Quantitative Macroeconomics
Final Project

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1 Simple Variant of Krusell-Smith Algorithm

1.1 Restate the proof

We start with two-period OLG model with both aggregate and idiosyncratic shocks, as well as pension PAYG system. What more agents work full-time during their first life period and and with participation λ and productivity $\eta_{i,2,t+1}$ during the second life period. Their lifetime utility maximized at the end of first period is given by:

$$U_t = (1 - \tilde{\beta} u(c_{1,t}) + \tilde{\beta}\mathbf{E}_t[u(c_{2,t+1})] \tag{1}$$

(Ass 4) with discount factor $\beta = \frac{\tilde{\beta}}{1-\tilde{\beta}} \in (0, 1]$ when $\tilde{\beta} \in (0, 1/2]$ and utility function $u(\cdot) = ln(\cdot)$. The budget constraint is given by:

$$c_{1,t} + a_{2,t+1} = (1 - \tau)w_t \tag{2}$$

$$c_{i,2,t+1} \leq a_{2,t+1}(1 + r_{t+1}) + \eta_{i,2,t+1}w_{t+1}(1 - \tau) + (1 - \lambda)b_{t+1}$$

where the second constraint will be indeed equality, $b_t = \tau$ is contribution to pension system and $w_t\frac{1+\lambda}{1-\lambda}$ is pension income. From First Order Condition of above Households' Problem one gets

$$1 = \beta\mathbf{E}[\frac{c_{1,t}(1 + r_{t+1})}{c_{i,2,t+1}}] \tag{3}$$

Firms face Cobb-Douglas production function F with technology shock ζ_t, technology level $\gamma_{t+1} = (1 + g)\gamma_t$, exogenous shock of cost of capital ϱ_t and full depreciation $\overline{\delta} = 1$. From firms' maximization profit:

$$\Pi = \zeta_t F(K_t), \gamma_t(1 + \lambda)) - (\overline{\delta} + r_t)\varrho_t^{-1}K_t - w_t(1 + \lambda) \tag{4}$$

one gets:

$$R_t = 1 + r_t = \alpha k_t^{\alpha-1}\zeta_t\varrho_t \tag{5}$$

$$w_t = (1 - \alpha)\gamma_t k_t^\alpha \zeta_t \tag{6}$$

where $k_t = \frac{K_t}{\gamma_t(1+\lambda)}$. With additional assumptions of Independence of shocks, one can prove the Proposition 3 by guessing and verifying as follows.

If households are ex-ante identical, then their decision on saving and consumption in first period should be the same:

$$a_{2,t+1} = s(1-\tau)w_t = s(1-\tau)(1-\alpha)\gamma_t k_t^\alpha \zeta_t = K_{t+1} \tag{7}$$

where s is saving rate and:

$$k_{t+1} = \frac{K_{t+1}}{\gamma_{t+1}(1+\lambda)} = \frac{s(1-\tau)(1-\alpha)\gamma_t k_t^\alpha \zeta_t}{\gamma_t(1+g)(1+\lambda)} = \frac{s(1-\tau)(1-\alpha)}{(1+g)(1+\lambda)}\zeta_t k_t^\alpha \tag{8}$$

Consumption then is:

$$c_{1,t} = (1-s)(1-\tau)w_t = (1-s)(1-\tau)(1-\alpha)\gamma_t\zeta_t k_t^\alpha \tag{9}$$

$$\begin{aligned} c_{i,2,t+1} = a_{2,t+1}R_{t+1} + (1-\tau)w_{t+1}\lambda\eta_{i,2,t+1} + w_{t+1}\tau(1+\lambda) = \\ s(1-\tau)(1-\alpha)\gamma_t\zeta_t k_t^\alpha \alpha k_{t+1}^{\alpha-1}\zeta_{t+1}\varrho_{t+1} \\ +(1-\alpha)\gamma_{t+1}\zeta_{t+1}k_{t+1}^\alpha(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1}))) \end{aligned} \tag{10}$$

Substituting to **3** using **5** and **8**:

$$\begin{aligned} 1 = \beta\mathbf{E}\Big[\frac{(1-s)(1-\tau)w_t = (1-s)(1-\tau)(1-\alpha)\gamma_t\zeta_t k_t^\alpha R_{t+1}}{s(1-\tau)(1-\alpha)\gamma_t\zeta_t k_t^\alpha \alpha k_{t+1}^{\alpha-1}\zeta_{t+1}\varrho_{t+1} + \frac{s(1-\tau)(1-\alpha)}{(1+g)(1+\lambda)}\varrho_t k_t^\alpha(1-\alpha)\gamma_t(1+g)\varrho_{t+1}k_{t+1}^{\alpha-1}(\lambda\eta...)}\Big] \\ = \beta\mathbf{E}\Big[\frac{(1-s)\alpha k_{t+1}^{\alpha-1}\zeta_{t+1}\varrho_{t+1}}{s\alpha\varrho_{t+1}\zeta_{t+1}k_{t+1}^{\alpha-1} + \frac{s(1-\alpha)\varrho_{t+1}k_{t+1}^{\alpha-1}}{1+\lambda}(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1})))}\Big] \\ \frac{s}{\beta(1-s)} = \mathbf{E}\Big[\frac{1}{1 + \frac{1-\alpha}{\alpha\zeta_{t+1}(1+\lambda)}(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1})))}\Big] = \Phi(\tau) \le 1 \end{aligned} \tag{11}$$

Which finally gives:

$$s = \frac{\Phi(\tau)\beta}{1+\beta} \le \frac{\Phi(\tau)\beta}{1+\beta} \tag{12}$$

1.2 Simulation of the first-order difference equation over T periods in logs

After logarithmizing (8), we get:

$$\ln(k_{t+1}) = \ln(\frac{1-\alpha}{1-\lambda}) + \ln(s(\tau)) + \ln(1-\tau) + \ln(\zeta_t) + \alpha \ln k_t \tag{13}$$

We also set $\alpha = 0.3$, $\beta = 0.99^{40}$, $\tau = 0$, $\lambda = 0.5$ and also $std(\ln\zeta = \sqrt{40 \cdot 0.02^2} \approx 0.13$, $std(\ln\varrho = \sqrt{40 \cdot 0.08^2} \approx 0.50$, $std(\ln\eta = \sqrt{40 \cdot 0.15^2} \approx 0.95$. Then we normalize by assuming $\mathbf{E}[\eta] = \mathbf{E}[\zeta] = \mathbf{E}[\varrho] = 1$. With such information we can simulate the economy for $T = 50000$ periods starting from steady state point.

Graph 1 presents capital path of simulation in continuous case, where steady state is computed directly substituting $\mathbf{E}[\zeta_{t+1}] = \mathbf{E}[\eta_{i,2,t+1}] = 0$ into **11** and values of both shocks are drawn from log-normal distribution. Graphs 2 presents capital path of simulation in discrete case. Technology shock η_t is drawn from binomial distribution, such that the mean and standard deviation are preserved. Probability of boom in economy was set to 0.125 and 0.5. Productivity shock is on the contrary discretized using multivariate normal distribution with 11 nodes. Then $\mathbf{E}[\Phi(\tau)]$ is calculated by discrete integration with proper weights.

One can see that in both cases, capital oscillates around steady state with dense cloud in continuous case and splints in discrete one.

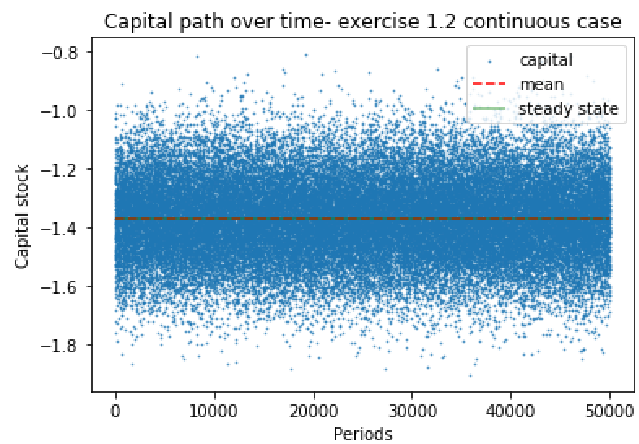


Figure 1: Continuous case

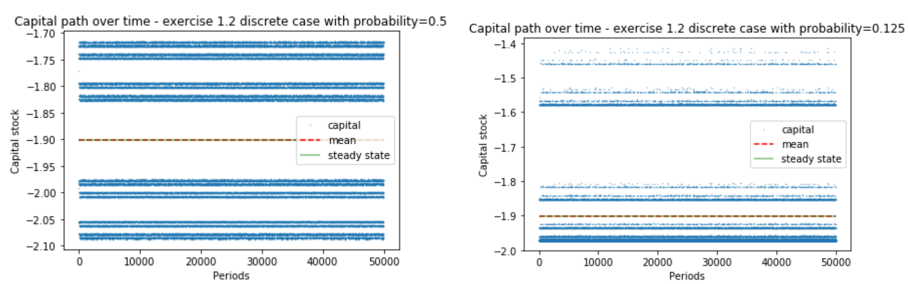


Figure 2: Discrete case

1.3 simple implementation of the Krusell-Smith algorithm (Krusell and Smith 1997; Krusell and Smith 1998)

For each aggregate state $z \in [z^r; z^b]$:

$$\ln(k_{t+1}) = \psi_0(z) + \psi_1(z) \ln k_t \tag{14}$$

1.3.1 Theoretical values of coefficients follow from 13

$$\psi_0(z) = \ln\left(\frac{1-\alpha}{1-\lambda}\right) + \ln(s(\tau)) + \ln(1-\tau) + \ln(\zeta_t) \tag{15}$$

$$\psi_1(z) = \alpha = 0.3 \tag{16}$$

The coefficient $\psi_1(z)$ is constant and simple to calculate. At the same time coefficient $\psi_0(z)$ varies in every random draw and both continuous and discrete case. In the first one in the average $\psi_0(z) = -0.96$ with corresponding $\Phi(\tau) = 0.5625$ and $s = 0.2734$. In the latter one $\psi_0(z) - 1.33$ with corresponding $\Phi(\tau) = 0.3478$ and $s = 0.1888$.

1.3.2 Following the numerical algorithm

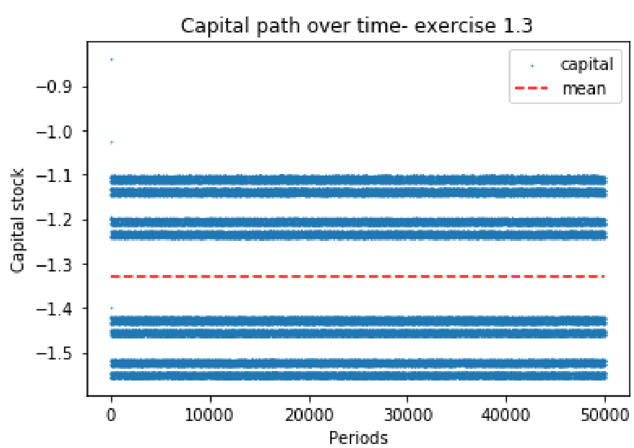
- Solving the household problem Starting with $\psi_0(z)$ and $\psi_1(z)$ guessed values, I solved Household problem finding a root of Euler equation $1 = \beta \mathbf{E}_t[\frac{c_{1,t}(1+r_{t+1})}{c_{i,2,t+1}}]$ for each value of z and every of 5 nodes of k. The matrix of saving rate policy function equals:

$$\begin{bmatrix} 0.273 & 0.289 \\ 0.531 & 0.872 \\ 0.716 & 0.943 \\ 0.849 & 0.882 \\ 0.948 & 0.835 \end{bmatrix}$$

- Fitting saving rates Then for each step in time I chose the most accurate savings rate for agents

- Simulating the economy With given parameters I simulated the economy for $T = 50000$ periods.

The capital path related to this simulation is presented in figure **??**, which is quite similar to the analytical solution



2 Complex Variant of Krusell-Smith Algorithm

In this section we investigate OLG economy with only technology shock altering the production function:

$$Y_t = z_t \cdot F(K_t, L_t) \tag{17}$$

with first-order Markov chain process given by transition matrix:

$$\Pi_z \left[\begin{array}{cc} 0.95 & 0.05 \\ 0.05 & 0.95 \end{array} \right]$$

with state vector $z \in \{0.97, 1.03\}$

The solution was analogical to the one presented in 1.2. With analytically obtained savings rate and values of z drawn randomly according to the process for T=100 and T=1000 periods.