# Assignment Week 4 Time Series Econometrics

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Group 1

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# 1 Exercise 4.3

We consider the binary density with a stochastically time-varying probability $\pi_t$, that is:

$$\log p(y_t|\theta_t) = \log\{\pi_t^{y_t}(1-\pi_t)^{1-y_t}\} = y_t\theta_t - \log(1+\exp\theta_t), \tag{1}$$

for $t = 1, ..., n$ where

$$\theta_t = \log[\pi_t/(1-\pi_t)]$$
$$\mathbb{E}(y_t) = \pi_t = \frac{\exp\theta_t}{(1+\exp\theta_t)}, \qquad \mathbb{V}ar(y_t) = \pi_t(1-\pi_t) = \frac{\exp\theta_t}{(1+\exp\theta_t)^2}.$$

The signal is modelled as $\theta_t = c + \alpha_t$ where $\alpha_t$ follows a stationary AR(1) process.

## 1.a Express the model in matrix form and give expressions for $p(Y_n|\theta)$ and $p(\alpha)$.

We consider the model:

$$y_t \sim p(y_t|\theta_t), \quad \theta_t = c_t + \alpha_t, \quad t = 1, ..., n \tag{2}$$

The matrix form can be formulated as follows:

$$Y_n \sim p(Y_n|\theta), \quad \theta = c + \alpha, \quad \alpha \sim p(\alpha) \tag{3}$$

Where $Y_n = (y_1', ..., y_n')'$, $\theta = (\theta_1', ..., \theta_n')'$, $\alpha = (\alpha_1', ..., \alpha_n')'$ and $c = (c_1', ..., c_n')'$. The signal vector $\theta$ is a linear combination of the stacked state vector. The state vector $\alpha_t$ follows a stationary AR(1) process:

$$\alpha_{t+1} = \phi\alpha_t + \eta_t, \quad \eta_t \sim N(0, Q_t) \tag{4}$$

$\phi$ is an $m$ x $m$ diagonal matrix and $\alpha_t$ and $\eta_t$ are $m$ x 1 vectors. The initial state distribution is $\alpha_1 \sim N(a_1, P_1)$ with $a_1 = \mathbb{E}(\alpha_t)$ and $P_1 = \mathbb{V}ar(\alpha_t)$. We can derive $\mathbb{E}(\alpha_t)$ as follows:

$$\begin{aligned}\mathbb{E}(\alpha_t) &= \mathbb{E}(\phi\alpha_{t-1} + \eta_t) \\ &= \phi\mathbb{E}(\alpha_{t-1}) + \mathbb{E}(\eta_t) \\ &= 0\end{aligned} \tag{5}$$

Because of the stationarity condition $\mathbb{E}(\alpha_t) = \mathbb{E}(\alpha_{t-1})$ and we know that $\mathbb{E}(\eta_t) = 0$. Now we will look at $\mathbb{V}ar(\alpha_t)$.

$$\begin{aligned}\mathbb{V}ar(\alpha_t) &= \mathbb{V}ar(\phi\alpha_{t-1} + \eta_t) \\ &= \phi\mathbb{V}ar(\alpha_{t-1})\phi' + \mathbb{V}ar(\eta_t) \\ &= \phi\mathbb{V}ar(\alpha_{t-1})\phi' + Q_t \\ &= Q_t(I_m - \phi\phi')^{-1}\end{aligned} \tag{6}$$

Because of the stationarity condition $\mathbb{V}ar(\alpha_t) = \mathbb{V}ar(\alpha_{t-1})$.

The state density in matrix form can be expressed as

$$\alpha \sim N(d, \Omega) \tag{7}$$

where $d = 0$ and we can define $\Omega$ as follows

$$\Omega = \Phi diag(P_1, Q_1^*, ..., Q_{n-1}^*)\Phi' \tag{8}$$

$Q_t^*$ is defined as $R_t Q_t R_t'$, but since $R_t = 1$ we can define $Q_t^* = Q_t$.
By repeated iteration of the AR(1) model we get the following:

$$\begin{aligned}
\alpha_t &= \phi\alpha_{t-1} + \eta_{t-1} \\
&= \phi(\phi\alpha_{t-2} + \eta_{t-2}) + \eta_{t-1} \\
&= \phi^2\alpha_{t-2} + \phi\eta_{t-2} + \eta_{t-1} \\
&= \phi^2(\phi\alpha_{t-3} + \eta_{t-3}) + \phi\eta_{t-2} + \eta_{t-1} \\
&\vdots \\
&= \phi^{t-1}\alpha_1 + \sum_{i=0}^{t-2}\phi^i\eta_i
\end{aligned} \tag{9}$$

The state equation will take the the matrix form of $\alpha = \Phi(\alpha_1^* + R\eta)$ with

$$\Phi = \begin{bmatrix} I & 0 & 0 & \dots & 0 & 0 \\ \phi & I & 0 & \dots & 0 & 0 \\ \phi^2 & \phi & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi^{t-2} & \phi^{t-3} & \phi^{t-4} & \dots & I & 0 \\ \phi^{t-1} & \phi^{t-2} & \phi^{t-3} & \dots & \phi & I \end{bmatrix} \tag{10}$$

$$\alpha_1^* = \begin{bmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} R = \begin{bmatrix} 0 & 0 & \dots & 0 \\ R_1 & 0 & \dots & 0 \\ 0 & R_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & R_n \end{bmatrix} \eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix} \tag{11}$$

For the binary likelihood $p(Y_n|\theta)$ and the Gaussian likelihood $p(\alpha)$ we get:

$$\begin{aligned}
p(Y_n|\theta) &= \prod_{t=1}^{n} p(y_t|\theta_t) \\
&= \prod_{t=1}^{n} \exp\{y_t\theta_t - \log(1 + \exp\theta_t)\} \\
&= \exp\left\{\sum_{t=1}^{n}[y_t\theta_t - \log(1 + \exp\theta_t)]\right\}
\end{aligned} \tag{12}$$

and since

$$log\ p_(\alpha) = -\frac{nm}{2}log\ 2\pi - \frac{1}{2}log|\Omega| - \frac{1}{2}(\alpha - d)'\Omega^{-1}(\alpha - d) \tag{13}$$

where $d$ is equal to zero we get the following expression for $p(\alpha)$:

$$p_(\alpha) = \frac{nm}{\sqrt{2\pi}} \times |\Omega|^{-1/2} \times \exp\left(-\frac{1}{2}\alpha'\Omega^{-1}\alpha\right) \tag{14}$$

### 1.b Provide the details for the algorithm of obtaining the mode of the smoothed density $p(\theta|Y_n)$ with respect to $\theta$.

1. Consider the model with non-Gaussian observation density:

$$Y_n \sim p(Y_n|\theta), \qquad \theta = c + \alpha, \qquad \alpha \sim p(\alpha).$$

2. Take an initial guess $g$ for the mode of signal $\theta$ (eg.$g = 0$).

3. Compute $z$ and $A$ using $g$. For this step, we need to derive $\dot{p}(Y_n|\theta)$ and $\ddot{p}(Y_n|\theta)$.

$$\begin{aligned}\dot{p}(Y_n|\theta)|_{\theta=g} &= \frac{\partial[Y_n\theta - \log(\mathbf{1} + \exp\theta)]}{\partial\theta}\Bigg|_{\theta=g} \\ &= Y_n - \frac{\exp g}{\mathbf{1} + \exp g}\end{aligned} \tag{15}$$

$$\begin{aligned}\ddot{p}(Y_n|\theta)|_{\theta=g} &= \frac{\partial^2[Y_n\theta - \log(\mathbf{1} + \exp\theta)]}{\partial\theta\partial\theta'}\Bigg|_{\theta=g} \\ &= -\frac{\exp g}{(\mathbf{1} + \exp g)^2}\end{aligned} \tag{16}$$

Where $\mathbf{1} = (1, ..., 1)'$. Now we can derive A and z:

$$A = -[\ddot{p}(Y_n|\theta)|_{\theta=g}]^{-1} \quad = \frac{(\mathbf{1} + \exp g)^2}{\exp g} \tag{17}$$

$$z = g + A[\dot{p}(Y_n|\theta)|_{\theta=g}] \quad = g + A\left[Y_n - \frac{\exp g}{(\mathbf{1} + \exp g)}\right] \tag{18}$$

4. Consider the linear Gaussian model with $Y_n = z$ and $H = A$. The smooth estimate of $\theta$ from this model obtained by KFS is set equal to $g^+$.

5. Replace $g$ by $g^+$ and go back to step 3.

6. This Newton-Raphon procedure terminates after some convergence has been reached.

# 2 Assignment 4.1: SV model (continued)

We have the SV model given by:

$$\begin{aligned}y_t &= \mu + \sigma_t\epsilon_t, & \mathcal{N}(0, 1) \\ \log\sigma_t^2 &= \xi + H_t, & \\ H_{t+1} &= \phi H_t + \sigma_\theta\theta_t, & \mathcal{N}(0, 1).\end{aligned}$$

**By adopting the QML estimates for the unknown coefficients obtained from the linear model for $x_t = \log(y_t - \bar{y}^2)$, consider the SV model for $y_t$ as given last week and compute the smoothed mode of $H_t$ using Kalman filter smoothing methods.**

For the computation of the smoothed mode of $h_t$, we used equations from section **10.6.5** of the Time Series Analysis by State Space Methods book.

$$A_t = 2\exp(\tilde{\theta}_t)/z_t^2, \qquad x_t = \tilde{\theta}_t + 1 - \exp(\tilde{\theta}_t)/z_t^2.$$

As mentioned in the last tutorial, to link our model with the model described in the literature we set $\sigma = \exp(\frac{1}{2}\zeta)$ and $\theta_t = \alpha_t = H_t$, with $\mu = 0$, $\zeta = \omega/(1-\phi)$ and $\sigma_\eta^2$ estimated in the previous week assignment.

We started our recursion with the guess $g = 0$ for the mode of signal $\theta$. Then we computed $z$ and $A$ using the guessed $g$ and we used the Kalman smoother to compute the smooth estimate of $\theta$ and set this estimate to $g^+$. After that we replaced $g$ by $g^+$ and we repeated the procedure, beginning with the computation of the new $z$ and $A$ with the new $g$. We calculated 15 iterations and compared the values of $g$ to find any converge pattern. Finally, we repeated the whole process again with use of *while* loop, where we set the convergence threshold as $\sqrt{(g_n - g_{n-1})^2} <= 0.00001$. Figure 1 shows the results from the first approach and Figure 2 shows the results of the second described approach.

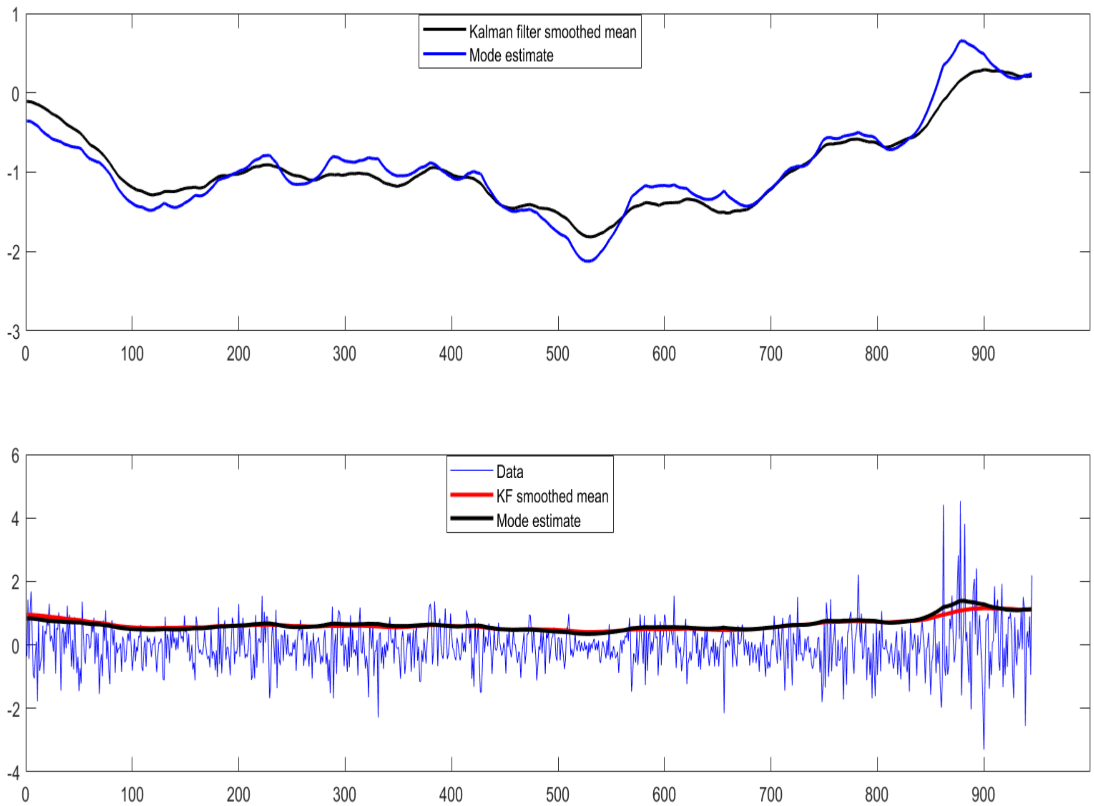


Figure 1: Mean estimate of $h_t$ computed by Kalman smoother from the approximated linear model and mode estimate of $h_t$ obtained from Newton-Raphson algorithm (for loop with 15 iterations).

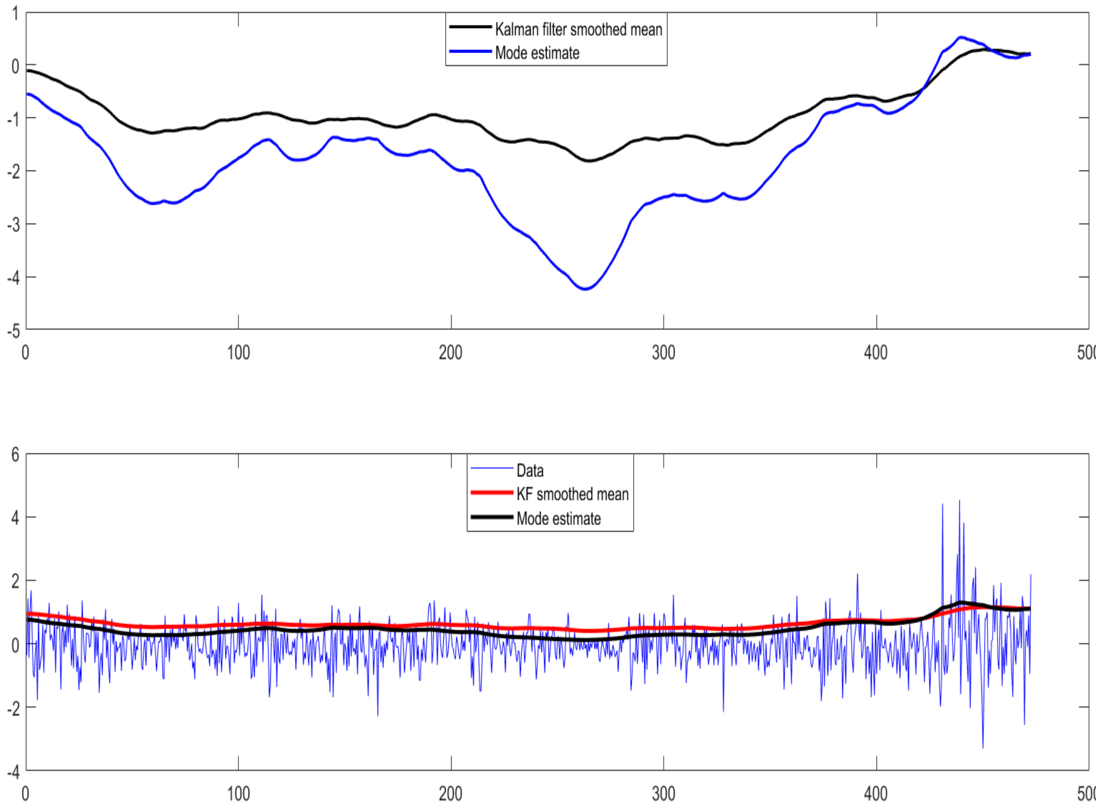


Figure 2: Mean estimate of $h_t$ computed by Kalman smoother from the approximated linear model and mode estimate of $h_t$ obtained from Newton-Raphson algorithm (while loop with the threshold 0.00001).