Assignment Week 1 Time Series Econometrics

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Group 1

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1 Exercise 1.3

1.a

Provide a detailed derivation (proof) of the Kalman filter for the Local Level model.
Next week

1.b

Discuss the initialization of the Kalman filter for the Local Level model
The Kalman filter is typically used to smooth noisy data and provide unobserved estimates for the parameters of interest given observed data. It updates the knowledge of the system every time a new observation is added to the time series. This means that the Kalman filter is recursive and optimizes the estimations every time a new y_t observation arrives. It takes in information which has some error, uncertainty, or noise. The Kalman filter then takes in this imperfect information and sorts out the useful parts of interest by removing the uncertainty and noise. In order to use the Kalman filter, the state space model is the most convenient representation of all linear gaussian time series models. It is defined as follows:

$$\alpha_{t+1} = T_t\alpha_t + R_t\zeta_t \qquad \zeta_t \sim \mathcal{NID}(0, Q_t) \tag{1}$$

$$y_t = Z_t\alpha_t + \epsilon_t \qquad \epsilon_t \sim \mathcal{NID}(0, H_t) \tag{2}$$

In this question the initialization of the Kalman filter is discussed for the Local Level model. The initial condition to be determined is defined by $\alpha_1 \sim N(a_1, P_1)$.

$$y_t = \mu_t + \epsilon_t \qquad \epsilon_t \sim \mathcal{NID}(0, \sigma_\epsilon^2) \tag{3}$$

$$\mu_{t+1} = \mu_t + \eta_t \qquad \eta_t \sim \mathcal{NID}(0, \sigma_\eta^2) \tag{4}$$

The local level model in state space form is given by: $T_t = 1$, $R_t = 1$, $Z_t = 1$, $Q_t = \sigma_\eta^2$ and $H_t = \sigma_\epsilon^2$, with $\alpha_t = \mu_t$ and $\zeta_t = \eta_t$, for all $t = 1, ..., n$. In this case, the state process is non-stationary, because $\phi \to 1$. a_1 is the unconditional mean and p_1 is the unconditional variance of the non-stationary process of μ_t. Here, $a_1 = 0$ and $p_1 = \dfrac{\sigma_\eta^2}{(1-\phi^2)} \to \infty$ as $\phi \to 1$. This is generally the solution for non-stationary element. In question 1c we will talk about the consequences of the Kalman filter when $P_1 \to \infty$.

The initial values chosen influence the convergence rate of the Kalman Filter, so the initialization is an important process. Assuming that the values of a_1 and P_1 are known is not realistic, so we have different options.

One way to deal with initialization is the diffuse density approach, when we fix a_1 and we let $P_1 \to \infty$. This approach will be discussed in question 1c.

Another way is to estimate α_1 by maximum likelihood from the first observation y_1. This gives $a_1 = E(\alpha_1)$ and $P_1 = VAR(\widehat{\alpha_1}) = \sigma_\epsilon^2$). These approaches can be used when you find the diffuse approach unpleasant because it make the assumption of an infinite variance, which can be seen as unnatural since all observed time series have finite variance.

1.c

What are the consequences of the Kalman filter when we let $P_1 \to \infty$?
When nothing is known about the distribution of α_1, which often occurs in practice, it is reasonable to represent α_1 as having a diffuse prior density. Here α_1 is fixed at an arbitrary value and $P_1 \to \infty$. The calibrated Kalman filter for the local level model is given by:

$$\begin{aligned} v_t &= y_t - a_t, & F_t &= P_t + \sigma_\epsilon^2, \\ a_{t|t} &= a_t + K_t v_t, & P_{t|t} &= P_t(1 - K_t), \\ a_{t+1} &= a_t + K_t v_t, & P_{t+1} &= P_t(1 - K_t) + \sigma_\eta^2, \end{aligned} \tag{5}$$

for $t = 1, ..., n$ where $K_t = P_t/F_t$. From (5) we have

$$v_1 = y_1 - a_1, \qquad F_1 = P_1 + \sigma_\epsilon^2$$

By substituting into the equations for a_2 and P_2 in (5), it follows that

$$a_2 = a_1 + \frac{P_1}{P_1 + \sigma_\epsilon^2}(y_1 - a_1) \tag{6}$$

$$P_2 = P_1\Big(1 - \frac{P_1}{P_1 + \sigma_\epsilon^2}\Big) + \sigma_\eta^2 = \frac{P_1}{P_1 + \sigma_\epsilon^2}\sigma_\epsilon^2 + \sigma_\eta^2 \tag{7}$$

When $P \to \infty$, $a_2 = y_1$ and $P_2 = \sigma_\epsilon^2 + \sigma_\eta^2$. Now we can continue under normal conditions with the Kalman Filter (5) for $t = 2, ..., n$. This is called the diffuse initialization of the Kalman filter. The resulting filter is called the diffuse Kalman filter. When we treat y_1 as fixed and take $\alpha_1 \sim N(y_1, \sigma_\epsilon^2)$ the same values of a_t and P_t for t=2,...,n can be obtained. We have $a_{1|1} = y_1$ and $P_{1|1} = \sigma_\epsilon^2$ and according to equation (8) it follows that for t=1 $a_2 = y_1$ and $P_2 = \sigma_\epsilon^2 + \sigma_\eta^2$.

$$\begin{aligned} a_{t+1} &= a_{t|t} = a_t + \frac{P_t}{P_t + \sigma_\epsilon}(y_t - a_t), \\ P_{t+1} &= P_{t|t} + \sigma_\eta^2 = \frac{P_t \sigma_\epsilon^2}{P_t + \sigma_\epsilon^2} + \sigma_\eta^2 \end{aligned} \tag{8}$$

Hence, it is shown that the filtering equations for $t = 2, ..., n$ are not affected by letting $P_1 \to \infty$. The state and disturbance smoothing equations are therefore also not affected for $t = n, ..., 2$ since these only depend on the Kalman filter output.

2 Assignment 1.1: LLM

2.a

Implement the Kalman filter for the Local Level model in a computer program with initial conditions $a_1 = 0$ and $p_1 = 10^7$ and for variances $\sigma_\epsilon^2 = 1$ and $\sigma_\eta^2 = q$ with $q = 0.1$.
In the attached Matlab script the Kalman filter has been implemented for the Local Level model. The initial conditions and values are defined at the top and can be changed easily according to the values we want to use per sub question.

2.b

Replicate the figure on Slide 67 of the lecture slides using the initial and variance values as provided above.
The upper part of Figure 1 shows the observation weights for the local level model. The x-axis shows the amount of lags indexed by j from time t and the y-axis shows the amount of weight attached. We can see that for the local level model, the larger the j, the smaller the weight attached. So if the lag increases, the weight attached to it decreases.

The lower part of Figure 1 shows the weights for the global level model. The global level model has a fixed μ, instead of a time varying μ. This means that every observation in the data has the same weight attached to it. We can see this in the graph: all the observations have the same weight.

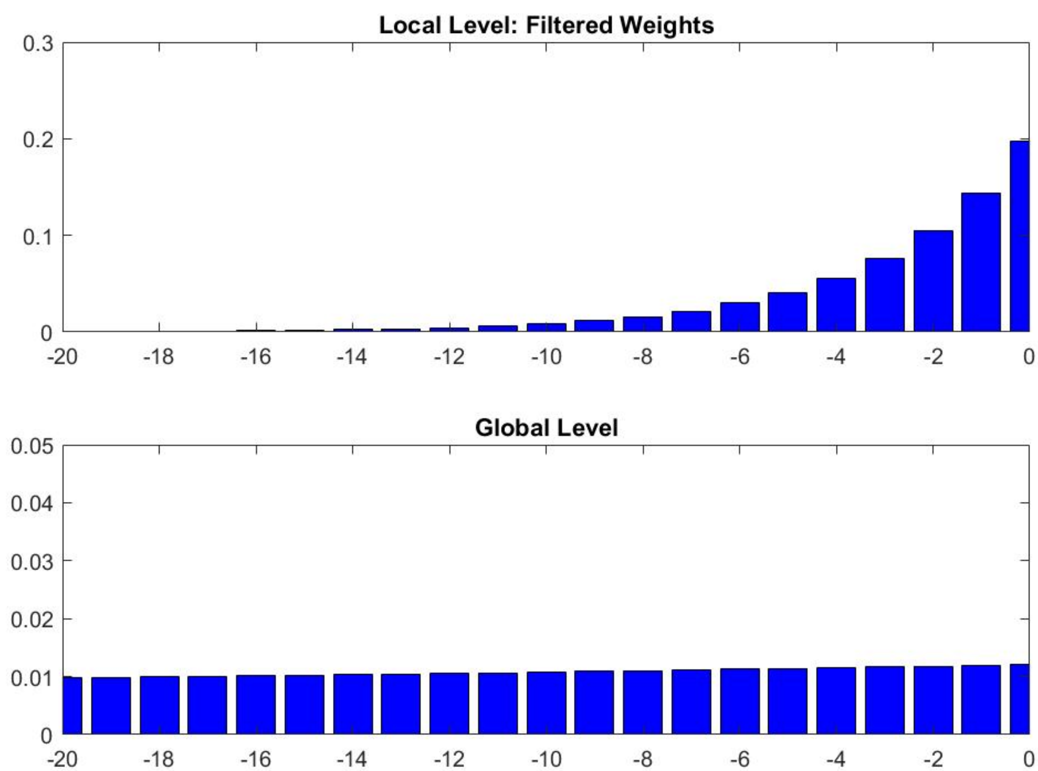

Figure 1: Observation weights for the local and global level model

2.c

Present 4 different weight functions (top figure on Slide 67 of the lecture slides) for $q = 10$, $q = 1$, $q = 0.1$ and $q = 0.001$.

The relative weights placed on the prior observations depend on the signal-to-noise ratio q. If the signal-to-noise ratio is small, more weight is placed on the prior knowledge. This is intuitive since then the observation is very noisy and not very informative. If signal-to-noise ratio is large, more weight is placed on the observation. We can see this in the graphs as well: for the smallest q of 0.001 more weight is placed on the previous observations, but for the highest q most weight is placed on the current observation.

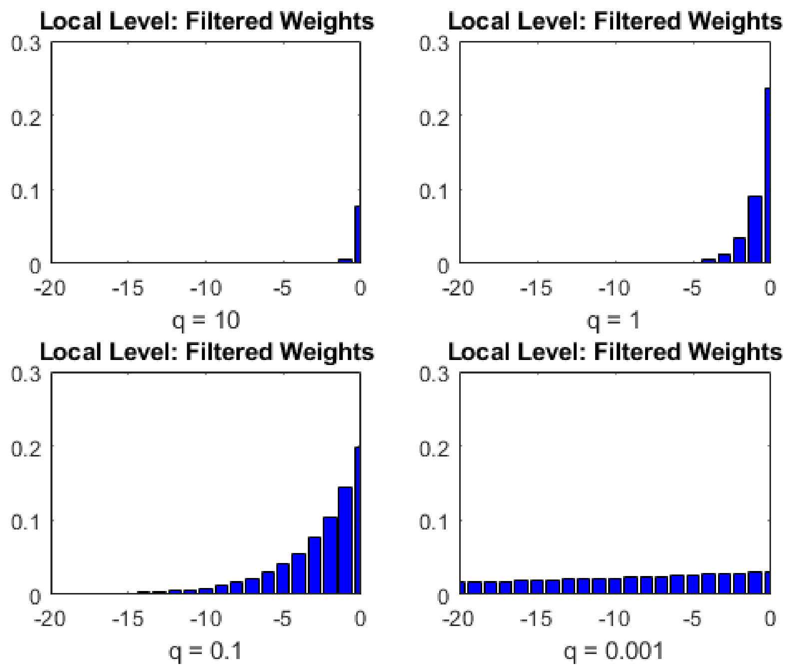

Figure 2: Observation weights for different q (signal-to-noise ratio).

2.d

Apply the Kalman filter to the Nile data with $\sigma_\epsilon^2 = 15099$ and $\sigma_\eta^2 = 1469.1$(Nile data is part of DK book data, see lecture slides).

Changing the variances to 15099 and 1469.1 will not change anything regarding to the Kalman filter. This has to do with the fact that the signal to noise ration q does not change. Previously, q was $\frac{0.1}{1} = 0.1$. Now the signal to noise ratio stays approximately the same, since $\frac{1459.1}{15099} \approx 0.097$.

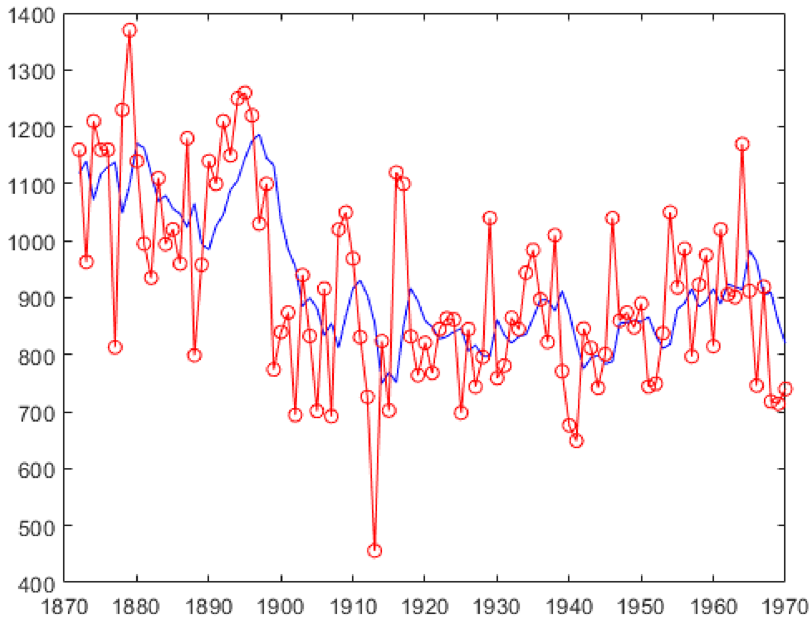

Figure 3: Signal extraction for Nile Data: filtered estimate of level with $\sigma_\epsilon^2 = 15099$ and $\sigma_\eta^2 = 1469.1$

2.e Replicate the figure on Slide 65 of the lecture slides.

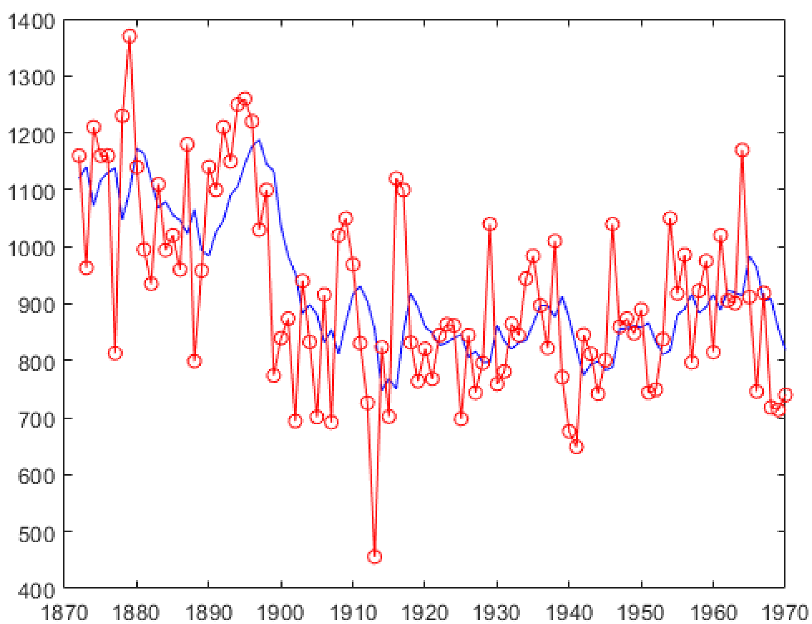

Figure 4: Signal extraction for Nile Data: filtered estimate of level with $\sigma_\epsilon^2 = 1$ and $\sigma_\eta^2 = 0.1$.

2.f

Repeat this assignment by initializing the Kalman filter at time $t = 2$ with $a_2 = y_1$ and $p_2 = \sigma_\epsilon^2 + \sigma_\eta^2$. Do the numerical results change very much ? Explain.
This approach is called the diffuse initialization of the Kalman filter. We can see in the graph below that it shows no difference in the Kalman filter. The Kalman filter depends only on the initial variance p_1, σ_ϵ^2 and σ_η^2, but does not depend on initial mean a_1. Therefore changes in a_1 do not influence numerical results at all and Kalman filter depends only on initial variance p_1, σ_ϵ^2 and σ_η^2.

The equation we are using to filter values with the Kalman filter for local level model is

$$k_t = \frac{p_t}{p_t + \sigma_\epsilon^2},$$

where p_t is the estimation variance vector. This variance is in next step recalculated by formula

$$p_{t|t} = k_t \sigma_\epsilon^2$$
$$p_{t+1} = p_{t|t} + \sigma_\eta^2$$

Let's consider t=1 and t=2. If we plug in our values from 2.d, where $p_1 = 10^7$, $\sigma_\epsilon^2 = 15099$ and $\sigma_\eta^2 = 1469.1$ we have

$$k_1 = \frac{10^7}{10^7 + 15099} = 0.99849 \tag{9}$$
$$p_2 = k_1 \sigma_\epsilon^2 + \sigma_\eta^2 \tag{10}$$
$$k_2 = \frac{k_1 \sigma_\epsilon^2 + \sigma_\eta^2}{k_1 \sigma_\epsilon^2 + \sigma_\eta^2 + \sigma_\epsilon^2} = \frac{0.99849 \sigma_\epsilon^2 + \sigma_\eta^2}{(0.99849 + 1)\sigma_\epsilon^2 + \sigma_\eta^2} \tag{11}$$

If we plug in our values from 2.f, where $p_2 = \sigma_\epsilon^2 + \sigma_\eta^2$ we have

$$k_2 = \frac{\sigma_\epsilon^2 + \sigma_\eta^2}{\sigma_\epsilon^2 + \sigma_\eta^2 + \sigma_\epsilon^2} = \frac{\sigma_\epsilon^2 + \sigma_\eta^2}{2\sigma_\epsilon^2 + \sigma_\eta^2}$$

In comparison with (11), it can be seen that the numerical values of the Kalman filter at time t=2 are almost the same because σ_ϵ^2 nor σ_η^2 have changed. The following values of the Kalman filter for $t = 3, \ldots, T$ depend on the previous values, therefore they are proportionally the same.

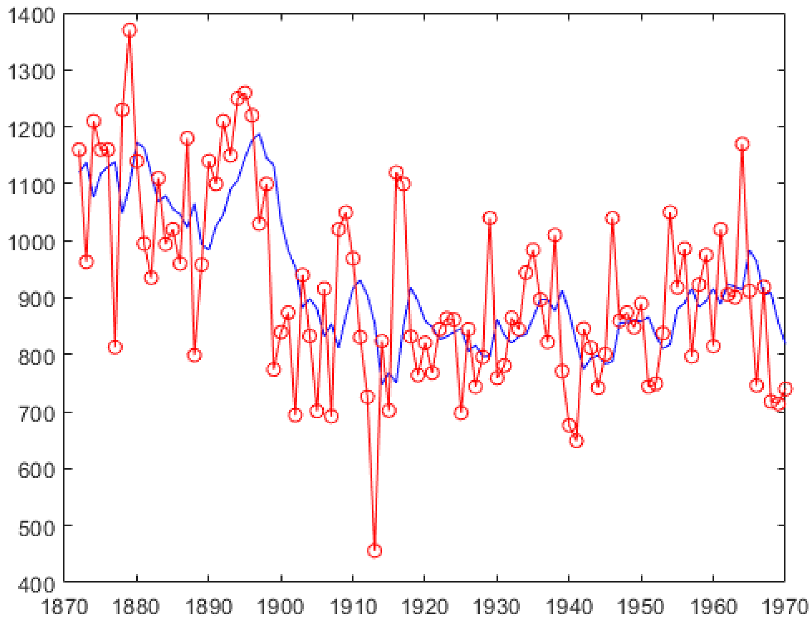

Figure 5: Filtered estimate of level with $t = 2$ with $a_2 = y_1$ and $p_2 = \sigma_\epsilon^2 + \sigma_\eta^2$.