is preserved. It can be shown that

$$PSS(S(t)) \frac{dS}{dt} = 1 - \dot{y}g_2/N_2 h_2; \quad \frac{dS}{dt} = -a_1 N_1 \frac{\partial f_1}{\partial x} \Big|_{x=S}. \quad (12.8.45)$$

After some calculations, we get

$$T(x, t) = -N_2 \bar{f}_1 / (g_1(1+w)) + w / (1+w) \cdot \bar{f}_2 N_2 / g_2, (\bar{f}_1, \bar{f}_2) = \int_0^S (f_1, f_2) dx. \quad (12.8.46)$$

$$a_1 = -N_2 / (g_1(1+w)), a_2 = w N_2 / (g_2(1+w)); w = (f_1 - y) / (y - f_2). \quad (12.8.47)$$

$y$  can be written as weighted average of two functions  $\Phi_1$  and  $\Phi_2$  where

$$\Phi_1 = -N_2 \bar{f}_1 / g_1 \text{ and } \Phi_2 = N_2 \bar{f}_2 / g_2; T = (\Phi_1 + w\Phi_2) / (1 + w). \quad (12.8.48)$$

As  $N_1 \rightarrow 0$ ,  $y \rightarrow \Phi_1$  and  $w \rightarrow 0$ ; as  $N_1 \rightarrow \infty$ ,  $y \rightarrow \Phi_2$  and  $w \rightarrow \infty$ .

Some of the highlights of the above solution procedure in [619] are: (1) Polynomial approximations of temperatures or other approximating functions have to satisfy conditions given in Eqs (12.8.41), (12.8.42) and there is no need of conditions of the form Eq. (12.8.4). (2) Temperature can be expressed as weighted average of two functions and weights have interpretation as bounds in some special cases. When  $T = \Phi_1$ ,  $w = 0$  and  $N_1 = 0$ . When  $T = \Phi_2$ ,  $w \rightarrow \infty$  and  $N_1 \rightarrow \infty$ . This helps in easyness of analytical-numerical solutions of the present problem as the solution of the heat equation can now be written as in Eq. (12.8.41) in which  $f_1$  and  $f_2$  are the solutions of two simple problems as given below.

$$\frac{\partial^2 f_1}{\partial x^2} = 0; f_1(S(t)) = 0; \left. \frac{\partial f_1}{\partial x} \right|_{x=0} = N_2(f_1|_{x=0} - 1), 0 \leq x \leq S(t), t \geq 0. \quad (12.8.49)$$

$$\frac{\partial f_2}{\partial t} = \frac{\partial^2 f_2}{\partial x^2}; f_2(S(t)) = 0 = \left. \frac{\partial f_2}{\partial x} \right|_{x=0} = N_2(f_2|_{x=0} - 1). \quad (12.8.50)$$

$a_1(t)$ ,  $a_2(t)$  and  $S(t)$  are to be determined using Eqs (12.8.45), (12.8.48). Analytical and numerical solutions of some simple problems by taking simple form of  $f_1$  and  $f_2$  have been compared.

In [620], an iterative solution procedure has been discussed for the following problem in dimensionless form.

$$\left. \begin{aligned} P(x) \frac{\partial T}{\partial t} &= \frac{\partial^2 T}{\partial x^2}, 0 < x < S(t), t > 0; T(0, t) = 1.0, \\ T(S(t), t) &= 0 \text{ on } x = S(t), \left. \frac{\partial T}{\partial x} \right|_{x=S} = -\alpha f(S) \dot{S}. \end{aligned} \right\} \quad (12.8.51)$$

$\alpha$  is inverse Stefan number,  $P(x)$  is known and  $f(S)$  is some function of phase-change boundary  $x = S(t)$ . Let  $G(x)$  be some function such that  $G(x)$ ,  $G'(x)$ ,  $G''(x)$  are integrable in the interval  $t > t^*$ . Multiply the heat equation (12.8.51) by  $G(x)$  and integrate with respect to  $x$  twice between  $x = x$  to  $x = S$ . We get an equation for  $T(x, t)$  which does not satisfy the condition  $T(0, t) = 1.0$ . Next, multiply the heat equation by  $G(x)$  and integrate once from  $x = x$  to  $x = S$  and then from  $x = 0$  to  $x = S$ . We get another equation for  $T(x, t)$  which satisfies the condition  $T(0, t) = 1.0$ . Let  $T_1(x, t)$  and  $T_2(x, t)$  be the two solutions of these two integrated equations. It can be shown that  $T_1(x, t) = T_2(x, t)$  at both  $x = 0$  and  $x = S(t)$  provided  $G(0)$  satisfies the following equation

$$\left. \begin{aligned} G(0) &= \frac{\partial}{\partial t} \int_0^S \xi P(\xi) G(\xi) T_1(\xi, t) d\xi - \int_0^S \xi T_1 G''(\xi) d\xi \\ &\quad - 2 \int_0^S T_1 G'(\xi) d\xi + G(S) f(S) \alpha \dot{S}, f(S) \text{ is known.} \end{aligned} \right\} \quad (12.8.52)$$

Let  $G(x) = A_0 + A_1 x + A_2 x^2 + \dots + A_n x^n$ ,  $A_n \neq 0$ . Multiply heat equation by this above polynomial of  $G(x)$  and perform double integrations on this product from  $x = x$  to  $x = S$ .

When the coefficient of  $A_0, A_1$ , etc., are collected, we get moments as first moment, second moments, etc. When  $P(x) = f(S) = 1$ , the equation obtained by taking first moment is

$$T_1(x, t) = \alpha S \dot{S} (1 - x/S) + \frac{\partial}{\partial t} \left[ x \int_S^x T_1 d\xi - \int_S^x \xi T_1 d\xi \right]. \quad (12.8.53)$$

When  $T(0, t) = 1.0$  is used in Eq. (12.8.53) we get

$$1 = \alpha S \dot{S} + \frac{\partial}{\partial t} \int_0^S \xi T_1 d\xi. \quad (12.8.54)$$

If the notations used in Eqs (12.8.39)–(12.8.50) are followed then for  $\alpha \rightarrow 0$ , the dimensionless time scale  $\tau/\alpha$  is diffusive time scale and the solution is  $f_2(x, t)$  in which heat equation is to be considered as transient and solution is obtained by considering appropriate formulation. When  $\alpha \rightarrow \infty$ , the dimensionless time scale is latent heat-controlled time scale and  $f_1(x, t)$  is solution of a steady-state heat equation with appropriate formulation.  $f_1(x, t)$  and  $f_2(x, t)$  can be calculated with the help of formulation given in Eqs (12.8.49), (12.8.50). The iterative solution procedure is given below.

$$\left. \begin{aligned} T_1(x, t) &= a_1(t)f_1(x, t) + a_2(t)f_2(x, t), \quad 0 \leq x < S(t) \\ a_1 &= \alpha S \dot{S} \leq 1, \quad 0 \leq a_2 = 1 - a_1 \leq 1, \end{aligned} \right\} \quad (12.8.55)$$

$$f_1(x, t) = (1 - x/S); \quad f_2^{n+1} \cdot \left( \frac{\partial}{\partial t} \int_0^S \xi f_2^n d\xi \right) = \frac{\partial}{\partial t} \left[ x \int_S^x f_2^n d\xi - \int_S^x \xi f_2^n d\xi \right], \quad (12.8.56)$$

$$T_1(x, t) = \alpha S \dot{S} (1 - x/S) + (1 - \alpha S \dot{S}) f_2^{n+1}. \quad (12.8.57)$$

The superscript  $n$  in Eq. (12.8.56) stands for the  $n$ th iterate. The approaches in [619, 620] are new and not discussed in other problems. Numerical work was done for planar and spherical problems also.

The solidification problem considered in [621] is related to a casting problem in which a molten liquid in a roller occupying the region  $0 \leq x \leq H$  is being chilled at both the surfaces  $x = 0$  and  $x = H$ . The cooling at  $x = 0$  is due to prescribed flux and at  $x = H$  it is due to convective heat transfer. Initially, the temperature of the melt is  $T_0 > T_m$ ,  $T_m$  is the freezing temperature. Period I of cooling is time taken for the temperature at  $x = 0$  to come down to  $T_m$  so that at the end of period I the solidification starts at  $x = 0$ . Let  $S_1(t)$  be the conduction penetration depth in a heat conduction problem during the period I. A conduction penetration depth also exists at the end  $x = H$  during period one which will contribute in period II.

Period II consists of heat flow zones of different lengths. Zone 1 has length  $S_2(t)$  which is solid thickness, i.e. the region for  $0 \leq x \leq S_2(t)$ . Zone 2 consists of the liquid region  $S_2(t) \leq x \leq S_1(t)$ .  $S_1(t)$  is conduction penetration depth due to cooling at  $x = 0$ . Zone 3 consists of the region,  $S_1(t) \leq x \leq S_3(t)$ ,  $S_3(t)$  is the conduction penetration depth resulting from the convection at surface  $x = H$ . Zone 3 is at the temperature  $T_0$ . Zone 4 consists of the region,  $S_3(t) \leq x \leq H$ . In period III, the two conduction fronts  $S_1(t)$  and  $S_3(t)$  have met, i.e. the region with temperature  $T_0$  has disappeared. Period III will have only three zones. Up to period III solidification does not start at  $x = H$  as per assumptions made in [621].  $S_1(t)$ ,  $S_2(t)$  and  $S_3(t)$  are zone moving boundaries.

This problem is far more complicated than the problems considered earlier. The highlight of the formulation is that in each period the initial temperature is changing in which

temperature gradients are present. We present below the heat balance integral equations for all three periods.

$$k \left[ \frac{d}{dt} \int_{S_{i-1}}^{S_i} T dx - T(S_i) \frac{dS_i}{dt} + T(S_{i-1}) \frac{dS_{i-1}}{dt} \right] = T_x(S_i) - T_x(S_{i-1}), \quad i = 1, 2, 3. \quad (12.8.58)$$

In each zone, a quadratic polynomial for approximate temperature is considered as given below.

$$T = A + B\eta_i + C\eta_i^2, \quad \eta_i = (x - S_{i-1})/(S_i - S_{i-1}), \quad i = 1, 2, 3. \quad (12.8.59)$$

Eqs (12.8.58), (12.8.59) have been made dimensionless and for scalings and assumptions made in the formulation, refer [621]. The formulation in each phase and obtaining the coupled system of differential equations for the moving boundaries is a lengthy process and it cannot be discussed here in detail. In short, start from period I and use Eq. (12.8.59) in Eq. (12.8.58). These solutions for temperatures in period I can be obtained by penetration depth method. For period II, a coupled system of integral equations is obtained when Eq. (12.8.59) is used in Eq. (12.8.58). Whenever phase change is taking place Stefan condition should be used.  $S_3(t)$  is obtained by penetration depth method. Similar procedure is adopted for period III and solutions in different zones can be obtained. To start numerical solution of the system of differential equations, approximate analytical solutions are required as the equations are singular at initial time step. Numerical work was done for all the three phases.

A two-phase solidification in one-dimensional radially symmetric spherical problem has been considered in [622]. When kinetic undercooling is considered in the solidification of supercooled drops, several laws have been proposed in the literature for the velocity of the solidification front which are based on experimental and theoretical considerations. In these laws, a ‘solidification wave speed parameter’, which we call  $A$  is involved. The main concern is to calculate parameter  $A$  by taking it unknown. The interface velocity can be calculated using Stefan condition and another equation for this velocity comes from the velocity formula proposed in [623] in which  $A$  for the present problem is taken as unknown. The cooling is taking place due to both radiation and convection at the outer surface. By considering presolidification formulation and postsolidification formulations for solid and liquid phases, analytical-numerical solution has been obtained for moving boundary using HBIM which also determines the parameter  $A$ .

For assumptions, formulation, scalings, approximations of temperatures and solution, refer [622]. The parameter  $A$  has been determined for  $\text{Al}_2\text{O}_3$  and  $\text{ZrO}_2$  drops.

Choosing the approximating polynomials in HBIM depends on many factors such as geometry of the problem, initial, boundary and interface conditions, extent of the region, coupling within the regions, etc. In every work there is some variety in scalings, boundary conditions, region under consideration, assumptions, etc. It is not possible to mention all these details. The reader if interested in these type of problems and their solutions using HBIM should refer original works after getting some feedback from the details presented here.

A variation of standard HBIM has been considered in [624] in connection with a simple freezing melting problem of one-phase considered in the region  $0 \leq x \leq S(t)$ . The formulation is as follows:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < S(t), \quad t > 0; \quad T(0, t) = F(t), \quad t > 0; \quad T(x, 0) = 1. \quad (12.8.60)$$

$$\left. \frac{\partial T}{\partial x} \right|_{S(t)} = \frac{1}{S_{te}} \frac{dS}{dt}; \quad T(S(t), t) = 1, S(0) = 0, S_{te} \text{ is the Stefan number.} \quad (12.8.61)$$

The heat equation can also be expressed in terms of a function  $v$  as given below.

$$\frac{\partial T}{\partial x} = v(x, t); \quad \frac{\partial v}{\partial x} = \frac{\partial T}{\partial t}. \quad (12.8.62)$$

Integrating both the equations in Eq. (12.8.62) from 0 to  $S(t)$  with respect to  $x$ , after incorporating the boundary and interface conditions, we get

$$1 - F(t) = \int_0^{S(t)} v(x, t) dx; \quad \frac{1}{S_{te}} \frac{dS}{dt} - v(0, t) = \frac{d}{dt} \left[ \int_0^{S(t)} T(x, t) dt \right] - \frac{dS}{dt}. \quad (12.8.63)$$

If the following approximation is done

$$\int_0^{S(t)} y dx \approx \frac{S}{2} [y(0) + y(S)], \quad (12.8.64)$$

then it is easy to get

$$\frac{1}{2S_{te}} \left[ \left( 1 + \frac{S_{te}}{4} \right) - \frac{S_{te}}{4} F(t) \right] \frac{dS^2}{dt} - \frac{1}{4} \frac{dF}{dt} S^2(t) = 1 - F(t). \quad (12.8.65)$$

From Eq. (12.8.65),  $S^2(t)$  or  $S(t)$  can be obtained on integration. The analytical solutions of  $S(t)$  when  $F(t)$  is constant and when  $F(t)$  is a linear function of time have been compared with the numerical solution.

A problem somewhat similar to that studied in [609] has been considered in [625]. The quasisteady-state solution has been studied using the standard HBIM. Time dependence has not been considered in heat transfer. With the help of defining suitable convection heat transfer rate and making several assumptions the melting rates in the melting problem have been calculated. In the two-dimensional quasistate case, the objective is to obtain flow due to natural convection in the thermal boundary layer in the liquid near the wall and in the cold boundary layer near the unmelted solid.

A solid having constant temperature is initially occupying the region  $-\infty < x \leq 0, 0 \leq y \leq H$  and is melting due to prescribed temperature at the wall  $x = 0$  whose temperature remains constant. To obtain this thermal boundary layer  $\lambda(t)$ ,  $\lambda(t) \geq 0$  near the heated wall which is advancing towards negative  $x$ -direction, i.e. on the left horizontally, the coordinate system is fixed at the wall. To obtain flow due to natural convection in the cold boundary layer  $x = \delta(t)$ ,  $0 \leq x \leq \delta(t)$  near the solid, the coordinate system is fixed at the solid-liquid interface. The mass conservation, momentum and energy equations are given below in the transformed coordinates.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \quad u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = v \frac{\partial^2 v}{\partial x^2} + g\beta(T - T_C); \quad u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = k \frac{\partial^2 T}{\partial x^2}, \quad (12.8.66)$$

$v$  is kinematic viscosity and  $\beta$  is coefficient of thermal expansion.  $T_C(y)$  is the temperature of the core region.  $y$ -axis is taken vertically upwards.



As mentioned earlier there is no fixed rule for choosing approximations for temperature and velocities and although conceptually HBIM remains the same, the details of solution procedure change from problem to problem and it is not possible to give these details. We give here just some hints for the solution details. To obtain the cold boundary layer  $\delta(t)$  in the region  $0 \leq x \leq \delta(t)$ , the following approximations have been considered in HBIM solution.

$$T = T_C \frac{x}{\delta} \left( 2 - \frac{x}{\delta} \right); v = V(y) \frac{x}{\delta} \left( 1 - \frac{x}{\delta} \right)^2; \frac{\partial^2 v}{\partial x^2} = \frac{g\beta}{\nu} (T_C - T), \quad (12.8.67)$$

$T_C = T_C(y)$  is the temperature of the core region between  $\delta(t)$  and  $\lambda(t)$ ,  $T_C(y)$  is unknown. When heat energy equation in Eq. (12.8.66) is integrated to use HBIM, it involves  $u|_{x=\delta} = U_\delta$  and  $u|_{x=0} = u_0$ .  $u_0$  is proportional to flux at  $x = 0$  and  $U_\delta$  which is entrainment velocity is obtained from mass conservation equation (cf. [625]).

For the warm boundary layer, the coordinate system is taken as  $(x_r, y)$ ,  $x_r$  is fixed at the wall horizontally and  $y$ -axis is perpendicular to  $x$ -axis upwards. The approximations of temperature and velocity are taken as

$$T_r = (\Delta T - T_C) \left( 1 + \frac{x_r}{\lambda} \right)^2 + T_C; v_r = -\frac{g\beta}{3\nu} (\Delta T - T_C) \lambda^1 \left( 1 + \frac{x_r}{\lambda} \right)^2, \quad (12.8.68)$$

$\Delta T$  is the temperature difference between the wall and the interface. On applying HBIM, the equation involves  $(u_r)_{-\lambda}$  and  $(u_r)_0$  which can be determined on the basis of conservation laws (cf. [625]). The temperature  $T_C(y)$  is the same at  $x = \delta(t)$  and at  $x_r = -\lambda(t)$ . This will give third equation to determine  $\delta(t)$ ,  $\lambda(t)$  and  $T_C(y)$ . For further details refer [625]. Numerical results are reported.

In the reference [626], the developments concerning modelling and approximate solutions of drug release from matrix systems have been reviewed. The topics covered are approximate analytical solutions using HBIM and other methods in planar and spherical matrix systems, heterogeneous matrix system, finite release medium, surface erodible matrix, finite dissolution rates, etc. Let  $A$  be the initial drug loading or in simple words, let  $A$  be the initial concentration of the drug (the amount of loading per unit volume) in the drug release matrix. Let  $\bar{C}$  be the drug solubility in the vehicle matrix. Drug release in a planar matrix can be modelled as given below in which  $\bar{C}_S$  is considered equilibrium drug solubility.

$$\left. \begin{aligned} \frac{\partial \bar{C}}{\partial t} &= D \frac{\partial^2 \bar{C}}{\partial x^2}; \quad 0 \leq x \leq a; \quad \bar{C}(a, t) = 0; \quad S(0) = a, \\ D \frac{\partial \bar{C}}{\partial x} \Big|_{S(t)} &= (A - \bar{C}_S) \frac{dS}{dt}, \quad t \geq 0, \quad \bar{C}(S(t), t) = \bar{C}_S, A \gg \bar{C}_S. \end{aligned} \right\} \quad (12.8.69)$$

$D$  is diffusion coefficient,  $x = S(t)$  is shrinking boundary of the initial boundary  $x = a$  of the drug release matrix which is in contact with the vehicle matrix. For obtaining the solution using HBIM, double integration with respect to  $x$  is done which has been also done in [601, 614], etc. This double integration method was originally proposed in [627]. Note that  $\bar{C}_S$  is unknown but when  $\bar{C}$  and  $S(t)$  are determined,  $\bar{C}_S = C(S(t), t)$  is obtained.

The most important quantity in these problems is the total amount  $M$  of drug depleted or released per unit area which is given by Fick's law. In planar matrix problem stated above

$$\frac{dM}{dt} = \frac{D \bar{C}_S}{S(t)}, \quad M = (1 + P) / \left( \sqrt{3P} \right) \left[ \bar{C}_S \sqrt{Dt} \right], \quad P = 5A/\bar{C}_S - 4 + \sqrt{\left( (A/\bar{C}_S)^2 - 1 \right)}. \quad (12.8.70)$$

The above result for  $M$  in the planar matrix case has been obtained using standard HBIM and double integration of diffusion equation. An approximate value of  $M$  can be obtained as  $M = \sqrt{(2A - \bar{C}_S)\bar{C}_S D t}$  by doing mass balance and using Fick's law.

In the one-dimensional spherical problem, the value of  $M$  in quasisteady-state case has been obtained by some research workers using HBIM and other methods which may be found in [626]. There are several types of drug release systems and solubility criteria for which refer [626].

*Direct selective laser sintering (SLS)* is an emerging technology of *solid freedom fabrication* via which three-dimensional parts are built from metal-based powder bed with CAD data [628]. A fabrication layer is created by selectively fusing a thin layer of the powder with scanning laser beam. After sintering of a layer, a new layer of the powder is deposited in the same manner and a three-dimensional part can be built in a layer-by-layer process.

The problem under investigation in [629] is the melting of a mixture of two metal powders by considering a two-phase problem in which solid and liquid regions are separated by a phase-change boundary  $x = S(t)$ . The powder with high melting point will never melt in the entire process. Due to melting of the low melting point powder, which rests above the high melting point powder, there is shrinkage of the region and the liquid surface acquires a velocity  $\dot{S}_0(t)$ . The initial extent of the solid powders is between  $0 \leq x \leq H$ , in which  $x$ -axis is taken vertically downwards and heat flux  $q''$  is applied at  $x = 0$ . There is a preheating period of time during which the surface  $x = 0$  attains the melting temperature  $T_m$  at  $x = 0$ . After the melting starts, flux  $q''$  is applied at  $x = S_0(t)$ . The formulations of the problems for the premelting duration and then in the melt and solid are not much different from the formulations in the earlier problems discussed here in the context of other problems. Therefore, the formulations, assumptions and dimensionalization will not be repeated here. However, anything which is new in [629] will be presented. A more general problem of this type has been considered later.

Let  $w$  be the velocity of the liquid which arises due to shrinkage.

$$w = \frac{dS_0}{dt} = \frac{\varepsilon_S - \varepsilon_L}{1 - \varepsilon_L} \frac{dS}{dt}, \quad \varepsilon \text{ is the porosity of unsintered powder.} \quad (12.8.71)$$

Eq. (12.8.71) is obtained by doing conservation of mass at  $x = S(t)$ . In the Stefan condition at  $x = S(t)$ , the discontinuity in the fluxes will be equal to  $(1 - \varepsilon_S)\varphi\rho_L l \frac{dS}{dt}$ .  $\varphi$  is the volume fraction of low melting point powder in the mixture. If  $\beta = q''H/(2K_S(T_m - T_i)) > 1$ , where  $q''$  is applied flux and  $T_i$  is the initial temperature then heating is fast and before the thermal penetration depth reaches the bottom of the powder, the melting starts. When  $\beta < 1$ , the time taken for melting to start takes longer, meanwhile the thermal penetration depth has reached the bottom of the mixture at  $x = H$ . The solution procedure for  $\beta < 1$  is slightly different from  $\beta > 1$ . This is because for  $\beta < 1$  at the time  $t = \bar{t}_m$  the penetration depth has reached  $x = H$  but at  $x = 0$  the temperature is not the melting temperature  $T_m$ .

For  $\beta < 1$ , a standard HBIM can be used for solutions of temperature and thermal penetration depth  $\delta(t) = H$ . When  $\delta(t) = H$ , time  $\bar{t}_m$  can be calculated and in the temperature approximation used in HBIM, for time  $t = \bar{t}_m$  can be substituted. Note that the approximate temperature profile should be chosen in such a way that when  $\delta(t) = H$ , the no flux condition prescribed at  $x = H$  is satisfied by the temperature. Now this approximate temperature at  $t = \bar{t}_m$  ( $\bar{t}_m$  is different from  $t_m$  for  $\beta > 1$ ) will serve as the initial temperature for calculating next phase of premelting solution for  $\beta < 1$ . When the time  $\bar{t}_m$  has been calculated so that the boundary condition at  $x = H$  is also satisfied in the solution, we are ready for the solution

for  $t > \bar{t}_m$ . This is now the solution of the standard heat conduction problem which gives us the time  $t = \bar{\bar{t}}_m, \bar{\bar{t}}_m > \bar{t}_m$  at which the melting starts at  $x = 0$ . For  $t > \bar{\bar{t}}_m$ , a two-phase melting problem is to be studied and such a solution has been discussed earlier. If the HBIM is not used for premelting time and some other method is used then this solution process is not required as  $t = \bar{\bar{t}}_m$  can be directly calculated. For  $\beta > 1$ , penetration depth method can be used directly for premelting time. When temperature for liquid is obtained it contains the unknowns  $S_0$  and  $S(t)$  and when solid temperature is obtained it contains  $S(t)$  as unknown. The condition (12.8.71) provides an additional condition. The solution procedure to determine unknowns is a standard one.

The two ranges of  $\beta$ ,  $\beta > 1$  and  $\beta < 1$  were suggested in [630]. Considerable numerical work is reported in [629] for different parameters and compared with some numerical solutions. For solution refer [629].

A problem of direct SLS has been considered in [631] with some difference from the problem in [629]. Applied flux is of *Gaussian type*, i.e.  $q''(t) = q_0'' \exp(-t^2/t_p^2)$ , where  $t_p$  is the half width of the laser beam pulse at  $1/e$ . When the flux is initially maximum at its peak, the powder which is occupying the region  $x \geq 0$ ,  $x$ -axis is taken vertically downwards, gets heated. At  $t = t_m$ , the surface  $x = 0$  attains the melting temperature  $T_m$ . For  $-\infty < t < t_m$ , it is premelting time period and temperature in the solid and  $t_m$  can be calculated using a standard HBIM. Initial temperature of the mixture is taken constant and as  $x \rightarrow \infty$ ,  $T_S \rightarrow T_i$ ;  $t > -\infty$ . As considered in [629], the powder starts melting for  $t \geq t_m$  and liquid acquires a velocity  $w = dS_0/dt$  due to shrinkage.

At the upper surface of the liquid which is  $x = S_0(t)$ , Gaussian flux is acting. As the heat flux is Gaussian it decreases after some time, and the melting will stop say at  $t = \bar{t}_S$ . For  $t_m < t < \bar{t}_S$  and  $\beta < 1$  ( $\beta$  as in [629]) it is a two-phase liquid-solid phase-change problem in which liquid occupies the region  $S_0(t) \leq x \leq S(t)$  and solid occupies the region  $x > S(t)$ . The formulation for this problem is of standard type. Condition (12.8.71) and Stefan condition as in [629] are to be used. Solution has been obtained in [631] using standard HBIM. Quadratic polynomials for approximate temperatures have been considered. When melting stops,  $w = 0$  and  $dS/dt = 0$ . From this condition  $\bar{t}_S$  can be calculated. For  $t > \bar{t}_S$ , resolidification of melted liquid starts and now a three-phase problem has to be solved. In the powder region for  $x > S(\bar{t}_S)$ , it is only heat conduction without phase change whose formulation is coupled at  $x = S(\bar{t}_S)$  with the liquid region whose only upper portion is a resolidified solid. Let  $T_R$  be the temperature of the resolidified solid. The regions of  $T_R$  and liquid are joined at the new phase-change boundary, say  $x = S_R(t)$ . The liquid region for  $t > \bar{t}_S$  is between  $S_0(\bar{t}_S) < x < S_R(\hat{t})$  and solid region for  $T_R$  is between  $S_R(\hat{t}) < x < S(\bar{t}_S)$ ,  $\hat{t} > \bar{t}_S$ . Standard HBIM has been used for obtaining solution in which quadratic polynomials have been considered for temperatures.

For complete formulations, scalings and assumptions refer [631]. Considerable numerical work has been reported for temperatures and moving boundaries for different parameters.

A supercooled water spray is impacting on a substrate whose temperature is below the freezing temperature. The water freezes immediately. In mild temperatures or when the solidified layer is sufficiently thick, a fluid layer may subsequently appear at the top. This type of accretion problem was considered in [436] and a perturbation solution for quasisteady-state was developed for a three-phase problem of heat transfer in substrate, rime ice and water with phase change at rime ice boundary.

In [632], a two-phase problem of rime ice and water has been formulated and both perturbation and HBIM solutions have been obtained and compared with the numerical

solution obtained with the help of semiimplicit finite difference moving grid method [68]. We discuss below only some important points in the solution procedure.

The temperature of the substrate remains constant. Let  $x = 0$  be the substrate surface and  $x$  be taken vertically upwards. The rime ice occupies the region  $0 \leq x \leq S(t)$ , where  $x = S(t)$  is rime ice-water phase-change boundary and water occupies the region  $S(t) \leq x \leq S(t) + h(t)$ ,  $h(t)$  is water head. If it is assumed that as soon as the supercooled drop of water falls on substrate, it becomes solid then for some  $t_m$ ,  $0 \leq t \leq t_m$ , there is only rime ice and no water. For  $0 \leq t \leq t_m$ , there is no water layer over ice which implies  $T_I(S(t_m), t_m) = T_m$ , where  $T_I$  is ice temperature and  $T_m$  is freezing temperature. For  $t > t_m$ , glaze ice water may form. Firstly only rime ice solution has been obtained in [632] by using perturbation method and HBIM. The mass balance equation at the rime ice and water interface is given by

$$\rho_I \frac{dS}{dt} + \rho_W \frac{dh}{dt} = \frac{dm}{dt}, \quad \frac{dm}{dt} \text{ is the rate at which mass enters,} \quad (12.8.72)$$

$\rho_I$  and  $\rho_W$  are the densities of ice and water, respectively. If  $\dot{m} = dm/dt$  is constant and only rime ice is considered then  $S(t) = \dot{m}t/\rho_I$ .

The boundary conditions at  $x = S(t)$  and  $x = S(t) + h(t)$  are of similar type (with only few parameters) as those given in Eqs (12.2.570)–(12.2.572). A quadratic polynomial for temperature in rime ice with coefficients as functions of time is considered. This polynomial is used in the integral of heat equation in HBIM. All the three unknown coefficients in temperature can be determined with the help of boundary conditions. For rime ice, at  $x = 0$  the temperature is that of substrate which is constant and at  $x = S(t) = \dot{m}t/\rho_I$ , the temperature derivative is prescribed at  $x = S(t)$ . Once temperature  $T_I$  in the rime ice is obtained, the time  $t_m$  at which  $T_I(S(t_m), t_m) = T_m$  can be determined and  $\psi(t) = \int_0^{S(t)} T_I(\xi, t) d\xi$  can be obtained which will be used later.

For  $t > t_m$ ,  $S(t)$  will be different from earlier  $S(t)$  which was for  $0 \leq t \leq t_m$  but new  $S(t)$  and  $h(t)$  still satisfy Eq. (12.8.72). The quadratic polynomial for temperature remains the same but coefficients will have different solutions for  $t > t_m$ . A suitable quadratic polynomial for  $T_W$  which is the temperature of water can also be considered. Two isotherm conditions at new  $S(t)$  are available for temperatures  $T_I$  and  $T_W$  and two boundary conditions, one at  $x = 0$  for  $T_I$  and another at  $x = S(t) + h(t)$  for  $T_W$ , are prescribed for all  $t$ . There is phase change and so Stefan condition is prescribed at rime ice and water common boundary. If  $a_2(t)$  is the coefficient of temperature of rime ice for  $t > t_m$  in the quadratic term (there are two more unknown coefficients) and  $C_2(t)$  is the similar coefficient in the expression of water temperature (with two more unknown coefficients) then the integral equations obtained by the integrals of heat equations in HBIM can be used to determine them. It is assumed here without further details that using the prescribed boundary conditions and if required generating extra boundary conditions, the other two unknown coefficients in each temperature are determined so that only one unknown coefficient remains in each temperature, i.e.  $a_2(t)$  and  $C_2(t)$ . But initial values of  $a_2(t)$  and  $C_2(t)$  at  $t = t_m$  are not known. Therefore these coefficients  $a_2(t)$  and  $C_2(t)$  are expressed in terms of  $\psi(t) = \int_0^{S(t)} T_I(\xi, t) dt$  and  $\varphi(t) = \int_{S(t)}^{S(t)+h(t)} T_W(\xi, t) d\xi$ . The initial values of  $\psi(t)$  and  $\varphi(t)$ , at  $t = t_m$ , are known.  $\psi(t)$  and  $\varphi(t)$  can be obtained numerically by solving coupled first-order differential equations for  $\psi$  and  $\varphi$ . The differential equations are obtained using Eq. (12.8.72) and Stefan condition at  $x = S(t)$ . For further details refer [632]. The scalings of various quantities have been done in [632] which have not been used in the above discussion. The introduction of  $\psi(t)$  and  $\varphi(t)$  is a good interesting idea.

### 12.8.4 Optimal Value of Exponent in a Polynomial Representation of Temperature in HBIM: Refinement RIM

The application of HBIM or refined HBIM, which will be called *RIM* in the subsequent discussions, requires an approximate expression for temperature in the form of a polynomial of degree  $n$ ,  $n \geq 1$ . The question which has been asked in [633] is: What is the best choice of exponent  $n$ ? This question has been addressed in the context of problems whose exact solutions are available for comparison with the solutions obtained by working out the approximate solutions in the context of HBIM and RIM. Now in further discussion by the method HBIM we mean standard HBIM.

In [633, 634], the earlier attempts in the direction of choice of exponent  $n$  have also been reviewed. It has been proposed in [634] that with correct scalings the standard approximating polynomial in heat transfer problems without phase change could be taken as

$$T(x, t) = a_n(1 - x/\delta)^n, \quad \delta(t) \text{ is the thermal penetration depth.} \quad (12.8.73)$$

Most authors assume  $n$  to be an integer. The solution method proposed in [633] optimizes the exponent  $n$  in Eq. (12.8.73) by minimizing the error. The RIM proposed in [626, 627] (different from the one proposed in [633]) and also used in [633] consists of performing double integration of linear heat equation in  $(x, t)$  with respect to  $x$ , first integration from 0 to  $x$  and second from 0 to  $\delta(t)$ . Initially for illustration and testing, simple heat conduction problems in the region  $0 \leq x < \infty, t > 0$  have been considered in [633] in which temperature, flux and convective-type boundary conditions could be prescribed at  $x = 0$ . Approximate temperature distribution in Eq. (12.8.73) is taken for obtaining solution in which the exponent  $n$  is an integer and not a function of time. The temperature given in Eq. (12.8.73) with  $a_n = 1$  is substituted in the linear heat equation and integrations are performed in HBIM and RIM (double integration). When  $\delta(t)$  is taken proportional to  $t^{1/2}$ , we get  $\delta(t) = \sqrt{2n(n+1)t}$  and  $\delta(t) = \sqrt{(n+1)(n+2)t}$ , in HBIM and RIM, respectively.

To obtain optimal value of  $n$ , a measure of accuracy is to be used. In [611], the error in HBIM is defined as

$$E_n(t) = \int_0^{\delta(t)} \left[ \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} \right]^2 dx \geq 0, k = 1 \text{ is taken after scaling.} \quad (12.8.74)$$

To find the optimal value of  $n$ ,  $E_n(t)$  is to be minimized, On evaluation of r.h.s. in Eq. (12.8.74), we get  $E_n(t) = e_n t^{-3/2}$  (this is possible as exact solutions are available) where  $e_n = e_n(n)$  are very complicated expressions in both HBIM and RIM which may have several minimum. Since  $\partial^2 T / \partial x^2$  at  $\delta(t) = 0$ , this requires  $n > 2$  and in the above two methods, we get  $n \approx 2.235$  and  $n \approx 2.218$ , respectively. This result is for temperature prescribed boundary condition. Obtaining these values of  $n$  requires numerical work and plotting of curves.

The process to determine the optimal value of  $n$  remains the same for flux and Newton boundary conditions but in these cases the approximate forms of temperatures are taken differently in [633] such as

$$T(x, t) = \frac{\delta(t)}{n} \left(1 - \frac{x}{\delta}\right)^n, \text{ for constant flux; } T(x, t) = \frac{\delta}{n + \delta} \left(1 - \frac{x}{\delta}\right)^n, \\ \text{for Newton cooling.} \quad (12.8.75)$$

The solutions of  $\delta(t)$  in HBIM and RIM will be different in terms of  $n$  in flux prescribed case but  $E_n$  is of the form  $E_n = e_n t^{-1/2}$  in both methods. It is assumed that  $\delta(t) \sim t^{1/2}$  in the flux case. The exponent  $n$  is not a function of time as  $e_n$  does not depend on time. However, for Newton boundary condition, we cannot assume that  $\delta \sim t^\alpha$  for some  $\alpha > 0$ . In this case  $e_n$  is a function of  $n$  and  $t$  and the minimum of  $E_n(t)$  can be obtained for particular values of  $t$  only. It was found that  $E_n$  in this particular case is decreasing with time and so minimum will occur at  $t = 0$  and  $n$  can be obtained. The form  $\delta(t) \sim t^\alpha$ ,  $\alpha > 0$  is dictated by some available analytical solution and if the behaviour of  $\delta(t)$  is not known as  $\delta(t) \sim t^\alpha$  for some  $\alpha$  then  $n = n(t)$  is to be taken.

In obtaining the optimal value of  $n$  in a one-phase Stefan problem (cf. [633]), consider a semiinfinite solid which is melting by prescribing temperature at  $x = 0$ ,  $0 \leq x < \infty$ ,  $t > 0$ . The formulation considered in [633] possesses exact solution and  $S(t) \sim t^{1/2}$ . The approximate temperature and  $S(t)$  are taken in the form,  $S(t) = \sqrt{2a_1 t/\beta}$ ,  $\beta$  is inverse Stefan number and

$$T(x, t) = a_1(1 - x/S) + (1 - a_1)(1 - x/S)^n, \quad a_1 \text{ is unknown constant.} \quad (12.8.76)$$

After some calculations, we get  $S dS/dt = a_1/\beta = F(n, \beta)$ . For  $F(n, \beta)$  refer [633].  $E_n(t)$  is calculated using Eq. (12.8.74) and it is of the form  $E_n(t) = e_n t^{-3/2}$ . The minimum of  $e_n$  depends on  $n$  and  $\beta$  and therefore it is to be found out numerically how  $n$  varies with  $\beta$ . For example, for  $\beta \in [0.7, 50]$ , it was found that  $n$  is decreasing and varies in the range [1.804, 1.765] for HBIM and in the range [1.809, 1.769] for RIM. For other results refer [633].

Three more problems, one of ablation due to constant flux, second concerning travelling wave solution and third of melting with time dependent prescribed flux, were also considered. The approximating temperatures considered in the three cases in the order as written above are given below.

$$\left. \begin{aligned} T(x, t) &= ((\delta - x)/(\delta - S))^n; \quad T(x, t) = a(1 - x/S) + b(1 - x/S)^n, \\ T(x, t) &= (1 - na_n)(1 - x/S) + a_n(1 - x/S)^n. \end{aligned} \right\} \quad (12.8.77)$$

$x = S(t)$  is the phase-change boundary and  $\delta$  is thermal penetration depth which are different for the three problems. All the Stefan problems considered in [633] are one-phase Stefan problems. For the ablation problem exact analytical solution is not known and  $E_n$  (error) cannot be put in the form  $e_n t^\alpha$ , for some  $\alpha$ .  $E_n$  is to be plotted for  $n = n(t)$  and  $t$  and then it is decided for which range of  $t$ ,  $n$  attains minimum. For example in RIM,  $\delta = n$  is obtained. If  $t_1 = 3n^2/(2(n+1)(n+2))$ , where  $t_1$  is the time at which ablation starts, then  $n = 3.822$ . Time  $t$  in  $n(t)$  depends on  $t_1$ . Numerical work is required at every stage. In every problem in addition to the information obtained through numerical work lot of other types of information are also required which is gathered through different types of reasoning. For details of travelling wave analytical solutions, refer [635]. The travelling wave solutions are of the form  $T(x, t) = \beta(e^{-d(x-dt)} - 1)$ ,  $S(t) = dt$ . As  $t$  is increasing, temperature is increasing. Approximate solution can be obtained only for small  $t$ . For  $t = 4$  the minimum error is 11.4% and 9.4% for HBIM and RIM, respectively. For further information about these problems and solution details refer [633]. We have discussed here the basic approach and clarified some main points.

In [636], the three heat conduction problems without phase change, which have been studied in [633], have been studied by using standard HBIM and RIM. RIM is used in [636] in the same way as in [633]. However, approximations of temperatures in the three problems

discussed are taken differently. In all the problems,  $0 \leq x < \infty, t > 0$  we have the respective temperatures as given below.

$$\left. \begin{aligned} T(x, t) &= a_0 + a_1(1 - x/\delta) + a_2(1 - x/\delta)^2 + a_3(1 - x/\delta)^3, \\ T(x, t) &= \frac{\delta}{n}(1 - x/\delta)^n; \quad T(x, t) = (\delta/(n + \delta))(1 - x/\delta)^n. \end{aligned} \right\} \quad (12.8.78)$$

$\delta$  is thermal penetration depth which is different in three problems. The first temperature approximation is for the problem when constant temperature is prescribed at  $x = 0$ , the second one is for constant flux prescribed case and third one is for Newton cooling  $\partial T/\partial x = T(0, t) - 1$ , at  $x = 0$ . In the third case  $n = n(t)$  as  $E_n(t)$  cannot be put in the form  $E_n(t) = e_n t^\alpha$ , for some  $\alpha > 0$ . RIM seems to be slightly more accurate.

The reference [637] is a good review article in which using HBIM, RIM and ARIM (discussed below) several one-phase one-dimensional solutions of Stefan problems have been discussed. As far as basic approaches in obtaining the solutions are concerned we have discussed almost all them which are mentioned in [637]. However, the references mentioned by us in using different approaches for obtaining solutions are different from [637]. Very few references are common. Most of our references pertain to those works in which the various approaches were originally proposed or proposed earliest. For example the RIM or the double integration of heat equation was first proposed in [627] and later used in many other references such as in [617, 626]. The application of exponential function in approximate temperature distribution may be found in [615]. Only that material in [637] which has not been covered by us will be discussed here. The alternate refined HBIM, 'ARIM' was earlier used in [614] but it was not reported as ARIM. As the derivation of ARIM was not given earlier, it is explained below.

$$\left. \begin{aligned} \int_0^S \int_0^x \frac{\partial T}{\partial t} d\xi dx &= \int_0^S \left( \int_0^x \frac{\partial T}{\partial t} d\xi \right) dx = \int_0^S \left( \frac{\partial T}{\partial x} - \frac{\partial T}{\partial x} \Big|_{x=0} \right) dx \\ &= T|_{x=S} - T|_{x=0} - S \frac{\partial T}{\partial x} \Big|_{x=0}. \end{aligned} \right\} \quad (12.8.79)$$

Using  $T(S(t), t) = 0$  and integrating the first double integral on extreme left in Eq. (12.8.79) by parts and replacing  $\xi$  by  $x$ , we get

$$S \frac{d}{dt} \left( \int_0^S T dx \right) - \frac{d}{dt} \left( \int_0^S x T dx \right) = -T|_{x=0} - S \frac{\partial T}{\partial x} \Big|_{x=0}. \quad (12.8.80)$$

Using Eq. (12.8.2), we get

$$\frac{d}{dt} \left( \int_0^S x T dx \right) = T|_{x=0} - \beta S \frac{dS}{dt}. \quad (12.8.81)$$

When  $\partial T/\partial x|_{x=0}$  is prescribed, the form of integrated heat equation in Eq. (12.8.80) has slight advantage as exact value of derivative can be used. Section 5 in [637] presented solutions of several one-phase Stefan problems in one dimension using different types of boundary conditions and using different types of approximations for temperatures. Some details of analytical-numerical solutions obtained with the help of HBIM, RIM and ARIM have been reported giving appropriate references of sources of solutions. The numerical solutions for

$S(t)$  and error estimates have been compared which were obtained with the help of HBIM and RIM and sometimes with ARIM using different boundary conditions. The bibliography contains many useful recent references.

This review article does not contain the works concerning the optimal value of  $n$  and error estimates as these works such as [633] are more recent.

In [633, 636], the exponent  $n$  such as in Eq. (12.8.76) was not having a fixed value such as 2 or 3 and its optimal value was obtained by minimizing the error defined in Eq. (12.8.74). We saw that in many problems in which the exact behaviours of  $S(t)$  are not known, time-dependent term  $E_n$  cannot be written as its product with  $e_n$  in which  $e_n$  is a function of only  $n$ . In such cases  $n$  is a function of time also (may depend on parameters) and optimal value of  $n$  will change with time.

It has been proposed in [638] that from the start itself treat  $n$  as a function of time and so in HBIM, RIM and ARIM the time derivative should involve also the differentiation of  $n$  with respect to  $t$  in the heat balance integrals. If  $n$  is taken independent of  $t$  then we have only  $dS/dt$  in the integral equation. When  $n = n(t)$ , we have  $dS/dt$  and  $dn/dt$  both in one equation. If HBIM and RIM are both used for the same problem formulation with  $n = n(t)$  then two differential equations are available. Each equation will have both  $dn/dt$  and  $dS/dt$  which can be solved to obtain  $S(t)$  and  $n(t)$ . The author in [638] calls it a *Combined Integral Balance Method* or in short 'CIM'. By considering a one-phase ablation problem similar to the one considered in Eq. (12.8.8), CIM has been used for solution in [638]. In [639], using CIM, a two-phase solidification problem has been studied. To save space, we are not discussing this valuable work reported in [638] and instead discussing first the two-phase problem studied in [639] in which the application of CIM is more general. However, all important points in the solution procedure reported in [638] will be presented.

The formulation of two-phase problem considered in [639] is similar to Neumann problem without natural convection and only changes are reported here. The initial temperature  $T_0$  of the melt is constant which is higher than the freezing temperature. The boundary  $x = 0$ ,  $0 \leq x < \infty$ , is cooled and flux is prescribed at  $x = 0$ . The solidification starts after some delay so that during the period  $0 \leq t \leq t_m$ , there is only cooling of the melt without phase change. Let  $T(x=0, t = t_m) = T_m$ ,  $T_m$  is the freezing temperature.

$$\left. \begin{aligned} K_L \frac{\partial T_L}{\partial x} \Big|_{x=0} &= Q, \text{ a constant, } 0 \leq t \leq t_m; \quad K_S \frac{\partial T_S}{\partial x} = Q, \quad t > t_m, \\ T_L &\rightarrow T_0 \quad \text{as } x \rightarrow \infty. \end{aligned} \right\} \quad (12.8.82)$$

Without mentioning the details of scalings done in [639], it will be assumed that all the variables are dimensionless. The solution of the presolidification phase is given by

$$T_L(x, t) = -1 + 2\sqrt{t/\pi} \exp(-x^2/4t) - x \operatorname{erfc}(x/(2\sqrt{t})). \quad (12.8.83)$$

$T_L(0, t_m) = 0$  gives  $t_m = \pi/4$ , provided zero is taken as dimensionless  $T_m$ . Using penetration depth method we have  $T_L(x = \delta(t), t) = -1$ ,  $\partial T_L/\partial x = 0$  at  $x = \delta(t)$ . After scalings new temperatures are  $\bar{T}_{L,S} = (T_0 - T_{L,S})/\Delta T$ , where  $\Delta T = T_0 - T_m$  and bar over  $T_{L,S}$  is dropped in further discussion. The approximation of  $T_L(x, t)$  for  $t < t_m$  is taken as

$$\left. \begin{aligned} T_L &= -1 + \delta/n(1 - x/\delta)^n, \quad n > 1; \quad T_L(x = \delta(t), t) = -1, \\ \partial T_L(x = \delta, t)/\partial x &= 0. \end{aligned} \right\} \quad (12.8.84)$$



Using standard HBIM in which the exponent  $n$  is prescribed as some integer,  $\delta(t)$  can be obtained analytically and/or numerically. The error  $E_n$  defined in Eq. (12.8.74) should be minimized to obtain optimal value of  $n$  if  $n$  is not prescribed. It may be noted that temperature  $T_L$  at  $t = t_m$  can be calculated from Eq. (12.8.83) as  $t_m$  has been determined but at the solidification stage  $\delta(t_m)$  is required. Therefore  $t_m$  and  $\delta(t_m)$  should be known as a function of the exponent  $n$ . We have seen in [633, 636] that  $n$  could be a function of time. If  $n = n(t)$ , whenever time derivative is required in the integral equation formulations using HBIM and RIM, the time derivatives of  $n$  and  $\delta(t)$  both should be considered. This will result in a single equation containing both  $d\delta/dt$  and  $dn/dt$  in HBIM and similarly another equation will result on using RIM. From these two equations  $\delta(t)$  and  $n(t)$  can be obtained analytically or numerically.

The heat balance integrals in HBIM and RIM formulations for the presolidification stage are given below.

$$\left. \begin{aligned} \text{HBIM: } & \frac{d}{dt} \int_0^\delta T_L dx + \frac{d\delta}{dt} = - \left. \frac{\partial T_L}{\partial x} \right|_{x=0} \\ \text{RIM: } & \frac{d}{dt} \int_0^\delta x T_L dx + \delta \frac{d\delta}{dt} = -T_L|_{x=\delta} + T_L|_{x=0} \end{aligned} \right\} \quad (12.8.85)$$

$$\text{HBIM: } \frac{d}{dt} \left[ \frac{\delta^2}{n(n+1)} \right] = 1, \quad \text{RIM: } \frac{d}{dt} \left[ \frac{\delta^3}{n(n+1)(n+2)} \right] = \frac{\delta}{n}. \quad (12.8.86)$$

Solving the two equations in Eq. (12.8.86), we get  $n = 4$  and  $\delta(t) = \sqrt{n(n+1)t}$ .  $T_L(0, t) = 0$  gives  $\delta = n$  and  $t_m = n/(n+1)$ .

The ablation problem studied in [638] has also a premelting period in time and the solution procedure discussed above holds also for the ablation problem in [638]. In [638] the interface is receiving heat by flux and in [639] the boundary is being cooled. For  $t > t_m$ , both solid and liquid phases are present in [639]. The approximate temperatures  $T_S$  and  $T_L$  are taken as follows.

$$T_S = a(t) (1 - x/S(t)) + (S(t) - Ka(t))/(mK)(1 - x/S(t))^m. \quad (12.8.87)$$

$$T_L = -1 + ((\delta - x)/(\delta - S))^n; S(t_m) = 0; \delta(t_m) = n(t_m) = 4; a(t_m) = 0. \quad (12.8.88)$$

In [639] initially it is the liquid region in a semiinfinite problem which dominates over the solid region. Hence  $n$  is taken as  $n = n(t)$  but  $m$  is taken as an unknown constant exponent. For the liquid region CIM has been used and HBIM is used for the solid. For the liquid region

$$\left. \begin{aligned} \text{HBIM: } & \frac{d}{dt} \int_{S(t)}^{\delta(t)} T_L dx + \frac{d\delta}{dt} = - \left. \frac{\partial T_L}{\partial x} \right|_{x=S}, \\ \text{RIM: } & \frac{d}{dt} \int_S^\delta x T_L dx - S \frac{d}{dt} \int_S^\delta T_L dx + (\delta - S) \frac{d\delta}{dt} = 1. \end{aligned} \right\} \quad (12.8.89)$$

HBIM formulation for the solid region and Stefan condition (second equation in Eq. 12.8.90) are given below.

$$\text{HBIM: } k \frac{d}{dt} \int_0^S T_S dx = \left. \frac{\partial T_S}{\partial x} \right|_{x=S} - \left. \frac{\partial T_S}{\partial x} \right|_{x=0}; \beta \frac{dS}{dt} = \frac{Ka}{S} - \frac{n}{\delta - S}, \quad (12.8.90)$$

$\delta$  is penetration depth in the liquid and  $\beta$  is inverse Stefan number. The condition  $a(t_m) = 0$  in Eq. (12.8.88) is an a priori condition obtained from the behaviour of  $S(t)$  at  $t = t_m$  which is  $\sim (t - t_m)^{3/2}$ . In several short-time solutions such as in [455] this type of behaviour has been reported.

Four equations in four unknowns, viz,  $S(t)$ ,  $\delta(t)$ ,  $a(t)$  and  $n(t)$  are obtained when Eqs (12.8.89), (12.8.90) are used after substituting temperatures. The equations will contain the exponent  $m$  which is unknown. Refer [639] for these equations.  $m$  is determined by using the method suggested in [633], i.e. minimizing  $E_n(t)$  defined in Eq. (12.8.74) and  $m = 2.332$  is obtained.

Convective-type boundary condition has also been considered to obtain solution of a two-phase problem. Solution procedure is the same as discussed earlier except at presolidification stage exact temperature will be different which is obtained by using Laplace transform. The expression of  $T_S$  in Eq. (12.8.87) will also change which takes care of the boundary conditions.

A two-phase solidification has also been studied [639] in a finite domain  $0 \leq x \leq h$ ,  $x$  is dimensionless. The scaling has been done in such a way that finite domain results can be compared with the infinite region case and this suggests  $h = 3.375$  to the first order. The boundary  $x = h$  is insulated and constant flux is prescribed at  $x = 0$ . HBIM has been used earlier for a two-phase solidification in a finite domain in [607] but the solution procedure adopted in [607] is different. In the present problem if  $h < \delta(t_m)$ ,  $t_m$  as defined above, refer Eqs (12.8.87), (12.8.88) and  $t_h$  is time at which  $\delta(t)$  in the presolidification stage satisfies  $\delta(t_h) = h$ , then  $t_h = h^2/(n(n+1))$  and  $\delta(t) = \sqrt{n(n+1)t}$  are obtained. In the presolidification stage, we must specify an alternate temperature profile for  $t \in [t_h, t_m]$ . The temperature  $T_L(h, t)$  is unknown and we take  $T_L(h, t) = f(t)$ ,  $f(t)$  is unknown but  $f(t_h) = -1$  (after scaling). Now take

$$T_L = f(t) + \frac{h}{n} \left(1 - \frac{x}{h}\right)^n; \quad \partial T_L / \partial x|_{x=h} = 0; \quad T_L(x = h, t) = f(t). \quad (12.8.91)$$

Using CIM both  $f(t)$  and  $n$  can be obtained. Note that when  $n = n(t_m)$ ,  $t_m$  is obtained by satisfying  $f(t_m) + h/n(t_m) = 0$ . At the solidification stage take  $T_L$  as given below but  $T_S$  remains the same as in Eq. (12.8.87).

$$T_L = f(t) \left[1 - \left(\frac{h-x}{h-S(t)}\right)^n\right], \quad t > t_m. \quad (12.8.92)$$

For further details of the solution and solution procedure in this case and for the case  $h > \delta(t_m)$ , refer [639]. We have discussed only some main points in the solution procedure using CIM. The complete solution requires many other details such as information gathered from numerical work and by plotting the curves, solutions obtained in other works, approximate behaviour of functions and approximate values and so on. It is a difficult task to obtain solutions of system of differential equations and reasoning out the correct solutions. Considerable numerical work was done using the *Keller box finite-difference scheme*, the details of which are given in [639, Appendix]. In the conclusion, authors found CIM, a more accurate method amongst the methods developed to optimize the exponent used in the approximate temperature distributions. The remark by the author that the CIM is simpler than obtaining numerical solutions is unacceptable. Fairly complicated problems having solid, mush and liquid phases were considered in finite domains in [68, 69] and using moving grid finite-difference scheme numerical solutions were obtained in which simple programming

is required. In [464], numerical solution of a coupled heat and mass transfer phase-change problem was obtained.

Developing new methods and new solution approaches whether simple or complicated are always welcome and appreciated in applied sciences and CIM is a good solution procedure.

### 12.8.5 Solutions Using HBIM in Enthalpy Formulation of Stefan Problems

For convenience we use the notation EHBIM for the application of HBIM using enthalpy formulation. We briefly review the enthalpy formulation of a two-phase one-dimensional solidification/melting problem. The enthalpy or enthalpy function  $H(T)$  is defined as

$$H(T) = h(T) + \rho(T)F(T)l = \int_0^T \rho(\alpha)C(\alpha)d\alpha + \rho(T)F(T)l. \quad (12.8.93)$$

In general the parameter  $\rho$ ,  $C$  and  $l$  defined earlier are taken as constant and this is so in [640]. Enthalpy per unit volume is the sum of sensible heat  $h(T)$  per unit volume and the latent heat per unit volume.  $F(T)$  depends on the liquid fraction in the mush. Consider a simple solidification problem as considered in [640]. The region  $0 \leq x < \infty$ , originally at  $t = 0$ , is occupied by a liquid having constant temperature  $\varepsilon > 0$ .  $-\varepsilon$  is the freezing temperature. Temperature range of mush is  $[-\varepsilon, \varepsilon]$  and if a temperature  $T_0 < -\varepsilon$  is prescribed at  $x = 0$  then solidification starts immediately at  $t > 0$ . According to the enthalpy formulation, we have

$$\left. \begin{aligned} \frac{\partial}{\partial t}(T + H/C) &= K \frac{\partial^2 T}{\partial x^2}; \quad H/C = 0 \text{ for } T < -\varepsilon, \\ H/C &= F(T)/\beta, \quad -\varepsilon < T \leq \varepsilon; \quad H/C = \beta \text{ for } T > \varepsilon, \end{aligned} \right\} \quad (12.8.94)$$

$\beta = l/C$ , and  $F(T)$  depends on the nature of phase change in the mushy region.  $F(-\varepsilon) = 0$  and  $F(\varepsilon) = 1$ . For  $t > 0$ , there is solid occupying the region  $0 \leq x \leq S_1(t)$ , the mush occupies the region  $S_1(t) < x < S_2(t)$  and for  $x > S_2(t)$  the liquid region exists in which there is no temperature gradient considered. The solid-mush (solidus boundary)  $x = S_1(t)$  and mush-liquid (liquidus boundary)  $x = S_2(t)$ , and temperatures are to be determined. We assume that all the variables are dimensionless. For scalings, refer [640].

The application of EHBIM requires the integration of differential equation given in Eq. (12.8.94) with respect to  $x$  from  $x = 0$  to  $x = S_2(t)$  which can be done in two parts: Integration from  $x = 0$  to  $x = S_1(t)$  and from  $x = S_1(t)$  to  $x = S_2(t)$ . In [612] a similar procedure was followed assuming that the total temperature range can be divided into parts of equal temperature range and moving boundaries are assigned to each part with the end points of intervals serving as isotherms.  $T_S = T_M = -\varepsilon$  at  $x = S_1(t)$  and  $T_M = T_L = \varepsilon$  at  $x = S_2(t)$ . In intervals  $0 \leq x \leq S_1(t)$  and  $S_1(t) \leq x \leq S_2(t)$ , the approximate temperatures are taken as

$$T_S = T_0 - \frac{x}{S_1(t)}(T_0 + \varepsilon) + a_0x \left[ 1 - \frac{x}{S_1(t)} \right], \quad a_0 \text{ is unknown.} \quad (12.8.95)$$

$$T_M = -\varepsilon + 2\varepsilon \frac{(x - S_1(t))}{S_2(t) - S_1(t)} + a_1(x - S_1(t)) \left[ 1 - \frac{(x - S_1(t))}{(S_2(t) - S_1(t))} \right], \quad (12.8.96)$$

$a_1$  is unknown. Subscripts  $S, M$  and  $L$  stand for solid, mush and liquid regions, respectively. Continuity of fluxes at  $x = S_1$  and  $x = S_2$  determines  $a_0$  and  $a_1$ .

To apply EHBIM using Eq. (12.8.1), perform integrations over the solid and mush regions separately in the enthalpy formulation given in Eq. (12.8.94). The substitutions  $z_0 = x/S_1(t)$  and  $z_1 = (x - S_1(t))/(S_2(t) - S_1(t))$  are made in the integrals in which  $T_S$  and  $T_M$  are taken as in Eqs (12.8.95), (12.8.96).  $a_0$  and  $a_1$  become known if boundary conditions are used. The integrations will be from 0 to 1 with respect to  $z_0$  and  $z_1$ , respectively.  $\int_0^1 F(z_1)dz_1$ , which will occur in second integral, will be a constant. Take  $S_1(t) = \delta_1\sqrt{kt}$  and  $S_2(t) = \delta_2\sqrt{kt}$ .  $F(T)$  in the mushy region used in Eq. (12.8.93) is to be prescribed which could be fairly general also.  $F(T)$  is just a notation as it could be a function of space variable also. On performing the integrations in the heat balance integrals, two coupled nonlinear equations in  $\delta_1$  and  $\delta_2$  are obtained which have to be solved numerically. For assumptions made in the formulation, calculation details and numerical results, refer [640].

By taking  $F(T) = 1 - z_1$ , analytical-numerical solutions have been obtained in [641] and an analytical solution in [642]. The numerical results obtained in [640] compare well with the results obtained in [641, 642]. By taking appropriate limits in the EHBIM formulation a one-phase isothermal phase-change problem can be obtained.

Application of EHBIM has been extended in [643] to include temperature gradients in the liquid- and temperature-dependent thermal properties. The solidification of a binary/eutectic alloy of a liquid which initially occupies the region  $x > 0$  is considered when cooling is done at  $x = 0$ . Let the initial temperature of the liquid be  $T_I$ , a constant, which is greater than the liquidus temperature taken as  $T_{ml}$ . For some time, only mush develops in which  $F_L$  is the liquid fraction and  $1 - F_L$  is the solid fraction. When the temperature of an elementary volume element reaches the solidus temperature which is taken zero then  $F_L = 0$  in this volume element. In a eutectic alloy if the eutectic temperature  $T_E$  is reached in the volume element then  $F_L > 0$  and rest of the liquid changes state isothermally. If the temperature at  $x = 0$  is below the solidus temperature at  $t = 0$  then solidification starts at  $x = 0$  and for  $t > 0$ , at any instant of time three phases are present. In the region where  $T < 0$  it is solid, for  $0 \leq T \leq T_{ml}$ , it is mushy region and for the region where  $T > T_{ml}$  it is liquid region.

Consider the enthalpy function given in Eq. (12.8.93). The energy equation in terms of enthalpy can be written as

$$\left. \begin{aligned} \frac{\partial H}{\partial t} &= \frac{\partial}{\partial x} \left( \frac{k \partial h}{\partial x} \right); F(T) = 1, \text{ for } T > T_{ml}, \\ F(T) &= f_L(T) \text{ for } 0 \leq T \leq T_{ml}; F(T) = 0, \text{ for } T \leq 0. \end{aligned} \right\} \quad (12.8.97)$$

$H$  and  $h$  are as given in Eq. (12.8.93). To apply EHBIM, the energy equation is to be integrated in the solid from  $x = 0$  to  $x = S_1(t)$ , in the mushy region from  $S_1(t)$  to  $S_2(t)$  and in the liquid region from  $x = S_2(t)$  to  $x = S_3(t)$  where  $x = S_3(t)$  is the penetration depth (not phase-change boundary) in the liquid which we have denoted by  $S_3(t)$ . There are three isotherms: (1)  $T = 0$  which is the temperature at  $x = S_1(t)$ , (2)  $T = T_{ml}$  which is the temperature at  $x = S_2(t)$  and (3)  $T = T_I$  which is the isotherm at  $x = S_3(t)$ . In a compact form, the three integrated equations in EHBIM can be written as

$$\frac{d}{dt} \left[ \int_{S_{i-1}}^{S_i} H dx + H_{i-1} S_{i-1} - H_i S_i \right] = k_i \frac{\partial h}{\partial S_i} - k_{i-1} \frac{\partial h}{\partial S_{i-1}}, \quad i = 1, 2, 3, \quad (12.8.98)$$

where  $i = 1, 2, 3$  stand for solid, mush and liquid phases, respectively, and the subscript  $i$  indicates evaluation at the  $i$ th isotherm.  $k_1$ ,  $k_2$  and  $k_3$  are the diffusivities of solid, mush and

liquid, respectively. In every subdomain the enthalpy  $h(T)$  is approximated as  $U_i$  by a quadratic profile given below.

$$\left. \begin{aligned} U_i &= h_{i-1} - Z_i(h_{i-1} - h_i) + A_i Z_i(1 - Z_i), \\ Z_i &= (x - S_{i-1})/(S_i - S_{i-1}), \quad i = 1, 2, 3, \end{aligned} \right\} \quad (12.8.99)$$

$A_1, A_2, A_3$  are unknowns which can be determined on satisfying the flux conditions at the three isotherm boundaries. At  $x = S_3(t)$ ,  $\frac{\partial T_L}{\partial x} = 0$ , and at  $x = S_2(t)$ ,  $K_2 \frac{\partial h_2}{\partial x} = K_3 \frac{\partial h_3}{\partial x}$ . At  $x = S_1(t)$ , we have Stefan-type condition

$$K_2 \left. \frac{\partial h_2}{\partial x} \right|_{S_1(t)+} - K_1 \left. \frac{\partial h_1}{\partial x} \right|_{S_1(t)-} = -\rho \phi l \frac{\partial S_1}{\partial t}, \quad (12.8.100)$$

where  $\phi$  is the liquid fraction at  $T = 0$ . If this value of  $\phi$  is nonzero then  $T = 0$  represents a eutectic phase change. In the case of eutectic phase change at  $T = 0$ ,  $f_L = \phi + Z_2(1 - \phi)$ . Having obtained  $A_1, A_2, A_3$ , the moving boundaries are obtained by assuming that  $\Delta S_i(t) = \delta_i \sqrt{t}$ ,  $i = 1, 2, 3$ ,  $\Delta S_i = S_i - S_{i-1}$  and substituting  $\Delta S_i$  together with  $U_i$  in Eq. (12.8.98). Three nonlinear coupled equations are obtained to determine  $\delta_1$ ,  $\delta_2$  and  $\delta_3$ , which have been obtained numerically with the help of Newton method. In the first step, approximate values of  $\delta_i$ ,  $i = 1, 2$ , are taken from the available short-time analytical and/or numerical solutions and substituted in the third equation of Eq. (12.8.98). First  $\delta_3$  is determined from the last equation and then  $\delta_1$  and  $\delta_2$  are determined from the other two equations and the whole process is repeated till convergence is obtained.

Considerable numerical work was done and compared with the available analytical and/or numerical solutions. Numerical results were also obtained by taking Scheil equation [644] for the liquid fraction  $f_L(T)$  in the mush, where  $f_L(T) = ((T_m - T)/(T_m - T_{ml}))^{1/(k_p - 1)}$ ,  $T_m$  is the equilibrium temperature and  $k_p$  is the equilibrium partition constant. A mushy region with superheat was also considered. For assumptions made in formulation, scalings and further solution details, refer [643].

A radially symmetric spherical one-phase problem of solidification in the region  $S(t) \leq r \leq r_1$ ,  $S(0) = r_1$  has been considered in [645].  $r = S(t)$  is the phase-change boundary.  $r = r_1$  is the outer radius of the sphere at which constant temperature  $T_0$  lower than the equilibrium temperature  $T_m$  has been prescribed. The formulation of the problem is of standard type as discussed here in many problems. If the substitution  $Z = r/r_1$  is made in the formulation then  $0 \leq Z \leq S^*(t)$ ,  $S^*(t) = S(t)/r_1$ . The interval  $0 \leq Z \leq 1$  is divided into  $n$  parts, with  $Z_0 = 0$ ,  $Z_n = 1$ . If  $\theta = (T - T_S)/(T_m - T_S)$  then  $0 \leq \theta \leq 1$  for  $0 \leq Z \leq 1$ . If  $\theta_i = i/n$ ,  $i = 0, 1, \dots, n$  then  $\theta_i$  can be taken as isotherm at the end point  $Z_i$  of the interval  $(Z_i, Z_{i+1})$ ,  $i = 0, 1, \dots, n-1$ . In each of the intervals  $(Z_i, Z_{i+1})$ , linear temperature profile is taken. Substituting them in the integrals which are obtained by using HBIM, a system of differential equations involving moving boundaries  $Z_1, Z_2, \dots, Z_n$  and their derivatives with respect to time is obtained which has been solved numerically using Newton method. The differential equations are singular at  $t = 0$ . Therefore starting values are required which have been obtained by expanding  $Z_i$  in a series of the form

$$Z_i = 1 - a_{i,0}t^{1/2} - a_{i,1}t - a_{i,2}t^{3/2} \dots; a_{i,n}, n = 0, 1, 2, \text{ are unknowns.} \quad (12.8.101)$$

Unknown coefficients can be determined by substituting  $Z_i$  in the system of differential equations obtained on using HBIM and solving the system of equations consisting of unknown constants. The required starting solution or values of  $Z_i$  can be obtained.

The enthalpy formulation of the above problem can be easily constructed which has been solved numerically in [645]. The above solution does not involve EHBIM. The two numerical solutions, one obtained by using HBIM and another obtained by numerical scheme using enthalpy formulation, have been compared.

### 12.8.6 Use of HBIM in Stefan-Like Multiphase Problems

Problems having more than two phases in which HBIM has been used have been considered earlier also such as in [621]. As far as application of HBIM is considered there is nothing new in the following two problems discussed. However, the formulations are more general as they involve several features of physical problems considered.

The melting of a two-metal powder alloy by SLS was considered in [629]. The melting point of one metal powder in the mixture was taken lower than the other and so only one of component melts. In [646], a fairly general mathematical model of melting of an alloy powder by DMLS (direct metal layer sintering) or SLS has been considered. The full presentation of the mathematical model, dimensionalization and solution is not possible here as it will occupy several pages. Initially at  $t = 0$ , the alloy powder occupies the region  $-\infty < x \leq S_0(t)$  which in [646] is taken as  $S_0(t) \leq x < \infty$  with  $x$ -axis taken downwards. The  $x$ -axis is taken by us vertically upward. At  $(x = S_0(t), t = 0)$ , a constant flux,  $q_0''$  is applied to melt the powder. The solidus temperature is taken  $T_{Sm}$ , a constant, and the powder melts in the temperature range  $(T_{Sm}, T_{em})$  so that a mushy region is formed above the solid powder. The initial temperature of the powder  $T_I$  is lower than the solidus temperature. Partial melting of the powder will result in the decrease of the volume of the powder bed because of the void spaces occupied by the gas due to porosity in the powder. This results in shrinkage of the powder bed and hence a shrinkage velocity  $W(t)$  develops in the lower part of the mushy region when solid powder melts. The upper part of the mush acquires only a constant shrinkage velocity  $W_0$ . Above the mushy region is the pure liquid region with no solid powder content which is also moving with the velocity  $W_0$ .

There are stages in which the melting progresses and to come to the melting stage discussed above, there are in between two stages. First stage is the heating of the powder with no mushy region formation. This stage lasts till the surface temperature at  $x = 0$  or at  $x = S_0(t)$  reaches the solidus temperature  $T_{Sm}$  at time  $t = t_S$ . Further heating results in the mush formation. The mush occupies the region  $S_1(t) \leq x < S_0(t), t > t_S$  and it is the region above the solid powder. This is the second stage. However, this second stage has been taken as the first stage in [646]. Following [646], we take the first stage when the lower part of the mushy region is formed which contains gases, solid and liquid. It is assumed that by releasing gases the powder bed has shrunk. When all the gases are released from the solid then afterwards its volume remains constant. The mushy region can be divided into two portions. The lower portion has porosity and is shrinking and the upper part of the mushy region contains only solid and liquid and no gases. Porosity in the lower part remains constant as it is defined as a ratio (cf. [646]) which remains constant. In the upper part of the mush there is no porosity and it occupies the region  $S_2(t) < x < S_0(t)$  and the lower part occupies the region  $S_1(t) < x < S_2(t)$ .

When the temperature of the solid bed reaches liquidus temperature  $T_{lm}$ , the liquid region appears at the top of the upper part of the mushy region and occupies the region  $S_3(t) < x < S_0(t)$ , where  $S_0(t)$  is the location of the heating surface which is moving downwards with a constant velocity  $W_0$  due to shrinkage. This is the third stage of melting. In the third stage, there will be four regions as described above. The formulations of different regions in the third stage have been given in [646] together with their solutions.

In following we present few equations to give some idea about the generality of the formulations in different regions. In the unsintered region  $0 \leq x < S_1(t)$ , the energy equation is considered as

$$(\rho C)_p(1 - \varepsilon) \frac{\partial T_S}{\partial t} = K_{\text{eff}} \frac{\partial^2 T_S}{\partial x^2}, \quad (12.8.102)$$

where  $\varepsilon$  is porosity and  $K_{\text{eff}}$  is effective conductivity. The expression of  $K_{\text{eff}}$  if written here will fill up about five lines. The expression of  $K_{\text{eff}}$  has been taken from [647]. Subscript  $p$  stands for the powder. In the lower part of the mushy region,  $S_1(t) < x < S_2(t)$ , we have energy equation and mass continuity equation.

$$\frac{\partial(\rho C T_M)}{\partial t} + \frac{\partial(\rho C W T_M)}{\partial x} = \frac{\partial}{\partial x} \left( K \frac{\partial T_M}{\partial x} \right) + \rho C \frac{\partial f}{\partial t}, \quad (12.8.103)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho W)}{\partial x} = 0, \quad f = (1 - \varepsilon)/(1 - \varphi_g), \quad (12.8.104)$$

$f$  is mass fraction of solid and  $\varphi_g$  is volume fraction of gas. The continuity equation (12.8.104) has been reformulated in terms of  $(1/f)$ , refer [646]. For dimensionalization, assumptions, remaining equations, boundary conditions and solutions, refer [646].

Standard HBIM has been used to obtain solutions of solid, lower and upper parts of mushy regions. In the solid region it is the solution by thermal penetration depth assumption in which  $S_1(t)$  is the penetration depth. Quadratic polynomials have been used for temperature approximations. At  $x = S_2(t)$ ,  $\varphi_g = 0$ ,  $f = 1 - \varepsilon$  and  $T_m = T_{sm} + \varepsilon(T_{lm} - T_{sm})$ , the subscript  $m$  stands for mush. It may be clarified that the formulation in the lower part of the mushy region can be understood only by going through the details given in [646] as it is of special type. For the liquid portion an exact analytical solution has been obtained. The polynomials for approximate temperatures are always chosen keeping in mind the prescribed boundary conditions and easyness in obtaining the solution. The analytical solutions of the four regions require considerable effort and presentation is good.

Considerable numerical work has been done using RKM. Effect of various processing parameters on the surface temperatures, interfacial locations and solid fractions on the surface were investigated.

The problem studied in [648] is about solidification of a liquid in a finite slab,  $x_c \leq x \leq x_i$ . The face  $x = x_i$  is insulated with no flux prescribed condition and at  $x = x_c$  a convective-type boundary condition is prescribed. The initial temperature of the liquid is greater than the liquidus temperature  $T_{Li}$ . In the first period of cooling, till the time the temperature at  $x_c$  becomes  $T_{Li}$  no mushy region appears. During the second period when the temperature at  $x_c$  is reduced to solidus temperature  $T_{Si}$ , the mush starts forming. When the temperature at  $x = x_i$  becomes liquidus temperature  $T_{Li}$ , all the three regions solid, mush and liquid exist and it is the third period. In the fourth period only solid and mush exist. Third period of time is the most important from the formulation points of view.

The solution obtained is an approximate solution but it is not obtained by HBIM. The authors first shift the  $x$ -axis to the solid-mush boundary  $x = S(t)$  and change the formulations of the three regions accordingly. In the solid approximate temperature is taken as

$$T_S = a_0 + a_1(x - S(t)) + a_2(x - S(t))^2. \quad (12.8.105)$$

Unknowns  $a_0$ ,  $a_1$  and  $a_2$  are determined by satisfying the boundary conditions at  $x = S(t)$  and satisfying the energy equation for solid only at  $x = S(t)$ . Similar procedure is repeated for the mushy region in which the energy equation contains a source term involving  $\partial f_S / \partial t$  where  $f_S$  is the solid fraction present in the mush. Finally, in the liquid region, HBIM is not used but thermal penetration depth  $x = \delta(t)$  is used. In determining  $\delta(t)$  and unknowns, the energy equation without integration is satisfied at the liquidus front and boundary conditions are used. For determining approximate temperature in the mush also the energy equation is satisfied at liquidus front. In the fourth period there is no liquid region and only solid and mush exist. In the mush penetration depth is used which is obtained by the procedure indicated earlier.

All the equations have been considered in [648] in dimensionless form and for all these details, assumptions, formulation and solution, refer [648].

The ablation problem considered in [649] is not a multiphase problem but it was left out in earlier discussion and so it is being considered here. The finite slab of solid  $0 \leq x \leq x_0$  is melting by prescribing flux at the surface  $x = x_0$  and the surface  $x = 0$  is insulated. It is assumed that the melting starts at  $t = 0$  and for any time  $t > 0$ , the region under consideration is  $0 \leq x \leq S(t)$ ,  $S(0) = x_0$ .  $S(t)$  is the moving boundary. The approximate temperature profile is taken as

$$T(x, t) = a(x/S)^4 + b(x/S)^2 + u_0, \quad u_0 = T(0, t). \quad (12.8.106)$$

$a$ ,  $b$ ,  $u_0$  and  $S(t)$  are four unknowns. The authors call it ‘Constrained Integral Method’. Differentiating the isotherm condition prescribed at  $x = S(t)$  another condition for flux at  $x = S(t)$  can be developed in which heat equation is also used. Isotherm condition plus this new equation determine  $a$  and  $b$ . To determine  $u_0$  and  $S(t)$ , zeroth moment and first moment of heat equation with respect to  $x$  are considered. If the initial temperature is a function of  $x$  then there is no way it can be satisfied exactly except when it has constant value.

If instead of flux, temperature is prescribed at  $x = 0$  then  $u_0$  in Eq. (12.8.106) can be taken as unknown flux. The temperature profile in Eq. (12.8.106) need not be symmetric in other problems. A one-phase problem of solidification of water has also been discussed.

A one-phase radially symmetric one-dimensional cylindrical problem has been studied in [650]. The method used in [612] has been followed after transforming the formulation in the region  $1 \leq Z \leq S(t)$ ,  $S(0) = 1$ . For the transformation, refer [467].

## 12.8.7 Application of HBIM in Multidimensional Problems

As far as application of HBIM to multidimensional problems is concerned the basic idea of considering the heat equation in the average sense remains the same as in one-dimensional problems. The approximate temperature distributions will now involve more than one space variable and also time and the phase-change boundary cannot be defined as  $x = S(t)$  or as  $t = \varphi(x)$ . The ordinary differential equations obtained earlier in an one-dimensional problem will now be partial differential equations in more than one variable. Therefore the solution approach could be different as in the higher dimensions it becomes very difficult to calculate the unknown quantities. Some earlier methods used in the one-dimensional problems which have been used for multidimensional problems also are being discussed in following paragraphs.

The solidification in a square region,  $-a \leq x \leq a$ ,  $-a \leq y \leq a$ , which is initially at the freezing temperature  $T_m$  is considered in [651]. Temperature  $T_0 < T_m$  is prescribed on



the whole boundary of the square. Let  $V(S, C_F)$  be the volume of the solid formed per unit time at any time  $\tau$ , where  $S(x, y, \tau) = 0$  and  $C_F(x, y) = 0$  are the phase-change boundary and the boundary of square region, respectively.  $X, Y, \tau$  are dimensionless and  $X = x/a$ ,  $Y = y/a$ ,  $\tau = tk/a^2$ ,  $C_F(x, y) = (x^2 - 1)(y^2 - 1)$ . Let  $\theta = (T - T_0)/(T_m - T_0)$  so that  $\theta = 0$  on  $C_F$  and  $\theta = 1$  on  $S$ . If  $v = V/a^2$ ,  $\sigma = \text{surface area}/a^2$  then

$$v(S, C_F) = \iint_{S=0}^{C_F} d\sigma; \quad -\beta \frac{dv}{d\tau} = \oint_S \frac{\partial \theta}{\partial \nu} dL, \quad \beta = \rho l k / (K (T_m - T_0)). \quad (12.8.107)$$

The second equation in Eq. (12.8.107) is the Stefan condition or the energy balance equation at  $S = 0$ ,  $L$  is arc length divided by  $a$  ( $a$  is half the length of the side of a square) along  $S$  and  $\nu$  is outward normal to  $S$ .

Integrating the heat equation over the solidified region and using Stefan and other conditions, we get

$$\beta \frac{dv}{d\tau} + \oint_{C_F=0} \frac{\partial \theta}{\partial \nu} dL = \iint_{S=0}^{C_F=0} \frac{\partial \theta}{\partial t} d\sigma, \quad (12.8.108)$$

integration is anticlockwise. Another equation is obtained by using the Stefan condition given as second equation in Eq. (12.8.107) and using boundary conditions and divergence theorem  $\text{div}(\theta \nabla \theta) = \nabla \theta \cdot \nabla \theta + \theta \nabla^2 \theta$ . We get

$$\beta \frac{dv}{d\tau} = \iint_{S=0}^{C_F=0} \left\{ \left( \frac{\partial \theta}{\partial X} \right)^2 + \left( \frac{\partial \theta}{\partial Y} \right)^2 + \theta \frac{\partial \theta}{\partial \tau} \right\} d\sigma. \quad (12.8.109)$$

If penetration depth method is used and it is assumed that after a long time solidified boundary will have rounded corners then the approximations for  $\theta$  and  $S$  can be taken as

$$\theta = (X^2 - 1)(Y^2 - 1)/\epsilon(\tau), \quad \epsilon = 0 \text{ when } \tau = 0; \quad S = (X^2 - 1)(Y^2 - 1) - \epsilon(\tau). \quad (12.8.110)$$

To determine  $\epsilon(\tau)$ , substitute  $\theta$  and  $S$  in Eq. (12.8.108) and perform integrations. After some lengthy calculations an ordinary first-order differential equation for  $\epsilon(\tau)$  can be obtained which can be solved numerically. To improve the accuracy, two parameters  $\epsilon(\tau)$  and  $g(\tau)$  instead of only one as discussed in [652] can be considered and the approximation for  $\theta$  is taken as

$$\theta = (X^2 - 1)(Y^2 - 1)(1 - g)/\epsilon + g(t)\{(X^2 - 1)(Y^2 - 1)/\epsilon(\tau)\}^2, \quad g(0) = 0. \quad (12.8.111)$$

To determine  $\epsilon(\tau)$  and  $g(\tau)$ ,  $S$  given in Eq. (12.8.110) and  $\theta$  in Eq. (12.8.111) are substituted in Eqs (12.8.108), (12.8.109) and resulting equations are solved for  $\epsilon(\tau)$  and  $g(\tau)$ . In this particular problem because of all around symmetry the approximate forms of moving boundary and temperature could be justified. However, guessing the moving boundary in a general problem is not possible in general and so is the case with temperature.

Numerical work done in [651] indicated that for  $\beta = 1.5613$  both the methods give almost same solution when  $\epsilon < 1/2$  but for  $\epsilon \sim 1$ , two parameter method having  $\epsilon(t)$  and  $g(t)$  as parameters are more accurate.  $X = (1 - (\epsilon(\tau))^{1/2})^{1/2} = Y$  and  $X$  remaining the same but  $Y = 0$  were the two cases taken up for numerical work.

The solution procedure used in [651, 653] was to study solidification in a cuboid  $(X^2 - 1)(Y^2 - 1)(Z^2 - 1) = 0$ . The Stefan condition in Eq. (12.8.107), heat balance integral equation (12.8.108) and the equation similar to Eq. (12.8.109) for cuboid can be easily derived if appropriate changes are made in the formulation of the problem discussed in [651]. For scalings the same procedure used in [651] is followed in [653]. The line integral in Eq. (12.8.107) will now be surface integral and the integrand will be  $\nabla\theta \cdot d\vec{S}$ , where  $d\vec{S}$  is the directed elementary surface area. The approximate temperature and phase-change interface will be just extensions to three-dimensional case. For example, temperature  $\theta$  in the present problem can be taken as  $\theta = (X^2 - 1)(Y^2 - 1)(Z^2 - 1)/\epsilon(\tau)$  and  $S(X, Y, Z, \tau) = (X^2 - 1)(Y^2 - 1)(Z^2 - 1) - \epsilon(\tau)$ . Numerical solutions for one parameter  $\epsilon(\tau)$  solution and two parameters  $\epsilon(\tau)$  and  $g(\tau)$  (as in [651]) solution have been reported.

To obtain results for an ellipsoid of sides  $2a$ ,  $2b$  and  $2c$  with a similar formulation as in [651, 653], replace  $\tau$  by  $\tau^*$  where  $\tau^* = (1/3)\tau(1 + (a/b)^2 + (a/c)^2)$  in earlier results. The results correspond to using the same form of profile but with  $X = 1 - \sqrt{(x^2/a^2 + y^2/b^2 + z^2/c^2)}$ . The results for a radially symmetric cylinder and sphere in one dimension can be obtained by taking  $\theta = X/\epsilon$  or  $\theta = X/\epsilon + g(X/\epsilon - X^2/\epsilon^2)$  and  $X = 1 - r_\lambda$ ,  $r_\lambda = r/a$  for cylinder and  $r_\lambda = 2r/a$  for a sphere.  $r = \epsilon(\tau)$  is the equation of solidification front. The results for an elliptic cylinder can be obtained by making appropriate changes.

The method of lines proposed in [654] for numerical solutions of Stefan problems was used in [655, 656] for approximate solutions of one-phase two-dimensional Stefan problems. The solutions have been obtained without using HBIM. A suitable quadratic polynomial (for constant  $x$ ) for approximate temperature was used. In Eq. (12.8.115), take  $A = 1$  and replace  $y/s$  by  $(1 - y/s)$ . In the moving boundary  $y = S(x, t)$ ,  $S(x, 0)$  is known. The unknown constants and moving boundary along  $y$  for given  $S(x, 0)$  were calculated without using heat balance integral equation but using boundary conditions and generating extra condition to obtain an unknown parameter. Forward difference in time direction and central difference in space direction were used for numerical solution. Numerical results for a three-dimensional problem were also obtained. In another method piecewise linear profile was considered for temperature.

In [657], method of lines has been used in conjunction with HBIM. The moving boundary  $y = S(x, t)$  is such that  $S(x, 0)$  is known. Because of this assumption, numerical computations get simplified considerably. The dimensionless formulation of one-phase problem is given below.

$$\begin{aligned} \frac{\partial T}{\partial t} &= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq S(x, t), \quad t > 0; \\ T(x, y, 0) &= 1 - \frac{y}{g(x, 0)}, \end{aligned} \quad (12.8.112)$$

$$\left. \begin{aligned} \frac{\partial T}{\partial x} &= 0, \text{ for } x = 0 \text{ and } x = 1; \quad T = 1 \text{ on } y = 0, \\ T &= 0 \text{ on } y = S(x, t), \quad S(x, 0) = g(x, 0). \end{aligned} \right\} \quad (12.8.113)$$

$y = S(x, t)$  is the unknown phase-change boundary and  $g(x, 0)$  is known.

Differentiating isotherm condition at  $y = S(x, t)$ , we get

$$\frac{\partial S}{\partial t} = -\frac{\partial T}{\partial y} + \frac{\partial T}{\partial x} \frac{\partial S}{\partial x}. \quad (12.8.114)$$

For a constant  $x$ , the temperature profile is taken as

$$T(x, y, t) = A(y/S)^2 + B(y/S) + 1, \quad A \text{ and } B \text{ are unknowns.} \quad (12.8.115)$$

Using the initial condition it can be easily seen that

$$T(x, y, t) = 1 - (1 + B)(y/S)^2 + B(y/S). \quad (12.8.116)$$

To determine  $B$  and  $S(t)$ , the zeroth and first moments of heat equation with respect to  $y$  are taken in applying standard HBIM which result in the following equations.

$$\int_0^S \frac{\partial T}{\partial t} dy = \int_0^S \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) dy; \quad \int_0^S y \frac{\partial T}{\partial t} dy = \int_0^S y \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) dy. \quad (12.8.117)$$

First equation in Eq. (12.8.117) can be reduced to Eq. (12.8.118) if Eq. (12.8.114) is used. If Eq. (12.8.116) is used further in Eq. (12.8.118) then an equation in  $S$ ,  $\frac{\partial S}{\partial t}$ ,  $B$ ,  $\frac{\partial B}{\partial t}$ ,  $\frac{\partial B}{\partial x}$  and  $\frac{\partial^2 B}{\partial x^2}$  is obtained.

$$\frac{\partial}{\partial t} \int_0^S T dy = \frac{\partial^2}{\partial x^2} \int_0^S T dy - \frac{\partial T}{\partial x} \Big|_{y=S} \frac{\partial S}{\partial x} + \frac{\partial T}{\partial y} \Big|_{y=0}^S. \quad (12.8.118)$$

Similarly after reducing the second equation in Eq. (12.8.117) to a suitable form and using Eqs (12.8.115), (12.8.116), we get one more equation which contains  $S$ ,  $\frac{\partial S}{\partial t}$ ,  $B$ ,  $\frac{\partial B}{\partial t}$ ,  $\frac{\partial B}{\partial x}$  and  $\frac{\partial^2 B}{\partial x^2}$ . These two equations can be solved numerically for  $\frac{\partial S}{\partial t}$  and  $\frac{\partial B}{\partial t}$ . Refer [657] for these equations and other details and also for finite difference numerical solution marching in time.

A two-dimensional one-phase problem in the region  $0 \leq x \leq 1$ ,  $0 \leq y \leq S(x, t)$  has been considered in [658]. The boundary conditions in this problem are different from [657] but HBIM is similar to that in [657].  $\partial T/\partial x = 0$ , for  $x = 0$  and  $x = 1$ ,  $\partial T/\partial y = 0$  on  $y = 0$ ,  $T(x, y, t) = S(x, t)$  on  $y = S(x, t)$  and  $T(x, y, 0) = 2g(x) - y^2/g(x)$ ,  $S(x, 0) = g(x)$ ,  $g(x)$  is known.  $y = S(x, t)$  is the phase-change boundary. Because of the changed boundary conditions the approximate temperature distribution is taken as

$$T(x, y, t) = C(1/S) + B(y/S) - A(y^2/S); \quad A(t), B(t) \text{ and } C(t) \text{ are unknowns.} \quad (12.8.119)$$

The temperature prescribed on  $y = S(x, t)$  is typically chosen to get some solution. The unknown coefficients  $C$  and  $B$  can be easily obtained in terms of unknown  $A$  and following the method employed in [657], two coupled equations in  $dA/dt$  and  $dS/dt$  are obtained which have been solved numerically by finite difference scheme marching in time. The equation for  $dA/dt$  will involve quantities like  $\partial A/\partial x$  and  $\partial^2 A/\partial x^2$  and similarly  $dS/dt$  will involve  $\partial S/\partial x$ ,  $\partial^2 S/\partial x^2$ . The approximate temperature in Eq. (12.8.119) and boundary conditions can be used to obtain desired approximate partial derivatives with respect to  $x$ .

A three-dimensional one-phase problem in  $0 \leq (x, y) \leq 1$  and  $0 \leq z \leq S(x, y, t)$  has been considered in [659]. Although in [659] the problem considered is an FBP in hydraulics, it can be considered as a solidification or melting problem as the formulation can be used for a Stefan problem also.  $\partial T/\partial x = 0$  on  $x = 0$  and  $x = 1$ ,  $t > 0$ ,  $\partial T/\partial y = 0$  on  $y = 0$  and  $y = 1$ ,  $t > 0$ ,  $\partial T/\partial z = 0$ , on  $z = 0$ ,  $t > 0$  and  $T(x, y, z, t) = S(x, y, t)$  on  $z = S(x, y, t)$ . Initial

temperature is  $T(x, y, z, 0) = zg(x, y) - z^2/g(x, y)$ ,  $S(x, y, 0) = g(x, y)$ .  $z = S(x, y, t)$  is the phase-change boundary. In this case

$$\frac{\partial S}{\partial t} = -\frac{\partial T}{\partial z} + \frac{\partial T}{\partial x} \cdot \frac{\partial S}{\partial x} + \frac{\partial T}{\partial y} \frac{\partial S}{\partial y}, \text{ on } z = S(x, y, t), \quad t > 0. \quad (12.8.120)$$

The approximate temperature is taken as

$$T(x, y, z, t) = C(x, y, t)(1/S(x, y, t)) + B(x, y, t)(z/S) - A(x, y, t)(z^2/S). \quad (12.8.121)$$

Following the solution procedure suggested in [657], for this problem also, coupled equations for  $dA/dt$  and  $dS/dt$  can be obtained by using standard HBIM which will also involve partial derivatives of  $A$  and  $S$  with respect to space variables  $x$  and  $y$ . Numerical work has been presented. In this problem also temperature at  $z = S(x, y, t)$  is typically taken.

Considerable work on growth of a gas bubble during solidification and nucleation in crystallization exists in the literature. The problem of bubble growth has been addressed from many angles and enough information about these studies can be obtained from the references listed in [660]. The analytical and numerical solutions of a problem of solidification with bubble which has grown during solidification have been obtained in [660]. The surface of the bubble is independent of time to a first approximation. Transient axisymmetric spherical heat conduction equations have been considered in liquid and solid regions as given below.

$$\frac{\partial T_L}{\partial t} = k_L \nabla^2 T_L; \quad \frac{\partial T_S}{\partial t} = k_S \nabla^2 T_S, \quad \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right). \quad (12.8.122)$$

$r$  is the spherical radial coordinate and  $\theta$  is the longitudinal coordinate or the polar angle in spherical coordinates, the angle which  $z$ -axis makes with radial direction. The origin ( $z = 0, r = 0$ ) is at the same level as solid-liquid interface at a time specified by  $t = 0$ .  $z$ -axis is taken in the upward direction and  $x$ -axis horizontally. In view of the cold temperature  $T_C$  at the bottom location  $z = z_C$  which is a negative value, the solid-liquid interface proceeds at a constant speed  $U$  in the  $z$ -direction upwards. The solidification front location at any time is given by  $z = v(\theta, t)$ ,  $U = dv/dt$ .

The bubble surface  $r = r_i$  is maintained at constant temperature  $T_i$  whilst the temperature at  $r = R$ , the outside radius, far from the bubble is linear in  $z$ -direction. In this case, we have  $T_C \leq T_m \leq T_i$ ,  $T_m$  is the melting temperature. As  $T_i \geq T_m$ , heat is transferred from the bubble to the solid resulting in the formation of a thin liquid layer  $\delta(\theta, t)$  around the bubble in the region  $r_i \leq r \leq S(\theta, t)$ .  $r = r_i(\theta)$  is the inner or the bubble surface and  $r = S(\theta, t)$  is the solid-liquid interface. At the solidification front which is  $z = v(t)$ , there exist two boundary conditions.

$$\left. \begin{aligned} T_S &= T_m \text{ for } \theta_{me} \leq \theta \leq \theta_{Rc}, \text{ on } z = v(t), \\ -k_L \frac{\partial T_L}{\partial \theta} &= hr_0(T_L - T_m), \theta_{ie} \leq \theta \leq \theta_{me}, \text{ on } z = v(t). \end{aligned} \right\} \quad (12.8.123)$$

The subscript 'me' stands for melting edge, 'Rc' for outer edge  $R$  at  $z = z_C$  and 'ie' stands for inner surface at the edge. For better understanding of these angles, refer Fig. 1 in [660]. The formulation has been considered in dimensionless form in which length is divided by  $r_0$ , where  $r_0$  is the initial radius of curvature at the axisymmetric axis, dimensionless temperature

is  $\psi = (T - T_m)/(T_i - T_m)$ ,  $t = \text{time}$ .  $k_L/r_0^2$ , temperature gradient  $\bar{g} = g(t)r_0/(T_i - T_m)$ ,  $g(t) = (T_m - T_C)/(S(t) - z_C)$ , energy function  $\bar{H} = H/((T_i - T_m)r_0^3)$ .

In addition to the boundary conditions in Eq. (12.8.123), other boundary conditions are as follows:

$$T_L = T_i = \text{constant at } r = r_i(\theta), \theta_{ie} \leq \theta \leq \pi. \quad (12.8.124)$$

$$T_L = T_m = T_S; -K_L \frac{\partial T_L}{\partial n} + K_S \frac{\partial T_S}{\partial n} = \rho l \frac{\partial S}{\partial t} A, \quad A = \left[ 1 + \left( \frac{1}{S} \frac{\partial S}{\partial \theta} \right)^2 \right]^{1/2}, \quad (12.8.125)$$

at  $r = S(\theta, t)$ ,  $\theta_{me} \leq \theta \leq \pi$ ,  $r = r_i + \delta(\theta, t) = S(\theta, t)$ .

$$\left. \frac{\partial}{\partial n} \right|_{r=S} = A \left( \frac{\partial}{\partial r} - \frac{1}{S^2} \frac{\partial S}{\partial \theta} \frac{\partial}{\partial \theta} \right), \quad n \text{ stands for outward normal.} \quad (12.8.126)$$

The linear temperature profile in the solid in the vertical direction at  $r = R$  is taken as

$$T_S = T_m + g(t)(z - v(\theta, t)), \quad g(t) \text{ is as given above, } r \geq R, \quad z_C \leq z \leq v. \quad (12.8.127)$$

For the application of HBIM, the heat equation in the liquid region is integrated from  $r = r_i(\theta)$  to  $r = S(\theta, t)$  and for the solid it is integrated from  $r = S(\theta, t)$  to  $r = R$ . For the sake of illustration only one equation is given below.

$$\frac{\partial \bar{H}_L}{\partial t} = S^2 \left. \frac{\partial \psi_L}{\partial r} \right|_S - r_i^2 \left. \frac{\partial \psi_L}{\partial r} \right|_{r_i} + \frac{1}{\sin \theta} \int_{r_i}^S \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi_L}{\partial \theta} \right) dr, \quad (12.8.128)$$

$\bar{H}_L = \int_{r_i(\theta)}^{S(\theta, t)} \psi_L r^2 dr$ . Similarly equation for  $\frac{\partial \bar{H}_S}{\partial t}$  can be obtained and total energy equation  $\frac{K_L}{K_S} \frac{\partial \bar{H}_L}{\partial t} + \frac{k_L}{k_S} \frac{\partial \bar{H}_S}{\partial t} = \nabla^2 (\bar{H}_L + \bar{H}_S)$  can be calculated in which Stefan condition has been used. The approximate temperature distributions are taken as given below.

$$\psi_L = a_L(\theta, t)[r - S(\theta, t)] + b_L(\theta, t)[r - S(\theta, t)]^2, \quad r = r_i(\theta) + \delta(\theta, t), \quad (12.8.129)$$

$$\psi_S = a_S(\theta, t)[r - S(\theta, t)] + b_S(\theta, t)[r - S(\theta, t)]^2. \quad (12.8.130)$$

However, when Eqs (12.8.129), (12.8.130) are used in total energy equation then also Stefan condition is satisfied. The unknowns  $a_L, b_L, a_S$  and  $b_S$  which are functions of  $\theta$  and  $t$  are determined by satisfying the boundary conditions. The boundary condition for  $\psi_S$  at  $r = R$  is satisfied by using Eq. (12.8.127). Therefore one of the coefficients  $a_S$  or  $b_S$  is determined with the help of a fictitious boundary condition which is generated beyond the solidification front. For this fictitious boundary condition and its effects on the temperature fields, calculation details, assumptions, etc., refer [660]. This problem is one of the very few problems studied in multidimensional spherical geometry. A good understandable reporting of the solution procedure which is fairly complicated is not possible here by us. The generation of a fictitious boundary condition is new and interesting.

Numerical work was done using finite difference schemes whose execution steps are mentioned. Considerable numerical work was done to compute temperatures and fusion zone shapes in liquid and solid and also compared with other available results.

### 12.8.8 Hybrid Method to Determine Optimal Value of Exponent in Polynomial Profile of Temperature in HBIM

The references [661, 662] contain works on heat transfer problems without phase change but address some important questions related to the application of HBIM. Both these references are concerned with the determination of optimal value of the exponent in the approximate parabolic temperature profiles. The proposed methods have been tested on very simple problems whose exact solutions are known or HBIM solutions can be easily obtained with good accuracy. A simple one-dimensional heat conduction problem in the infinite region  $0 \leq x < \infty$  has been considered in [661]. At  $x = 0$ , a constant temperature or a constant flux could be prescribed. If  $\delta(t)$  is the thermal penetration depth then  $T(\delta, t) = T_\infty$ ,  $K\partial T/\partial x|_{x=\delta(t)} = 0$  and  $\partial^2 T/\partial x^2|_{x=\delta} = 0$ . The approximate temperature profile in HBIM is generally taken as

$$T_a(x, t) = A + B(1 + x)^n \text{ or } \theta = (T(x, t) - T_\infty)/(T(0, t) - T_\infty) = (1 - x/\delta)^n, \quad (12.8.131)$$

the exponent  $n$  is independent of time in [661].

A pertinent question arises ‘What should be the value of  $n$  to get the best possible accurate results?’ This question was asked and answered earlier in several references such as in [636, 639] in which the problems discussed were Stefan problems. In [661], it is proposed that the solution should be tested for accuracy by considering the local *thermal entropy generation rate*  $\dot{E}_T$  or in short *TEG* in the absence of internal heat generation and minimizing an appropriate difference  $\Delta\dot{E}_T$  as given below.

$$\dot{E}_T = \frac{K}{T^2} (\text{grad } T)^2; \Delta\dot{E}_T = [\dot{E}_T(T, T_x)_a - \dot{E}_T(T, T_x)_e] \rightarrow \text{minimum}. \quad (12.8.132)$$

The local approach at  $x = 0$  and  $\Delta\dot{E}_T = 0$  implies that

$$(\partial T/\partial x)_{a(x=0)}^2/T_{a(x=0)}^2 = (\partial T/\partial x)_{e(x=0)}^2/T_{e(x=0)}^2, \quad (12.8.133)$$

The subscript  $a$  stands for approximate solution and  $e$  stands for exact solution. Eq. (12.8.133) implies that if  $T$  at  $x = 0$  is prescribed then flux at  $x = 0$  should satisfy Eq. (12.8.133) and if flux is prescribed at  $x = 0$  then temperature at  $x = 0$  should be known exactly and satisfy Eq. (12.8.133). It may be noted that only local error is minimized and not global error. The  $L^2$ -norm error in Eq. (12.8.74) is global error.

How to obtain solutions of these missing boundary conditions at  $x = 0$  which are required in Eq. (12.8.133). The solutions which use half-time fractional derivatives have been suggested. For example, if temperature is prescribed at  $x = 0$  then

$$-K \frac{\partial T}{\partial x} \Big|_{x=0} = q''(0, t) = \frac{K}{\sqrt{k}} \left[ \frac{\partial^{1/2} T(0, t)}{\partial t^{1/2}} - \frac{T_\infty}{\sqrt{\pi t}} \right], \text{ if } T(0, t) \text{ is prescribed}, \quad (12.8.134)$$

or,

$$T(0, t) = \frac{\sqrt{k}}{K} \frac{\partial^{-1/2} (q''(0, t))}{\partial t^{-1/2}} + T_\infty, \text{ if } -K \frac{\partial T}{\partial x} \Big|_{x=0} = q''(t) \text{ (prescribed)}. \quad (12.8.135)$$

The fractional derivatives can be calculated using Riemann-Liouville fractional derivatives which have been defined in [392]. The optimal value of exponent  $n$  has been obtained using *TEG* criterion. The approximate solutions for two types of boundary conditions have been obtained and compared with exact solutions. It has been mentioned that when the error is defined by Eq. (12.8.74) in  $L^2$ -norm it is difficult to calculate it. This is supported by the solutions obtained in [636]. If a similarity variable solution is used which is not always possible then calculation of the error in  $L^2$ -norm becomes easy and the equation to determine  $n$  is obtained more easily.

In [662], a collocation method has been suggested first to obtain an optimal value of exponent  $n$  which is considered a function of  $t$  but not of  $x$ , i.e.  $n = n(t)$ . The problem considered is the same heat transfer problem as in [661]. The approximate temperature  $T_a(x, t)$  is taken in the form in which  $m$  is the collocation point.

$$T_a(x, t) = T_\infty + (T(0, t) - T_\infty)(1 - m)^n, \quad x = m\delta, \quad 0 \leq m(x) \leq 1, \quad n = n(t). \quad (12.8.136)$$

When Eq. (12.8.136) is substituted in the heat equation, a nonlinear first-order ordinary differential equation is obtained which is difficult to solve and so another approach based on the similarity variable  $\eta$  was suggested. In this case the pointwise error  $D_{pw}(n, \eta) = \theta_e - \theta_a = 0$  was used to determine  $n(t)$ .  $\theta_e$  is exact solution and  $\theta_a$  is approximate solution.  $\theta$  is dimensionless temperature. As the similarity solutions for both the prescribed boundary conditions at  $x = 0$  are available exactly in this problem,  $n = n(t)$  can be obtained numerically for some ranges of  $\eta$ . For further details refer [662].

Hybrid of HBIM has been discussed in [639] but the approach in [663] to obtain hybrids of HBIM is different and more general. However, questions concerning the error estimates in a combination of hybrid methods even in a simple one-phase one-dimensional Stefan problem and the optimal value of an exponent in a series form of the approximate temperature profile seem to be difficult to answer. The determination of the solution depends heavily on numerical solutions of unknown quantities after the analytical calculations are over.

A hybrid HBIM discussed in [663] is a combination of some HBIM solutions of a partial differential equation. Let  $F\left(u(x, t), \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) = 0$  be the partial differential equation which holds in a certain specified region and boundary and initial conditions are prescribed.  $u(x, t)$  could be temperature or any other physical-dependent variable. If  $u(x, t)$  is the exact solution of equation  $F = 0$  then it is exactly satisfied but if  $v(x, t)$  is an approximate solution then  $F(v) \neq 0$  and there will be some residual.

In the earlier discussion on HBIM some variations of HBIM were considered. One variation was concerned with taking moments of heat equation with respect to the space variable. In the earlier discussion these moments were taken up to  $x^2$ . The hybrid method proposed in [663] is about combining the solutions of any number of specific approximate solutions using HBIM which are obtained by taking moments of successive higher order. Combining the solutions is done in some particular manner which is being discussed below with respect to the HBIM solutions of a simple one-phase one-dimensional Stefan problem given below.

$$\left. \begin{aligned} \frac{\partial T}{\partial t} &= \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < S(t), \quad t > 0; \quad \left. \frac{\partial T}{\partial x} \right|_{x=S(t)} = -\beta \frac{dS}{dt}, \\ T(S(t), t) &= 0, \quad T(0, t) = f(t); \quad T(x, 0) = 0, \end{aligned} \right\} \quad (12.8.137)$$

$S(t)$  is the phase-change front and  $1/\beta$  is Stefan number.

For the above problem  $\text{HBIM}_{(0,0)}$  solution implies the solution obtained by taking the zeroth moment of heat equation or in other words using the standard HBIM. Our concern here is to explain the hybrid method and so solutions will not be presented. We assume (refer [663]) that  $S(t)$  can be obtained as  $\sqrt{t}F_0(n, \beta)$  if approximate temperature is taken as

$$v(x, t; n, a_1, a_2) = a_1(x/S - 1)^n + a_2(1 - x/S), \quad a_1, a_2, n \in \mathbb{R}. \quad (12.8.138)$$

If  $f(t) = 1$  in Eq. (12.8.137) then  $a_2 = 1 - a_1$  in Eq. (12.8.138).  $\text{HBIM}_{(1,1)}$  is the solution obtained by taking first moment and for the above simple Stefan problem,  $S(t) = \sqrt{t}F_1(n, \beta)$  is obtained in this case for some  $F_1$ .  $\text{HBIM}_{(2,2)}$  and  $\text{HBIM}_{(3,3)}$  solutions for  $S(t)$  can be obtained for this present problem as  $\sqrt{t}F_2(n, \beta)$  and  $\sqrt{t}F_3(n, \beta)$ , respectively.  $F_0(n, \beta)$ ,  $F_1$ ,  $F_2$  and  $F_3$  are given in [663] together with differential equations to determine  $S(t)$ .  $\text{HBIM}_{(k,k)}$ ,  $k = 0, 1, 2, \dots$ , stands for HBIM solution when  $k$ th moment of heat equation is taken with respect to  $x$ . The hybrid method  $\text{HBIM}_{(k,m)} = \text{HBIM}_{(m,k)}$ ,  $m, k \in \{0, 1, 2, \dots\}$  is a combination of solutions  $\text{HBIM}_{(k,k)}$  and  $\text{HBIM}_{(m,m)}$ .

How  $\text{HBIM}_{(k,m)}$  solution from  $H_{(k,k)}$  and  $H_{(m,m)}$  solutions is obtained will be explained now. After this step how approximate temperature  $v_{k,m}(x, t)$  and approximate interface  $S_{k,m}(x, t)$  are constructed in  $\text{HBIM}_{(k,m)}$  will be discussed. Having obtained  $\text{HBIM}_{(0,0)}$  and  $\text{HBIM}_{(1,1)}$ , we construct  $\text{HBIM}_{(0,1)}$  in which  $v_{(0,0)}$ ,  $v_{(1,1)}$ ,  $S_{(0,0)}$  and  $S_{(1,1)}$  are known as their solutions have been obtained.  $\text{HBIM}_{(0,1)}$  is CIM which has been discussed in [639]. So  $\text{HBIM}_{(1,2)}$  will be discussed. Suppose for each  $t^i \in \mathbb{R}^+$ , there exists at least one nontrivial  $x_j^i$ ,  $J \in \mathbb{Z}^+$ , such that  $v_{11}(x_j^i, t) = v_{22}(x_j^i, t)$  with  $x_0^i = \min(x_j^i)$ ,  $J \in \mathbb{Z}^+$ . Let  $S_{(1,2)}(t)$  and  $v_{(1,2)}(x, t)$  be the estimates of  $S(t)$  and  $v(x, t)$  in  $\text{HBIM}_{(1,2)}$  as discussed in splicing algorithm given below. In [663], the solutions of  $\text{HBIM}_{(1,1)}$  and  $\text{HBIM}_{(2,2)}$  are given. The constructions of  $\text{HBIM}_{(0,1)}$  and  $\text{HBIM}_{(1,2)}$  are not discussed here as CIM can be used and construction of  $\text{HBIM}_{(0,1,2)}$  is given here. However, the procedure to obtain  $\text{HBIM}_{(0,1)}$  and  $\text{HBIM}_{(1,2)}$  is the same as that used for  $\text{HBIM}_{(0,1,2)}$  given below. The authors in [663] call this procedure *splicing algorithm*.

### Splicing Algorithm

Step 1: Apply the  $\text{HBIM}_{(0,1)}$  and  $\text{HBIM}_{(1,2)}$  to determine respective moving front estimates  $S_{(0,1)}(t)$  and  $S_{(1,2)}(t)$ .

Step 2: Obtain respective temperature approximations  $v_{(0,1)}(x, t)$  and  $v_{(1,2)}(x, t)$ .

Step 3: For every  $t^i \in \mathbb{R}^+ \cup \{0\}$  under consideration, determine  $x_i \in [0, S_{(1,2)}(t^i)]$  such that  $v_{(0,1)}(x, t^i) = v_{(1,2)}(x, t^i)$ .

Step 4: Select the nontrivial minimum of the  $x_i$  and denote it by  $x_0$ .

Step 5: Write the spliced temperature field due to  $\text{HBIM}_{(0,1,2)}$  as

$$\left. \begin{aligned} v_{(0,1,2)}(x, t^i) &= v_{(0,1)}(x, t^i), 0 \leq x \leq x_0^i \\ &= v_{(1,2)}(x, t^i), x_0^i \leq x \leq S_{(1,2)}(t^i). \end{aligned} \right\} \quad (12.8.139)$$

What should be the polynomial expression for  $v(x, t)$ ? For this the authors suggest two methods. A Hermite polynomial over the interval  $(0, S(t))$  can be constructed. This Hermite polynomial has been discussed in [663] and a cubic *Hermite interpolating polynomial*



expression has been reported also but it has not been used in the solutions of HBIM<sub>(0,0)</sub>, HBIM<sub>(1,1)</sub>, etc. Depending on the nature of boundary conditions, several generating sets can be developed. Based on the Lagrange basis  $\{p_i\}$  where  $\{p_i\} = \{x/S(t), (1 - x/S(t))\}$ , interpolating polynomials of the following form have been used.

$$v(x, t) = \sum_{j=1}^m a_j(x, S(t)) (1 - x/S)^{r_j} + \sum_{k=1}^m d_k(x, S(t)) (x/S)^{n_k}, \quad (12.8.140)$$

$r_j$  and  $n_k$  have to be determined within the frame work of HBIM or in respect of the boundary conditions given. In [663],  $d_k = 0$  have been taken.

The feasibility of the algorithm for HBIM<sub>(0,1,2)</sub> has been discussed which can be extended to HBIM<sub>(0,1,2,3)</sub>. The positivity of  $v(x, t)$  has also been discussed. Refer [663] for all these results.

What should be the optimal value of the exponent  $r_m$  in Eq. (12.8.140)? In Eq. (12.8.140), the exponent  $r_m$  is not only a function of  $t$  but it depends on  $\beta$  also. Therefore  $r_m$  has to be plotted versus  $\beta$  for all HBIM solutions and their combinations. Note that optimal value of  $r_m$  is different in different ranges of  $\beta$  for these solutions. To resolve this difficulty, taking a cue from the significance of the first moment in the theory of probability, which is the expected value, the optimal value of  $r_m$  is taken as the value of  $r_m$  projected by HBIM<sub>(1,1)</sub>. This value will be different in different ranges of  $\beta$ . For example, if the Hermite polynomial is quadratic then for  $\beta \ll 1$ ,  $n = 2.00$  and for  $\beta \gg 1$ ,  $n = 1.500$ .

The flux prescribed case was also investigated but the problem considered is a heat conduction problem with no phase change. In this case Hermite polynomials have been considered for  $v(x, t)$ . For both the problems numerical work has been reported and compared with the exact solutions.

### *Investigation of Some Physical Quantities Associated With Phase-Change Problems*

Solidification in finite bodies with prescribed heat flux condition has been considered in [664] with the objectives of obtaining bounds for the freezing time and removed energy. It has been shown earlier that planar, cylindrical and spherical one-dimensional problems can be formulated with the help of a single formulation and so the results obtained in [664] can be interpreted for all the three problems in three regions. The back wall of the body is insulated, whereas the outer surface is subjected to a constant monotonically decreasing outflux of heat at rate  $q(\tau)$  per unit area (per unit length in the cylindrical case) in dimensionless form. Both solid and liquid regions have been considered with no natural convection in the liquid. The problem formulation is similar to that of Neumann problem. The formulation considered is dimensionless.

On using standard HBIM and doing some calculations, the following equation can be obtained in which we have ignored several in between steps.

$$\int_{\tau_1}^{\tau^*} q(\tau') d\tau' = (1 - a^{n+1}) / (n + 1) + \int_a^1 \xi^n (\theta_I(\tau_1, \xi) - \theta_S(\tau^*, \xi)) d\xi, \quad (12.8.141)$$

where  $n = 0, 1, 2$  for planar, cylindrical and spherical problems,  $a$  is ratio of inner to the outer radius of the body,  $\tau_1$  is the time at which solidification begins and  $\tau^*$  is the time at which solidification ends,  $\theta_I$  is the dimensionless initial temperature,  $\theta_S$  is solid temperature and  $\xi$

is the spacial coordinate. Using the superharmonic behaviour of  $\theta_S$  and the harmonic solution  $\bar{\theta}_S$  of the steady-state heat equation, upper and lower bounds of the total heat energy released during solidification have been obtained. The lower bound for total solidification time has also been obtained when  $q(\tau) = \text{constant}$  and  $\theta_I$  is also a constant. Numerical results have been compared with known solutions.

One-dimensional freezing of a tissue over a planar and a spherical cryosurgical probe has been considered in [665]. The probe is at a temperature lower than the freezing temperature. The transient heat equation in the frozen tissue is of the type as in Neumann problem. In the unfrozen tissue an extra term of the form  $-\beta T - \gamma$  is occurring in the transient heat equation in which  $\beta$  is the blood perfusion parameter and  $\gamma$  is metabolic heat production parameter. In the planar case the tissue region extends from zero to infinity and in spherical case the region is the exterior of the probe. First a quasisteady-state solution has been obtained in which Stefan number is not contributing and so it is not a good solution. However, this solution is used in the transient solution. For the transient solution HBIM is not used but the solution has been obtained by taking approximate temperatures in the quadratic polynomial forms as taken in HBIM solutions. In the solidified tissue, a quadratic polynomial in  $(\xi/S(\tau) - 1)$  which contains three unknowns has been taken in which  $\xi$  is the dimensionless distance and  $\xi = S(t)$  is the phase-change interface. Three unknowns are there in this polynomial for solid. In the unfrozen tissue the dimensionless temperature  $\theta(\xi, \tau)$  is taken as

$$\theta(\xi, \tau) = b_0 + b_1 \exp(-b_2 \xi^n / S^n), n \text{ is an unknown exponent.} \quad (12.8.142)$$

The unknowns  $b_0$ ,  $b_1$  and  $b_2$  could be functions of time. As the number of six unknowns exceed the number of boundary conditions, an extra boundary condition is developed by considering the behaviours of  $S(\infty)$  and temperatures at infinity. In this process  $n$  is also determined. To obtain this condition quasisteady-state solution is used. For further details refer [665].

## 12.8.9 Approximate Methods Other Than HBIM: Methods of Weighted Residual

### *Galerkin Method*

There are several approximate methods other than HBIM which have been studied in the literature. However, in the context of analytical-numerical solutions, our concern is in those solutions in which analytical solution is predominant and on the basis of this approximate method is chosen here. The HBIM and its variations are the most widely investigated and applied in physical problems. In HBIM solutions, analytical approaches and approximate analytical solutions form basis for numerical solutions. We shall see later that there is not much scope for obtaining analytical solutions of Stefan problems using methods of weighted residuals (MWR). Some approximate methods other than Galerkin method which will also be discussed here are oriented more towards numerical solutions. Such approximate solutions could have been left out but are being discussed here very briefly for the sake of providing general information about them. Having discussed HBIM and its variations at some length and devoting enough space for them it is not possible to discuss the solution procedures of other approximate solutions and survey the literature existing for them. However, some illustrative examples have been included after explaining the essential features of some of the methods. We begin with the Galerkin method which is suitable for engineering problems and has been used widely. As in Galerkin method, there is an underlying central idea in the MWR and their variations, which is being explained below.

The MWR are also approximate methods. We are as in HBIM, trying to satisfy the given formulation of the problem only partially or in an average sense. The dependent variable is expressed as a series of trial functions. The coefficients of trial functions in the series are unknowns. The operator in the differential equation is multiplied by a weighting function  $W(X)$ ,  $X = (x_1, x_2, \dots, x_n)$  and integrated over the domain of the problem formulation. Time is not considered at the moment. During integration the dependent variable is expressed in terms of the series of trial functions. If the series of trial functions represents the exact solution then the integration as indicated above will give zero residual otherwise residual is nonzero. Therefore, our attempt is to minimize this residual. Both weighting function and trial functions are to be chosen suitably. Ultimately, a system of algebraic equations in unknown constants is obtained which is to be solved numerically. In some simple problems if only a single trial function is used then an approximate analytical solution may be obtained which may give considerable error. In HBIM and its variations, analytical solution procedures are predominating over the numerical solutions in the sense that the analytical solution procedures need deeper study. In MWR not much is to be done towards analytical solution procedures and it is more of a numerical solution procedure after some steps. In MWR, the numerical solution involves solution of a system of algebraic equations in unknown coefficients and solutions of differential equations are generally not required. Therefore MWR is very suitable for engineering problems. Irregular geometrics, nonhomogeneous boundary conditions and multidimensional problems can be handled with variety of numerical methods such as finite-difference, finite volume, finite element and are being used for numerical solutions of MWR.

The Galerkin method proposed by a Russian engineer Galerkin (1915) is a variation of general MWR. Let us consider a simple problem in which the dependent variable is temperature  $T(x, y)$  which in MWR is expressed as a series of the form

$$T(x, y) = T_0 + \sum_{i=1}^n C_i T_i, \quad \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad (x, y) \in V, \quad V \subset R^2, \quad (12.8.143)$$

$T = T_0$  on the boundary  $S$  of volume  $V$ . The functions  $T_i(x, y)$  called trial functions are chosen in such a way that they satisfy the homogeneous boundary conditions  $T_i = 0, i = 1, 2, \dots, n$ , so that  $T(x, y)$  in Eq. (12.8.143) satisfy the boundary conditions for all choices of constants  $C_i$ . If  $T_i$  are not the exact solutions of the differential equation then there will be nonzero residual  $R(C_i, x, y) = \nabla^2 T_0 + \sum_{i=1}^n C_i \nabla^2 T_i$ . By choosing  $C_i$  suitably the weighted integral of residual  $R$  is forced to be zero in an average sense. The series in Eq. (12.8.143) is the  $n$ -term approximation of an infinite series. If  $W_J, J = 1, 2, \dots, n$ , are the weighting functions then

$$\begin{aligned} (W_J, R) &= \int_V W_J \left( \nabla^2 T_0 + \sum_{i=1}^n C_i \nabla^2 T_i \right) dv = (W_J, \nabla^2 T_0) + \sum_{i=1}^n C_i (W_J, \nabla^2 T_i). \\ &= -A_J + \sum_{i=1}^n B_{Ji} C_i, \end{aligned} \quad (12.8.144)$$

$$A_J = -(W_J, \nabla^2 T_0) \text{ and } (W_J, \nabla^2 T_i) = B_{Ji}. \quad (12.8.145)$$

$(W_J, R)$  is the scalar product of  $W_J$  and  $R$  (cf. [Appendix A](#)) which is forced to be zero.  $W_J$  and  $T_i$  are known functions and  $C_i, i = 1, 2, \dots, n$  are the only unknowns which are to be obtained by solving the system of algebraic equations given below.

$$\sum_{i=1}^n B_{ji} C_i = A_J, \quad J = 1, 2, \dots, n. \quad (12.8.146)$$

The weighting functions can be chosen in many ways and each choice corresponds to a different criterion of applying MWR. One of the best known approximate method is Galerkin method. In this method the weighting functions are chosen to be trial functions, i.e.  $W_i = T_i$ . The trial functions must be chosen as members of a *complete set of functions*. The set of functions  $\{T_i\}$  is complete if any function  $f$  of a given class of functions say, of continuous functions, can be expressed as  $f = \sum_{i=1}^{\infty} a_i T_i$ ,  $a_i$  are constants. This representation of  $f$  is exact. A continuous function is zero if it is orthogonal to every member of a complete set of functions. In Galerkin method, the residual is made zero by choosing weighting function  $W_i$  as  $T_i$ ,  $i = 1, 2, \dots, n$  and invoking orthogonality.

In general, in any given formulation the differential equation can be written in the operator form as

$$L(f) = 0; f_n = n\text{-term approximation of } f, f_n = \sum_{i=1}^n c_i \phi_i(x_1, x_2, \dots, x_n). \quad (12.8.147)$$

$L(f_n)$  must be continuous for each  $n$  in the context of Galerkin method. Residual  $L(f_n)$  is not zero and we want to make the residual zero. An equivalent set of conditions to make the residual zero is:  $L(f_n)$  is continuous,  $\{\phi_i\}_{i=1}^{\infty}$  is a complete set of functions and  $L(f_n)$  is orthogonal to each  $\phi_i$ ,  $i = 1, 2, \dots$ . If these conditions are satisfied then  $\lim_{n \rightarrow \infty} L(f_n) = 0$ . The trial functions must form a complete set of functions and must be linearly independent. Any function in a given class of continuous functions can be represented exactly by an infinite series of a complete set of functions. In Eq. (12.8.145), if  $\phi_i(T_i)$ ,  $i = 1, 2, \dots$ , are complete set of functions satisfying boundary conditions,  $L(f_n)$  is continuous and  $L(f_n)$  is orthogonal to each  $\phi_i$ ,  $i = 1, 2, \dots$ , so that

$$(\phi_J, L(f_n)) = 0 \text{ for } J = 1, 2, \dots, n, \quad (12.8.148)$$

then by taking  $n$  sufficiently large  $L(f_n)$  can be made zero. Using Eq. (12.8.148) and  $f_n$  in Eq. (12.8.147), a system of  $n$ -algebraic equations is obtained to determine  $c_1, c_2, \dots, c_n$ . In the variational method to be discussed later, it is not necessary to satisfy *natural boundary conditions* as whilst deriving Euler-Lagrange equation, some boundary conditions are automatically satisfied. The remaining boundary conditions, if any, which are called *essential boundary conditions* should be satisfied. In Galerkin method the trial functions satisfy all the boundary conditions (this condition can be relaxed). We clarify this point by constructing trial functions in a simple problem of heat conduction without phase change.

A problem considered in [359] is about the application of Galerkin method to the solution of the equation  $L(T(x, y)) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + g/k = 0$ , to be satisfied in the region,  $|x| \leq a$ ,  $|y| \leq b$ .  $T = f(x, y)$  on the boundary of the rectangle. The trial functions  $T_i$ ,  $i = 1, 2, 3, \dots$ , satisfying the boundary conditions can be constructed with the help of a set of linearly independent functions  $\phi_1, \phi_2, \phi_n, \dots$ , given below.

$$\phi_1 = w(x, y), w(x, y) = (a^2 - x^2)(b^2 - y^2); \phi_2(x, y) = x^2 w(x, y), \quad (12.8.149)$$

$$\phi_3 = y^2 w(x, y), \phi_4 = x^2 y^2 w(x, y), \phi_5 = xyw(x, y), \phi_6, \phi_7, \dots \quad (12.8.150)$$

$$T_1(x, y) = c_1 \phi_1(x, y); \quad T_2(x, y) = c_1 \phi_1(x, y) + c_2 \phi_2(x, y), \quad (12.8.151)$$

$$T_n(x, y) = w(x, y) \left\{ c_1 + c_2 x^2 + c_3 y^2 + c_4 x^2 y^2 + \dots + c_n x^{2i} y^{2j} \right\}. \quad (12.8.152)$$

It is better to consider trial functions which are constant on the boundary. If the boundary condition is  $T(x, y) = f(x, y)$  on the boundary then take trial function  $T(x, y) = f(x, y) + \sum_{i=1}^N a_i T_i(x, y)$ ,  $T_i(x, y) = 0$  on the boundary for all  $i$ .

If only one term approximation  $T_1(x, y)$  of  $T(x, y)$  is considered then to determine the unknown constant  $c_1$  we use the orthogonality condition and obtain  $c_1$  as given below when  $f(x, y) = -g/k$ .

$$\int_{-b}^b \int_{-a}^a (L(T_1(x, y)), \phi_1(x, y)) dx dy = 0, \quad c_1 = 5(g/k)/(8(a^2 + b^2)). \quad (12.8.153)$$

For a two-term approximation of  $T(x, y)$ , consider  $T_2(x, y) = c_1 \phi_1(x, y) + c_2 \phi_2(x, y)$  and take scalar products of  $T_2$  with each  $\phi_1$  and  $\phi_2$ . Note that the above set of  $\phi_1, \phi_2, \dots$  is not a orthogonal set of functions. Evaluation of integrals becomes difficult as  $n$  increases and invariably it is to be done numerically. When  $T_2(x, y)$  is considered and scalar products of  $T_2$  with each  $\phi_1$  and  $\phi_2$  are taken, two equations containing two unknowns  $c_1$  and  $c_2$  are obtained. More terms can be added in the series of trial functions and the process is continued till sufficiently close values of constants are obtained or until calculations become unwieldy.

Instead of considering  $T_n(x, y)$  given in Eq. (12.8.152), it is possible to consider  $T_n(x, y)$  as

$$T_n(x, y) = C_1 + C_2 x + C_3 y + C_4 x^2 + C_5 xy + C_6 y^2 + \dots + C_n y^m. \quad (12.8.154)$$

$T_n(x, y)$  should satisfy the boundary conditions prescribed for the region and in this process some coefficients in Eq. (12.8.154) are eliminated and remaining coefficients are determined by Galerkin method. The requirement of complete set of functions should be met.

A simple one-dimensional one-phase Stefan problem whose numerical solution has been obtained in [666] is briefly discussed below. This problem is being discussed for illustration of a different type of application of Galerkin method in which the completeness criterion is satisfied in a weak sense. As mentioned earlier we do not intend to do literature survey and so for illustration some sample problems have been taken up. The temperature  $T(\xi, \tau)$  in dimensionless form and phase-change boundary  $x = S(t)$  satisfy the following equations.

$$\frac{\partial T}{\partial \tau} = \frac{\xi}{S(\tau)} \frac{dT}{d\tau} \frac{\partial T}{\partial \xi} + \frac{1}{S^2} \frac{\partial^2 T}{\partial \xi^2}, \quad 0 < \xi < 1, \tau > 0, \quad \xi = x/S(t), \tau = t, \quad (12.8.155)$$

subjected to the conditions

$$T = -1, \xi = 0; \quad T = 0, \xi = 1, \tau > 0; \quad \frac{1}{S} \frac{\partial T}{\partial \xi} = \frac{dS}{d\tau}, \xi = 1, \tau > 0, \quad (12.8.156)$$

along with one of the initial conditions given below.

$$T(\xi, 0) = (\xi - 1) \text{ for } 0 \leq \xi \leq 1, \text{ or } T(\xi, 0) = 0; \quad S(0) = 0.25. \quad (12.8.157)$$

To apply Galerkin method Eq. (12.8.155) is multiplied by a weight function  $W(\xi)$  and integrated from  $\xi = 0$  to  $\xi = 1$ . If  $W(\xi)$  is continuous and  $W(0) = W(1) = 0$ , it is not difficult to obtain

$$\int_0^1 \left( \frac{\partial T}{\partial \tau} W(\xi) - \frac{\xi}{S} \frac{dS}{d\tau} \frac{\partial T}{\partial \xi} W(\xi) + \frac{1}{S^2} \frac{\partial T}{\partial \xi} \frac{\partial W}{\partial \xi} \right) d\xi = 0. \quad (12.8.158)$$

In Galerkin method or in any general MWR, the differential equation is not satisfied pointwise in general in the classical sense but the differential equation could be satisfied in weak sense. The solution of Eq. (12.8.158) in [666] has been obtained with the help of finite element method. The interval  $[0, 1]$  is divided into  $N$  subintervals of equal length and if  $\xi_J = Jh$ ,  $J = 0, 1, 2, \dots, N$  where  $h = 1/N$  then we have  $N$  meshes of equal size and in the  $N$ -subintervals approximation of temperature can be written as given below.

$$\left. \begin{aligned} T_N(\xi, \tau) &= \sum_{J=0}^N T_J^h(\tau) \phi_J^h(\xi), \\ T_N(\xi_{i-1} \leq \xi \leq \xi_i, \tau) &= T_{i-1}^h(\tau) \phi_{i-1}^h(\xi) + T_i^h(\tau) \phi_i^h(\xi). \end{aligned} \right\} \quad (12.8.159)$$

$$\left. \begin{aligned} \phi_J^h(\xi) &= (\xi - \xi_{J-1})/h \text{ for } \xi_{J-1} \leq \xi \leq \xi_J, \\ \phi_J^h(\xi) &= (\xi_{J+1} - \xi)/h \text{ for } \xi_J \leq \xi \leq \xi_{J+1}, \\ \phi_J^h(\xi) &= 0, \text{ otherwise.} \end{aligned} \right\} \quad (12.8.160)$$

$T_J^h(\tau)$  and  $\phi_J^h(\xi)$  are approximations of  $T(\xi, \tau)$  in the  $J$ th subinterval when the region  $0 \leq \xi \leq 1$  is subdivided into  $N$  equal parts. If we take  $T_0^h(\tau) = -1$  and  $T_N^h(\tau) = 0$  in Eq. (12.8.159) then boundary conditions for temperature at  $\xi = 0$  and  $\xi = 1$  are satisfied. When the interval  $0 \leq \xi \leq 1$  is discretized and the weighting functions  $W(\xi)$  in different meshes are taken as  $W(\xi) = \phi_J^h(\xi)$ ,  $J = 1, 2, \dots, N-1$ , a coupled system of  $N-1$  differential equations is obtained as follows.

$$a_1 \frac{dT_{i-1}^h}{d\tau} + a_2 \frac{dT_i^h}{d\tau} + a_3 \frac{dT_{i+1}^h}{d\tau} = b_1 T_{i-1}^h + b_2 T_i^h + b_3 T_{i+1}^h, \quad i = 1, 2, \dots, N-1. \quad (12.8.161)$$

For the expressions of  $a_1, a_2, \dots$ , refer [666]. The initial condition for  $T_i^h(\tau)$  is taken as  $T_i^h(0) = ih - 1$ ,  $i = 1, 2, \dots, N-1$ ,  $S(0) = 0.25$ . To obtain the phase-change boundary  $S(\tau)$  the following equation can be used.

$$\frac{dS}{d\tau} = \frac{1}{S} T_N^h(\tau) \left( \phi_{N-1}^h(\xi) \right)'_{\xi=1} = -\frac{1}{Sh} T_{N-1}^h(\tau), \quad \text{for } \xi = 1, \quad (12.8.162)$$

dash denotes differentiation with respect to  $\xi$ . Numerical integration of Eq. (12.8.162) is simple. The procedure is repeated until further iterations make no change.

Finite-difference method can also be used for numerical computations and for numerical procedure [666].

The transient heat equation in  $T(x, t)$  is a two-dimensional problem if considered in  $x$  and  $t$  independent variables. A two-dimensional problem whether in  $x$  and  $t$  or in two space variables  $x$  and  $y$  can be treated as one-dimensional problem using MWR provided the dependence on

$x$  or  $y$  is specified which is to be decided according to the problem formulation. Suppose  $y$  dependence is assumed then  $x$  dependence is used in a  $N$ -term trial function and using MWR,  $y$  dependence is determined as done in Eq. (12.8.161). The independent variable  $\xi$  in Eq. (12.8.161) was introduced to fix the moving boundary and if it was not required then we could have continued with  $x$  and  $t$  independent variables. The choice of obtaining coupled system of differential equations in time variable is apparent in this case because of Eq. (12.8.158). In terms of space variables  $x$  and  $y$ , the choice of obtaining coupled system of differential equations in  $x$  or  $y$  depends on the boundary conditions. In Eq. (12.8.158) the dependence on  $\xi$  has been specified. This point will be further discussed.

In Eq. (12.8.158) the weight function  $W(x)$  satisfies essential boundary conditions and the same  $W(x)$  is used in Galerkin method. It is generally thought that in all MWR the trial function must satisfy all the boundary conditions. This is not true. In the variational methods which are also MWR, the natural boundary conditions need not be satisfied by trial functions as the Variational Principle forces them to be satisfied. However, the trial functions should satisfy essential boundary conditions. The same approach is used in MWR. It is very difficult to construct trial functions which satisfy all the boundary conditions. Galerkin method is closely related to variational and adjoint variational methods to be discussed later.

Our interest here is on the applications of Galerkin method in obtaining analytical-numerical solutions of Stefan problems but not much material is available concerning Stefan or Stefan-like problems in whose solutions Galerkin method has been used. The convergence aspects are important in MWR and in particular Galerkin method. For rigorous treatment of Galerkin method and some other approximate methods, mathematical proofs, results regarding convergence and error bounds and applications to many problems of mathematical physics, refer [667, 668]. However, Stefan and Stefan-like problems have not been discussed in these references. A systematic treatment of MWR and variational methods can be found in [495] with hundreds of references related to many aspects of problems in heat and mass transfer, fluid mechanics, chemical reaction systems, etc. Convergence and error bounds have also been discussed but the survey is up to 1970.

A two-phase continuous casting Stefan problem has been considered in [669] and finite element numerical solution has been obtained by combining it with a characteristic Galerkin method with adaptive error control. A convection-dominated nonlinearly degenerate diffusion problem has been considered in which boundary conditions could be of first, second or convective type. The concept of *parabolic duality* has been introduced and several stability estimates have been proved. A priori error estimates and error bounds have also been derived. An exact analytical solution in the form of travelling wave solution has been considered for comparing the numerical method's performance. By taking realistic physical parameters, the application of results has been shown for casting of steel. The authors have made good contribution by studying the problem from different angles. Without enlarging the scope of the contents of this chapter further details cannot be discussed.

Galerkin method in finite element solution of a problem of cooling process for spin casting mould and cast parts has been considered in [670]. A three-dimensional problem has been considered in which the latent heat released is taken as the specific heat in order to simplify the formulation. Therefore the problem considered is a heat conduction problem without phase change. Experimental verification of numerical solution has also been done.

An analytical-numerical solution of a Stefan-like problem using Galerkin method has been presented in [671]. A one-dimensional heat and mass transfer problem of geophysical interest has been modelled by taking into account coupling between two-phases present which are

the ice sheets and the ocean. The interaction between an ice sheet and ice free part of the earth-atmosphere system has been modelled as an FBP. In the modelling of the dynamical system, atmosphere-hydrosphere-cryosphere are participating, in which the spacial degrees of freedom along the vertical and longitudinal directions have been lumped. At the phase-change boundary, the boundary conditions are of the type as in Neumann problem but the parameters involved are different from those in the Stefan problem except latent heat. The derivation of the energy equation and its linearization involves several new parameters and lengthy discussion. Therefore it is not possible to present the formulation here and only some steps in the method of solution are given below.

In the application of Galerkin method in obtaining analytical solution of the problem considered in [671] let  $\theta(x, t)$  be the dimensionless temperature satisfying energy equation in which  $x$  is the sine of the latitude and  $t$  is time.  $\theta$  is expressed as

$$\theta = \sum_{n=0}^N \theta_n P_n(x) + u(x), \quad n \text{ is even}; \quad \theta_n = \int_0^1 (2n+1) \theta P_n(x) dx, \quad (12.8.163)$$

$P_n(x), n = 2, 4, \dots, P_n(x)$  are *Legendre polynomials* forming complete set of functions for a symmetric problem in  $x$  and  $u(x)$  is orthogonal to all Legendre polynomials. Take  $N = 2$ ,  $\theta = \theta_0(t) + \theta_2(t)P_2(x) + u(x)$ , and

$$u(x) = a_0 + a_2 x^2/2, \quad x < x_S; \quad u(x) = b_0 + b_2 x + c_2 \frac{x^2}{2}, \quad x > x_S, \quad (12.8.164)$$

$x_S$  is the sine of the latitude of the ice boundary. Using orthogonality conditions of  $u(x)$  and the two boundary conditions at the phase-change boundary four equations are obtained having  $b_0$  as a free parameter. Substituting  $\theta$  in the energy equation and multiplying it successively by  $P_0, P_2$  and  $P_4$  and integrating over the domain of the problem, we get three first-order differential equations for  $\theta_0, \theta_2$  and  $b_0$ . Instead of solving this system of equations but still using them, a characteristic equation for the problem is obtained in terms of a parameter  $w$  by taking time dependence as  $\exp(wt)$ . The analysis is centred on the stability of climatic regime with respect to small excursions of the ice boundary. What has been presented above is only a broad outline of the solution procedure. For further details refer [671].

A steady-state, radiative and conductive coupled heat transfer problem without phase change in a three-dimensional semitransparent grey medium with opaque diffusive walls has been considered in [672]. For the numerical solution, meshless local *Petrov-Galerkin approach* has been used. *Moving least-square (MLS) approximation* [673] is employed for constructing shape functions. Weight function has been constructed by using quadratic spline functions. The introduction in [673] is useful as it contains several references about variations of Galerkin method and their applications in conjunction with numerical methods.

The general form of the radiative steady-state heat transfer in an anisotropic scattering participating medium with spatially-dependent refractive index has been considered in [674]. There is no phase change. Galerkin method converts the integro-differential form of radiative heat transfer equation into a set of algebraic equations that can be easily solved. After obtaining the analytical form of radiation energy density and the net radiant heat flux through the medium, reflection and transmission coefficients at the medium boundary have been determined.



## Orthogonal Collocation Method

In the collocation method the weighting function  $W_J(X)$  is taken as

$$W_J(X) = \delta(X - X_J) \text{ and so } \int_V W_J R dv = R|_{X_J}, \quad R \text{ is residual,} \quad (12.8.165)$$

$X_J, J = 1, 2, N$ , are some known points in the region,  $V < R^n$ , whose choice is dictated by the problem formulation or could be taken arbitrarily. It is assumed that at the collocation points residual is zero. The unknown constants in the trial function are determined by solving a system of algebraic equations in one-dimensional problems but in multidimensional problems the system of differential equations is to be solved. Once the unknown constants in the trial functions are known, the trial function at any required point in the region under consideration could be obtained by interpolation.  $X_J$  are called collocation points or interior collocation points. The arbitrariness in the choice of collocation points can be removed if *orthogonal collocation* technique is used which is described below. It is assumed that when the number of collocation points becomes very large, the exact solution is obtained.

One common form of trial functions  $y_i(x), x \in R$  and the polynomials  $P_m(x)$  obtained with the help of these trial functions are given below

$$P_m(x) = \sum_{J=0}^m C_J x^J; \quad y_i = x^{i-1}, \quad i = 1, 2, \dots; \quad m = 0, 1, 2, \dots \quad (12.8.166)$$

The unknown coefficients  $C_J$  are determined by requiring  $P_m(x)$  to be orthogonal to all polynomials of order less than  $m$  with respect to some weighting function  $W(x) \geq 0$ , i.e.

$$\int_a^b W(x) P_n(x) P_m(x) dx = 0, \quad n = 0, 1, 2, \dots, m-1, m \neq n. \quad (12.8.167)$$

$P_0$  is taken as unity,  $P_1(x) = x$  and  $P_2(x) = 1 - 3x^2$  provided the interval of integration is  $1 \leq x \leq 1$  and  $W(x) = 1$ . If the  $m$ th degree polynomial  $P_m(x)$  is the last polynomial in the approximating function for the dependent variable used in Eq. (12.8.167) then collocation points are taken as the roots of  $P_m(x) = 0$ . The interval of integration in Eq. (12.8.167) can be changed to any other suitable interval by a simple transformation.  $P_n(x)$  is completely known for  $n = 0, 1, \dots, m$  if the above procedure is used.

The polynomials  $P_n(x), n = 1, 2, \dots, m$ , so obtained are called *Legendre polynomials*. If the interval of integration is  $0 \leq x \leq 1$  and the problem is symmetrical about  $x = 0$  then instead of  $P_m(x), P_m(x^2)$  should be considered and similarly  $W(x^2)$  is used instead of  $W(x)$ . Orthogonality condition can now be written as

$$\int_0^1 W(x^2) P_J(x^2) P_i(x^2) x^{p-1} dx = C_i S_{iJ}, \quad J = 1, 2, \dots, i-1, \quad (12.8.168)$$

where  $p = 1, 2, 3$ , for planar, cylindrical and spherical geometry, respectively. The first coefficient in every polynomial is taken as unity. A possible form of trial function/approximating function in the symmetrical case is

$$y(x) = y(1) + (1 - x^2) \sum_{i=1}^N a_i P_{i-1}(x^2) \text{ or } y(x) = \sum_{i=1}^{N+1} d_i x^{2i-2}, \quad 0 \leq x \leq 1. \quad (12.8.169)$$

The integer  $N$  stands for interior collocation points.  $P_{N-1}(x^2)$  is a polynomial of degree  $(N-1)$  in  $x^2$ . When  $y(x)$ ,  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  are evaluated at the collocation points  $x_J$ , we get

$$y(x_J) = \sum_{i=1}^{N+1} d_i x_J^{2i-1}, \quad \bar{y} = \bar{Q}\bar{d}, \quad Q_{Ji} = x_J^{2i-2}, \quad \bar{d} = \bar{Q}^{-1}\bar{y}. \quad (12.8.170)$$

$$\left. \frac{dy}{dx} \right|_{x_J} = \sum_{i=1}^{N+1} \left. \frac{d(x^{2i-2})}{dx} \right|_{x_J} d_i, \quad \frac{\bar{dy}}{dx} = \bar{C}\bar{d}, \quad C_{Ji} = \left. \frac{d(x^{2i-2})}{dx} \right|_{x_J}, \quad \frac{\bar{dy}}{dx} = \bar{C}\bar{Q}^{-1}\bar{y}. \quad (12.8.171)$$

$$\nabla^2 y|_{x_J} = \sum_{i=1}^{N+1} \nabla^2(x^{2i-2})|_{x_J} d_i, \quad \overline{\nabla^2 y} = \bar{D}\bar{d}, \quad D_{Ji} = \nabla^2(x^{2i-2})|_{x_J}, \quad \overline{\nabla^2 y} = \bar{D}\bar{Q}^{-1}\bar{y}. \quad (12.8.172)$$

Bar over various quantities indicates that they are matrices and  $Q^{-1}$  stands for inverse of a matrix. The inverse of  $\bar{Q}$  exists as  $1, x^2, x^4, \dots$ , form a linearly independent set of functions.  $\bar{d}$  is uniquely determined. For example  $\bar{D}$  is a  $N \times N$  matrix and  $D_{Ji}$  is the element in the  $J$ th row and  $i$ th column.  $\bar{d}$  is a  $N \times 1$  column matrix with  $i$ th element as  $d_i$ .

A simple procedure to calculate accurately the integrals in Eq. (12.8.169) has been reported in [495]. Using quadrature formula

$$\int_0^1 f(x^2)x^{p-1}dx = \sum_{i=1}^{N+1} W_i f(x_i); \quad \sum_{i=1}^{N+1} W_i x_i^{2i-2} = (2i-2+p)^{-1} = f_i. \quad (12.8.173)$$

If  $f(x) = f = x^{2i-2}$  and  $W_i = x^{p-1}$  in the second equation in Eq. (12.8.173) then it is easy to obtain the relation  $\bar{G}\bar{Q} = \bar{f}$ ,  $\bar{G} = 1 \times N$  matrix,  $\bar{Q}$  is a  $N \times N$  matrix as in Eq. (12.8.170) and  $\bar{f} = [f_1, f_2, \dots, f_N]$ . If  $W(x) = 1 - x^2$  in Eq. (12.8.168) and functions  $f$  are polynomials of degree  $2N$  in  $x^2$  and interior collocation points are the roots of  $P_N(x^2)$ , then the integration in Eq. (12.8.173) is exact and we get  $f_i$  as in Eq. (12.8.173) on r.h.s.

As an application of orthogonal collocation method, the formulation of a Stefan problem considered in [675] is given below. We take all the parameters as unity for convenience.

$$\left. \begin{aligned} \frac{\partial C_A}{\partial t} &= \frac{\partial^2 C_A}{\partial r^2} + \frac{2}{r} \frac{\partial C_A}{\partial r}, \quad S(t) < r < 1, \quad t > 0; \quad C_A(1, t) = 1, \\ C_A(S, t) &= 0; \quad \frac{dS}{dt} = - \left. \frac{\partial C_A}{\partial r} \right|_{r=S}, \end{aligned} \right\} \quad (12.8.174)$$

$r$  is the radial coordinate in a spherical problem and  $r = S(t)$  is the phase-change boundary,  $S(0) = 1$ . It is required that the integration region in collocation method should be a fixed region. By using the following transformation the above problem can be transformed into a fixed region problem,

$$x = (r - S(t))/(1 - S(t)), \quad 0 \leq x \leq 1; \quad r = S(t) \Rightarrow x = 0. \quad (12.8.175)$$

It is easy to get transformed formulation. The trial function is taken as

$$C_A = x + \sum_{n=1}^{N+1} d_n x(1-x) P_{n-1}(2x-1), \quad P_n \text{ is Legendre polynomial.} \quad (12.8.176)$$

It may be noted that trial function should satisfy the prescribed boundary conditions as boundary conditions are not satisfied separately afterwards. The internal collocation points are the roots of the equation  $P_N(2x-1) = 0$ . In [675], the matrix equations to determine unknown coefficients  $d_n$  are not obtained following the method suggested in Eqs (12.8.170)–(12.8.172). In any case  $d_n$  are obtained uniquely if  $\bar{Q}$  is not singular. Having obtained  $d_n$ ,  $C_A(x_J)$ ,  $J = 1, 2, \dots, N$  are substituted in the differential equation in Eq. (12.8.174) and a system of  $N$  first-order differential equations is obtained in time variable at the points  $x_J$  which are solved numerically using initial condition.  $x_J$  are the collocation points which are the roots of the equation  $P_N(2x-1) = 0$ . For the numerical solution several numerical methods are available such as RKM.

Collocation method can also be used on combining it with least squares method. The *least squares method* is very briefly described below. In the least squares method if the residual is denoted by  $R(C_i, X)$  then the weighting function is  $\partial R / \partial C_J$  and therefore

$$I(C_i) = \int_V R^2(C_i, X) dv, \quad (12.8.177)$$

is minimized with respect to constants  $C_i$ . These days computer codes are available to obtain the minimization numerically. The advantage in least squares method is that the minimization gives best possible error bounds.

To improve the values of the constant coefficients obtained in the collocation method in which if the collocation points are fixed, least squares method may be used. In the application of least squares method the residual can be evaluated at some additional points, i.e. more than the allotted collocation points. The residual at each point can be squared and added and then Eq. (12.8.177) can be used to improve the values of the coefficients by minimization. This way we are improving the values of the same constants using the same collocation points.

In the boundary collocation method, the collocation points are taken on the boundary of the region. This means that the problem has to be formulated in terms of boundary integrals. This integral formulation has been discussed in Section 12.2 which is the analytical part in analytical-numerical solutions. The second part is the numerical solution or the numerical procedures which have been discussed earlier very briefly but not in the context of collocation method. Further discussion about numerical procedures in the context of boundary collocation is not possible here.

In [676], comparison of boundary collocation methods in the context of singular and non-singular axisymmetric heat transfer problems has been reported. Numerical results have been compared by considering some testing problems. The introduction gives useful information about the boundary collocation methods which includes method of fundamental solutions (MFS) with fixed locations of source points, MFS with variable locations of source points, direct Trefftz method, indirect Trefftz method, direct and indirect BEMs. As the formulations of testing problems are not given in [676] it is not clear whether Stefan condition was included in the formulation of heat transfer in melt zone or not. Numerical results have been presented and references of many other related works are given in [676].

Some of the following guidelines for choosing the trial functions in approximate methods could be used. If the dependent variable is  $T(x, t)$  then the trial function could be expanded in spacial variable  $x$  as follows

$$T(x, t) = f(x) + \sum_{i=1}^N A_i(t) X_i(x), \quad (12.8.178)$$

$X_i$  satisfy boundary conditions. If the dependent variable is  $T(x, y, z)$  then the trial function could be taken in the form

$$T(x, y, z) = Z_1(z)T_1(x, y) + Z_2(z)T_2(x, y), \quad (12.8.179)$$

provided the solution at  $z = 0$  is  $T_1(x, y)$  and at  $z = \infty$  it is  $T_2(x, y)$ . The splicing functions  $Z_i$  may be obtained by MWR. The solutions  $T_1(x, y)$  and  $T_2(x, y)$  could be exact solutions, numerical solutions or MWR solutions. Discontinuous trial functions can also be used as in reciprocal variational principles (cf. [495]). In some problems, the trial function may easily satisfy the differential equation but it is not easy to satisfy boundary conditions. In such cases boundary methods can be used. Spline functions and finite element functions are completely general trial functions which are available for wide variety of geometrics.

### Variational Methods and Variational Principles

Before going into details of variational methods, we first consider a simple problem of finding a continuous curve  $y = y(x)$  in  $(x, y)$  plane  $R^2$  whose length between the points  $(a, b)$  and  $(c, d)$  is shortest. If  $S$  is the arc length then

$$S = \int_{(a,b)}^{(c,d)} ds = \int_{(a,b)}^{(c,d)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx, \quad y(a) = b, y(c) = d. \quad (12.8.180)$$

To find the shortest curve, the functional  $S$  in Eq. (12.8.180) is to be minimized. Eq. (12.8.180) describes a mapping from some space of continuous functions to which  $y(x)$  belongs into the set of real numbers. This type of mapping or a function is called a *functional*. A similar type of question about shortest length can be asked about a curve on a surface or about minimum or maximum thermal energy or fluid velocity, maximum rate of solidification/melting, etc. In essence, we are concerned with the maxima or minima of functionals or in other words extremum of functionals. In the above problem it was easy to construct the functional whose minimum gives the answer. What about the functionals of Stefan problems which have been considered in this chapter? So the first thing is to answer 'how to construct a functional for a given problem'. If the given problem can be solved exactly by using its classical or weak formulation then it is not necessary to find the extremum of a functional as generally it is not possible to find the extremum exactly. Only approximate values of required minimum of functionals are obtained numerically by some well-defined procedures. For a nonlinear problem invariably an exact analytical solution is not possible and we have to find an approximate solution.

A variational method is concerned with obtaining the *extremum* of a functional provided extremum exists or we say provided a variational principle exists. This is the requirement for the application of variational methods. The other related material will be discussed after

the minimum of the functional in Eq. (12.8.180) is obtained. It is assumed that  $y = y(x)$  possesses second-order partial derivatives and the minimum of the functional is attained at  $y = \phi(x)$ . We consider a family of neighbouring curves of  $\phi(x)$  which are characterized by a small scalar parameter  $\epsilon$  and any member of this family can be expressed as  $y = \phi(x) + \epsilon\eta(x)$ , where  $\eta(x)$  is an arbitrary function belonging to a family of curves to which  $y(x)$  belongs. Since  $y(a) = \phi(a) = b$ ,  $\eta(a) = 0$  and similarly  $\eta(b) = 0$ . Define  $F(x, y, y')$  and  $G(\epsilon)$  as follows.

$$F(x, y, y') = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2}, \quad G(\epsilon) = \int_a^b F(x, \phi + \epsilon\eta, \phi' + \epsilon\eta') dx. \quad (12.8.181)$$

It is assumed that  $F$  has continuous second-order partial derivatives with respect to its arguments  $x, y, y'$ .  $G(0)$  is minimum of  $G(\epsilon)$  which is attained at  $y = \phi(x)$ .  $G(\epsilon) \geq G(0)$ . If  $G(\epsilon)$  takes the minimum value at  $\epsilon = 0$  then  $G'(0) = 0$  which is a necessary condition.

$$G'(0) = \lim_{\epsilon \rightarrow 0} \frac{G(\epsilon) - G(0)}{\epsilon} = \lim_{\epsilon \rightarrow 0} \int_a^b \frac{F(x, \phi + \epsilon\eta, \phi' + \epsilon\eta') - F(x, \phi, \phi')}{\epsilon} dx. \quad (12.8.182)$$

$$\left. \begin{aligned} F(x, \phi + \epsilon\eta, \phi' + \epsilon\eta') &= F(x, \phi, \phi') + \epsilon\eta(x)F_{\phi}(x, \phi + \epsilon\eta, \phi') \\ &+ \epsilon\eta'(x)F_{\phi'}(x, \phi, \phi' + \epsilon\eta') + \text{terms of } O(\epsilon^2). \end{aligned} \right\} \quad (12.8.183)$$

$F_{\phi}$  and  $F_{\phi'}$  denote derivatives of  $F$  with respect to  $\phi$  and  $\phi'$  respectively,  $0 < \theta_1 < 1$  and  $0 < \theta_2 < 1$ . The first term in the integrand in Eq. (12.8.182) is expanded as in Eq. (12.8.183) and the term  $\eta'(x)F_{\phi'}$  is integrated by parts. When the limit  $\epsilon \rightarrow 0$  is taken it is easy to get

$$G'(0) = \int_a^b \left\{ \frac{d}{dx} F_{y'}(x, \phi, \phi') - F_{y'}(x, \phi, \phi') \right\} \eta(x) dx. \quad (12.8.184)$$

Since  $\eta(x)$  is completely arbitrary except  $\eta(a) = \eta(b) = 0$ , and  $F$  has continuous derivatives,  $G'(0) = 0$  implies

$$\frac{d}{dx} F_{y'}(x, \phi, \phi') - F_{y'}(x, \phi, \phi') = 0. \quad (12.8.185)$$

Let

$$\delta G = G'(\epsilon)\epsilon = \int_a^b \left\{ F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') \right\} \epsilon \eta dx = 0. \quad (12.8.186)$$

If  $y(x) = \phi(x) + \epsilon\eta(x) = \phi(x) + \delta y(x)$  then in shorter notation we write

$$\delta G = G'(\epsilon)\epsilon = \delta \int_a^b F dx = \int_a^b \delta F dx = \int_a^b \left( F_y - \frac{d}{dx} F_{y'} \right) \delta y dx = 0. \quad (12.8.187)$$

$\delta G$  is known as the *first variation of the functional* corresponding to  $F(x, y, y')$ . Eq. (12.8.185) is obtained by equating first variation of the functional  $F$  to zero and it is called *Euler's equation* or *Euler-Lagrange equation*. The calculus of variations is that branch of science or of mathematics which deals with the variations of functionals. The necessary condition

for a functional to have an extremum is that its first variation should be zero. We also say that the functional is stationary if its first variation is zero. The condition  $\delta G = 0$  does not tell whether the stationary value of the functional is maximum or minimum or the solution of Euler's equation will maximize or minimize the functional. For this information sufficient conditions are to be derived and these will be discussed later.

A simple functional has been considered in Eq. (12.8.180). We may have integrand  $F$  in a functional of the type given below with constraint  $H$  given below in which  $m$  is prescribed.

$$F = F(t, x_1, x_2, \dots, x_n, \dot{x}_1, \dot{x}_2, \dot{x}_n); \quad H(t, x_1, x_2, \dots, x_n) = m, \dot{x} = dx/dt. \quad (12.8.188)$$

$F$  could be more general and there could be several constraints of different types. For the shortest length problem considered in Eq. (12.8.180), after calculating derivatives occurring in Eq. (12.8.185), the Euler's equation is obtained as

$$\frac{d}{dx} \left\{ \phi' / \sqrt{1 + \phi'^2} \right\} = 0 \text{ and } \phi(x) = bx/a \text{ is the solution.} \quad (12.8.189)$$

When constraints of the form given in Eq. (12.8.188) are imposed on the solution then Lagrange multipliers are used to determine Euler's equations and its solution.

Consider a simple problem in  $x(t)$  and  $y(t)$  variables in which  $t$  is an independent variable and the functional is

$$G = \int_a^b F(t, x, y, \dot{x}, \dot{y}) dt; \quad H(t, x, y) = m. \quad (12.8.190)$$

It is assumed that functions  $x = \phi_1(t)$ ,  $y = \phi_2(t)$  and  $F$  are nice smooth functions.  $\phi_1$  and  $\phi_2$  are the two functions which give the stationary value of the above functional satisfying the constraint, and  $\eta_1(t)$  and  $\eta_2(t)$  as defined in Eq. (12.8.182) satisfy the boundary conditions prescribed for  $\phi_1$  and  $\phi_2$ .  $\eta_1(t)$ ,  $\eta_2(t)$  belong to the class of functions to which  $\phi_1$  and  $\phi_2$  belong. Define  $G(\epsilon_1, \epsilon_2)$  and  $\bar{H}(\epsilon_1, \epsilon_2)$  as given below.

$$G(\epsilon_1, \epsilon_2) = \int_a^b F(t, \phi_1 + \epsilon_1 \eta_1, \phi_2 + \epsilon_2 \eta_2, \dot{\phi}_1 + \epsilon_1 \dot{\eta}_1, \dot{\phi}_2 + \epsilon_2 \dot{\eta}_2) dt. \quad (12.8.191)$$

$$\bar{H}(\epsilon_1, \epsilon_2) = \int_a^b H(t, \phi_1 + \epsilon_1 \eta_1, \phi_2 + \epsilon_2 \eta_2) dt - \bar{m} = 0. \quad (12.8.192)$$

$$\begin{aligned} \delta G(\epsilon_1, \epsilon_2) &= \frac{\partial G}{\partial \epsilon_1}(0, 0) \epsilon_1 + \frac{\partial G}{\partial \epsilon_2}(0, 0) \epsilon_2 \\ &= \int_a^b \left[ F_x(t, \phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) - \frac{d}{dt} F_{\dot{x}}(t, \phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) \right] \epsilon_1 \eta_1(t) dt \\ &\quad + \int_a^b \left[ F_y(t, \phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) - \frac{d}{dt} F_{\dot{y}}(t, \phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) \right] \epsilon_2 \eta_2(t) dt. \end{aligned} \quad (12.8.193)$$

The Euler's equations are obtained when  $\delta G = 0$  subjected to the condition  $\delta \bar{H} = 0$ .

$$\delta \bar{H}(\epsilon_1, \epsilon_2) = \int_a^b (H_x(t, \phi_1, \phi_2) \epsilon_1 \eta_1(t) + H_y(t, \phi_1, \phi_2) \epsilon_2 \eta_2(t)) dt. \quad (12.8.194)$$

$\eta_1(t)$  and  $\eta_2(t)$  satisfy the constraint  $H = m$  given in Eq. (12.8.188). Multiply Eq. (12.8.194) by  $\lambda(t)$ , which is called Lagrange multiplier and add it to  $\delta G$ . Consider the stationary value of  $\delta\bar{H} + \delta G$  with respect to  $(\epsilon_1 = 0, \epsilon_2 = 0)$ . First differentiate  $G + \lambda\bar{H}$  with respect to  $\epsilon_1$  and evaluate it at  $(\epsilon_1 = 0, \epsilon_2 = 0)$ . Then differentiate  $G + \lambda\bar{H}$  with respect to  $\epsilon_2$  and evaluate it at  $(\epsilon_1 = 0, \epsilon_2 = 0)$ . There will be two integrals one each for differentiation with respect to  $\epsilon_1$  and  $\epsilon_2$ , respectively. Since  $\lambda(t)$  is arbitrary, we choose it such that the following equation holds.

$$F_x - \frac{d}{dt}F_{\dot{x}} + \lambda\bar{H}_x = 0. \quad (12.8.195)$$

This leaves us with only the second integral and since  $\eta_2(t)$  is arbitrary, we have

$$F_y - \frac{d}{dt}F_{\dot{y}} + \lambda\bar{H}_y = 0. \quad (12.8.196)$$

The function  $\lambda(t)$  can be eliminated from Eqs (12.8.195), (12.8.196) which gives one equation in  $\phi_1$  and  $\phi_2$ . Another equation in  $\phi_1$  and  $\phi_2$  is provided by the constraint  $H(t, \phi_1, \phi_2) = m$ . The treatment in [228] is slightly different but finally the equations obtained are Eqs (12.8.195), (12.8.196).

The shortest curve on a cylindrical surface of given length can be found using the above method. Variety of functionals have been considered in [228, 495, 668, 677, 678] with attention to their formulation and solution aspects. The spaces to which the functionals and dependent variables should belong have been discussed along with necessary and sufficient conditions for the existence of variational principles. The criteria for knowing that the stationary value of functional leads to an extremum or not have been discussed more rigorously in [678].

If the stationary value of a functional exists, i.e. the Euler's equation can be obtained, then we say that variational principle exists for the functional or simply variational principle exists. Euler's equation is the necessary condition for variational principle to exist. We usually have the functional given and its Euler's equation is to be obtained. The inverse problem is that if the problem formulation with differential equation and boundary conditions is known then how to construct the functional for this given problem. What is the importance of the functional and variational principle in this case? Why not work directly with the formulation. We have seen earlier that when the problem is nonlinear, in particular a Stefan problem, then only approximate solutions can be obtained except in some very simple cases. There exist good methods by which fairly accurate minimum of a functional can be obtained. It is difficult to estimate the errors in approximate solutions obtained in several other methods but this is possible in variational method and it gives the least error amongst MWR. If weighting function in Galerkin method is chosen suitably then variational method is the same as the Galerkin method. This is shown below.

Let  $P(T) = 0$  be some nonlinear equation and  $I(T)$  be the variational formulation for this equation for which variational principle exists, i.e. the functional can be made stationary. As  $P(T) = 0$  is the Euler's equation which is obtained by variational method, we have

$$\delta I = \int_V \delta(T)P(T)dv = 0 \Rightarrow P(T) = 0. \quad (12.8.197)$$

This is how the Euler's equation is obtained. To find the solution by variational method,  $T$  is expressed in a series of trial functions in which the unknown coefficients  $a_J$  which occur in the series are to be determined and to determine them, the following equations are used.

$$\int_V \frac{\partial T}{\partial a_J} P(T) dv = 0, \quad J = 1, 2, n. \quad (12.8.198)$$

If in the Galerkin method, weighting function is taken as  $\frac{\partial T}{\partial a_J}$  then it is the same as the variational method.

In the least squares method also the solution can be obtained by the application of variational method and this has been shown in [679] through an example. Consider the following partial differential equation with some boundary and initial conditions.

$$G(T, T_x, T_t, T_{xx}, T_{xt}, T_{tt}, x, t) = 0, \quad x \in [x_0, x_1], \quad t \in [t_0, t_1], \quad (12.8.199)$$

and the functional corresponding to  $G = 0$ .

$$I = \int_{t_0}^{t_1} \int_{x_0}^{x_1} [G]^2 dx dt. \quad (12.8.200)$$

The first variation of  $I$ , i.e.  $\delta I$ , can be calculated using the procedure similar to that outlined in Eq. (12.8.191). There is no constraint in this problem. The lengthy expression of  $\delta I$  is not given here and refer [679] for the full expression of  $\delta I$  but it may be mentioned that the Euler's equation is a fourth-order partial differential equation. In the process of finding  $\delta I$ , in addition to other terms we get two terms, one regarding boundary conditions at  $x_0$  and  $x_1$  and another term for conditions at  $t = t_0$  and  $t = t_1$ . The conditions to be satisfied only at  $t_0$  and  $t_1$  are being discussed here to explain some important points in the method (cf. [679]). Consider the following expression.

$$2 \int_{x_0}^{x_1} \left[ \left\{ G \frac{\partial G}{\partial T_t} - \left( G \frac{\partial G}{\partial T_{tt}} \right)_t \right\} \delta T + \left\{ \frac{\partial G}{\partial T_{tt}} \delta T_t + G \frac{\partial G}{\partial T_{tx}} \delta T_x \right\} \right]_{t_0}^{t_1} dx. \quad (12.8.201)$$

Depending on whether  $T$  is prescribed or  $T_x$  is prescribed, some conditions can be derived at  $t = t_0$  and  $t = t_1$  so that expression in Eq. (12.8.201) equals zero. For example, if  $T$  and  $T_x$  are prescribed at  $t = t_1$  then  $\delta T = 0 = \delta T_x$  and  $\delta T_t$  is arbitrary, so  $\partial G / \partial T_{tt} = 0$  is the natural boundary condition which is and should be prescribed in the problem. This explains how natural boundary conditions arise corresponding to prescribed boundary conditions. An important point here is that all this makes sense provided the variational principle exists for the problem. These conditions which we derive using variations in prescribed boundary conditions such as  $T, T_t$  and  $T_x$  give rise to *natural boundary conditions*. Whilst finding the approximate solutions, the trial functions need not satisfy the natural boundary conditions as in the variational formulation they are naturally satisfied whilst obtaining extremum of functionals provided correct functionals are chosen. Boundary conditions other than natural boundary condition which are called essential boundary conditions are to be satisfied by trial functions. For transient heat transfer problems a true variational principle based on theory of calculus of variations does not exist. This point will be further discussed later.



In an example considered in [679], we have

$$\left. \begin{aligned} G &= \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2}, \quad 0 \leq x \leq q(t), \quad 0 \leq t \leq \infty, \\ I(G) &= \int_{t=0}^{\infty} \int_{x=0}^{q(t)} G^2 dx dt, \quad T = T_0 \left(1 - \frac{x}{q(t)}\right)^2, \end{aligned} \right\} \quad (12.8.202)$$

$T(0, t) = T_0$ ,  $T(q(t), t) = 0$ , and approximate solution is taken as in Eq. (12.8.202).  $q(t)$  is the penetration depth in this heat transfer problem and  $q$  is taken as generalized coordinate in the Lagrangian function  $L(t, q, \dot{q}, \ddot{q})$ .  $q(0) = 0$  and so  $\delta q(0) = 0$  but  $\delta q(t)$  as  $t \rightarrow \infty$  is not zero as  $q(t)$ ,  $t \rightarrow \infty$  is not prescribed. We first take  $t_0 \leq t \leq t_1$  and assume that temperature is prescribed at  $t = t_0$  and  $t = t_1$ . It can be easily checked that when  $G$  is replaced by the approximate temperature given in Eq. (12.8.202), after integration with respect to  $x$ , we get

$$\begin{aligned} I &= 4T_0^2 \int_{t_0}^{t_1} \left( \frac{k^2}{q^3} - \frac{k}{3} \frac{\dot{q}}{q^2} + \frac{\dot{q}^2}{30q} \right) dt; \\ \delta I &= \int_{t_0}^{t_1} \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \ddot{q}} \right) \right) \delta q dt \\ &\quad + \left[ \left( \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} \right) \delta q + \frac{\partial L}{\partial \ddot{q}} \delta \dot{q} \right]_{t_0}^{t_1}, \\ L &= 4T_0^2 \left( \frac{k^2}{q^3} - \frac{k}{3} \frac{\dot{q}}{q^2} + \frac{\dot{q}^2}{30q} \right). \end{aligned} \quad (12.8.203)$$

For minimization  $\delta I = 0$  and following the arguments used earlier, we get the differential equation and natural boundary conditions as follows.

$$\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} = 0 \text{ and } \frac{\partial L}{\partial \ddot{q}} = 0; \quad 2q^3 \ddot{q} - q^2 \dot{q}^2 + 90k^2 = 0. \quad (12.8.204)$$

$\partial L / \partial \ddot{q} = 0$  as the Lagrangian corresponding to  $T(x, t)$  in Eq. (12.8.202) does not have  $\ddot{q}$ . Note that at  $t_1 \rightarrow \infty$ ,  $\delta q$  is not zero. As  $L$  is not a function of  $\ddot{q}$  from second equation in Eq. (12.8.204) we get  $\frac{\partial L}{\partial \dot{q}} = 0$  which gives  $\dot{q}/5q - k/q^2 = 0$ , as for  $t \rightarrow \infty$ ,  $\delta q(t)$  is not prescribed and we get this as the natural boundary condition which should be prescribed in the formulation. For further details of this above problem and several other problems refer [679].

Now we discuss the method of obtaining the functional appropriate for a differential equation so that the first variation of the functional when made stationary gives the given differential equation as an Euler's equation. We consider the following wave equation in which for convenience all the physical parameters are taken as unity.

$$\left. \begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad (x, y) \in R_0 \subset R^2, \quad t_0 \leq t \leq t_1, \\ u(x, y, t) &= u_0 \text{ on } C_1; \quad \mathbf{n} \cdot \nabla u = q_0 \text{ on } C_2, \\ u(x, y, 0) \text{ and } \frac{\partial u}{\partial t} &\text{ at } t = 0 \text{ are prescribed.} \end{aligned} \right\} \quad (12.8.205)$$

$C_1 \cup C_2 = C$ ,  $C$  is the boundary of  $R_0$ . It is assumed that  $u$  and its partial derivatives,  $R_0$  and  $C$ , are such that (smooth enough) all the following operations in deriving Euler's equation are valid. To obtain the variational integral, consider the following functionals  $I(u)$  and  $I(\epsilon)$ .

$$I(u) = \int_{t_0}^{t_1} \int_{R_0} \left( \left( \frac{\partial u}{\partial t} \right)^2 + \nabla u \cdot \nabla u \right) dx dy dt - \int_{t_0}^{t_1} \left( \int_{C_2} q_0 u ds \right) dt. \quad (12.8.206)$$

$$I(\epsilon) = \int_{t_0}^{t_1} \int_{R_0} \left\{ \left( \frac{\partial(\phi + \epsilon \eta)}{\partial t} \right)^2 + \nabla(\phi + \epsilon \eta) \cdot \nabla(\phi + \epsilon \eta) \right\} dx dy dt - \int_{t_0}^{t_1} \left( \int_{C_2} q_0(\phi + \epsilon \eta) ds \right) dt. \quad (12.8.207)$$

Let  $u = \phi$  be the exact solution and all the functions  $\phi + \epsilon \eta$  satisfy the given boundary condition which  $\phi$  satisfies on  $C_1$  and conditions at  $t_0$  as they are essential boundary conditions. Therefore  $\eta(x, y, t) = 0$  on  $C_1$  but is arbitrary in  $R_0$ . On  $C_2$ , the prescribed boundary condition need not be satisfied by  $\phi$  as it is a natural boundary condition which is automatically satisfied by trial functions in the variational formulation. On  $C_2$ ,  $\eta(x, y, t)$  is arbitrary as  $\phi$  is not prescribed on  $C_2$ .

$$I'(0) = \int_{t_0}^{t_1} \int_{R_0} \left( \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi \right) \eta dx dy dt + \int_{t_0}^{t_1} \int_{C_2} (\mathbf{n} \cdot \nabla \phi - q_0) \eta ds dt. \quad (12.8.208)$$

In deriving Eq. (12.8.208), we take  $\eta(x, y, t_0) = \eta(x, y, t_1) = 0$  when integration by parts with respect to time is done for the first term in the first integral in Eq. (12.8.207). We use the Green's identity given below to convert  $\nabla \phi \cdot \nabla \eta$  in the required form.

$$\iint_{R_0} \nabla u \cdot \nabla v dx dy = - \iint_{R_0} v \nabla^2 u dx dy + \int_C v \frac{\partial u}{\partial n} ds. \quad (12.8.209)$$

The second integral in Eq. (12.8.208) is zero because  $\eta$  is arbitrary on  $C_2$  and we take it to be zero on  $C_2$ . As  $\eta$  is arbitrary in  $R_0$ , the first integral in  $I'(0) = 0$  implies the term which multiplies  $\eta$  should be zero as the second integral is already zero and so we get the required wave equation as Euler's equation. Next we take another  $\eta$  in the class of admissible arbitrary functions to be nonzero on  $C_2$  and since first integral is zero,  $I'(0) = 0$  implies the boundary condition on  $C_2$  which is prescribed.

The above procedure to obtain transient heat equation fails as the Euler's equation cannot be obtained by considering the stationary value of a functional using methods of calculus of variations. This point will be further clarified later by specifying the reason. The steady-state heat equation with natural boundary conditions, as flux prescribed or convective type, can be obtained in one, two and three dimensions using the above procedure in which calculus of variations is used and extremum of a function is obtained.

Variational functionals can be derived independently by considering some physical laws such as rate of entropy production. The variational principle in this case is the minimum rate of entropy production. The stationary value of the functional gives the Euler's equation. Minimum of total energy may also be used as variational principle. If we have correctly calculated the total energy of the system then the minimum of functional will tell us the correct differential equation for the problem. In these variational principles also we can derive the Euler's equation which is obtained by making the functional stationary. In finding approximate solutions, the functional could be used to find its minimum by using a suitable minimization method and obtain analytical-numerical solution which avoids solving differential equations. At every stage of minimization we can ascertain the error which is the advantage.

What are the conditions under which a functional exists for a linear or nonlinear equation such that the stationary value gives required equation which is the Euler's equation of the functional? Next question we ask is: How to obtain the functional? In simple words we may answer that the functional corresponding to a nonlinear operator equation  $P(w) = 0$  exists if the Fréchet derivative of  $P(w)$  exists. Fréchet derivative and Gâteaux derivative have been defined in Appendix D. A simple definition is being given here.

The Fréchet derivative  $P'_w\phi$  of the operator  $P(w)$  in the direction  $\phi$  is defined as

$$P'_w\phi = \lim_{\epsilon \rightarrow 0} \frac{P(w + \epsilon\phi) - P(w)}{\epsilon} = \left( \frac{\partial}{\partial \epsilon} P(w + \epsilon\phi) \right)_{\epsilon=0}. \quad (12.8.210)$$

If this limit on r.h.s. exists then  $P'_w\phi$  is called the Fréchet derivative of operator  $P$  in the direction  $\phi$  and  $P'_w$  in which dash denotes differentiation with respect to  $w$  is called Fréchet derivative of  $P$ . On using the definition of  $P'_w$  and assuming that the integral given below is independent of path of integration it can be shown that

$$\int_V \psi P'_w\phi dv = \int_V \phi P'_w\psi dv, \quad (12.8.211)$$

$\phi(w)$  and  $\psi(w)$  are linear continuous real-valued functions. Eq. (12.8.211) implies that  $P'_w$  is a symmetric operator.

By considering a functional  $F$  defined on a subset of  $R^3$  in the form  $F(w) = \int Q(w)dv$  and defining the Fréchet derivative of a functional as in the case of an operator, it can be shown that  $\text{grad } F = \int Q'_w(\phi)dv = \int \phi P(w)dv$  (integrate by parts) and so  $P(w) = \text{grad } F(w)$  and  $\delta F = \int P(w)\delta w dv$ .  $\delta F = 0 \Rightarrow P(w) = 0$  which is Euler's equation of  $F(w)$ . This implies that if  $P(w)$  is the Fréchet derivative of a functional then the Euler's equation of the functional is  $P(w) = 0$ .

If Eq. (12.8.211) holds then the  $P'_w$  is called self-adjoint operator. The definitions of symmetric and adjoint operator include any suitable operator  $N$ . This self-adjointness properly is the condition that proves that there exists a functional  $F(w)$  whose gradient is  $P(w)$ . If

$$\int \psi P'_w\phi dv = \int \phi \bar{P}'_w\psi dv, \quad (12.8.212)$$

then  $\bar{P}'_w$  is adjoint to  $P'_w$ . If the operator equation is written in the form of an equation  $g(w; w_{iJ}; w_{iJK}) = 0$  then the condition of symmetry requirement can be obtained on using Eq. (12.8.212). In this case also the necessary condition for a variational principle to exist for the operator equation  $g = 0$  can be determined (provided  $g$  is a symmetric operator). This necessary condition is the Euler's equation corresponding to the functional for  $g$ .

For linear self-adjoint operators variational principle exists. If the operator  $P$  is as defined below

$$P(T) = \frac{\partial T}{\partial t} - \nabla^2 T + \vec{u} \cdot \nabla u - Ku, \quad (12.8.213)$$

then the operator adjoint to  $P$  denoted by  $P^*$  is obtained as

$$P^*(T^*) = -\frac{\partial T^*}{\partial t} - \nabla \cdot (\vec{u} T^*) - \nabla^2 (T^*) - K T^*. \quad (12.8.214)$$

In order to obtain an adjoint equation  $P^*(T^*) = h$  in  $V$  and  $A_j^*(T^*) = 0$  on boundary  $S$  of  $V$  corresponding to the linear equation  $P(T) = f$  in  $V$  and  $A_i(T) = 0$  on  $S$ , we consider the inner product

$$(T^*, P(T)) = (T, P^*(T^*)) + A(T, T^*), \quad (12.8.215)$$

$A(T, T^*)$  has the property that  $A(T, T^*) = 0$  for all  $T$  satisfying  $A_i(T) = 0$  on  $S$  when  $A_j^*(T^*) = 0$  on  $S$ .  $A(T, T^*)$  is a boundary term arising on integration by parts.

The above discussion has arisen because for linear nonself-adjoint operators also variational principles exist. In this case we consider both original equations  $P(T) = f$  and  $P^*(T^*) = h$  and define the functional

$$F(T, T^*) = (T, P^*(T^*) - h) - (T^*, f). \quad (12.8.216)$$

For the determination of  $h$  see [495]. Let the functions  $\phi_1$  and  $\phi_2$  be the solutions of the boundary value problems  $P(T) = 0$  in  $V$  and  $A_i(T)$  on  $S$  and its adjoint problem, respectively. Then consider the functions  $\phi_1 + \epsilon_1 \eta_1$  and  $\phi_2 + \epsilon_2 \eta_2$  where  $\eta_1$  and  $\eta_2$  are arbitrary functions belonging to the same class of functions to which  $\phi_1$  and  $\phi_2$  belong and  $\eta_1$  and  $\eta_2$  are zero on  $S$ ,  $\epsilon_1$  and  $\epsilon_2$  are scalars. Calculate  $(\partial F / \partial \epsilon_1)_{\epsilon_1 = \epsilon_2 = 0}$  and  $(\partial F / \partial \epsilon_2)_{\epsilon_1 = \epsilon_2 = 0}$ . Equating each of them to zero, we get Euler's equations, the solutions of which make the functional stationary.

The stationary value is minimum or maximum can be obtained by calculating the second variation  $\delta^2 F$ . If  $\delta^2 F$  is always positive then a minimum variational principle may exist and if  $\delta^2 F$  is always negative then a maximum variational principle may exist. These are not sufficient conditions. If  $\delta^2 F$  could be positive or negative depending on the functions, derivatives, parameters, etc., involved, then the minimum or maximum does not exist or an extremum variational principle does not exist. Only stationary values of functional can be obtained in this case. If the operator  $P$  is linear self-adjoint operator then  $P(w) = \text{grad } F(w)$  and  $F(w)$  can be expressed as

$$F(w) = \int w \int_0^1 P(\lambda w) d\lambda dv = \frac{1}{2} \int w P(w) dv. \quad (12.8.217)$$

Consider a linear steady-state heat conduction equation  $K \nabla^2 T + f(X, T) = 0$  in  $V \subset \mathbb{R}^3$  with convective-type boundary condition on  $S_1$ , flux  $q_2$  prescribed on  $S_2$  and temperature  $T_3$  prescribed on  $S_3$ ,  $S_1 \cup S_2 \cup S_3 = S$ , where  $S$  is the surface of  $V$ . By taking the functional as

$$\begin{aligned} F(T) = & \int \left[ \frac{1}{2} K \nabla T \cdot \nabla T + \int_{T_0}^T f(X, T') dT' \right] dV + \int_{S_2} q_2 T ds \\ & + \frac{1}{2} \int_{S_1} h(T - T_3)^2 ds \end{aligned} \quad (12.8.218)$$

and obtaining its first variation using calculus of variations, the heat equation is recovered as Euler's equation. The natural boundary conditions on  $S_2$  and  $S_1$  which are  $K \vec{n} \cdot \nabla T + q_2 = 0$  on  $S_2$  and  $K \vec{n} \cdot \nabla T + h(T - T_3) = 0$  on  $S_1$  are satisfied. To obtain the stationary value of the functional  $F(T)$  in Eq. (12.8.218) we proceed as suggested for the functional of wave equation in Eq. (12.8.205). Integration by parts is to be done using divergence theorem. The

temperature-prescribed boundary conditions are essential boundary conditions and are to be satisfied by testing functions. The remaining steps are on the same lines as suggested earlier.

A sufficient condition for a functional  $F(\bar{T}, \eta)$  to be minimum in which  $\bar{T}$  is the solution of the Euler's equation and  $\eta$  is as defined in  $\bar{T} + \epsilon \eta$  is that there exists a positive number  $A$  for which

$$\delta^2 F(\bar{T}, \eta) \geq A \|\eta\|, \text{ for all } \eta \text{ in some normed space.} \quad (12.8.219)$$

$\delta^2$  stands for second variation. Necessary and sufficient conditions for minimum principle to exist are satisfied for the functional in Eq. (12.8.218) (cf. [668]) provided  $\partial F(x, \bar{T})/\partial \bar{T} \geq 0$ .

When the procedure to obtain transient heat conduction equation using calculus of variation approach is followed then we do not succeed in getting transient heat equation as Euler's equation if the functional is taken as in Eq. (12.8.218). It was suggested in [680] to consider the functional as

$$G\left(T, \frac{\partial T}{\partial t}\right) = \int_V \left( \rho C T \frac{\partial T}{\partial t} + \frac{K}{2} \nabla T \cdot \nabla T \right) dV, \quad (12.8.220)$$

in which  $\partial T/\partial t$  is kept fixed and only  $X$  in  $T(X, t)$  varies so that  $\eta = \eta(x)$  and  $\eta$  is not of a function of  $t$ . Transient heat equation is obtained as the Euler's equation but only for a fixed  $t$ . In this case no variational principle exists. In [680], it has been called *restricted variational principle*.

Biot's variational technique [681–683] has also been used in obtaining analytical-numerical solutions of heat transfer problems without phase change and also with phase change. This method has been explained in [684] in a simple way and one application in a heat transfer problem without phase change has been discussed. Biot's technique is different from finding stationary value of a functional which has been considered here earlier. It also depends on different physical quantities associated with heat transfer.

Biot's technique uses the sum  $\partial P + \partial F$ , the sum of first variations of  $P$  and  $F$ , where  $P$  is thermal potential and  $F$  is related to the concept of a dissipation function. If  $T(x, t)$  is temperature and  $\vec{H}$  is the heat-flow vector then

$$P = \frac{1}{2} \iiint_V \rho C T^2 dV; \quad \frac{\partial \vec{H}}{\partial t} = \vec{q}, \quad \vec{q} \text{ is flux vector; } F = \iiint_V \frac{1}{K} \frac{\partial \vec{H}}{\partial t} dV, \quad (12.8.221)$$

$$\left. \begin{aligned} \delta P + \delta F &= \int_V \left( \rho C \text{grad } T + \frac{1}{K} \frac{\partial \vec{H}}{\partial t} \right) \cdot \delta \vec{H} dV, \quad \rho C T = -\text{div } \vec{H}, \\ \rho C \frac{\partial T}{\partial t} + \text{div } \vec{q} &= 0. \end{aligned} \right\} \quad (12.8.222)$$

The variational statement is the first equation or statement in Eq. (12.8.223).

$$\left. \begin{aligned} \delta P + \delta F &= \iint_S T \vec{n} \cdot \delta \vec{H} ds, \quad (\text{variational statement}), \\ \delta P &= \int_V \rho C T \delta T dV = \int_V \delta \vec{H} \cdot \text{grad } T dV + \iint_S T \vec{n} \cdot \delta \vec{H} ds. \end{aligned} \right\} \quad (12.8.223)$$

Integration by parts has been done in deriving second equation in Eq. (12.8.223).

Therefore using the variational statement and calculating  $\delta P + \delta F$  from the second equation in Eq. (12.8.223), and from Eq. (12.8.221) we get

$$\int_V \left( \text{grad } T + \frac{1}{K} \frac{\partial \vec{H}}{\partial t} \right) \cdot \delta \vec{H} dV = 0 \Rightarrow \text{grad } T + \frac{1}{K} \frac{\partial \vec{H}}{\partial t} = 0. \quad (12.8.224)$$

$\delta \vec{H}$  is arbitrary as heat flow is not prescribed and only heat flux is prescribed, the second equation in Eq. (12.8.224) is obtained and by taking divergence of it, the heat equation is obtained. In applications,  $\vec{H}$  is expressed in terms of generalized coordinates. For example, in heat transfer problems the generalized coordinate could be the penetration depth of heat transfer and phase-change boundary in Stefan problems.

Biot's technique is a quasivariational method in which there are no variational integrals and the variational principle is described in terms of a variational equation as in Eq. (12.8.223).

In [685, 686], the applications of Biot's technique have been used to obtain solutions of some Stefan problems. In [685], solidified thickness has been obtained when warm fluid flows over a cold plate. Heat flux variable  $Q$  which is the integral of flux with respect to time is a function of generalized coordinate and is also a function of space coordinates and time. A quadratic polynomial for  $Q$  has been assumed. The three unknown coefficients in the polynomial are three generalized coordinates which are determined by boundary conditions. Approximate analytical solution has been compared with known results in limiting cases. Numerical work has been done.

The problem considered in [686] consists of solidification in which a cold wall is in contact with a liquid in the semiinfinite region  $0 \leq x < \infty$  at freezing temperature. Altogether eight problems corresponding to different boundary conditions at  $x = 0$  and at the phase-change boundary have been studied. In the cold wall, temperature has been obtained by penetration depth method. Generalized coordinates have been considered in obtaining solutions by Biot's technique. Two coupled differential equations are obtained for each problem which have been solved numerically in each problem.

Melting and solidification of thin wires has been considered in [687] by considering a variational problem of minimization of total energy which is the sum of surface free energy and the external field energy. Instantaneous equilibrium shape of the melt, which is an unknown boundary, has been considered as a flat one-dimensional surface. The minimization of total energy provides Euler's equation which is in terms of the flat melt surface with a constraint. At any given instant the total volume occupied by the melt is constant and it is the prescribed constraint. The problem is one-phase in a radial coordinate in a circular cylinder. Lagrange multiplier has been considered. This steady-state problem has been studied using perturbation method. Numerical solution of two-dimensional transient melting problem has also been obtained by finite-difference method. Numerical results are not given in this part of the paper.

A one-dimensional heat and mass transfer problem has been studied in [688] which is concerning evaporation of a liquid and the vapour percolating into a porous media and forming a mixture of liquid and vapour. The liquid occupies the region  $0 \leq x \leq S(t)$ , and the mixture occupies the region  $S(t) \leq x < \infty$ .  $S(t)$  is an unknown moving boundary. The formulations of heat transfer in liquid and the two equations of heat and mass transfer in the porous medium  $S(t) \leq x < \infty$  are simple and such formulations have been considered earlier. They are not

reported here and so is the case with the boundary conditions and interface conditions. The heat transfer penetration depth  $S_1(t)$  will be different from mass transfer penetration depth  $S_2(t)$ . The moving boundaries  $S(t), S_1(t)$  and  $S_2(t)$  are taken proportional to  $t^{1/2}$  and are unknowns. For  $0 \leq x < S(t)$  the solution is a particular case of Neumann solution and so the constant of proportionality in  $S(t)$  is known. For the temperature  $T_2(x, t)$  and the moisture  $\theta_2(x, t)$  in the region  $S \leq x < \infty$ , suitable parabolic profiles satisfying boundary conditions have been considered as in HBIM. For determining the unknown proportionality constants in moving boundaries, a variational functional  $J(S_1, S_2)$  in the form of local potential has been considered.

$$J(S_1, S_2) = \int \left[ \frac{1}{2} \left( \frac{\partial T_2}{\partial x} \right)^2 + T_2 \frac{\partial T_2}{\partial t} + T_2 \frac{\partial \theta_2}{\partial t} + \frac{1}{2} \left( \frac{\partial \theta_2}{\partial x} \right)^2 + \theta_2 \frac{\partial \theta_2}{\partial t} \right] dx. \quad (12.8.225)$$

In Eq. (12.8.225), all thermo-physical parameters have been taken by us as unity for convenience. Stationary values of  $J(S_1, S_2)$  have been obtained by considering the variations in  $T_2$  and  $\theta_2$  as

$$T_2 = \bar{T}_2 + \delta T_2 \quad \text{and} \quad \theta_2 = \bar{\theta}_2 + \delta \theta_2; \quad \bar{T}_2 \text{ and } \bar{\theta}_2 \text{ are exact solutions.} \quad (12.8.226)$$

If  $S_1 < S_2$  then the limits of integration in Eq. (12.8.225) will be from  $x = S$  to  $x = S + S_2$  and if  $S_2 < S_1$  then the limits of integration will be from  $x = S$  to  $x = S + S_1$ . Solutions will be different in the two cases. Unknown constants  $(\lambda_1, \lambda_2)$  which are constants of proportionality and occur in  $S_1 = \lambda_1 t^{1/2}$  and  $S_2 = \lambda_2 t^{1/2}$  have been determined from stationary conditions of  $J(S_1, S_2)$ . Figures have been presented for the numerical work done.

A good description of Russian works in heat and mass transfer problems using variational methods can be found in [689].

Vast literature exists on various aspects of variational methods and variational principles concerning theory and applications. A comprehensive treatment of even theoretical aspects, such as Fréchet derivatives, adjoint operators and admissible functions, error bounds, proofs of various theorems, would require couple of volumes to be written. Many of the aspects, such as variational formulation using a convolution-type functional (cf. [689]), reciprocal variational principle, error bounds, have not been even touched in the above discussion presented here. As mentioned earlier, after discussing HBIM in somewhat detail, there is not much space available for other approximate methods which are more oriented towards numerical methods than quasianalytical solutions. Our objective has not been to survey all the methods and their applications which is not possible by us in this volume. Further, numerical methods and their solutions are not the subject matter of this chapter. Often an inquisitive research worker working in any area of science and engineering, in some context, becomes aware of applications of approximate methods and wants to just know: ‘What some of these methods are like’. The application of these methods comes afterwards depending on interest and necessity. What we have discussed above is just to satisfy the curiosity of an inquisitive research worker a little bit.

The book by Carslaw and Jaeger [24] generally occupies a place on the table of a research worker in heat transfer. In a similar way the book by Finlayson [495] is worth occupying a place on the table of a research worker working on application aspects of approximate methods.

## 12.9 REVIEW OF SOME LIMITED NUMBER OF PERTURBATION METHODS AND SOLUTIONS OF STEFAN AND STEFAN-LIKE PROBLEMS

### 12.9.1 Introduction

The existing literature concerning perturbation methods and solutions is so vast that it is not possible even to update the list of references. Invariably, when some analytic solution cannot be found easily, an attempt is made to find a perturbation parameter, if it is readily available, otherwise either through some assumptions or simplification of the problem, a suitable perturbation parameter is introduced. In many research articles it is difficult to make out from the title of the research paper that in the text a perturbation solution has been obtained. Without discussing the physical significance of the perturbation parameter and how it has been obtained, there does not seem to be a much point in discussing the solutions. In general, it is not possible to describe the perturbation parameter briefly as it requires the problem formulation, scalings, assumptions and further simplification of resulting equations. Therefore, we have decided to review only some limited number of research papers in which there is some variety in obtaining solutions.

A good number of books such as [690–700] and other sources are available about perturbation methods in which different type of perturbation methods such as regular perturbation, singular perturbation, method of strained coordinates, method of matched asymptotic expansions are explained and illustrated with the help of some solutions of problems. There does not seem to be much usefulness in explaining these methods in detail here. However, to make this section self-contained, some perturbation methods and the variations to them will be briefly described in the context of some solutions which are being presented here. Although perturbation solutions obtained for Stefan and Stefan-like problems are comparatively much less discussed, still the available material is voluminous. The determination of first few terms of the perturbation series and the determination of higher-order terms depend on special type of analytical and numerical techniques whose description requires full derivation which is not possible here as the available space is limited. For the quasianalytical solutions and methods presented in this section which are analytical-numerical solutions, there is no single source where they can be found.

The published research works on perturbation solutions obtained by HPM and HAM are comparatively recent additions to the literature on Stefan problems. Even on the applications of these methods not many works are available on Stefan and Stefan-like problems using HPM or HAM and so some heat transfer solutions without phase change are also being included for better understanding. We first discuss solutions obtained by HPM and HAM.

### 12.9.2 Homotopy Analysis and Homotopy Perturbation Methods and Solutions

He in [536] refers to his method as HPM and Liao in [539] refers to his method as HAM. Both HPM and HAM are based on an important property of topology known as homotopy. Although without knowing the definition of homotopy and some other related definitions and theorems, it is possible to use the procedures to obtain solutions in the applications of HPM and HAM, it seems better if a reader is equipped with the central idea in these methods. Very briefly, homotopy is explained below.



**Continuous map:** A map  $f : X \rightarrow Y$  of a topological space  $X$  to a topological space  $Y$  is called continuous with respect to the given topologies if the preimage of any open set in  $Y$  is open in  $X$ .

This definition is just an extension of  $(\in, \delta)$  definition of a continuous function defined on the real line with the concept of standard topology.

**Homotopy:** A homotopy is a family of maps  $f_\lambda : X \rightarrow Y, \lambda \in [0, 1]$ , such that the associated map  $F : X \times [0, 1] \rightarrow Y$  given by  $F(x, \lambda) = f_\lambda(x)$  is continuous.

If  $f_\lambda$  is a homotopy then corresponding to each  $\lambda \in [0, 1]$ , there exists a continuous map  $F(x, \lambda)$  defined as above. Any two members of the family  $f_\lambda$  are said to be homotopic via  $f_\lambda$ . The maps  $f_0$  and  $f_1$  are homotopic via the homotopy  $f_\lambda$  and these are expressed as  $f_0 \simeq f_1$  or as  $f_0 \sim f_1$ . Using the continuity of  $F(x, \lambda)$ , an alternate way of defining homotopy could be as follows.

Two continuous maps  $f$  and  $g$  from topological space  $X \rightarrow Y$  are called homotopic if  $f$  can be continuously deformed into  $g$  in the class of continuous maps or in other words if there exists a one-parameter family of continuous maps beginning with  $f$  and ending with  $g$ . In terms of the map  $F(x, \lambda)$ ,  $f$  and  $g$  are homotopic if there exists a continuous map  $F(x, \lambda) : X \times [0, 1] \rightarrow Y$  such that  $F(x, 0) = f(x)$  and  $F(x, 1) = g(x)$  for all  $x \in X$ . As mentioned above with each  $\lambda$ ,  $F(x, \lambda)$  associates a map  $f_\lambda$  such that  $F(x, \lambda) = f_\lambda(x)$  for all  $x \in X$  and all  $\lambda \in [0, 1]$ . All these maps are continuous and depend continuously on  $\lambda$ . The family of maps  $f_\lambda$  obtained in this way is called *homotopy*. These minor details can be found in any book on topology, refer [701].

How do we use this concept of homotopy in obtaining perturbation solutions of linear and nonlinear problems? Suppose  $N(T(x, y, z, t)) = 0$  is a nonlinear equation in which  $t$  is time. The initial condition and boundary condition are prescribed. We write  $T(x, y, z, t)$  as  $T(\vec{r}, t)$ . Let  $\bar{T}(\vec{r}, t)$  be the exact solution of this initial value problem. Also  $\theta(\vec{r}, t; \lambda) = \theta_\lambda(\vec{r}, t), \lambda \in [0, 1]$  be a family of continuous mappings  $\theta_\lambda : X \rightarrow Y$ .  $\theta_\lambda$  for each  $\lambda$  belongs to space of solution of  $T(\vec{r}, t)$ . Some authors call this parameter  $\lambda$  as *embedding parameter*. To construct homotopy, we define a continuous mapping  $F(\vec{r}, t; \lambda) : X \times [0, 1] \rightarrow Y$  such that  $F(\vec{r}, t; 0) = \theta_0(\vec{r}, t)$  and  $F(\vec{r}, t; 1) = \theta_1(\vec{r}, t)$  and for each  $\lambda$ ,  $F(\vec{r}, t; \lambda)$  associates a  $\theta_\lambda(\vec{r}, t)$  as  $\lambda$  varies from 0 to 1.

Till now nothing has been done towards obtaining the solution except repeating the definition of homotopy. Now the homotopy  $\theta_\lambda$  will be associated with the solution of the given nonlinear problem. We start with some initial guess of the solution of  $T(\vec{r}, t)$  denoted as  $\bar{T}_0(\vec{r}, t)$  and define the continuous map  $F(\vec{r}, t; \lambda) : X \times [0, 1] \rightarrow Y$  such that  $F(\vec{r}, t; 0) = \theta_0(\vec{r}, t)$  for all  $(\vec{r}, t) \in X$  and  $\theta_\lambda(\vec{r}, t; \lambda = 0)$ . Take  $\theta_0(\vec{r}, t)$  = initial approximation or initial guess of  $\bar{T}(\vec{r}, t) = \bar{T}_0(\vec{r}, t)$ .  $\bar{T}_0$  is the exact solution. Initial guess does not mean here the approximation at time  $t = 0$ .  $F(\vec{r}, t; 1) = \theta_1(\vec{r}, t)$  for all  $(\vec{r}, t) \in X$  is defined such that  $\theta_1(\vec{r}, t) = \theta_\lambda(\vec{r}, t; \lambda = 1) = \bar{T}(\vec{r}, t)$  (exact solution). We have now defined a homotopy  $\theta_\lambda : X \rightarrow Y, \lambda \in [0, 1]$  such that  $\theta_0$  varies from the initial guess of  $\bar{T}_0$  to the exact solution  $\bar{T}$  which is obtained at  $\theta_1$  and this variation is continuous as  $\lambda$  varies from 0 to 1. In topology this kind of variation is called *deformation*.  $F(\vec{r}, t; \lambda)$  can be regarded as a homotopy between  $\theta_0$  and  $\theta_1$  through the variation of  $\lambda$ .  $\theta_0$  and  $\theta_1$  are homotopic belonging to the family of homotopy  $\theta_\lambda$  and  $F(\vec{r}, t; \lambda)$  can be identified as a homotopy.

How to construct  $F(\vec{r}, t; \lambda)$ ? This question will be answered a little later. We first answer the question: 'How to obtain the solution from the constructed homotopy'? For this  $\theta_\lambda(\vec{r}, t)$  is expanded as a power series in  $\lambda$  which is the perturbation series in  $\lambda$ . The coefficients of different powers of  $\lambda$ , which are functions of  $(\vec{r}, t)$ , can be obtained by the usual methods

used in perturbation solutions. If the series converges at  $\lambda = 1$  then according to the definition of homotopy or deformation, the series at  $\lambda = 1$  represents the exact solution of the given problem. The construction of  $F(\vec{r}, t; \lambda)$  will be considered now in HPM and for this some specific phase-change problems are being considered. In [Section 12.5.2](#), some solutions obtained by HPM have been reported but in that section, the context was the solution of inverse problems and so the emphasis was on solutions of inverse problems. In the present section, the emphasis is on perturbation solutions. Of course this does not change neither the method nor the solution.

The continuous function  $F(\vec{r}, t; \lambda)$  will now be denoted by  $H(\vec{r}, t; \lambda)$  so that it signifies the homotopy in HPM and HAM. The solution of the movement of the shoreline in a sedimentary ocean base has been obtained in [\[702\]](#) in which fractional derivative with respect to time has been used in the formulation. This type of problem was earlier considered in [\[420\]](#) and fractional derivatives have been defined in Eqs (12.2.249), (12.2.250). We skip the physics of the problem, parameter description and scalings and consider the following mathematical model. If  $\eta(x, t)$  is the height of the shoreline above the datum which is unknown,  $x = S(t)$  is the moving shoreline position and  $q(t)$  is the prescribed sediment line flux then we have the following formulation.

$$D_t^\alpha \eta(x, t) = v \frac{\partial^2}{\partial x^2} \eta(x, t), \quad 0 < x < S(t), \quad 0 < \alpha \leq 1; \quad v \frac{\partial \eta}{\partial x} \Big|_{x=0} = -q(t). \quad (12.9.1)$$

$$-v \frac{\partial \eta}{\partial x} \Big|_{x=S(t)} = \gamma S(t) D_t^\alpha S(t); \quad \eta(S, t) = 0, \quad S(0) = 0; \quad v \text{ and } \gamma \text{ are constants.} \quad (12.9.2)$$

The operator under consideration in this problem is given below.

$$L(\eta(x, t)) = v \frac{\partial^2}{\partial x^2} \eta(x, t) - D_t^\alpha \eta(x, t), \quad D_t^\alpha \text{ is the Caputo fractional derivative.} \quad (12.9.3)$$

The homotopy corresponding to this operator according to Eq. (12.5.12) is the following.

$$H(u(x, t); p) = v \left( \frac{\partial^2 u(x, t)}{\partial x^2} - \frac{\partial^2 \eta_0(x, t)}{\partial x^2} \right) - p \left( D_t^\alpha u(x, t) - \frac{v \partial^2 \eta_0(x, t)}{\partial x^2} \right). \quad (12.9.4)$$

For  $p = 0$ , the solution of the equation  $H(u, 0) = 0$  is  $u(x, t) = \eta_0(x, t)$ , where  $\eta_0(x, t)$  is some initial approximation of the exact solution  $\eta(x, t)$ . For  $p = 1$ ,  $H(u; 1) = 0$  is the original Eq. (12.9.1) and as per assumption  $u(x, t) = \eta(x, t)$  is the exact solution. To obtain the perturbation solution, we write

$$u(x, t) = \sum_{J=0}^{\infty} p^J \eta_J; \quad \lim_{p \rightarrow 1} u(x, t) = \eta(x, t) = \sum_{J=0}^{\infty} \eta_J; \quad S(t) = \sum_{J=0}^{\infty} p^J S_J. \quad (12.9.5)$$

We get the solution only if both the infinite power series for  $u(x, t)$  and  $S(t)$  in Eq. (12.9.5) converge. Convergence of series in HPM and HAM will be discussed later. As it is usually done, instead of an infinite series, we consider a finite series by taking some finite value of  $J$  in Eq. (12.9.5), substitute the finite sums of  $u(x, t; p)$  and  $S(t)$  in terms of finite series of  $p$  in Eqs (12.9.1), (12.9.2) and compare the terms of various powers of  $p$  on both sides

of the equations. The parameter  $p$  in this problem depends on  $S(t)$  also. Therefore,  $\eta_i(x, t)$  is expressed as

$$\eta_i(x, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n \eta_i}{\partial x^n} \right|_{S_0} (x - S_0)^n, \quad i = 0, 1, 2 \dots \quad (12.9.6)$$

The solution of  $\eta_0, \eta_1$  and  $S_0, S_1$  is reported in [702]. For calculations' details and parameters refer [702]. We give below  $S_0$  and  $\eta_0$  which determine short-time behaviour.

$$S_0 = a_0 t^{\alpha/2}, a_0 = \sqrt{\frac{q}{\gamma}} \left[ \frac{\Gamma(1 - \alpha/2)}{\Gamma(1 + \alpha/2)} \right]^{1/2}; \quad \eta_0 = a_1 (a_0 t^{\alpha/2} - x), a_1 = q/v. \quad (12.9.7)$$

The calculations using Caputo fractional derivative explained in [388] have been done. Numerical results for the first-order analytical solution were obtained using MATHEMATICA software and compared with some exact analytical solutions obtained in [703] by drawing figures.

The essence of HPM lies in the construction of homotopy as in Eqs (12.5.12), (12.9.4). In both these equations the so-called embedding parameter  $p \in [0, 1]$ . When  $p = 0$ , we start with some approximation of the exact solution which is the first member of the family of continuous functions or of homotopy. It is assumed that the last member of this family is the exact solution of the linear nonlinear problem under consideration. This last member is obtained for  $p = 1$  by continuous deformation, provided the perturbation series at  $p = 1$  converges. In the initial stages of applications of any new method as in this homotopy approach, the prime objective is to obtain some solution and show that it is reasonably accurate by comparing it with other solutions. The questions concerning existence, uniqueness, continuity, convergence and error bounds are answered slowly in due course of time. This is also the case with HPM and HAM.

A Stefan problem with fractional time derivative and fractional space derivative has been considered in [704]. The problem is that of solidification of a liquid which is initially at the freezing temperature unity. It is a one-dimensional one-phase problem and not a problem in a finite slab as mentioned in the title. A finite slab problem is a two-phase problem. The formulation is as given below.

$$v \frac{\partial^\beta T(x, t)}{\partial t^\beta} = \frac{\partial^\alpha T(x, t)}{\partial x^\alpha}, \quad 0 \leq x < S(t), t > 0, \quad 0 < \beta \leq 1 < \alpha \leq 2. \quad (12.9.8)$$

$$T(x, 0) = 1; \quad T(0, t) = 0; \quad {}_0 D_t^\beta S(t) = \left. \frac{\partial T}{\partial x} \right|_{x=S(t)}, \quad T(S(t), t) = 1, \quad S(0) = 0. \quad (12.9.9)$$

In the above equations it is assumed that all the equations are dimensionless and for scalings refer [704].  $x = S(t)$  is the freezing front and  ${}_0 D_t^\beta$  is *new modified Riemann-Liouville fractional derivative*. Caputo and Riemann-Liouville fractional derivatives have been defined in Eqs (12.2.49), (12.2.282), respectively. This new fractional derivative was proposed by Jumarie [705, 706] and is defined as

$${}_0 D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t (t - \xi)^{n-\alpha} (f(\xi) - f(0)) d\xi, \quad t \in [0, 1], \\ n - 1 \leq \alpha < n, n \geq 1. \quad (12.9.10)$$

$f : R \rightarrow R$  is a continuous function but may not be differentiable in the classical sense. If a function  $f$  is continuous and has fractional derivatives of order  $k\alpha$ , for any positive integer  $k$  and any  $\alpha, 0 < \alpha \leq 1$ , then

$$f(t+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\alpha k!} f^{(\alpha k)}, \quad 0 \leq t \leq 1, \quad 0 < \alpha \leq 1, \quad h \in (0, 1), \quad (12.9.11)$$

$$f^{\alpha}(t) = \lim_{h \rightarrow 0} \frac{\Delta^{\alpha}[f(t) - f(0)]}{h^{\alpha}}, \quad \Delta^{\alpha}f(t) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[t + (\alpha - k)h]. \quad (12.9.12)$$

Another result which is useful is

$$\int_0^t f(t)(dt)^{\alpha} = \alpha \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad 0 < \alpha \leq 1. \quad (12.9.13)$$

From Eq. (12.9.10), the fractional derivative of a constant is zero. If the integer-order exponent of derivative is greater than  $\alpha$  then the definition of fractional derivative is not required. Note that only finitely many terms are considered of the infinite series in calculations.

The application of HPM to obtain solution of Eqs (12.9.8), (12.9.9) is not much different from what has been discussed concerning the solution of problem in Eqs (12.9.1), (12.9.2). In the present problem fractional space and time derivatives both have been considered so whilst calculating the fractional derivatives the above definition is to be used. Expansions of the form in Eqs (12.9.5), (12.9.6) have been considered. It may be noted that for a function which is not differentiable, Taylor series expansions of the form in Eq. (12.9.6) are not possible. Therefore temperature function is considered as smooth as in Fourier heat equation. We are considering non-Fickian heat condition and therefore fractional derivatives have been considered.

Only first-order approximate solution has been obtained which does not involve complicated calculations of fractional derivatives. The series expansion for  $S(t)$  in [704] is the same as in Eq. (12.9.5). To calculate  $S_0$  in the series, in [704] it has been assumed that  $S_0 = a_0 t^{3/2}$  in which  $a_0$  is to be determined. Similar type of assumption for  $S_0$  has been made in [702] also. The advantage in considering the new definition of fractional derivative has not been shown explicitly in the calculations. For approximate solution refer [704]. When  $\beta = 1$  and  $\alpha = 2$ , exact solution is available. Numerical results have been obtained for the perturbation solution and compared with the exact solution by considering three cases, ( $\beta = 1, \alpha = 2$ ), ( $0 < \beta \leq 1, \alpha = 2$ ) and ( $\beta = 1, 1 < \alpha \leq 2$ ).

An inverse one-dimensional heat conduction problem in the slab  $0 \leq x \leq b$  has been considered in [707]. The heat equation is linear, initial condition is prescribed and at  $x = 0$ , temperature is prescribed. At  $x = b$ , an unknown temperature can be prescribed or an unknown flux can be prescribed. An additional condition in the form of temperature at some point  $x_p \in (0, b)$  is given. The application of HPM is as in [702]. There is no phase change in [707]. Third-order approximate perturbation solution has been obtained. The perturbation solution up to first-order approximation is subjected to the boundary condition at  $x = 0$  and the temperature prescribed condition at  $x = x_p$ . Higher-order solutions do not include these conditions. This approximate solution has been compared numerically with exact solutions of two test problems whose solutions have been constructed by choosing suitable data. The agreement is good.

The problem in [708] is concerned with obtaining the temperature in a fin in the presence of surface emissivity and conductive-radiative heat transfer. A steady-state one-dimensional problem has been considered in the region  $0 \leq x \leq 1$  with thermal conductivity varying as linear function of temperature. We skip the details of formulation, scalings, description of parameters and give below the dimensionless steady-state heat equation and boundary conditions as reported in [708].

$$\frac{d^2\theta}{dx^2} + A\theta \frac{d^2\theta}{dx^2} + A \left( \frac{d\theta}{dx} \right)^2 - A\theta_a \frac{d^2\theta}{dx^2} - N_r[1 + B(\theta - \theta_S)][\theta^4 - \theta_S^4] = 0, \quad (12.9.14)$$

$$\frac{d\theta}{dx} = 0 \text{ at } x = 0; \theta = 1 \text{ at } x = 1, \theta \text{ is dimensionless temperature,} \quad (12.9.15)$$

$A, B, \theta_a$  and  $\theta_S$  are dimensionless constant temperatures and  $N_r$  is a dimensionless parameter.

Eq. (12.9.14) is of the form  $L(\theta) + N(\theta) - f = 0$  (cf. Eq. 12.5.12) in which  $\frac{d^2\theta}{dx^2}$  is linear operator  $L$ ,  $N(\theta)$  is nonlinear operator and  $f$  is nonhomogeneous or source term. The construction of homotopy operator can be done as described in [536] and perturbation solution can be obtained in terms of a power series of embedding parameter  $p \in [0, 1]$ . The initial approximation of  $\theta_0$  is taken as some unknown constant. When  $\theta = \sum_{j=0}^{\infty} p^j \theta_j$  is substituted in Eq. (12.9.14) and coefficients of different powers of  $p$  are collected then nonlinear powers of  $\theta_0, \theta_1$ , etc., and their derivatives are bound to occur. In obtaining higher-order approximate perturbation solutions either further approximations are done for nonlinear terms or highly nonlinear equations are to be solved. Up to second-order approximation, nonlinear terms in  $\theta_1$  and its derivative do not occur in the solution in [708].  $\theta_1, \theta_2$  have been obtained and unknown constant value of  $\theta_0$  is determined when the second-order approximate solution is subjected to the prescribed boundary conditions. Numerical work has been done to calculate temperature, surface emissivity parameter  $B$ , radiation-conduction parameter  $N_r$ , etc.

In [709] also steady-state heat transfer in a one-dimensional model of a fin having dimensionless length  $0 \leq x \leq 1$  has been considered. Thermal conductivity depends linearly on temperature. The dimensionless energy equation is taken in the form of following equation.

$$\frac{d^\alpha \theta}{dx^\alpha} + \beta \theta \frac{d^2\theta}{dx^2} + \beta \left( \frac{d\theta}{dx} \right)^2 - \psi \theta^4 = 0, \quad 1 < \alpha \leq 2, 0 \leq x \leq 1, \quad (12.9.16)$$

$\theta(x)$  is dimensionless temperature, and  $\beta$  and  $\psi$  are dimensionless thermal conductivity and thermo-geometric fin parameters, respectively. In the first term, fractional derivative has been considered to examine the effect of non-Fickian heat flux law. For the derivation of Eq. (12.9.16), scaling and description of parameters, refer [709]. In dealing with nonlinear operators and using HPM, when the dependent variable is expanded in a power series of embedding parameter  $p \in [0, 1]$ , the power series in nonlinear terms is to be expanded to obtain its square, cube, etc. In [708] it was done using elementary procedure. Another method of obtaining it is as given below.

If  $N$  is a nonlinear operator then in HPM

$$N(\theta) = N \left( \sum_{n=0}^{\infty} p^n \theta_n \right) = \sum_{n=0}^{\infty} p^n H_n(\theta) = \sum_{n=0}^m p^n H_n(\theta_0, \theta_1, \theta_2, \dots, \theta_m), m > 0, \quad (12.9.17)$$

$$H_n(\theta_0, \theta_1, \theta_2, \dots, \theta_m) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[ N \left( \sum_{i=0}^m p^i \theta_i \right) \right]_{p=0}, \quad n = 0, 1, 2, \dots, m. \quad (12.9.18)$$

For example if the second term in Eq. (12.9.16) is considered then

$$H_0(\theta) = \theta_0(x) D_{xx}(\theta_0(x)), \quad H_1(\theta) = \theta_1(x) D_{xx}(\theta_0(x)) + \theta_0(x) D_{xx}(\theta_1(x)). \quad (12.9.19)$$

To calculate fractional derivatives in addition to Caputo derivative and Riemann-Liouville fractional derivatives, other methods have also been proposed in [392]. Because of the presence of fractional derivative in the first term in Eq. (12.9.16), a transform known as Sumudu transform, refer [710–712], has been applied to Eq. (12.9.16). It is not possible to discuss here in detail the application of Sumudu transform to Eq. (12.9.16) and then further its use in HPM. Given below is the definition of Sumudu transform and some of its properties. If  $S_d$  denotes the Sumudu transform of a suitable function  $f(t)$  (cf. [709, 710]) then

$$F(u) = S_d(f(t)) = \int_0^\infty f(ut) e^{-t} dt, \quad u \in (-\tau_1, \tau_2), \quad \tau_1, \tau_2 > 0. \quad (12.9.20)$$

The inverse transform is defined as

$$S_d^{-1}[F(u)] = \sum \text{residues} \left[ \frac{\exp(ut) F\left(\frac{1}{u}\right)}{u} \right], \quad |F(1/u)/u| < MR^{-k}, \quad (12.9.21)$$

$M$  and  $k$  are positive constants and  $F(1/u)/u$  is assumed to be a meromorphic function in some circular region of radius  $R$ . The following two properties are useful in calculations.

$$\left. \begin{aligned} S_d[f^{(n)}(t)] &= \frac{S_d[f(t)]}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}}, \\ S_d[D_t^\alpha f(t)] &= u^{-\alpha} S_d[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0+), \end{aligned} \right\} \quad (12.9.22)$$

$m-1 \leq \alpha \leq m$ .  $D_t^\alpha$  is the Caputo derivative as in Eq. (12.9.16).

Sumudu transform of Eq. (12.9.16) can be taken after expanding  $\theta$  in a power series of  $p$  and then it is inverted. Further details can be seen in [709] in which  $\theta_0$ ,  $\theta_1$  and  $\theta_2$  have been obtained.  $\theta_0 = A$ , a constant, and  $\theta_1(x) = \psi A^4 x^\alpha / \sqrt{(\alpha+1)}$ . Numerical work has been done to show the effect of parameters on temperature. In the presence of fractional derivatives, to determine  $\theta_0$ ,  $\theta_1$ ,  $\theta_2$  we have to solve equations for them. These equations contain fractional derivatives of  $\theta_0, \theta_1$ , etc., which have to be solved using some transform such as the one used in [709]. In the earlier references [703, 704] fractional derivatives were occurring in the differential equations for first- and second-order solutions. These equations have to be solved by using some transform but solutions could be obtained with the help of an appropriate guessing of solutions. In [704] this problem did not arise as it is a simple problem and only  $\theta_0$  and  $\theta_1$  have been obtained in which it is not difficult to obtain fractional derivatives using Caputo derivatives.

For the problem considered in [709], the application of Sumudu transform and its inversion has been explained. The work is appreciable.

### 12.9.3 Homotopy Analysis Method (HAM): Brief Theoretical Background and Some Solutions

We first describe HAM and afterwards its comparison with HPM will be presented. In [540], Liao has reported many results on various aspects of HAM such as solution procedure, rate of convergence, choice of base functions, its relation to other approximate methods. Several solutions of problems using HAM have also been reported. We cannot do any reporting of HAM better than the contents of [540] but for the sake of understanding of solution procedure in some problems to be discussed here some description of HAM is necessary. As in HPM, in HAM also, the concept is to construct a homotopy.

Consider a nonlinear operator equation  $N(\theta(x, t)) = 0$ . Let  $L$  be a linear operator which is one of the terms in the operator  $N$ . There is no rule for choosing  $L$  except that the order of the highest derivative in  $L$  cannot exceed the highest-order derivative in  $N$  excluding  $L$ .  $L$  is also chosen keeping in mind the boundary and initial conditions, choice of base functions, asymptotic behaviour of the solution, etc. These are some of the guidelines.  $\theta$  could be a function of several variables and may represent temperature or any other physical variable. In HAM, the homotopy  $H_A(\phi(x, t); p, h, H(x, t))$  is defined as

$$H_A(\phi(x, t); p, h, H) = (1 - p)L[(\phi; p, h, H) - \theta_0(x, t)] - phH(x, t)N(\phi; p, h, H), \quad (12.9.23)$$

$p$  is embedding parameter as in HPM,  $h$  is a nonzero parameter and  $H(x, t)$  is a nonzero auxiliary function. When  $h = -1$  and  $H(x, t) = 1$ , then  $H_A$  is the same homotopy as in HPM. It is assumed that when  $f = 0$ ,  $L(f) = 0$ . If  $p = 0$  in Eq. (12.9.23), then

$$H_A = 0 \Rightarrow L[(\phi; 0, h, H) - \theta_0(x, t)] = 0 \Rightarrow (\phi; 0, h, H) = \theta_0(x, t), \quad (12.9.24)$$

$\theta_0(x, t)$  is the initial approximation of  $\theta(x, t)$ . When  $p = 1$

$$H_A(\phi; 1, h, H) = 0 \Rightarrow hHN(\phi; 1, h, H) = 0 \Rightarrow N(\phi; 1, h, H) = 0, \quad (12.9.25)$$

as  $h \neq 0$  and  $H \neq 0$ . It is clear that when  $p$  varies from 0 to 1, the function  $\phi(x, t; p, h, H)$  varies continuously from  $\theta_0$  to the exact solution of the problem. Through the equation  $H_A(\phi; p, h, H) = 0$  subjected to the initial condition  $\phi(t; p = 0) = \theta_0(x, t)$ , a family of functions has been constructed by the variation of the parameter  $p$ . In topology this type of variation is called deformation. The role of  $h$  and  $H$  will be discussed after constructing the solution of a Stefan problem for the sake of better understanding.

#### *mth-Order Deformation*

Let

$$H_A^0 = (1 - p)L[(\phi(x, t; p)) - \theta_0(x, t)] - phHN(\phi(x, t; p)) = 0, \quad (12.9.26)$$

with the initial condition  $\phi(x, t; p = 0) = \theta_0(x, t)$ .  $\phi(x, t; p)$  is the short form of  $\phi(x, t; p, h, H)$ . This system is called zero-order deformation. As in HPM, in HAM also, the perturbation solution is constructed in the form of the series in term of parameter  $p$ . We consider

$$\phi(x, t; p) = \phi_0(x, t) + \sum_{m=1}^{\infty} p^m \phi_m = \phi_0(x, t) + \frac{1}{m!} \sum_{m=1}^{\infty} \frac{\partial^m \phi}{\partial p^m} \Big|_{p=0} p^m, \quad (12.9.27)$$

$$\phi_m = \frac{1}{m!} \sum_{m=1}^{\infty} \frac{\partial^m \phi}{\partial p^m} \Big|_{p=0}; \quad \phi(x, t; 1) = \theta(x, t). \quad (12.9.28)$$

When  $\phi$  in Eq. (12.9.26) is replaced by the series in Eq. (12.9.27) and the zero-order deformation is differentiated  $m$  times,  $m = 1, 2, \dots$ , with respect to  $p$  and limit  $p = 0$  is taken, we get

$$L(\phi_m - \psi_m \phi_{m-1}) = hHR_m(\vec{\phi}_{m-1}), \quad \vec{\phi}_m = \{\phi_0(x, t), \phi_1(x, t), \phi_m(x, t)\}, \quad (12.9.29)$$

$$\psi_m = 0 \text{ when } m \leq 1, = 1 \text{ otherwise; } R_m(\vec{\phi}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N(\phi(x, t; p))}{\partial p^{m-1}} \right|_{p=0}. \quad (12.9.30)$$

Eq. (12.9.29) is linear. Starting from the initial approximation  $\phi_0$ , determine  $\phi_1$  and then  $\phi_2$  using  $\phi_1$  and  $\phi_0$ . In this way  $\phi_m$  can be determined using Eq. (12.9.29).

A simple one-dimensional one-phase inverse Stefan problem has been considered in [713]. The formulation given below has been considered earlier also but we repeat it here.

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \quad 0 < x \leq S(t), t > 0; \quad T(x, 0) = \phi(x), 0 \leq x \leq S(0); S(0) = 0. \quad (12.9.31)$$

$$T(S(t), t) = T_M; \quad -\frac{\partial T}{\partial x}(S(t), t) = \frac{dS}{dt}; \quad T(0, t) = T_0(t) \text{ or } -\left. \frac{\partial T}{\partial x} \right|_{x=0} = q(t). \quad (12.9.32)$$

$x = S(t)$  is the phase-change boundary which is known. For convenience all the parameters have been taken as unity by us. Any one of the two boundary conditions in Eq. (12.9.32) can be prescribed and  $T_0(t)$  or  $q(t)$  is unknown. Take  $H(x, t) = 1, L = \frac{\partial^2}{\partial x^2}, N = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$  and initial approximation as  $\phi(x)$ . Calculate the  $m$ th-order deformation equation as given in Eqs (12.9.29), (12.9.30). If  $T(x, t) = \sum_{j=0}^{\infty} p^j T_j(x, t)$  then it is not difficult to obtain the following system of equations after substituting the finite truncated series of  $T(x, t)$  in Eq. (12.9.29) and using  $T_0(x, t)$  for  $\phi_0(x, t)$ ,  $T_1(x, t)$  for  $\phi_1(x, t)$  and so on. On comparing different powers of  $p$ , we get

$$\frac{\partial^2 T_1}{\partial x^2} = h \left( \frac{\partial T_0}{\partial t} - \frac{\partial^2 T_0}{\partial x^2} \right); \quad \frac{\partial^2 T_n}{\partial x^2} = \frac{\partial^2 T_{n-1}}{\partial x^2} + h \left( \frac{\partial T_{n-1}}{\partial t} - \frac{\partial^2 T_{n-1}}{\partial x^2} \right), \quad n \geq 2. \quad (12.9.33)$$

The interface conditions are satisfied approximately as follows.

$$T_0(S(t), t) + T_1(S(t), t) = T_M; \quad \left. \frac{\partial T_0}{\partial x} \right|_{x=S(t)} + \left. \frac{\partial T_1}{\partial x} \right|_{x=S(t)} = -\frac{dS}{dt}. \quad (12.9.34)$$

For rest of the equations authors have taken

$$T_n(S(t), t) = 0; \quad \left. \frac{\partial T_n}{\partial x} \right|_{x=S(t)} = 0, \quad n \geq 1. \quad (12.9.35)$$

The parameter  $p$  in this case should depend also on  $S(t)$  as in Eq. (12.9.6). In [713],  $p$  is taken independent of  $S(t)$ . Three test problems have been considered by taking  $S(t)$  and  $\phi(x)$  as known. On the basis of this information corresponding temperature can be calculated and then subsequently all other required quantities and exact solution is available. All the HAM analytical solutions in three test cases compare well with the known exact solutions when  $h = 1$  is taken. In these particular problems since the exact solutions are known, optimal values of  $h$



can be determined by considering the mean squared error of  $T_x(0, 0)$  (calculated by HAM and exact solution) or some other quantity such as temperature as a function of  $h$  and minimizing the error. Such graphs of  $h$  versus some required quantities are known as  $h$ -curves. For  $h=1$ , the perturbation solution found in one problem is the same as the exact solution but this is not so for other values of  $h$ . Numerical work was done for all the three test problems and the inverse problems have been solved for both unknown temperature and unknown flux boundary conditions. The numerical work has been presented in the form of tables and figures which provide information about rate of convergence of the relative percentage error with respect to variation of  $h$  from  $h = 0.6$  to  $h = 1.7$  versus absolute errors of the partial sums by increasing the number of terms in the series in Eq. (12.9.27). Ten terms of the infinite perturbation series were calculated. The equations in Eqs (12.9.29), (12.9.30) can be generalized if specific heat and thermal conductivity are functions of temperature. The  $h$ -curve does not provide any value of  $h$  which will give fastest convergence rate.

Having discussed the above solution procedure using HAM, we are better equipped to discuss some other aspects of HAM. The parameter  $h$  in Eq. (12.9.23) is associated with the rate of convergence of the perturbation series at  $p = 1$  which is of interest. The parameter  $h$  depends very much on the auxiliary function  $H(x)$  and thermo-physical parameters. In moving boundary problems,  $H(x)$  also depends on the moving boundary. In [540, 714, 715], Liao has proposed some rules to be followed whilst using HAM. We prefer to call them suggestions rather than rules as even if these rules are violated, the solutions of problems can be obtained using HAM. How good or bad solutions without following these rules would be in terms of absolute errors or squared residual errors may differ from problem to problem. To understand the proposed rules mentioned in [540], a nonlinear problem discussed in [715] is being taken up. We skip the details of formulation as they are not relevant in the following discussion. Consider the nonlinear equation

$$F'''(\xi) + (A - 1)F(\xi)F''(\xi) - 2A(F'(\xi))^2 = 0; F(0) = 0, F'(0) = 1, F'(\infty) = 0. \quad (12.9.36)$$

It has been proved in an earlier work (cf. [715]) that as  $\xi \rightarrow \infty$ ,  $F(\xi) \sim \xi^\alpha$ ,  $\alpha = (1 - A)/(1 + A)$ ,  $0 < A < 1$ . In view of the initial conditions and asymptotic behaviour, the base functions (or basis) and  $F(\xi)$  should be of the following form.

$$\{(1 + \xi)^\alpha, 1, (1 + \xi)^{m\alpha - n} | m\alpha - n < 0, m \geq 1 \text{ and } n \geq 1 \text{ are integers}\}, \quad (12.9.37)$$

$$F(\xi) = a + (1 + \xi)^\alpha \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} b_{m,n} (1 + \xi)^{m\alpha - n}, \text{ } a \text{ and } b_{m,n} \text{ are coefficients.} \quad (12.9.38)$$

This choice of an appropriate basis is called ‘the rule of solution expression’ (RSE).

Another appropriate choice to be made is of initial approximating function in the perturbation series expansion so that the prescribed initial conditions are satisfied. In the present problem it is taken as  $F_0(\xi) = ((1 + \xi)^\alpha - 1)/\alpha$ . The linear operator in [715] has been taken as

$$L[\phi(\xi; p)] = (1 + \xi)^3 \frac{\partial^3 \phi}{\partial \xi^3} + 2(1 - \alpha)(1 + \xi)^2 \frac{\partial^2 \phi}{\partial \xi^2} - \alpha(1 - \alpha)(1 + \xi) \frac{\partial \phi}{\partial \xi}, \quad (12.9.39)$$

$p$  is the embedding parameter and the nonlinear operator is

$$N[\phi(\xi; p)] = \frac{\partial^3 \phi}{\partial \xi^3} + (A - 1)\phi \frac{\partial^2 \phi}{\partial \xi^2} - 2A \left( \frac{\partial \phi}{\partial \xi} \right)^2. \quad (12.9.40)$$

The linear operator  $L$  has the property that  $L[c_0 + c_1(1 + \xi)^\alpha + c_2(1 + \xi)^{\alpha+1}] = 0$ .

The  $m$ th-order deformation equations can be obtained as in Eqs (12.9.29), (12.9.30). As  $h$  is a constant, it can be retained as in Eq. (12.9.29) as  $h$  but the auxiliary function is a function of  $\xi$ , it has to be prescribed and it has to be one of the base functions.  $H(\xi)$  is taken as

$$H(\xi) = (1 + \xi)^\gamma, \quad \gamma \text{ is to be suitably chosen.} \quad (12.9.41)$$

The choice of  $\gamma$  depends on RSE according to which  $\gamma$  has to be an integer. If  $\gamma \geq 2$  then the solution will have one term which is of the form  $(1 + \xi)^{\alpha+1}$  which violates RSE. If  $\gamma \leq 0$  then coefficients of some terms such as  $(1 + \xi)^{\alpha-1}$  are always zero and solution cannot be improved when the number of terms is increased in the series. When the order of approximation tends to infinity, i.e. when the number of terms in the series expansion of  $\phi(x, t; p)$  as in Eq. (12.9.27) is increased the solution should improve and this rule is called 'rule of coefficient ergodicity' (RCE). Therefore  $H(\xi) = 1 + \xi$ . For the solution of the above problem by HAM refer [715].

The parameter  $h$  is the *rate of convergence controlling parameter* in the infinite series solution in Eq. (12.9.27). Enough freedom is there to choose it but its optimal value can be determined only after obtaining numerical results for different values of  $h$  and observing 'h-curve'. Starting from  $m = 1$  in Eq. (12.9.27), the number of terms is increased and  $h$  curve is plotted for the HAM solution for the desired quantity which could be temperature or its derivative or some other quantity. Another method is to minimize the mean squared residual error (MSRE) and this method can be called OHAM which is a short form of *optimal homotopy analysis method*, proposed in [714]. There is no rule which can be proposed to choose the correct value of  $h$  from mathematical analysis point of view. In determining the value of  $h$  in the 'h-curve' method, the approximate perturbation solution is plotted versus  $h$  and some interval for  $h$ , say, between (0, 2) can be obtained over which the solution does not change much. We can call this interval as the region of convergence of the solution with respect to  $h$ . The 'h-curve' gives only a region and not the optimal value of  $h$ . In OHAM, a good approximate value of  $h$  in the region of convergence is obtained. In minimizing the MSRE, the residual error for the exact solution is zero. For approximate solution the integrand should be taken as the operator in the equation. For example, in [713], the integrand will be square of  $\left( \frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} \right)$  and approximate solution by HAM which contains  $h$  is substituted for  $T$ . This integrand is to be integrated over the time-dependent region for minimization with respect to  $h$ . The minimum is obtained by differentiating this residual with respect to  $h$  and equating the derivative to zero which determines  $h$ . For an approximate solution, the minimum is not zero. In [713],  $h$  curves have been drawn and MSRE has also been used.

In [716], three ordinary second-order differential equations have been considered whose exact solutions are known. Approximate solutions have been obtained using HAM in which  $H(x) = 1$  is taken. Six terms of the infinite series have been obtained. The value of auxiliary parameter  $h$  has been obtained by using MSRE. For minimization software such as Mathematica 8.0 is available.

A highly nonlinear Stefan-like problem concerning thermal explosion in two-phase polydisperse fuel spray of a combustible mixture of gas with fuel droplets has been considered in [717]. Such problems involve different time scales and such processes are modelled as singular perturbed systems (SPS) of ordinary differential equations. For example, consider the following SPS.

$$\frac{dy}{dt} = f(y, z, \gamma), \quad y(t_0) = y_0 \quad \text{and} \quad \gamma \frac{dz}{dt} = g(y, z, \gamma), \quad z(t_0) = z_0, \quad (12.9.42)$$

where  $y \in R^m$  (called slow variable) and  $z \in R^n$  (called fast variable) are vectors in Euclidean space,  $t \in (t_0, \infty)$ ,  $0 < \gamma < \gamma_0 \ll 1$ ,  $\gamma$  is a small parameter associated with the inverse of the adiabatic temperature after explosion in the system.  $f$  and  $g$  are functions from  $R^m \times R^n \rightarrow R^{m,n}$  and are smooth functions for all  $y$ ,  $z$  and  $\gamma$ . When  $\gamma = 0$  the system degenerates as the second equation in Eq. (12.9.42) becomes independent of time. Also rate of change of  $z$  becomes infinity so it becomes a fast subsystem whilst the first equation describes a slow subsystem.

In the analytical-numerical solution obtained in [717], HAM has been used in conjunction with the *method of invariant manifold* or method of integral manifold (MIM) (cf. [717]). Invariant manifold is briefly defined below.

A smooth surface in the phase space  $P \in R^m \times R^n \times R$  is called invariant manifold of the system (12.9.42), if any phase trajectory  $(y(t, \gamma), z(t, \gamma))$  such that  $(y(t_1, \gamma), z(t_1, \gamma)) \in P$  for any  $t > t_1$ . If this condition holds only for  $t \in [t_1, T]$  for some  $T$ , then  $P$  is called a local invariant manifold.

If such invariant manifolds exist then the dimension of the system is reduced and analysis is simplified as discussed below. Therefore the aim is to find an invariant manifold of dimension  $m$  which is the dimension of the slow variable  $y$  which can be represented as a graph  $z = G(y, \gamma)$ . The exact location of the slow manifold is difficult to determine. MIM [718] states that the system described in Eq. (12.9.42) by the fast and slow subsystems has a unique invariant manifold which can be represented as a power series with respect to the small parameter  $\gamma$ , i.e.

$$z = G(y, \gamma) = \sum_{i=0}^{\infty} \gamma^i G_i(y). \quad (12.9.43)$$

In the above asymptotic representation of the invariant manifold,  $G_0(y)$  is  $O(\gamma)$ , and it is the approximation of the slow invariant manifold. The dynamics of this slow invariant manifold is given by

$$\frac{dy}{dt} = f(y, G(y, \gamma), \gamma). \quad (12.9.44)$$

Up to second-order approximation of the exact slow manifold can be found rather easily by using HAM as this method is applicable for small  $\gamma$  also. In this case  $g(y, z, \gamma) = 0$  in Eq. (12.9.42) which determines the slow surface and the analysis of the original system is reduced to the analysis on the slow surface. On the slow surface, the changes in slow and fast variables are comparable. Ultimately for MIM also HAM is the best choice and so the analytical-numerical solution in [717] is based on HAM.

The formulation of this interesting and difficult problem has been only partially explained in [717] by referring to some of the material given in other works. Full description of the formulation would require several pages. We skip all the preliminary formulation details. The formulation involves a system of three coupled equations. Let  $\theta_g$ ,  $M$  and  $\eta$  be the dimensionless temperature, mass of the droplet and concentration of mixture of gas fuel, respectively. The formulation consists of the equations of the following type.

$$\gamma \frac{d\theta_g}{dt} = f_1(\eta, \theta_g, M \text{ and some parameters}). \quad (12.9.45)$$

$$\frac{dM}{dt} = f_2(\theta_g, M \text{ and some parameters}); \quad \frac{d\eta}{dt} = f_3(\eta, M, \theta_g \text{ and other parameters}). \quad (12.9.46)$$

$M$  depends on the radius  $R$  of the droplet and radius  $R$  is a function of time,  $R = S(t)$  is the moving boundary. However, the velocity of this moving boundary  $S(t)$  has been obtained by some other physical considerations and not analytically. Initial conditions are prescribed for all the variables.

The coupled system of Eqs (12.9.45), (12.9.46) has been solved using HAM and  $m$ th-order deformation equations have been given in [717] in which the linear operator is the time derivative. The convergence controlling parameter  $h$  has been determined with the help of  $h$ -curve as well as by minimizing the integral of  $\{N(f(t))\}^2$  or in other words minimizing the MSRE where  $N(f(t)) = 0$  are the equations in Eq. (12.9.45), (12.9.46). The auxiliary function  $H$  occurring in Eq. (12.9.26) has been taken unity. In the application of MIM, it is required that the  $m$ th-order deformation equations are written as

$$F_0(\theta_0, \dots, \theta_{m-1}, M_0, \dots, M_{m-1}, \eta_0, \dots, \eta_{m-1}) + \gamma F_1(\theta_{m-1}) + \gamma^2 F_2(\theta_{m-1}) = 0. \quad (12.9.47)$$

The algorithm for HAM and for the combination of HAM and MIM has been presented explicitly with all the steps explained clearly. Analytical-numerical solutions have been obtained for two cases: (1)  $\gamma = 0$  and  $\gamma^2 = 0$  and (2)  $\gamma \neq 0$  and  $\gamma^2 = 0$  by substituting these values in the  $m$ th-order deformation equations. The solutions so obtained are the solutions obtained in the combination of HAM and MIM. Solutions up to 50th-order approximation in HAM were considered.  $h = -1$  ensures the convergence of the series. For further details refer [717]. Solutions for  $\theta_g$ ,  $M$  and  $\eta$  are presented graphically. Complete mathematical model of the above problem was solved numerically also by RKM and the analytical-numerical solution is in good agreement with the numerical solution. This study is appreciable.

Boundary layer flow with heat transfer over a permeable flat plate in a Darcian medium with radiation effects has been considered in [719]. There is no moving boundary in this problem. The initial two-dimensional steady-state flow has been simplified by making assumptions and it results in the following one-dimensional coupled equations in terms of a similarity variable  $\eta$ .

$$f''' + A_1(ff'' - f'^2) - f' = 0, \quad f = f(\eta), f, \theta \text{ and } \eta \text{ are dimensionless.} \quad (12.9.48)$$

$$A_2\theta'' + A_3f\theta' = 0, \quad \theta = \theta(\eta), A_1, A_2 \text{ and } A_3 \text{ are parameters.} \quad (12.9.49)$$

$$f(0) = f_w, f'(0) = 1, f'(\infty) = 0, \theta(0) = 1, \theta(\infty) = 0. \quad (12.9.50)$$

$f$  and  $\theta$  are similarity functions for velocity and temperature, respectively. The description of  $\eta$  is not straightforward and refer [719] for it. In view of the boundary conditions at infinity, the base functions can be taken as  $\{\eta^k \exp(-n\eta), k \geq 0, n \geq 0\}$ . In the application of HAM the linear operators  $L_1$  and  $L_2$  are taken as the linear operators as they are occurring in Eqs (12.9.48), (12.9.49), respectively, without any change and so is the case with nonlinear operators.

$$L_1(c_1 + c_2e^{-\eta} + c_3e^{\eta}) = 0; \quad L_2(c_4e^{-\eta} + c_5e^{\eta}) = 0, \quad (12.9.51)$$

$c_1, c_2, \dots, c_5$  are arbitrary constants. The solution procedure is as described earlier in which  $H(x) = 1$  was taken. Up to 50th-order approximate solutions have been obtained. The  $h$ -curves have been drawn by considering the interval  $-2.5 \leq h \leq 0.5$ . MSRE has also been used to find optimal value of  $h$ . Numerical work has been reported and approximate solution has been compared with the numerical solution.

Having discussed HAM and HPM and also discussed their application in some problems, one thing is very clear that both these methods are applicable to obtain the approximate solutions of the given problem without restricting the range of parameter values. This is a great advantage in using these methods as other perturbation methods are valid for restricted ranges of parameter values. Invariably when approximate perturbation solutions other than HAM and HPM are studied, the original problem formulation is approximated to consider a meaningful range of perturbation parameter. The restriction on perturbation parameter values also puts restriction on other parameter values. In both HAM and HPM the approximate solution can be obtained by considering the problem formulation in its original form.

In [720], He has obtained some solutions and indicated that solutions obtained by HPM are more accurate and easily obtained than obtained by HAM. This is possible in some problems. In any case HPM is a particular case of HAM. In HPM, there is no convergence controlling parameter and the only way is to find out convergence is to compute more and more terms of the infinite perturbation series and compare the solutions with purely numerical solutions. There is no rigorous proof for the convergence of the infinite series in both the methods. In both HPM and HAM it is assumed that for  $p = 1$  (refer Eqs 12.9.5, 12.9.27), the series converges. If the computer softwares as indicated in earlier references were not available, it was neither possible to calculate 50th-order approximate solution as in [717] nor it was possible to draw  $h$ -curve and obtain the optimal value of  $h$ . The availability of these softwares makes HAM advantageous.

Umpteen number of cross-references can be found in each of the references [702–720] but very few of them are concerned with phase-change problems. We wanted to concentrate only on Stefan problems but in their absence, some heat transfer problems had to be discussed to explain the methods clearly. The future direction of research in HPM and HAM should be on their applications to Stefan and Stefan-like problems.

## 12.9.4 Regular and Singular Perturbation Methods and Solutions

### *Choosing the Perturbation Parameters*

Whilst reviewing perturbation methods and their solutions, it was found that the contribution of earlier research workers in developing this topic plays an important role and so these works find substantial space in this section. Connected with perturbation solutions there are many other considerations and some of which are: (i) choosing appropriate dimensionless perturbation parameters whose effects are to be investigated in the problems considered. This also requires the ranges of the parameters in which they may vary (2) uniform validity of perturbation solutions throughout the region under consideration and also in the desired ranges of parameters. This may require different types of perturbation solutions such as regular and/or singular perturbation solutions (3) convergence and improvements of perturbation series solutions. As mentioned earlier, several books, monographs and review articles exist on perturbation methods and solutions such as [690–700]. Enough theoretical and procedural background and other related aspects have been discussed in these references and also in some other sources covering variety of problems in science and engineering. The solutions of

different orders of perturbation solutions (sometimes even the first-order solution) invariably require application of various advanced tools of applied mathematics to get an analytical solution and/or an analytical-numerical solution. This is so for solutions of Stefan and Stefan-like problems also using perturbation methods. Therefore the presentation of a perturbation solution in an explicit form is not possible. We do not intend to survey the literature and present solutions as it would require a big volume separately. The discussion in this section is just a brief account of some old and new solutions of Stefan problems covering some aspects of methods and solutions.

Choosing an appropriate parameter is the first requirement of a perturbation solution. The perturbation parameter is generally denoted by  $\epsilon$ , unless denoted otherwise, and in the case of multiple parameters they are denoted by  $\epsilon_1, \epsilon_2, \dots$  and so on. The effect of these perturbation parameters on the solution is to be investigated. This involves recasting of problem formulation in dimensionless form in such a way that  $\epsilon$  now depends on other parameters which affect  $\epsilon$  in a significant way. This should also be supported by the actual physical situation. For example, in some Stefan problems without any convection in the liquid, the perturbation parameter is taken as  $\epsilon = \text{latent heat/sensible heat} = l/C_p(T_m - T_0)$ ,  $T_m$  is the freezing temperature,  $T_0$  is the cooling temperature and  $C_p$  is specific heat at constant pressure.  $\epsilon$  is the Stefan number, also denoted by  $Ste$ . Some authors take Stefan number as  $\epsilon^{-1}$ . In most of the solidification and melting problems, the solidification or melting rate is the most important quantity and the controlling parameters are latent heat and/or sensible heat. The range of  $\epsilon$  depends on of what order latent heat and specific heat are with respect to  $\epsilon$ . If the boundary condition is of convective type and if the effect of heat transfer coefficient  $h$  is to be investigated then we consider Biot's number  $Bi = hR/K$ ,  $R$  is some characteristic length and  $K$  is thermal conductivity. In imperfect mold contact problem, Stefan number could be greatly influenced by Biot's number and we have to take into account of what order Bio's number is with respect to  $\epsilon$ . Several such examples will be considered whilst discussing perturbation solutions.

### *Regular Perturbation Methods and Solutions in Planar, Cylindrical and Spherical Geometries: One- and Multidimensional Problems*

In perturbation solutions, for example, in an one-phase one-dimensional transient solidification problem, the dependent variables temperature and freezing front are expressed in a power series or in an asymptotic series in  $\epsilon$  such as

$$T(x, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n T_n(x, t) \text{ and } S(t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n S_n(t); T_n, S_n \text{ are unknown.} \quad (12.9.52)$$

If two perturbation parameters have been considered then double power series expansions of the following form are considered.

$$T(x, t; \epsilon_1, \epsilon_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} T_{mn} \epsilon_1^m \epsilon_2^n. \quad (12.9.53)$$

In Eq. (12.9.52), the series are Taylor series expansions about  $\epsilon = 0$ . We discuss only series in Eq. (12.9.52) as similar arguments hold for perturbation solutions with multiple perturbation parameters. To obtain the solution, the perturbation series in Eq. (12.9.52) is substituted in the complete formulation and coefficients of different powers of  $\epsilon$  are collected. By equating these

coefficients of  $\epsilon^n$ ,  $n = 0, 1, 2, \dots$ , and equating them to zero, we get equations which may be differential or algebraic to determine the unknown coefficients.

It is invariably assumed, without actually proving it, that there exists some interval  $|\epsilon| < \epsilon_0$ ,  $\epsilon_0$  is small, and some region  $\Omega$  in the  $(x, t)$  plane in which the series in Eq. (12.9.52) converges. As there is generally no rigorous proof of convergence, it is only possible to show the convergence of perturbed solution through the computed solution and comparing it with either an exact solution or comparing it with a purely numerical solution. Often after two or three terms of the series in Eq. (12.9.52), the solutions of equations resulting from the coefficients of  $\epsilon^n$  equated to zero for  $n > 3$  are difficult to determine analytically. The term convergence is used for an infinite series and although it is a misnomer to use it for finite series we use it just for very few terms of the series. If  $|\epsilon^n T_n| < |\epsilon^{n-1} T_{n-1}|$  where  $n$  could be 3 or 4 and the perturbation solution agrees uniformly with the exact analytical or numerical solution in some interval  $|\epsilon| < \epsilon_0$  and for some region  $\Omega_0 < \Omega$  of the  $(x, t)$  plane then we say that the solution has been obtained by *regular perturbation method* or regular perturbation series expansion has been used to determine the solution. We have to specify the range or ranges and the region for which perturbation solution is valid. Generally, we are looking for uniformly valid perturbation solution in the whole region  $\Omega$  considered in the formulation. For example take  $\Omega = \{(x, t); 0 \leq x \leq 1, 0 \leq t \leq t_1\}$ , where  $t_1$  is total solidification time of a liquid in a slab  $0 \leq x \leq 1$ . It may happen that the perturbation series solution is uniformly valid only for  $0 \leq t \leq t_0 < t_1, 0 \leq x < \frac{1}{2}$  and  $|\epsilon| \leq \epsilon_0 < \epsilon_0$ . Even then the solution is a regular perturbation series solution provided the range of  $\epsilon$  and the region of convergence is clearly specified. It may be clarified here that generally it is required that the solution should be uniformly valid for the entire region under consideration in the formulation. If this is not so then it is said to be a singular solution and singular perturbation method is to be used to obtain a uniformly valid solution. The term asymptotic expansion is also used for such expansions when parameter is very small or it is very large.

In a one-dimensional spherical or cylindrical Stefan problem in the region  $0 \leq r \leq 1$  when regular perturbation series expansions are used for inward solution then the first-order coefficient  $T_1(r, t)$  becomes singular at  $r = 0$  and so is the case with higher-order terms. Later on when we report some solutions this singularity can be easily noticed. Therefore regular perturbation series method is not valid uniformly for  $0 \leq r \leq 1$ . If we exclude a small neighbourhood of  $r = 0$  then regular perturbation solution can be used provided the solution is uniformly valid in the region  $r_0 \leq r \leq 1, r_0 > 0$ . When the solution is not uniformly valid in any part of the region then the expansion is called a singular perturbation expansion and the methods associated with obtaining uniformly valid solutions by using singular perturbation expansions are called *singular perturbation methods*. The objective of singular perturbation expansion is to obtain uniformly valid solution. In some solutions obtained by the series expansions of the type of regular perturbations, the cause of nonuniformity of the solution which is generally referred as singularity can be easily noticed. For example, in cylindrical and spherical problems a term of  $(1/r)$  comes in the solution. In Eq. (12.9.42), for  $\gamma = 0$ , the system of equations becomes degenerate and therefore regular perturbation expansion cannot be used in the neighbourhood of  $\gamma = 0$  if  $\gamma$  is the perturbation parameter. In a differential equation if the highest-order derivative is multiplied by  $\epsilon$  the regular perturbation series solution breaks down when  $\epsilon = 0$ . If any coefficient in the solution by regular perturbation series expansion violates the physics of the problem such as instead of temperature being strictly positive or negative it shows negative or positive value, respectively, due to some value of this coefficient in some region then a singularity has occurred. If any of the boundary conditions is not satisfied by the regular perturbation solution then it is a singular solution.

If in the problem formulation itself a discontinuity in the dependent variable is prescribed for some value of an independent variable then singular perturbation method is to be used for the solution.

A general outline of the procedure to be followed in obtaining the perturbation solution has been mentioned above. Depending on the boundary conditions, the spatial dimension of the problem and perturbation parameters, some variations to solution procedures have to be adopted. Our interest here is only in these different procedures and not in the presentation of solutions, each of which requires lot of space, as invariably, except the zeroth-order solution, higher-order solutions are not easy to obtain analytically. For further information about complete formulation and assumptions, scalings, and numerical results, refer original references. In all the problems which are discussed below, solutions have been obtained in dimensionless form in which we have not mentioned the procedure for scalings.

A simple one-dimensional problem of the freezing of a liquid initially at freezing temperature occupying the region  $x \geq 0$  has been considered in [721]. Convective-type boundary condition (cf. Eq. 1.4.38) is prescribed at  $x = 0$  and the perturbation parameter  $\epsilon$  is the ratio of sensible heat to the latent heat which is taken small. By using Landau's transformation, the freezing front  $x = S(t)$  is fixed at  $x = 1$ . The temperature  $T(\delta, S(\tau); \epsilon)$  is then expanded in a Taylor's series in  $\epsilon$  about  $\epsilon = 0$ ,  $\delta = x/S(t)$  and  $\tau$  is dimensionless time,  $\tau = h^2 t / \rho C K$ ,  $h$  is heat transfer coefficient.  $\epsilon$  is generally small for liquids.

$$T(\delta, S(\tau); \epsilon) = T_0(\delta, S(\tau)) + \epsilon T_1(\delta, S) + \epsilon^2 T_2(\delta, S) + \dots; T(\delta = 1, \tau) = 0, \quad (12.9.54)$$

$$\epsilon \tau(S(t); \epsilon) = \tau_0(S(t)) + \epsilon \tau_1(S(t)) + \epsilon^2 \tau_2(S(t)) + \dots; S(0) = 0. \quad (12.9.55)$$

$T$  and  $\tau$  are two dependent variables and  $\delta$  and  $S(t)$  are two new independent variables. The problem formulation is then rewritten in terms of independent variables  $\delta$  and  $\tau$  which is not given here as it can be easily obtained. The total solidification time when  $S(t) = 1$  can be calculated from Eq. (12.9.55). The standard solution procedure as indicated earlier is followed to determine unknown functions  $T_0$ ,  $T_1$ ,  $T_2$ , and  $\tau_0$ ,  $\tau_1$ ,  $\tau_2$ ,  $\tau_3$  and the series expansion in Eqs (12.9.54), (12.9.55) are substituted in the changed formulation. For numerical work  $S(t)$  is taken as input.

Two one-dimensional one-phase problems have been considered in [722]. In the first problem, constant temperature is prescribed at  $x = 0$ . In this case dimensionless time  $\tau = Kt/(\rho C x_S^2)$  where  $x_S$  is some characteristic length such as frozen thickness in steady-state case. For freezing of a saturated liquid considered in [722], a constant temperature is prescribed at  $x = 0$ . The exact analytical solution of this problem is a particular case of Neumann solution. If  $S(t) = 2\lambda t^{1/2}$  then  $\epsilon$  (Stefan number) can be expressed as a power series in  $\lambda$  by using the equation which is used to determine  $\lambda$ . From this equation,  $\lambda$  can be expressed by reversion as a power series in Stefan number  $\epsilon$ . Once  $\lambda$  is expressed in terms of  $\epsilon$  then the exact solution can be expressed in terms of a power series of  $\epsilon$ . Now this exact solution can be used for comparison with perturbed solution which is to be found as a series expansion in  $\epsilon$  using the procedure described in [721], i.e. taking  $\delta$  and  $S(t)$  as independent variables.

The second problem in [722] is that of solidification of a warm fluid flowing over a cooled plate. It is a one-phase problem. The energy balance condition at  $x = S(t)$  will have an



additional term of the form  $h_1(T_1 - T_m)$ , where  $T_1$  is temperature of the warm fluid and  $h_1$  is the heat transfer coefficient which has been taken in terms of the known heat transfer coefficient  $h_0$  and some other known parameters.  $h_0$  is the heat transfer coefficient in the boundary condition at  $x = 0$ . In this case in addition to small Stefan number which is perturbation parameter, the solution depends also on the Biot's number  $Bi = h_0 x_S / K$  which appears in the formulation.  $x_S$  is the frozen thickness in steady-state case. Series expansions as in Eqs (12.9.54), (12.9.55) have been considered. In this case  $T_2$  in Eq. (12.9.55) is obtained in an integral form. Numerical results agree with the experimental results and the steady-state solution.

As far as mathematical formulation is concerned in [723] it is similar to the one considered for the second problem in [722] and therefore solution procedure remains the same. Some parameters are different such as latent heat is the sum of latent heat as in [722] plus a term which is proportional to  $(T_1 - T_m)$ . To improve the solution, firstly Shank's transformation [396] as given below was used.

$$\epsilon = \tau = \tau_0 + \epsilon \tau_1 / (1 - \epsilon \tau_2 / \tau_1). \quad (12.9.56)$$

Much better results were obtained when the following transformation proposed in [724] was used.

$$\epsilon = \tau = \frac{\tau_0}{2} \left\{ \left[ 1 + \frac{4\epsilon\tau_1}{\tau_0} \left( 1 + \frac{\tau_1}{\tau_0} \right) / \left( 1 - \frac{\tau_2}{\tau_1} \right) \right]^{1/2} + 1 \right\}. \quad (12.9.57)$$

When the transformation given in Eq. (12.9.57) is used for temperature, the second-order term in the expansion of temperature does not satisfy the necessary condition (the total sum of two terms should be strictly positive or negative). Therefore instead of  $T(x, t)$ ,  $(T(x, t))^{-1}$  was expanded in a perturbation series and correct solution was then obtained. Numerical work was done and solution is compared with an exact solution.

Using standard regular perturbation method, solutions of three problems corresponding to three different boundary conditions were obtained in [725]. The three different boundary conditions prescribed at  $x = 0$  are that of temperature prescribed, flux prescribed and of convective type. The problem formulation is similar to that in [721, 722]. Numerical work was done and compared with the perturbed solution. Landau transformation has not been used and perturbation expansions used for small Stefan number  $\epsilon$  as in [721] are of the following form.

$$g(S) = \frac{dS}{dt} = g_0(S) + \epsilon g_1(S) + \epsilon^2 g_2(S) + \dots \quad (12.9.58)$$

$$\tau(S) = \tau_0(S) + \epsilon \tau_1(S) + \epsilon^2 \tau_2(S) + \dots, T(S, x) = T_0(S, x) + \epsilon T_1(S, x) + \dots \quad (12.9.59)$$

$\tau$  is dimensionless time. The solutions have been obtained in terms of  $T(S, x)$  and  $g(S)$ .

The boundary condition at  $x = 0$  in [726] is of convective-radiative type, refer Eq. (1.4.38) and at  $x = L$ , no flux boundary condition is imposed in the considered one-phase one-dimensional phase-change problem. At  $t = 0$ , a liquid at the fusion temperature occupies the region  $0 \leq x \leq L$ . In the radiative term, temperature  $T^4$  is first approximated as

$T^4 \approx 4T_a^3T - 3T_a^4$ , where  $T_a$  is the ambience or environment temperature. The moving boundary is not fixed. The solution procedure is the same as in [725]. The solution has been further improved by approximating  $T^4$  as  $T^4 \approx T_a^4 + 4T_a^3(T - T_a) + 6T_a^2(T - T_a)^2 + \dots$ . Numerical work is reported.

The problem and its solution presented in [723] is again reported by a different author in [727] and published in a different journal. A two-phase melting problem of a solid in the region  $0 \leq x \leq d$  has been considered in [728]. At  $x = 0$ , constant heat flux  $H$  is prescribed and the edge  $x = d$  is insulated. Both solid and liquid phases have been considered with transient heat conduction. Heat flux after scaling is the perturbation parameter  $\epsilon$ , and  $\epsilon = Hd/(2K_S V)$ , where  $K_S$  is the thermal diffusivity of the solid and  $-V$  is the initial temperature of the solid. Instead of using problem formulation in the standard and original form such as in Neumann problem, the HBIM has been used to obtain the average temperatures in the two regions separately. The heat flux  $H$  is equated to the rate of change of heat in the system. This equation in dimensionless form is used in place of two energy equations in solid and liquid. Accordingly the boundary conditions, etc., are fixed. The formulation of this type is new and not discussed here earlier. As done in a standard regular perturbation method, temperatures and moving boundary are expanded in powers of  $\epsilon$  and zeroth-, first- and second-order solutions have been obtained for temperatures and moving boundary. However, for large dimensionless time, because of the presence of terms like  $\tau^2 e^{-3\tau}$  ( $\tau$  is dimensionless time) the second-order term in solid temperature becomes positive which is physically incorrect. In essence singular perturbation method should be used for large time.

The solidification of a liquid at freezing temperature in a slab  $0 \leq x \leq d$  has been considered in [729]. The solidification takes place at both the ends, i.e. at  $x = 0$  as well as at  $x = d$  due to the prescribed boundary conditions of the form

$$K \left. \frac{\partial T}{\partial x} \right|_{x=0} = h(T|_{x=0} - T_{in}); \quad -K \left. \frac{\partial T}{\partial x} \right|_{x=d} = h'(T|_{x=d} - T'_{in}), \quad (12.9.60)$$

$$T_{in} = T_a + A \sin \alpha t \leq T_m; \quad T'_{in} = T'_a + A \sin \alpha t \leq T_m, \quad (12.9.61)$$

where  $A$  is constant amplitude;  $\alpha$ ,  $T_a$ ,  $T'_a$  are constants and  $T_m$  is the freezing temperature. The boundary conditions and various parameters are all directly related to applications in PCM systems and refer [729] for this. If  $\delta_1(t)$  and  $\delta_2(t)$  are the two moving boundaries starting at  $x = 0$  and  $x = d$ , respectively, then the time  $t_m$  when they meet satisfies  $\delta_1(t_m) + \delta_2(t_m) = d$ . It has been assumed that each side of the slab is independent and can be considered separately. In terms of dimensionless quantities  $\tau$  and  $\delta$  for time and moving boundaries, respectively, we have (refer [729])

$$\delta_1(\tau_m) + (h/h')\delta_2(\tau'_m) = h d/k; \quad \delta_1 = 0, t = 0 \text{ and } \delta_2 = d, \tau = 0. \quad (12.9.62)$$

Perturbation series for  $T$ ,  $\delta_1$  and  $\tau$  for l.h.s. phase change is taken as

$$T = T_0(x, t) + \epsilon T_1(x, t) + \beta T_2(x, t) + \dots; \quad \delta_1 = \bar{\delta}_0(t) + \epsilon \bar{\delta}_1(t) + \beta \bar{\delta}_2(t) + \dots \quad (12.9.63)$$

$$\tau = \tau_0 + \epsilon \tau_1 + \beta \tau_2 + \dots, \quad (12.9.64)$$

$\epsilon$  is the Stefan number as in [721] and  $\beta = A/(T_m - T_a)$ . For r.h.s phase change also similar expansions can be used for  $T$  and  $\delta_2$ . The expansions are done in the vicinity of  $\bar{\delta}_0(t)$ . If heat balance in the region  $0 \leq x \leq \delta_1(t)$  is done and perturbation solution is used for the phase change of left-hand side then  $\delta_1(\tau)$  can be obtained from the heat balance equation for the

region  $0 \leq x \leq \delta_1(\tau)$  in which perturbed solution is used and from Eq. (12.9.62)  $\delta_2(t'_m)$  is obtained. However, to obtain temperature and time  $\tau$  for r.h.s. perturbation solution is to be obtained for right-hand side phase change. When the face  $x = d$  is insulated exact solutions are available and analytical-numerical solutions by perturbation method can be compared with the exact solution.

Three one-phase one-dimensional solidification problems in three regions, planar  $x \geq 0$ , spherical and cylindrical regions  $r \geq 1$ , have been considered in [730]. The boundary conditions at  $x = 0$  in planar case and at  $r = 1$  in spherical and cylindrical problems and at the phase-change boundary are given below.

$$T(x = 0, \text{ or } r = 1; t) = f(t), f(t = 0) = 1; T(x = S(t) \text{ or } r = S(t); t) = 0, \quad (12.9.65)$$

$$\frac{dS}{dt} = -\alpha \left( \frac{\partial T}{\partial x} \right)_{x=S(t)}; \quad \frac{dS}{dt} = \alpha \left( \frac{\partial T}{\partial r} \right)_{r=S(t)}, \text{ for sphere and cylinder,} \quad (12.9.66)$$

$\alpha$  is Stefan number as in [721]. Temperature  $T(x, S(t))$  is expanded in a series of  $\alpha$  for small  $\alpha$  in which  $S(t)$  is taken as another independent variable instead of time, i.e.  $t = t(S(t))$ .  $dS/dt$  can be obtained with the help of the Stefan condition in which perturbed solution for temperature is substituted. A cubic equation in  $dS(t)/dt$  is obtained in which only the coefficients of cubic and square terms are not singular. By neglecting the singular terms, an approximate solution has been obtained for  $dS/dt$  which is singular at  $t = 0$  but  $S(t)$  is not singular. The same procedure has been used for spherical and cylindrical problems and terms which are singular at  $r = 1$  are ignored.  $dS/dt$  can be integrated when  $f(t = 0) = 1$  for both sphere and cylinder. In planar case this condition is not required for integration.

When the liquid temperature is more than the freezing temperature and convection boundary condition is prescribed then there is presolidification time  $0 \leq t \leq t_m$  for solidification to start. The liquid temperature at  $t = t_m$  can be determined by solving a heat transfer problem without phase change. For  $t \geq t_m$ , a two-phase solidification problem has been considered in [731] in the region  $0 \leq x \leq d$ . The solution of temperature for  $0 \leq t \leq t_m$  has also been obtained when at  $x = 0$  a convective boundary condition with heat transfer coefficient  $h$  is prescribed and the face  $x = d$  is insulated. For the heat equation in the solid a convective term has been introduced due to shrinkage which creates a void between the mold and the solid and so the solid region is given by  $\gamma_S \leq x < S, x = S(t)$  is the moving boundary, liquid region is taken as  $S < x \leq d, \gamma = 1 - \rho_L/\rho_S, t \geq t_m$ . The boundary condition is prescribed at  $x = \gamma_S$  in [731]. The Stefan number  $\epsilon$  in this problem is the ratio of sensible heat to the latent heat. The diffusion time scales in solid and liquid are of the same order and time scale is taken as  $d^2/k_L$ . However, time scale in the void is defined as  $d\rho_L C_L/h$  and Biot number  $Bi$  is  $hd/K_S$ .

Two cases have been considered,  $Bi \approx O(\epsilon^{-1/2})$ , i.e.  $h$  is large and another case is  $Bi = O(1)$ ,  $h$  is moderate in this case. When  $Bi$  is large, the series expansions for temperatures and moving boundary have been expressed in power series of  $\epsilon^{1/2}$  and when  $Bi$  is small they are expanded in power series of  $\epsilon$ . The perturbed solution gives a short-time solution of the problem. By using Landau transformation

$$\xi = x/S, \quad \eta = (x - S)/(d - S), \quad (12.9.67)$$

moving boundary is fixed at  $\xi = 1$  and the problem is formulated in terms of  $(\xi, \tau)$  for solid and  $(\eta, \tau)$  for the liquid. For both large and small heat transfer coefficients only zeroth-order

analytical solution could be obtained for the liquid temperature and first-order approximations could be obtained for solid temperatures and  $S(t)$ . Near  $\eta = 1$ , the higher-order terms become comparable to lower-order terms so it is a short-time solution.

For large-time solution, the dimensionless time  $\tau$  used in the short-time solution is replaced by  $\bar{\tau} = \epsilon \tau$ ,  $\bar{\tau} \approx O(1)$ . The formulation is then written in terms of  $\bar{\tau}$ ,  $\xi$  and  $\eta$  coordinates and perturbed solution is obtained as obtained for short-time solution. The perturbation solution has also been discussed for one-phase problem and for a problem in which  $Bi$  is time dependent due to heat transfer coefficient which is taken as  $h = K_v/\gamma S(\tau)$ ,  $K_v$  is the thermal conductivity of the void between mold and solid. Finite difference numerical solution has been obtained for the original problem and the approximate perturbed solution has been compared with it.

It has been shown earlier that one-dimensional planar, spherical and cylindrical problems can be formulated in terms of a single set of differential equations and boundary conditions with appropriate minor changes. This has been used in [732] to obtain solutions for solidification in the three different geometries with the help of one solution procedure. In the liquid phase, temperature and concentration gradients are present and liquid phase occupies the region  $0 \leq x < S(t)$ ,  $S(0) = 1$ .  $x = S(t)$  is phase-change boundary. In the solid only temperature gradient is present and for solid the region is  $S \leq x < 1$ . The boundary conditions are time independent. Refer [732] for boundary conditions and assumptions. The only coupling considered between temperature and concentration is through the temperature at  $x = S(t)$  which is defined as

$$T_L = 1 - \alpha \bar{c}_L, \quad \alpha \text{ is the compositional ratio; } \bar{c}_S|_{S(t)} \approx 0. \quad (12.9.68)$$

As discussed in some earlier problems, at  $x = S(t)$  both heat and mass balance is to be done. The heat balance at  $x = S(t)$  in [732] does not involve  $\bar{c}_L$ , which is the liquid concentration. This equation is like Stefan condition involving Stefan number which is defined as the ratio of latent heat and sensible heat which is considered large and therefore perturbation parameter is  $\epsilon = (\text{Stefan number})^{-1/2} \ll 1$ . Mass balance at  $x = S(t)$  which is taken independent of temperature involves *Lewis number* taken as the ratio of thermal conductivity and mass diffusivity.  $\alpha$  in Eq. (12.9.68) is taken very small,  $\alpha \ll 1$  and  $\alpha = \bar{\alpha} \epsilon^q$ ,  $q \geq 2$  and so the dependence of temperature on concentration is removed up to  $O(\epsilon^2)$ .

Landau transformation given in Eq. (12.9.67) is first used to fix the boundary  $x = S(t)$  at  $\xi = 1$  and variable  $\eta$  is used for the liquid region. The solution procedure consists of using standard perturbation expansions for temperatures and moving boundary. Up to  $O(\epsilon)$  analytical-numerical solutions have been obtained for planar and spherical cases. The solution for cylindrical case is not reported. Numerical work has been reported.

Swelling of glassy polymer due to penetration of solvent into it by diffusion beyond a threshold concentration of the solvent in the polymer has been studied in [733]. At the moving boundary  $x = S(t)$ , mass balance is done and the penetration law describing the swelling kinetics is considered. We skip the details of physics of the problem and scalings, etc., and give below the nondimensional formulation of the problem. It may be made clear here that the formulation involves some phenomenological and thermo-physical quantities. We have taken all these quantities equal to unity as our interest is mainly in the method of solution. Refer [733] for details of parameters.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < S(t), t > 0; u = 1 \text{ at } x = 0; S(0) = 0; u \text{ is concentration,} \quad (12.9.69)$$

$$(\lambda + u) \frac{dS}{dt} = -\frac{\partial u}{\partial x}, \text{ at } x = S(t); \quad \frac{dS}{dt} = u, \text{ at } x = S(t), \quad (12.9.70)$$

$\lambda$  is a control parameter which is analogous to  $(1/\epsilon)$ , where  $\epsilon$  is Stefan number and it varies from  $O(1)$  to large number. The first condition in Eq. (12.9.70) arises due to mass balance and second condition describes the penetration law describing swelling kinetics. For the numerical solution, the formulation given in Eqs (12.9.69), (12.9.70) has been considered in [733] after using Landau transformation. However, for perturbation solution when  $\lambda \simeq O(1)$  this formulation is not suitable. Similar to parameter group theory method [359], the following transformations have been used so that perturbation solution for small  $\epsilon$  can be carried out which will give a short-time asymptotic solution. For  $\lambda \approx O(1)$ , we set

$$t = \epsilon \tau, \quad x = \epsilon^a y, \quad S(t) = \epsilon^a \delta(\tau), \quad u(x, t) = \epsilon^b v(y, \tau); \quad \epsilon \ll 1. \quad (12.9.71)$$

When new variables given in Eq. (12.9.71) are substituted in the earlier formulation, we get  $\epsilon^b v = 1$  at  $y = 0$  which implies  $b = 0$  to get a consistent boundary condition when  $\epsilon \rightarrow 0$ .  $a = 1/2$  does not give proper balancing of diffusion equation so  $a = 1$  is taken. Standard perturbation expansions in a Taylor series of  $\epsilon$  were considered for  $v(y, \tau)$  and  $\delta(\tau)$  and up to second-order solutions were obtained which are better than the solutions obtained in [734]. Short-time solution is also required to execute the numerical scheme.

To obtain large-time solutions when  $\lambda \approx O(1)$  and  $\alpha \ll 1$ , we take

$$t = \tau/\alpha^2, \quad x = \alpha^a y, \quad S(t) = \alpha^a \delta(\tau), \quad u(x, t) = \alpha^b v(y, \tau), \quad (12.9.72)$$

$\alpha \ll 1$  is an artificial parameter and  $a, b \leq 0$ .  $\alpha$  is taken as a perturbation parameter. In this case  $b = 0$  and  $a = -1$  are taken on observing the transformed equations and up to second-order approximations have been obtained. For  $\lambda$  large, take  $\epsilon = 1/\lambda$  and use the changed variables as in Eq. (12.9.71). In this case  $a = 1$  and  $b = 0$  are taken for a consistent solution. Up to second-order perturbation solutions have been obtained by considering perturbation expansions in powers of  $\epsilon$ ,  $\epsilon \ll 1$ .

The HBIM and variation to it named CIM which were earlier discussed (refer [638, 639]) can be used for all time and all values of the parameters. Using CIM an approximate solution was obtained in [733] by taking a three-term polynomial for concentration in terms of powers of  $(1 - x/S)$ . In the third term in the polynomial, the exponent is  $m$  whose optimal value has been determined. The second-order accurate Keller box finite difference numerical, analytical-numerical and CIM solutions have been compared. The presentation of work is appreciable.

The dynamic linear stability analysis has been studied in [735] which is different from the stability analysis of a planar phase-change interface in earlier references which were concerned with the perturbations in steady-state solidification velocity. The study in [735] is concerned with planar solidification of binary melts and discusses stability relative to the width of mushy region. The initial problem formulation of the mush in terms of temperature, concentration and solid fraction in the mush together with boundary conditions at the time-dependent solidus and liquids is somewhat similar to that discussed in [437] in Section 12.2. Perturbation solutions have been obtained for dimensionless concentration  $\bar{c}_m$ , solid fraction  $\phi_m$  and  $\psi_m$  where  $\psi_m = \delta u_S/D_L$ ,  $\delta$  is the thickness of the mush,  $u_S$  is quasistationary solidification velocity, and  $D_L$  is the mass diffusivity of the liquid.  $\psi_m$  is unknown. The perturbation parameter is  $P_1 \ll 1$ , which may be regarded as relative concentrational supercooling. The perturbation solution procedure is a standard one. The solution procedure for obtaining broad mushy region

and narrow mushy region widths continues further which is a lengthy process and cannot be presented here. As far as perturbation solution is concerned it ends here. For stability analysis refer [735]. In essence, solidification with a narrow mushy region is found to be absolutely unstable. An increase in the mushy region width leads to an instability changeover from the 'hard' type to an oscillatory 'soft' one. The steady-state solidification regime with a broad mushy region was found to be absolutely stable. Numerical work done by taking different parameter values has been presented. The reference [736] mentioned in [735] is directly related to the formulation. This study has some novelty.

### Multidimensional Perturbation Solutions

The steady-state two-dimensional shape of solidification interface subjected to a spatially periodic heat flux has been investigated in [737]. The dimensionless formulation is given below.

$$T_{xx} + T_{yy} = 0, 0 < x < A, 0 < y < h(x); T = 0 \text{ at } y = 0 \text{ and } T = 1 \text{ at } y = h(x), \quad (12.9.73)$$

$$T_x = 0 \text{ at } x = 0, \text{ and } A; (1 + h_x^2)^{-1/2} (T_y - h_x T_x) = 1 + M \cos(\pi x/A), \text{ at } y = h(x). \quad (12.9.74)$$

$A = a/\gamma$ ,  $a$  is half-wave length of heat flux variation along  $y = h(x)$ ,  $y = h(x)$  is the stationary solidification interface,  $\gamma$  is a characteristic length,  $M = \Delta q/q_m$ ,  $q$  is the convective heat flux from the liquid to the interface,  $q_m$  is the mean value of  $q$  and  $\Delta q$  is the amplitude of periodic variation in  $q$ .  $M$  is heat flux nonuniformity parameter and is also the perturbation parameter which is considered small.  $y$ -axis is taken vertically upwards. Temperature  $T(x, y)$  and interface  $h(x)$  are expanded in power series of  $M$  and up to second-order perturbation solutions have been obtained. Once  $h(x)$  is determined, the height of the interface  $h_l$  at  $x = 0$  and  $h_r$  at  $x = A$  can be obtained. The quantity  $h_d$ , the difference between these heights is given by

$$h_d = (h_r - h_l)/2M = (A/\pi) \tan h(\pi/A) + O(M^3). \quad (12.9.75)$$

**Analysis for Large  $A$ :** Large  $A$  essentially means that the characteristic length scale along the  $x$ -direction is  $O(A)$ . So to obtain perturbation solution, the perturbation parameter will be  $A^{-1}$  and the formulation should now be considered in terms of the new variable  $\bar{x} = A^{-1}x$ . The heat equation is transformed to  $A^{-2} T_{\bar{x}\bar{x}} + T_{yy} = 0$ . For large  $A$ , the contribution of the first term will not be accounted properly and the regular perturbation solution will not be uniformly valid for large value of  $A$ . Therefore singular perturbation methods have to be used in this case. The method of matched asymptotic expansions has been used in [737]. This method will be explicitly described later whilst discussing spherical problems. Very briefly, this means that inner solution near the boundaries  $x = 0$  and  $x = A$  and which takes into account the boundary conditions matches the outer solution which is the solution away from the sidewalls.

In this problem the inner solution is obtained by expanding  $T(\bar{x}, y)$  and  $h(\bar{x})$  in a power series of  $A^{-1}$  which should satisfy the boundary conditions. The inner solution which has been constructed in [737] satisfies the boundary conditions. When  $h_d$  for inner solution is calculated for small  $M$  then  $h_d = 1 - \pi^2/3A^2$  and from Eq. (12.9.75) also we get the same  $h_d$  for small  $M$ . So both the solutions match and therefore the solution obtained for large  $A$  is uniformly valid for all values of  $A$ .

In [738], the two-dimensional region considered is  $z \geq 0$ ,  $-\infty < x < \infty$ . The face  $z = 0$  is subjected to a temperature prescribed boundary condition of the form

$$T_w = T_0 + \epsilon(T_m - T_0)f(x/L), \text{ on } z = 0, \quad (12.9.76)$$

where  $\epsilon$  is a small number,  $T_w < T_m$  for all  $x$ ,  $L$  is some characteristic length. Solidification of a liquid at fusion temperature  $T_m$  has been considered in the above region. In dimensionless form, the following formulation is considered.

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2} = \frac{\partial \theta}{\partial t}; \quad \theta = \epsilon f(x) \text{ on } z = 0; \quad \theta = 1 \text{ on } z = S(x, t), \quad (12.9.77)$$

$$\frac{\partial \theta}{\partial z} - \frac{\partial \theta}{\partial x} \frac{\partial S}{\partial x} = \beta \frac{dS}{dt} \text{ on } z = S(x, t); \quad S(x, t) \text{ is the phase-change interface,} \quad (12.9.78)$$

$\beta$  is the ratio of latent heat to specific heat and  $S(0) = 0$ . Regular perturbation solution in a power series of  $\epsilon$  about  $\epsilon = 0$  has been considered. Zeroth-order solution is simple. First-order problem formulation has been obtained by following the standard formulation procedure for first-order temperature and moving boundary and then their solutions using complex Fourier transform have been obtained. Obtaining solution of transformed equations is difficult and lengthy. The interface conditions are not satisfied at  $z = S(x, t)$  but they are satisfied at  $z = S_0$ ,  $S(x, t) = S_0(t) + \epsilon S_1(x, t) + \dots$ . For scalings and solution refer [738].

The problem considered in [739] is similar to that in [738] except that in the boundary condition flux is prescribed at the wall  $z = 0$  which is a slightly varying function of  $x$  along the wall as the temperature is in [738]. The dimensionless formulation is given below.

$$\frac{\partial \theta}{\partial z} = 1 + \epsilon f(x) \text{ at } z = 0; \quad \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2} = \frac{1}{\beta} \frac{\partial \theta}{\partial \tau}; \quad \theta = 0, z = S(x, t), \quad (12.9.79)$$

$$\frac{\partial \theta}{\partial z} - \frac{\partial \theta}{\partial x} \frac{\partial S}{\partial x} = \frac{\partial S}{\partial \tau}, \quad z = S(x, t); \quad \beta \text{ is Stefan number } \gg 1. \quad (12.9.80)$$

The perturbation parameter is  $\epsilon$  which is small and regular perturbation expansions are of standard type as given below.

$$\theta(z, x, t) = \theta_0(z, t) + \epsilon \theta_1(z, x, t) + \dots; \quad S(x, t) = S_0(t) + \epsilon S_1(x, t) + \dots \quad (12.9.81)$$

Exact analytical solution of even the zeroth-order solution cannot be determined. In the zeroth-order and higher-order formulations, the obtained temperature differential equations contain time derivative of temperature multiplied by  $\beta^{-1}$ . Therefore,  $\theta_0$  and  $\tau$  are expanded in series of the following form for large  $\beta$ .

$$\theta_0 = \theta_{00}(S_0, z) + \beta^{-1} \theta_{01}(S_0, z) + \dots; \quad \tau = \tau_0(S_0) + \beta^{-1} \tau_1(S_0) + \dots, \quad (12.9.82)$$

$S_0$  is used as a time-like variable. Now  $\theta_0$ ,  $\tau$  and  $S_0$  can be obtained by substituting series in Eq. (12.9.81) in the problem formulated for  $\theta_0$  (not given here) which is obtained when different powers of  $\epsilon$  are collected in the given formulation of the problem. To obtain  $\theta_1$ , again for  $\theta_1$  and  $S_1(x, \tau)$  series expansions of the form in Eq. (12.9.82) are used. Even this solution is not straight forward and complex Fourier transform method was used to obtain it. In [738], it was possible to find analytical solutions for  $\theta_0(z, t)$  and  $S_0(t)$ . For the solutions of  $\theta_1(z, x, t)$  and  $S_1(x, t)$  complex Fourier transform was used in [738]. However, independent variables remained  $x$  and  $\tau$ . In [739] it became necessary to take  $\theta_0(z, S_0(t))$  and  $\tau(S_0(t))$  as dependent variables to obtain even a short-time solution. It was observed that by an appropriate

choice of scalings, a short-time solution of this simplified problem is the same as that of the large  $\beta$  solution in which  $\theta(z, x, t)$  and  $S(x, t)$  are expressed as in Eq. (12.9.81).

The analytical-numerical solution was also obtained by using HBIM which holds for short time as well as for large  $\beta$  and perturbation solution was compared with it. Numerical solution has been presented in [739].

The problems studied in the references [740–743] are one-phase problems concerned with the solidification of a liquid at the melting temperature contained in a metallic mold of finite thickness  $h_0$  whose height is taken vertically along the  $y$ -axis. For some convenience the author has taken the mold lying down and  $x$ -axis is taken horizontally. The mold occupies the region  $(-\infty < x < \infty, -h_0 \leq y \leq 0)$  and the solidified thickness occupies the region  $0 < y \leq S(x, t), -\infty < x < \infty$ . The lower surface of mold is taken  $y = -h_0$  and at  $y = 0$ , it is in contact with the solid thickness formed by cooling at  $y = -h_0$ . The cooling at the bottom could be through the boundary condition of temperature prescribed type or flux prescribed type. It has been observed in continuous casting that due to imperfect contact of the mold and the solid some gaps are formed in the casted material. To make improvements in the casted material it was suggested in [744] to consider the mold surfaces as sinusoidal surfaces and then study the phase-change problem. With this background it is now simple to formulate the problem and to have a clear understanding of the formulation which is being given below without scaling. Let  $T_d$  and  $T_S$  be the temperatures of mold and solid, respectively, and  $y = S(x, t)$  be the phase-change interface.

$$k_d \nabla^2 T_d = \frac{\partial T_d}{\partial t}, -h_0 < y < 0, -\infty < x < \infty; k_S \nabla^2 T_S = \frac{\partial T_S}{\partial t}, 0 < y < S, \quad (12.9.83)$$

$$T_S|_{y=S} = T_m; K_S \frac{\partial T_S}{\partial y} \Big|_{y=S} = l\rho \frac{dS}{dt}; K_d \frac{\partial T_d}{\partial y} \Big|_{y=0} = K_S \frac{\partial T_S}{\partial y} \Big|_{y=0}, \quad (12.9.84)$$

$$K_d \frac{\partial T_d}{\partial y} \Big|_{y=0} = \frac{1}{R_0} [T_S - T_d]_{y=0}; k_d \frac{\partial T_d}{\partial y} \Big|_{y=-h_0} = Q; S(x, 0) = L \in_1 \cos(mx), \quad (12.9.85)$$

where  $Q$  is known heat flux,  $y = -h_0$  and  $y = 0$  are the mean values of  $y$  at the mold surfaces. In applying boundary conditions to obtain the solution,  $y = -h_0$  and  $y = 0$  should be taken as

$$y = -(h_0 + L \in_2 \cos mx) \text{ and } y = L \in_1 \cos mx, \text{ respectively.} \quad (12.9.86)$$

$$\in_1 = a_1/L; \in_2 = a_2/L; L = \lambda/2\pi = 1/m; \lambda \text{ is wavelength,} \quad (12.9.87)$$

$a_1$  and  $a_2$  are amplitudes of the upper and lower mold surfaces,  $h_0$  is the mean thickness of the mold,  $R_0$  is thermal contact resistance in the boundary condition at  $y = 0$ .

It has been assumed that both the temperatures  $T_d$  and  $T_S$  and  $S(x, t)$  can be approximated as given below.

$$T(x, y, t) = T_0(y, t) + T_1(y, t) \cos(mx); T_1(y, t) \ll T_0(y, t), \quad (12.9.88)$$

$$S(x, t) = S_0(t) + S_1(t) \cos(mx); S_1(t) \ll S_0(t), mS_1(t) \ll 1. \quad (12.9.89)$$

$T(x, y, t)$  stands for both  $T_d$  and  $T_S$ . The amplitude of this perturbation is small as compared to its wavelength  $mS_1 \ll 1$  and so  $\partial S/\partial x \ll 1$  which implies that heat flux in the  $x$ -direction is negligible to the first order. For small perturbations,  $y$  can be expanded near the interface in a power series around  $y = S_0(t)$ ,  $S_0(t)$  is to be determined. To obtain  $T_0, T_1, S_0$  and  $S_1, T(x, t)$  and  $S(x, t)$  given in Eqs (12.9.88), (12.9.89) are substituted in Eqs (12.9.83)–(12.9.86) and on



separating periodic and nonperiodic terms, a system of equations to determine  $T_0, T_1, S_0$  and  $S_1$  is obtained in which higher-order terms of  $T_2$  and  $S_2$ , etc., have been neglected.

The dimensionless formulation has been presented in [740] and for this formulation, assumptions made and system of equations to determine zeroth- and first-order solutions, refer [740]. Analytical solution has been obtained only for a particular case in which thermal diffusivities of both solid and mold are infinite and zeroth- and first-order perturbation solutions have been obtained in this particular case. Numerical solution of dimensionless truncated equations as in Eqs (12.9.88), (12.9.89) using finite-difference scheme was obtained by *Lagrangian method* in which grid size and time steps are varying. Numerical work has been presented and numerical solution has been compared with analytical solution for the particular case mentioned above.

The problem description and mathematical formulation in [741] are similar to that in [740] except one change which pertains to the boundary condition at  $y = -h_0$ . In [741] flux is prescribed as follows.

$$K_d \frac{\partial T_d}{\partial y}(x, -h_0, t) = Q_0(t) + Q_1(t) \cos mx, \quad Q_1 \ll Q_0. \quad (12.9.90)$$

Some of the dimensionless parameters have been defined differently. The two conditions in Eqs (12.9.86) which define the sinusoidal shape of the surfaces  $y = -h_0$  and  $y = 0$  have not been prescribed. The suggested procedure to obtain perturbed solution and numerical solution is the same as in [740]. Analytical solutions for zeroth- and first-order temperatures and  $S(t)$  have been obtained only in the particular case when thermal diffusivities of solid thickness and mold are infinite. Numerical work has been presented.

In the one-phase problem considered in [742], mold thickness has not been considered and at  $y = 0$ , flux of the form given in Eq. (12.9.90) has been prescribed. Isotherm condition and Stefan condition are prescribed at  $x = S(t)$ . The perturbation solution procedure is the same as in [740, 741]. Attempts have been made to obtain zeroth- and first-order solutions for temperature in the solid and  $S(t)$ . However, in these solutions all the boundary conditions are not satisfied. For example, in obtaining solution of  $T_0$ , a pure heat conduction problem without phase change has been considered. Similarly in determining other quantities like  $S_0, S_1$  and  $T_1$  not all the conditions and differential equations are satisfied. In the limiting case of zero Stefan number,  $T_0, T_1$  and  $S_0, S_1$  have been obtained. Numerical solution as in [740] of truncated equations has been obtained and compared with the approximate perturbed solution in which all the conditions are not satisfied.

The geometry and formulation of the problem in [743] is similar to that in [740] except that at  $y = -h_0$ , temperature  $T^*$  is prescribed. In Eq. (12.9.88),  $T_1(y, t) \ll T_0(y, t) - T^*$  for both solid and mold for the present problem. The zeroth- and first-order solutions have been obtained when thermal diffusivities are infinitely large. Numerical solution of the truncated equations has been presented and compared with the analytical solution of limiting case.

In [738] a one-phase solidification problem of slightly varying temperature along the wall  $z = 0$  was considered in the two-dimensional region  $z \geq 0, -\infty < x < \infty$ . In [745] two one-phase inward solidification problems in a cylindrical geometry  $(r, \theta), 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$ , have been considered separately for temperature and flux prescribed boundary conditions.

$$T^* = T_0(1 + \epsilon^* f(\theta)), r = a; \quad K \frac{\partial T^*}{\partial r} = -Q(1 + \epsilon f(\theta)), r = a, \quad (12.9.91)$$

$f(\theta)$  is assumed to be periodic or it is such that it can be expanded in a Fourier series.

We skip the dimensionalization of variables in the formulation and discuss only the main points in obtaining perturbed solution. The boundary conditions after scalings at the interface  $R = 1 - S(\tau)$ ,  $S(0) = 1$ , are satisfied at  $R = 1 - S_0(\tau)$  where

$$S(\tau) = S_0 + \epsilon S_1(\theta, S_0) + \dots; T = T_0(R, S_0) + \epsilon T_1(R, \theta, S_0) + \dots, \quad (12.9.92)$$

$$\tau = \tau_0 + \epsilon \tau_1(S_0) + \dots, \quad (12.9.93)$$

$\beta$  is large Stefan number.  $S_0(\tau)$  is time-like variable. The solutions of  $T_0$  and  $S_0$  are reported in [746] and these solutions obtained in [746] will be discussed later in connection with the spherical problems. The leading terms in both cylindrical and spherical problems are generally the same in solutions. To obtain solutions of  $T_1$  and  $S_1$ , series expansions of the following form have been used.

$$T_1(R, \theta, S_0) = \sum_{n=1}^{\infty} (d_{1n}(R, S_0) \cos n\theta + \bar{d}_{1n}(R, S_0) \sin n\theta). \quad (12.9.94)$$

Similar expansions for  $S_1(\theta, \tau)$  and  $f(\theta)$  are assumed.  $f(\theta)$  is either the prescribed temperature or flux in nondimensional form. The determination of unknown coefficients in  $T_1$  and  $S_1$  is a very lengthy process in which some approximations are also done and the unknown coefficients  $d_{1n}$  and  $\bar{d}_{1n}$  are further expanded in a series of powers of  $\beta^{-1}$  as in [739] where  $\beta$  is the ratio of latent heat and sensible heat. Refer [745] for the calculations.

In [747], in the one-phase axially symmetric cylindrical solidification problem, the boundary values of either prescribed temperature or prescribed heat flux vary slightly with respect to coordinate  $z$  and periodically along  $\theta$  in the cylinder in  $(r, \theta, z)$  coordinates. The prescribed boundary conditions are of the following form.

$$T^* = T_0(1 + \epsilon^* f(z^*/a)) \text{ at } r = a, \text{ or } K \frac{\partial T^*}{\partial r} = -Q(1 + \epsilon f(z^*/a)) \text{ at } r = a, \quad (12.9.95)$$

$\epsilon$  and  $\epsilon^*$  are small numbers. The method of solution is similar to the one discussed in [745] except that in Eq. (12.9.94) as  $f(z)$  is periodic in  $z$ ,  $\cos nz$  and  $\sin nz$  will be there in place of  $\cos n\theta$  and  $\sin n\theta$ , respectively. If  $f(z)$  is not periodic then Fourier transform should be taken as in [748]. Numerical results for a particular case are reported.

In [748] the surface  $r = a$  of the cylinder in coordinate system  $(r, \theta, z)$  is slightly varying along  $z$ -direction and the perturbed surface of the cylinder is given by

$$r = a(1 - \epsilon f(z^*/a)), \quad \epsilon \text{ is small and } f(z) \text{ is a periodic function.} \quad (12.9.96)$$

At the outer cylindrical boundary, the following boundary conditions in a one-phase solidification are prescribed.

$$\text{either } T^* = T_0 \text{ or } K \vec{n} \cdot \nabla T^* = -Q. \quad (12.9.97)$$

An alternate approach to consider the boundary conditions has been used in which the boundary conditions are approximated as

$$T^* - \epsilon af(z^*/a) \frac{\partial T^*}{\partial r} = T_0 \text{ at } r = a, \text{ or } K \left( \frac{\partial T^*}{\partial r} - \epsilon af(z^*/a) \frac{\partial^2 T^*}{\partial r^2} \right) = -Q \text{ at } r = a. \quad (12.9.98)$$

The formulation is independent of coordinate  $\theta$ . The solution procedure is similar to that used in [737]. The perturbation parameter is  $\epsilon$ . The calculations are too lengthy to be presented here.

The inward and outward perturbation solutions for small Stefan number  $\epsilon$  (ratio of latent heat to sensible heat) for one-phase radially symmetric cylindrical and spherical solidification problems have been obtained in [749] by using regular perturbation expansions. The problem formulation consists of linear parabolic equations with convective boundary conditions at  $r = a$ . The interfaces  $r = S(t)$  in spherical and cylindrical problems are fixed by using transformations given below.

$$\delta = (r - 1)/(S(t) - 1) \text{ for sphere and } \delta = \ln r / \ln(S(t)) \text{ for cylinder.} \quad (12.9.99)$$

For outward spherical and cylindrical problems,  $1 \leq r \leq S(t)$ ,  $S(0) = 1$ . For inward spherical problem  $0 < S(t) \leq r \leq 1$ ,  $S(0) = 1$ . At  $S(t) = 0$  or at  $r = 0$  there is singularity in both spherical and cylindrical problems and therefore a small region around  $r = 0$  is not included. We skip the dimensionalization process and assume that all the quantities are dimensionless.

A new time variable  $Y(\tau)$  in spherical outward and inward solidification problems is defined in [749] as follows.

$$Y(\tau) = S(t) - 1 \text{ for outward problem and } Y(\tau) = 1 - S(t) \text{ for inward problem.} \quad (12.9.100)$$

$Y(\tau)$  is now time-like variable, and  $\delta$  and  $Y(\tau)$  are now independent variables.  $\tau = \tau(Y(t)) \geq 0$ ,  $\tau(Y) = 0$  at  $Y = 0$ . In the cylindrical problem  $Y(\tau) = \ln(S(t))$ ,  $\tau = \tau(Y(\tau))$  and  $\tau(Y) = 0$  at  $Y = 0$ . To obtain the solution, the formulation is changed using the transformations given in Eqs (12.9.99), (12.9.100). The boundary conditions at  $r = 1$  are of convective type for both outward and inward spherical and cylindrical problems in which sign of prescribed flux changes from positive to negative for outward and from positive to negative for inward problem. If  $\phi(\delta, Y)$  is dimensionless temperature in outward spherical and cylindrical problems and  $\tau = \tau(Y)$  is dimensionless time then the following regular perturbation expansions can be used.

$$\phi(\delta, Y) = \phi_0(\delta, Y) + \epsilon \phi_1(\delta, Y) + \epsilon^2 \phi_2 + \dots; \quad \tau = \tau_0(Y) + \epsilon \tau_1(Y) + \epsilon^2 \tau_2(Y) + \dots \quad (12.9.101)$$

Perturbation solutions up to second order have been obtained for temperatures and  $\tau(Y)$  which are not reported here as they are lengthy. The singularity in the analytical solution is present at  $r = 0$  in the inward solidification problems. Numerical work has been presented.

The problem considered in [750] is similar to that in [745] except that the geometry is spherical and not cylindrical. This changes the solution procedure as expansions of the form in Eq. (12.9.94) are not suitable for spherical problems in  $(r, \theta)$  coordinates (independent of  $\phi$  in  $(r, \theta, \phi)$  coordinates). Instead of using expansions of the form given in Eq. (12.9.94), the following type of expansions are to be used for this problem.

$$T_1(R, \theta, S_0) = \sum_{n=1}^{\infty} \phi_{1n}(R, S_0) P_n(\mu) = \sum_{n=1}^{\infty} T_{1n}; \quad S_1 = \sum_{n=1}^{\infty} e_{1n}(S_0) P_n(\mu), \quad (12.9.102)$$

$$f(\theta) = \sum_{n=1}^{\infty} F_n P_n(\mu); \quad \mu = \cos \theta; \quad P_n \text{ are Legendre polynomials,} \quad (12.9.103)$$

the notations  $T_0$ ,  $T_1$ ,  $S_1$ ,  $S_0$  and  $f(\theta)$  are the same as defined in [745]. Approximate zeroth-order solutions have been obtained in [750] for spherical problem.  $S_0$  is now the timelike variable and  $\tau = \tau(S_0)$ .  $\phi_{1n}$  and  $e_{1n}$  are further expanded in the following series as no analytical solution could be obtained even for  $\phi_{11}$  and  $e_{11}$ .  $\beta$  is large Stefan number.

$$\phi_{1n} = \phi_{10n}(R, S_0) + \frac{1}{\beta} \phi_{11n}(R, S_0) + \cdots; \quad e_{1n} = e_{10n}(S_0) + \frac{1}{\beta} e_{11n}(S_0) + \cdots. \quad (12.9.104)$$

All these expansions are substituted in the boundary value formulations obtained for  $T_{1n}$ ,  $n = 1, 2, \dots$ , and then unknown quantities are determined step by step. For calculations refer [750].

A short-time solution in the three-dimensional region,  $z \geq 0$ ,  $|x| < \infty$ ,  $|y| < \infty$ , has been presented in [751]. The problem is one-phase and solidification can take place either due to prescribed temperature or due to prescribed flux condition of convective type at  $z = 0$  as given below. For formulation which is of usual type and scaling refer [751].

$$T = T_0 + (T_m - T_0)g(x, y), \quad \text{on } z = 0, \quad g(x, y) > -1, \quad T_0 < T_m, \quad (12.9.105)$$

or

$$K \frac{\partial T}{\partial x} = h(x, y)(T - T_0), \quad \text{on } z = 0, \quad h(x, y) > 0. \quad (12.9.106)$$

If  $z = f(x, y, t)$  is the phase-change interface then define  $\eta = z/f(x, y, t)$ . The problem is now considered in the coordinates  $(x, y, \eta, t)$  and equations are transformed accordingly. For small time, the following expansions have been used.

$$T(x, y, \eta, t) = T_0(x, y, \eta) + tT_1 + t^2T_2; \quad f(x, y, t) = 2t^{1/2}f_0 + 2t^{3/2}f_1 + \cdots. \quad (12.9.107)$$

When convective-type cooling (Newton cooling) is prescribed then temperature is expanded as

$$T(x, y, z, t) = 1 + \sum_{n=1}^{\infty} G_n(x, y, t)[z - f(x, y, t)]^n, \quad T_m = 1 \text{ after scaling.} \quad (12.9.108)$$

The series in Eq. (12.9.108) is a polynomial in  $z$ , therefore the solution will be valid only for large Stefan number  $\beta$  which is the ratio of latent heat and sensible heat.

In the regular perturbation solutions discussed above our interest is not in merely presenting some problems and procedures but also to discuss the methods to overcome the difficulties in some problem which come in the way to obtain some analytical solution. The most difficult and interesting part is applications of mathematical methods, sometime fairly sophisticated, in obtaining the perturbation analytical or semianalytical solutions of different orders (different terms in series). This could not be taken up due to space limitations.

### Singular Perturbation Method of Strained Coordinates

It has been observed earlier in several problems that the regular perturbation method breaks down as the solution is not uniformly valid throughout the region under consideration. This happens due to singularity which may arise due to various reasons mentioned earlier. Although the solutions which are not uniformly valid are also obtained with lot of difficulty, their utility is limited. For example, in cylindrical or spherical regions when regular perturbation expansion is used, except the zeroth-order term in temperature and zeroth- and first-order terms in  $\tau$ , all higher-order approximations are singular as  $S(t)$  approaches the centre  $r = 0$ . To obtain uniformly valid solutions which are obtained by using singular perturbation methods, perturbation series expansions are still used as used earlier but the solution procedure changes and it requires lot of different other procedures. The presentation of solution procedure in the regular perturbation method could be explained without giving much of the details of the solutions of zeroth-, first- or second-order solutions but this is not possible in singular perturbation solutions as these solutions are required to explain further procedures. To explain the solution procedure in singular perturbation methods several of which exist in the literature, solutions have to be reproduced which we could do here for illustration only for one solution due to space limitations.

In Stefan problems concerning perturbation solutions, the application of method of matched asymptotic expansion has been used more often and the next method used often is the method of strained coordinates. We plan to review the solution procedure for only these two methods. However, several singular perturbation methods exist in the literature as these solutions are being developed since a long time. We cite two review articles [752, 753] which contain lot of information and provide enough cross-references. The information about the applications of singular methods related to Stefan problems is very little in [752] and nil in [753].

In this method two new variables  $(\phi, \psi)$  are defined. One independent variable and one dependent variable existing in the problem are expanded in terms of Taylor series expansions of perturbation parameter  $\epsilon$  in which the coefficients of different powers of  $\epsilon$  are functions  $\phi$  and  $\psi$ . These coefficients are called 'straining functions'. The perturbation series expansions are substituted into the given formulation and sequence of formulations (boundary value problems) for different order approximate solutions are generated as in regular perturbations. To determine the straining functions a rule is applied which envisages that the higher-order approximations are no more singular than the first order. The method will now be explained explicitly with the help of the solution presented in [754] for a one-phase inward radially symmetric spherical solidification in  $0 \leq r \leq 1$ . The formulation is given below in dimensionless form in which transformation  $(t, r) \rightarrow (S(t), r)$  has been used.

$$\left. \begin{aligned} \frac{1}{r} \frac{\partial^2}{\partial r^2}(ru) &= \epsilon \frac{\partial u}{\partial S} \frac{\partial u}{\partial r} \Big|_{r=S(\tau)} ; u(S(\tau), r=1) = 0; u(S(\tau), r=S(\tau)) = 1, \\ \frac{dS}{d\tau} &= \frac{\partial u}{\partial r} \Big|_{r=S(\tau)} ; u = (T - T_0)/(T_m - T_0), r = R/R_0, \epsilon = C(T_m - T_0)/l, \\ \tau &= K(T_m - T_0)t/(\rho l R_0^2); r = S(\tau) \text{ is the moving boundary, } S(0) = 1. \end{aligned} \right\} \quad (12.9.109)$$

$T$  is the temperature and  $T_0$  is the prescribed temperature at  $R = R_0$ ,  $R_0$  is the radius of the sphere.  $\epsilon = C(T_m - T_0)/l$ ,  $l$  is latent heat. In temperature  $u = u(r, S(t))$  instead of writing  $t$ ,  $S(t)$  is written in the formulation. This point will be clarified later.

Two new variables  $\phi$  and  $\psi$  are introduced and  $u, r$  and  $S(\tau)$  are expanded as given below. The perturbation parameter is  $\epsilon$  which is small.

$$u = u_0(\phi, \psi) + \epsilon u_1(\phi, \psi) + \epsilon^2 u_2(\phi, \psi) + \dots \quad (12.9.110)$$

$$r = \phi + \epsilon \sigma_1(\phi, \psi) + \epsilon^2 \sigma_2(\phi, \psi) + \dots; S(\tau) = \psi + \epsilon \sigma_1(\psi, \psi) + \epsilon^2 \sigma_2(\psi, \psi) + \dots \quad (12.9.111)$$

$\sigma_1(\phi, \psi)$  and  $\sigma_2(\phi, \psi)$  are called straining functions.

$$\phi = \phi(S(\tau), r), \psi = \psi(S(\tau), r = S(\tau)), r = r(\phi, \psi), S(\tau) = S(\psi). \quad (12.9.112)$$

In order to satisfy boundary conditions at  $r = 1$  and at  $S(0) = 1$  it is assumed that

$$\lim_{\phi \rightarrow 1} \sigma_i(\phi, \psi) = \lim_{\psi \rightarrow 1} \sigma_i(\psi, \psi) = 0; \phi \rightarrow 1 \text{ as } r \rightarrow 1, S(t) \rightarrow 1 \text{ as } \psi \rightarrow 1. \quad (12.9.113)$$

The variables are now changed from  $(r, S(t)) \rightarrow (\phi, \psi)$  and for doing this we proceed as given below.

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial \phi} + \frac{\partial u}{\partial S} \cdot \frac{\partial S}{\partial \phi} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial \phi}; \frac{\partial u}{\partial r} = \frac{\partial u}{\partial \phi} \cdot \left( \frac{\partial r}{\partial \phi} \right)^{-1}. \quad (12.9.114)$$

$$\frac{\partial u}{\partial \psi} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial \psi} + \frac{\partial u}{\partial S} \frac{\partial S}{\partial \psi} \text{ or } \frac{\partial u}{\partial S} = \left( \frac{\partial u}{\partial \psi} - \frac{\partial u}{\partial r} \frac{\partial r}{\partial \psi} \right) \left( \frac{\partial S}{\partial \psi} \right)^{-1}. \quad (12.9.115)$$

In a similar way using the chain rule of differentiation for other derivatives, the formulation given in Eq. (12.9.109) can be converted in  $(\phi, \psi)$  coordinates and their derivatives. Next the series expansions in Eqs (12.9.110), (12.9.111) are substituted in the new formulation in terms of  $(\phi, \psi)$  coordinates and their derivatives and terms of different powers of  $\epsilon$  are collected. We obtain a sequence of boundary value problems to obtain  $u_0, u_1, u_2$  and  $\sigma_1, \sigma_2$ , etc. For the zeroth-order problem

$$\frac{\partial^2(\phi u_0)}{\partial \phi^2} = 0; u_0(\psi, \phi = 1) = 0; u_0(\psi, \phi = \psi) = 1 \text{ as } r = S(t) \Rightarrow \phi = \psi, \quad (12.9.116)$$

and the solution is  $u_0 = (1/\phi - 1)/(1/\psi - 1)$ .

The first-order problem is

$$\frac{d^2}{d\phi^2} \left( \phi u_1 + \frac{\psi}{1 - \psi} \frac{\sigma_1}{\phi} \right) = -\frac{(1 - \phi)}{\psi(1 - \psi)^3}; u_1(\psi, \phi = 1) = 0; u_1(\psi, \phi = \psi) = 0. \quad (12.9.117)$$

If  $u_1$  is not supposed to be more singular than  $u_0$  then we should take  $u_1 \equiv 0$  which implies that a second-order differential equation is to be solved for  $\sigma_1(\phi, \psi)$ . To obtain the solution of a second-order differential equation, we have to impose an extra derivative condition. For example if  $\frac{d}{d\phi}(\sigma_1/\phi) = 0$  as  $\phi \rightarrow 1$  is imposed then

$$\sigma_1 = -\phi(1 - \phi)^3/(6\psi^2(1 - \psi)^2). \quad (12.9.118)$$

Taking  $\sigma_1$  as given in Eq. (12.9.118), we get  $u_1 = 0$ .  $\sigma_1$  in Eq. (12.9.118) is the simplest solution satisfying  $\sigma_1 = 0$  for  $\phi \rightarrow 1$  and satisfying  $\frac{d}{d\phi}(\sigma_1/\phi) = 0$ . In the second-order problem  $u_2$  and  $\sigma_2$  will appear. To make  $u_2 = 0$ , a second-order differential equation for  $\sigma_2$  is to be solved whose solution will give  $\sigma_2(\phi, \psi)$ .  $\sigma_2(\phi, \psi)$  is given in [754]. One extra condition to determine  $\sigma_2$  is to be imposed which may give a simple solution for  $\sigma_2$  satisfying  $\sigma_2 = 0$  when  $\phi \rightarrow 1$ . Having determined  $\sigma_1$  and  $\sigma_2$ ,  $u(\phi, \psi)$ ,  $r(\phi, \psi)$  and  $S(\psi)$  are known but time  $\tau$  is not known. To determine  $\tau$ , we use the interface condition in the following form.

$$\frac{d\tau}{d\psi} \frac{\partial u_0}{\partial \phi} \Big|_{\phi=\psi} = 1 + \epsilon \left[ \frac{d\sigma_1}{d\psi}(\psi, \psi) + \frac{\partial \sigma_1}{\partial \phi} \Big|_{\phi=\psi} \right] + \text{higher-order terms.} \quad (12.9.119)$$

The r.h.s. in Eq. (12.9.119) is  $(\partial r / \partial \phi)|_{\phi=\psi}$  which has been written as a perturbation series. From Eq. (12.9.119),  $\tau$  in terms of  $\psi$  can be obtained on integration. The expression of  $\tau$  in terms of  $\psi$  is given in [754].

It is clear that  $\phi$  and  $\psi$  are not determined explicitly. To obtain  $\phi$  and  $\psi$ , we have to solve numerically implicit equations for  $\phi$  and  $\psi$ . First, the values of  $\epsilon$  and  $S(t)$  are fixed. For a given value of  $S$ , using Eq. (12.9.111)  $\psi$  is determined numerically. Note that  $\phi$  is not determined and we have to choose values of  $\phi$  between 0 and 1. With the values of  $\epsilon, S, r, \phi$  and  $\psi$ ,  $u(\phi, \psi; \epsilon)$  is obtained as  $\sigma_1$  and  $\sigma_2$  are now known. It is generally not possible to obtain exact analytical solutions for  $\sigma_1$  and  $\sigma_2$  and in such a situation all the quantities are to be obtained numerically.

If Shanks transformation [396] is used as discussed in Eq. (12.9.56) then accuracy of the solution can be improved. For example to improve the accuracy of  $S(\tau)$ , the following improved  $S^*(\tau)$  can be used.

$$S^*(\tau) = \psi \sigma_1(\psi, \psi) - [\psi \sigma_2(\psi, \psi) - \sigma_1^2(\psi, \psi)] / (\sigma_1(\psi, \psi) - \sigma_2(\psi, \psi)). \quad (12.9.120)$$

Using Shanks transformation,  $\tau$  and  $r$  can also be improved. Numerical work for the analytical solution is presented in [754].

In [755] a perturbation solution using strained coordinates in which up to second-order terms have been obtained for a one-phase inward radially symmetric cylindrical problem is discussed. The formulation is similar to that in [754] except that the heat equation in cylindrical region is different. Solution procedure is as described above but solution is different. Perturbation parameter is the small Stefan number. In this problem,  $u_0 = \ln \phi / \ln \psi$ .  $\sigma_1, \sigma_2$  and  $u_0, u_1, u_2$  have been obtained analytically. Shanks transformation [396] has been used to improve the solution.

The inward one-phase radially symmetric solidification problems of both sphere and cylinder have been considered in [756]. The boundary conditions prescribed at the boundary  $r = R_0, 0 \leq r \leq R_0$ , in both the problems are of radiative-convective type.  $r$  is the radial coordinate. Method of strained coordinates has been used for perturbation solution in which the perturbation parameter  $\epsilon$  is Stefan number which is considered small. If the prescribed boundary conditions are of convective type, Biot's number also appears in the solution. Method of solution is the same as in [754] but in this problem even the zeroth-order solution is to be obtained numerically. Only  $\sigma_1(\phi, \psi)$  has been reported which has to be obtained numerically. The perturbation solution has been compared with the numerical solution obtained by using enthalpy method which we have described earlier. The agreement is good for  $\epsilon = 0.5$  and is not very bad even for  $\epsilon = 2.0$ . For other parameter values, refer [756].

### Euler's Transformation

Regular perturbation expansion for inward solidification of a one-phase radially symmetric spherical problem with temperature prescribed boundary condition was studied in [757]. The formulation is similar to the one considered in [754]. The phase-change boundary is defined by  $r = S(t)$ ,  $S(0) = 1$  and  $S(t)$  is considered as a time-like variable.  $T(S(t), r; \epsilon)$  and  $g(S(t); \epsilon) = \frac{\partial T}{\partial r} \big|_{r=S(t)}$  are expanded as given below.

$$T(S, r; \epsilon) = T_0(S, r) + \epsilon T_1(S, r) + \epsilon T_2(S, r); \quad g(S, \epsilon) = g_0(S) + \epsilon_1 g_1(S) + \dots \quad (12.9.121)$$

All the variables are in nondimensional form, refer [757]. When unknown coefficients of different powers of  $\epsilon$  in Eq. (12.9.121) are determined then  $T(S, r; \epsilon)$  and  $g(S; \epsilon)$  can be expressed in the form

$$T/T_0 = 1 + \epsilon a_1/S + \epsilon^2 a_2/S^2 + \dots; \quad g/g_0 = 1 + \epsilon b_1/S + \epsilon^2 b_2/S^2 + \dots \quad (12.9.122)$$

The two series in Eq. (12.9.122) can be expressed in the form

$$\beta = 1 + d_1 \eta + d_2 \eta^2 + \dots; \quad \eta = 1/S. \quad (12.9.123)$$

$a_1, a_2, \dots$  and  $b_1, b_2, \dots$  and also  $d_1, d_2, \dots$  are known functions of  $r, \epsilon$  and  $S$ . As  $r \rightarrow 0$  or  $S(t) \rightarrow 0$ ,  $T$  and  $g$  in Eq. (12.9.122) tend to infinity which implies that these series expansions can be used when  $S$  is large. When  $S$  is small,  $\eta$  is large and for large  $\eta$  the series in Eq. (12.9.123) may not converge if  $\epsilon$  is also not very small.

In Euler's transformation, a new variable  $\xi$  is defined as

$$\xi = \eta/(p(\epsilon) + \eta), \quad p(\epsilon) \text{ is to be obtained; } \xi \rightarrow 1 \text{ as } \eta \rightarrow \infty. \quad (12.9.124)$$

$T$  and  $g$  in Eq. (12.9.122) can be obtained in terms of  $p$  and  $\xi$ . For example,

$$T/T_0 = 1 + p a_1 \epsilon (1 + \xi) \xi + p \epsilon (a_1 + p^2 a_3 \epsilon) \xi^3 + p \epsilon (a_1 + 3p^2 a_3 \epsilon) \xi^4 + \dots \quad (12.9.125)$$

To determine  $p(\epsilon)$  an overall energy balance for the complete freezing of the sphere is considered.

$$\int_0^1 \left[ \left( \frac{\partial T}{\partial r} \bigg|_{r=1} \right) / g \right] dS + \epsilon \int_0^1 r^2 T(S(t) = 0, r) dr = (1 + \epsilon)/3. \quad (12.9.126)$$

Depending on how many integral powers of  $\xi$  in  $T/T_0$  and  $g/g_0$  are considered,  $p(\epsilon)$  can be determined. Different values of  $p(\epsilon)$  will be obtained if different powers of  $\xi$  are considered in  $T$  and  $g$ . For further details refer [757].

### Singular Perturbation Method of Matched Asymptotic Expansions

For a clear understanding of this method which has been used often, it is necessary to discuss problem formulation, scalings and the solution as done earlier to explain the method of strained



coordinates. A two-phase one-dimensional freezing problem in a slab,  $0 \leq x \leq d$ , was considered in [758].

$$k_S \frac{\partial^2 T_S}{\partial x^2} = \frac{\partial T_S}{\partial t}, 0 < x < S(t), t > 0; k_L \frac{\partial^2 T_L}{\partial x^2} = \frac{\partial T_L}{\partial t}, S(t) < x < d, \quad (12.9.127)$$

$$T_S(0, t) = T_a; \frac{\partial T_L}{\partial x}(d, t) = 0; T_S = T_L = T_m \text{ on } x = S(t), S(0) = 0, \quad (12.9.128)$$

$$T_L(x, 0) = T_i < T_m; K_S \left. \frac{\partial T_S}{\partial x} \right|_{x=\bar{S}} - K_L \left. \frac{\partial T_L}{\partial x} \right|_{x=L} = \rho l \frac{dS}{dt}. \quad (12.9.129)$$

The subscripts and other notations are as explained earlier in this book. The following dimensionless variables are introduced.

$$\theta = k_S(T_S - T_m)/k_L(T_i - T_m), \phi = (T_L - T_m)/(T_i - T_m), \xi = x/S(t), \quad (12.9.130)$$

$$\sigma = S/d, \quad \eta = (x - S(t))/(d - S(t)), \tau = \epsilon k_L t/d^2, \epsilon = C(T_i - T_m)/l, T_i > T_m. \quad (12.9.131)$$

The transformed formulation is

$$\frac{\partial^2 \theta}{\partial \xi^2} = \epsilon (k_L/k_S) \left( \sigma^2 \frac{\partial \theta}{\partial \tau} - \xi \sigma \frac{d\sigma}{d\tau} \frac{\partial \theta}{\partial \xi} \right); \theta(0, \tau) = \frac{K_S(T_a - T_m)}{K_L(T_i - T_m)} = \theta_a. \quad (12.9.132)$$

$$\frac{\partial^2 \phi}{\partial \eta^2} = \epsilon \left[ (1 - \sigma)^2 \frac{\partial \phi}{\partial \tau} - (1 - \sigma)(1 - \eta) \frac{d\sigma}{d\tau} \frac{\partial \phi}{\partial \eta} \right]; \frac{\partial \phi}{\partial \eta}(1, \tau) = 0. \quad (12.9.133)$$

$$(1 - \sigma) \left. \frac{\partial \theta(\xi, \tau)}{\partial \xi} \right|_{\xi=1} - \sigma \left. \frac{\partial \phi(\eta, \tau)}{\partial \eta} \right|_{\eta=0} = \sigma(1 - \sigma) \frac{d\sigma}{d\tau}; \quad (12.9.134)$$

$$\theta(\xi, \tau)|_{\xi=1} = \phi(\eta, \tau)|_{\eta=0} = 0. \quad (12.9.134)$$

$$\phi(\eta, 0) = 1, \sigma(0) = 0, \text{ are the initial conditions.} \quad (12.9.135)$$

When  $\epsilon \rightarrow 0$ ,  $\tau$  also goes to zero. It is clear from Eqs (12.9.132), (12.9.133) that time derivatives of  $\theta$  and  $\phi$  will not take part in any solution when  $\tau \rightarrow 0$  or in other words the solution is not uniformly valid when  $\epsilon$  is small or  $\tau$  is small. To construct solution for large time which is called outer solution which is away from the singularity, the following expansions are used.

$$\begin{aligned} \theta(\xi, \tau, \epsilon) &= \sum_{n=0}^m \epsilon^{n/2} \theta_n(\xi, \tau); \phi(\eta, \tau, \epsilon) = \sum_{n=0}^m \epsilon^{n/2} \phi_n(\eta, \tau); \\ \sigma(\tau; \epsilon) &= \sum_{n=0}^m \epsilon^{n/2} \sigma_n(\tau). \end{aligned} \quad (12.9.136)$$

We skip the formulations of boundary value problems for  $\theta_0, \theta_1, \phi_0, \phi_1, \sigma_0, \sigma_1$  and present here the solution directly in which the initial conditions are not used as this solution is valid when  $\tau$  is large.

$$\theta_0 = \theta_a(1 - \xi), \theta_1 = 0; \phi_0 = 0, \phi_1 = 0; \sigma_0 = (A - 2\theta_a\tau)^{1/2}, \sigma_1 = 0. \quad (12.9.137)$$

A in  $\sigma_0$  is some unknown constant which will be determined on matching the inner solution which is valid for small time with the outer solution which is given in Eq. (12.9.137). The outer solution can be expressed as

$$\theta(\xi, \tau) = \theta_a(1 - \xi) + O(\epsilon); \phi(\eta, \tau) = O(\epsilon); \sigma(\tau) = (A - 2\theta_a\tau)^{1/2} + O(\epsilon). \quad (12.9.138)$$

In the outer solution, the initial conditions are not satisfied and this long-time solution in Eq. (12.9.138) can be regarded as a quasisteady-state solution. To match the short-time solution with the long-time solution so that the solution is uniformly valid for all time, we want to see how the short-time solution looks like on a large-time scale. A new time variable  $\bar{\tau}$  is defined as  $\bar{\tau} = \tau/\epsilon$ . When  $\tau$  is small and of order  $\epsilon$ ,  $\bar{\tau}$  is of the order of unity. It is this short-time solution which develops into the quasistate solution. Eqs (12.9.132)–(12.9.135) are now rewritten in terms of  $\bar{\theta}(\xi, \bar{\tau}; \epsilon)$ ,  $\bar{\phi}(\eta, \bar{\tau}; \epsilon)$  and  $\bar{\sigma}(\bar{\tau}; \epsilon)$ , and expansions of the form in Eq. (12.9.136) are used for them. Since the long-time solution was obtained up to  $O(\epsilon)$  terms, the inner solution also is to be obtained up to  $O(\epsilon)$  terms as in Eq. (12.9.138). The solutions in this case are not easy due to the presence of the transient terms like  $\partial\bar{\phi}_0/\partial\bar{\tau}$  and  $\partial\bar{\phi}_1/\partial\bar{\tau}$ . In inner solutions higher-order terms in  $\bar{\theta}$ ,  $\bar{\phi}$  and  $\bar{\sigma}$  may be required as their expansions of the form in Eq. (12.9.136) are in terms of  $\bar{\tau} = t/\epsilon$  and matching of the inner and outer solutions is done in terms of powers of  $\epsilon$ . We skip the details of solution and present the  $O(\epsilon)$  solution.

$$\bar{\theta}_0 = \theta_a(1 - \xi), \bar{\theta}_1 = 0; \bar{\theta}(\xi, \bar{\tau}; t) = \theta_a(1 - \xi). \quad (12.9.139)$$

The expressions of  $\bar{\sigma}_2$  and  $\bar{\phi}_0$  are in the form of infinite series and are given below.  $\bar{\phi}_1$  is too complicated and not given here.

$$\begin{aligned} \bar{\sigma}_0(\bar{\tau}) &= 0; \bar{\sigma}_1(\bar{\tau}, \epsilon) = (-2\theta_a\bar{\tau})^{1/2}; \\ \bar{\sigma}_2(\bar{\tau}; \epsilon) &= \sum_{m=0}^{\infty} \left( \frac{2}{E^2} \exp(-E^2\bar{\tau}) - \frac{\sqrt{\pi} \operatorname{erf}(E\bar{\tau}^{1/2})}{E^3\bar{\tau}^{1/2}} \right), \\ E &= (m + 1/2)\pi, \bar{\phi}_0 = \sum_{m=0}^{\infty} \left[ \frac{2}{E} \sin(E\eta) \right] \exp(-E^2\bar{\tau}); \bar{\sigma}_1(\bar{\tau}) = (2\theta_a\bar{\tau})^{1/2}. \end{aligned} \quad (12.9.140)$$

For  $\bar{\phi}_1$  refer [758] as it is too complicated and not needed in the solution of  $\bar{\sigma}_2$ . On matching inner and outer solutions of the same order of  $\epsilon^{1/2}$ , we get  $A = 0$ . If  $A = 0$  is taken in the outer solution in which terms of  $\epsilon$  and higher powers of  $\epsilon^{1/2}$  are neglected then it is a uniformly valid solution till the whole liquid gets solidified.

The solution presented above is a simple solution and explains the method of matched asymptotic only to some extent. This is because to match the inner and outer solutions; firstly, analytical solutions are required and then their expansions in powers of  $\epsilon^{1/2}$  are required. The analytical solutions could be in the form of integrals whose integrands may be some complicated functions or the solutions are obtained in infinite series of Bessel functions or some other sophisticated functions. For matching, series solutions in powers of  $\epsilon^{1/2}$ ,  $\epsilon$  or sometimes some other rational power of  $\epsilon$  are required. Just to indicate how difficult this matching could be, we give below one more solution in which matching is quite difficult.

Solidification in a cylindrical annular region has been considered in [759] and this early work has been widely referred. The radially symmetric region  $b \leq r \leq a$  at time  $t = 0$  is

occupied by a superheated liquid. We continue with some of the notations used for different variables as in [758]. Solidification starts at the outer boundary  $r = a$  whose temperature is suddenly dropped to  $T_a < T_m$ . The inner boundary  $r = b$  is either insulated (called type II boundary condition) or maintained at the temperature  $T_i$  (type I) which is also the initial temperature of the liquid. The quantities whose definitions are different from those given in Eqs (12.9.5), (12.9.131) are being given below.

$$\eta = (S(t) - r)/(S(t) - b), \xi = (a - r)/(a - S(t)), \sigma = (a - S(t))/a. \quad (12.9.141)$$

If  $\theta$  and  $\phi$  are solid and liquid temperatures after scaling then the formulation of the problem in terms of dimensionless quantities can be written as given below.

$$\frac{\partial^2 \theta}{\partial \xi^2} - \frac{\sigma}{1 - \sigma \xi} \frac{\partial \theta}{\partial \xi} = \epsilon \left[ \sigma^2 \frac{\partial \theta}{\partial \tau} - \sigma \xi \frac{d\sigma}{d\tau} \frac{\partial \theta}{\partial \xi} \right], \quad 0 < \xi < 1, \tau = \epsilon k_{st}/a^2. \quad (12.9.142)$$

$$\begin{aligned} & \frac{\partial^2 \phi}{\partial \eta^2} - \frac{(1 - \sigma - \beta)}{[(1 - \sigma) - (1 - \sigma - \beta)\eta]} \frac{\partial \phi}{\partial \eta} \\ & = \alpha \epsilon \left[ (1 - \sigma - \beta)^2 \frac{\partial \phi}{\partial \tau} - (1 - \sigma - \beta)(1 - \eta) \frac{d\sigma}{d\tau} \frac{\partial \phi}{\partial \eta} \right], \quad (12.9.143) \\ & 0 < \eta < 1; \beta = b/a, \alpha = k_S/k_L. \end{aligned}$$

$$\theta(0, \tau) = -1, \theta(1, \tau) = 0; \phi(0, \tau) = 0; \frac{\partial \phi}{\partial \eta}(1, \tau) = 0 \text{ or } \phi(1, \tau) = \phi_i. \quad (12.9.144)$$

$$\phi(\eta, 0) = \phi_i, \sigma(0) = 0; (1 - \sigma - \beta) \frac{\partial \theta(1, \tau)}{\partial \xi} - \sigma \frac{\partial \theta}{\partial \eta}(0, t) = \sigma(1 - \sigma - \beta) \frac{d\sigma}{d\tau}. \quad (12.9.145)$$

It is not possible to explain the matching of inner and outer solutions without presenting at least partially the perturbed solutions. It is clear from Eqs (12.9.142), (12.9.143) that when  $\epsilon \rightarrow 0$  or  $\tau \rightarrow 0$ , the time derivative terms do not contribute in Eq. (12.9.143) so the perturbation solution without matching will not be uniformly valid for all time. We skip the details and present first the outer solution. The solution procedure is the same as discussed above for the solution obtained in [758]. Matching requires some more considerations. First-order solutions are reported below.

$$\begin{aligned} & \theta_0 = P_1 P_2 - 1, P_1 = 1/\ln(1 - \sigma_0), P_2 = \ln(1 - \sigma_0 \xi); \\ & \text{for both type I and II boundary conditions.} \end{aligned} \quad (12.9.146)$$

$$\begin{aligned} & \phi_0 = 0, \text{ for type II; } \phi_0 = [\phi_i / \ln((1 - \sigma_0)/\beta)] \ln \frac{(1 - \sigma_0)}{(1 - \eta)(1 - \sigma_0) + \beta \eta}, \\ & \text{for type I condition.} \end{aligned} \quad (12.9.147)$$

$$(1 - \sigma_0)^2 (1/P_1 - 1/2) = 2(A_0 + \tau), \text{ for type II, } A_0 \text{ is unknown.} \quad (12.9.148)$$

For type I boundary condition, the equation to determine  $\sigma_0$  is given in [759] which is lengthy. It contains an unknown constant. We are interested here in presenting the matching procedure for the solution of the problem with type II boundary condition whose solution is somewhat simple and procedure can be described here clearly. Along this type II case, solutions are being presented of Type I boundary condition if they are simple.

$$\theta_1 = P_1 \sigma_1 \left[ \frac{P_1 P_2}{(1 - \sigma_0)} - \frac{\xi}{1 - \sigma_0 \xi} \right], \text{ for both types I and II.} \quad (12.9.149)$$

$$\phi_1 = 0, \text{ for type II. For type I condition } \phi_1 \text{ is lengthy.} \quad (12.9.150)$$

$$\sigma_1 = P_1 A_1 / (1 - \sigma_0), \text{ for type II; } A_1 \text{ is unknown.} \quad (12.9.151)$$

$$\begin{aligned} \theta_2 = P_1 \sigma_2 \left( \frac{1}{\sigma_0} - \frac{1}{\sigma_0(1 - \sigma_0 \xi)} + \frac{P_1 P_2}{(1 - \sigma_0)} \right) - \frac{P_1^3}{4(1 - \sigma_0)^2} [(1 - \sigma_0 \xi)^2 P_2 \\ + 2\sigma_0 \xi - \sigma_0^2 \xi^2 + \sigma_0^2 P_1 P_2 - 2\sigma_0 P_1 P_2 - (1 - \sigma_0)^2 P_2], \end{aligned} \quad (12.9.152)$$

for both type I and II boundary conditions.

$$\phi_2 = 0 \text{ and } \sigma_2 = \frac{P_1 \sigma_0 (1 - \sigma_0/2)}{2(1 - \sigma_0)} (P_1 - 1) - \frac{P_1 A_2}{(1 - \sigma_0)}, \text{ for type II.} \quad (12.9.153)$$

The functions  $\bar{\theta}(\xi, \bar{\tau})$ ,  $\bar{\phi}(\eta, \bar{\tau})$  and  $\bar{\sigma}(\bar{\tau})$  for the inner solution are obtained as indicated in [758]. For the insulated type II boundary condition, the solution is given below.

$$\bar{\theta}_0 = \xi - 1; \bar{\sigma}_0 = 0; \text{ for both type I and II boundary conditions.} \quad (12.9.154)$$

$$\begin{aligned} \bar{\phi}_0 = \sum_{k=0}^{\infty} a_k J_0[G_k(1 - \eta + \beta\eta)] - \frac{J_1(\beta G_k)}{Y_1(\beta G_k)} Y_0[G_k(1 - \eta + \beta\eta)] e^{-[\lambda_k^2/\alpha(1-\beta)^2]\bar{\tau}}, \\ G_k = \lambda_k/(1 - \beta), \lambda_k \text{ are the roots of } J_0(G_k) = Y_0(G_k) \frac{J_1(\beta G_k)}{Y_1(\beta G_k)}, \end{aligned} \quad (12.9.155)$$

$a_k$  are given in terms of a big expression in [759] which are not given here as it does not contain terms of  $\bar{\tau}$  which are required for matching.  $J_0, J_1$  are Bessel functions of first kind and  $Y_0$  is the Bessel function of second kind. The subscripts 0 and 1 are for zeroth- and first-order solutions, respectively.

$$\bar{\theta}_1 = (1/2)\xi(\xi - 1)(2\bar{\tau})^{1/2} \text{ and } \bar{\sigma}_1 = (2\bar{\tau})^{1/2}, \text{ for both types.} \quad (12.9.156)$$

$\bar{\phi}_1$  could not be determined analytically and fortunately it is not required in calculating  $\bar{\sigma}_1$ .

$$\bar{\theta}_2 = \frac{1}{6}(4\bar{\tau} - 1)\xi^3 + (\bar{\sigma}_2 - \bar{\tau})\xi^2/2 + \frac{1}{6}(1 - \bar{\tau})\xi - \frac{\bar{\sigma}_2 \xi}{2}, \text{ for types I and II.} \quad (12.9.157)$$

$$\bar{\sigma}_2 = \frac{\bar{\tau}}{3} + \sum_{k=0}^{\infty} a_k d_k e^{-[\lambda_k/\alpha(1-\beta)^2]\bar{\tau}} + \frac{(1 - \beta)}{2\lambda_k} \frac{(\pi\alpha)^{1/2}}{(\bar{\tau})^{1/2}} \operatorname{erf} \frac{\lambda_k}{(1 - \beta)\alpha^{1/2}} (\bar{\tau})^{1/2}, \quad (12.9.158)$$

for type II condition. For  $d_k$  refer [759].

For matching,  $\sigma$  in the outer solution which is the sum of three terms,  $\sigma_0 + \epsilon^{1/2}\sigma_1 + \epsilon\sigma_2 + O(\epsilon^{3/2})$ , should be expressed in the form  $\sigma = e_0 + e_1 \epsilon^{1/2} + e_2 \epsilon + O(\epsilon^{3/2})$ ,  $e_0, e_1, e_2$  contains terms of  $\tau^{1/2}, \tau, \tau^{3/2}$ , etc., and the same procedure should be adopted for  $\bar{\sigma}$ . When  $\tau \ll 1$ , Binomial expansions can be used to obtain  $\sigma$  in the form given below.

$$\sigma = (2\tau)^{1/2} + \tau/3 + (2)^{3/2}\tau^{3/2}/9 - \epsilon^{1/2}A_1/(2\tau)^{1/2} + \epsilon(1/2 + A_2)/(2\tau)^{1/2}. \quad (12.9.159)$$

Similarly in  $\bar{\sigma}$ , firstly terms are expanded in a series of powers of  $\epsilon^{1/2}$  and then they are arranged in a series by collecting coefficients of different powers of  $\epsilon^{1/2}$ . Matching is done

for inner and outer solutions as  $\bar{\tau} \rightarrow \infty$  and  $t \rightarrow 0$ , respectively, and then  $\bar{\tau}$  is replaced by  $\tau/\epsilon$ . For the inner solution

$$\bar{\sigma} = (2\tau)^{1/2} + \frac{\tau}{3} + \epsilon^{3/2} (1 - \beta)(\pi\alpha)^{1/2} (\Sigma a_k d_k / \lambda_k) / (\tau)^{1/2}, \quad (12.9.160)$$

$\phi_i$  is dimensionless initial temperature of liquid.

Comparing inner and outer solutions, we get

$$A_1 = 0, \quad A_2 = -1/2. \quad (12.9.161)$$

In the isothermal case, matching is difficult as some of the short-time expansions are obtained with difficulty. Having matched the moving interface, temperatures of outer and inner regions should also be matched. In the outer solution  $A_1$  and  $A_2$  are the only unknown constants which have been determined in [759].

The singularity in [758, 759] arises due to degeneracy of the heat equation from transient to the steady state. However, our purpose of showing process of matching with the help of solution is fulfilled as whatever may be nature of singularity, the matching is done in the same manner.

We reconsider the problem in [746]. If in this one-phase spherical problem  $S(t)$  is redefined as  $1 - S(t)$  and  $\epsilon = 1/\beta$  then solutions of temperature  $\theta(S, R)$  and  $\tau(S)$  which were not mentioned earlier are obtained as given below in which  $R$  is dimensionless radial coordinate.

$$\theta = (1 - S)(1 - R)/RS + \frac{(1 - R)}{6\beta(1 - S)RS} \left\{ 1 - \left( \frac{1 - R}{S} \right)^2 \right\} + O(\beta^{-2}), \quad (12.9.162)$$

$$\tau = \tau(S_0) = (3S^2 - 2S^3)/6 + S^2/6\beta - S^2/(45\beta^2(1 - S)) + O(\beta^{-3}). \quad (12.9.163)$$

These solutions break down when  $R \rightarrow 0$  or  $S \rightarrow 1$ . The next three references [760–762] are concerned with obtaining the solutions of one-phase radially symmetric spherical problem as discussed in [746] which are uniformly valid in  $0 \leq R \leq 1$  for large Stefan number  $\beta$ . The series solutions in Eqs (12.9.162), (12.9.163) are obtained when perturbation expansions are considered in integral powers of  $(1/\beta)$ . To know what is happening as  $R \rightarrow 0$ , in [746],  $S(t)$  was written as

$$S = 1 - \Delta S_0, \quad S_0 = O(1), \quad \Delta \ll 1, \quad (12.9.164)$$

$S_0$  is the region in which the solutions in Eqs (12.9.162), (12.9.163) are not valid. When Eq. (12.9.164) is substituted in Eq. (12.9.163) it is found that in  $\tau'(S)$  (dash for derivative), the second term overtakes the first term when  $\Delta \sim \beta^{-1/2}$ ,  $\beta$  large. It may be emphasized here that uniformity of solution is not merely of mathematical interest but it also provides insight into the nature of freezing process near the centre. Temperature has also different characteristics when  $R \sim O(1)$  and  $R \sim O(\Delta)$ . Therefore when  $\Delta \sim (\beta^{-1/2})$ , heat flow should be examined in two separate regions. Further the solution for inner region should ensure that  $\theta|_{R \rightarrow 0} = \text{isotherm temperature}$ . The perturbation expansions are therefore considered in powers of  $(\beta^{-1/2})$  in the following form in the inner region.

$$\theta = T_0 + \beta^{-1/2} T_1 + \beta^{-1} T_2 + O(\beta^{-3/2}), \quad \theta = \theta(r, S_0); r = \beta^{-1/2} R, r \sim O(1), \quad (12.9.165)$$

$$\tau = \frac{1}{6} + \tau_0(S_0)\beta^{-1} + \tau_1(S_0)\beta^{-3/2} + \tau_2(S_0)\beta^2, \tau = \tau(S_0), S_0 \sim O(1). \quad (12.9.166)$$

On satisfying the differential equation and interface conditions, the unknown functions in the series of  $\theta$  have been determined. To match the solution of  $\tau(S_0)$  in Eq. (12.9.165) with the solution in Eq. (12.9.163),  $S_0 \rightarrow \infty$  should be taken in the inner solution which will determine  $\tau(S_0)$ .

For the outer solution,  $R$  is the relevant radial coordinate and  $\bar{\theta}$  is taken as

$$\bar{\theta} = \beta^{-1/2}\bar{T}_0 + \beta^{-1}\bar{T}_1 + O(\beta^{-3/2}), \bar{\theta} = 0 \text{ at } R = 1. \quad (12.9.167)$$

After matching with the inner solution as  $R \rightarrow 0$  the behaviour of  $\bar{T}_0$  has been obtained as  $S \rightarrow \infty$ . To obtain the analytical solution of  $\theta$  at the terminal times, Fourier sine transform has been used. The first-order term in the analytical solution of  $\theta(r, S_0)$  has logarithmic singularity as  $R \rightarrow 0$  and therefore the solution fails in a small neighbourhood of  $R = 0$ . A similar problem in cylindrical coordinates has also been studied in [746] with the same solution procedure.

The total freezing time  $t_E$  given by the outer solution in [746] is  $t_E = 1/6 + 1/6\beta + O(\beta^{-1})$ . By considering expansion in powers of  $(\beta^{-1/2})$ , one more term in the solution of  $t_E$  was provided in [746] but this solution also breaks down when  $t_E - t = O(\beta^{-1}e^{-\beta^{-1/2}})$ . In [760] the authors have attempted to correct the solution in the region of this breakdown time interval. Their procedure requires lot of space to be described as the approach is different from what has been discussed earlier and cannot be explained without giving solutions obtained in the various steps involved. However, the initial approach breaks down when  $\xi = O(1)$ , where  $\xi = \beta^{-1/2} \ln(1/x)$ ,  $x = \beta(t_E - t)$ , as nonuniformity sets in the limiting behaviour of some functions (cf. [760]). This nonuniformity was removed by considering  $\xi$  as the independent variable instead of  $x$  and some of the arbitrary functions defined in the earlier approach in [760] were redefined in terms of  $(\xi, \beta^{-1/2})$ . The final temperature solution at  $r = 0$  gives the correct isotherm temperature and also satisfies the no flux condition at  $r = 0$ . To understand the difficulties involved in obtaining the final result one has to go through the full paper to appreciate it. This paper has been widely referred.

Instead of temperature-prescribed boundary condition at  $R = 1$ , constant flux is prescribed at  $R = 1$  in [761]. Other details in the formulation remain the same as in [746]. The outer solution is obtained in powers of  $(1/\beta)$  where  $\beta$  is the ratio of latent heat to specific heat and  $\beta$  is large. It was found from the obtained solution of time variable  $\tau$  that the solution breaks down when  $(1 - S(\tau)) \sim O(\epsilon)$ ,  $\epsilon = \beta^{-1/4}$  and  $R = 1 - S(\tau)$ ,  $S(0) = 0$ . It will be assumed here that all the quantities are in dimensionless form. The outer solution is obtained by considering expansions in power of  $(1/\beta)$  with flux prescribed condition at  $R = 1$ . In this solution  $\tau = \tau(S)$ ,  $S$  is time-like variable. To obtain a uniformly valid solution in the region  $0 \leq R \leq 1$ , in the inner region the time-like variable  $S_0 = (1 - S)/\epsilon$  should be considered,  $S_0 = O(1)$  near the total solidification time. In [761] the region  $0 \leq R \leq 1$  has been divided into three regions. Region III is the outer region in which  $1 - S = O(1)$  and the variables are  $(R, S)$ . Inner region is subdivided into two regions I and II. In region II,  $R = O(1)$  and the variables are  $(R, S_0)$  and, in region I,  $R = O(\epsilon)$  and variables are  $(r, S_0)$ ,  $r = R/\epsilon$ . The temperature solution in region I satisfies only the isotherm interface condition which is matched with the outer solution together with the solution of  $\tau(S_0)$ . Therefore the solution of region III is expanded for small  $\epsilon$  and written in terms of variables  $(r, S_0)$ . For the solution in region II, the solutions of regions I and III are expanded for small  $\epsilon$  and in terms of the variables  $(R, S_0)$ . Getting a clue from the forms of these solutions, the temperature  $\theta(R, S_0)$  in region II is expanded as

$$\theta = \bar{T}_0(R, S_0)/\epsilon + \bar{T}_1(R_1 S_0). \quad (12.9.168)$$

$\theta(R, S_0)$  in Eq. (12.9.168) satisfies only the transient heat equation. The structure of  $\theta$  in Eq. (12.9.168) is the same as in [760] but because of flux prescribed boundary condition,  $\bar{T}_0$  satisfies only steady-state heat equation whose solution is obtained and  $\bar{T}_1$  could be obtained only numerically. In [761] only first-order term for  $\bar{T}_0$  could be obtained as a transient equation for  $\bar{T}_0$  was to be solved. Approximate time for total solidification was obtained and at this total time for small  $\epsilon$  outer boundary becomes very cold which is expected.

A two-phase one-dimensional radially symmetric spherical problem has been considered in [762]. A superheated liquid with constant temperature  $V > 0$  at time  $t = 0$  occupying the region  $0 \leq r \leq 1$  is subjected to cooling at  $r = 1$  in the form of prescribed temperature  $T_S = -1 < 0$ , where 0 is the freezing temperature.  $r = S(t)$  is the moving interface,  $S(0) = 1$ . Heat equations and interface conditions are as prescribed in Neumann problem. Solutions in terms of perturbation expansions of the parameter  $\beta$  where  $\beta$  is the ratio of latent heat to specific heat have been considered. Four cases of different time scales have been considered. The first problem is that of large  $\beta \gg 1$  and short-time scale  $t = O(1)$ . In the inner region  $r$  is scaled as  $1 - r = O(\beta^{-1/2})$  so that  $r = 1 - \beta^{-1/2}\bar{r}$  and  $S(t) = 1 - \beta^{-1/2}\bar{S}(t)$ . The solid and liquid temperatures  $\bar{T}_S$  and  $\bar{T}_L$  are expanded as follows.

$$\left. \begin{aligned} \bar{T}_S &\sim \bar{T}_{S0}(\bar{r}, t) + \beta^{-1/2}\bar{T}_{S1}(\bar{r}, t) + O(\beta^{-1}), \\ \bar{T}_L &\sim \bar{T}_{L0}(\bar{r}, t) + \beta^{-1/2}\bar{T}_{L1}(\bar{r}, t) + O(\beta^{-1}), \end{aligned} \right\} \quad (12.9.169)$$

$$\bar{S}(t) = \bar{S}_0(t) + \beta^{-1/2}\bar{S}_1(t) + O(\beta^{-1}), \quad \beta \rightarrow \infty.$$

For the outer region spacial scaling for  $r$  is done as  $1 - r = O(1)$ . Only liquid region has been considered in which  $T_L = \bar{T}_L(r, t)$ ,  $\bar{T}_L = 0$  at  $r = 1$ ,  $\frac{\partial \bar{T}_L}{\partial r} = 0$  at  $r = 0$  and  $\bar{T}_L = V$  at  $t = 0$ .  $\bar{T}_L$  satisfies the transient heat equation. On matching the inner and outer expansions, we get

$$\bar{S}_0(t) = (2t)^{1/2}, \quad S(t) \sim 1 - (2t)^{1/2}/\beta^{1/2} + \text{higher-order terms}. \quad (12.9.170)$$

The total solidification time  $t$  is obtained as

$$t/\beta \sim O(1 - S(t)^2). \quad (12.9.171)$$

Summary of findings in each case has been reported in a concise form separately. In the second case  $t = O(\beta)$ . Time  $t$  is rescaled as  $t = \beta\bar{t}$  where  $\bar{t} = O(1)$ . In the region  $0 < r < S(\bar{t})$ , only liquid region has been considered. At the interface  $\bar{T}_L = 0$ ,  $r = S(\bar{t})$ . If  $\rho = r/S(\bar{t})$  then solution of  $\bar{T}_L(r, \bar{t})$  is sought in the form of

$$\bar{T}_L \sim e^{-\beta g(\bar{t})} \left( \bar{T}_0(\rho) + \frac{1}{\beta} \bar{T}_1(\rho, \bar{t}) + \dots \right), \quad \text{as } \beta \rightarrow \infty. \quad (12.9.172)$$

The solution of  $\bar{T}_0(\rho)$  has been obtained when  $\beta \rightarrow \infty$ .  $\bar{T}_0(\rho)$  satisfies the transient heat equation, isotherm condition and no flux condition at  $r = 0$ . This solution involves unknown  $S(\bar{t})$  as  $g(\bar{t})$  is obtained from  $g'(\bar{t}) = n^2 \pi^2 k / S(\bar{t})$ .

In the outer solution liquid phase does not contribute and the solution is the same as obtained in [746]. When time scales are  $t - t_E = O(1)$  and  $t - t_E = O(\exp(-2\sqrt{2\pi}\beta^{1/2}))$ ,

solutions have already been presented in [760, 763]. For the solutions of cases  $\beta \rightarrow 0$  and  $k \ll 1$  ( $k$  is thermal diffusivity), refer [762].

In [763] the spherical problem considered is the same as in [746, 757, 760] and is similar for cylinder as in [746] for cylindrical case. Most of the results obtained for spherical problem in [763] are the same as in [746, 760] and those obtained for cylinder in [763] are in [746]. However, the solution approach is different. When  $t = O(1)$ ,  $t = (t_E - t_k^*)/t_k^*$ ,  $t_k^* = a^{*2} \rho C/k$  which is time taken for heat to diffuse through a distance  $a^*$ , the method of images has been used in [763] to obtain the solid-phase solution.  $t_E$  is the total freezing time. A heat source of unknown strength is placed at the origin  $r = 0$  and the region  $0 \leq r \leq 1$  is completely occupied by the solid. The solid temperature satisfies the heat equation, the boundary condition at  $r = 1$  and the flux condition at  $r = 0$ . The strength of heat source can be obtained using the condition that at  $r = S(t)$ , temperature is isotherm temperature. When a heat source of strength  $4\pi\beta^{-1/2}\phi(t)$  is situated at the origin then the asymptotic form of temperature  $\theta$  can be taken as

$$\theta = -1 + \beta^{-1/2}[\phi(t)/r - \psi(t)] + \text{terms of } O(r) \text{ as } r \rightarrow 0, \quad (12.9.173)$$

$\beta$  is large Stefan number,  $\phi(t)$  and  $\psi$  are unknowns and for obtaining them boundary conditions are to be used. For the solution, finite Fourier transform has been used. In the terminal core, a similarity variable  $\xi = r/2t^{1/2}$  has been used for solution.

For cylinder also the same procedure is followed. A heat source of strength  $2\pi(\ln \beta)^{-1}\phi(t)$  is placed at the origin and  $\theta$  is taken as

$$\theta = -1 + (\ln \beta)^{-1}[\phi(t) \ln r^{-1} - \psi + 1] + \text{term of } O(r) \text{ as } r \rightarrow 0. \quad (12.9.174)$$

$\phi$  and  $\psi$  are unknowns and they are determined by using boundary conditions. Finite Hankel transform (cf. [389]) has been used for solution. For other details refer [763].

A one-phase solidification problem in the one-dimensional region  $x \geq 0$  has been considered in [764]. The boundary condition at  $x = 0$  is of black body radiation type. In this problem there will be two important parameters  $\beta$  and  $\lambda$ ,  $\beta$  is the ratio of latent heat to specific heat and  $\lambda$  is environmental thermal head which is  $(T_m - T_0)/T_0$ ,  $T_m$  is the freezing temperature and  $T_0$  is ambience temperature. Small-time solution is simple to obtain, refer [24]. Large-time solution has been assumed in a typical way which has not been discussed enough by authors. It involves two unknown constants. To determine these constants, firstly perturbation solution for large  $\beta$  is obtained. If  $\epsilon = 1/\lambda$  then the interface in the large  $\beta$  solution has three layers of singular structure. If  $\tau = t/\epsilon$  then the three regions of interest are  $\tau = O(\epsilon^{-9})$ ,  $O(\epsilon^{-8})$  and  $O(1)$ . How these different layers arise refer [763] for it as for the further discussion the complete solution is to be reported here which is not possible. In short, these three regions are dictated by the equation to obtain the moving interface which is finally obtained by matching these solutions which requires lot of details.

The singular perturbation solution in [765] pertains to a welding problem. The automatic welding has been theoretically modelled as a moving point source of heat with conduction dominated heat transfer. A three-dimensional one-phase melting problem has been considered in which the workspace is assumed to be an infinite half space  $y > 0$ .  $(x, y, z)$  are Cartesian coordinates. The welding torch is stationary in the frame, so that the material moves relative to the torch with the velocity  $-V\vec{k}$ , where  $\vec{k}$  is the unit vector in the  $z$ -direction. The problem has been formulated in spherical polar coordinates  $(r, \theta, \phi)$  and is independent of  $\phi$  as  $\partial T/\partial \phi = 0$



at  $\varphi = 0$  has been taken. Steady-state problem in dimensionless form has been considered whose formulation is given below.

$$(\nabla^2 - 1)\bar{T}(s, \theta; \alpha, \gamma) = 0; \lim_{s \rightarrow 0} s^2 \frac{\partial \bar{T}}{\partial s} = -\alpha, \alpha \text{ is the speed-power parameter,} \quad (12.9.175)$$

$$\bar{T}|_{s=s_i=0} = 0; s = s_i(\theta, \alpha, \gamma) \text{ is the dimensionless interface,} \quad (12.9.176)$$

$$[\nabla \bar{T} \cdot \vec{n}]_{s=s_i} = 2\gamma \bar{T}_i \vec{k} \cdot \vec{n}; \bar{T}_i = \bar{T}|_{s=s_i} = e^{s_i \cos \theta}; \bar{T} = O(e^{s \cos \theta}) \text{ as } s \rightarrow \infty, \quad (12.9.177)$$

$$\left. \begin{array}{l} \gamma \text{ is latent heat parameter, } \bar{T} = Te^z/T_m, T_m \text{ is the melting temperature,} \\ T \text{ is the dimensionless temperature,} \end{array} \right\} \quad (12.9.178)$$

$s = r/\beta$ , where  $\beta$  is diffusion length which is diffusivity divided by  $V/2$ ,  $r$  is polar radius.

In the inner region  $s \leq s_i$ , which is the molten region, the inner limit is defined as  $\bar{s} = s/\alpha$  which remains fixed as  $\alpha \rightarrow 0$ . Therefore  $\bar{s}_i = s_i/\alpha = O(1)$  as  $\alpha \rightarrow 0$  and the inner expansions of temperature can be considered in the form

$$\bar{T}_{\text{in}}(\bar{s}, \theta; \alpha, \gamma) \simeq g_0(\bar{s}) + \sum_1^\infty \alpha^n g_n(\bar{s}, \theta, \gamma); \bar{s}_i(\theta, \alpha, \gamma) \simeq \alpha \left[ 1 + \sum_1^\infty \alpha^n R_n(\theta, \gamma) \right], \quad (12.9.179)$$

$\gamma$  is fixed and  $g_n, R_n \simeq O(1)$  as  $\alpha \rightarrow 0$ .

In the outer region,  $s$  and  $\gamma$  are fixed as  $\alpha \rightarrow 0$  and the expansions in the outer region can be considered in the form

$$\bar{T}_{\text{out}}(s, \theta; \alpha, \gamma) \simeq h_0(s) + \sum_1^\infty \alpha^n h_n(s, \theta, \gamma), \quad h_n \approx O(1) \text{ as } \alpha \rightarrow 0. \quad (12.9.180)$$

The zeroth-order terms  $g_0(s)$  and  $h_0(s)$  are the representations of the same function in different regions and so they should match. It is found from the obtained analytical solutions (see [765]) that matching of the two solutions is possible only in some intermediate region or in other words in a boundary layer region which could be defined as a region in which

$$s = O(\eta(\alpha)) \text{ as } \alpha \rightarrow 0, \quad O(\alpha) < O(\eta(\alpha)) < O(1). \quad (12.9.181)$$

Define,  $s_\eta = s/\eta(\alpha) = \alpha \bar{s}/\eta(\alpha)$  fixed as  $\alpha \rightarrow 0$ ,  $\lim_\eta = \lim_{\alpha \rightarrow 0}$ ,  $s_\eta$  fixed.

Matching of the two solutions can be done in the following limit.

$$\lim_\eta \{ \bar{T}_{\text{out}}(s_\eta, \theta; \alpha, \gamma) - T_{\text{in}}(s_\eta, \theta, \alpha, \gamma) \} / \alpha^n = 0. \quad (12.9.182)$$

For further details about assumptions, scalings and solution, refer [765]. Numerical work was done to find the vertical extent of the fusion zone. It was found that latent heat has an effect on the penetration that is  $O(\alpha^2)$  as  $\alpha \rightarrow 0$ .

In [766], axisymmetric solidification in a cylindrical ampoule of finite height  $L$  in the  $z$ -direction taken vertically and of radius  $R$  is considered. Dimensionless transient heat equations in solid and liquid regions for temperature and concentration have been considered in  $(r, z)$  coordinates as given below.

$$\left. \begin{aligned} \varepsilon^4 \bar{\alpha}_L \frac{\partial T_L}{\partial t} &= (T_L)_{rr} + (T_L)_r/r + \varepsilon^2 (T_L)_{zz}, \\ \varepsilon^4 \bar{\alpha}_S \frac{\partial T_S}{\partial t} &= (T_S)_{rr} + (T_S)_r/r + \varepsilon^2 (T_S)_{zz}. \end{aligned} \right\} \quad (12.9.183)$$

$$\left. \begin{aligned} \varepsilon^4 \bar{\alpha}_L \beta_L \frac{\partial \bar{c}_L}{\partial t} &= (\bar{c}_L)_{rr} + (\bar{c}_L)_r/r + \varepsilon^2 (\bar{c}_L)_{zz}, \\ \bar{\alpha}_S \beta_S (\bar{c}_S)_t &= (\bar{c}_S)_{rr} + (\bar{c}_S)_r/r + \varepsilon^2 (\bar{c}_S)_{zz}. \end{aligned} \right\} \quad (12.9.184)$$

$$\bar{\alpha} = \alpha \varepsilon^2; \beta_L = k_L/D_L, \beta_S = k_S/D_S; D_S/D_L = O(\varepsilon^4), D_S \ll D_L, \quad (12.9.185)$$

$\beta$  is Lewis number, which is ratio of thermal conductivity and mass diffusivity,  $k$  is thermal diffusivity and  $D$  is mass diffusivity. The subscripts  $S$  and  $L$  stand for solid and liquid, respectively, and  $\bar{c}$  is concentration. The perturbation parameter  $\varepsilon = R/L$  is small.  $\alpha_L$  and  $\alpha_S$  are Stefan numbers in the solid and liquid regions, which are defined as  $\alpha_L = T_m \rho C/l$ ,  $T_m$  is melting temperature and  $\alpha_S = \alpha_L k_L/k_S$  which are also small as latent heat  $l$  is considered large. The formulation is considered in dimensionless form. For scalings and assumptions, see [766].

The solid-liquid interface location is  $z = h(r, t)$  which is taken as

$$z = h(r, t) = H(t) + \varepsilon^2 \bar{H}(r, t), \quad H(t) \text{ is the mean planar location of interface,} \quad (12.9.186)$$

$\varepsilon^2 \bar{H}(r, t)$  is a small nonplanar interface correction. At the top of the ampoule  $z = 1$ , and at the bottom  $z = 0$ , we have the following boundary conditions.

$$T_L = \theta_h(t) \text{ and } \bar{c}_L = 0, \text{ on } z = 1; \quad T_S = 0 \text{ and } \bar{c}_S = 0, \text{ on } z = 0. \quad (12.9.187)$$

The boundary conditions at  $r = 1$  for solid and liquid portions of ampoule and initial conditions are given below.

$$(T_L)_{r=1} = -\varepsilon^2 B_L [T_L - \theta(Lz - V\tau t)]; \quad (T_S)_{r=1} = -\varepsilon^2 B_S [T_S - \theta(Lz - V\tau t)], \quad (12.9.188)$$

$$\left. \frac{\partial \bar{c}_L}{\partial r} \right|_{r=1} = 0, \quad \left. \frac{\partial \bar{c}_S}{\partial r} \right|_{r=1} = 0; \quad \bar{c}_L(r, z, t = 0) = 1 \text{ and } T_L(r, z, t = 0) = 1, \quad (12.9.189)$$

$B_L$  and  $B_S$  are Biot numbers for liquid and solid, respectively, and are  $O(1)$ ,  $V$  is the velocity of the translating external heater moving vertically,  $\theta$  is the temperature of the side walls of the ampoule,  $\tau = (lL^2/T_m k_L)$  and  $t$  is time which is scaled through a factor of  $\tau$ .

Temperature at the interface is continuous but unknown and Gibbs-Thompson effect due to curvature has been considered at  $z = h(r, t)$ . Contact angle effect on the interface has also been taken into account which is  $O(\varepsilon)$ .

Stefan condition for heat balance and mass balance condition for concentration at the interface have been taken into consideration. Linear-phase diagram has been considered for temperature and concentration relationship at the interface and  $\bar{c}_S = k_{SC} \bar{c}_L$ ,  $k_{SC}$  is segregation coefficient taken as 0.17. For solution of outer problem, regular perturbation expansion in integral powers of  $\varepsilon$  has been considered. In the zeroth-order solution it is found that  $\bar{c}_S$  is a function of  $z$  only and so it can be assumed that at any height  $z = z_0$ ,  $0 < z_0 < 1$ ,  $\bar{c}_S = k_{SC} \bar{c}_L$  remains the same.  $z_0$  could be interface also. When calculations are carried out, the zeroth-order temperature and concentration solutions were found to depend on the knowledge of zeroth-order solution of  $\bar{c}_L$  at  $z = z_0$  which is the same for any  $z_0$ .

If  $\bar{c}_L$  is known then  $H(t)$  also can be obtained numerically by solving a first-order ordinary differential equation as the equation for  $H(t)$  contains only  $\bar{c}_L$  (zeroth-order solution). Note that the relation  $\bar{c}_S = k_{SC}\bar{c}_L$  corresponds to a well-mixed assumption of the growth of the solute. To the  $O(\varepsilon^4)$  approximation, it was found that  $\bar{c}_L = (1 - H(t))^{k_{SC}-1}$  which is the Scheil's equation.

In order to find  $\bar{H}(r, t)$ , a boundary layer analysis must be performed near  $z = H(t)$ . This analysis is also required at  $z = 0$  and  $z = 1$  to satisfy terms of  $O(\varepsilon^2)$ . Considering only the boundary layer analysis at  $z = H(t)$ , a new boundary layer variable is defined as

$$\eta = (z - H(t))/\varepsilon, \quad 0 \leq \eta \leq 1. \quad (12.9.190)$$

To obtain the solution of the inner problem, the problem formulation is transformed in terms of the variables  $(t, \eta, r)$  and the temperatures and concentration are expanded in powers of  $\varepsilon$ . We need to match the outer solutions for the temperature and concentration both in the liquid ( $\eta \rightarrow \infty$ ) and in the solid ( $\eta \rightarrow -\infty$ ). On carrying out calculations, a second-order nonhomogeneous equation for  $\bar{H}(r, t)$  is obtained which has been solved numerically. Depending on some functions occurring in the equation for  $\bar{H}(r, t)$ , this equation for  $\bar{H}(r, t)$  could be a Bessel equation, a modified Bessel equation or an Euler's equation. For further details refer [766].

Numerical work was done to calculate  $H(t)$  and  $H(t) + \bar{H}(r, t)$  for different heater speeds and to calculate core melt and solid temperature profiles. Due to inclusion of curvature effects on the melting temperature, the shape of the interface is described by a superposition of a Bessel function onto a parabolic profile. The Bessel function profile is the response of the interface to meet the requirement of prescribed contact angle. We noted that not many problems in such generality have been studied.

The problem considered in [767] is similar to the directional solidification problem considered in [766] with a difference that the solute is everywhere well-mixed, except in a thin stagnant layer of thickness  $\delta$  ahead of the solidification interface that lies within the interface boundary layer of thickness  $\varepsilon$ . Most of the details of formulation and outer and inner solution procedures are similar to those discussed in the context of [766]. Some significant changes are present in the relation between concentrations  $\bar{c}_L$  and  $\bar{c}_S$ . In the outer zeroth-order solution it is assumed that  $\bar{c}_L$  (zeroth order)  $= (1 - H(t))^{k_{\text{eff}}-1}$ . When  $k_{\text{eff}} = 1$ , we have diffusion-controlled growth of the solute and concentration of the solute remain the initial concentration. When  $k_{\text{eff}} = k_{SC}$ , we have a well-mixed growth of the solute. At the interface  $z = H(t) + \varepsilon^2 \bar{H}(r, t)$  relation between  $\bar{c}_S$  and  $\bar{c}_L$  is taken as  $\bar{c}_S = k_{\text{eff}}\bar{c}_L = k_{SC}\bar{c}_L$  (initial). In the boundary layer region  $0 \leq \eta \leq \delta$ ,  $\eta$  defined as in Eq. (12.9.190). It is found that the zeroth-order solution for  $\bar{c}_L$  is as given below.

$$\bar{c}_L = (1 - H(t))^{k_{\text{eff}}-1} \left\{ \begin{array}{l} [k_{\text{eff}} + (1 - k_{\text{eff}}) \exp(\beta_L \bar{\alpha}_L H_t (\delta - \eta))], 0 \leq \eta \leq \delta, \\ = (1 - (H(t)))^{k_{\text{eff}}-1}, \eta > \delta, \end{array} \right\} \quad (12.9.191)$$

$$k_{\text{eff}} = k_{SC} / [k_{SC} + (1 - k_{SC}) \exp(-\bar{\alpha}_L \beta_L H_t \delta)]. \quad (12.9.192)$$

$H_t$  is the time derivative of  $H(t)$ ,  $k_{\text{eff}}$  is time-dependent as it is defined through the velocity term  $H(t)$ .  $H_t$  is taken as positive, i.e. there is no melting. When  $\delta \rightarrow \infty$ , we recover the pure diffusion profile and as  $\delta \rightarrow 0$ , the Scheil profile is obtained.

Eqs (12.9.191), (12.9.192) can be used only when  $\delta < 1/\varepsilon$ . Therefore in the solution of temperatures and concentrations, terms of  $O(\varepsilon)$  have to be considered and this will change

the solution. Refer [767] for further details. Stagnant film concept has been used to include the effects of melt flow without calculating either the details of the flow or the associated transport of solute by convection. In the case of undercooling it was found in [766, 767] that the phase-change boundary becomes oscillatory which is consistent with the earlier studies that instability develops due to undercooling. Parabolic profile of the interface develops due to radial and axial diffusive transport of heat.

The directional solidification of a diffusion-dominated binary system has been considered in [768]. Approximate analysed solutions have been obtained by matching the inner and outer perturbation solutions in which the perturbation parameter  $\varepsilon = W/L$ ,  $2W$  is the width of the ampoule in the horizontal  $x$ -direction and  $L$  is the height in the vertical  $z$ -direction. A two-dimensional steady-state problem has been considered in which  $U$  and  $V$  are the flow velocities of the melt in the  $x$ - and  $z$ -directions, respectively. The heat transfer between the ampoule and melt can be asymmetric. As in [766], moving heater profiles have been considered in the boundary conditions along the vertical edges. The phase-change interface is the sum of the planar interface and a perturbation and is of the form  $z = H(x) = H_0 + \varepsilon^2 \bar{H}(x)$ ,  $H_0$  is the planar interface. In the solution procedure, to obtain the solution of  $\bar{H}(x)$ , a boundary layer at  $z = H_0$  has been considered. Temperature and concentration are coupled at the interface. The flow field is decoupled from temperatures and concentration in the boundary layer but it is taken as coupled in the outer and inner solutions. Concentration gradients are present only in the liquid. Although the solution procedure is essentially the same as in [766], the solution procedure requires discussion different from the one presented above as in [766]. For further details refer [768].

Considerable numerical work has been reported to indicate the effect of asymmetric heating on the second-order correction to the location of the interface. The effect of velocity of the moving heater applied on the side walls has also been studied numerically on the location of the interface concerning second-order corrections at the top and bottom of the ampoule.

## 12.10 BRIEF REVIEWS OF SOME SUPPLEMENTARY REFERENCES CONCERNING THIS CHAPTER

As mentioned earlier, the directions in which the study of Stefan problem has progressed cannot be covered even for its limited aspects discussed in this book. By including supplementary references, we have just tried to provide a little more information about the studies progressed in different directions related to the material discussed in these 12 chapters. All the published works whether included here or not are important and provide some new insight and useful information. However, due to space limitations, brief notes on only some of the supplementary references somewhat directly related to the material discussed in this book are presented here. Purely numerical solutions obtained without much contents of analytical solution have not been discussed. These notes have been reported here after going through the full-length paper but contents are reported very briefly. Some more references not discussed in this section but they are connected with this chapter are listed in [839–853].

The one-dimensional melting of a finite and an infinite solid has been considered in [769]. Depending on the prescribed boundary conditions at the bottom and top of a solid slab, different types of melting problems can be formulated. Problem concerning two moving boundaries and four phases has also been studied in [769]. Approximate analytical solutions have been obtained using HBIM and perturbation method depending on the suitability of the method in any phase. Numerical results for temperatures in different zones and moving boundaries have been obtained and compared with exact solutions obtained in some particular cases. The results have application in de-icing.

ADM discussed in Section 12.4.1 has been used in [770] for the analytical-numerical solution of a steady-state one-dimensional heat transfer problem (without phase change) concerning fin in which heat generation terms are of convective-radiative type. The energy equation considered is highly nonlinear. Recursive relations have been obtained for higher-order terms in ADM. Five terms in the series for temperature have been obtained analytically. Numerical results have been compared with the results obtained by Galerkin method.

In an one-dimensional transient Stefan problem in [771] the latent heat is taken as  $(S(t))^\alpha$  and the prescribed flux is proportional to  $t^{(\alpha-1)/2}$ ,  $\alpha > 0$ . Using the invariance property, an exact similarity solution has been obtained in [771] in terms of Kummer function (cf. [389]) and numerical computations have been done. Existence and uniqueness of the solution is discussed.

The variational method proposed by He (cf. [494]) has been used in [772] to obtain analytical solutions of some nonlinear steady-state heat transfer problems without phase change in which thermal conductivity is a function of temperature and heat generation term is of convective-radiative type. In the transient problem thermal conductivity is taken as zero. Numerical work is presented. Energy equation is highly nonlinear.

In an earlier work (cf. [398]) it was pointed out that in the one-dimensional one-phase solidification of a supercooled liquid it is not correct to consider this problem as a limiting case of a two-phase problem by taking thermal conductivity of the solid as zero. Correct formulation was discussed earlier to obtain the reduction of two-phase problem to one-phase problem. In [773], the correct formulation has been further discussed when the phase-change temperature is a variable temperature in a one-dimensional planar problem. One-dimensional radially symmetric cylindrical and spherical problems have also been discussed. Numerical work is presented.

The total extinction time required for evaporation of a radially symmetric spherical droplet occupying the region  $0 \leq r \leq 1$  in a hot gas has been numerically calculated in [774] using Keller-box finite difference scheme coupled with boundary immobilization. The complete numerical solution for extinction time requires some modifications of analytical solutions to take care of the inconsistency at  $t = 0$  in the initial temperature and flux at  $r = 1$  and this has been discussed. To continue with the numerical solution till the droplet is evaporated completely, analytical solution has been developed near  $r = 0$ . This has been done also to maintain the second-order accuracy of temperature, interface and total evaporation time.

The solidification in a PCM storage with fins has been considered in [775]. On making some assumptions in the formulation, in an one-dimensional approach a coupled system of equations results between temperature and moving boundary. Temperature is obtained in a series involving moving boundary and moving boundary is obtained proportional to square root of temperature. Numerical solution has been obtained for a similar two-dimensional problem using enthalpy scheme and results are presented for several cases.

An analytical-numerical solution has been obtained in [776] for a PCM-air heat exchanger in which air flows across the tube bank and the PCM melts and flows inside tubes. Energy equation at the phase-change boundary has not been considered. The formulation involves heat transfer rate, PCM melt fraction and temperature. After assuming suitable expressions for temperature and heat transfer rate, differential equation has been obtained for melt fraction. On obtaining an analytical solution for one tube, the solution is extended for multiple tubes with some assumptions. Considerable numerical work has been done and analytical solution has been compared with numerical solution.

A simple problem of two-dimensional flow coupled with heat transfer has been considered in [777]. There is no phase change. The problem is reduced to a nonlinear second-order differential equation after making some assumptions and defining a similarity variable. The solution of this ordinary nonlinear differential equation has been obtained with the help of HAM. The set of base functions have been suitably chosen which are of the form  $\{\eta^p \exp(-s\eta), p \geq 0, s \geq 0\}$ .  $\eta$  is a dimensionless coordinate which is a similarity variable. The solution obtained by HAM has been compared with analytical solution obtained using DTM-Pade' (see reference [40] in [777]). Considerable numerical work has been done for different parameters.

The main consideration in [778] is to study the behaviour of the two-phase mushy region in the one-dimensional solidification of a binary melt. By fixing the reference frame at the solidus boundary, the solidification process becomes stationary. The steady-state formulation in the mushy region involves concentration of the impurity, bulk fraction of the solid phase, the diffusion equations for the concentration and the coupling equation between temperature and concentration. The two steady-state equations in the mushy region formulation have been considered in sufficient generality by taking several factors into consideration such as temperature-dependent solute-diffusion coefficient. The Soret effect has also been included. The effects of solid and liquid phases on mushy region have been taken into account in some approximate way by making some assumptions and not by solving their differential equation formulations. The effect of parameters has been analysed and the theory demonstrates that the mushy region can be treated as a self-similar object. The mushy region formulation includes the effects of many parameters and is fairly general. Numerical results have been presented and the effect of different parameters is discussed. Numerical scheme is not discussed. In [779] constitutional supercooling has been considered and discussed in the mushy region solidification in a binary alloy solidification. The emphasis is on the calculation of supercooling and an appropriate model formulation which throws some light on the complex structure of the nonequilibrium mushy region. In both [778, 779] several thermo-physical parameters concerning temperature and concentration have been considered based on several theoretical considerations which have not been discussed in this book. Both these references provide rich information about thermodynamical and metallurgical aspects of the problem.

In the one-dimensional problem considered in [780], molten droplet is falling on a cold solid surface. The solid-liquid phase change is taking place. The velocity of liquid which is moving along the cold surface depends on the supercooling and the phase-change velocity and the later is an unknown function of supercooling. The energy balance at the moving interface does not involve the temperature gradient of liquid but it is replaced by a convective heat transfer term. We skip some details here and mention that a second-order ordinary differential equation for the phase-change velocity is obtained. The solution of moving boundary has been obtained in terms of a similarity variable which is space variable divided by the unknown function of supercooling. Analytical and numerical solutions have been obtained. An analytical solution of a simple one-dimensional problem has been obtained first and this self-similar solution is taken as a base for numerical solution of a two-dimensional binary alloy solidification in [780]. A criterion for equilibrium phase change to nonequilibrium phase change has been discussed.

A discrepancy in the prediction of geometric models of crystal growth and experimental observations exists. The authors in [781] have suggested a simplified description of interfacial evolution which reproduces the experimental behaviour. In authors' view the incompleteness of the geometrical model of crystal growth results because they are essentially kinematic in nature. They cannot be applied when field variables are modified by the interface motion itself

in which the phase boundary is controlled solely by local surface properties and molecular kinematics. Refer [781] as to how the geometric model has been modified and formulated mathematically. This is an interesting work which deals with the long-time evolution aspects of crystal growth and interfacial evolution.

In [782], a variational formulation of a transient nonlinear heat conduction problem in which temperature-dependent specific heat, thermal conductivity and a negative heat source term has been considered. However, the functional for the variational formulation, whose stationary value will give nonlinear heat equation, requires the solution of an equation which is obtained by considering a different functional whose integrand is the product of temperature and flux. The integral of this function is a state variable. Sufficient conditions for the existence of the minimum of the functional have been discussed. For obtaining approximate solutions fictitious heat sources are introduced. Error estimation has been discussed and an approximate solution is obtained.

A three-dimensional cylindrical continuous casting problem has been considered in which all the three components of natural convection in the melt are present. Perturbation solution has been obtained in [783] in which Grashof number is the perturbation parameter. The analytical study suggests that due to natural convection the solidification rate grows downstream as  $z^{5/2}$  ( $z$  is axial coordinate) and is not aximuthally uniform. Numerical results have also been obtained.

The reference [784] is a good review article discussing various theoretical aspects concerning classical and weak formulations of the Stefan problem. It is a highly informative article for those who are interested in weak formulations of the topics covered in Chapters 1–11. In [784] a considerable number of useful cross-references are reported which systematically tell us about the gradual growth of the subject. The existence and uniqueness of a one-dimensional two-phase classical Stefan problem of solidification in a finite region  $0 < x < b$  have been discussed in [785]. The solid region is shrinking due to different densities of solid and liquid and is enclosed by two moving boundaries,  $rS(t)$  on the left and  $S(t)$  on the right,  $0 \leq r \leq 1$ ,  $r$  is constant. This type of problem, in a simpler form, was considered in [363]. In [785], solid temperature is a function of  $rS(t)$ . Heat equation in the liquid region remains the usual linear heat equation. Existence and uniqueness of the two-phase problem in weak sense has been proved under the assumptions that  $S(t) \in C'(0, d)$ ,  $S(0) = 0$ ,  $\dot{S}(0) > 0$  and  $\dot{S}(t) \geq 0$ ,  $d$  is the total solidification time. Maximum principle and Schauder fixed point theorem have been used in proofs.

A two-phase classical Stefan problem in  $\Omega \subset R^n$  with Gibbs-Thomson kinetic condition has been considered in [786] in which thermo-physical parameters are constant. Local existence and uniqueness of the solution in weak sense under regularity and higher-order compatibility conditions for the initial data have been discussed. Similar proofs were developed in [141] through phase-field equations. Global existence and asymptotic behaviour for the radially symmetric problem in  $\Omega \subset R^n$  have been proved in [787]. The Gibbs-Thomson kinetic condition has been considered at the interface. Source terms have been considered in both the solid and liquid phases. When source terms are nonzero then local-in-time existence of the solution has been proved. When source terms are not present, global existence of the solution has been proved and also asymptotic behaviour of the solution as  $t \rightarrow \infty$  has been studied in [787].

The question of obtaining the solution of a two-phase Stefan problem (when isotherm temperature is a constant) as the limit  $\varepsilon \rightarrow 0$  of a two-phase problem with kinetic condition prescribed as  $T_\varepsilon(S(t), t) = \varepsilon S_\varepsilon(t)$  was earlier considered in Section 4.5.3. Some results in

this case were also reported (refer [123]). In [788], the two-phase problem is considered in  $0 \leq x \leq 1$  and  $S(0) = b$ ,  $0 < b < 1$ . In the limit  $\epsilon \rightarrow 0$ , the moving boundary converges to the isothermal case as  $C^{1+\delta/2}[0, T]$ ,  $T > 0$ ,  $0 < \delta < 1$ .

In [789, 790], methods of numerical solutions of AFSs have been suggested. The analytical-numerical method proposed earlier for plane problems in [527, 528] has been extended for cylindrical and spherical problems.

The problem considered in [791] is concerned with the two-dimensional solidification of a binary alloy in which eutectic state is imposed on the eutectic front such that no species equation is needed in the pure solid state. Velocity components of the interdendritic liquid have been accounted in the formulation. The emphasis is on mushy zone in which mushy zone and liquid region have been regarded as a single body having irregular shape. The differential formulation for control volume is quite general in which most of the quantities are taken in average sense. A simplified form of energy equation has been obtained in which the specific heat is the weighted average of specific heats of solid and liquid. The weights are taken as some appropriate liquid fractions. Latent heat and product of liquid fraction in the mushy region has been taken in the form of average density. Numerical solution has been obtained. To trace the interface of the mushy zone and pure liquid, a interpolation method has been proposed.

In [792], a numerical solution of a two-dimensional solidification in a square plate has been considered. Moving boundary is fixed at the boundaries of the plate. Numerical solution using finite difference scheme has also been obtained for a cuboid. The numerical procedure is different but to describe it we require lot of space.

Small- and large-time solutions of a one-dimensional one-phase solidification problem have been obtained in [793] by assuming series solution of temperature in terms of a function of a suitable similarity variable  $\eta$  in long-time solution.  $\eta = x/(t + \delta)^{1/2}$ , where  $\delta$  is unknown, which is not determined and depends on the initial condition. The method of solution does not present anything new. Moving boundary is expressed in terms of a series of functions of time.

Solidification in a quarter of the plane has been considered in [794] by coordinate transformation in terms of similarity variables  $(\xi, \eta)$ .  $\xi$  and  $\eta$  are the ratio  $(x/t^{1/2})$  and  $(y/t^{1/2})$ , respectively. As  $\eta \rightarrow \infty$ , Neumann solution is obtained and so  $\theta$  which is the temperature of problem in  $(\xi, \eta)$  is written as  $\theta(\xi, \eta) = \text{erf}\xi/\text{erf}\lambda + \theta_1(\xi, \eta)$ ,  $\lambda$  is as in Neumann solution. It has been assumed that the temperature  $\theta_1$  can be expressed as a product of two functions of the form  $f(\xi)$  and  $g(\eta)$ . In obtaining the solutions of  $f(\xi)$  and  $g(\eta)$ , the series of repeated error functions has been suitably modified. Recursive relations for the unknown coefficients have been obtained. Numerical results have been compared with some known solutions. The solution method is interesting.

An inverse problem in  $\Omega \subset R^2$  has been considered in [795] in which the temperature at the boundary of  $\Omega$  is unknown. In the transient two-dimensional two-phase problem, the diffusion coefficient is a function of space variables and time. It is the problem of growing solid and vanishing liquid in which the region  $\Omega_S(0) \subset \Omega \subset R^2$  is occupied by solid. Freezing temperature is the phase-change temperature. It has been proved that under certain assumptions, the weak solution is unique.

Various types of mathematical models of mushy region have been considered in [796]. Mushy regions in stationary and nonstationary situations, mushy regions with vanishing heat capacity, nuclear-controlled solidification, etc., have been considered. It is a good review article.



In [797], some Stefan and Stefan-like problems have been reviewed with respect to their formulations in terms of classical solvability in several space dimensions, regularization of supercooling, dynamical contact angle, etc. A big list of references provides useful information about the developments on these topics.

The integral method of boundary characteristics in solving a Stefan problem with Dirichlet condition has been considered in [798]. On the basis of multiple integrations of the heat equation, a sequence of identical inequalities with boundary characteristics in the form of  $n$ -fold integrals of the surface temperature have been obtained. It was found that the accuracy in this method is much higher in obtained solution of temperature and moving boundary than the result obtained by other approximate methods.

In [799] an inverse two-dimensional problem of binary alloy solidification to determine numerically heat transfer coefficient at the boundary of the region when temperatures at selected points in the region are known has been considered. Solidus and liquidus temperatures are concentration dependent and Scheil model is used for concentration (from the abstract).