Chapter 9

Inverse Stefan Problems

9.1 INTRODUCTION

To understand the basic features of an *inverse problem*, which could be or not be a Stefan problem, we first consider a simple heat conduction problem of finding the temperature $T(x,t)$ satisfying the following heat equation, and the initial and the boundary conditions

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \; (x,t) \in \Omega_{t_*} = \Omega \times \{0 < t < t_* < \infty\}, \quad \Omega = [0,1], \tag{9.1.1}$$

$$T(x,0) = 0, \quad x \in \Omega, \tag{9.1.2}$$

$$\frac{\partial T}{\partial x}(0,t) = 0, \quad 0 < t < t_*, \tag{9.1.3}$$

$$T(1,t) = f(t), \quad 0 < t < t_*. \tag{9.1.4}$$

For convenience, thermophysical parameters have been taken to be unity in the above formulation. Under fairly general smoothness conditions on $f(t)$, the solution of Eqs (9.1.1)–(9.1.4) exists. From the solution $T(x,t)$ of Eqs (9.1.1)–(9.1.4), $T(0,t)$ can be obtained. Let

$$T(0,t) = g(t), \quad 0 < t < t_*. \tag{9.1.5}$$

Here, $g(t)$ could be known exactly through an analytical solution or known only approximately, for example, through an experiment or from a numerical solution. We shall call the problem (9.1.1)–(9.1.4), a *direct problem*. Consider another problem consisting of Eqs (9.1.1)–(9.1.3), (9.1.5) and from the solution of this problem (if it exists) obtain $T(1,t) = f(t)$. This is called an *inverse problem* corresponding to the direct problem (9.1.1)–(9.1.4). Several other inverse problems can be formulated corresponding to the direct problem (9.1.1)–(9.1.4). For example, an inverse problem may consist of Eqs (9.1.1), (9.1.3), (9.1.4) and the condition

$$T(x,t_0) = T_0, \quad x \in \Omega, \; 0 < t_0 < t_*, \tag{9.1.6}$$

and we are required to find $T(x,0)$.

It is clear from the above discussion that the 'input data' of an inverse problem contains some information about the input data of the direct problem. By interchanging the roles of

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some known and unknown quantities, a direct problem can be treated as an inverse problem and vice versa. However, in practical problems, in most cases, there is a quite natural distinction between a direct and an inverse problem. For example, a direct Stefan problem is concerned with finding the temperature field and the free boundary for an exactly known data and an inverse Stefan problem is generally concerned with controlling the free boundary with the help of boundary or initial data. Input data in a direct Stefan problem can be identified with the causes and the temperature and the free boundary are effects. In inverse problems, the objective is to determine causes for a desired or an observed effect. In the context of inverse Stefan problems, a cause could be an unknown thermophysical parameter, an initial condition or a boundary condition and the desired effect could be a specified free boundary or a known temperature distribution at some future time in an evolutionary system.

9.2 WELL-POSEDNESS OF THE SOLUTION

The term 'solution' cannot be used loosely in the context of solutions of either direct or inverse problems. The function space to which we expect the solution to belong should be specified. If the thermal conductivity in the heat equation is a discontinuous function of temperature, then we cannot expect $T(x,t)$ to belong to $C^1(\Omega_{t_*})$ $\left(\Omega_{t_*} = \Omega \times (0, t_*)\right)$. Admissibility of the input data should be specified together with the topology to be used for measuring continuity. We should specify the properties which a solution should possess. For example, we may ask the following questions which are relevant to both direct and inverse problems:

Does a solution exist for all admissible data? (9.2.1)

Is the solution unique for all admissible data? (9.2.2)

Does the solution depend continuously on the data? (9.2.3)

If the answers to the above equations are all positive, then irrespective of the problem being direct or inverse, we say, that the problem is *well-posed* or *correctly-posed or properly-posed* in the sense of Hadamard [225]. If any answer of Eqs (9.2.1)–(9.2.3) is negative, then the problem is said to be *ill-posed* or *incorrectly-posed* or *improperly-posed*. Direct problems may or may not be ill-posed but inverse problems are generally ill-posed. In everyday life also, finding the cause which has given a desired effect may have a nonunique answer. We shall now discuss Eqs (9.2.1)–(9.2.3) in the context of the solution of an ill-posed problem.

Nonexistence of the Solution

For an exact data, the existence of the solution is an important requirement and it can often be achieved by relaxing the requirement of the solution to belong to a desired function space. When the data is only approximately known, such as, through experiments then the problem has to be 'regularized' (cf. Section 9.3) and hence changed anyway. The regularized problem is well-posed (cf. Section 9.3).

Nonuniqueness of the Solution

The nonuniqueness of the solution is considered to be much more serious than nonexistence of the solution [226]. If a problem has several solutions, then the solution of interest can be picked up by requiring the solution to satisfy some additional conditions of quantitative or qualitative nature. For example, the solution should have the smallest norm. The qualitative information could be about the smoothness of the solution.

Continuous Dependence of the Solution on the Input Data

The rigorous definition of *continuous dependence* of the solution on the data will be given later after formulating the direct and inverse problems as operator equations. We first discuss the significance of the continuous dependence of the solution on the data.

Continuous dependence of a solution on the data is also called *stability* of the solution and nonstability may create serious numerical difficulties. Real life problems are highly nonlinear which are generally studied numerically. If one wants to study the solution of a problem by traditional numerical methods without 'regularization' and the solution does not depend continuously on the data, then the numerical method becomes unstable as a small error at any step in the numerical procedure goes on magnifying at subsequent steps. As a result of this, either the numerical solution cannot be obtained or the numerical solution obtained is erroneous. A partial remedy for this is the use of 'regularization methods', although one should keep in mind, and this is important, that no mathematical trick can make an inherently unstable problem stable. All that a regularization method can do is to recover partial information about the solution as stably as possible.

Let us now consider the inverse problem of obtaining $f(1,t)$ in Eq. (9.1.4) from the solution of Eqs (9.1.1)–(9.1.3), (9.1.5) and show that the inverse problem is unstable. If $T(x,t) \in L^2(\Omega)$ for a.a. t, then the solution of Eq. (9.1.1) can be obtained by using 'exponential Fourier transform' which is defined as

$$\hat{T}(x,\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha t} T(x,t)dt, \quad \alpha \in R, \quad i = \sqrt{-1}. \tag{9.2.4}$$

On taking the exponential Fourier transform of Eq. (9.1.1), we get

$$\frac{\partial^2 \hat{T}}{\partial x^2} = i\alpha \hat{T}, \quad x \in \Omega, \quad \alpha \in R. \tag{9.2.5}$$

In view of the boundary condition (9.1.3), the solution of Eq. (9.2.5) can be taken as

$$\hat{T}(x,\alpha) = A \cosh(x\sqrt{i\alpha}), \tag{9.2.6}$$

and on satisfying Eq. (9.1.5), we get

$$\hat{T}(x,\alpha) = \cosh(x\sqrt{i\alpha})\hat{g}(\alpha). \tag{9.2.7}$$

From Eq. (9.1.4), it is easy to obtain the equation

$$\hat{f}(x,\alpha) = \cosh(x\sqrt{i\alpha})\hat{g}(\alpha), \tag{9.2.8}$$

and the inverse transform of $\hat{f}(\alpha)$ is given by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\alpha t} \cosh(\sqrt{i\alpha})\hat{g}(\alpha)d\alpha, \quad 0 < t < t_*. \tag{9.2.9}$$

The integral in Eq. (9.2.9) will exist only if $\hat{g}(\alpha)$ decays very rapidly as $|\alpha| \to \infty$. This is because

$$\left|\cosh(\sqrt{i\alpha})\right| = \left(\sinh^2\sqrt{\frac{\alpha}{2}} + \cos^2\sqrt{\frac{\alpha}{2}}\right)^{1/2}, \tag{9.2.10}$$

goes to infinity exponentially as $|\alpha| \to \infty$ and the integral in Eq. (9.2.9) will exist only if $g(t)$ is very smooth so that it decays very fast as $|\alpha| \to \infty$. Even then arbitrarily small errors in the data $g(t)$ can lead to arbitrarily large errors in the calculation of f which is confirmed by the following example.

If the data is obtained through experiments or numerical methods are employed for the solution, errors are bound to develop. Suppose the data is perturbed as follows [226]:

$$\left.\begin{aligned} \hat{g}^{\delta}(\alpha) &= g(\alpha),\ \alpha \notin [\alpha_0, \alpha_0+1],\ \alpha \in R \text{ is arbitrary,} \\ &= g(\alpha) + \delta,\ \alpha \in [\alpha_0, \alpha_0+1]. \end{aligned}\right\} \tag{9.2.11}$$

Then $\|\hat{g}^{\delta}(\alpha) - \hat{g}(\alpha)\|_{L^2(R)} = \|g^{\delta} - g\|_{L^2(R)} = \delta$. The corresponding error in the solution of $f(t)$ can be calculated using Eqs (9.2.9), (9.2.11)

$$\begin{aligned} \|f - f^{\delta}\|^2_{L^2(R)} = \|\hat{f} - \hat{f}^{\delta}\|^2_{L^2(R)} &= \delta^2 \int_{\alpha_0}^{\alpha_0+1} \left|\cosh\sqrt{i\alpha}\right|^2 d\alpha, \\ &\geq \delta^2 \int_{\alpha_0}^{\alpha_0+1} \sinh^2\sqrt{\frac{\alpha}{2}} d\alpha \geq \delta^2 \sinh^2\sqrt{\frac{\alpha_0}{2}}. \end{aligned} \tag{9.2.12}$$

If $\alpha_0 > 0$, we have

$$\|f - f^{\delta}\| \geq \frac{\delta}{2} \exp\sqrt{\frac{\alpha_0}{2}}. \tag{9.2.13}$$

Since $-\infty < \alpha < \infty$, $\alpha_0 > 0$ could be large and even if δ is small the error in the output function f could be very large. It is interesting to note that some 'a priori' information about f can stabilize this ill-posed inverse problem. It has been shown in [227] that under suitable assumptions, *Hölder stability* (cf. Section 9.5) of the solution $f(t)$ can be obtained. The inherent reason for the ill-posedness of the above inverse problem lies in the strong smoothing properties of the operator describing the direct problem, i.e. the operator mapping f onto g.

Eq. (9.2.9) suggests that in general we should consider an operator equation of the form

$$Az = u; \quad u \in U, \ \ z \in Z. \tag{9.2.14}$$

Here, A is a linear bounded operator, $A : Z \to U$ and Z and U are suitable metric spaces or Hilbert spaces. u is called an 'input data' or an admissible data and z is called a 'solution' for the input data u, if there exists a $z \in Z$ such that $Az = u$. z need not be unique. If $N(A) = 0$ (if only zero of Z is mapped onto the zero of U under the transformation A), then z is unique and A is invertible, i.e. A^{-1} exists. An equation similar to Eq. (9.2.9) can be obtained expressing $g(t)$ in terms of $f(t)$ if $g(t)$ is unknown and $f(t)$ is known. In this case also the operator equation will be of the same form as in Eq. (9.2.14). Therefore, in an abstract setting, whether we are dealing with a direct problem or an inverse problem, the operator equation can be taken to be of the form (9.2.14). In principle, A could be a nonlinear operator but unless mentioned otherwise, A will be taken as a linear operator. Z and U will be taken as Hilbert spaces.

In some situations, we may like to identify an inverse problem distinctly from its direct problem. Suppose the direct problem can be formulated as an operator equation of the form (9.2.14). If A is invertible, then the inverse problem corresponding to the direct problem can be stated in the form

$$Gu = A^{-1}u = z; \quad u \in U, \ \ z \in Z. \tag{9.2.15}$$

In Eq. (9.2.15), z in now an input data and u is the solution. When there is no confusion, for both direct and inverse problems, we shall consider the formulation in the form of an operator equation (9.2.14).

Definition 9.2.1. Stability of the solution.

Consider the operator equation (9.2.14). Suppose the concept of solution is defined and to every element $u \in U$ there is a corresponding unique element $z \in Z$ given by the relation $z = A^{-1}u$. Let u_1 and u_2 be any two elements of U and $z_1 = A^{-1}u_1$ and $z_2 = A^{-1}u_2$ with z_1 and z_2 in Z. If for every given positive real number $\varepsilon > 0$, there exists a positive real number $\delta(\varepsilon)$ such that

$$\|u_1 - u_2\| \leq \delta(\varepsilon) \Rightarrow \|z_1 - z_2\| \leq \varepsilon, \tag{9.2.16}$$

then we say that the solution of Eq. (9.2.14) depends continuously on the data or the problem formulated in Eq. (9.2.14) is stable on the spaces (Z, U). If A^{-1} is defined for all of U, then the well-posedness of the problem is equivalent to the continuity of A^{-1}. For a long time, it was an accepted point of view in mathematical literature that every mathematical problem to be studied has to be well-posed [228]. With the development of regularization methods it is possible to obtain an approximate solution of an ill-posed problem by approximating an ill-posed problem by a well-posed problem. The most important of regularization methods is the *Tikhonov regularization* (cf. [229, 230]) but before discussing this regularization method, let us explore the possibility of obtaining an approximate solution of the operator equation (9.2.14) by some other methods.

We first consider the case when the input data is exactly known. The operator A is exactly known, is continuous, bounded and has an inverse A^{-1} which is not in general continuous. Let M be a compact subset of Z. Suppose that the input data in Eq. (9.2.14) is $u_0 \in U$ and an exact solution $z_0 \in Z$ exists but it is difficult to calculate it. Can z_0 be calculated approximately? Usually, we take for M a set of elements depending on finite number of parameters varying within finite limits in such a way that M is a closed set contained in a finite-dimensional space. Note that if Z is infinite-dimensional, then A^{-1} need not be defined on all of U, i.e. $AZ \neq U$, and secondly A^{-1} defined on $AZ \subset U$ need not be continuous. If the input data belongs to $\hat{N} = AM \subset U$, then there exists a $z_0 \in M$ and

$$\|Az_0 - u_0\|_U = \inf_{z \in M} \|Az - u_0\|_U = 0. \tag{9.2.17}$$

9.2.1 Approximate Solutions

To construct an approximate solution $z_0 \in M$ for a given u_0 (as discussed above), we construct a sequence $\{z_n\}$ of elements of M such that the sequence of numbers $\|Az_n - u_0\|_U \to 0$ as $n \to \infty$. If $\{Az_n\} \to u_0$ as $n \to \infty$ in the norm of U and if $\{z_n\} \to z_0$ in the norm of M, then some z_n for sufficiently large n can be taken as a good approximation to z_0.

If the conditions of Proposition 9.2.1 (given below) are satisfied, then the sequence $\{z_n\}$ converges and an approximate solution can be obtained. This method of finding an approximate solution is stable and the calculation of A^{-1} which is often difficult can be avoided.

Proposition 9.2.1. *Suppose that a compact (in itself) subset M of a Hilbert space Z is mapped onto a subset $\hat{N} = AM$ of a Hilbert space U. If the mapping $M \to \hat{N}$ is continuous and one-to-one, the inverse mapping $\hat{N} \to M$ is also continuous.*

For the proof of the continuity of A^{-1}, see [231]. It may be noted if the conditions of Proposition 9.2.1 are satisfied, then the problem stated in Eq. (9.2.14) is well-posed on the spaces $(M, \hat{N})$. To have a well-posed problem on the spaces (Z, U), A^{-1} should be defined on all of U and should be continuous.

Generally, the input data is available with some error. Suppose u^δ is the approximate input data which is available in the place of an exact data u_0 and we know that

$$\|u_0 - u^\delta\|_U \le \delta, \delta > 0; \quad z_0 = A^{-1}u_0, \text{ and } z^\delta = A^{-1}u^\delta. \tag{9.2.18}$$

Let $u_0, u_\delta \in \hat{N} = AM \subset U$ and the conditions in Proposition 9.2.1 are satisfied. Then there exists a sequence $\{z_n^\delta\} \to z_\delta$ such that $\|Az_n^\delta - u_\delta\| \to 0$ and some z_n^δ for large n can be taken as an approximate solution corresponding to $u_\delta \cdot z_n^\delta$ will also serve as an approximate solution of Eq. (9.2.14) even if u_0 is not exactly known. As $\delta \to 0$, $z^\delta \to z_0$.

Definition 9.2.2. Quasi-solution.

Most of the time, we do not have an effective criterion to determine whether the element u_δ in Eq. (9.2.18) belongs to the set $\hat{N} = AM$ or not but we know that $u_0 \in \hat{N}$. In such a case, we cannot write $z_\delta = A^{-1}u_\delta$ and take z_δ as the solution of Eq. (9.2.14) for a given u_δ. However, if we know that M is a compact set and A is a *completely continuous operator*, then there does exist an element $\hat{z} \in M$ such that

$$\left\|A\hat{z} - u\right\|_U = \inf \|Az - u\|_U, \quad u \in U. \tag{9.2.19}$$

$\hat{z}$ is called a *quasi-solution*. The 'Euler equation' for determining the minimum in Eq. (9.2.19) or the equation to obtain a quasi-solution has the form

$$A^*Az = A^*u, \quad A^* : U \to Z, \quad \langle Az, u\rangle = \langle z, A^*u\rangle, \tag{9.2.20}$$

where A^* is the *conjugate operator* of A. Eq. (9.2.20) is called the *normal equation* of the operator equation (9.2.14). If M is compact and $u \in \hat{N}$, then $\hat{z}$ is an exact solution. Quasi-solution may not be unique. It is possible to state the conditions under which a quasi-solution is unique and stable.

Let Q be a subset of a Hilbert space U and u an element of U. An element q of the set Q is called a 'projection' of the element u on Q, if

$$\|u - q\|_U = \inf_{p \in Q} \|u - p\|_U . \tag{9.2.21}$$

Proposition 9.2.2. *Let M be a compact set of Z and let A be completely continuous on M. A quasi-solution of $Az = u$ exists on M for $u \in U$. If the projection of each element u of U onto the set $\hat{N} = AM$ is unique, then the quasi-solution of the equation $Az = u$ is unique and depends continuously on $u \in U$.*

For some theoretical results and approximate determination of quasi-solutions, see [231]. Fundamental to the theory of approximate solutions of ill-posed problems is the notion of regularizing algorithms which are based on a regularizing family of operators which will be discussed in Section 9.3.

9.3 REGULARIZATION

9.3.1 The Regularizing Operator and Generalized Discrepancy Principle

In Section 9.2.1 some stable methods were described to obtain approximate solutions of Eq. 9.2.14 for a given exact input data as well as for a given approximation of the input data. It was assumed that the class of possible solutions of Eq. (9.2.14) is a compact subset M of Z and A^{-1} is continuous on $\hat{N} = AM$ or A is completely continuous on Z. In a number of problems, the set M (the set Z can also be considered) is not compact and the approximation u_δ of the input data u_0 in Eq. (9.2.14) may take u_δ ($\|u_0 - u_\delta\| \leq \delta$) outside the set $\hat{N}$ (the set U can also be considered). Such problems are *genuinely ill-posed* and a new approach for the solutions of such problems was developed in [229, 230].

As mentioned earlier, there is no trick by which a genuinely ill-posed problem can be made well-posed. But we can approximate an ill-posed problem by a family of neighbouring well-posed problems. This is done by constructing a family of *regularization operators* $\{R_\alpha\}$, where each R_α is a continuous operator, dependent on a parameter α, and $R_\alpha : U \to Z$.

To motivate the definition of a regularizing operator, we make the following assumptions:

(1) The operator $A : Z \to U$ is only approximately known and its approximation $A_\eta : Z \to U$ is also a linear bounded operator just as A is. Further,

$$\left\|A_\eta - A\right\| \leq \eta, \quad A_0 = A. \tag{9.3.1}$$

(2) A^{-1} is not continuous on the set AZ.

(3) The set of all possible solutions $M \subset Z$ is not compact.

(4) The input data $u_0 \in U$ is only approximately known and its approximation $u_\delta \in U$ is such that

$$\|u^\delta - u_0\| \leq \delta > 0. \tag{9.3.2}$$

u^δ is called *noisy data*, and δ is the error, also called *noise level*.

(5) A solution $z_0 \in M$ of Eq. (9.2.14) for the exact data $u_0 \in U$ exists such that $z_0 = A^{-1}u_0$.

The initial information consists of $\{u_\delta, A_\eta, \delta, \eta\}$. Note that even if a $z_\delta \in Z$ exists such that $Az_\delta = u_\delta$, z_δ may not be a stable solution as we have not assumed the continuity of A^{-1}. From the initial information we are required to obtain an element $z_\beta \in Z$, $\beta = (\delta, \eta)$, such that as $\beta \to 0$, $z_\beta \to z_0 = A^{-1}u_0$. This is possible provided a regularizing operator for A^{-1} exists. In general terms, regularization is the approximation of an ill-posed problem by a family of neighbouring well-posed problems. The precise definition of a regularizing operator is given below, firstly, for the case when A is exactly known.

Definition 9.3.1. Regularization operator.

Let $A : Z \to U$ be a linear bounded operator between Hilbert spaces Z and U, and let $\alpha_0 \in (0, +\infty]$. The initial information consists of $\{A, u^\delta, \delta\}$, and for every $\alpha \in (0, \alpha_0)$

$$R_\alpha : U \to Z \tag{9.3.3}$$

is a continuous (not necessary linear) operator. The family $\{R_\alpha\}$ is called a *regularization* or a *regularization (regularizing) operator* for A^{-1} or A^{-1} is said to be *regularizable*, if, for all $u \in D(A^{-1})$ (domain of A^{-1}), there exists a 'parameter choice rule' $\alpha = \alpha(\delta, u^\delta)$ such that

$$\lim_{\delta\to 0} \sup \left\{ \|R_\alpha u^\delta - A^{-1}u\|;\ u^\delta \in U,\ \|u - u^\delta\| \le \delta \right\} = 0, \tag{9.3.4}$$

holds. Here

$$\alpha : R^+ \times U \to (0, \alpha_0) \tag{9.3.5}$$

is such that

$$\lim_{\delta\to 0} \sup \left\{ \alpha(\delta, u^\delta);\ u^\delta \in U,\ \|u - u^\delta\| \le \delta \right\} = 0. \tag{9.3.6}$$

For a specific $u \in D(A^{-1})$, a pair (R_α, α) is called a (convergent) regularization method for solving Eq. (9.2.14) if Eqs (9.3.4), (9.3.6) hold (cf. [226]).

If the operator A is only approximately known with an approximation A_η satisfying Eq. (9.3.1), then in Definition 9.3.1, replace A by A_η and the mapping in Eq. (9.3.5) should be defined as

$$\alpha = \{(\beta, u^\delta);\ \beta = (\delta, \eta),\ \delta > 0,\ \eta > 0,\ \|u^\delta - u\| \le \delta\} \to (0, \alpha_0). \tag{9.3.7}$$

In view of Eq. (9.3.7), we shall have $\beta = (\delta, \eta) \to 0$ in Eqs (9.3.4), (9.3.6).

A *regularization method* consists of constructing a family of regularization operators $\{R_\alpha\}$ and a parameter-choice rule which is convergent in the sense that if the *regularization parameter* α is chosen according to that rule, then the regularized solutions converge in the norm of Z as the error δ in the input data tends to zero (when the operator is exactly known), or as $\beta = (\delta, \eta) \to 0$ when the operator is only approximately known. This convergence is assumed for any collection of noisy input data compatible with the noise level δ and any $u \in U$.

If a regularization method exists for the problem defined by Eq. (9.2.14), then there exists $z_\alpha^\delta \in Z$, $\alpha = (\delta, u^\delta)$ such that

$$z_\alpha^\delta = R_\alpha u^\delta, \tag{9.3.8}$$

and as $\delta \to 0$, we have $z_\alpha^\delta \to z_0 \in Z$, $Az_0 = u_0$. If instead of A only A_η is known and a regularization method exists, then there exists $z_\alpha^\beta \in Z$, $\alpha = \alpha(\beta, u^\delta)$, such that

$$z_\alpha^\beta = R_\alpha u^\delta, \tag{9.3.9}$$

and as $(\delta, \eta) \to 0$, we have $z_\alpha^\beta \to z_0 \in Z, Az_0 = u_0$.

If the operators $\{R_\alpha\}$ are linear (linearity was not imposed earlier on R_α), then the family $\{R_\alpha\}$ is called a family of linear regularization operators and the corresponding method of obtaining a regularized solution is called a *linear regularization method*. For nonlinear problems, the operator A could be nonlinear. The theory of linear ill-posed problems is very well developed and for both linear and nonlinear regularization methods, we refer the reader to [226].

The parameter-choice rule depends not only on u^δ but also on the exact input data u_0. Since u_0 is generally not known, to take into account the dependence of α on u_0, we should have some qualitative information about the input data such as smoothness properties. The parameter α depends on the operator A also. It may be noted that the regularizing operator $\{R_\alpha, \alpha\}$ is not unique.

Is it possible to construct a stable approximate solution of an ill-posed problem if δ is unknown but u_δ is known and it is known that $\|u_\delta - u_0\| \to 0$ as $\delta \to 0$? The answer is 'negative' for an ill-posed problem but 'positive' for a well-posed problem.

Definition 9.3.2. An a priori parameter-choice rule.

If the parameter α depends only on δ and not on u^δ, then α is called an *a priori parameter-choice rule* and we write $\alpha = \alpha(\delta)$. If α is not an a priori parameter-choice rule, then it is called an *a posteriori parameter-choice rule*.

An a priori parameter-choice rule does not depend on the actual computation and can be devised before the actual computations start.

Definition 9.3.3. Least-squares solution of $Az = u$.

Let $A : Z \to U$ be a bounded linear operator and Z and U be Hilbert spaces. An element $\overset{0}{z} \in Z$ is called a *least-squares solution* of $Az = u$ for a given $u \in U$, if

$$\|A \overset{o}{z} - u\| = \inf_{z \in Z} \{\|Az - u\|\}. \tag{9.3.10}$$

If A is absolutely continuous, then the infimum exists which can be obtained by solving the normal equation (9.2.20).

Definition 9.3.4. Best-approximate solution of $Az = u$.

An element $\hat{z} \in Z$ is called a *best-approximate solution* of $Az = u$ if $\hat{z}$ is a least-squares solution and

$$\|\hat{z}\| = \inf\{\|\overset{0}{z}\|;\ \overset{0}{z} \text{ is least-squares solution of } Az = u\}. \tag{9.3.11}$$

9.3.2 The Generalized Inverse

In [226], the operator-theoretic approach for constructing regularizing operators is based on the notion of the *Moore–Penrose generalized inverse* (MP-generalized inverse) which we shall denote by $\hat{A}$. The MP-generalized inverse $\hat{A}$ of $A \in \mathcal{L}(Z, U)$ (the set of linear bounded operators) is defined by restricting the domain and range of A in such a way that the resulting restricted operator is invertible; its inverse is then extended to its maximal domain. In Definition 9.3.1, the generalized inverse $\hat{A}$ can be used in the place of A^{-1}.

In simple terms (for the rigorous definition, see [226]), an operator $\hat{A}$ is the *MP-generalized inverse* of $A \in \mathcal{L}(Z, U)$ if and only if it has the following properties:

$$\begin{aligned}
&(i)\ \ A\hat{A}A = A.\\
&(ii)\ \ \hat{A}A\hat{A} = \hat{A}.\\
&(iii)\ A\hat{A} = (A\hat{A})^*.\\
&(iv)\ \hat{A}A = (\hat{A}A)^*.
\end{aligned} \tag{9.3.12}$$

The MP-generalized inverse always exists, is linear and unique. If A is nonsingular, then A^{-1} and $\hat{A}$ are the same. We mention below some of the results associated with the MP-generalized inverse and the operator A (cf. [226]).

Proposition 9.3.1. *MP-generalized inverse $\hat{A}$ has a closed graph $g_r(\hat{A}) = \{(u, \hat{A}u);\ u \in D(\hat{A})\}$. Furthermore, $\hat{A}$ is bounded (i.e. continuous) if and only if $R(A)$ (range of A) is closed.*

Proposition 9.3.2. *Let $u \in D(\hat{A})$. Then $Az = u$ has a unique best-approximate solution $\hat{z}$, which is given by*

$$\hat{z} = \hat{A}u. \tag{9.3.13}$$

Note that $\hat{z}$ is not a well-posed solution. The set of all least-squares solution is $\hat{z} + N(A)$.

Proposition 9.3.3. *Let $u \in D(\hat{A})$. Then, $z \in Z$ is the least-squares solution of $Az = u$ if and only if the normal equation (9.2.20) holds.*

Let $\hat{z} = \hat{A}u$ be a minimum norm solution (best-approximate solution) of $Az = u$ then $A^*A\hat{z} = A^*u$ (from Proposition 9.3.3). So we have $\hat{A} = (A^*A)^\wedge A^*$.

Proposition 9.3.4. *For all $\alpha > 0$, let R_α be a continuous (possibly nonlinear) operator. Then the family $\{R_\alpha\}$ is a regularization for $\hat{A}$ if,*

$$R_\alpha \to \hat{A},\ \textit{pointwise on } D(\hat{A}) \textit{ as } \alpha \to 0. \tag{9.3.14}$$

In this case, there exists, for every $u \in D(\hat{A})$, an 'a priori' parameter-choice rule α such that (R_α, α) is a convergent regularization method for solving $Az = u$.

The converse of Proposition 9.3.4 holds in the following sense: if $\{R_\alpha, \alpha\}$ is a convergent regularization method, then

$$\lim_{\delta \to 0} R_{\alpha(\delta, u^\delta)} u = \hat{A}u \tag{9.3.15}$$

holds for all $u \in D(\hat{A})$.

The regularizations are pointwise approximations of MP-generalized inverse of A.

9.3.3 Regularization Methods

The following questions can be asked about a regularization operator:

(i) Under what conditions can an ill-posed problem be regularized?

(ii) How can a regularization operator be constructed?

(iii) How can one obtain an approximate solution with the help of a regularizing operator?

(iv) How does one perform the steps in the regularizing algorithm in an optimal way?

We shall discuss here briefly the questions (i) and (ii) raised above in the same order. For detailed discussion, see [226, 230–232]. The third question has been discussed in Sections 9.6 and 9.7 by constructing some approximate solutions. The fourth question is more relevant to the actual numerical computations.

(i) Just as there is no unique regularizing operator, there is no unique method to construct it. Associated with $\{R_\alpha\}$ is also a parameter-choice rule. The conditions under which the regularization of an ill-posed problem is possible depend on the regularization method used and the properties of the operator A. However, a general statement can be made that if the operator A in Eq. (9.2.14) is linear, continuous and bijective (one–one and onto), the resulting problem is regularizable.

(ii) Depending on the problem under consideration, several regularization methods have been developed in the literature. We describe here some of them briefly. The applications of these methods will be discussed in connection with the regularization of some specific inverse Stefan problems considered in Section 9.7.

We describe below some regularization methods.

(a) A variational method: Tikhonov regularization

We first introduce *Tikhonov-regularization method* and the motivation behind it will be discussed later. Let $A : Z \to U$, where Z and U are Hilbert spaces, and let $D \subset Z$ be a closed convex set of constraints of the problem ($D = Z$ if there are no constraints), and $0 \in D$. Consider the problem of minimization of a functional $P^\alpha(z)$, called a *smoothing functional*, defined as

$$P^\alpha(z) = \|A_\eta z - u^\delta\|_U^2 + \alpha\,\|z\|_Z^2\,. \tag{9.3.16}$$

Here, $\alpha > 0$ is a regularization parameter and A_η and u^δ are defined as in Eqs (9.3.1), (9.3.2), respectively. The variational problem is to find

$$\inf_{z\in D} P^\alpha(z). \tag{9.3.17}$$

Proposition 9.3.5. *If A and A_η are linear bounded operators and $\alpha > 0$, then the variational problem given by Eq. (9.3.17) is solvable and has a unique solution $z_\beta^{\alpha(\beta)} \in D$, $\beta = (\delta, \eta)$, and*

$$\|z_\beta^\alpha\| \le \|u^\delta\|/(\sqrt{\alpha}). \tag{9.3.18}$$

For $\alpha > 0$, $P^{\alpha}(z)$ is strictly convex as $((P^{\alpha}(z))''z, z) = (2A_{\eta}^{*}A_{\eta}\, z + 2\alpha z, z) \geq 2\alpha \, \|z\|^{2}$ for $z \in Z$ and coercive as $\lim\limits_{\|z\| \to \infty} P^{\alpha}(z) = +\infty$. Therefore there exists a unique element $z_{\beta}^{\alpha(\beta)} \in Z$ which minimizes $P^{\alpha}(z)$ (see Theorem 7.2.1). If z_{β}^{α} is an interior point of D, then

$$\left[P^{\alpha}(z_{\beta}^{\alpha})\right]' z = 0, \quad \forall\, z \in Z. \tag{9.3.19}$$

Here, dash denotes *Fréchet derivative* (see Appendix D). It can be seen that

$$\left[P^{\alpha}\left(z_{\beta}^{\alpha}\right)\right]' z = 2(A_{\eta}^{*}A_{\eta}z_{\beta}^{\alpha} - A_{\eta}^{*}u^{\delta} + \alpha z_{\beta}^{\alpha},\, z), \quad \forall\, z \in Z. \tag{9.3.20}$$

On using Eq. (9.3.20) in Eq. (9.3.19), we get

$$A_{\eta}^{*}A_{\eta}z_{\beta}^{\alpha} + \alpha z_{\beta}^{\alpha} = A_{\eta}^{*}u^{\delta}. \tag{9.3.21}$$

Note that z_{β}^{α} may not be a 'regularized solution' of $Az = u$ unless some further conditions are imposed on A and A_{η}. We have also not given any parameter-choice rule for α. Since $0 \in D$, and z_{β}^{α} gives the infimum, we have

$$P^{\alpha}\left(z_{\beta}^{\alpha}\right) \leq P^{\alpha}(0), \tag{9.3.22}$$

and so Eq. (9.3.18) holds. Proposition 9.3.6 (given after Eq. 9.3.33) suggests that if the operators A and A_{η} satisfy certain conditions and a parameter-choice rule for α is defined, then a regularized solution can be obtained.

We shall now discuss the motivation for considering the minimization of the functional given in Eq. (9.3.16) which comes from the notion of a 'stabilizing functional'. Let $F(z)$ be a continuous, nonnegative functional $F : Z \to U$ (take $U = R$) defined on a subset Z_1 of Z which is everywhere dense in Z. We assume that the following conditions hold:

1. z_0 belongs to the domain of Z_1 and $Az_0 = u_0$, $u_0 \in U$ where u_0 is the exact data.

2. For every positive real number d, the set of elements z of Z_1 for which $F(z) \leq d$ is a compact subset of Z_1.

A functional $F(z)$ satisfying the above conditions is called a *stabilizing functional*.

Let

$$Z_{1,\delta} = Q_{\delta} \cap Z_1; \quad Q_{\delta} \equiv \left\{z : \|Az - u^{\delta}\| \leq \delta\right\}. \tag{9.3.23}$$

Here, we want to consider only those elements of Q_{δ} on which $F(z)$ is defined. It can be proved (cf. [231]) that there exists a $z^{\delta} \in Z_{1,\delta}$ such that $z^{\delta} = \inf\limits_{z \in Z_{1,\delta}} F(z)$ and $z^{\delta} = \overline{R}_{\delta}(u^{\delta})$, where, $\overline{R}_{\delta}$ is a regularizing operator and as $\delta \to 0$, $z^{\delta} \to z_0$ and $Az_0 = u_0$.

We shall now obtain the infimum of $F(z)$ over Z_1. Let

$$z_{F}^{\delta} = \inf_{z \in Z_1} F(z). \tag{9.3.24}$$

For simplicity, we take z_F^δ to be the unique minimum but in general it be a set M_0. If z_F^δ belongs to $Z_{1,\delta}$, then we take z_F^δ to be the stable solution on $Z_{1,\delta}$ and $z_F^\delta = z^\delta$. If z_F^δ does not belong to $Z_{1,\delta}$, then $\|Az_F^\delta - u^\delta\| > \delta$. In this case z_F^δ can be obtained by solving a constrained minimization problem which is discussed below.

The stabilizing functional $F(z)$ (defined above), is called *quasi-monotonic* if, for every element $z_M \in Z_1$ that does not belong to the set M_0, every neighbourhood of it includes an element z_1 of Z_1 such that $F(z_1) < F(z_M)$. It can be proved (cf. [231]) that if $F(z)$ is quasi-monotonic on the set $Z_{1,\delta}$ and $M_0 \cap Z_{1,\delta}$ is empty then the infimum of $F(z)$ is attained at an element z_M^δ for which $\|Az_M^\delta - u^\delta\| = \delta$. We can use this result to find the minimum in Eq. (9.3.24) on the set Z_1 under the constraint that the minimizer $z \in Z$ which we are seeking satisfies the condition

$$\|Az - u^\delta\| = \delta. \tag{9.3.25}$$

The method of *Lagrange multipliers* (cf. [82]) can be used to study this minimization problem with a constraint and a functional of the form $P^\alpha(z)$ as in Eq. (9.3.16) can be considered in which we take $\alpha F(z)$ instead of $\alpha\,\|z\|^2$ and the parameter α is determined from the condition

$$\|Az_\alpha^\delta - u^\delta\| = \delta. \tag{9.3.26}$$

Here, z_α^δ is the minimizer of $P^\alpha(z)$ and A is exactly known. Proposition 9.3.6 given later suggests there is no necessity of considering a constrained minimization problem and z_α^δ can be regarded as the result of applying a regularization R_α such that $z_\alpha^\delta = R_\alpha u^\delta$. The parameter α is determined by a parameter-choice rule.

If instead of the operator A only its approximation A_η is known, then once again, we consider a constrained minimization problem for the functional $P^\alpha(z, u_\delta, A_\eta)$, where

$$P^\alpha(z, u_\delta, A_\eta) = \left\|A_\eta z - u_\delta\right\|^2 + (\alpha - \eta^2)F(z). \tag{9.3.27}$$

The parameter α is to be determined from the condition

$$\|A_\eta z_\alpha^\delta - u^\delta\|^2 = \delta^2 + \eta^2 F(z), \tag{9.3.28}$$

where z_α^δ minimizes the functional given in Eq. (9.3.27).

The choice of a stabilizing functional $F(z)$ is often prompted by the nature of the problem and in a number of cases, more than one choice is possible. The stabilizing functional considered in (9.3.16) is $\|z\|_Z^2$. If the Hilbert space $W^{p,2}(\Omega)$, $\Omega = \{x; a \le x \le b\}$ is considered as the solution space, then we can consider the stabilizing functional $F(z)$ as

$$F(z) = \int_a^b \sum_{r=0}^{p} q_r(x) \left(\frac{d^r(z)}{dx^r}\right)^2 dx, \quad z = z(x), \quad z \in W^{p,2}(\Omega)\,. \tag{9.3.29}$$

Here, $q_0(x), q_1(x), \ldots, q_{p-1}(x)$ are given nonnegative continuous functions and $q_p(x)$ is a given positive continuous function. Stabilizers of the form where $q_r(x) \ge 0$, for

$r = 0, 1, \ldots, p-1$, and $q_p(x) > 0$ are called *stabilizers of pth order*. If all the functions $q_r(x)$ are constant, then we have a stabilizer of pth order with constant coefficients or a *Tikhonov stabilizer*.

To determine the parameter α in the Tikhonov regularization, a ‘generalized discrepancy principle’ was first suggested in [233, 234] and later on modified in [235]. We give below few definitions which are related to the definition of the generalized discrepancy principle.

Definition 9.3.5. Incompatibility measure.

Let the set D, the operators A, A_η and u_0, u^δ be the same as defined in Proposition 9.3.5. Then an *incompatibility measure* $\mu_\beta(A_\eta, u^\delta)$ of Eq. (9.2.14) is defined as

$$\mu_\beta(A_\eta, u^\delta) = \inf_{z\in D} \|A_\eta z - u^\delta\|, \quad \beta = (\eta, \delta). \tag{9.3.30}$$

If $u^\delta \in \overline{A_\eta D}$ (bar stands for the closure in the space U), then $\mu_\beta(A_\eta, u^\delta) = 0$.

If $\|u^\delta - u_0\| \le \delta$, $Az_0 = u_0$, $z_0 \in D$, $\|A_\eta - A\| \le \eta$, then $\mu_\beta(u^\delta, A_\eta) \to 0$ as $\beta \to 0$. This can be proved by using the following result.

$$\mu_\beta\left(u^\delta, A_\eta\right) = \inf_{z\in D} \|A_\eta z - u^\delta\| \le \|A_\eta z_0 - u^\delta\| \le \|A_\eta z_0 - Az_0 + Az_0 - u^\delta\| \le \delta + \eta\|z_0\|. \tag{9.3.31}$$

Definition 9.3.6. Discrepancy.

The norm

$$\|A_\eta z_\beta^\alpha - u^\delta\| \tag{9.3.32}$$

is called *discrepancy*. Here, $z_\beta^\alpha = R_\alpha\left(u^\delta\right)$, R_α is a regularizing operator, $\alpha = (\beta, u^\delta)$ and $\beta = (\eta, \delta)$.

Definition 9.3.7. Generalized discrepancy.

The *generalized discrepancy* denoted here by $\rho_\beta(\alpha)$ is defined as

$$\rho_\beta(\alpha) = \|A_\eta z_\beta^\alpha - u^\delta\|^2 - \left(\delta + \eta\|z_\beta^\alpha\|\right)^2. \tag{9.3.33}$$

Here, z_β^α is the solution of Eq. (9.3.21). In earlier works [233, 234], the definition of generalized discrepancy included a term of incompatibility measure also (which can also be computed with an error) but later on in [235] it was pointed out that in the definition of generalized discrepancy the term of incompatibility measure can be taken to be zero even if $u^\delta \notin \overline{A_\eta D}$.

Definition 9.3.8. Generalized discrepancy principle.

The *generalized discrepancy principle* consists of the following rules:

(1) If the condition $\|u^\delta\| > \delta$ is not fulfilled, i.e. $\|u^\delta\| \le \delta$ then we take $z_\beta^\alpha = 0$ (z_β^α is the solution of Eq. 9.3.21) as an approximate solution of $Az = u_0$. If $z_\beta^\alpha = 0$, then the relation $\|A_\eta z_\beta^\alpha - u^\delta\| \le \delta$ implies $\|u^\delta\| \le \delta$.

(2) If the condition $\|u^\delta\| > \delta$ is satisfied, then we proceed as follows:
  (a) If some $\alpha^* > 0$ exists such that $\rho_\beta(\alpha^*) = 0$, then we take $z_\beta^{\alpha^*}$ as the approximate solution of $Az = u_0$.
  (b) If $\rho_\beta(\alpha) > 0$ for all $\alpha > 0$, then we take $z_\beta = \lim\limits_{\alpha\to 0} z_\beta^\alpha$ as the approximate solution of $Az = u_0$.

Proposition 9.3.6. *Let A be a bijective operator (one–one and onto), $A : Z \to U$, and A_η be a bounded linear operator, $A_\eta : Z \to U$, such that $\|A - A_\eta\| \leq \eta$. Further, $\|u_0 - u^\delta\| \leq \delta$, $z_0 = Au_0$, $z_0 \in D \subset Z$, D is a closed convex set and $0 \in D$, $\beta = (\eta, \delta)$ and $\alpha = (\beta, \delta)$. Then $\lim\limits_{\beta\to 0} z_\beta^{\alpha^{**}} = z_0$, where $z_\beta^{\alpha^{**}}$ is the solution of Eq. (9.3.21) and it is chosen according to the generalized discrepancy principle stated above. The solutions obtained as above with the help of the generalized discrepancy principle are regularized solutions of the equation $Az = u_0$.*

If A is not bijective, then the approximate regularized solution z_β (see Definition 9.3.8) converges to $\bar{\bar{z}}_0$ on D, where, $\bar{\bar{z}}_0$ is the solution of the normal equation (9.2.21). Tikhonov-regularization method is one of the many regularizing methods used for ill-posed problems. For self-adjoint operators, a class of linear regularization methods can be constructed using *spectral theory* [236]. Tikhonov-regularization method is a particular case of the continuous regularization methods constructed with the help of the spectral theory approach (cf. [226]).

In his original papers [229, 230], Tikhonov considered a regularization method by considering a more general functional of the form

$$\|Az - u\|^2 + \alpha \|Mz\|^2, \quad z \in D(M). \tag{9.3.34}$$

Here, M is an operator which could be a differential operator such as a second derivative operator and $D(M)$ is the domain of the operator. In this case, the least-squares solution $\hat{z}_M$ of $Az = u$ minimizes a different (semi) norm, namely,

$$\|M\hat{z}_M\| = \inf\{\|M \overset{0}{z}\|, \overset{0}{z} \text{ is the least-squares solution of } Az = u\}. \tag{9.3.35}$$

This leads to the notion of a weighted generalized inverse of A (cf. [226]).

(b) Maximum entropy regularization

Suppose the input data are defined by a random variable X which could be discrete or continuous and $x_1, x_2, \ldots, x_n$ be its values (in the discrete case) with probabilities given by

$$P^*(\{x_i\}) = p_i^*, \quad \sum p_i^* = 1. \tag{9.3.36}$$

P^* is called a 'prior distribution'. Due to some additional information available, a new probability distribution is assigned to the random values which is given by

$$P(\{x_i\}) = p_i, \quad \sum p_i = 1. \tag{9.3.37}$$

Let $I(p, p^*) = I(p_1, p_2, \ldots, p_n; p_1^*, p_2^*, \ldots, p_n^*)$ denote the information of P relative to P^*. $I(p, p^*)$ must satisfy some conditions such as continuity, invariance under changes of labelling of the possible values (see [226, 237] for further conditions). An appropriate form of $I(p, p^*)$ satisfying all the axioms can be taken as

$$I(p_1, p_2, \ldots, p_n; p_1^*, p_2^*, \ldots, p_n^*) = \gamma \sum_{i=1}^{n} p_i \log(p_i/p_i^*), \tag{9.3.38}$$

where γ is a constant. The *maximum entropy method* consists of maximizing the entropy

$$E(p, p^*) = -\gamma \sum p_i \log\left(p_i/p_i^*\right), \tag{9.3.39}$$

subject to the constraint that the sum of all the probabilities equals one. In the continuous case, instead of Eq. (9.3.37), we shall have

$$\int_a^b x(t)dt = 1, \tag{9.3.40}$$

where $x(t)$ is the probability density function of X. The data y is given by an (possibly nonlinear) operator equation

$$F(x) = y, \tag{9.3.41}$$

defined on a suitable function space, e.g. on $L^2\,[a, b]$. The entropy functional to be maximized in the continuous case is defined as

$$E(x, x^*) = -\gamma \int_a^b x(t) \log\left(x(t)/x^*(t)\right) dt, \tag{9.3.42}$$

subject to the constraint (9.3.40). Using Lagrange multipliers, we are led to a problem of minimizing the functional

$$\|F(x) - y\|^2 + \alpha \int_a^b x(t) \log\left(x(t)/x^*(t)\right) dt. \tag{9.3.43}$$

(c) Regularizing methods for equations of convolution integral type

The solutions of many physical problems are obtained by solving integral equations of the first kind and among them one often encounters an equation of the convolution type which is written as $v(t) * z(t) = u(t)$ and which stands for

$$\int_{-\infty}^{\infty} v(t-\tau)z(\tau)d\tau = u(t). \tag{9.3.44}$$

Regularizing operators for operator equations of *convolution integral* type can be constructed by using integral transforms such as Laplace, Fourier and Mellin. Some of these regularizing operators will be discussed in Section 9.6 in the context of some inverse Stefan problems.

(d) Regularization by projection

The numerical solutions of ill-posed problems can be obtained by regularization methods using numerical schemes such as finite-difference or finite element, collocation, Galerkin or

Ritz approximation. The numerical calculations can be realized only in finite-dimensional spaces. One approach in this direction is the *regularization by projection* (cf. [238]) in which we try to find approximately the best-approximate solution $\hat{z}$ of $Az = u$, in a finite-dimensional subspace of the space Z.

Let there be a sequence $\{Z_n\}_{n=1}^{\infty}$ of finite-dimensional subspaces of Z such that

$$Z_1 \subset Z_2 \subset Z_3 \subset \cdots, \tag{9.3.45}$$

whose union is dense in Z. If z_n is the least-squares minimum norm solution of $Az = u$ in Z_n, then

$$z_n = \hat{A}_n u, \tag{9.3.46}$$

where, $A_n = AO_n$, O_n is the orthogonal projector onto Z_n (A is a bounded linear operator and the range of O_n is closed, so the range of A_n is closed). Since the range of A_n is closed, $\hat{A}_n$, the generalized inverse of A_n is bounded (see Proposition 9.3.1). $\hat{A}_n$ is linear and bounded, hence, continuous. Therefore z_n is a stable approximation of $\hat{z}$. However, as the following proposition suggests, $z_n \to \hat{z}$ only under certain conditions.

Proposition 9.3.7. *Let $u \in D(\hat{A})$ ($\hat{A}$ is the generalized inverse of A) and let z_n be as above. Then*

(i) *$z_n \to \hat{z}$ if and only if $\{\|z_n\|\}$ is bounded. Here, $\hat{z} = \hat{A}u$,*

(ii) *$z_n \to \hat{z}$ if and only if $\lim_{n\to\infty} \sup \|z_n\| \le \|\hat{z}\|$.*

Proposition 9.3.8. *Let $u \in D(\hat{A})$ and z_n be as above. If the condition*

$$\lim_{n\to\infty} \sup \|(\hat{A}_n)^* z_n\| = \lim_{n\to\infty} \sup \|(\hat{A}_n^*) z_n\| < \infty, \tag{9.3.47}$$

holds, then $z_n \to \hat{z}$.

In the *dual least-squares method* described below, the convergence $z_n \to \hat{z}$ is always guaranteed. Consider a sequence $\{U_n\}_{n=1}^{\infty}$ of finite-dimensional subspaces of $\overline{R(A)} = N(A^*)^{\perp} \subset U$, whose union is dense in $N(A^*)^{\perp}$ (cf. [238]). Let z_n be the best-approximate solution of the equation

$$A_n z = u_n, \quad A_n := Q_n A, \quad u_n = Q_n u, \tag{9.3.48}$$

where Q_n is the orthogonal projector onto U_n.

Proposition 9.3.9. *Let $u \in D(\hat{A})$ and z_n be as above. Then $z_n = O_n \hat{z}$, where O_n is the orthogonal projector onto $Z_n := A^* U_n$, $\hat{A}_n u = O_n \hat{z}$. Moreover, $z_n \to \hat{z}$ as $n \to \infty$.*

It can be shown that $\{\hat{A}_n\}$ with A_n defined in Eq. (9.3.48) is a regularizing operator for $\hat{A}$. z_n is a stable approximation to $\hat{z}$ where $\hat{z}$ is the best-approximate solution of $Az = u$.

In the noise free case, $\hat{A}_n u$ is the best approximation in the space Z_n and $\hat{A}_n u = O_n \hat{z}$ and no further regularization is required. For the noisy data it can be shown (see [226]) that

$$\|z_n^{\delta} - \hat{z}\| \le \left\|z_n - \hat{z}\right\| + \delta/\mu_n, \tag{9.3.49}$$

where μ_n is the smallest singular value of A_n. Since the singular value of A_n decreases rapidly as n increases, the projection method should be used in conjunction with some regularization method, e.g. Tikhonov regularization. In this way spaces with larger dimensions can be used.

In obtaining the numerical solution of an ill-posed problem by Tikhonov regularization, we have to work in a finite-dimensional space Z_n as described in Eq. (9.3.46). The minimization of $P^{\alpha}(z)$ (see Eq. 9.3.16) over the space Z_n gives an approximation of $\hat{z}$. This problem is equivalent to minimizing $P_n^{\alpha}(z)$ over Z, where

$$P_n^{\alpha}(z) = \|A_n z - u^{\delta}\|^2 + \alpha \|z\|^2 . \tag{9.3.50}$$

Here, $A_n = AO_n$ and O_n is the orthogonal projector onto the subspace Z_n. If $Z_{\alpha,n}^{\delta}$ is the minimizer of $P_n^{\alpha}(z)$, then

$$z_{\alpha,n}^{\delta} = \left(A_n^* A_n + \alpha I\right)^{-1} A_n^* u^{\delta}. \tag{9.3.51}$$

It α is chosen according to the parameter-choice rule, then $z_{\alpha,n}^{\delta}$ is a regularized solution.

9.3.4 Rate of Convergence of a Regularization Method

In obtaining numerical solutions by regularization methods, the rate at which a regularization method converges plays an important role or in other words, an *optimal regularization method* should be used for faster convergence. The rate of convergence of the regularization method (R_α, α) for which $z_\alpha = R_\alpha u$ holds is defined (u is exactly known) as the rate with which

$$\|z_\alpha - \hat{z}\| \to 0 \ \text{ as } \alpha \to 0, \tag{9.3.52}$$

or the rate (if u is approximately known) with which

$$\|z_\alpha^{\delta} - \hat{z}\| \to 0 \ \text{ as } \delta \to 0. \tag{9.3.53}$$

Here,

$$z_\alpha^{\delta} = R_\alpha u^{\delta}, \quad \alpha = (\delta, u^{\delta}), \text{ and } \hat{z} = \hat{A}u. \tag{9.3.54}$$

In both the cases, for simplicity, it has been assumed that A is exactly known.

Let $\mathcal{M} \subset Z, A : Z \to U, Az = u, z \in Z$ and $u \in U$, $\|u - u^{\delta}\| < \delta$, $\delta > 0$, $\hat{z} = \hat{A}u$. $\hat{A}$ is the generalized inverse of A. We make an a priori assumption that

$$\hat{z} \in \mathcal{M}. \tag{9.3.55}$$

Under the assumption in Eq. (9.3.55), the *worst-case error* for a regularization method $\hat{R}$ for $\hat{A}$ under the information $\|u - u^{\delta}\| \le \delta$, is given by

$$\triangle(\delta, \mathcal{M}, \hat{R}) = \sup \left\{ \|\hat{R}u^{\delta} - z\|; z \in \mathcal{M},\ u^{\delta} \in U,\ \|Az - u^{\delta}\| \le \delta \right\} . \tag{9.3.56}$$

An ‘optimal method’ R_0 in a class of methods $\mathcal{R}$ would be one for which

$$\triangle(\delta, \mathcal{M}, R_0) = \inf\left\{\triangle(\delta, \mathcal{M}, \hat{R}) : \hat{R} \in \mathcal{R}\right\}. \tag{9.3.57}$$

The optimality of a method is to be understood with respect to an a priori information (9.3.55) and the class of methods considered.

For some $\mu > 0$, let

$$Z_{\mu,\hat{\rho}} = \left\{z \in Z : z = \left(A^*A\right)^{\mu} \omega,\ \|\omega\| \leq \hat{\rho}\right\} \tag{9.3.58}$$

and

$$Z_\mu = \bigcup_{\hat{\rho}>0} Z_{\mu,\hat{\rho}} = R\left(\left(A^*A\right)^{\mu}\right), \ R \text{ stands for the range.} \tag{9.3.59}$$

Definition 9.3.9. Let $R(A)$ (range of A) be nonclosed, and (R_α, α) be a regularization operator for $\hat{A}$. For μ, $\hat{\rho} > 0$ and $u \in AZ_{\mu,\hat{\rho}}$, let α be a parameter-choice rule for solving $Az = u$. We call (R_α, α) optimal in $Z_{\mu,\hat{\rho}}$, if

$$\triangle\left(\delta, Z_{\mu,\hat{\rho}}, R_\alpha\right) = \delta^{\frac{2\mu}{2\mu+1}}\ \hat{\rho}^{\frac{1}{2\mu+1}}, \tag{9.3.60}$$

holds for all $\delta > 0$. We call (R_α, α) of optimal order in $Z_{u,\hat{\rho}}$ if there exists a constant $p \geq 1$ such that

$$\triangle\left(\delta, Z_{\mu,\hat{\rho}}, R_\alpha\right) \leq p\, \delta^{\frac{2\mu}{2\mu+1}}\ \hat{\rho}^{\frac{1}{1+2\mu}}, \tag{9.3.61}$$

for all $\delta > 0$

The Tikhonov regularization with an ‘a priori’ parameter-choice rule given by

$$\alpha \sim \left(\delta/\hat{\rho}\right)^{\frac{2}{1+2\mu}} \tag{9.3.62}$$

is of optimal order in $Z_{\mu,\hat{\rho}}$.

The best possible convergence rate is obtained for $\mu = 1$ and

$$\|z_\alpha^\delta - \hat{z}\| = \mathrm{O}(\delta^{2/3}), \tag{9.3.63}$$

as soon as $\hat{z} \in Z_{1,\hat{\rho}}$. This is the maximum convergence rate possible in Tikhonov regularization. For further results on convergence, see [226].

9.4 DETERMINATION OF UNKNOWN PARAMETERS IN INVERSE STEFAN PROBLEMS

If some of the thermophysical parameters are unknown in problems of heat conduction with phase-change or without phase-change, then some additional information is required for their determination. This additional information is generally in the form of some overspecified boundary conditions and such problems are generally ill-posed. For example, in heat conduction problems without phase-change, if the data is overspecified, then we shall be dealing with a *noncharacteristic Cauchy problem*. The determination of unknown parameters in parabolic heat transfer problems by the *method of overspecified boundary conditions* has been the subject matter of several studies and many such references can be found in [239, 240].

9.4.1 Unknown Parameters in the One-Phase Stefan Problems

The following problem of determining thermal conductivity and some other parameters by prescribing an overspecified boundary condition has been considered in [241]:

$$\rho C T_t = \nabla \cdot (K(T)\nabla T), \quad 0 < x < S(t), \quad S(0) = 0, \quad t > 0, \tag{9.4.1}$$

$$T(0,t) = T_0 < T_m, \quad t > 0, \text{ and } T(x,0) = T_m; \quad 0 < x < \infty, \tag{9.4.2}$$

$$T(S(t),t) = T_m, \quad t > 0, \text{ and } K(T_m)T_x(S(t),t) = \rho l \dot{S}(t); \quad t > 0, \tag{9.4.3}$$

$$K(T_0)T_x(0,t) = q_0/t^{1/2}, \quad t > 0, \quad q_0 > 0. \tag{9.4.4}$$

Here, $K(T) = K_0\{1 + \beta(T - T_0)/(T_m - T_0)\}$; $\beta > 0$, $T_0 > 0$ and $T_m > 0$ are constant. The overspecified boundary condition is given by Eq. (9.4.4) in which $q_0 > 0$ is known. Note that the flux in Eq. (9.4.4) is infinite at $t = 0$ which it should be if $S(t)$ is proportional to $t^{1/2}$ (cf. [242]). We make 'a priori' assumptions that $T(x,t)$ and $S(t)$ (both unknown) can be expressed in the form

$$T(x,t) = T_0 + (T_m - T_0)\Phi_\delta(\eta)/\Phi_\delta(\lambda), \quad \eta = x/(2a\sqrt{t}), \quad \delta > -1. \tag{9.4.5}$$

$$S(t) = 2\sigma\sqrt{t} = 2\lambda a\sqrt{t}, \quad a = \sqrt{K_0/(\rho C)}. \tag{9.4.6}$$

Several combinations of unknown parameters have been considered in Eqs (9.4.1)–(9.4.4) but the coefficient β is taken unknown in all the cases and in addition to it two parameters from K_0, σ, ρ, C and l have been taken as unknown. $\Phi_\delta(x)$, $0 \le x < \infty$, is the *modified error function* which is the unique solution of a boundary value problem consisting of Eqs (9.4.7), (9.4.8):

$$\left[(1 + \delta\, y(x))\, y'(x)\right]' + 2xy'(x) = 0, \quad \delta > -1, \tag{9.4.7}$$

$$y(0+) = 0, \quad y(+\infty) = 1. \tag{9.4.8}$$

Here, dash denotes differentiation with respect to x. If will be assumed that $\delta > -1$ is a given real number and $\delta \neq 0$. For $\delta = 0$, $\Phi_0(x) = \mathrm{erf}(x)$. It can be seen that

$$\Phi_\delta(0) = 0, \; \Phi_\delta(\infty) = 1, \quad \Phi_\delta'(x) > 0 \text{ and } \Phi_\delta''(x) \le 0, \quad 0 \le x < \infty. \tag{9.4.9}$$

$T(x,t)$ and $S(t)$ given in Eqs (9.4.5)–(9.4.6) should satisfy Eq. (9.4.1) together with the first condition in both Eqs (9.4.2), (9.4.3). The pair $(T(x,t), S(t))$ will be a solution of the system Eqs (9.4.1)–(9.4.4) if the following conditions are satisfied:

$$\beta = \delta\Phi_\delta(\lambda), \tag{9.4.10}$$

$$(1 + \delta\Phi(\lambda))\, \Phi_\delta'(\lambda)/(\lambda\Phi_\delta(\lambda)) = 2l/(C(T_m - T_0)), \tag{9.4.11}$$

$$\sqrt{K_0\rho C}\; \Phi_\delta'(0)/\Phi_\delta(\lambda)) = 2q_0/(T_m - T_0). \tag{9.4.12}$$

The unknown coefficients are to be determined from Eqs (9.4.10) to (9.4.12). The unknowns, for example, can be taken as: (1) β, λ, K_0, or (2) β, λ, l, or (3) β, λ, C. Ten such cases have been investigated in [241]. To illustrate the method used in [241], we consider the following two cases.

Case I. The parameters $q_0 > 0$, $\delta > -1$ $(\delta \neq 0)$, $T_0 > 0$, $T_m > T_0$, $\rho > 0$, $C > 0$ and $l > 0$ are taken to be known and parameters $\beta > 0$, $\lambda > 0$ and $K_0 > 0$ are taken as unknown. $\beta > 0$ and $K_0 > 0$ can be determined from the following two equations:

$$\beta = \delta\, \Phi_\delta(\lambda) \tag{9.4.13}$$

and

$$K_0 = 4q_0^2\Phi_\delta^2(\lambda)/\left\{\rho C(T_m - T_0)^2\left(\Phi_\delta'(0)\right)^2\right\}, \tag{9.4.14}$$

provided λ can be obtained from the equation

$$C(T_m - T_0)F_1(x) = 2lF_2(x), \tag{9.4.15}$$

$$F_1(x) = 1 + \delta\Phi_\delta(x), \quad \text{and} \quad F_2(x) = x\,\Phi_\delta(x)/\Phi_\delta'(x). \tag{9.4.16}$$

The functions $F_1(x)$ and $F_2(x)$ possess the following properties:

$$F_1(0+) = 1,\ F_1(+\infty) = 1 + \delta; \quad F_1' > 0 \text{ for } \delta > 0; \quad F_1' < 0 \text{ for } -1 < \delta < 0, \tag{9.4.17}$$

$$F_2(0+) = 1,\ F_2(+\infty) = +\infty; \quad F_2' > 0 \text{ for } \delta > -1. \tag{9.4.18}$$

In view of Eqs (9.4.17), (9.4.18), Eq. (9.4.15) has a unique solution $\lambda > 0$.

Case II. The parameters $\beta > 0$, $\lambda > 0$ and $l > 0$ are unknown and all other parameters are known. In this case β is given by Eq. (9.4.13) and λ is the solution of the equation

$$\Phi_\delta(x) = (T_m - T_0)\sqrt{\rho C K_0}\,\Phi_\delta'(0)/(2q_0)\,. \tag{9.4.19}$$

In view of Eq. (9.4.9), if

$$(T_m - T_0)\sqrt{\rho C K_0}\,\Phi_\delta'(0) < 2q_0, \tag{9.4.20}$$

then Eq. (9.4.19) has a unique solution $\lambda > 0$. The parameter l is given by the equation

$$l = C\,(T_m - T_0)\,\Phi_\delta'(\lambda)[1 + \delta\Phi_\delta(\lambda)]/(2\lambda\Phi_\delta(\lambda)). \tag{9.4.21}$$

The condition (9.4.20) is a necessary and sufficient condition for the existence of the solution in case II. The main consideration in the success of the above method is an a priori assumption that it is possible to obtain both $T(x,t)$ and $S(t)$ in the form of Eqs (9.4.5), (9.4.6) which constitute a similarity solution. As mentioned earlier, Neumann-type exact analytical solutions (cf. Section 2.3) for the phase-change problems are extremely few. If some other combinations of unknown parameters are considered in the above problem (ten such cases are possible), then for determining λ, we get equations which will involve functions different from F_1 and F_2 given in Eq. (9.4.16). The necessary and sufficient conditions for the existence of the unique value of $\lambda > 0$ in these cases can be derived from the equation obtained for determining λ in any particular case.

Another important criterion in the success of this method for the problem (9.4.1)–(9.4.4), and some other related problems discussed below, is that it yields equations of the type (9.4.19) or (9.4.24) (given below) which contain only one unknown (σ/a). Note that both σ and a may be unknown but we consider σ/a as a single unknown. For illustration, consider the derivation of Eq. (9.4.24). When Eq. (9.4.23) is substituted in Eq. (9.4.4), we get Eq. (9.4.24). The condition obtained on satisfying Eq. (9.4.3) ($T_m = 0$ in this case) with the help of Eqs (9.4.22), (9.4.23) has been split into two equations as in Eq. (9.4.25) so as to get an equation of the form Eq. (9.4.24).

Determination of two unknown parameters when $S(t)$ is known

In [239], the method described above has been used for the simultaneous determination of two unknown parameters (K is a constant now). Consider the formulation given in Eqs (9.4.1)–(9.4.4) with $T_m = 0$ and $K(T) = K =$ constant, and $S(t)$ known. If $S(t)$ is known, then only one boundary condition is required at $x = S(t)$ and thus we have two extra conditions, namely, Eq. (9.4.4) and one of the conditions in Eq. (9.4.3). There could be six pairs of unknowns: (i) (K, ρ), (ii) (K, C), (iii) (K, l), (iv) (l, C), (v) (l, ρ) or (vi) (C, ρ). We present here the solution for only one pair (l, ρ) but the method of solution for other pairs remains the same. Let

$$S(t) = 2\sigma t^{1/2}, \quad \sigma > 0, \text{ and } \sigma \text{ is known,} \tag{9.4.22}$$

$$T(x,t) = T_0 - \frac{T_0}{f(\sigma/a)} f(x/2at^{1/2}), \quad a^2 = K/\rho C, \quad f(y) = \operatorname{erf}(y). \tag{9.4.23}$$

If $\xi = \sigma/a$, then ξ is the solution of

$$f(\xi) = \xi K T_0/(q_0 \sigma \pi^{1/2}), \quad \xi > 0, \tag{9.4.24}$$

and

$$\rho = K\xi^2/(C\sigma^2), \quad l = q_0 C\sigma \exp(-\xi^2)/(K\xi^2). \tag{9.4.25}$$

A unique solution of Eq. (9.4.24) exists if $KT_0/(2q_0\sigma) < 1$. It is not necessary to take q_0 to be known. If q_0 is unknown together with any one of the remaining parameters, even then a solution can be obtained.

9.4.2 Determination of Unknown Parameters in the Two-Phase Stefan Problems

We consider the two-phase Neumann problem formulated in Eqs (1.3.1)–(1.3.7) and take $\rho_S = \rho_L$ and $T_m = 0$ and to match this problem with the problem studied in [243], take the region $0 \le x < S(t)$ to be liquid and the region $x > S(t)$ to be solid. The overspecified boundary condition is given by

$$K_L \frac{\partial T_L}{\partial x}(0,t) = -q_0/t^{1/2}, \quad q_0 > 0. \tag{9.4.26}$$

An exact analytical solution of the problem (1.3.1)–(1.3.7) has been given in Eqs (1.3.11)–(1.3.17). Along with $S(t)$ one more thermophysical parameter can be taken to be unknown as an extra condition (9.4.26) is available. The method of solution remains the same as explained in Section 9.4.1 but now the necessary and sufficient conditions for the existence and uniqueness of the solution become fairly lengthy. We mention here only some observations about the nature of solutions. If $S(t)$ is unknown and any one of the six parameters ρ, l, C_S, C_L, K_S or K_L is unknown, then the following results have been proved in [243]:

(i) If $S(t)$ and ρ are unknown, then a unique solution of the Neumann-type exists for the two-phase problem.

(ii) If $S(t)$ is unknown and one of the remaining five parameters is unknown (ρ is excluded), then a unique solution of the Neumann-type is possible for the two-phase problem provided in each case a complementary condition (cf. [243]) is satisfied.

If $S(t)$ is known and q_0 in the overspecified condition (9.4.26) is known, then any two of the six thermophysical parameters can be taken as unknown. There will be 15 such pairs. If ρ and K_L are unknown, then a unique solution of the Neumann-type exists. If (l, K_S), or (l, C_S), or (K_S, C_S) are unknown, then the free boundary problem has infinitely many solutions whenever some complementary conditions are satisfied. In the remaining 11 cases, unique solutions of the Neumann-type can be obtained provided some complementary conditions are satisfied. For complementary conditions, see [243].

If it is not possible to obtain similarity solutions of the types (1.3.11)–(1.3.13), the above method of finding unknown parameters will not work. Short-time analytical solutions based on series expansions of temperatures and the free boundary have been obtained when similarity solutions are not possible (cf. [244, 245]). It is difficult to prove the existence and uniqueness of short-time solutions of Stefan problems but these analytical solutions have been compared in some cases with the numerical solution [246]. Short-time analytical solutions can also be used to obtain approximate analytical solutions of some inverse Stefan problems.

9.5 REGULARIZATION OF INVERSE HEAT CONDUCTION PROBLEMS BY IMPOSING SUITABLE RESTRICTIONS ON THE SOLUTION

In many inverse problems of mathematical physics, an a priori information about the smoothness of the solution stabilizes the problem. The inverse Stefan problems are generally studied as control problems in which for obtaining the regularized solutions, procedures different from those discussed in this section are adopted. These procedures will be discussed in Sections 9.6 and 9.7. To give some idea of the type of smoothness conditions to be imposed on the solution which may stabilize the problem, some heat conduction problems with and without phase-change are being discussed here. We first consider the following one-dimensional noncharacteristic Cauchy problem in heat conduction which has been studied in [247]:

$$T_t - a(x)T_{xx} - b(x)T_x - e(x)T = q(x,t), \quad x \in (0,d), \quad t \in I, \tag{9.5.1}$$

$$T(0,t) = \phi(t), \quad t \in I, \tag{9.5.2}$$

$$T_x(0,t) = \psi(t), \quad t \in I, \tag{9.5.3}$$

where, $I = R$ or $I = R^+$, and in the latter case, an initial condition

$$T(x,0) = g(x), \quad x \in [0,d], \tag{9.5.4}$$

should be prescribed. It may be noted that a boundary condition is required at $x = d$ but instead of that an overspecified boundary condition (9.5.3) is prescribed at $x = 0$. The inverse problem consists of obtaining $T(d,t)$. Instead of Eqs (9.5.3), (9.5.4), we can consider

$$T_x(0,t) = 0, \quad \text{and} \quad T(x,0) = 0, \tag{9.5.5}$$

and take $q(x,t) = 0$ in Eq. (9.5.1). This is possible by considering a suitable well-posed problem in the region $0 \le x < \infty$ with Eqs (9.5.3), (9.5.4) and $q(x,t) \ne 0$ in Eqs (9.5.1) and subtracting its solution from the solution of the problem (9.5.1)–(9.5.4) which is considered in $0 \le x \le d$.

For further discussion, we shall consider the problem (9.5.1)–(9.5.4) in which we take

$$q(x,t) = 0, \quad \psi(t) = 0 \quad \text{and} \quad g(x) = 0. \tag{9.5.6}$$

The functions, a, b, c and ϕ are given and the continuous dependence of $T(x,t)$ on $\phi(t)$ is to be shown under suitable restrictions. The following smoothness conditions and other restrictions will be assumed

$$a(x) \in W^{2,\infty}[0,d], \quad b(x) \in W^{1,\infty}[0,d], \quad e(x) \in L^{\infty}[0,d], \tag{9.5.7}$$

$$e(x) \le 0, \quad \lambda \le a(x) \le \gamma, \quad \gamma > 0; \quad x \in [0,d], \tag{9.5.8}$$

$$\phi \in L^2(R); \quad f(t) = T(d,t) \in L^2(R). \tag{9.5.9}$$

In order to obtain stability estimates for $T(x,t)$, the problem is first formulated in terms of the Fourier transform of $T(x,t)$ with respect to time, denoted by $\hat{T}$, and the stability estimate for $\hat{T}$ are obtained (cf. [247]). The stability estimate for $T(x,t)$ which shows the exact Hölder type dependence of $\|T(x,t)\|_{L^2}$ on $\|\phi\|_{L^2}$ can then be obtained which is given by

$$\|T(x,t)\| \le M\|\phi\|^{1-A(x)/p}\left(\|\phi\|^{A(x)/p} + \|T(d,t)\|^{A(x)/p}\right), \tag{9.5.10}$$

$$A(x) = \int_0^x a(y)^{-1/2}dy, \quad p = A(d). \tag{9.5.11}$$

The $L^2(R)$-norm has been considered in Eq. (9.5.10) and M is a suitable constant depending on λ, γ, d and the norms of other coefficients.

In the problem considered in [248], $b = e = q = 0$ in Eq. (9.5.1), $1 \le a(x) \le \nu$, $\nu > 0$, $a \in L^{\infty}[0,d]$, $0 \le t \le t_1 < \infty$, $\phi \in C^0[0,t_1]$, $\|\phi\|_{L^\infty(0,t_1)} \le \varepsilon$. Further, $T(x,t)$ satisfies an a priori bound

$$\|T(x,t)\|_{L^2(R)} \le \sqrt{dt_1}E, \quad \varepsilon \le E. \tag{9.5.12}$$

Under these assumptions, by considering a weak formulation of the problem, the stability estimates for $T(x,t)$ have been obtained which are of Hölder type in the interior and of logarithmic type at the boundary which are given below. The continuous dependence on the Cauchy data and the coefficient $a(x)$ has also been considered.

$$|T(x,t)| + d\,|T_x(x,t)| \le \beta(x,t)\varepsilon^{\gamma(x,t)}E^{1-\gamma(x,t)}, \quad 0 \le x < d, \quad 0 \le t < t_1, \tag{9.5.13}$$

$$\gamma(x,t) = \exp\left(\frac{-pt_1/t}{1-x/d}\,\frac{x}{d}\right), \tag{9.5.14}$$

$$\beta(x,t) = \left(\frac{pt_1/t}{(1-x/d)}\right)^{p((t_1/t)/(1-x/d))}. \tag{9.5.15}$$

Here, p is a computable constant that depends only on ν and t_1/d^2. For the stability estimate at the boundary, we further require that

$$\max_{t\in[0,t_*]}\left(d\int_0^d T_x^2\,(x,t)\,dx\right) \le E^2. \tag{9.5.16}$$

For $t \in (0,t_1)$,

$$|T(d,t)| = E\cdot\frac{t_1}{t}o\left(\left(\log\left(\log\frac{E+\varepsilon}{\varepsilon}\right)\right)^{-1/2}\right), \text{ as } \varepsilon/E \to 0. \tag{9.5.17}$$

The following problem of determining an unknown source control $q = q(t)$ has been considered in [249]:

$$T_t = a\,(x,t,T,T_x)_x + q(t)T + F(x,t,T,T_x,q(t)), \text{ in } Q_{t_*};$$

$$Q_{t_*} = \{(x,t) : 0 < x < 1,\ 0 < t < t_*\}, \tag{9.5.18}$$

$$T(x,0) = \phi\,(x) \ge 0, \quad 0 < x < 1, \tag{9.5.19}$$

$$T(0,t) = f(t) \ge 0, \text{ and } T\,(1,t) = g(t) \ge 0; \quad 0 < t < t_*, \tag{9.5.20}$$

$$\int_0^{S(t)} \Phi(x,t)T(x,t)dx = G(t) > 0, \quad 0 < t < t_*, \quad 0 < S\,(t) \le 1. \tag{9.5.21}$$

The functions a, $F\,(\ge 0)$, ϕ, f, g, S, $\Phi\,(> 0)$ and G are known. The functions a and F are smooth functions of their arguments. (T,q) is called a solution if there exists an α, $0 < \alpha < 1$, such that $T \in C^{1+\alpha}\left(\bar{Q}_{t_*}\right) \cap C^{2+\alpha}\left(Q_{t_*}\right)$, $q(t) \in C^{\alpha/2}\,[0,t_*]$ and the pair (T,q) satisfies Eqs (9.5.18)–(9.5.21). The existence, uniqueness, and continuous dependence of the solution on the data has been shown with the help of some a priori estimates, compactness arguments, and the strong maximum principle. For the conditions which the data has to satisfy, see [249]. Some problems of recovering a source term or a nonlinear coefficient in the inverse problems of parabolic type have been discussed in [250].

One typical structural restriction for the one-phase melting Stefan Problems is the nonnegativity of the solution [251] but the most commonly investigated a priori information concerns norm bounds. In the latter case if the problem is linear, then the stability estimates can be derived by estimating the size of solutions for the data fulfilling such norm bounds. There are two major techniques to obtain stability estimates: (1) complex variable methods [252] and (2) the Fourier transform technique with its own limitations such as in this technique the domain should be cylindrical and the time interval should be infinite.

By using an extension of the complex variable technique, the stability analysis of a one-dimensional one-phase inverse Stefan problem has been done in [253]. The region considered is $D_{t_*} = \{(x,t); 0 < x < S(t), 0 < t \leq t_*\}$. The free boundary $x = S(t)$ is assumed to be known and is Lipschitz continuous. The noncharacteristic Cauchy problem considered is as follows:

$$\left.\begin{aligned} T_{xx} - T_t = q(x,t) \text{ in } D_{t_*}; \quad T(x,0) = g(x), \\ T(S(t),t) = f_1(t); \quad T_x(S(t),t) = f_2(t), \quad S(0) = b. \end{aligned}\right\} \tag{9.5.22}$$

The inverse problem consists of determining $T_x(0,t)$ which is assumed to be bounded. The interior estimates of nonuniform Hölder type as well as uniform estimates of logarithmic type have been obtained for the temperature and its gradient under suitable assumptions on the data.

9.6 REGULARIZATION OF INVERSE STEFAN PROBLEMS FORMULATED AS EQUATIONS IN THE FORM OF CONVOLUTION INTEGRALS

The regularization of a one-dimensional one-phase inverse problem concerning melting of ice has been considered in [254]. The formulation of the Stefan problem is as follows:

$$T_{xx} - T_t = 0, \text{ in } D_{t_*} = \{(x,t) : 0 < x \leq S(t),\ 0 < t < t_*\}, \tag{9.6.1}$$

$$T(S(t),t) = 0, \text{ and } T_x(S(t),t) = -\mu \dot{S}(t); \quad 0 < t < t_*, \tag{9.6.2}$$

$$T(0,t) = v(t) \geq 0, \quad 0 < t < t_*; \quad T(x,0) = T_0(x) \geq 0, \quad 0 \leq x \leq b, \tag{9.6.3}$$

$$S(0) = b, \quad b > 0. \tag{9.6.4}$$

The region $S(t) < x < \infty$ is occupied by ice cold water at the melting temperature zero. For simplicity, some of the thermophysical parameters have been taken to be unity. In Eqs (9.6.1)–(9.6.4), $S(t)$ is a given nondecreasing C^1-function and $T_0(x)$ is a given C^1-function with a bounded derivative. The problem is to obtain a regularized solution for $v(t)$ satisfying Eqs (9.6.1)–(9.6.4). One of the conditions in Eq. (9.6.2) is an overspecified condition as $S(t)$ is known.

Using standard methods (see [9]), an integral equation can be obtained to determine $v(t)$ (see Eq. 9.6.9) which can be studied by Tikhonov regularization but it is difficult to obtain error estimates in this way. Therefore, an equation in the form of *convolution integral* is obtained as follows. Consider the identity

$$\frac{\partial}{\partial \xi}\left(G\frac{\partial T}{\partial \xi} - T\frac{\partial G}{\partial \xi}\right) - \frac{\partial}{\partial \tau}(TG) \equiv 0, \tag{9.6.5}$$

where

$$G(x,t;\xi,\tau) = Q(x,t;\xi,\tau) - Q(-x,t;\xi,\tau), \tag{9.6.6}$$

and

$$Q(x,t;\xi,\tau) = \frac{1}{2\sqrt{\pi\,(t-\tau)}} \exp\left(-\frac{(x-\xi)^2}{4\,(t-\tau)}\right). \tag{9.6.7}$$

On integrating Eq. (9.6.5) over the domain $0 < \varepsilon < \tau < t - \varepsilon$, $0 < \xi < S(\tau)$ and letting $\varepsilon \to 0$, after some rearrangement, we get

$$T(x,t) = \int_0^b T_0(\xi) G(x,t;\xi,0) d\xi + \int_0^t v(\tau) \frac{\partial G}{\partial \xi}(x,t;0,\tau) d\tau + \int_0^t \frac{\partial T}{\partial \xi}(S(\tau),\tau)\, G(x,t;S(\tau),\tau) d\tau. \tag{9.6.8}$$

From the condition $T(S(t),t) = 0$, we get

$$\frac{1}{2\sqrt{\pi}} \int_0^t \frac{S(t)}{(t-\tau)^{\frac{3}{2}}} \exp\left(-\frac{S^2(t)}{4(t-\tau)}\right) v(\tau) d\tau = -\int_0^b G(S(t),t;\xi,0) T_0(\xi) d\xi + \mu \int_0^t G(S(t),t;S(\tau),\tau) \dot{S}(\tau) d\tau. \tag{9.6.9}$$

Eq. (9.6.9) is a linear Volterra integral equation of the first kind which can be transformed into a convolution integral. Using a lemma given in [9, pp. 217], Eq. (9.6.9) can be converted into an equivalent equation of the form

$$\frac{m}{2\sqrt{\pi}} \int_0^t (t-\tau)^{\frac{3}{2}} \exp\left(-\frac{m^2}{4(t-\tau)}\right) v(\tau) d\tau = \int_0^t Q_\xi(m,t;S(\tau),\tau) U_0(\tau) d\tau - \int_0^t Q(m,t;S(\tau),\tau)\{U_0(\tau)\dot{S}(\tau) + U_1(\tau)\} d\tau, \tag{9.6.10}$$

where m is some real number $> S(t)$, $\forall\, t > 0$,

$$U_0(t) = \lim_{x \to S(t)-0} g(x,t), \text{ and } U_1(t) = \lim_{x \to S(t)-0} g_x(x,t), \tag{9.6.11}$$

$$g(x,t) = -\int_0^b G(x,t;\xi,0) T_0(\xi) d\xi + \mu \int_0^t G(x,t;S(\tau),\tau) \dot{S}(t) d\tau + T(x,t), \quad 0 < x < S(t), \quad t > 0. \tag{9.6.12}$$

Eq. (9.6.10) can be written as

$$(z * v)(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} z(t-\tau) v(\tau) d\tau = F(t), \quad \forall\, t \text{ in } R, \tag{9.6.13}$$

where

$$z(t) = \begin{cases} \sqrt{\pi}\, t^{-3/2} \exp(-m^2/4t), & t > 0, \\ 0, & t \le 0, \end{cases} \tag{9.6.14}$$

and $F(t)$ is the r.h.s. of Eq. (9.6.10) multiplied by $(2/m)(2\pi)^{-1/2}$. A family of regularized solutions $\{v_\varepsilon\}$, $0 < \varepsilon < 1$, can be constructed in which v_ε is stable with respect to variations in the function $F(t)$.

The following proposition describes the regularized solution and gives an estimate of the error involved in it.

Proposition 9.6.1. *Suppose the exact solution v_0 of Eq. (9.6.13) corresponding to F_0 (given) is in $H^1(R) \cap L^1(R)$ and $\|F - F_0\| < \varepsilon$. There exists a regularized solution v_ε of Eq. (9.6.13) which is given by*

$$v_\varepsilon(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \Psi(x) e^{itx} dx, \tag{9.6.15}$$

where

$$\Psi(x) = \overline{\hat{z}(x)}.\, \hat{F}(x)/(\varepsilon + \|\hat{z}(x)\|^2), \tag{9.6.16}$$

and, $\hat{z}(x)$ and $\hat{F}(x)$ stand for the Fourier transforms (cf. Eq. 9.6.15) of z and F, respectively. The error in v is given by

$$\|v_\varepsilon - v_0\| \le \alpha \left(\ln(1/\varepsilon)\right)^{-1}, \tag{9.6.17}$$

where α is any constant $\ge m\sqrt{\pi}(3 + 2m).\max((\|\hat{v}_0\|^2 + 1)^{1/2}, \|v_0'\| + M)$, with $M = \left(8/m^3\right)(2/\pi)^{1/2} e^{-2}$. Here, $\|\cdot\|$ stands for the $L^2(R)$-norm. Furthermore, if $\hat{v}_0/\left|\hat{z}\right| \in L^2(R)$ then we have

$$\|v_\varepsilon - v_0\| \le \beta\sqrt{\varepsilon}, \tag{9.6.18}$$

where β is any constant $\ge 1 + \left\|\hat{v}_0/\hat{z}\right\|$ and $\hat{z}(t)\hat{v}_0(t) = \hat{F}_0(t)$, $t \in R$.

The proof of Proposition 9.6.1 depends on obtaining suitable estimates of

$$\left\|\hat{v}_\varepsilon - \hat{v}_0\right\|_{L^2}^2, \; \|v_\varepsilon - v_0\|_{L^2}^2 \text{ and } \varepsilon \left\|t\left(\hat{v}_\varepsilon - \hat{v}_0\right)\right\|_{L^2}^2 + \left\|t\hat{z}\left(\hat{v}_\varepsilon - \hat{v}_0\right)\right\|_{L^2}^2. \tag{9.6.19}$$

A numerical example has been considered in which $F(t) = F_0(t) + \varepsilon e^{-t/2}$ and $S(t)$ and $T_0(x)$ are known. Regularized numerical solutions using finite-difference discretization have been obtained for different values of ε and the numerical results indicate convergence as $\varepsilon \to 0$.

A two-phase one-dimensional inverse Stefan problem has been considered in [255]. If some minor changes are incorporated in the Neumann problem (1.3.1)–(1.3.7), then the problem considered in [255] can be obtained. The changes are as follows:

(i) The region $0 \le x \le S(0) = b$ is initially occupied by warm liquid and the region $b \le x \le d$ is initially occupied by ice and the Stefan problem is concerned with the melting of ice. (ii) $T_L(x,0) = T_1(x)$, $0 \le x \le b$, and $T_S(x,0) = T_2(x)$, $b \le x \le d$. (iii) $T_L(0,t) = v(t)$, and $\partial T_S/\partial x = 0$, at $x = d$. (iv) $T_m = 0$. (v) Initial and boundary temperatures satisfy compatibility conditions at $x = b$ and d. For a given $S(t)$, the problem is to determine $v(t)$ satisfying all other equations of the problem. With the help of suitable Green's functions, the problem is first reduced to a system of integral equations by obtaining temperatures $T_L(x,t)$ and $T_S(x,t)$ in the form of equations similar to Eq. (9.6.8). The Green's functions $G(x,t;\xi,\tau)$ for the liquid and $W(x,t;\xi,\tau)$ for the solid are given below

$$G(x,t;\xi,\tau) = E(x-\xi, k_L^2(t-\tau)) - E(x+\xi, k_L^2(t-\tau)), \tag{9.6.20}$$

$$W(x,t;\xi,\tau) = E(x-\xi, k_S^2(t-\tau)) - E(x+\xi-2a, k_S^2(t-\tau)), \tag{9.6.21}$$

$$\left.\begin{aligned} E &= \frac{1}{2\sqrt{\pi t}}\exp\left(-x^2/4t\right), & t>0,\ x\in R,\\ &= 0, & t\le 0,\ x\in R. \end{aligned}\right\} \tag{9.6.22}$$

The three boundary conditions at $x = S(t)$, give rise to three integral equations involving $v(t)$, $(\partial T_L/\partial x)(S(t),t)$ and $(\partial T_S/\partial x)(S(t),t)$. A crucial step in the regularizing method for this problem is to convert the integral equation for $(\partial T_S/\partial x)(S(t),t)$ into a linear Volterra integral equation of the second kind, and the integral equation for $v(t)$ into an equation in the form of a convolution integral. The integral equation for $v(t)$ contains $(\partial T_L/\partial x)(S(t),t)$. If the solution of the Volterra integral equation of the second kind is known or in other words $(\partial T_S/\partial x)(S(t),t)$ is assumed to be known (note that $S(t)$ is known so we shall be solving a heat conduction problem in the solid region without phase-change), then with the help of the Stefan condition $(\partial T_L/\partial x)(S(t),t)$ can be determined which is then substituted in the integral equation for $v(t)$. For the details of obtaining convolution integral, see [255]. In Eq. (9.6.10), $S(t)$ was replaced by some constant m, $m > S(t)$. In the present two-phase problem, since $S(t) < d$, $\forall\ t > 0$, we can replace $S(t)$ in the convolution integral by the constant d in the convolution integral. A family of regularized solutions $\{v_\varepsilon\}_{\varepsilon>0}$ has been obtained for the present two-phase problem and a proposition similar to Proposition 9.6.1, with some changes, has been proved by obtaining estimates of various expressions. A numerical example has been considered and the regularized numerical solutions have been obtained for different value of ε by the finite-difference discretization of integrals.

A family of regularized solutions has been obtained in [256] for the following one-phase two-dimensional inverse Stefan problem

$$T_{xx} + T_{yy} - T_t = 0, \quad x\in R, \quad 0<y<S(x,t), \quad t>0, \tag{9.6.23}$$

$$T(x,S(x,t),t) = 0, \quad x\in R, \quad t>0, \tag{9.6.24}$$

$$T(x,0,t) = v(x,t) \ge 0, \quad x\in R, \quad t>0, \tag{9.6.25}$$

$$\frac{\partial T}{\partial n}(x,S(x,t),t) = -\frac{\partial T}{\partial x}(x,S(x,t),t)S_x + \frac{\partial T}{\partial y}(x,S(x,t),t) = \frac{\partial S}{\partial t}, \tag{9.6.26}$$

$$S(x,0) = b(x) > 0, \quad x\in R, \tag{9.6.27}$$

$$T(x,y,0) = T_0(x,y) \ge 0, \quad x\in R, \quad 0<y<b(x). \tag{9.6.28}$$

Here, $y = S(x,t)$, $x \in R$, is the equation of the phase-change boundary which is a known smooth function, and $\vec{n}$ stands for the unit outward normal on the interface. The region $y > S(x,t)$ is at the melting temperature zero. $T_0(x,y)$ is also a known smooth function. The problem is to determine $v(x,t)$. The method of finding a family of regularized solutions depends on obtaining an equation in the form of a convolution integral. By integrating the identity

$$\operatorname{div}(T\nabla G - G\nabla T) = -\frac{\partial}{\partial\tau}(TG), \tag{9.6.29}$$

where

$$G(x,y,t;\xi,\eta,\tau) = W(x,y,t;\xi,\eta,\tau) - W(x,-y,t;\xi,\eta,\tau), \tag{9.6.30}$$

and

$$W(x,y,t;\xi,\eta,\tau)=\frac{1}{4\pi\,(t-\tau)}\ \exp\left(-\frac{(x-\xi)^2+(y-\eta)^2}{4\,(t-\tau)}\right), \tag{9.6.31}$$

over the domain $-\beta<\xi<\beta, 0<\eta<S\,(\xi,\tau)$, $1/\beta<\tau<t-1/\beta$ and taking the limit as $\beta\to\infty$, $T(x,y,t)$ can be obtained in the form of an equation similar to Eq. (9.6.8)

$$\begin{aligned}T(x,y,t)&=\int_0^t\int_{-\infty}^{\infty}\frac{y}{(t-\tau)}W(x,y,t;\xi,0,\tau)v(\xi,\tau)d\xi d\tau\\&+\int_{-\infty}^{\infty}\int_0^{b(\xi)}T_0(\xi,\eta)G(x,y,t;\xi,\eta,0)d\eta d\xi\\&+\int_{-\infty}^{\infty}\int_0^{t}G(x,y,t;\xi,S(\xi,\tau),\tau)\frac{\partial S}{\partial\tau}\,(\xi,\tau)\,d\tau d\xi,\quad x\in R,\quad t>0,\quad 0<y<S(x,t).\end{aligned} \tag{9.6.32}$$

On taking the limit $y\to S(x,t)-0$ in Eq. (9.6.32), we get an integral equation to determine $v(x,t)$. For the details of obtaining an equation in the form of a convolution integral, the reader is referred to [256]. A proposition similar to Proposition 9.6.1 defining the regularized solutions and giving error estimate can be proved for the two-dimensional problem also provided some changes are made in the assumptions made in Proposition 9.6.1. For example, the norm to be considered is $L^2(R^2)$-norm, and the Fourier transform in the place (9.6.15) will now be a two-dimensional Fourier transform defined as:

$$\hat{v}(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi(\omega,\eta)e^{i(x\omega+t\eta)}d\omega d\eta, \tag{9.6.33}$$

$$\Psi(\omega,\eta)=\overline{\hat{z}\,(\omega,\eta)}\,\hat{F}(\omega,\eta)\Big/\left(\varepsilon+\left\|\hat{z}\,(\omega,\eta)\right\|^2\right), \tag{9.6.34}$$

$$z(x,t)=\frac{1}{t^2}\exp\left(-\frac{x^2+m^2}{4t}\right),\qquad m>S(x,t),\quad m\in R^+,\quad t>0. \tag{9.6.35}$$

For some other changes in Proposition 9.6.1 which are required for proving the results in the present problem, see [256].

A numerical example has been considered in which $F(x,t)=F_0(x,t)+\varepsilon e^{-|x|-t}$ ($F(x,t)$ is the r.h.s. of the convolution integral in this problem), $S(x,t)=\arctan(x^2+t+1)$, $b(x)=\arctan(x^2+1)$, $T_0(x,y)=|\cos 2x|\,.(b(x)-y)^2$ and $m=2$. The numerical method consists of finite-difference discretization of integrals. Regularized solutions for different values of ε have been obtained. The numerical solutions for different ε indicate convergence as $\varepsilon\to 0$.

The regularization of a two-dimensional two-phase inverse Stefan problem in the region $-\infty<x<\infty$, $0\le y\le a$ has been considered in [257]. The region $0\le y<S(x,t)$, is occupied by warm water and the region $S(t)<y\le a$ consists of ice. The initial temperatures in the two regions are functions of x and y. The phase-change interface is given by $y=S(x,t)$ and $S(x,0)=b(x)$, $x\in R$. Initially the warm water occupies the region $0\le y<b(x)$, $x\in R$. The method of obtaining a family of regularized solutions is similar to that considered in [254–256]. A numerical example has also been considered and regularized solutions have been obtained for different values of ε. The convergence is indicated as $\varepsilon\to 0$.

9.7 INVERSE STEFAN PROBLEMS FORMULATED AS DEFECT MINIMIZATION PROBLEMS

A stable and regular propagation of the free boundary in the classical Stefan problem requires application of controls. An application of controls could be through thermophysical parameters, geometry of the mould, initial temperature or through boundary conditions. In the control of the free boundary, which is our interest in this section, there exist mainly two approaches. The first is only to search for a control which generates a solution to the corresponding direct Stefan problem with a free boundary that is close to the prescribed one. It is assumed in this approach that the solution of the direct problem exists. This approach may yield an approximation but it will not be a good solution as it is well known that the inverse Stefan problems are ill-posed. In the second approach the aim is to regularize the problem in some way and then obtain an approximate solution. If the direct Stefan problem is well-posed, then as a consequence of the stability of the regularized solution, this second approach includes the first one. Both the approaches will be discussed here briefly with reference to some specific Stefan problems.

We consider the following one-dimensional one-phase melting problem which will be studied with the help of the two approaches described above:

$$T_{xx}(x,t) - T_t(x,t) = q(x,t), \text{ in } D_{t_*}(S) = \{(x,t) : 0 < x < S(t), 0 < t < t_*\}, \tag{9.7.1}$$

$$T(x,0) = T_0(x), \quad 0 \le x \le b = S(0), \tag{9.7.2}$$

$$\left.\begin{aligned} (A_i T)(t) = \phi(t), \quad 0 < t < t_*, \quad i = 1,2, \\ (A_1 T)(t) = T(0,t), \text{ and } (A_2 T)(t) = \eta\, T(0,t) - T_x(0,t), \end{aligned}\right\} \tag{9.7.3}$$

$$T(S(t),t) = r_1(S(t),t), \quad 0 < t < t_*, \tag{9.7.4}$$

$$T_x(S(t),t) = r_2(S(t),\dot{S}(t),t), \quad 0 < t < t_*. \tag{9.7.5}$$

If $T(x,t)$ and $S(t)$ are unknowns to be determined, then we have a direct Stefan problem to be solved and it is well known that problem (9.7.1)–(9.7.5) is well-posed under sign restrictions and smoothness of the data (cf. [59]).

In the above inverse problem, $q(x,t)$, $T_0(x)$, b, r_1, r_2 and $S(t)$ are known and $\phi(t)$ is the control which is to be determined in such a way that for $\phi = \phi^*$, the exact solution of Eqs (9.7.1)–(9.7.5) gives exactly prescribed $S(t) = S^*(t)$. Let Q be a free boundary operator defined as

$$Q : \{\phi \in C[0,t_*] : \phi \ge 0\} \to C[0,t_*],\ \phi :\to S; \quad Q\phi^* = S^*. \tag{9.7.6}$$

For a given $\phi(t)$, $S(t)$ and $T(x,t)$ can be obtained and the direct problem (9.7.1)–(9.7.5) is solved. Even if ϕ^* exists, it is generally difficult to obtain it exactly. An approximation $\hat{\phi}$ of ϕ^* can be possibly obtained such that $Q\hat{\phi}$ is 'close to' S^*. This closeness to S^* is generally expressed in terms of a norm which involves a function ϕ belonging to some suitable space of functions and which minimizes $\left\|Q\phi - S^*\right\|$ or in words we choose a ϕ which minimizes the defect. The questions related to the existence, uniqueness and stability have to be addressed. This approach can be called a 'direct approach' which has been used in [258–260]. The operator Q is in general a nonlinear operator. The defect minimization problem corresponding to Eqs (9.7.1)–(9.7.5) with $r_1 = 0$ and r_2 as given in Eq. (9.7.8) has been studied in [261] with the help of a linear auxiliary problem. This auxiliary problem can be stated in terms of a linear operator F (defined in Eq. 9.7.14) such that

$$Q(\phi^*) = S^* \Leftrightarrow F\phi^* = r(S^*), \tag{9.7.7}$$

$$r = r_2 = -\lambda\dot{S}(t) + \mu(t), \quad \mu \geq 0. \tag{9.7.8}$$

To obtain the solidification problem studied in [258], we take $q(x,t) = 0$, $r_1 = 0$, $r_2 = -\dot{S}(t)$, $T_x = \phi(t)$ at $x = 0$ and $\phi \leq 0$, in (9.7.1)–(9.7.5). For this inverse problem, existence of a ϕ which gives S^* has not been proved in the literature. Therefore, in [258] the inverse problem has been formulated as a nonlinear approximation problem which does not require the existence of the solution to be known and which can be solved by a highly stable iterative Newton-like procedure developed in [262]. The nonlinear approximation problem can be described as follows:

Find $\hat{\phi} \in W$ such that

$$\|Q\hat{\phi} - S^*\| = \inf\left\{\left\|Q\phi - S^*\right\|; \phi \in W, \phi \leq 0\right\}. \tag{9.7.9}$$

Here, W represents the set of admissible controls and $\|\cdot\|$ is some norm in $C[0,t_*]$. The operator Q is the same as in Eq. (9.7.6). It is difficult to prove the existence of an optimal control if W is an infinite-dimensional space. Therefore, a finite-dimensional subspace V of $C[0,t_*]$ has been considered and let $W = \{\phi \in V : \phi \leq 0\}$. The set W is the maximal subset of V for which the existence of a solution of the present direct Stefan problem is known under the conditions that

$$T_0(x) \in C^1[0,b], \quad T_0(x) \geq 0, \ T_0(b) = 0. \tag{9.7.10}$$

By virtue of the fact that a unique solution (T,S) of the direct Stefan problem exists under the above conditions, the solution operator $Q : W \to C[0,t_*]$, $\phi :\to S(t)$ is Lipschitz continuous, and $S(t)$ depends monotonically on ϕ and is a monotonically nondecreasing function (cf. [258]). By considering the uniform norm $\|S\| = \sup\limits_{0\leq t\leq t_*} |S(t)|$ in W, the existence of an optimal solution $\hat{\phi} \in W$ has been proved.

Let V_n be an ascending sequence of subspaces, i.e. $V_n \subset V_{n+1}, n = 1,2,\ldots$, and the union of V_n be dense in $C[0,t_*]$. Let W_n be the cone $\{\phi \in V_n : \phi \leq 0\}$ and

$$\rho_n(S^*) = \inf_{\phi\in W_n} \left\|Q(\phi) - S^*\right\|. \tag{9.7.11}$$

Because of the continuity of Q, ρ_n converges to zero as $n \to \infty$ but it is difficult to prove that the sequence $\{\hat{\phi}_n\}$ of optimal controls $(\hat{\phi}_n \in W_n)$ converges to the solution ϕ^* of the inverse problem.

If B is the Banach space $C[0,t_*]$ with the uniform norm

$$\|\phi\|_\infty := \sup\{|\phi(t)| : 0 \leq t < t_*\}, \tag{9.7.12}$$

and A is the cone $A : \{\phi \in C[0,t_*] : \phi < 0\}$, then it has been proved in [259] that the operator $Q : A \subset B \to B$ is Fréchet differentiable. This property was found useful in obtaining the numerical solution of a one-dimensional inverse problem in [259] by the generalized Gauss-Newton method. Numerical solution of a two-dimensional inverse Stefan problem has been considered in [260].

The problem considered in [261] is the inverse problem (9.7.1)–(9.7.5) with $r_1 = 0$ and r_2 given in Eq. (9.7.8). A melting problem has been considered. $S(t)$ is known and the control $\phi \geq 0$ is to be determined. For the defect minimization, instead of the operator Q, a linear operator F (cf. Eq. 9.7.14) has been considered. Let $S^* \in C^1[0, t_*]$, $S^* > 0$, $b = S^*(0)$, $T_0 \geq 0$, $T_0(b) = 0$, $q \leq 0$, and $\mu \geq 0$ (the last condition can be relaxed). For other regularity conditions on the data, see [261]. As mentioned earlier, for numerical computations, one has to deal with finite-dimensional spaces. Therefore the defect is minimized in the space $X_n \cap A_n$ where X_n is an n-dimensional subspace of $C\,[0, t_*]$ and $A_n \subset \{\phi : \phi \geq 0\}$, $\phi \in C\,[0, t_*]$. The minimization problem can be stated as follows:

$$\text{Minimize } \|F\phi - r\|_{L^p}\,, \text{ for } \phi \in X_n \cap A_n, \quad 1 < p < \infty, \tag{9.7.13}$$

$$F : C[0, t_*] \to C[0, t_*], \quad \phi \to T_x(S^*(t), t), \tag{9.7.14}$$

where

$$T_x\left(S^*(t), t\right) = \left.\begin{array}{ll} \lim\limits_{x \to S(t)-0} T_x(x, t)\,, & t > 0, \\ \lim\limits_{t \to 0+} T_x\left(S^*(t), t\right), & t = 0. \end{array}\right\} \tag{9.7.15}$$

$T(x, t)$ solves the problem (9.7.1)–(9.7.5) in the following sense:

$$T \in C(\overline{D_{t_*}(S^*)}); \quad T_{xx},\ T_t \in C(D_{t_*}(S^*)),$$

$T_x(x, t)$ is continuous in (x, t) for $t \in [0, t_*]$, $x \in \left(0, S^*(t)\right)$ for $i = 1$ and $x \in \left[0, S^*(t)\right)$ for $i = 2$ (see Eq. 9.7.3 for $i = 1, 2$) and Eqs (9.7.1)–(9.7.4) are satisfied pointwise. Note that the isotherm condition (9.7.4) ($r_1 = 0$) is satisfied exactly in this problem and Eq. (9.7.5) is to be satisfied in the sense of Eq. (9.7.13).

Some of the results established in [261] are stated below.

1. F in (9.7.14) is well-defined and continuous.
2. If $X_n \cap A_n$ is closed in X_n and not void, then the minimization problem has a solution ϕ_n which is unique if $1 < p < \infty$.
3. For $t \in [0, t_*]$, we have

$$\left|(Q\phi)(t) - S^*(t)\right| \leq P(t) \int_0^t |(F\phi)(\tau) - r(\tau)|\, d\tau. \tag{9.7.16}$$

   In Eq. (9.7.16), $P(t)$ depends on $\max q(x, t)$, $\max\left\{(\mu(t))_+, (\mu(t) + (F\phi)(t) - r(t))_+\right\}$, λ and b. This result justifies the choice of the linear auxiliary problem.

4. If $\phi_n \geq 0$, then for $t \in [0, t_*]$, we have

$$\left|(Q\phi_n)(t) - S^*(t)\right| \leq P(t) t^{1-1/p} I_n, \tag{9.7.17}$$

   where I_n is the minimum of $\|F\phi_n - r\|_{L^p}$; $\phi_n \in X_n \cap A_n$, $1 < p < \infty$. It has been shown that $I_n \to 0$ as $n \to \infty$. To obtain a regularized solution, a numerical method using polynomial splines has been applied and its order of convergence has been discussed. A numerical example has been considered.

The inverse Stefan problem considered in [263] can be obtained if the following changes are made in the Stefan problem (9.7.1)–(9.7.5).

(1) The condition (9.7.3) is replaced by the condition

$$T_x(0,t) \le 0, \quad 0 < t < t_*, \tag{9.7.18}$$

and $q(x,t) = 0$, in Eq. (9.7.1).

(2) $S(t),\ T_0(x),\ r_1$ and r_2 are specified and $T(x,t)$ is the control. (9.7.19)

The above control problem with Eqs (9.7.18), (9.7.19) can be formulated as an operator equation as follows:

Let X be the space defined as:

$$X = \{T(x,t) \in C^{2,1}(D_{t_*}) \cap C^{1,0}(\overline{D}_{t_*}) \mid \mathcal{L}T = T_{xx} - T_t = 0 \text{ in } D_{t_*} = D_{t_*}(S)\}. \tag{9.7.20}$$

Let the space X be equipped with the norm

$$\|T\|_X = \max\{\|T\|_{\overline{D}_{t_*}}, \|T_x\|_{\overline{D}_{t_*}}\}, \tag{9.7.21}$$

where $\|\cdot\|_{\overline{D}_{t_*}}$ is the supremum norm on $\overline{D}_{t_*}$. It can be proved that X is complete in this norm and X is a Banach space. For $T \in X$, we define an operator $\hat{Q}$ as:

$$\hat{Q}T = \{T(S(t),t),\ T_x(S(t),t),\ T(x,0)\}, \quad T \in X. \tag{9.7.22}$$

The initial and boundary values are given by the trace operator

$$\hat{Q} : X \to C[0,t_*] \times C[0,t_*] \times C[0,b], \tag{9.7.23}$$

and $C[0,t_*] \times C[0,t_*] \times C[0,b]$ is equipped with the product norm

$$\|(\rho_1,\rho_2,T_0)\| = \max\{\|\rho_1\|_{[0,t_*]}, \|\rho_2\|_{[0,t_*]}, \|T_0\|_{[0,b]}\}. \tag{9.7.24}$$

Let A and P be defined as:

$$A = \{T(x,t) \in X : T_x \le 0,\ 0 < t < t_*\}, \tag{9.7.25}$$

$$P = \{r_1(S(t),t),\ r_2(\dot{S}(t),S(t),t),\ T_0(x)\}. \tag{9.7.26}$$

It can be shown that the set A is closed, convex and nonempty. The inverse problem can be stated as:

$$\text{Find } T(x,t) \in A \text{ such that } \hat{Q}T = P. \tag{9.7.27}$$

As discussed earlier, one of the methods to obtain an approximate numerical solution of the problem stated in Eq. (9.7.27) is to minimize the defect over a finite-dimensional space, i.e.

$$\text{minimize } \|\hat{Q}T - P\| \text{ on } X_n \cap A, \quad n \in N. \tag{9.7.28}$$

Here, $X_n \subset X$ and $X_n = \text{span}\{v_1, v_2, \ldots, v_n\}$; $\{v_n\} \subset X$ is a complete family of functions in X, i.e. $\bigcup_{n \in N} X_n$ is dense in X. The problem (9.7.28) has always a solution as it is a finite-dimensional linear approximation problem of Chebyshev-type [264]. The following proposition has been proved in [263].

Proposition 9.7.1. *Let $T^* \in A$ be a solution of $\hat{Q}T = P$. Then T^* is unique and*

$$I_n^* \leq \inf\left\{\left\|T - T^*\right\|_X : T \in X_n \cap A\right\}, \tag{9.7.29}$$

where

$$I_n^* = \inf\{\|\hat{Q}T - P\| : T \in X_n \cap A\}, \quad n \in N. \tag{9.7.30}$$

$I_n^* \to 0$*, as* $n \to \infty$.

The temperature giving the infimum is not a regularized solution. The problem stated in Eq. (9.7.28) has been regularized in [263] by forcing the finite-dimensional solution to lie in a compact set. For achieving this, we make the following assumptions.

(A1) There exists a unique solution $T^* \in A$ of $\hat{Q}T = P$ such that $T_x^*(0,t) \in C^1[0,t_*]$. Let M be a known constant such that $\left\|T_{xt}^*(0,t)\right\|_{[0,t_*]} < M$.

(A2) $\bigcup_{n \in N} X_n$ is dense in X, and $Y = \left\{T \in X : T_x(0,t) \in C^1[0,t_*]\right\}$ be a subset of X with the norm

$$\|T\|_Y = \max\{\|T\|_X,\ \|T_{xt}(0,t)\|_{[0,t_*]}\}. \tag{9.7.31}$$

Let

$$A_M = \{T \in Y \cap A : \|T_{xt}(0,t)\|_{[0,t_*]} \leq M,\ |T_x(0,0) - T_0'(0)| \leq M\}. \tag{9.7.32}$$

If $b = 0$, then no initial temperature is to be prescribed and the second condition in Eq. (9.7.32) will not be there. Under the above assumptions, A_M is convex and closed and has a nonempty interior in Y. A family of regularized solutions can be obtained by considering the problem:

$$\text{Minimize } \|\hat{Q}T - P\| \text{ on } X_n \cap A_M, \quad n \in N. \tag{9.7.33}$$

Let $\hat{T}_n$ be any solution of Eq. (9.7.33) which exists and Eq. (9.7.33) be a finite-dimensional linear approximation problem of Chebyshev-type with linear constraints and let $\hat{I}_n$ be the minimum in Eq. (9.7.33). It has been proved that under assumptions (A1) and (A2) given above, $\hat{I}_n \to 0$ as $n \to \infty$ and as the following proposition suggests, $\hat{T}_n$ converges to T^* (cf. [263]).

Proposition 9.7.2. *Let $S \in C^2[0,t_*]$; for $b = 0$, assume that there exist a constant λ such that $\dot{S}(t) \geq \lambda t$, $\lambda > 0$, $0 \leq t \leq t_*$, and the assumptions (A1) and (A2) hold. Let $\hat{T}_n$ be any solution of Eq. (9.7.33). Then $\hat{T}_n$ converges to T^* in $C(\overline{D}_{t_*})$ and $\hat{T}_{nx}(0,t)$ converges to $T_x^*(0,t)$ in $C\,[0,t_*]$ for $n \to \infty$.*

The heat polynomials

$$v_i(x,t) = i! \sum_{n=0}^{[i/2]} \frac{x^{i-2n}\, t^n}{(i-2n)!\, n!}, \quad (x,t) \in R^2, \quad i = 0, 1, 2, \ldots, \tag{9.7.34}$$

have been taken as the complete set of functions in the numerical example considered in [263]. The minimization problem (9.7.33) has been solved by transforming it into an equivalent semi-infinite linear programming problem similar to the usual Chebyshev approximation problem [264]. After finite-difference discretization, we get a finite-dimensional linear programming problem (with constraints) and this problem can be solved by any linear programming package. The controls $\hat{T}_{nx}(0,t)$ (or $\hat{T}_x(0,t)$) are now taken as an input data for the corresponding direct Stefan problem. Numerical solutions of three inverse problems have been presented.

A one-phase two-dimensional inverse Stefan problem in a bounded domain has been considered in [265] with the temperature on a portion of the fixed boundary serving as a control. The two boundary conditions at the free boundary which are prescribed in the form of temperature and its normal derivative are known, and the initial temperature is also known. By imposing some a priori constraints and suitably defining the term 'solution', it has been established that the solution depends continuously on the data. The defect is defined in terms of the maximum of the three norms of the difference of the three prescribed quantities and the respective computed quantities and the maximum norm is to be minimized. In the numerical example considered, the two-dimensional heat polynomials have been taken as the complete set of functions. The basic approach in proving various results in [265] is similar to that used in [263] but for the numerical solution, instead of a semiinfinite linear programming problem considered in [263], a finite linear programming approach has been considered. If the numerical procedure adopted in [263] is followed for a two-dimensional problem it would require solving a problem with thousands of constraints.

The problem considered in [266] is an extended formulation of the one-phase inverse Stefan problem considered in Eqs (9.7.1)–(9.7.5). It consists of finding an unknown flux prescribed at $x = 0$ for a given $S(t)$. The formulation of the problem is as follows:

$$\begin{aligned} &d(x,t)T_t = (a(x,t)T_x)_x + b(x,t)T_x + e(x,t)T + q(x,t), \quad (x,t) \in D_{t_*}, \\ &\qquad D_{t_*} = \{(x,t) : 0 < x < S(t),\ 0 < t < t_*\}, \end{aligned} \tag{9.7.35}$$

$$a(0,t)T_x(0,t) = f(t) \in C^{\alpha}[0,t_*], \quad 0 \le \alpha \le 1, \tag{9.7.36}$$

$$T(x,0) = \phi(x), \quad 0 \le x \le S(0), \quad a(0,0)\phi'(0) = f(0), \tag{9.7.37}$$

$$T(S(t),t) = \mu\,(S(t),t)\,, \quad 0 < t < t_*, \tag{9.7.38}$$

$$a\,(S(t),t)\ T_x(S\,(t),t) = -\lambda(S(t),t)\dot{S}(t) + \nu(S(t),t), \quad 0 < t < t_*. \tag{9.7.39}$$

Here, $S(t) \in C^1[0,t_*]$ and is known and the inverse Stefan problem consists of determining $f(t)$ and $T(x,t)$. Under suitable assumptions on the data (cf. [266]), a solution of the inverse problem (9.7.35)–(9.7.39) exists and if it exists, then it is stable. The method described below determines the exact solution if it exists, otherwise, a quasi-solution can be obtained. Let the problem (9.7.35)–(9.7.39) be called Problem $\left(P^0\right)$ and let its temperature solution be denoted by $T^0(x,t)$. An auxiliary Problem (P_1) consists of Eqs (9.7.35)–(9.7.38), and let its temperature solution be denoted by $T_1(x,t)$. An auxiliary Problem (P_2) consists of

Eqs (9.7.35)–(9.7.37), (9.7.39) and its temperature solution will be denoted by $T_2(x,t)$. We introduce the following notations:

$$J(f) = \int\int_{D_{t_*}} (T_1 - T_2)^2 dxdt = \|T_1 - T_2\|^2_{L^2}, \quad L^2 = L^2(D_{t_*}), \tag{9.7.40}$$

$$G = \left\{ f(t) \in W_2^1[0,t_*] : a(0,0)\phi'(0) = f(0) \right\}, \tag{9.7.41}$$

$$E = \{f(t) \in W_2^1[0,t_*] : \|f(t)\|_{W_2^1} \le \beta\}, \tag{9.7.42}$$

$$G_\beta = G \cap E = \{f(t) \in W_2^1[0,t_*] : a(0,0)\phi'(0) = f(0),\ \|f(t)\|_{W_2^1} \le \beta\}, \tag{9.7.43}$$

$$J^*(\beta) = \inf_{f \in G_\beta} J(f). \tag{9.7.44}$$

Under some smoothness assumptions on the data, unique solutions of Problem (P_1) and Problem (P_2) exist and these solutions are sufficiently smooth (cf. [266]).

Proposition 9.7.3. *The necessary and sufficient conditions for the existence of a solution of Problem* (P^0) *is that there exists* $f^*(t) \in G_\beta$ *such that* $J^*(\beta) = J(\beta^*) = J(f^*(t)) = 0$.

It has been proved that for the minimization of $J(f)$ on G_β, gradient methods can be used and the convergence is fast (see [267] for gradient methods). Let T_1^β and T_2^β be the solutions of Problems (P_1) and (P_2), respectively, for some β. As $\beta \to \beta^*$, the results in Eq. (9.7.45) hold

$$\|T_1^\beta - T^0\|_{W_2^{2,1}} \to 0, \quad \text{and} \quad \|T_2^\beta - T^0\|_{W_2^{2,1}} \to 0. \tag{9.7.45}$$

If the solution of Problem (P^0) does not exist on G_β, then a quasi-solution can be obtained as follows:

Let $T_\alpha^\beta = \alpha T_1^\beta + (1-\alpha)\, T_2^\beta$, $0 \le \alpha \le 1$. Then for any fixed α, $0 \le \alpha \le 1$,

$$\|T_\alpha^\beta - T^0\|_{W_2^{2,1}} \to 0 \ \text{ as } \beta \to \beta^*. \tag{9.7.46}$$

We call $T_{\alpha_\beta}^\beta$ a *quasi-solution* of Problem (P^0) on G_β for $\beta < \beta^*$ if $T_{\alpha_\beta}^\beta$ minimizes the residual $\Phi_\beta(\alpha)$ in $L_2\,[0,t_*]$, with respect to the parameter α, where,

$$\Phi_\beta(\alpha) = \|(1-\alpha)\{T_2^\beta(S(t),t) - \mu(S(t),t)\}\|^2_{L_2} + \left\| \alpha \left\{ \left(a(x,t) \frac{\partial T_1^\beta}{\partial x} \right)\Bigg|_{x=S(t)} + \lambda(S(t),t)\dot{S} - \nu(S(t),t) \right\} \right\|^2_{L_2}. \tag{9.7.47}$$

Proposition 9.7.4. *As* $\beta \to \beta^*$, $T_{\alpha_\beta}^\beta\,(x,t)$ *converges in the norm of* $W_2^{2,1}$ *to the solution* $T^0(x,t)$ *of Problem* $\left(P^0\right)$.

The inverse Stefan problem considered in [268] is a particular case of the problem considered in Eqs (9.7.1)–(9.7.5) but the control is different from that considered in other problems. Let $D_{t_*} = \{(x,t) : a < S(t) < x < d, \quad 0 < t < t_*\}$. We take $q(x,t) = 0$ in Eq. (9.7.1), $r_1 = 0$ in Eq. (9.7.4), $r_2 = \lambda\dot{S}(t)$ in Eq. (9.7.5) and instead of Eq. (9.7.3), consider

$$T_x(d,t) + \alpha T(d,t) = -v(t). \tag{9.7.48}$$

Here, λ and α are constants. The free boundary $x = S(t)$ is a known monotonically decreasing C^1-function with $S(0) = b > a$. The temperature at $x = d$ is manipulated by a heating (cooling) system according to Eq. (9.7.48). In Eq. (9.7.48), $v(t)$ depends on an unknown function $u(t)$ and this dependence can be expressed as

$$v'(t) + \gamma v(t) = u(t), \text{ a.a. } t \in [0, t_*], \quad v(0) = 0, \quad u \in U, \ \gamma \text{ (constant)}, \tag{9.7.49}$$

$$U = \left\{ u \in L^\infty(0, t_*) : 0 \le u(t) \le M, \text{ a.a. } t \in [0, t_*] \right\}, \quad M \text{ (constant)}. \tag{9.7.50}$$

The inverse problem is to find $u \in U$ such that

$$T_u(S(t), t) = 0, \text{ for every } t \in [0, t_*]. \tag{9.7.51}$$

It is understood here that for a given u, the temperature T_u satisfies all other equations exactly except Eq. (9.7.51). The optimal control problem is to find u which minimizes the integral

$$\int_0^{t_*} (T_u(S(t), t))^2 \, dt; \quad u \in U. \tag{9.7.52}$$

After obtaining necessary conditions for optimality, a descent algorithm for obtaining a solution for this control problem has been presented. Particular attention has been devoted in [268] to find a starting control by a local variations method described in [269].

Analysis and control of Stefan problems by considering weak enthalpy formulations have been studied in [167]. The report [270] also contains several references on control and identification of free boundary problems of parabolic, hyperbolic and elliptic types (see also the cross references in [167, 270]). Our main concern here is the classical Stefan problem.

In most of the references in [254–268], numerical methods employed to obtain numerical solutions have also been justified and attempts have been made to obtain regularized solutions. In view of the ill-posedness of the inverse problems, justification of the numerical methods becomes necessary so that it is certain that the solution we have obtained could be an approximate solution but is not a bad solution (unstable). For reasons of scope and emphasis in this volume, discussion of numerical solutions is limited.

The Tikhonov regularization method can be used for inverse heat transfer problems with or without phase-change. A given problem is to be formulated first as a regularizing functional. In [254–257], it is possible to obtain regularizing functionals for the operator equations to which the problems are reduced and then Tikhonov regularization could have been used but for the purpose of calculating error estimates, a different type of regularization was done. The determination of an optimal value of the Tikhonov regularization parameter α (see Eq. 9.3.16) requires lot of computational effort. The numerical solution of a one-dimensional inverse heat transfer problem (without phase-change) by Tikhonov regularization has been presented in [271]. The original problem is reformulated in terms of obtaining the solution of a Volterra integral equation of the first kind and a finite-difference discretization has been employed to obtain a stable solution with the help of a Tikhonov regularizing functional. In [272], a one-dimensional heat conduction problem without phase-change has been considered in which both the ends of a plate are considered as free boundaries. On using the transformation given in Eq. (9.7.53), a problem on the fixed domain $0 \le \xi \le 1$ can be formulated where ξ is given by

$$\xi = \frac{x - S_1(t)}{S_2(t) - S_1(t)}, \quad t > 0. \tag{9.7.53}$$

Here, S_1 and S_2 ($S_2 < S_1$) are the free boundaries but no phase-change is taking place. The fixed domain formulation of the problem on discretization by finite-difference method gets reduced to a system of nonlinear equations with a tri-diagonal matrix. A regularizing functional in which a stabilizing functional is of the form (9.3.29) has been considered. Terms up to second order temperature derivatives have been included in the stabilizing functional. The choice of the regularization parameter has also been discussed.

A one-dimensional two-phase solidification problem similar to the Neumann problem (1.3.1)–(1.3.7) but formulated in the region $0 \le x \le b$, has been considered in [273]. The initial temperature T_0 of the melt, $S(t)$, T_m, b, $(\partial T_S/\partial x)\,(S(t), t) = q_{yS}\,(t)$, $(\partial T_L/\partial x)(S(t), t) = q_{yl}\,(t)$ are given and the problem is to find $(\partial T_S/\partial x)\,(0, t) = q_{0S}\,(t)$ and $(\partial T_L/\partial x)\,(b, t) = q_{0l}\,(t)$. The temperature gradients $q_{yS}(t)$, $q_{yl}\,(t)$ and $S(t)$ satisfy the Stefan condition

$$K_S q_{yS} - K_L q_{yl} = \rho l \dot{S}(t), \quad x = S(t). \tag{9.7.54}$$

In essence, we have two independent inverse Stefan problems to solve. The aim in this problem is to calculate the boundary fluxes at the fixed boundaries that will give the desired freezing front velocity on which depends the liquid feeding to the mould and hence the desired cast structure.

For the numerical computations, a boundary element method with constant elements has been used in conjunction with the *sensitivity analysis* discussed in [274]. By using transformations of the type (9.7.53) (after appropriate modifications), fixed domain formulations can be obtained for both solid and liquid phases. The temperatures in the solid and liquid regions can be expressed in terms of integrals using appropriate Green's functions (see Eq. 9.6.6). Note that since $S(t)$ is known, we are solving only parabolic heat equations. These integral representations are required in the numerical method which uses boundary elements. It may be noted that the integral representations of temperatures in the present case are different from Eq. (9.6.8) as in the fixed domain formulations, the heat equations will get transformed. Temperatures at the fixed boundaries of the transformed regions can be obtained from their integral representations and on discretization of integrals the matrix equations which contain unknown nodal values of temperatures can be obtained. For the description of sensitivity analysis, we consider here only the solid phase. Let q_{0S}^m be the unknown flux during the time interval (t_{m-1}, t_m), i.e. during the time step m and all $q_{0S}^{m_1}$, for $m_1 < m$ are known. To stabilize the solution of the inverse problem, it is assumed that q_{0S} is constant at the future $(r-1)$ time steps. This assumption is used temporarily until q_{0S}^m is calculated. The *sensitivity coefficients* for this problem are defined as

$$\left\{T_S^{m+i-1};\ q_{0S}^m\right\} = \frac{\partial\left\{T_S^{m+i-1}\right\}}{\partial q_{0S}^m}, \quad i = 1 \text{ to } r. \tag{9.7.55}$$

Here, the notation $\{T_S\}$ stands for a matrix. The error in the prescribed temperature T_m at $x = S(t)$ and its calculated value is to be minimized with respect to q_{0S} and this gives an equation to determine q_{0S}^m, $m > 1$ at time steps other than the initial time step. Similarly q_{0l}^m

can be obtained. Starting solutions have been obtained with the help of some approximate analytical solutions developed in [273].

A problem of estimating unknown free boundary in a two-dimensional heat conduction problem with the help of some temperature measurements along a portion of the fixed boundary of the region has been considered in [275]. The formulation of the direct problem is as follows:

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=\frac{\partial T}{\partial t}, \quad 0<x<b, \quad 0<y<S(x,t), \quad 0<t<t_*, \tag{9.7.56}$$

$$\frac{\partial T}{\partial x}=0, \text{ at } x=0, \quad 0<t<t_*, \tag{9.7.57}$$

$$\frac{\partial T}{\partial x}=0, \text{ at } x=b, \quad 0<t<t_*, \tag{9.7.58}$$

$$\frac{\partial T}{\partial y}=q_0/K, \text{ at } y=0, \quad 0<t<t_*, \tag{9.7.59}$$

$$T=T_1, \text{ at } y=S(x,t), \quad 0<t<t_*, \tag{9.7.60}$$

$$T=T_0, \text{ at } t=0. \tag{9.7.61}$$

Here, $y = S(x,t)$ is the equation of the free boundary. In the direct problem considered in [275], $S(x,t)$ is taken as known and the temperature $T(x,y,t)$ is to be determined. In the inverse problem corresponding to this direct problem, $S(x,t)$ is unknown, and in this case we should have two conditions at $y = S(x,t)$. Instead of imposing one more condition at $y = S(x,t)$, we take some temperature readings denoted by $\hat{T}_m(t)$ along $y = 0$. Let

$$T(x_m,0,t)=\hat{T}_m(t), \quad m=1 \text{ to } M, \quad 0<t<t_*. \tag{9.7.62}$$

The inverse problem consists of determining $T(x,t)$ and $S(x,t)$ satisfying Eqs (9.7.56)–(9.7.62). For the solution of the inverse problem, *conjugate gradient method* has been used in conjunction with a boundary element method. For the boundary integral formulation of the present problem and the discretization of equations, see [275, 276]. The method of solution by the conjugate gradient method for the present inverse problem requires the solution of three problems, namely, the direct, sensitivity and adjoint. The solution of the inverse problem has been obtained in such a way that the following functional is minimized

$$J\{S(x,t)\}=\int_{t=0}^{t_*}\sum_{m=1}^{M}\{T_m(t)-\hat{T}_m(t)\}^2 dt. \tag{9.7.63}$$

Here, $T_m(t)$ are the temperatures at $(x_m, 0, t)$ which are obtained by solving a direct problem in which an approximate $\hat{S}(x,t)$ is used in the place of exact $S(x,t)$. Note that in the inverse problem we start with an approximate value of $S(x,t)$. The data $\hat{T}_m(t)$ can have measurement errors. The numerical results indicate that the solution of the inverse problem obtained by the above method remains stable as measurement errors are increased and the number of observed temperature locations are decreased.

A sequential algorithm for the identification of the position of the moving boundary in the one and two-dimensional Stefan problems from discrete measurements of temperatures and fluxes at the fixed boundaries has been presented in [277]. In most cases, the direct measurements of the position of the phase-change boundary is impracticable. Identification of the interface position is, therefore, to be incorporated in the feedback control algorithm. In a two-phase Stefan problem, the physical phenomenon occurring in the liquid phase is difficult to model accurately because of some factors such as surface tension, thermal and solutal convection, and external forces. This drawback can be eliminated by having all the measurements made in the solid phase and considering a one-phase problem consisting of only the solid phase.

The two-dimensional one-phase problem considered in [277] is similar to the problem described in Eqs (9.7.56)–(9.7.61) except that in the condition (9.7.60), T_1 is now the fusion (melting) temperature. In the direct Stefan problem described in [277], $S(x,t)$ is known, and either temperature or flux is known at $y = 0$. The temperature is to be determined everywhere. If the temperature is prescribed at $y = 0$, then after calculating the temperature in the direct problem (since $S(x,t)$ is known we are solving a heat conduction problem without phase-change), the flux can be determined at $y = 0$ so that both T and $\partial T/\partial y$ are known at $y = 0$. In the inverse problem both $S(x,t)$ and $T(x,t)$ are to be determined. Therefore, an overspecified boundary condition is imposed at $y = 0$ in terms of either the calculated temperature (if the flux is prescribed) or the calculated flux (if the temperature is prescribed). As mentioned in the earlier problems, for the solution of an inverse problem, a direct problem with some known value of $S(t)$ is first solved to determine the approximate temperature values everywhere. If at $y = 0$, the temperature is taken as prescribed in the direct problem, then this temperature is a model input and the model output will be the flux at $y = 0$. In this way we have a pair $(T, \partial T/\partial y)$. The numerical solution of $\partial T/\partial y$ should be compared for correctness with the prescribed $\partial T/\partial y$ which is an overspecified boundary condition. Similarly we may have another pair $(\partial T/\partial y, T)$. For the given input data, sensitivity coefficients for the output data with respect to $S(t)$ have been calculated. It may be noted that if the flux is prescribed at $y = 0$, then $T(x,0,t)$ is the output and $\partial T/\partial y$ is the input. For a given $S(t)$, output/input sensitivity coefficients have also been calculated. In the one-dimensional case, it was found that the best approach consists of taking prescribed temperature as the model input [277].

Let the value of $S(t)$ at time $t_{m+1} = (m+1)\,\Delta t$ (Δt is the time step), $m = 0, 1, 2, \ldots$ be denoted by S_{m+1}. The value of S_{m+1} has been obtained in [277] through the minimization of a penalized least-squares criterion evaluated in the time interval $\left[t_{m+1}, t_{m+r}\right]$. The length of the observation horizon is $\tau = r\Delta t$. The functional $J_\alpha\left(S_{m+1}\right)$ to be minimized is taken as

$$J_\alpha(S_{m+1}) = \frac{1}{r}\sum_{i=1}^{r} e_{m+i}^{T_r} e_{m+i} + \alpha G(S_{m+1}), \tag{9.7.64}$$

where

$$G(S_{m+1}) = \left[S_{m+1} - S_m\right]^{T_r}\left[S_{m+1} - S_m\right], \tag{9.7.65}$$

$$e_{m+i} = \hat{Z}_{m+i} - Z_{m+i}(S_{m+i}; \hat{U}_{m+1}, \ldots, \hat{U}_{m+i}). \tag{9.7.66}$$

Here, T_r stands for the transpose of a matrix, $\hat{Z}$ and $\hat{U}$ are the estimated quantities, α is the regularization coefficient, Z stands for the model output, and U stands for the model input. The matrices Z and U are defined below

$$Z_{m+i} = \left[z\left\{0, (m+i)\Delta t\right\}, z\left\{\Delta x, (m+i)\,\Delta t\right\}, \ldots, z\left\{(N-1)\,\Delta x, (m+i)\,\Delta t\right\}\right]^{T_r}, \tag{9.7.67}$$

$$U_{m+i} = \left[u\left\{0, (m+i)\,\Delta t\right\}, u\left\{\Delta x, (m+i)\,\Delta t\right\}, \ldots, u\left\{(N-1)\,\Delta x, (m+i)\,\Delta t\right\}\right]^{T_r}. \tag{9.7.68}$$

In Eqs (9.7.67), (9.7.68), N stands for the number of nodal points in the x-direction in the one-dimensional case. These matrices in the two-dimensional case can be similarly written. Noisy data can also be considered. For optimization, Gauss–Newton algorithm has been considered.

Till now we have discussed ill-posedness of the inverse problems and not of any direct problem. An example showing the ill-posedness of a one-dimensional one-phase oblation problem has been given in [278]. The formulation of this problem is similar to that given in Eqs (10.1.34)–(10.1.38) (see Chapter 10) except that f, λ and μ in [278] are not constant but functions of time. The heat source, initial temperature, flux at $x = 0$, flux at $x = S(t)$ and the latent heat are known functions of time and a parameter α, $\alpha > 0$. The prescribed quantities (cf. [278]) are taken in such a way that it is possible to obtain an exact analytical solution of the problem in terms of α and variables x and t, and for any $\alpha > 0$ the prescribed quantities and their derivatives of any order are less than unity. Furthermore, the difference in the derivatives of any order of these data are uniformly close. An exact analytical solution of the free boundary can be obtained in the form $S(t; \alpha) = \alpha t$. The difference in the two values of $S(t; \alpha)$ increases with α and t and the solution becomes unstable. This example shows that the well-posedness of many problems is conditional and if the prescribed data take the solution beyond the limits of well-possedness, the solution becomes unstable. The regularization of this problem has been achieved by defining a suitable solution space.