Assignment Week 3 Time Series Econometrics

Rose Barzilai | 2596415 | r.n.barzilai@student.vu.nl
Zuzana Leova | 2591033 | z.leova@student.vu.nl
Charlotte Taman | 2100584 | c.e.j.taman@student.vu.nl
Femke Vedder | 2555497 | f.m.l.vedder@student.vu.nl

Group 1

February 26 2018

1 Figure 2.4 of Nile Data

1.a Graph

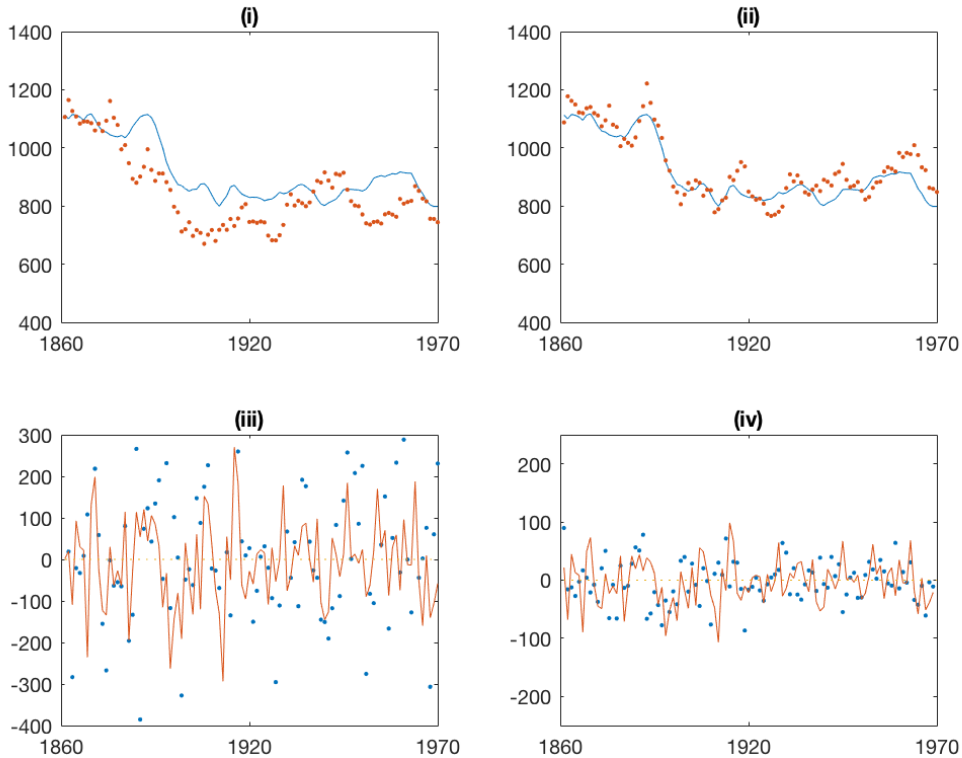

Figure 1: Simulation: (i) smoothed state $\hat{\mu}_t$ (solid line) and sample μ_t^+(dots); (ii) smoothed state $\hat{\mu}_t$ (solid line) and sample $\tilde{\mu}_t$(dots); (iii) smoothed observation error $\hat{\epsilon}_t$ (solid line) and sample $\tilde{\epsilon}_t$ (dots); (iv) smoothed state error $\hat{\eta}_t$(solid line) and sample $\tilde{\eta}_t$ (dots).

1.b Explanation

In this figure the local level model is used:

$$\begin{aligned} y_t &= \mu_t + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_\epsilon^2) \\ \mu_{t+1} &= \mu_t + \eta_t, \quad \eta_t \sim N(0, \sigma_\eta^2) \end{aligned} \tag{1}$$

This figure represents the difference between simulating a sample from the local level model unconditionally and conditional on the observations. Figure (i) presents the smoothed state $\hat{\mu_t}$ and a sample generated by the local level model unconditionally. What we can see from this figure, is that the two series have nothing in common.

Figure (ii) represents the smoothed state together with a sample generated conditional on the observations. The generated sample here is much closer to $\hat{\alpha_t}$.

Figure (iii) and (iv) represents the smoothed disturbances together with a sample from the corresponding disturbances conditional on the observations.

For simulation from the linear Gaussian state space model unconditionally, no data is known. If the model is well specified, the disturbances can be simulated from specified distributions. Now we can simulate y and μ.

Unconditional simulation:

1. Sample μ_1 from $\mu_1 \sim \mathcal{N}(a_1, P_1)$ to obtain μ_1^+. Here $a_1 = 0$ and $P_1 = 10^7$;
2. Sample η_t from $\eta_t \sim NID(0, \sigma_\eta^2)$ to obtain η_t^+ for $t = 1, ..., n-1$;
3. Compute $\mu_{t+1}^+ = \mu_t^+ + \eta_t^+$ for $t = 1, ..., n-1$;
4. Sample ϵ_t from $\epsilon_t \sim N(0, \sigma_\epsilon^2)$ for given σ_ϵ^2 and $t = 1, ..., n$ to obtain ϵ_t^+;
5. Compute $y_t^+ = \mu_t^+ + \epsilon_t^+$ for $t = 1, ..., n$.

The values y_t and μ_t are realizations of the state space model.

Conditional simulation is used when we have the data y. So we can obtain the simulated level of μ given the data y, which is called simulation smoothing.

Conditional simulation:

1. To obtain y^+ and μ^+ we need to performs unconditional simulation (given σ_ϵ^2 and σ_η^2);
2. Apply Kalman Filter smoothing on the data y and obtain $\hat{\mu} = \mathbb{E}(\mu|y)$;
3. Apply Kalman Filter smoothing on the simulated data y^+ and obtain $\hat{\mu}^+ = \mathbb{E}(\mu|y^+)$;
4. Compute $\tilde{\mu} = \hat{\mu} + \mu^+ - \hat{\mu}^+$.

2 Exercise 3.1 Kalman Filter proof for a Linear Gaussian Model

2.a Provide the appropriate arguments for each step in the derivation of the Kalman Filter

The linear Gaussian state space model is given by:

$$\begin{aligned} \alpha_{t+1} &= T_t\alpha_t + R_t\zeta_t, \quad \zeta_t \sim \mathcal{NID}(0, Q_t) \\ y_t &= Z_t\alpha_t + \epsilon_t, \qquad \epsilon_t \sim \mathcal{NID}(0, H_t) \end{aligned} \tag{2}$$

We will derive each step of the Kalman filter and provide the appropriate arguments connected to each step. We will start with the prediction error v_t.

Prediction error v_t:

$$\begin{aligned} v_t &\overset{(1)}{=} y_t - \mathbb{E}(y_t|Y_{t-1}) \\ &\overset{(2)}{=} y_t - \mathbb{E}(Z_t\alpha_t + \epsilon_t|Y_{t-1}) \\ &\overset{(3)}{=} y_t - Z_t\mathbb{E}(\alpha_t|Y_{t-1}) + \mathbb{E}(\epsilon_t) \\ &\overset{(4)}{=} y_t - Z_t a_t \end{aligned}$$

The top argument holds by definition: the prediction error is the difference between the actual observation y_t and the expectation of y_t given the past data. For the second step we fill in the equation for y_t. The third step holds by the linearity of expectations and because ϵ_t and Y_{t-1} are uncorrelated, also Z_t can be taken outside the expectation since it is a fixed matrix. Now we are left with the expectation of α_t given the past data, which is equal to a_t. Hence, the prediction error v_t is equal to $y_t - Z_t a_t$. We can rewrite this by filling in y_t:

$$\begin{aligned} v_t &= Z_t \alpha_t + \epsilon_t - Z_t \alpha_t \\ &= Z_t(\alpha_t - a_t) + \epsilon_t \end{aligned}$$

The expectation of the prediction error:

$$\begin{aligned} \mathbb{E}[v_t] &\overset{(1)}{=} \mathbb{E}[Z_t(\alpha_t - a_t) + \epsilon_t] \\ &\overset{(2)}{=} Z_t \mathbb{E}(\alpha_t - a_t) + \mathbb{E}(\epsilon_t) = 0 \end{aligned}$$

Let $x = \alpha_t$ and $y = Y_{t-1}$. Now it follows from Lemma I that for $e = x - \mathbb{E}[x|y] = \alpha_t - a_t$:
$\mathbb{E}(\alpha_t - a_t) = 0$. Here we applied the reversed law of iterated expectations: $\mathbb{E}[a_t] = \mathbb{E}[\mathbb{E}[\alpha_t|Y_{t-1}]] = \alpha_t$

Variance of the prediction error F_t:

$$\begin{aligned}
F_t &\overset{(1)}{=} \mathbb{V}ar(v_t) \\
&\overset{(2)}{=} \mathbb{V}ar(Z_t(\alpha_t - a_t) + \epsilon_t) \\
&\overset{(3)}{=} Z_t \mathbb{V}ar(\alpha_t - a_t) Z_t' + H_t \\
&\overset{(4)}{=} Z_t \mathbb{V}ar(\alpha_t | Y_{t-1}) Z_t' + H_t \\
&\overset{(5)}{=} Z_t P_t Z_t' + H_t
\end{aligned}$$

F_t is the variance of the prediction error. When substituting this error as derived earlier we obtain the equation in step 2. In the third step we obtain the variance of ϵ_t, which is H_t. Moreover, since Z_t is a fixed matrix we can apply the following rule: $\mathbb{V}ar(M'X) = M'\mathbb{V}ar(X)M$, where M is a fixed matrix. In the fourth step Lemma I is applied again: $\mathbb{V}ar(\alpha_t - a_t) = (\alpha_t | Y_{t-1}) = P_t$

To derive a_{t+1} Lemma II will be used:

$$\begin{aligned}
a_{t+1} &\overset{(1)}{=} \mathbb{E}(\alpha_{t+1} | Y_t) \\
&\overset{(2)}{=} \mathbb{E}(\alpha_{t+1} | Y_{t-1}, v_t) \\
&\overset{(3)}{=} \mathbb{E}(\alpha_{t+1} | Y_{t-1}) + cov(\alpha_{t+1}, v_t) \mathbb{V}ar(v_t)^{-1} v_t
\end{aligned} \tag{3}$$

The first step shows the definition of a_{t+1}. Since Y_t is consists of the past observations and the current observation, we can replace Y_t by Y_{t-1} and y_t. Because v_t is a linear combination of y_t, we can replace y_t by v_t and no information is lost. After this we can apply Lemma II to obtain the third step.

To apply Lemma II for the Kalman Filter let $x = \alpha_{t+1}$, $y = Y_{t-1}$ and $z = v_t$. The following four elements are used:

$$\begin{aligned}
\mathbb{E}(\alpha_{t+1} | Y_{t-1}) &\overset{(1)}{=} (T_t \alpha_t + R_t \zeta_t | Y_{t-1}) \\
&\overset{(2)}{=} T_t \mathbb{E}(\alpha_t | Y_{t-1}) + R_t(\zeta_t) \\
&\overset{(3)}{=} T_t a_t
\end{aligned}$$

Now we need to derive the covariance between α_{t+1} and v_t:

$$\begin{aligned}
cov(\alpha_{t+1}, v_t) &\overset{(1)}{=} \mathbb{E}[\alpha_{t+1} v_t'] - \mathbb{E}[\alpha_{t+1}]\mathbb{E}[v_t'] \\
&\overset{(2)}{=} \mathbb{E}[(T_t \alpha_t + R_t \eta_t)(Z_t(\alpha_t - a_t) + \epsilon_t)'] - 0 \\
&\overset{(3)}{=} T_t \mathbb{E}[\alpha_t(\alpha_t - a_t)'] Z_t' + T_t \mathbb{E}[\alpha_t \epsilon_t'] + R_t \mathbb{E}[\eta_t(\alpha_t - a_t)'] Z_t' + R_t \mathbb{E}[\eta_t \epsilon_t'] \\
&\overset{(4)}{=} T_t \mathbb{E}[\alpha_t(\alpha_t - a_t)'] Z_t + 0 + 0 + R_t \mathbb{E}[(\epsilon_t \eta_t')'] \\
&\overset{(5)}{=} T_t \mathbb{V}ar(\alpha_t | Y_{t-1}) Z_t + R_t G_t' \\
&\overset{(6)}{=} T_t P_t Z_t + R_t G_t'
\end{aligned}$$

The conditioning on Y_{t-1} in step 5 can be used because Y_{t-1} provides no additional information for the prediction error $(\alpha_t - a_t)$. Therefore, with and without conditioning is the same.

The third element we need to use is the variance of the predicition error. We derived this earlier:

$$F_t = \mathbb{V}ar(v_t) = Z_t P_t Z_t' + H_t$$

And the last element is v_t. Now we can add all these element together in equation (3). This gives us:

$$\begin{aligned} a_{t+1} = \mathbb{E}(\alpha_{t+1}|Y_{t-1}, v_t) &\overset{(1)}{=} \mathbb{E}(\alpha_{t+1}|Y_{t-1}) + cov(\alpha_{t+1}, v_t)\mathbb{V}ar(v_t)^{-1}v_t \\ &\overset{(2)}{=} T_t a_t + (T_t P_t Z_t' + R_t G_t')F_t^{-1}v_t \\ &\overset{(3)}{=} T_t a_t + K_t v_t \end{aligned}$$

with $K_t = (T_t P_t Z_t' + R_t G_t')F_t^{-1}$.

2.b Please derive the updating equation for P_{t+1}.

To derive P_{t+1} again Lemma II is used:

$$\begin{aligned} P_{t+1} &\overset{(1)}{=} \mathbb{V}ar(\alpha_{t+1}|Y_t) \\ &\overset{(2)}{=} \mathbb{V}ar(\alpha_{t+1}|Y_{t-1}, v_t) \\ &\overset{(3)}{=} \mathbb{V}ar(\alpha_{t+1}|Y_{t-1}) - cov(\alpha_{t+1}, v_t)\mathbb{V}ar(v_t)^{-1}cov(\alpha_{t+1}, vt)' \end{aligned} \tag{4}$$

We already derived the covariance between α_{t+1} and v_t and the variance of v_t. We still need to derive the $\mathbb{V}ar(\alpha_{t+1}|Y_{t-1})$:

$$\begin{aligned} \mathbb{V}ar(\alpha_{t+1}|Y_{t-1}) &\overset{(1)}{=} \mathbb{V}ar(T_t\alpha_t + R_t\eta_t|Y_{t-1}) \\ &\overset{(2)}{=} T_t P_t T_t' + R_t Q_t R_t' \end{aligned}$$

In the second step we obtain the variance of η_t, which is Q_t. Moreover, since T_t and R_t are fixed matrices we can apply the following rule: $\mathbb{V}ar(M'X) = M'\mathbb{V}ar(X)M$, where M is a fixed matrix.

No we can add everything together in equation (4) and we get:

$$\begin{aligned} P_{t+1} &\overset{(1)}{=} \mathbb{V}ar(\alpha_{t+1}|Y_{t-1}, v_t) \\ &\overset{(2)}{=} \mathbb{V}ar(\alpha_{t+1}|Y_{t-1}) - cov(\alpha_{t+1}, v_t)\mathbb{V}ar(v_t)^{-1}cov(\alpha_{t+1}, vt)' \\ &\overset{(3)}{=} T_t P_t T_t' + R_t Q_t R_t' - (T_t P_t Z_t + R_t G_t')F_t^{-1}(T_t P_t Z_t + R_t G_t')' \\ &\overset{(4)}{=} T_t P_t T_t' + R_t Q_t R_t' - (T_t P_t Z_t + R_t G_t')F_t^{-1}F_t F_t^{-1}(T_t P_t Z_t + R_t G_t')' \\ &\overset{(5)}{=} T_t P_t T_t' + R_t Q_t R_t' - K_t F_t K_t' \end{aligned}$$

Where again the Kalman Filter is $(T_t P_t Z_t + R_t G_t')F_t^{-1}$

2.c Provide the update equation for a_{t+1} in terms of a_t and y_t.

$$\begin{aligned} a_{t+1} &\overset{(1)}{=} T_t a_t + K_t v_t \\ &\overset{(2)}{=} T_t a_t + K_t[y_t - Z_t a_t] \\ &\overset{(3)}{=} T_t a_t + K_t y_t - K_t Z_t a_t \\ &\overset{(4)}{=} [T_t - K_t Z_t] a_t + K_t y_t \end{aligned}$$

3 Exercise 3.1 An alternative Kalman Filter

The linear Gaussian state space model is given in equation (9). The prediction error and its variance are given by the following equations:

$$\begin{aligned} v_t &= y_t - Z_t a_t \\ F_t &= Z_T P_t Z_t' + H_t \end{aligned}$$

We derive the filtered state as follows:

$$\begin{aligned} a_{t|t} = \mathbb{E}(\alpha_t|Y_t) &\overset{(1)}{=} \mathbb{E}(\alpha_t|Y_{t-1}, v_t) \\ &\overset{(2)}{=} \mathbb{E}(\alpha_t|Y_{t-1}) + cov(\alpha_t, v_t)\mathbb{V}ar(v_t)^{-1} v_t \\ &\overset{(3)}{=} a_t + Z_t P_t v_t/(P_t + \sigma_\epsilon^2) \\ &\overset{(4)}{=} a_t + P_t Z_t' F_t^{-1} v_t \\ &\overset{(5)}{=} a_t + M_t v_t \end{aligned}$$

with $M_t = P_t Z_t' F_t^{-1}$. The first equation is substituted by the definition of the filtered state, the next holds because the vector Y_t is separated into Y_{t-1} and y_t. For y_t, v_t is used and no information is lost. For the second step, Lemma II is used. For Lemma II it is supposed that x, y and z are jointly Normally distributed vectors with $E(z) = 0$ and $\Sigma = 0$. Then

$$\begin{aligned} \mathbb{E}(x|y,z) &= \mathbb{E}(x|y) + \Sigma_{xz}\Sigma_{zz}^{-1} z \\ \mathbb{V}ar(x|y,z) &= \mathbb{V}ar(x|y) + \Sigma_{xz}\Sigma_{zz}\Sigma_{xz}' \end{aligned} \tag{5}$$

The $cov(\alpha_t, v_t)$ is derived as follows

$$\begin{aligned} cov(\alpha_t, v_t) &\overset{(1)}{=} \mathbb{E}(\alpha_t v_t) - \mathbb{E}(\alpha_t)\mathbb{E}(v_t) \\ &\overset{(2)}{=} \mathbb{E}[\alpha_t(y_t - Z_t a_t)] \overset{(3)}{=} \mathbb{E}[\alpha_t(Z_t\alpha_t + \epsilon_t - a_t)] \\ &\overset{(4)}{=} Z_t\mathbb{E}(\alpha_t^2) + \mathbb{E}(\alpha_t\epsilon_t) - a_t\mathbb{E}(\alpha_t) \\ &\overset{(5)}{=} Z_t\mathbb{V}ar(\alpha_t) + (\mathbb{E}(\alpha_t))^2 - a_t\mathbb{E}(\mathbb{E}(\alpha_t|Y_{t-1})) \\ &\overset{(6)}{=} Z_t P_t + a_t^2 - a_t^2 = Z_t P_t \end{aligned}$$

Again, for the second step v_t was filled in as derived before. Also $\mathbb{E}(v_t) = 0$, therefore $\mathbb{E}(\alpha_t)\mathbb{E}(v_t) = 0$. For the third step, the observation equation was substituted for y_t. In the fifth step we applied basic operations and the fact that $\mathbb{E}(\alpha_t \epsilon_t) = 0$. The last equality holds because of Lemma I $\mathbb{V}ar(\alpha_t) = \mathbb{E}(\mathbb{V}ar(\alpha_t|Y_{t-1})) + \mathbb{V}ar(\mathbb{E}(\alpha_t|Y_{t-1})) = P_t$ and $(\mathbb{E}(\alpha_t))^2 = (\mathbb{E}(\mathbb{E}(\alpha_t|Y_{t-1})))^2 = a_t^2$.

The derivation of the predicted state is as follows:

$$\begin{aligned}
a_{t+1} &\overset{(1)}{=} \mathbb{E}(\alpha_{t+1}|Y_t) \\
&\overset{(2)}{=} \mathbb{E}(T_t\alpha_t + R_t\zeta_t|Y_t) \\
&\overset{(3)}{=} T_t\mathbb{E}(\alpha_t|Y_t) + R_t\mathbb{E}(\zeta_t|Y_t) = T_t a_{t|t}
\end{aligned}$$

In the second step the state equation was substituted for α_{t+1}. The third step holds because of the linearity of expectations and because Y_t and ζ_t are uncorrelated. Morever, $\mathbb{E}(\zeta_t) = 0$.

Variance of the filtered state:

$$\begin{aligned}
P_{t|t} &\overset{(1)}{=} \mathbb{V}ar(\alpha_t|Y_t) \\
&\overset{(2)}{=} \mathbb{V}ar(\alpha_t|Y_{t-1}, v_t) \\
&\overset{(3)}{=} \mathbb{V}ar(\alpha_t|Y_{t-1}) - cov(\alpha_t, v_t)\mathbb{V}ar(v_t)^{-1}cov(\alpha_t, v_t)' \\
&\overset{(4)}{=} P_t - Z_tP_tF_t^{-1}(Z_tP_t)' \\
&\overset{(5)}{=} P_t - P_tZ_t'F_t^{-1}F_tF_t^{-1}(Z_tP_t)' \\
&\overset{(6)}{=} P_t - M_tF_tM_t'
\end{aligned}$$

With $M = P_tZ_t'F_t^{-1}$. The first step holds by definition. In the second step Y_t is separated in Y_{t-1} and y_t. For y_t, we used v_t without loss of information. In the third step, Lemma II is applied. The last step holds because $Pt = \mathbb{V}ar(\alpha_t|Y_{t-1})$. The $cov(\alpha_t, v_t)$ and $\mathbb{V}ar(v_t)$ are derived above.

Variance of the predicted state:

$$\begin{aligned}
P_{t+1} &\overset{(1)}{=} \mathbb{V}ar(\alpha_{t+1}|Y_t) \\
&\overset{(2)}{=} \mathbb{V}ar(T_t\alpha_t + R_t\zeta_t|Y_t) \\
&\overset{(3)}{=} T_t\mathbb{V}ar(\alpha_t|Y_t)T_t' + R_t\mathbb{V}ar(\zeta_t)R_t' \\
&\overset{(4)}{=} T_tP_{t|t}T_t' + R_tQ_tR_t'
\end{aligned}$$

The first step holds by definition. Then the state equation is substituted. In the third step we applied the following rule: $\mathbb{V}ar(M'X) = M'\mathbb{V}ar(X)M$, where M is a fixed matrix. We derived $\mathbb{V}ar(\alpha_t|Y_t)$ before, so therefore we can substitute it with $P_{t|t}$.

4 Assignment 3.1: SV Model

4.a The financial series is in actual levels. Transform the series to returns, present graphs and descriptive statistics.

The given data in the `sv.dat` file is already given in returns. So there is no need to apply the log return transformation where $r_t = log(P_t/P_{t-1})$.

Figure 2 shows a line plot and a histogram of the returns used for this assignment.

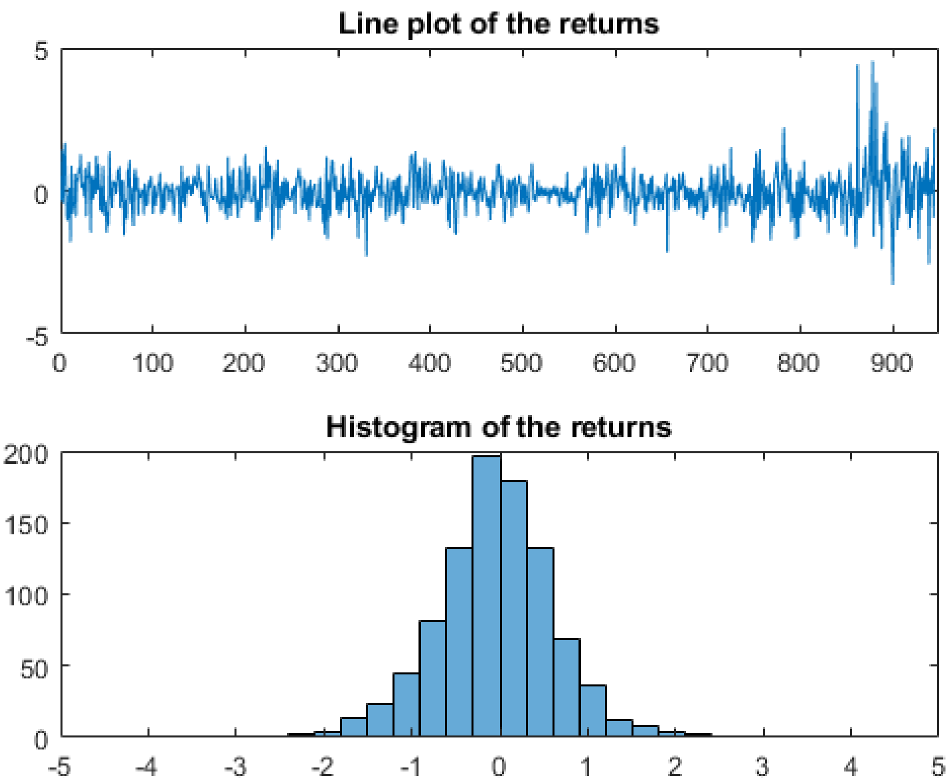

Figure 2: A graphical representation of the returns.

Table 1 shows us the descriptive statistics of the returns.

	Returns
Mean	-0.0353
Median	-0.0457
Min	-3.2961
Max	4.5345
SD	0.7111
Skewness	0.6042
ExKurtosis	4.8619

Table 1: Descriptive statistics of the returns

4.b The SV model can be made linear by transforming the returns data to $x_t = \log(y_t^2)$. This is the basis of the QML method. Compute x_t and present a graph.

To compute x_t we used the following formula: $x_t = \log((y_t - \mu_y)^2)$. By demeaning y_t we avoid taking logs of zeros. The resulting model has therefore the linear form:

$$x_t = \log((y_t - \mu_y)^2) = log(\sigma_t \epsilon_t)^2 = \log \sigma_t^2 + log\epsilon_t^2 = h_t + u_t, \tag{6}$$

where $h_t = \log \sigma_t^2$ and $u_t = \log \epsilon_t^2$ After the computing and plotting x_t we obtained the following plot, shown in Figure 3. The mean of y_t is given by -0.0353.

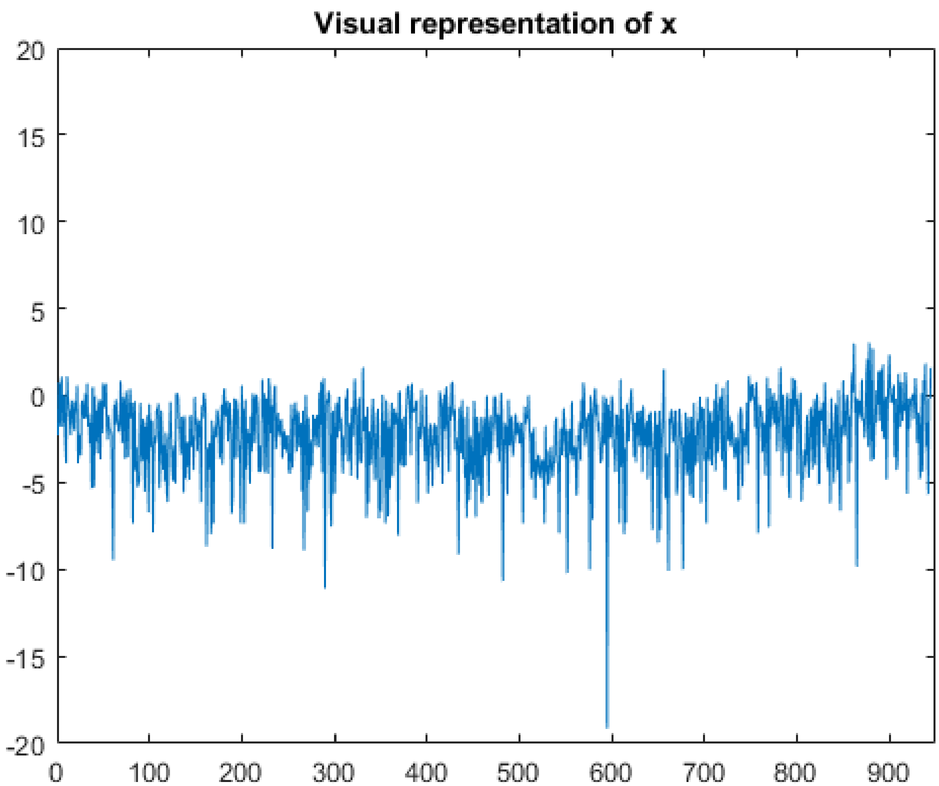

Figure 3: Plot of the transformed returns.

4.c The disturbances in the model for x_t will not be normally distributed. But we can assume that they are normal with mean and variance corresponding to those of the log_{χ^2} distribution. Estimate the unknown coefficients by the QML method using the Kalman filter and present the results in a Table. Hint: the $\log_{\chi^2}$ (1) distribution has mean -1.27 and variance $\pi^2/2 = 4.93$.

The SV model is given by the following model:

$$\begin{aligned} h_{t+1} &= \omega + \phi h_t + \sigma_\eta \eta_t, & \eta_t &\sim \mathcal{NID}(0, \sigma_\eta^2) \\ x_t &= h_t + u_t, & u_t &\sim \mathcal{NID}(-1.27, 4.93) \end{aligned} \tag{7}$$

The linear Gaussian state space model is given by:

$$\begin{aligned} \alpha_{t+1} &= T_t \alpha_t + R_t \zeta_t, \quad \zeta_t \sim \mathcal{NID}(0, Q_t) \\ y_t &= Z_t \alpha_t + \epsilon_t, \qquad \epsilon_t \sim \mathcal{NID}(0, H_t) \end{aligned} \tag{8}$$

To use our code for kalman filter for the state space model from previous assignment, we need to substitute parameters of general state space model for parameters for SV model such as: $\alpha_t = h_t$, $y_t = x_t$, $Z_t = 1$, $\epsilon_t = u_t$, $T_t = \phi$, $R_t = \sigma_\eta$, $H_t = 4.93$ $Q_t = \sigma_\eta^2$ and $\zeta_t = \eta_t$.

In order to estimate the coefficients of σ_η^2, ω and ϕ by the QML method using the Kalman filter, it is convenient to include mean adjustments in the state space model. This is caused by the fact the our model includes the ω parameter in the state equation, and the disturbances u_t in the observation equation are approximated by the normal distribution with mean -1.27.

For this estimation we will use the Kalman filter for models with mean adjustments. This is given by the following form:

$$\begin{aligned} \alpha_{t+1} &= T_t \alpha_t + d_t + R_t \zeta_t, \quad \zeta_t \sim \mathcal{NID}(0, Q_t) \\ y_t &= Z_t \alpha_t + c_t + \epsilon_t, \qquad \epsilon_t \sim \mathcal{NID}(0, H_t) \end{aligned} \tag{9}$$

In the SV model the d_t is equal to ω, which is a constant, and c_t is equal to -1.27. So here we have added a constant (-1.27) to the observation equation and another constant (ω) to the state transition equation. Because disturbances in both equations are normally distributed ($u_t = \log \epsilon_t^2 \sim \mathcal{NID}(-1.27, 4.93)$) and $\eta_t \sim \mathcal{NID}(0, \sigma_\eta^2)$, we can estimate the unknown coefficients σ_η^2, ω and ϕ by QML method, where the log-likelihood function is maximized with a use of equations from the Kalman filter. The log-likelihood is given by

$$\log L = -\frac{n}{2} \log(2\pi) - \frac{1}{2} \sum_{t=1}^{n} \left(\log F_t + \frac{v_t^2}{F_t} \right) \tag{10}$$

The procedure was implemented with use of Matlab function *fmincon* and interior-point algorithm that enabled to find a minimum of constrained nonlinear multivariate function with the following parameter constrains:

$$\sigma_\eta^2 > 0 \tag{11}$$

$$\omega > 0 \tag{12}$$

$$0 < \phi < 1. \tag{13}$$

For the exact specification of initial values and parameter restrictions together with QML estimation, please see *WK3_Assignment_3_1.m*. For the computation of log-likelihood function with Kalman filtering procedure see function *llik_fun.m*. Because we need to maximize log-likelihood value but the optimization procedure in Matlab searching for the minimum of the function in the restricted space, this function is put into *fmincon* procedure with opposite (minus) sign. We obtained the following estimates: $\sigma_\eta^2 = 0.0047$, $\omega = 1.0017e - 07$ and $\phi = 0.9975$.

4.d Take the QML estimates as your final estimates. Compute the smoothed mean of h_t based on the approximate model for x_t by using the Kalman filter and smoother.

Because we have estimated the parameters of our model in the previous section, we can compute the smoothed estimate of h_t with use of the same Kalman filtering equations, previously used for the estimation, and Kalman smoother that we used for our assignment from previous week. For convenience, we created a new function called *kf_smooth.m*, where we have implemented the same Kalman filter as for the estimation part. For the smoothing part, we used standard equations for smoothing recursion, adjusted with the mean adjustments d_t and c_t for equations for $\hat{\alpha}_t$ and V_t in the following way

$$r_{t-1} = Z_t' F_t^{-1} v_t + L_t' r_t, \qquad N_{t-1} = Z_t' F_t^{-1} Z_t + L_t' N_t L_t, \tag{14}$$

$$\hat{\alpha}_t = \alpha_t + P_t r_{t-1} + d_t, \qquad V_t = P_t - P_t N_{t-1} P_t - c_t, \tag{15}$$

for $t = n, \ldots, 1$ with initialization $r_n = 0$ and $N_t = 0$. The resulting smoothed mean of h_t is displayed in Figure 4.

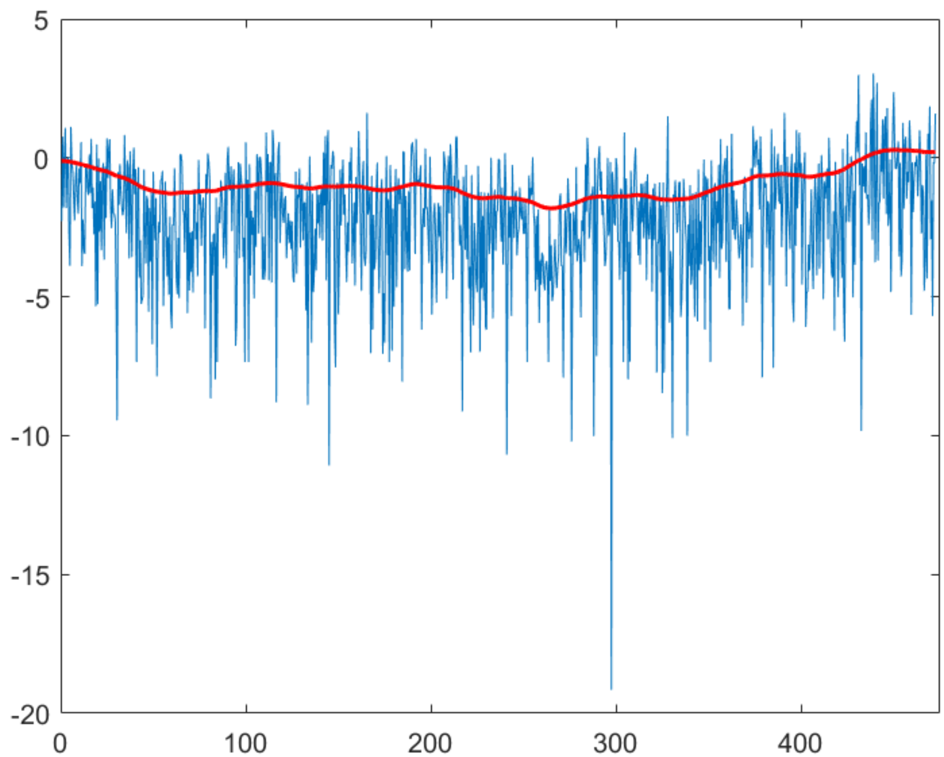

Figure 4: The $\log((y_t - \mu_y)^2)$ time series with the smoothed estimate of h_t