Assignment Week 6 Time Series Econometrics

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Group 1

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1 Exercise 6.2: SIS and SISR

1.a Algorithm for SISR is given in slides. Give, in a similar style, the general algorithm for the SIS.

The general algorithm for the SIS is given is done by the following steps:

1. At time t draw N values $\alpha_t^{(i)}$ from $g(\alpha_t|\alpha_{1:t-1}, Y_t)$ for i = 1,...,N.
2. Compute the corresponding weights $\tilde{w_t}^{(i)}$ for $i = 1, \ldots, N$ with the following equation:
$$\tilde{w_t}^{(i)} = \tilde{w}_{t-1}^{(i)} \frac{p(\alpha_t^{(i)}|\alpha_{t-1}^{(i)})p(y_t|\alpha_t^{(i)})}{g(\alpha_t^{(i)}|\alpha_{1:t-1}^{(i)}, Y_t)} \tag{1}$$
After this, normalize the weights, again for $i = 1, \ldots, N$, with the following equation:
$$w_t^{(i)} = \frac{\tilde{w_t}^{(i)}}{\sum_{j=1}^{N} \tilde{w_t}^{(j)}} \tag{2}$$
3. Compute the estimate $\hat{x_t}$ with the following equation:
$$\hat{x_t} = \sum_{i=1}^{N} x_t(\alpha_{1:t}^{(i)}) w_t^{(i)} \tag{3}$$
4. Take t as $t = t + 1$ and go back to step 1.

1.b Modify your SIS algorithm to incorporate the resampling.

1. At time t draw N values $\tilde{\alpha_t}^{(i)}$ from $g(\alpha_t|\alpha_{t-1}^{(i)}, Y_t)$ for $i = 1, \ldots, N$ and store $\tilde{\alpha}_{t-1:t}^{(i)} = \left\{\alpha_{t-1}^{(i)}, \tilde{\alpha}_t^{(i)}\right\}$.
2. Compute the corresponding weights $\tilde{w}_t^{(i)}$ for $i = 1, \ldots, N$ with the following equation:
$$\tilde{w}_t^{(i)} = \tilde{w}_{t-1}^{(i)} \frac{p(\tilde{\alpha_t}^{(i)}|\alpha_{t-1}^{(i)})p(y_t|\tilde{\alpha_t}^{(i)})}{g(\tilde{\alpha_t}^{(i)}|\tilde{\alpha}_{t-1}^{(i)}, Y_t)} \tag{4}$$
After this, normalize the weights, again for $i = 1, \ldots, N$, with the following equation:
$$w_t^{(i)} = \frac{\tilde{w_t}^{(i)}}{\sum_{j=1}^{N} \tilde{w_t}^{(j)}} \tag{5}$$
3. Compute the variable of interest x_t: given the set of particles $\left\{\tilde{\alpha_t}^{(1)}, \ldots, \tilde{\alpha_t}^{(N)}\right\}$ and compute the estimate:
$$\hat{x_t} = \frac{1}{N} \sum_{i=1}^{N} x_t(\tilde{\alpha_t}^{new(i)}), \tag{6}$$
where $w_t^{new(i)} = \frac{1}{N}$
4. Resample: draw N new independent particles $\alpha_t^{(i)}$ from $\left\{\tilde{\alpha_t}^{(1)}, \ldots, \tilde{\alpha_t}^{(N)}\right\}$ with replacement and with corresponding probabilities $\left\{w_t^{(1)}, \ldots, w_t^{(N)}\right\}$.
5. Take t as $t = t + 1$ and go back to step 1.

1.c The bootstrap filter algorithm is a specific SISR algorithm. Can you give a similarly specific SIS?

The general algorithm for the bootstrap SIS is given by the following steps:

1. At time t draw N values ${\alpha_t}^{(i)}$ from $p(\alpha_t|\alpha_{t-1}^{(i)})$ for i = 1,...,N.
2. Compute the corresponding weights ${w_t}^{(i)}$ for i=1,...,N with the following equation:

$$ {w_t}^{(i)} = \frac{1}{N} p(y_t|\tilde{\alpha_t}^{(i)}) \tag{7} $$

   After this, normalize the weights, again for $i = 1, \ldots, N$, with the following equation:

$$ w_t^{(i)} = \frac{\tilde{w_t}^{(i)}}{\sum_{j=1}^{N} \tilde{w_t}^{(j)}} \tag{8} $$

3. Compute the estimate $\hat{x_t}$ with the following equation:

$$ \hat{x_t} = \sum_{i=1}^{N} w_t^{(i)} x_t(\tilde{\alpha_t}^{(i)}) \tag{9} $$

4. Take t as $t = t + 1$ and go back to step 1.

1.d Consider the local level model. Give expressions for $p(y_t|\alpha_t)$ and $p(\alpha_{t+1}|\alpha_t)$.

The local level model is given by:

$$ y_t = \alpha_t + \epsilon_t, \quad \alpha_{t+1} = \alpha_t + \eta_t \tag{10} $$

with initialization and disturbances $\alpha_1 \sim N(a_1, P_1), \epsilon_t \sim N(0, \sigma_\epsilon^2), \eta_t \sim N(0, \sigma_\eta^2)$ for t=1,...,n and with fixed parameters $\sigma_\epsilon^2 = 15099$ and $\sigma_\eta^2 = 1469.1$. $p(y_t|\alpha_t)$ and $p(\alpha_{t+1}|\alpha_t)$ are both normally distributed: $p(y_t|\alpha_t) \sim N(\alpha_t, \sigma_\epsilon^2)$ and $p(\alpha_{t+1}|\alpha_t) \sim N(\alpha_t, \sigma_\eta^2)$. Therefore we have:

$$ p(y_t|\alpha_t) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \exp\left[-\frac{\epsilon_t^2}{2\sigma_\epsilon^2} \right] $$

$$ p(\alpha_{t+1}|\alpha_t) = \frac{1}{\sqrt{2\pi\sigma_\eta^2}} \exp\left[-\frac{\eta_t^2}{2\sigma_\eta^2} \right] $$

2 Assignment 6.1: LLM and Nile data

Consider the Nile time series Week 1 and 2 that we have modelled by the local level model,

$$ y_t = \alpha_t + \epsilon_t, \quad \alpha_{t+1} = \alpha_t + \eta + t \tag{11} $$

with initialization and disturbances

$$ \alpha_1 \sim N(a_1, P_1), \quad \epsilon_t \sim N(0, \sigma_\epsilon^2), \quad \eta_t \sim N(0, \sigma_\eta^2), \tag{12} $$

for $i = 1, \ldots, m$ and with fixed parameters $\sigma_\epsilon^2 = 15099$ and $\sigma_\eta^2 = 1469.1$

2.a Calculate the filtered state and filtered state variance using sequential importance sampling (SIS) with N=100 (number of particles). Produce a graph where you plot the data, the filtered state from your SIS and the filtered state from the Kalman filter. In another graph, plot the filtered state variance from the SIS and from the Kalman filter. Discuss the performance of SIS.

Figure 1 displays the Nile data, the filtered state from the Kalman filter and the filtered state using sequential importance sampling with 100 particles. In figure 2 the filtered stated variance from the Kalman filter and the filtered state variance from SIS are plotted. The problem of SIS is that as t increases, the weight distribution can become highly skewed. For t large, it is possible for all but one particle to have negligible weights. The sample has become degenerate, due to a peaked $p(y_t|\alpha_t)$ relative to $p(\alpha_t|\alpha_{t-1})$. A few of the values $\alpha_t^{(1)}, ..., \alpha_T^{(N)}$ lead to non-negligible values of $w_t^{(i)}$. This can be seen from our plots: As t increases the differences between the filtered state from the Kalman filter and the filtered state of SIS become bigger. Also the variances differ more over time.

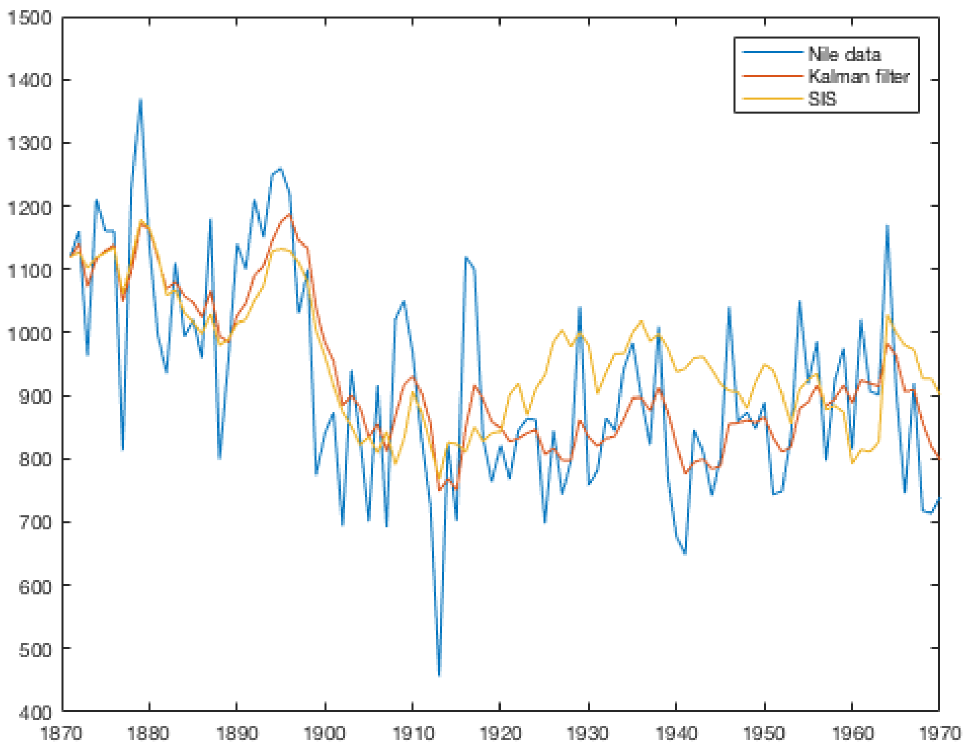

Figure 1: Nile data, filtered state from the Kalman filter and filtered state using sequential importance sampling with $N = 100$.

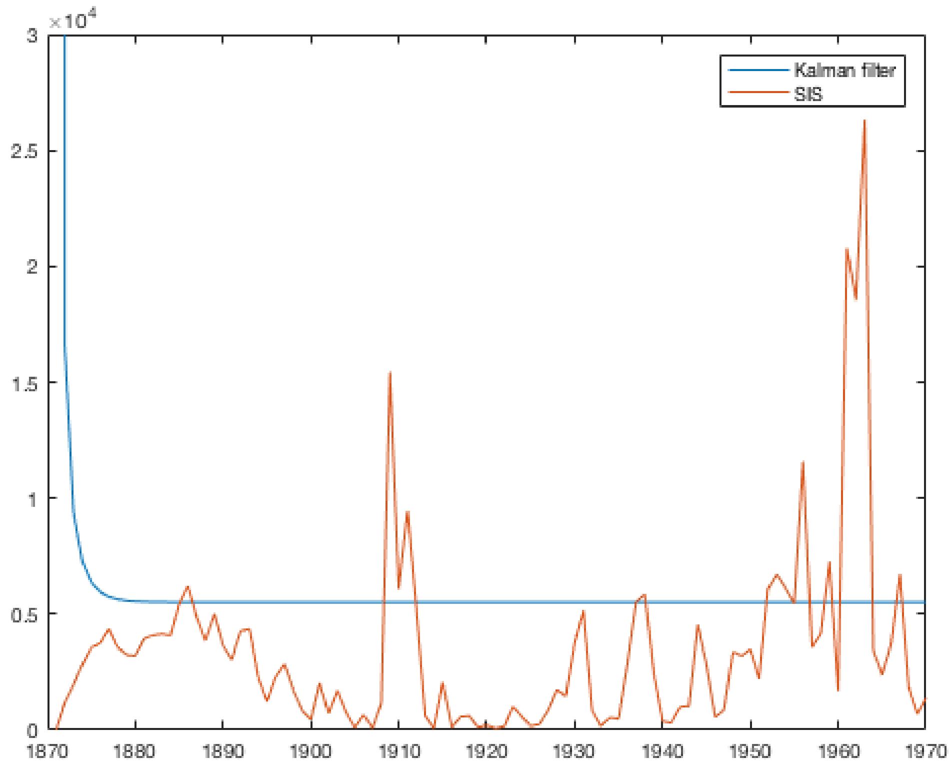

Figure 2: Filtered state variance from the Kalman filter and filtered state variance using sequential importance sampling with $N = 100$.

2.b Plot the efficient sample size (ESS) of the SIS as a function of time t. Interpret your finding.

To test whether a sufficient number of particles has remained, the following formula can be used:

$$ESS = \left(\sum_{i=1}^{N}(w_t^{(i)})^2\right)^{(-1)} \tag{13}$$

ESS is a value between 1 and N and it measures the weight stability. The upper figure of figure 3 presents the efficient sample size of the SIS as a function over time. The ESS decreases very fast, this means that the particle filter is unstable and therefore resampling is needed. In this way particles with a negligible effect on the estimate are removed.

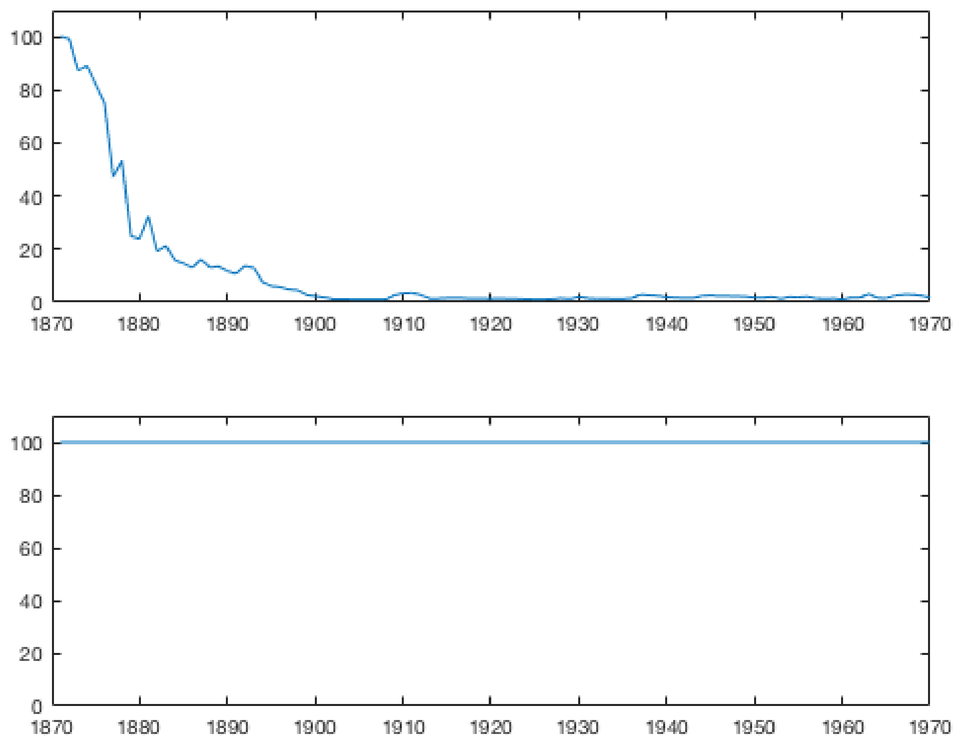

Figure 3: The efficient sample size: of the SIS (upper figure) and the bootstrap filter (bottom graph) as a function of time t.

2.c Reproduce all plots above, but now for the bootstrap filter with $N = 100$ particles. Discuss your findings.

Figure 4 presents the Nile data, filtered state from the Kalman filter and filtered state using the bootstrap filter with $N = 100$. Figure 5 presents the variances of the filtered state from the Kalman filter and filtered state using the bootstrap filter with $N = 100$. The differences between the filtered state of the Kalman filter and the bootstrap filter are very small, although it can be seen that a problem of degeneracy becomes more apparent with time progress. Also the variances of the Kalman filter and the bootstrap filter are close to each other. We can conclude that resampling takes a crucial role in the bootstrap filter. The bottom figure of figure 3 presents the ESS of the bootstrap filter. This graph confirms our findings. The number of relevant particles remains high throughout the time series.

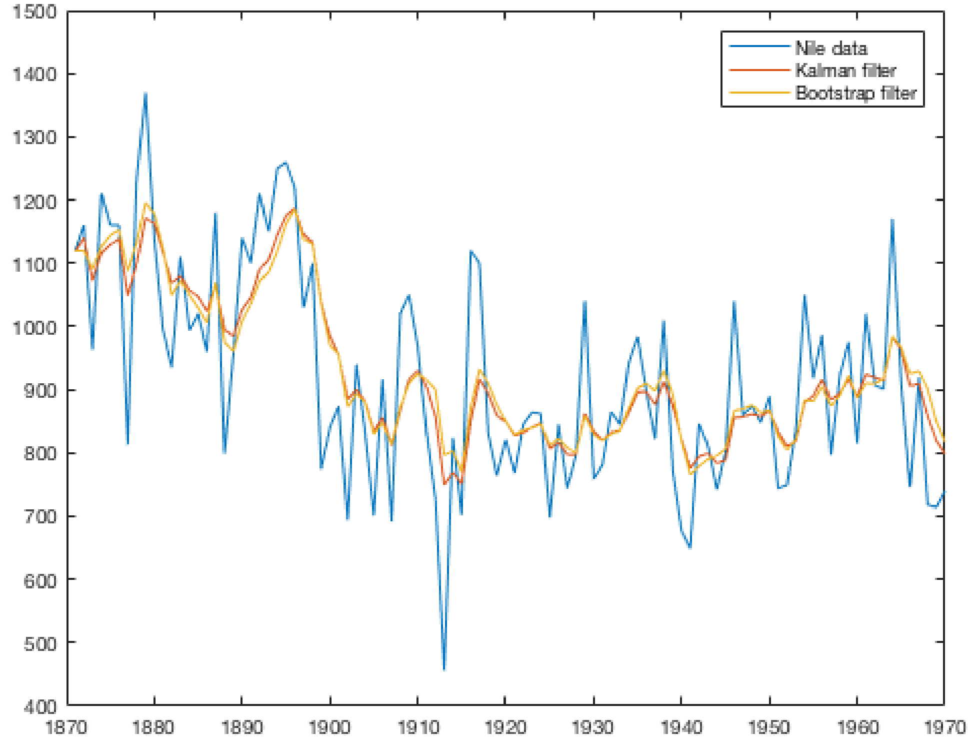

Figure 4: Nile data, filtered state from the Kalman filter and filtered state using the bootstrap filter with N=100.

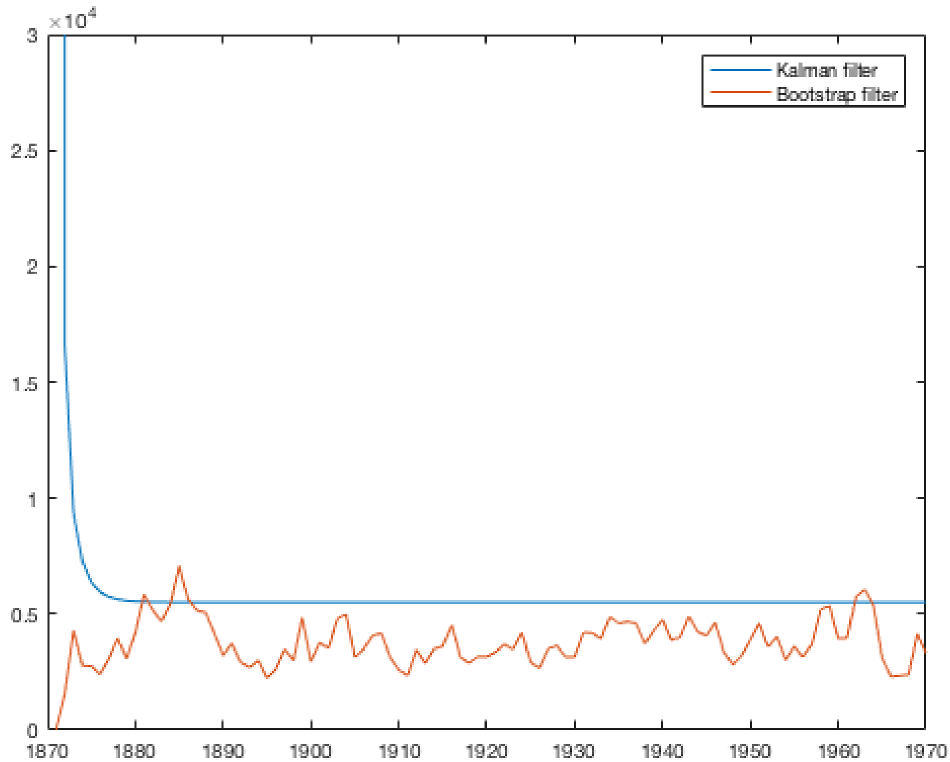

Figure 5: Filtered state variance from the Kalman filter and filtered state variance using the bootstrap filter with $N = 100$.

2.d Now set the number of particles at $N = 15$. Create four graphs where you plot all N particle paths over time from $t = 1$ up to time $t = 5, 10, 15, 20$ from the bootstrap filter together with the filtered stated from the Kalman filter. What can you say about the approximation performance?

Figure 6 presents the filtered state using the Kalman filter and the bootstrap filter from $t = 1$ up to time $t = 5, 10, 15, 20$. The particle paths have the same dynamics as the Kalman filter and they do not deviate from the state filtered by Kalman filter. This again shows the crucial role of resampling in the bootstrap filter.

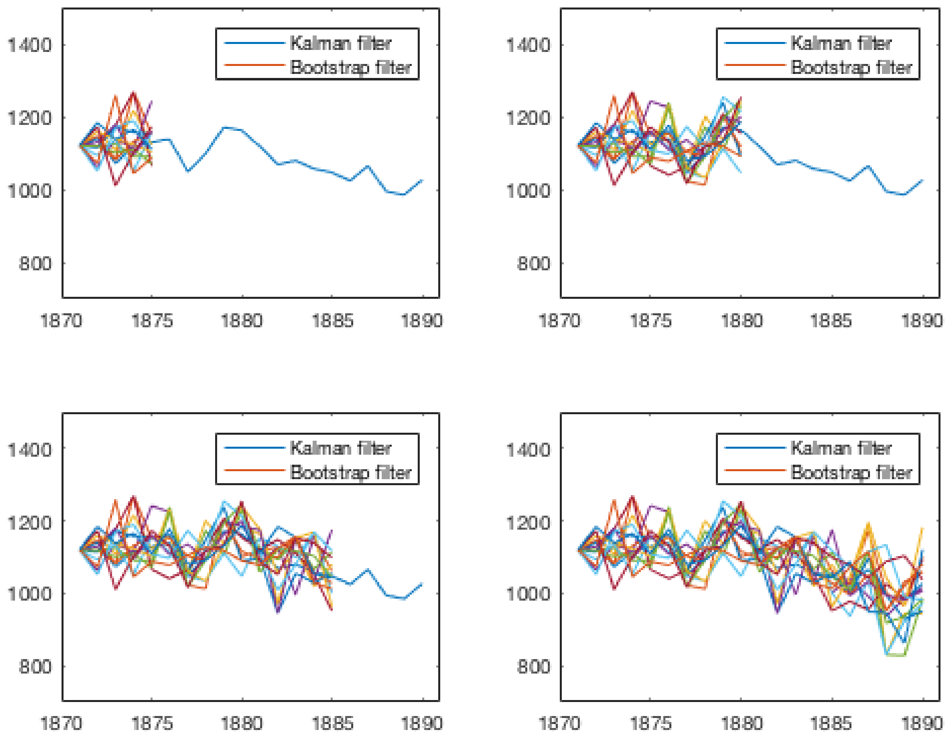

Figure 6: $N = 15$ particle paths over time from $t = 1$ up to $t = 5, 10, 15, 20$ from the bootstrap filter together with the filtered state from the Kalman filter.