6.3 注释

张志聪

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注释 1. 命题 *3.4 (ii)* $\Rightarrow$ *(iii)* 省略的步骤。

$\alpha = k\epsilon_i + \epsilon_j$ 代入上式，

$$\begin{aligned}
(\mathscr{U}(k\epsilon_i+\epsilon_j), \mathscr{U}(k\epsilon_i+\epsilon_j)) &= (k\mathscr{U}\epsilon_i + \mathscr{U}\epsilon_j, k\mathscr{U}\epsilon_i + \mathscr{U}\epsilon_j) \\
&= k(\mathscr{U}\epsilon_i, k\mathscr{U}\epsilon_i + \mathscr{U}\epsilon_j) + (\mathscr{U}\epsilon_j, k\mathscr{U}\epsilon_i + \mathscr{U}\epsilon_j) \\
&= k\overline{(k\mathscr{U}\epsilon_i + \mathscr{U}\epsilon_j, \mathscr{U}\epsilon_i)} + \overline{(k\mathscr{U}\epsilon_i + \mathscr{U}\epsilon_j, \mathscr{U}\epsilon_j)} \\
&= k\overline{(k\mathscr{U}\epsilon_i, \mathscr{U}\epsilon_i) + (\mathscr{U}\epsilon_j, \mathscr{U}\epsilon_i)} + \overline{(k\mathscr{U}\epsilon_i, \mathscr{U}\epsilon_j) + (\mathscr{U}\epsilon_j, \mathscr{U}\epsilon_j)} \\
&= k\overline{k + (\mathscr{U}\epsilon_j, \mathscr{U}\epsilon_i)} + \overline{k(\mathscr{U}\epsilon_i, \mathscr{U}\epsilon_j) + 1} \\
&= k[\overline{k} + (\mathscr{U}\epsilon_i, \mathscr{U}\epsilon_j)] + \overline{k}(\mathscr{U}\epsilon_j, \mathscr{U}\epsilon_i) + 1 \\
&= k\overline{k} + k(\mathscr{U}\epsilon_i, \mathscr{U}\epsilon_j) + \overline{k}(\mathscr{U}\epsilon_j, \mathscr{U}\epsilon_i) + 1
\end{aligned}$$

又

$$\begin{aligned}
(k\epsilon_i+\epsilon_j, k\epsilon_i+\epsilon_j) &= k(\epsilon_i, k\epsilon_i+\epsilon_j) + (\epsilon_j, k\epsilon_i+\epsilon_j) \\
&= k\overline{(k\epsilon_i+\epsilon_j, \epsilon_i)} + \overline{(k\epsilon_i+\epsilon_j, \epsilon_j)} \\
&= k\overline{(k\epsilon_i, \epsilon_i) + (\epsilon_j, \epsilon_i)} + \overline{(k\epsilon_i, \epsilon_j) + (\epsilon_j, \epsilon_j)} \\
&= k\overline{k} + 1
\end{aligned}$$

于是

$$\begin{aligned}
k\overline{k} + k(\mathscr{U}\epsilon_i, \mathscr{U}\epsilon_j) + \overline{k}(\mathscr{U}\epsilon_j, \mathscr{U}\epsilon_i) + 1 &= k\overline{k} + 1 \\
k(\mathscr{U}\epsilon_i, \mathscr{U}\epsilon_j) + \overline{k}(\mathscr{U}\epsilon_j, \mathscr{U}\epsilon_i) &= 0
\end{aligned}$$

注释 2. *(ii)* $\Rightarrow$ *(iii)* 的证明方法，对正交变换是否成立，即命题 *2.1* 中 *(iv)* 推 *(iii)*

设 $\mathscr{A}k\epsilon_i + \epsilon_j \in V(i \neq j)$，

$$\begin{aligned}(\mathscr{A}k\epsilon_i + \epsilon_j, \mathscr{A}k\epsilon_i + \epsilon_j) &= (k\mathscr{A}\epsilon_i + \mathscr{A}\epsilon_j, k\mathscr{A}\epsilon_i + \mathscr{A}\epsilon_j) \\ &= (k\mathscr{A}\epsilon_i, k\mathscr{A}\epsilon_i) + (k\mathscr{A}\epsilon_i, \mathscr{A}\epsilon_j) + (\mathscr{A}\epsilon_j, k\mathscr{A}\epsilon_i) + (\mathscr{A}\epsilon_j, \mathscr{A}\epsilon_j) \\ &= k \cdot k(\mathscr{A}\epsilon_i, \mathscr{A}\epsilon_i) + k(\mathscr{A}\epsilon_i, \mathscr{A}\epsilon_j) + k(\mathscr{A}\epsilon_j, \mathscr{A}\epsilon_i) + (\mathscr{A}\epsilon_j, \mathscr{A}\epsilon_j) \\ &= k \cdot k + 2k(\mathscr{A}\epsilon_i, \mathscr{A}\epsilon_j) + 1\end{aligned}$$

又

$$\begin{aligned}(k\epsilon_i + \epsilon_j, k\epsilon_i + \epsilon_j) &= (k\epsilon_i, k\epsilon_i) + (k\epsilon_i, \epsilon_j) + (\epsilon_j, k\epsilon_i) + (\epsilon_j, \epsilon_j) \\ &= k \cdot k + 1\end{aligned}$$

于是

$$k \cdot k + 2k(\mathscr{A}\epsilon_i, \mathscr{A}\epsilon_j) + 1 = k \cdot k + 1$$
$$(\mathscr{A}\epsilon_i, \mathscr{A}\epsilon_j) = 0$$

又因为

$$\begin{aligned}(\mathscr{A}\epsilon_i, \mathscr{A}\epsilon_i) &= (\epsilon_i, \epsilon_i) \\ &= 1\end{aligned}$$

所以 $\mathscr{A}\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ 是标准正交基。

注释 3. $(\mathscr{A}\alpha, \beta) = (\alpha, \mathscr{A}^*\beta)$ 显然等价于

$$(\mathscr{A}\epsilon_i, \epsilon_j) = (\epsilon_i, \mathscr{A}^*\epsilon_j) \quad (i, j = 1, 2, \cdots, n)$$

的原因。

任意 $\alpha, \beta \in V$，可表示为

$$\alpha = x_1\epsilon_1 + x_2\epsilon_2 + \cdots + x_n\epsilon_n$$
$$\beta = y_1\epsilon_1 + y_2\epsilon_2 + \cdots + y_n\epsilon_n$$

于是

$$
\begin{aligned}
(\mathscr{A}\alpha,\beta) &= (\mathscr{A}x_1\epsilon_1+x_2\epsilon_2+\cdots+x_n\epsilon_n, y_1\epsilon_1+y_2\epsilon_2+\cdots+y_n\epsilon_n)\\
&= (x_1\mathscr{A}\epsilon_1+x_2\mathscr{A}\epsilon_2+\cdots+x_n\mathscr{A}\epsilon_n, y_1\epsilon_1+y_2\epsilon_2+\cdots+y_n\epsilon_n)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i(\mathscr{A}\epsilon_i, y_j\epsilon_j)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\overline{(y_j\epsilon_j, \mathscr{A}\epsilon_i)}\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\overline{y_j(\epsilon_j, \mathscr{A}\epsilon_i)}\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\overline{y_j}\overline{(\epsilon_j, \mathscr{A}\epsilon_i)}\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\overline{y_j}(\mathscr{A}\epsilon_i, \epsilon_j)
\end{aligned}
$$

又

$$
\begin{aligned}
(\alpha,\mathscr{A}\beta) &= (x_1\epsilon_1+x_2\epsilon_2+\cdots+x_n\epsilon_n, \mathscr{A}^*y_1\epsilon_1+y_2\epsilon_2+\cdots+y_n\epsilon_n)\\
&= (x_1\epsilon_1+x_2\epsilon_2+\cdots+x_n\epsilon_n, y_1\mathscr{A}^*\epsilon_1+y_2\mathscr{A}^*\epsilon_2+\cdots+y_n\mathscr{A}^*\epsilon_n)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i(\epsilon_i, \mathscr{A}^*y_j\epsilon_j)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\overline{(y_j\mathscr{A}^*\epsilon_j, \epsilon_i)}\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\overline{y_j(\mathscr{A}^*\epsilon_j, \epsilon_i)}\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\overline{y_j}\overline{(\mathscr{A}^*\epsilon_j, \epsilon_i)}\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\overline{y_j}(\epsilon_i, \mathscr{A}^*\epsilon_j)
\end{aligned}
$$

- 左边推右边

  取 $i=1, j=1$，可得 $(\mathscr{A}\epsilon_1,\epsilon_1)=(\epsilon_1,\mathscr{A}^*\epsilon_1)$。
  类似地，推导出剩余项。

- 右边推左边

  是平凡的结果。

综上，命题得证。