15.7 习题

张志聪

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15.7.1

- (a)

  利用引理 15.6.6 和习题 15.6.16 中的 $\exp(z+w)=\exp(z)\exp(w)$。

$$\begin{aligned}\sin(x)^2+\cos(x)^2 &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2+\left(\frac{e^{ix}+e^{-ix}}{2}\right)^2\\ &= \frac{e^{2ix}+e^{-2ix}-2e^{ix-ix}}{-4}+\frac{e^{2ix}+e^{-2ix}+2e^{ix-ix}}{4}\\ &= \frac{e^{2ix}+e^{-2ix}}{-4}+\frac{-2}{-4}+\frac{e^{2ix}+e^{-2ix}}{4}+\frac{2}{4}\\ &= 1\end{aligned}$$

- (b)

$$\begin{aligned}sin'(x) &= \left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)'\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)x^{2n}}{(2n+1)!}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\\ &= cos(x)\end{aligned}$$

$$
\begin{aligned}
cos'(x) &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right)' \\
&= \sum_{n=1}^{\infty} \frac{2n(-1)^n x^{2n-1}}{(2n)!} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}
\end{aligned}
$$

令 $m = n - 1$，即 $n = m + 1$，利用命题 7.4.3（级数的重排序），

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^{m+1} x^{2(m+1)-1}}{(2(m+1)-1)!} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^{m+1} x^{2m+1}}{(2m+1)!} \\
&= -\sum_{m=0}^{\infty} \frac{(-1)^{m} x^{2m+1}}{(2m+1)!} \\
&= -sin(x)
\end{aligned}
$$

- (c)

$$
\begin{aligned}
sin(-x) &= \frac{e^{-ix} - e^{ix}}{2i} \\
&= -\frac{e^{ix} - e^{-ix}}{2i} \\
&= -sin(x)
\end{aligned}
$$

$$
\begin{aligned}
cos(-x) &= \frac{e^{i(-x)} + e^{-i(-x)}}{2} \\
&= \frac{e^{-ix} + e^{ix}}{2} \\
&= cos(x)
\end{aligned}
$$

- (d)

$$
\begin{aligned}
cos(x)cos(y) - sin(x)sin(y) &= \frac{e^{ix} + e^{-ix}}{2}\frac{e^{iy} + e^{-iy}}{2} \\
&- \frac{e^{ix} - e^{-ix}}{2i}\frac{e^{iy} - e^{-iy}}{2i} \\
&= \frac{e^{ix}e^{iy} + e^{ix}e^{-iy} + e^{-ix}e^{iy} + e^{-ix}e^{-iy}}{4} \\
&- \frac{e^{ix}e^{iy} - e^{ix}e^{-iy} - e^{-ix}e^{iy} + e^{-ix}e^{-iy}}{-4} \\
&= \frac{2e^{ix}e^{iy} + 2e^{-ix}e^{-iy}}{4} \\
&= \frac{e^{ix}e^{iy} + e^{-ix}e^{-iy}}{2} \\
&= \frac{e^{i(x+y)} + e^{-i(x+y)}}{2} \\
&= cos(x+y)
\end{aligned}
$$

$$
\begin{aligned}
sin(x)cos(y) + cos(x)sin(y) &= \frac{e^{ix} - e^{-ix}}{2i}\frac{e^{iy} + e^{-iy}}{2} \\
&+ \frac{e^{ix} + e^{-ix}}{2}\frac{e^{iy} - e^{-iy}}{2i} \\
&= \frac{e^{ix}e^{iy} + e^{ix}e^{-iy} - e^{-ix}e^{iy} - e^{-ix}e^{-iy}}{4i} \\
&+ \frac{e^{ix}e^{iy} - e^{ix}e^{-iy} + e^{-ix}e^{iy} - e^{-ix}e^{-iy}}{4i} \\
&= \frac{2e^{ix}e^{iy} - 2e^{-ix}e^{-iy}}{4i} \\
&= \frac{e^{i(x+y)} - e^{-i(x+y)}}{2i} \\
&= sin(x+y)
\end{aligned}
$$

- (e)

$$
\begin{aligned}
cos(0) &= \frac{e^{i\times 0}+e^{-i\times 0}}{2i} \\
&= \frac{1+1}{2} \\
&= 1
\end{aligned}
$$

由 (a) 可知，$sin(0) = 1 - cos(0)^2 = 1 - 1 = 0$

- (f)

$$
\begin{aligned}
cos(x) + isin(x) &= \frac{e^{ix}+e^{-ix}}{2} + i\frac{e^{ix}-e^{-ix}}{2i} \\
&= \frac{e^{ix}+e^{-ix}}{2} + i^2\frac{e^{ix}-e^{-ix}}{2i^2} \\
&= \frac{e^{ix}+e^{-ix}}{2} + \frac{e^{ix}-e^{-ix}}{2} \\
&= \frac{2e^{ix}}{2} \\
&= e^{ix}
\end{aligned}
$$

同理可得，

$$
\begin{aligned}
cos(x) - isin(x) &= \frac{e^{ix}+e^{-ix}}{2} - i\frac{e^{ix}-e^{-ix}}{2i} \\
&= \frac{e^{ix}+e^{-ix}}{2} - i^2\frac{e^{ix}-e^{-ix}}{2i^2} \\
&= \frac{e^{ix}+e^{-ix}}{2} - \frac{e^{ix}-e^{-ix}}{2} \\
&= \frac{2e^{-ix}}{2} \\
&= e^{-ix}
\end{aligned}
$$