Module-1

Complex Variables

1.1 Analytic Functions:

Def: A function $f(z)$ is said to be analytic at a point z_0 if $f(z)$ is derivable at every point z in some ϵ-neighbourhood of z_0.

i.e., If $f'(z)$ exists for all z such that $|z - z_0| < \epsilon$ for some $\epsilon > 0$, then $f(z)$ is said to be analytic at z_0.

Note: ① An analytic function is also known as regular or holomorphic function.

② If $f(z)$ is analytic at every point z on the complex plane $\mathbb{C}$, then $f(z)$ is said to be an entire function

③ If $f'(z)$ does not exists at $z = z_0$, then z_0 is called a singular point of $f(z)$.

Cauchy-Riemann (C-R) Equations:

Let $w = f(z) = u(x,y) + iv(x,y)$ be a complex valued function. Then the following are known as C-R equations:

$$\text{(i)}\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and (ii)}\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Note: If $f(z) = u + iv$ satisfies C-R equations and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous, then $f(z)$ is analytic.

Polar form of Cauchy-Riemann Equations:

Let $z = re^{i\theta}$ be a complex number (in polar form). Then the following are called polar form of C-R equations

$$\text{(i)}\ \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\text{and (ii)}\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

* Here $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$.

Problems

1: Show that $f(z) = z^2$ is analytic for all z.

Sol: Given $f(z) = z^2$.

i.e., $f(z) = (x+iy)^2$

$= (x^2 - y^2) + i(2xy)$

$= u + iv$, say.

here $u = x^2 - y^2$ and $v = 2xy$.

Now, $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = -2y$

$\frac{\partial v}{\partial x} = 2y$, $\frac{\partial v}{\partial y} = 2x$

Therefore, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Also, all $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous.

Hence $f(z) = z^2$ is analytic for all z.

2. Show that $f(z) = e^z$ is analytic everywhere in the complex plane.

Sol: Given $f(z) = e^z$

i.e., $f(z) = e^{x+iy} = e^x \cdot e^{iy}$

$= e^x(\cos y + i \sin y)$

$= e^x \cos y + i\, e^x \sin y$

$= u + iv$ say.

here $u = e^x \cos y$ and $v = e^x \sin y$.

clearly $\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$

and $\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial y}$.

Also, all $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous.

Hence $f(z) = e^z$ is analytic everywhere.

(3) Determine where the function $f(z) = \log z$ is non-analytic.

Sol: Let $z = re^{i\theta}$ (i.e., $x + iy = re^{i\theta}$).

Then $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$

and $\log z = \log(x+iy) = \log(re^{i\theta})$

$= \log r + i\theta$

$= \frac{1}{2}\log(x^2+y^2) + i \tan^{-1}(y/x)$

here $u = \frac{1}{2}\log(x^2+y^2)$ and $v = \tan^{-1}(y/x)$.

clearly $\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} = \frac{\partial v}{\partial y}$

and $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} = -\left(\frac{-y}{x^2+y^2}\right) = -\frac{\partial v}{\partial x}$

Therefore, C-R equations are satisfied.

And the partial derivatives are continuous except at $(0,0)$.

Hence $f(z) = \log z$ is analytic everywhere except at $z = 0$.

(4) Find all values of k such that $f(z) = e^{x}(\cos ky + i\sin ky)$ is analytic.

Sol: Answer: $f(z)$ is analytic when $k=1$.

(5) Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin although C-R equations are satisfied at the origin.

Sol: Given $f(z) = \sqrt{|xy|}$

i.e., $u(x,y) + iv(x,y) = \sqrt{|xy|}$.

here $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$.

At the origin,

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{Lt} \frac{u(x,0) - u(0,0)}{x - 0} = 0$$

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{Lt} \frac{u(0,y) - u(0,0)}{y - 0} = 0$$

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{Lt} \frac{v(x,0) - v(0,0)}{x - 0} = 0$$

and $$\frac{\partial v}{\partial y} = \underset{y \to 0}{Lt} \frac{v(0,y) - v(0,0)}{y - 0} = 0$$

Hence C-R equations are satisfied at the origin.

Now, along the path $y = mx$, we get

$$f'(0) = \underset{z \to 0}{Lt} \frac{f(z) - f(0)}{z - 0}$$

$$= \underset{z \to 0}{Lt} \frac{\sqrt{|xy|} - 0}{x + iy}$$

$$= \underset{x \to 0}{Lt} \frac{\sqrt{|mx^2|} - 0}{x(1 + im)} = \underset{x \to 0}{Lt} \frac{\sqrt{|m|}}{1 + im}$$

This limit depends on m and hence not unique.

Therefore, $f(z)$ is not derivable at $z=0$.

Hence $f(z)$ is not analytic at the origin.

⑥ Show that $f(z) = \begin{cases} \dfrac{xy^2(x+iy)}{x^2+y^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

is not analytic at the origin although C-R equations are satisfied at the origin.

⑦ Prove that $f(z) = z^n \ (n \in \mathbb{Z}^+)$ is analytic

Sol: Hint $\boxed{z = re^{i\theta}}$

So, $f(z) = r^n e^{in\theta}$

$$= \underbrace{r^n \cos n\theta}_{u(r,\theta)} + i \underbrace{r^n \sin n\theta}_{v(r,\theta)}.$$

clearly, $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

Hence $f(z) = z^n \ (n \in \mathbb{Z}^+)$ is analytic.

1.2 Harmonic Functions:

Def: A function ϕ is said to be harmonic if it satisfies the Laplace equation

i.e., If $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$, then $\phi(x, y)$ is called an harmonic function.

Note: The real and imaginary parts of an analytic function are harmonic.

But, if $u(x,y)$ and $v(x,y)$ are harmonic, then the function $f(z) = u(x,y) + iv(x,y)$ need not be analytic.

Example: If $u = x^2 - y^2$ and $v = \frac{-y}{x^2+y^2}$, then both u and v are harmonic. But $f(z) = u + iv$ is not an analytic (regular) function.

Sol: clearly $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial y^2} = 0$.

Therefore, both $u(x,y)$ and $v(x,y)$ are harmonic.

But, $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$.

Since u and v are not satisfies C-R equations, we have $f(z) = u + iv$ is not an analytic function of z.

Conjugate Harmonic Function

Def: Let $f(z) = u(x,y) + i\,v(x,y)$ be an analytic function. Then $v(x,y)$ is said to be an harmonic conjugate of $u(x,y)$.

Construction of an Analytic function whose real or imaginary part is Known:

Milne-Thomson's method:

Let $f(z) = u(x,y) + i v(x,y)$. $\longrightarrow (1)$

Since $z = x + iy$ and $\bar{z} = x - iy$, we have

$$x = \frac{z+\bar{z}}{2} \text{ and } y = \frac{z-\bar{z}}{2i}$$

Therefore, $f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + i v\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$

$\longrightarrow (2)$

Now, considering this relation as a formal identity in two independent variables z and $\bar{z}$.

Taking $\bar{z} = z$ in ②, we get

$$f(z) = u(z,0) + i\,v(z,0)$$

Therefore, eqn ③ is same as ① if we replace x by z and y by 0.

Thus, to express any function in terms of z, replace x by z and y by 0.

Now, $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$

(by C-R equations)

Let $\dfrac{\partial u}{\partial x} = \phi_1(x,y)$ and $\dfrac{\partial u}{\partial y} = \phi_2(x,y)$

Then $f'(z) = \phi_1(x,y) - i\,\phi_2(x,y) \longrightarrow$ ④

Now, to express $f'(z)$ completely in terms of z, we replace x by z and y by 0 in ④.

$$\therefore f'(z) = \phi_1(z,0) - i\,\phi_2(z,0)$$

Integrating, we get

$$f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + C,$$

where C is a complex constant.

Problems:

1. Find an analytic function whose real part is $u = x^2 - y^2 - x$.

Sol: Let $f(z) = u(x,y) + i\,v(x,y)$.

given $u(x,y) = x^2 - y^2 - x$

Then $\frac{\partial u}{\partial x} = 2x - 1$ and $\frac{\partial u}{\partial y} = -2y$.

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

(by C-R equations)

$$= (2x-1) + i(2y)$$

By Milne-Thomson's method, $f'(z)$ expressed interms of z by replacing x by z

and y by 0.

Hence $f'(z) = 2z - 1$

Integrating, we get

$$f(z) = 2\int z\,dz - \int dz + c$$

$\Rightarrow$ $f(z) = z^2 - z + c$, where c is a complex constant.

<u>observation</u>:

$$f(z) = z^2 - z + c$$
$$= (x+iy)^2 - (x+iy) + c$$
$$= (x^2 - y^2 - x) + i(2xy - y) + c$$

So, $f(z) = (x^2 - y^2 - x) + i(2xy - y) + c$

or, $f(z) = (x^2 - y^2 - x) + i(2xy - y + c_1)$

$\boxed{c = ic_1}$

(2) Find an analytic function $f(z)$ whose imaginary part is $v(x,y) = \log(x^2+y^2) + x - 2y$.

<u>Answer:</u> $f(z) = 2i\log z - (2-i)z + c$.

③ Show that the function $u(x,y) = e^x \cos y$ is harmonic and determine its harmonic conjugate $v(x,y)$ and hence find the analytic function $f(z) = u + iv$.

Sol: Given $u(x,y) = e^x \cos y$

clearly $\frac{\partial u}{\partial x} = e^x \cos y$ | $\frac{\partial u}{\partial y} = -e^x \sin y$

$\frac{\partial^2 u}{\partial x^2} = e^x \cos y$ | $\frac{\partial^2 u}{\partial y^2} = -e^x \cos y$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence u is an harmonic function.

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function.

we have $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

(by C-R equations)

$$= e^x \sin y \, dx + e^x \cos y \, dy$$

$$\Rightarrow dv = d(e^x \sin y)$$

Integrating, we get

$$v = e^x \sin y + c$$

Therefore, $f(z) = u + iv$

$$= e^x \cos y + i e^x \sin y + ik$$
$(c = ik)$

$$= e^x (\cos y + i \sin y) + ik$$

$$= e^x e^{iy} + ik$$

$$= e^{x+iy} + ik$$

$$\therefore \quad f(z) = e^z + ik.$$

which is the required analytic function.

(4) Find the analytic function whose imaginary part is $e^x (x \sin y + y \cos y)$ and hence find the real part.

Sol: Let $f(z) = u(x,y) + iv(x,y)$ be the required analytic function.

Given $v(x,y) = e^x (x \sin y + y \cos y)$

we have $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

$$= \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x}$$

$$= e^x[(x+1)\cos y - y\sin y] + i e^x[(x+1)\sin y + y\cos y] \longrightarrow (1)$$

By the Milne-Thamson's method, we have

$$f'(z) = (z+1)e^z$$

(Replacing $x = z$ and $y = 0$ in (1))

Integrating, we get

$$f(z) = ze^z + c$$, where c is a complex constant.

which is the required analytic function.

Now,

$$f(z) = (x+iy)e^{x+iy} + c$$

$$= (x+iy)\, e^x . e^{iy} + c$$

$$= e^x(x\cos y - y\sin y) + i e^x(x\sin y + y\cos y) + c$$

$$= [e^x(x\cos y - y\sin y) + c] + i e^x(x\sin y + y\cos y)$$

$\therefore$ $u(x,y) = e^x[x\cos y - y\sin y] + c$ is the required real part.

Fall Sem. 2023-24

Programme Name & Branch: B. Tech.

Course Name & Code: Complex Variables and Linear Algebra (BMAT201L)

Practice all the questions

1. Show that the function $f(z)=\begin{cases}\dfrac{xy^2(x+iy)}{x^2+y^4}, & if \quad z\neq 0\\ 0, & if \quad z=0\end{cases}$

   is not analytic at origin although C-R equations are satisfied at the origin.

   Hint: Verify along the path $x=my^2$.

2. Determine whether the function $f(z)=z\,\overline{z}$ is analytic at $z=0$.

3. If $w=f(z)=x^2+ay^2-2xy+i(bx^2-y^2+2xy)$ is analytic, then find the values of a and b.

   Answer: $a=-1,\ b=1.$

4. Find the analytic function $f(z)=u+iv$ in terms z of whose real part is $u=e^{2x}(x\cos 2y-y\sin 2y)$ an analytic function of z and hence find v.

   Answer: $f(z)=z\,e^{2z}+c$ and $v=e^{2x}(x\sin 2y+y\cos 2y)$

5. If $w=f(z)=\phi+i\psi$ represents the complex potential function for an electric field and $\psi(x,\ y)=x^2-y^2+\dfrac{x}{x^2+y^2}$, then find $\phi(x,\ y)$.

   Answer: $\phi(x,\ y)=-2xy+\dfrac{y}{x^2+y^2}+k$

1.3

Applications

Complex Potential Function:

Let $\omega = f(z) = \phi(x,y) + i\psi(x,y)$ be an analytic function. Then it is called complex potential function. Its real part $\phi(x,y)$ is called velocity potential function (potential function) and imaginary part $\psi(x,y)$ is called stream function (flux function).

Problems:

1. If the potential function is $\log\sqrt{x^2+y^2}$, find the flux function and the complex potential function.

Sol: Let $\omega = f(z) = \phi(x,y) + i\psi(x,y)$ be the complex potential function.

Then $\phi(x,y)$ is the potential function and $\psi(x,y)$ is the flux function.

Given $\phi(x,y) = \log(\sqrt{x^2+y^2})$

i.e., $\phi(x,y) = \frac{1}{2}\log(x^2+y^2)$

Now, $f'(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$

$= \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$ (by C-R equations)

$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

By Milne-Thomson's method, $f'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Therefore, $f'(z) = \frac{z}{z^2} = \frac{1}{z}$

Integrating, we get

$f(z) = \log z + c$, where c is a complex constant

Hence, the required complex potential function is $w = f(z) = \log z + c$

Now, $\omega = \log z + c$

$= \log(x+iy) + ik$ (where $c = ik$)

Let $x = r\cos\theta$ and $y = r\sin\theta$.

Then, $\omega = \log(r\cos\theta + ir\sin\theta) + ik$

$\Rightarrow \phi + i\psi = \log[r(\cos\theta + i\sin\theta)] + ik$

$= \log r + \log e^{i\theta} + ik$

$= \log(\sqrt{x^2+y^2}) + i\theta + ik$

$= \log(\sqrt{x^2+y^2}) + i\tan^{-1}(y/x) + ik$

Therefore, the riquired flow function is $\psi(x,y) = \tan^{-1}(y/x) + k$.

2. If $\omega = \phi + i\psi$ represents the complex potential function for an electric field and $\psi(x,y) = x^2 - y^2 + \frac{x}{x^2+y^2}$, find the function $\phi(x,y)$.

Answer : $\phi(x,y) = -2xy + \frac{y}{x^2+y^2} + c$

(3) In a two-dimensional flow of a fluid, the velocity potential is $\phi(x,y) = x^2 - y^2$. Find the stream function ψ.

Sol: Since $\phi(x,y) = x^2 - y^2$ is velocity potential, it must satisfy the Laplace equation.

here $\frac{\partial \phi}{\partial x} = 2x$, $\frac{\partial \phi}{\partial y} = -2y$

$$\frac{\partial^2 \phi}{\partial x^2} = 2 \text{ and } \frac{\partial^2 \phi}{\partial y^2} = -2$$

Clearly, $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

$\therefore$ ϕ satisfies the Laplace equation.

Let $w = f(z) = \phi(x,y) + i\psi(x,y)$ be a complex potential function, where $\psi(x,y)$ is the stream function.

Then $f'(z) = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x}$

$= \frac{\partial \phi}{\partial x} - i \frac{\partial \psi}{\partial y}$ (by C-R equations)

$= 2x + i2y$

By Milne-Thomson's method, replacing x by z and y by 0.

$\therefore f'(z) = 2z$

Integrating, we get

$w = f(z) = z^2 + c$, where c is a Complex Constant.

$\Rightarrow \phi + i\psi = (x+iy)^2 + c$

$= (x^2 - y^2) + i2xy + ic_1$

$(c = ic_1)$

$= (x^2 - y^2) + i(2xy + c_1)$

Hence the required stream function is $\psi(x,y) = 2xy + c_1$

④ If $w = \phi + i\psi$ represents the complex potential for an electric field and $\psi = 3x^2y - y^3$, find ϕ.

<u>Answer:</u> $\phi(x,y) = x^3 - 3xy^2$

⑤ In a two-dimensional flow of a fluid, the stream function is $\psi = \frac{-y}{x^2+y^2}$. Find the velocity potential (or, potential function).

<u>Answer:</u> $\phi(x,y) = \frac{x}{x^2+y^2}$.

Fall Sem. 2023-24

Tutorial– 1

Programme Name & Branch: B. Tech.

Course Name & Code: Complex Variables and Linear Algebra (BMAT201L)

Slot: C1+TC1+TCC1

Practice all the questions

1. Show that for the function $f(z)=\sqrt{|x\,y|}$ C-R equations are satisfied at the origin but $f'(z)$ does not exist.

2. Show that the function $f(z)=\begin{cases}\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, & if \quad z\neq 0\\ 0, & if \quad z=0\end{cases}$

   is not analytic at origin although C-R equations are satisfied at the origin.

3. If $f(z)=\dfrac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(\dfrac{kx}{y}\right)$ is analytic function, then find the values of k .

Answer: k=-1

4. If $f(z)$ is analytic function, show that $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$.

5. If $f(z)=\phi+i\psi$ represents the complex potential function for an electric field and $\psi(x,\ y)=3x^3y-y^3$, then find $\phi(x,\ y)$.

Answer: $\phi(x,\ y)=x^3-3xy^2$

6. If $f(z)=u+iv$ is an analytic function of z and $u-v=(x-y)(x^2+4xy+y^2)$, then find f(z) in terms of z.

Answer: $f(z)=-iz^3+c$.

Fall Sem. 2023-24

Tutorial– 2

Programme Name & Branch: B. Tech.

Course Name & Code: Complex Variables and Linear Algebra (BMAT201L)

Slot: C1+TC1+TCC1

Practice all the questions

1. Find the image of the rectangle bounded by the lines $x=0,\ y=0,\ x=2$ and $y=1$ in the z-plane under the transformation $w=z+2+3i$.

   Answer: The rectangle bounded by the lines $u=2,\ v=3,\ u=4$ and $v=4$ in the w-plane.

2. Find the image of the region $y>\frac{1}{2}$ in the z-plane under the transformation $w=(1-i)\,z$.

Answer: $u+v>1.$

3. Find the image of the region $|z-2i|=2$ in the z-plane under the transformation $w=\frac{1}{z}$.

Answer: $v=-\frac{1}{4}.$

4. Find the image of the region $1<x<2$ in the z-plane under the transformation $w=\frac{1}{z}$.

Answer: The infinite strip $1<x<2$ transformed into the region in between the circles $\left(u-\frac{1}{2}\right)^2+v^2=\frac{1}{4}$ and $\left(u-\frac{1}{4}\right)^2+v^2=\frac{1}{16}$ in the $w-$plane.

5. Find the image of the region bounded by the lines $x=1,\ y=1$ and $x+y=1$ in the z-plane under the transformation $w=z^2$.

Answer: The image of the rectangular region bounded by the lines $x=1,\ y=1$ and $x+y=1$ in the z-plane is the region in the $w-$plane which is bounded the parabolas $v^2=4(1-u),\ v^2=4(1+u)$ and $u^2=2\left(\frac{1}{2}-v\right)$.

6. Find the image of the $y-$axis under the transformation $w=e^z$.

Answer: Find the image of the $y-$axis (i.e., $x=0$) is the unit circle $u^2+v^2=1$.

7. Find the image of the region $0<\operatorname{Re} z<1$ and $-2<y<2$ under the transformation $w=(1+2i)\,z$.

8. Determine the bilinear transformation that maps the points $1,\ i,\ -1$ in z-plane into the points $2,\ i,\ -2$ respectively in w-plane.

   Answer: $w=\dfrac{-6z+2i}{iz-3}$

9. Determine the bilinear transformation that maps the points $1,\ i,\ -1$ in z-plane into the points $2,\ i,\ -2$ respectively in w-plane and find the image of $x=0$ under this transformation.

   Answer: $w=\dfrac{1-z}{1+z}$ and the image of $x=0$ under this transformation is $u^2-v^2=1$.

10. Find the invariant points of the bilinear transformation $w=\dfrac{2z+3}{z-4}$.

    Hint: The roots of the bilinear transformation $z=\dfrac{2z+3}{z-4}$ are the invariant points or fixed points.

    Answer: $z=3+2\sqrt{3},\ 3-2\sqrt{3}$.

Module 3: Complex Integration

3.1 Functions given by Power Series

Def: A series of the form $\sum_{n=0}^{\infty} a_n z^n$ is called a complex power series or Power series.

Def: If $\sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < R$ and diverges for $|z| > R$, then R is called the radius of convergence of the Power series and $|z| = R$ is called the Circle of Convergence of the Power series.

Taylor's Series

Taylor's Theorem:

Let $f(z)$ be analytic at all points within a Circle $|z - z_0| = R$. Then at each point z within $|z - z_0| = R$,

$$f(z) = f(z_0) + (z - z_0)\frac{f'(z_0)}{1!} + (z - z_0)^2 \frac{f''(z_0)}{2!} + \dots + (z - z_0)^n \frac{f^{(n)}(z_0)}{n!} + \dots \rightarrow (1)$$

The expansion in (1) on the RHS is called the Taylor's series expansion of $f(z)$ in power of $z - z_0$ or Taylor's series expansion of $f(z)$ about $z = z_0$.

Examples:

① Obtain the Taylor's series expansion of $f(z) = \frac{1}{z}$ about the poin $z = 1$ (or, inpowers of $(z-1)$).

Sol: The given function $f(z) = \frac{1}{z}$ is analytic at $z = 1$ and not analytic at $z = 0$. Therefore, the Taylor's series expansion of $f(z) = \frac{1}{z}$ about $z = 1$ is valid in $|z-1| < 1$.

Let $z - 1 = w$. Then $z = 1 + w$.

$$
\begin{aligned}
\therefore\ f(z) = \frac{1}{z} &= \frac{1}{1+w} \\
&= (1+w)^{-1} \\
&= 1 - w + w^2 - w^3 + \cdots \quad \text{for } |w| < 1 \\
&= 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots \\
&\qquad \text{for } |z-1| < 1.
\end{aligned}
$$

② Find the Taylor's series expansion of the function $f(z) = \frac{z^2 - 1}{(z+2)(z+3)}$ in the region $|z| < 2$.

Sol: Given $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$

or, $f(z) = 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}$ (Resolving into partial fractions)

$$= 1 + \frac{3}{2}\frac{1}{\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right)$$

$$- \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right)$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\cdot z^n - \frac{8}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}\cdot z^n$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^n\left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right]z^n$$

which is the require Taylor's series.

(3) Expand $f(z) = \dfrac{z-1}{z+1}$ in Taylor's series about (i) $z=0$ and (ii) $z=1$

Sol: $f(z) = \dfrac{z-1}{z+1}$

$$= \frac{z+1-2}{z+1} = 1 - \frac{2}{z+1}$$

$$= 1 - 2(1+z)^{-1}$$

$$= 1 - 2\left(1 - z + z^2 - z^3 + z^4 - \dots\right) \text{ for } |z|<1$$

$$= -1 + 2\left(z - z^2 + z^3 - z^4 + \dots\right) \text{ for } |z|<1$$

$$= -1 + 2\sum_{n=1}^{\infty} (-1)^{n-1} \cdot z^n \text{ for } |z|<1.$$

(ii) Let $z - 1 = w$. Then $z = w+1$.

$$\therefore f(z) = \frac{z-1}{z+1}$$

$$= \frac{w}{(1+w)+1}$$

$$= \frac{w}{2+w} = \frac{w}{2\left(1+\frac{w}{2}\right)} = \frac{w}{2}\left(1+\frac{w}{2}\right)^{-1}$$

$$= \frac{w}{2}\left(1 - \frac{w}{2} + \left(\frac{w}{2}\right)^2 - \left(\frac{w}{2}\right)^3 + \dots\right) \text{ if } \left|\frac{w}{2}\right|<1$$

$$= \frac{w}{2} - \left(\frac{w}{2}\right)^2 + \left(\frac{w}{2}\right)^3 - \left(\frac{w}{2}\right)^4 + \dots \text{ if } \left|\frac{w}{2}\right|<1$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{w}{2}\right)^n \text{ if } |w|<2$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{z-1}{2}\right)^n \text{ if } |z-1|<2$$

(4) Expand $f(z) = \frac{1}{(z+1)^2}$ in Taylor's series about $z = -i$

Sol: Let $z + i = w$. Then $z = w - i$

$$\therefore f(z) = \frac{1}{(z+1)^2}$$

$$= \frac{1}{((w-i)+1)^2}$$

$$= \frac{1}{[1-i+w]^2}$$

$$= \frac{1}{(1-i)^2\left(1+\frac{w}{1-i}\right)^2}$$

$$= \frac{1}{(1-i)^2}\left[1+\frac{w}{1-i}\right]^{-2}$$

$$= \frac{1}{(1-i)^2}\left[1-\frac{2w}{1-i}+3\left(\frac{w}{1-i}\right)^2-4\left(\frac{w}{1-i}\right)^3+\cdots\right]$$

$$= \frac{-1}{2i}\left[1-2\left(\frac{w}{1-i}\right)+3\left(\frac{w}{1-i}\right)^2-4\left(\frac{w}{1-i}\right)^3+\cdots\right]$$

$$= \frac{i}{2}\left[1-\frac{2}{1-i}(z+i)+\frac{3}{(1-i)^2}(z+i)^2-\frac{4}{(1-i)^3}(z+i)^3+\cdots\right]$$

which is the required expansion.

or, Given $f(z) = \frac{1}{(z+1)^2}$. So, $f'(z) = \frac{-2}{(z+1)^3}$, $f''(z) = \frac{6}{(z+1)^4}$

and so on $f^{(n)}(z) = (-1)^n \frac{(n+1)!}{(z+1)^{n+2}}$

By Taylor's theorem, we have

$$f(z) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n \longrightarrow (1)$$

To find Taylor's series about $z=-i$, taking

$a=-i$ in (1), we get

$$f(z) = f(-i) + \sum_{n=1}^{\infty} \frac{f^{n}(-i)}{n!}(z+i)^n$$

$$= \frac{i}{2} + \frac{f'(-i)}{1!}(z+i) + \frac{f''(-i)}{2!}(z+i)^2 + \cdots$$

$$= \frac{i}{2} - \frac{2!}{(-i+1)^3(1!)}\cdot(z+i) + \frac{3!}{(-i+1)^4(2!)}(z+i)^2 - \cdots$$

$$= \frac{i}{2} - \frac{2}{(1-i)^2(1-i)}(z+i) + \frac{3}{(1-i)^2(1-i)^2}(z+i)^2 - \cdots$$

$$= \frac{i}{2} - \frac{2}{-2i(1-i)}(z+i) + \frac{3}{(-2i)(1-i)^2}(z+i)^2 - \cdots$$

$$= \frac{i}{2} - \frac{(-i)2}{(-2)(1-i)}(z+i) - \frac{3(-i)}{2(1-i)^2}(z+i)^2 - \cdots$$

$$= \frac{i}{2} - \frac{i}{(1-i)}(z+i) + \frac{i}{2}\frac{3}{(1-i)^2}(z+i)^2 - \cdots$$

$$\therefore f(z) = \frac{i}{2}\left[1 - \frac{2}{(1-i)}(z+i) + \frac{3}{(1-i)^2}(z+i)^2 - \cdots\right]$$

3.1

Laurent Series

If $f(z)$ is an analytic function in a ring shaped region R bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2 $(r_1 > r_2)$ with centre at a, then for all z in R,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \dots \quad \text{①}$$

where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)\,dw}{(w-a)^{n+1}}$

and $b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)\,dw}{(w-a)^{-n+1}}$,

where the integrals are taken around C_1 and C_2 in the anticlockwise sense.

The RHS of ① is called the Laurent's Series expansion of $f(z)$ about $z=a$ (or, around $z=a$).

Examples

① Find the Laurent's series for $f(z) = \frac{1}{(z-2)(1-z)}$ in the annular region $1 < |z| < 2$.

Sol: Here $|z| = 1$ and $|z| = 2$ are two concentric circles with centre at o and radii 1 and 2 respectively. clearly $f(z) = \frac{1}{(1-z)(z-2)}$ is analytic in the region $1 < |z| < 2$.

Now,
$$f(z) = \frac{1}{(1-z)(z-2)}$$
$$= \frac{-1}{(z-1)(z-2)}$$
$$= \frac{1}{z-1} - \frac{1}{z-2}$$
$$= \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{2\left(1-\frac{z}{2}\right)}$$
$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$
$$= \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) + \frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right)$$

valid for $1 < |z| < 2$.

(2) Expand $f(z) = \frac{1}{z^2-3z+2}$ as a Laurent's series in the region $0 < |z-1| < 1$.

(3) Find the Laurent Series for $f(z) = \frac{7z-2}{(z+1)z(z-2)}$

in the region $1 < |z+1| < 3$.

② Expand $f(z)=\frac{1}{z^2-3z+2}$ as a Laurent's series in the region $0<|z-1|<1$.

③ Find the Laurent's series for

$$f(z)=\frac{7z-2}{(z+1)\,z\,(z-2)}$$ in the region $1<|z+1|<3$.

Sol: Given $f(z)=\frac{7z-2}{(z+1)\,z\,(z-2)}$

$$=\frac{1}{z}-\frac{3}{z+1}+\frac{2}{z-2}$$

(Resolving into partial fractions)

Let $z+1=\omega$. Then $z=\omega-1$

For $1<|z+1|<3$, $\left|\frac{1}{|z+1|}\right|<1$ and $\left|\frac{z+1}{3}\right|<1$

i.e., $\left|\frac{1}{\omega}\right|<1$ and $\left|\frac{\omega}{3}\right|<1$.

Now, $f(z)=\frac{1}{\omega-1}-\frac{3}{\omega-1+1}+\frac{2}{\omega-1-2}$

$$=\frac{+1}{\omega(1-\frac{1}{\omega})}-\frac{3}{\omega}-\frac{2}{3}\frac{1}{(1-\frac{\omega}{3})}$$

$$=\frac{1}{\omega}\left(1-\frac{1}{\omega}\right)^{-1}-\frac{3}{\omega}-\frac{2}{3}\left(1-\frac{\omega}{3}\right)^{-1}$$

$$=\frac{1}{\omega}\left(1+\frac{1}{\omega}+\frac{1}{\omega^2}+\cdots\right)-\frac{3}{\omega}-\frac{2}{3}\left(1+\frac{\omega}{3}+\frac{\omega^2}{9}+\cdots\right)$$

$$=\left(-\frac{2}{\omega}+\frac{1}{\omega^2}+\frac{1}{\omega^3}+\cdots\right)-\frac{2}{3}\left(1+\frac{\omega}{3}+\frac{\omega^2}{9}+\cdots\right)$$

$$= -\frac{2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots$$

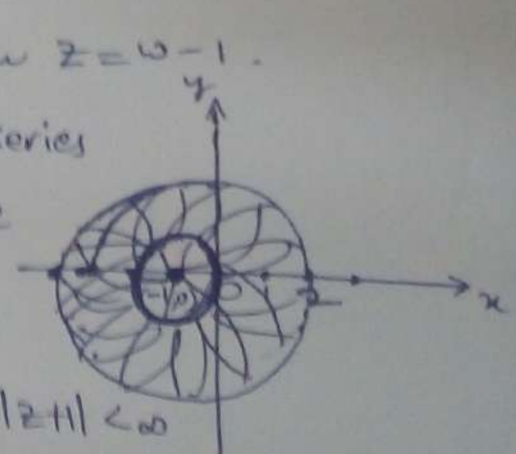
$$-\frac{2}{3}\left(1 + \frac{z+1}{3} + \frac{(z+1)^2}{9} + \cdots\right)$$

(4) obtain the Laurent series of the function $f(z) = \frac{7z-2}{(z+1)z(z-2)}$ about $z=-1$.

Sol: The function $f(z) = \frac{7z-2}{(z+1)z(z-2)}$ is not analytic at $z=-1, 0, 2$.

Let $z+1 = w$. Then $z = w-1$.

Clearly, the Laurent series of $f(z)$ is valid in the regions $0<|z+1|<1$, $1<|z+1|<3$ and $3<|z+1|<\infty$ about $z=-1$.



(5) Obtain the Laurent series of the function $f(z) = \frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ about $z=-2$.

3.2 Singularities

Def: A singular point (or, singularity) of a function $f(z)$ is the point at which the function $f(z)$ is not analytic.

e.g. $f(z) = \frac{1}{z-1}$ is not analytic at $z=1$.

So, $z=1$ is a singular point of $f(z)$.

Types of Singularities

1. Isolated Singularity:

A singular point $z=a$ of a function $f(z)$ is said to be isolated singularity of $f(z)$ if there exists a neighbourhood of the point $z=a$ which contains no other singularity.

e.g. Let $f(z) = \frac{1}{(z-1)(z+1)}$, then $z=-1, 1$ are two ~~(isotatate)~~ isolated singular points of $f(z)$.

Alternate Def: Suppose $z=a$ is a singular point of a function $f(z)$ and no other singular point of $f(z)$ exists in a circle with centre at a, then $z=a$ is said to be an isolated singular point of $f(z)$.

In such a case $f(z)$ can be expressed by Laurent's series around $z = a$.

i.e., $f(z)$ can be expressed

as a Laurent's series in the region $\boxed{0 < |z-a| < R}$

Note:

The portion $$\frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \frac{b_3}{(z-a)^3} + \cdots \longrightarrow (1)$$

of the Laurent's series of $f(z)$, involving the negative powers of $z-a$, is called the principal part of $f(z)$ at $z=a$. The co-efficient b_1 in equation (1), is the residue of the function $f(z)$ at $z = a$.

Pole: If the principart of $f(z)$ contains a finite number of terms (i.e., at least one non-zero term) say m (i.e., $b_n = 0$ for $n > m$, and $m \geq 1$), then the singular point $z = a$ is called a pole of order m of $f(z)$.

Simple pole is a pole of order One.

For example, if $f(z) = \frac{z}{(z-1)(z+1)^3}$, then

$z=1$ is a simple pole and $z=-1$ is a pole of order 3.

<u>Alternate Def:</u> If there exists a positive integer n such that $\lim_{z \to a} (z-a)^n f(z) = l \neq 0$, then $z=a$ is called a pole of order n.

<u>Examples:</u>

① Find the poles of the function

$$f(z) = e^z.$$

<u>Sol:</u> $f(z) = e^z$

$$= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$$

Here, $f(z)$ contains no principal part.

Hence there are no poles for $f(z)$.

In fact, the function $f(z) = e^z$ is analytic everywhere in $\mathbb{C}$.

② Find the poles of the function

$$f(z) = e^{1/z}.$$

Sol: $f(z) = e^{1/z}$

$$= 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots$$

Here, principal part of $f(z)$ contains infinite number of terms. So, there are no poles for $f(z)$.

(3) Find the pole of $f(z) = (z-2) + \frac{1}{z-2}$

Sol: Clearly $z = 2$ is a pole of $f(z)$ of order 1 (since principal part of $f(z)$ contains only one term).

(4) Find the poles of the function

$$f(z) = \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} - \cdots$$

Sol: Clearly $f(z)$ has a pole $z=0$ of order 2.

② <u>Essential Singularity</u>: If the principal part of $f(z)$ contains an infinite number of terms (i.e., the series $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ contains an infinite number of terms), then $z=a$ is called essential singularity of $f(z)$.

e.g. For the function $f(z) = e^{1/z}$

$$= 1 + \frac{1}{1!\,z} + \frac{1}{2!\,z^2} + \cdots,$$

~~(clearly)~~ $z=0$ is an essential singularity.

③ <u>Removeble Singularity</u>:

If the principal part of $f(z)$ contains no terms (i.e., $b_n = 0 \ \forall n$), then the singular point $z=a$ is called a removeble singularity of $f(z)$.

In this case $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$.

For example, if $f(z) = \frac{1-\cos z}{z}$

$$= \frac{1}{z}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\right]$$

$$= \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \cdots,$$

then $z = 0$ is a removeble singularity.

Calculus of Residues

To evaluate the residue of a function $f(z)$ at $z = a$.

i) Suppose $z = a$ is a simple pole, then the Laurent series of $f(z)$ is $\sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a}$

i.e, $$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} \longrightarrow (1)$$

where $$b_1 = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{(w-a)^{-1+1}} dw = \frac{1}{2\pi i} \int_{C_2} f(w)\, dw$$

Multiplying eq.n (1) with $(z-a)$ on both sides, we get

$$(z-a)\, f(z) = \sum_{n=0}^{\infty} a_n (z-a)^{n+1} + b_1$$

Taking limit as $z \to a$ on both sides, we get

$$\underset{z \to a}{Lt}\, (z-a)\, f(z) = 0 + b_1$$

Hence $b_1 = \underset{z \to a}{Lt}\, (z-a)\, f(z)$ is called the residue of $f(z)$ at a simple pole $z = a$.

ii) Suppose $z = a$ is a pole of order n, then the Laurent series of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \dots + \frac{b_n}{(z-a)^n} \longrightarrow (1)$$

Multiplying eqn (1) with $(z-a)^n$ on both sides, we get

$$(z-a)^n f(z) = \sum_{n=0}^{\infty} a_n (z-a)^{2n} + b_1 (z-a)^{n-1} + b_2 (z-a)^{n-2} + \dots + b_{n-1} (z-a) + b_1 \longrightarrow (2)$$

Differentiating eqn (2), $(n-1)$ times with respect to z and then taking limit as $z \to a$ on both sides, we get

$$\lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right] = b_1 (n-1)!$$

Hence $b_1 = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]$ is

the residue of $f(z)$ at pole $z=a$ of order n.

Examples:

1. Find the poles and the corresponding residues at each pole of the function

$$f(z) = \frac{z+1}{z^2(z-2)}.$$

sol: Given $f(z) = \frac{z+1}{z^2(z-2)}$.

The poles of $f(z)$ are $z=0$ and $z=2$.

Here $z=0$ is a pole of order 2 and

$z=2$ is a simple pole (pole of ~~ord~~ order one).

Now,

i) $\text{Res.at}(z=2) = \underset{z\to 2}{\mathrm{Lt}}\,(z-2)\,f(z)$

$$= \underset{z\to 2}{\mathrm{Lt}}\,(z-2)\,\frac{(z+1)}{z^2(z-2)}$$

$$= \underset{z\to 2}{\mathrm{Lt}}\,\frac{(z+1)}{z^2} = \frac{3}{4}.$$

ii) $\text{Res.}\,(z=0) = \frac{1}{(2-1)!}\underset{z\to 0}{\mathrm{Lt}}\,\frac{d^{2-1}}{dz^{2-1}}\left[(z-0)^2 f(z)\right]$

$$= \underset{z\to 0}{\mathrm{Lt}}\,\frac{d}{dz}\left[z^2\left(\frac{z+1}{z^2(z-2)}\right)\right]$$

$$= \underset{z\to 0}{\mathrm{Lt}}\,\frac{d}{dz}\left(\frac{z+1}{z-2}\right) = \underset{z\to 0}{\mathrm{Lt}}\,\frac{(z-2)-(z+1)}{(z-2)^2}$$

$$= \underset{z\to 0}{\mathrm{Lt}}\,\frac{-3}{(z-2)^2} = \frac{-3}{4}.$$

2. Find the poles and the corresponding residues at each pole of $f(z) = \frac{1-e^z}{z^4}$.

Sol: $z=0$ is a pole of order 4.

$$\text{Res.}\,(z=0) = \frac{1}{(4-1)!}\underset{z\to 0}{\mathrm{Lt}}\,\frac{d^3}{dz^3}\left((z-0)^4 f(z)\right)$$

$$= \frac{1}{3!} \lim_{z\to 0} \frac{d^3}{dz^3}(1-e^z) = \frac{-1}{6}.$$

3. Find the poles and the corresponding residues of (a) $f(z) = \dfrac{z^2+2z}{(z+1)^2(z^2+4)}$ (b) $f(z) = \dfrac{z^2}{z^4-1}$.

Sol: (a) poles are $z=-1$, $2i$ and $-2i$.

Here $z = 2i$ and $-2i$ are simple poles and $z=-1$ is a pole of order 2.

i) $\text{Res.}(z=2i) = \dfrac{1-7i}{25}$ (ii) $\text{Res.}(z=-2i) = \dfrac{1+7i}{25}$

and (iii) $\text{Res.}(z=-1) = \dfrac{2}{25}$.

(b) poles are $z = 1, -1, i, -i$ and all are simple poles.

i) $\text{Res.}(z=1) = \dfrac{1}{4}$ (ii) $\text{Res.}(z=-1) = \dfrac{-1}{4}$

iii) $\text{Res.}(z=i) = \dfrac{-i}{4}$ (iv) $\text{Res.}(z=-i) = \dfrac{i}{4}$.

3.3 Integration of a complex function along a contour:

A contour is a continuous chain of or finite number of smooth arcs.

If a contour is closed and does not intersect itself, it is called a closed contour.

<u>Note:</u> The length of a contour is the sum of the lengths of the smooth arcs containing the contour.

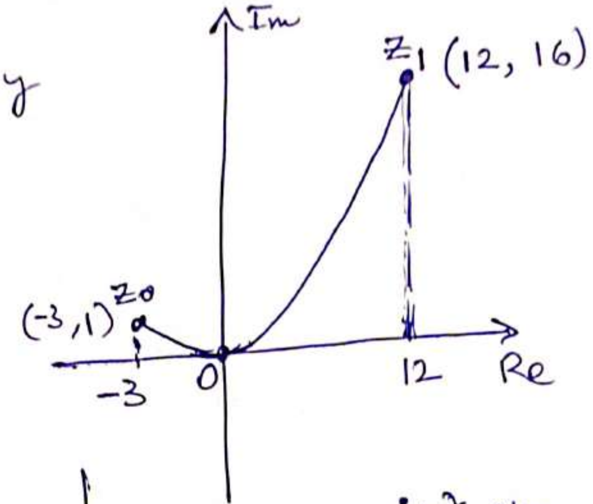
Line Integral:

Consider a contour C parametrized by $z(t) = x(t) + i\,y(t)$ for $a \le t \le b$. We define the integra of the complex function $f(z)$ along C $\left(\text{i.e., } \int_C f(z)\,dz\right)$ and is called the line integral. It is also called a contour integral.

Examples

1. Evaluate $\int_C \bar{z}\, dz$, where C is given by

$x = 3t,\ y = t^2$ with $-1 \le t \le 4$.



Sol: Here C is given by

$$z(t) = 3t + it^2$$

So, $dz = 3\,dt + 2it\,dt$

and $f(z(t)) = \overline{z(t)}$

$$= 3t - it^2$$

$$\therefore \int_C \bar{z}\, dz = \int_{-1}^{4} (3t - it^2)(3 + 2it)\,dt$$

C: $z(t) = 3t + it^2$ with $-1 \le t \le 4$

$$\boxed{\begin{aligned} z_0 &= -3 + i \\ z_1 &= 12 + 16i \end{aligned}}$$

$$= \int_{-1}^{4} (2t^3 + 9t + 3t^2 i)\,dt$$

$$= \left[\frac{t^4}{2} + \frac{9t^2}{2}\right]_{-1}^{4} + \left[3i\,\frac{t^3}{3}\right]_{-1}^{4}$$

$$= 195 + 65i$$

2. Evaluate $\int_C \frac{1}{z}\, dz$, where C is the circle

$x = \cos t,\ y = \sin t$ with $0 \le t \le 2\pi$.

Sol: C is given by $z(t) = \cos t + i \sin t = e^{it}$

so, $f(z(t)) = \frac{1}{e^{it}}$ and $dz = ie^{it}dt$.

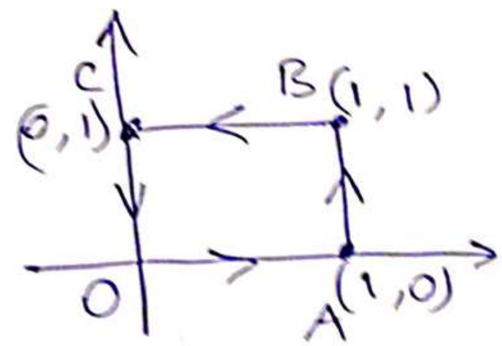
$$\therefore \int_C f(z)\,dz = \int_C \frac{1}{z}\,dz$$

$$= \int_0^{2\pi} e^{-it}(ie^{it})\,dt$$

$$= i \int_0^{2\pi} dt = 2\pi i$$

3. Evaluate $\int_C (3z+1)\,dz$ where C is the boundary of the square with vertices at the points $z=0$, $z=1$, $z=1+i$, $z=i$ and orientation of C is anticlockwise

Sol: C is the square $OABC$



$$\therefore \int_C (3z+1)dz$$

$$= \int_{OA} (3z+1)\,dz + \int_{AB} (3z+1)\,dz + \int_{BC} (3z+1)\,dz + \int_{CO} (3z+1)dz$$

Along OA: $y = 0 \Rightarrow dy = 0$

and x varie from 0 to 1.

$$\therefore \int_{OA} (3z+1)\, dz = \int_0^1 (3x+1)\, dx = \frac{5}{2}$$

<u>Along AB:</u>

$$x = 1 \Rightarrow dx = 0$$

and y varies from 0 to 1.

$$\therefore \int_{AB} (3z+1)\, dz = i \int_0^1 [3(1+iy)+1]\, dy$$

$$= 4i - \frac{3}{2}$$

<u>Along BC:</u>

$$y = 1 \Rightarrow dy = 0$$

and x varies from 1 to 0.

$$\therefore \int_{BC} (3z+1)\, dz = -\int_0^1 [3(x+i)+1]\, dx = -\frac{5}{2} - 3i$$

<u>Along CO:</u>

$$x = 0 \Rightarrow dx = 0$$

and y varies from 1 to 0

$$\therefore \int_{OC} (3z+1)\, dz = \int_1^0 [3iy+1]\, i\, dy = \frac{3}{2} - i$$

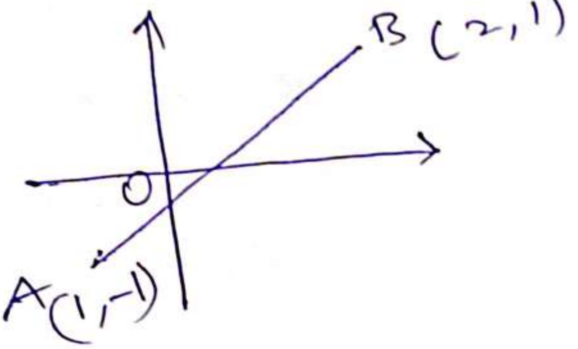
Hence, $\displaystyle\int_C (3z+1)\, dz = \frac{5}{2} + 4i - \frac{3}{2} - \frac{5}{2} - 3i + \frac{3}{2} - i$

$$= 0$$

4. Evaluate $\displaystyle\int_{i-1}^{2+i} (2x+iy+1)\,dz$ along $1-i$ to $2+i$

Sol: Along $1-i$ to $2+i$ is the straight line AB joining $(1,-1)$ and $(2,1)$.

$\therefore$ The equation of AB is



$$y-1 = \frac{(-1-1)}{(1-2)}(x-2)$$

$\Rightarrow y = 2x-3.$

So, $dy = 2dx$ and x varies from 1 to 2.

$$\therefore \int_{1-i}^{2+i} (2x+iy+1)\,dz = \int_{1-i}^{2+i} (2x+iy+1)(dx+idy)$$

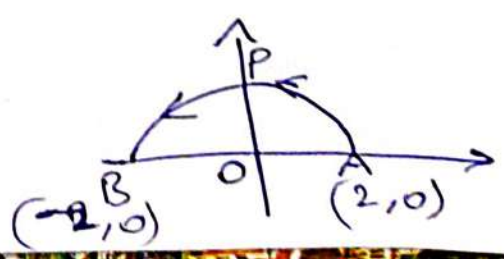
$$= \int_{x=1}^{2} \left[(-2x+7)dx + i(6x-1)dx\right]$$

$$= 4+8i.$$

5. Evaluate

$\displaystyle\int_C \left(\frac{z+2}{z}\right)dz$, where C is the semi-circle $z = 2e^{i\theta}$ (θ varies from 0 to π).

Answer: $-4+2\pi i$



Cauchy-Goursat Theorem :

If a function $f(z)$ is analytic at all points interior to and on a simple closed curve C, then $\int_C f(z)\,dz = 0$.

i.e., If $f(z)$ is analytic on some simply connected domain D, then for every close contour C within D, $\int_C f(z)dz=0$

Cauchy's Integral Formula:

let $f(z)$ be an analytic function within and on a closed contour C. If $z=a$ is any point within C, then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz,$$

where the ~~ing~~ integral is taken in the positive sense around C.

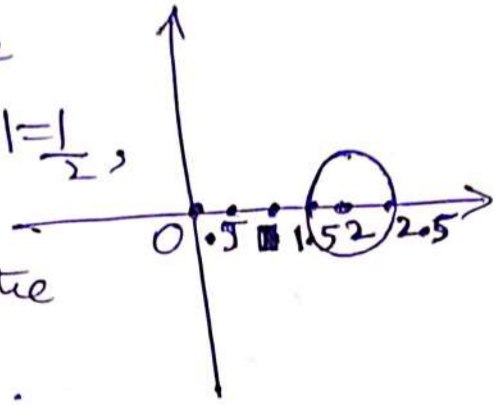
Examples:

1. Evaluate $\int_C \frac{z}{(z-1)(z-2)}\, dz$, where C is $|z-2| = \frac{1}{2}$

Sol: clearly

$$\frac{z}{(z-1)(z-2)} = \frac{2}{z-2} - \frac{1}{z-1}$$

and $z = 1$ is out side of the circle $|z-2| = \frac{1}{2}$,



$z = 2$ is inside of the circle $|z-2| = \frac{1}{2}$.

$$\therefore \int_C \frac{dz}{z-1} = 0$$

and

$$\int_C \frac{2}{z-2}\, dz = 2\pi i\, f(2) \qquad (\text{since } f(z) = 2)$$

$$= 2\pi i\,(2)$$

$$= 4\pi i$$

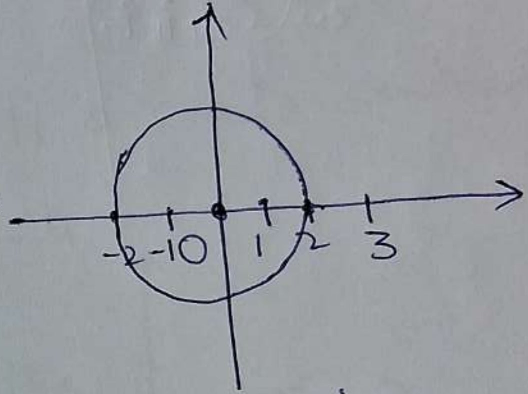
Hence $\int_C \frac{z}{(z-1)(z-2)}\, dz = 4\pi i$

Generalisation of Cauchy's Integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

2. Evaluate $\int_C \frac{z+1}{(z-1)(z-3)}\, dz$, where C is $|z|=2$

Sol: clearly $z=+1$ is inside and $z=3$ is ouside of the circle $C: |z|=2$



$\therefore \int_C \frac{\left(\frac{z+1}{z-3}\right)}{z-1}\, dz = 2\pi i \cdot f(1)$

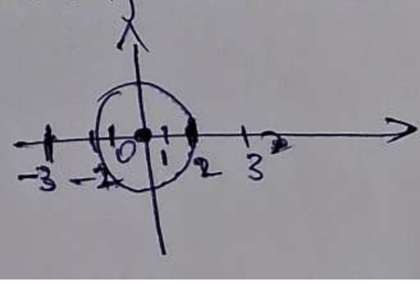
$= 2\pi i \left(\frac{2}{-2}\right)$

$= -2\pi i$

here $f z = \frac{z+1}{z-3}$

3. Evaluate $\int_C \frac{e^{2z}}{(z-1)^3}\, dz$ where C is $|z|=2$

Sol: Clearly $z=1$ is inside of the circle $|z|=2$



$$\therefore \int_C \frac{e^{2z}}{(z-1)^3}\,dz = \frac{2\pi i}{2!} f''(1)$$

$$= \pi i (4e^2)$$

$$= 4\pi i e^2$$

here $f(z) = e^{2z}$

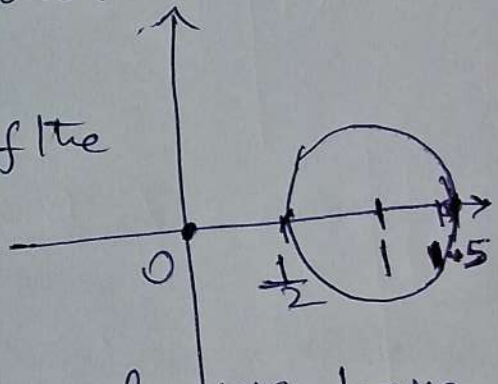
so, $f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

4. Evaluate $\int_C \frac{z^3 e^{-z}}{(z-1)^3}\,dz$, where C is $|z-1| = \frac{1}{2}$.

<u>Sol:</u> Clearly $|z-1| = \frac{1}{2}$ is a circle centred at $z=1$ and radius 0.5 units

Here $z=1$ is inside of the circle $|z-1| = \frac{1}{2}$.



By Cauchy's integral formula, we have

$$\int_C \frac{e^{-z} \cdot z^3}{(z-1)^3}\,dz = \frac{2\pi i}{2!} f''(1)$$

$$= \frac{\pi i}{e}.$$

here $f(z) = z^3 e^{-z}$.

so, $f''(1) = \frac{1}{e}$

Cauchy's Residue Theorem

If $f(z)$ is analytic within and on a closed contour C except at a finite number of points $z_1, z_2, \dots, z_n$ within C and $R_1, R_2, \dots, R_n$ are the residues of $f(z)$ at $z_1, z_2, \dots, z_n$ respectively, then

$$\int_C f(z)\,dz = 2\pi i\,(R_1 + R_2 + \dots + R_n)$$

Examples:

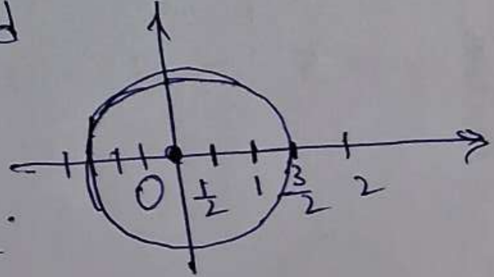
1. Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$ where C is the circle $|z| = \frac{3}{2}$ using residue theorem.

Sol: Given $f(z) = \frac{4-3z}{z(z-1)(z-2)}$.

Clearly, the poles of $f(z)$ are

$z = 0, 1, 2$ and

$z = 0, 1$ are lies inside the circle $|z| = \frac{3}{2}$.



Now,

i) $\text{Res.}(z=0) = \underset{z\to 0}{Lt}\ (z-0)\, f(z)$

$$= \underset{z\to 0}{Lt}\ z\left[\frac{4-3z}{z(z-1)(z-2)}\right]$$

$$= \frac{4}{2} = 2$$

ii) $\text{Res.}(z=1) = \underset{z\to 1}{Lt}\ (z-1)\, f(z) = \underset{z\to 1}{Lt}\ (z-1)\left[\frac{4-3z}{z(z-1)(z-2)}\right]$

$$= -1$$

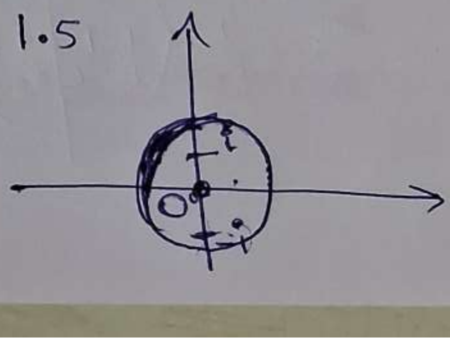
$\therefore$ By Residue theorem,

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\, dz = 2\pi i\,\big(2+(-1)\big) = 2\pi i$$

2. Evaluate $\displaystyle\int_C \frac{1}{(z^2+1)(z^2-4)}\, dz$ where C is the circle $|z| = \frac{3}{2}$.

<u>Sol:</u> Given $f(z) = \dfrac{1}{(z^2+1)(z^2-4)}$

Clearly $f(z)$ has four simple poles at $z=\pm i$ and $z=\pm 2$. But $z=\pm i$ are inside of the circle $|z| = 1.5$ and $z=\pm 2$ are out side of the circle.



Now,

i) $\text{Res.}(z=i) = \lim_{z\to i} (z-i)\, f(z)$

$= \lim_{z\to i} (z-i) \dfrac{1}{(z+i)(z-i)(z^2-4)}$

$= \dfrac{-1}{10i}$

ii) $\text{Res.}(z=-i) = \lim_{z\to -i} (z+i) \dfrac{1}{(z+i)(z-i)(z^2-4)}$

$= \dfrac{1}{10i}$

$\therefore$ By Residue theorem,

$$\int_C \frac{1}{(z^2+1)(z^2-4)}\,dz = 2\pi i\left(\frac{-1}{10i}+\frac{1}{10i}\right)$$

$$= 0.$$

3. Evaluate $\int_C \dfrac{2e^z}{(z+1)^3}\,dz$ where C is the circle $|z-1|=3$ using Residue theorem.

Answer: $\dfrac{4\pi i}{e}$

4. Evaluate $\int_C \dfrac{z^3\,dz}{(z-1)^2(z-3)}$ where C is the circle $|z|=2$ using Residue theorem.

Answer: $\dfrac{-7\pi i}{2}$.

3.4 Evaluation of Real Integrals:

1. Integrals around the unit circle

Integrals of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$,

where f is a rational function in $\cos\theta$ and $\sin\theta$.

To evaluate this type of integrals, we take the unit circle $|z|=1$ as the contour

on $|z|=1$, $z = e^{i\theta} \Rightarrow \frac{dz}{d\theta} = ie^{i\theta}$

$\Rightarrow dz = iz\,d\theta$

or, $d\theta = \frac{dz}{iz}$

Also, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$

and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2} = \frac{z-\frac{1}{z}}{2} = \frac{z^2-1}{2zi}$.

here, $|z|=1 \Rightarrow \theta$ varies from 0 to 2π.

$$\therefore \int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta = \int_C f\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{dz}{iz}$$

Examples

1. Evaluate $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$ using contour integra

Sol: Let $z = e^{i\theta}$. Then $d\theta = \frac{dz}{iz}$ and $\sin\theta = \frac{z^2-1}{2i}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_C \frac{1}{5+4\left(\frac{z^2-1}{2iz}\right)} \frac{dz}{iz}, \text{ where } C \text{ is } |z|=1$$

$$= \int_C \frac{1}{2z^2+5iz-2}\, dz$$

$$= \int_C f(z)\, dz \longrightarrow (1)$$

where $f(z) = \frac{1}{2z^2+5iz-2}$

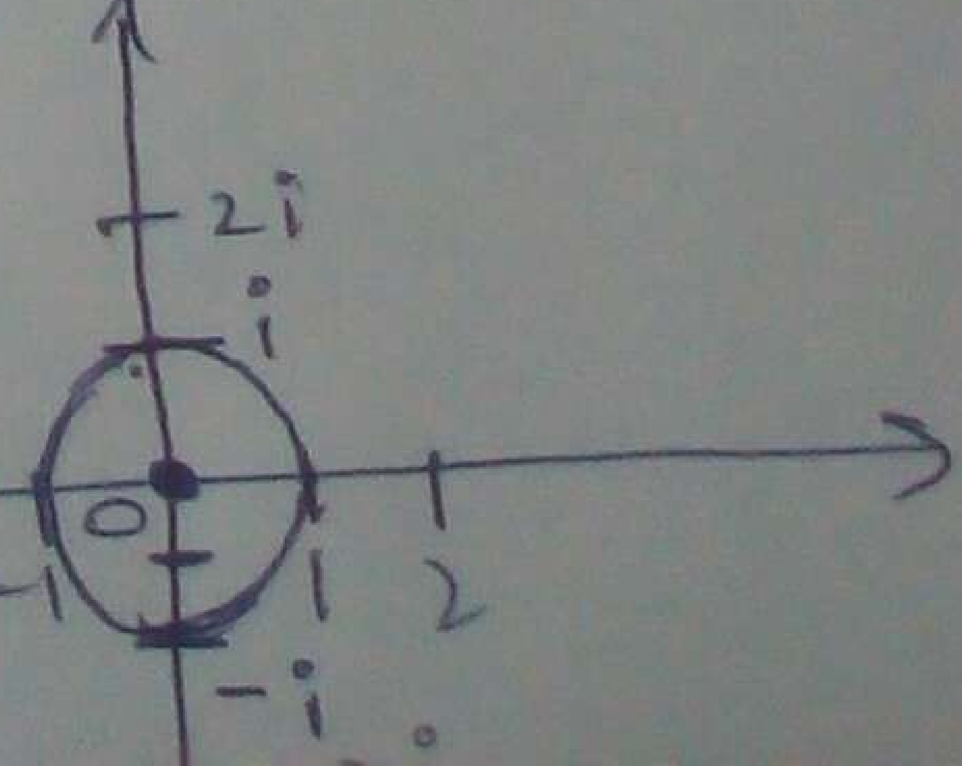
To find the poles of $f(z)$, taking $2z^2+5iz-2=0$

then $z = \frac{-5i \pm \sqrt{-25+16}}{4}$

$$= \frac{-5i \pm 3i}{4} = -\frac{i}{2},\ -2i \text{ are}$$

poles of order one

Clearly, $z = -\frac{i}{2}$ lies inside of $|z|=1$ and $z=-2i$ lies outside of $|z|=1$.



Now,

$$\text{Res.}\left(z=-\frac{i}{2}\right)=\left[\text{Res.} f(z)\right]_{z=-\frac{i}{2}}$$

$$=\lim_{z\to -\frac{i}{2}}\left(z+\frac{i}{2}\right)f(z)$$

$$=\lim_{z\to -\frac{i}{2}}\left(z+\frac{i}{2}\right)\frac{1}{2\left(z+\frac{i}{2}\right)(z+2i)}$$

$$=\frac{1}{2\left(-\frac{i}{2}+2i\right)}=\frac{1}{3i}$$

$\therefore$ By Cauchy's Residue theorem, we have

$$\int_C f(z)\,dz=2\pi i\left(\left[\text{Res.} f(z)\right]_{z=-\frac{i}{2}}\right)$$

$$=2\pi i\left(\frac{1}{3i}\right)=\frac{2}{3}\pi$$

Hence, from (1), we get

$$\int_0^{2\pi}\frac{1}{5+4\sin\theta}\,d\theta=\frac{2}{3}\pi.$$

② Evaluate $\int_0^{2\pi} \frac{1}{a+b\cos\theta}\, d\theta$ $(a>b,\ b>0)$ $(a>b>0)$
using contour integration.

Sol: $$\int_0^{2\pi} \frac{1}{a+b\cos\theta}\, d\theta = \int_C \frac{1}{a+b\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$
where C is $|z|=1$.

$$= \frac{2}{i}\int_C \frac{1}{bz^2+2az+b}\, dz$$

$$= \frac{2}{i}\int_C f(z)\, dz \longrightarrow ①$$

where $f(z) = \frac{1}{bz^2+2az+b}$

Clearly, the poles of $f(z)$ are

$z = \frac{-a+\sqrt{a^2-b^2}}{b}$ $= \alpha$ (say) and $z = \frac{-a-\sqrt{a^2-b^2}}{b}$ $= \beta$ (say)

of order one.

Also, $z = \frac{-a+\sqrt{a^2-b^2}}{b}$ lies inside of $|z|=1$

and $z = \frac{-a-\sqrt{a^2-b^2}}{b}$ lies outside of $|z|=1$.

Since $a>b$, we can write $bz^2+2az+b$
$= b(z-\alpha)(z-\beta)$

Now,

$$\left[\text{Res. } f(z)\right]_{z=\alpha} = \lim_{z\to\alpha} (z-\alpha) f(z)$$

$$= \lim_{z\to\alpha} (z-\alpha)\frac{1}{b(z-\alpha)(z-\beta)}$$

$$= \frac{1}{b(\alpha-\beta)}$$

$$= \frac{1}{2\sqrt{a^2-b^2}}$$

$\therefore$ By Cauchy's Residue theorem, we have

$$\int_C f(z)\,dz = 2\pi i\left(\frac{1}{2\sqrt{a^2-b^2}}\right) = \frac{\pi i}{\sqrt{a^2-b^2}}$$

Hence, from ①, we get

$$\int_0^{2\pi} \frac{1}{a+b\cos\theta}\,d\theta = \frac{2}{i}\left(\frac{\pi i}{\sqrt{a^2-b^2}}\right) \quad (a>b>0).$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

3. Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta$ using contour Integration.

Answer: $\boxed{\frac{\pi}{12}}$ $\boxed{\frac{\pi}{12}}$

4. Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{5-3\cos\theta}\,d\theta$ using contour Integration.

Answer: $\boxed{\frac{2\pi}{9}}$

5. Using contour Integration, evaluate the real integral $\displaystyle\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta$

Hint: $$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta$$

$$\left(\text{Since } \int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx \quad \text{if } f(2a-x) = f(x)\right)$$

Answer: 0 .

6. Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{a+b\sin\theta}\, d\theta \quad (a>b>0)$

Answer: $\dfrac{2\pi}{\sqrt{a^2-b^2}}$.

7. Evaluate $\displaystyle\int_0^{\pi} \frac{a}{a^2+\sin^2\theta}\, d\theta \quad (a>0)$

Sol: $\int_0^{\pi} \frac{a}{a^2+\sin^2\theta} d\theta = \int_0^{\pi} \frac{2a}{2a^2+(1-\cos 2\theta)} d\theta$

Taking $2\theta = \phi$, then we get $2d\theta = d\phi$.

And $\theta \to 0, \phi \to 0$

$\theta \to \pi, \phi \to 2\pi$

$$\therefore \int_0^{\pi} \frac{a}{a^2+\sin^2\theta} d\theta = \int_0^{2\pi} \frac{a}{2a^2+1-\cos\phi} d\phi$$

Put $z = e^{i\phi}$, then $d\phi = \frac{dz}{iz}$ and $\cos\phi = \frac{z^2+1}{2z}$

and hence

$$\int_0^{\pi} \frac{a}{a^2+\sin^2\theta} d\theta = -\frac{2a}{i} \int_C f(z)\,dz, \text{ where}$$

$$f(z) = \frac{1}{z^2-(4a^2+2)z+1}$$

Now, the poles of $f(z)$ are given by

$$z = (2a^2+1) + 2a\sqrt{a^2+1} = \alpha \text{ (say)}$$

and $z = (2a^2+1) - 2a\sqrt{a^2+1} = \beta$ (say).

Clearly $|\alpha| > 1$ and $|\beta| < 1$.

$$\therefore \left[\text{Res}. f(z)\right]_{z=\beta} = \lim_{z \to \beta} (z-\beta) f(z)$$

$$= \frac{1}{-4a\sqrt{a^2+1}}$$

Therefore, by Residue theorem, we get

$$\int_0^{\pi} \frac{a}{a^2+\sin^2\theta}\, d\theta = -\frac{2a}{i}\left[2\pi i\, \left[\text{Res. } f(z)\right]_{z=\beta}\right]$$

$$= \frac{\pi}{\sqrt{a^2+1}}.$$

Note: Let $z = e^{i\theta}$, then $d\theta = \frac{dz}{iz}$

and $\cos 2\theta = \frac{e^{i2\theta}+e^{-i2\theta}}{2}$

$$= \frac{\left(z^2+\frac{1}{z^2}\right)}{2} = \frac{z^4+1}{2z^2},$$

$$\sin\theta = \frac{e^{i2\theta}-e^{-i2\theta}}{2i} = \frac{z^4-1}{2iz^2}.$$

2. Integrals of the type $\int_{-\infty}^{\infty} f(x)\,dx$:

Integration around semi-circles

To solve these, consider $\int_C f(z)\,dz$, wher c is the closed contour consisting of the semi-circle $C_R : |z| = R$, together with the real axis from $-R$ to R.

If $f(z)$ has no singular points on the real axis, by Residue theorem, we have $\int_C f(z)\,dz = 2\pi i \left(\begin{matrix}\text{Sum of} \\ \text{the} \\ \text{residues}\end{matrix}\right)$

i.e., $\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \left(\begin{matrix}\text{Sum of the} \\ \text{residues at} \\ \text{interior poles}\end{matrix}\right)$

So, we find the value of $\int_{-\infty}^{\infty} f(x)\,dx$, provided

$\int_{C_R} |f(z)|\,dz \to 0$ when $R \to \infty$.

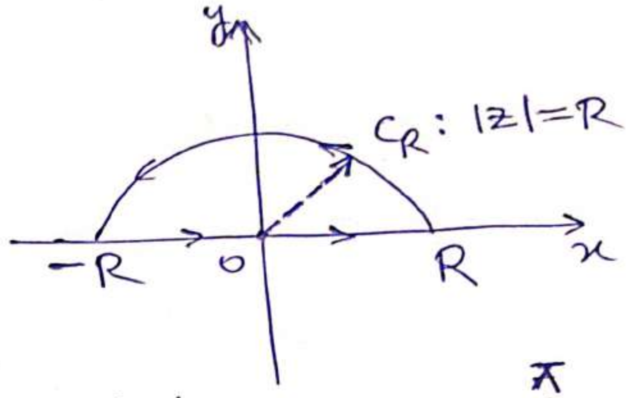
Examples:

(1) Evaluate $\int_0^{\infty} \frac{1}{1+x^2}\,dx$ using contour integration

Sol: since $\frac{1}{1+x^2}$ is an even function, we have

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \left(\frac{1}{2}\right)\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx .$$

consider $\int_C \frac{1}{1+z^2}\,dz$, where C is the closed contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R.



Therefore,

$$\int_C \frac{1}{1+z^2}\,dz = \int_{C_R} \frac{1}{z^2+1}\,dz + \int_{-R}^{R} \frac{1}{1+x^2}\,dx$$

$$\left(\text{Since } \left|\int_{C_R} \frac{dz}{z^2+1}\right| \le \int_{C_R} \frac{|dz|}{|z^2+1|} \le \frac{1}{R^2-1}\int_0^{\pi} R\,d\theta = \frac{R\pi}{R^2-1} \to 0 \text{ as } R\to\infty\right)$$

which implies that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_C f(z)\,dz, \text{ where } f(z) = \frac{1}{1+z^2}. \longrightarrow (1)$$

Clearly, $z = \pm i$ are the poles of $f(z)$ and $z = i$ lies inside and $z = -i$ lies outside of the semi-circle of the contour C.

$\therefore$ By Residue theorem, we have

$$\int_C f(z)\,dz = 2\pi i \left[\operatorname{Res} f(z)\right]_{z=i}$$

$$= 2\pi i \left[\frac{1}{2i}\right] = \pi$$

Hence,

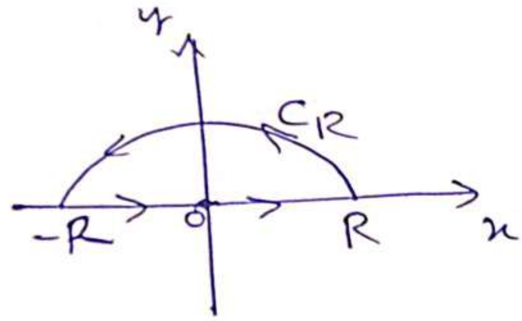
$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \frac{\pi}{2} \quad (\text{from } (1)).$$

② Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx$ using Residue theorem.

Sol: To evaluate the given integral, we consider

$$\int_C \frac{z^2}{(z^2+1)(z^2+4)} dz = \int_C f(z)\,dz,$$ where c is the closed contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R.



Here, $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$

has simple poles at

$z = \pm i$, $z = \pm 2i$. But $z = i$ and $z = 2i$ are the only poles inside the semi-circle of the contour C.

Now (i) $[\text{Res. } f(z)]_{z=i} = \lim_{z\to i} (z-i) f(z) = \frac{-1}{6i}$

and (ii) $[\text{Res. } f(z)]_{z=2i} = \lim_{z\to 2i} (z-2i) f(z) = \frac{1}{3i}$

$$\therefore \int_C f(z)\,dz = 2\pi i \left[\frac{-1}{6i} + \frac{1}{3i} \right]$$

i.e., $$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \left(\frac{1}{6i}\right) = \frac{\pi}{3}$$

Taking $R \to \infty$, we get

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{3} \quad \left(\int_{C_R} f(z)\,dz = 0 \text{ as } R \to \infty \right)$$

Hence $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\, dx = \frac{\pi}{3}$.

(3) Evaluate $\int_{0}^{\infty} \frac{1}{x^4+1}\, dx$ using Residue theorem.

Hints: Here $f(z) = \frac{1}{z^4+1}$

Now,
$$z^4+1=0 \Rightarrow z=(-1)^{1/4}$$
$$\Rightarrow z = [\cos\pi + i\sin\pi]^{1/4}$$
$$\Rightarrow z = [\cos(2n+1)\pi + i\sin(2n+1)\pi]^{1/4}$$
$$\Rightarrow z = \cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4}$$
$$\Rightarrow z = e^{i\frac{(2n+1)\pi}{4}} \quad (n=0,1,2,3)$$

Therefore, the poles of $f(z)$ are
$$z = e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}} \text{ and}$$
among these poles $z = e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}$ are only inside C.

$$\therefore \int_C f(z)\, dz = 2\pi i\left[-\frac{e^{i\frac{\pi}{4}}}{4} - \frac{e^{i\frac{3\pi}{4}}}{4}\right]$$
$$= -\frac{2\pi i}{4}\left[\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\right]$$
$$= -\frac{\pi i}{2}\left[\frac{2i}{\sqrt{2}}\right] = \frac{\pi}{\sqrt{2}}$$

i.e, $\int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz = \frac{\pi}{\sqrt{2}}$

Taking $R\to\infty$, we get $\int_{-\infty}^{\infty} f(x)\, dx = \frac{\pi}{\sqrt{2}}$.

Thus $\int_{0}^{\infty} \frac{1}{x^4+1}\, dx = \frac{\pi}{2\sqrt{2}}$.

④ Evaluate $\int_0^{\infty} \frac{1}{x^6+1}\,dx$ using Residue theorem.

<u>Answer:</u> $\frac{\pi}{3}$.

<u>Hint:</u> here poles are $z = e^{i\frac{(2n+1)\pi}{6}}$ $(n=0,1,2,3,4,5)$

3. <u>Integrals of the Type $\int_{-\infty}^{\infty} f(x)e^{imx}\,dx$:</u>

① Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx$ using Residue theorem.

Sol: Consider the function $f(z) = \frac{e^{iz}}{z^2+1}$

clearly, the poles of $f(z)$ are $z = \pm i$.

Here $z=i$ is the only pole inside of the semi circle of the contour C.

$$\therefore \int_C f(z)\,dz = 2\pi i \left(\text{Res } f(z)\right)_{z=i}$$

$$= 2\pi i \left(\frac{e^{-1}}{2i}\right) = \frac{\pi}{e}$$

Hence

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = \frac{\pi}{e}$$

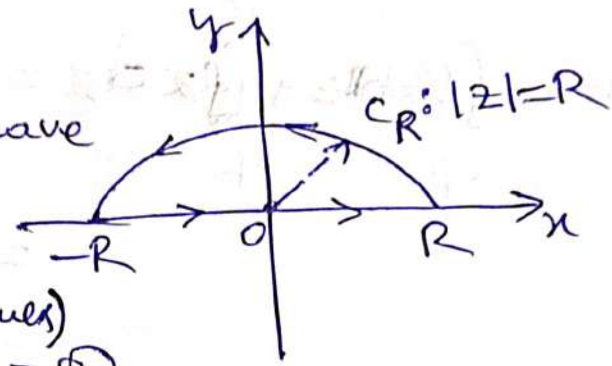
Taking $R \to \infty$, we get $\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{e}$

equating the real parts,

we get $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{e}$

② Evaluate $\displaystyle\int_0^\infty \frac{x\sin(mx)}{x^4+16}\,dx$ using Residue theorem

Sol: Consider the integral $\displaystyle\int_C f(z)\,dz$, where C is the closed contour consisting of the semi-circle C_R of radius R together with the real axis from $-R$ to R



By Residue theorem, we have

$$\int_C f(z)\,dz = 2\pi i\,(\text{Sum of the residues}) \longrightarrow (1)$$

Let $f(z) = \dfrac{z e^{imz}}{z^4+16}$.

Poles of $f(z)$ are given by $z^4+16=0$

$$\Rightarrow z = [16(-1)]^{1/4}$$

$$\Rightarrow z = 2\,(\cos\pi + i\sin\pi)^{1/4}$$

$$\Rightarrow z = 2\left[\cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4}\right]$$

for $n = 0, 1, 2, 3$.

$\therefore\ z = 2e^{i\frac{\pi}{4}} = \sqrt{2}+i\sqrt{2}$, $z = 2e^{i\frac{3\pi}{4}} = -\sqrt{2}+i\sqrt{2}$,

$z = 2e^{i\frac{5\pi}{4}} = -\sqrt{2}-i\sqrt{2}$, $z = 2e^{i\frac{7\pi}{4}}$
$= -\sqrt{2}+i\sqrt{2}$

among these poles, $z = 2e^{i\frac{\pi}{4}}$, $z = 2e^{i\frac{3\pi}{4}}$ are only inside of C.

Now, $$\left[\text{Res} f(z)\right]_{z=2e^{i\frac{\pi}{4}}} = \frac{ie^{-m\sqrt{2}}\, e^{im\sqrt{2}}}{-16}$$

and $$\left[\text{Res } f(z)\right]_{z=2e^{i\frac{3\pi}{4}}} = \frac{ie^{-m\sqrt{2}} \cdot e^{-im\sqrt{2}}}{16}$$

$\therefore$ From (1), we get

$$\int_C f(z)\,dz = 2\pi i \frac{\left[ie^{-m\sqrt{2}}\left(-2i\sin(m\sqrt{2})\right)\right]}{16}$$

i.e., $$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = \frac{i\pi}{4} e^{-m\sqrt{2}} \cdot \sin(m\sqrt{2})$$

Taking $R \to \infty$, we get

$$0 + \int_{-\infty}^{\infty} f(x)\,dx = i\frac{\pi}{4} e^{-m\sqrt{2}} \cdot \sin(m\sqrt{2})$$

equating the imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{x\sin mx}{x^4+16}\,dx = \frac{\pi}{4} e^{-m\sqrt{2}} \cdot \sin(m\sqrt{2}).$$

(3) Evaluate $\int_0^{\infty} \frac{\sin x}{x^2+1}\,dx$ using Residue theorem.

4. Integrals by Indentation:

Integration around the contours having poles on the real axis.

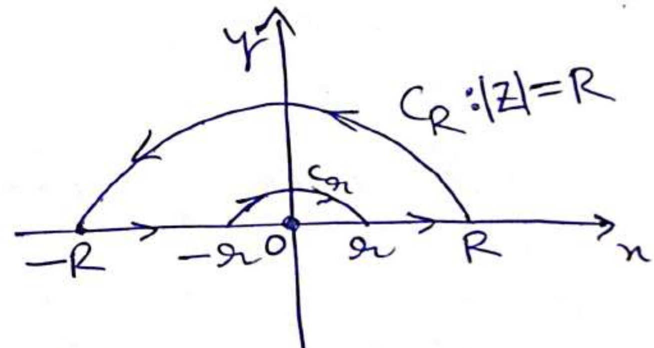
When the integrand has a simple pole on the real axis, we remove this from the region by indenting the contour (i.e., by taking a small semi-circle with its centre at the pole inside C). This procedure is called Indenting at a point.

Examples

① Evaluate $\int_0^\infty \frac{\sin x}{x}\,dx$

Sol: Consider

$$f(z) = \frac{e^{iz}}{z}.$$



Clearly $f(z)$ has a simple poles at $z=0$ which lies on the real axis. This pole will be avoided by indentation.

Draw a small semi-circle $C_r : |z| = r$ which contours the singular point $z=0$ inside a semi circle $C_R : |z| = R$.

Therefore, by Cauchy's theorem, we have

$$\int_C f(z)\,dz = 0$$

i.e., $$\int_{C_R} f(z)\,dz + \int_{-R}^{-r} f(x)\,dx + \int_{C_r} f(z)\,dz + \int_{r}^{R} f(x)\,dx = 0$$

Taking $R \to \infty$ and $r \to 0$, we get

$$0 + \int_{-\infty}^{0} f(x)\,dx + (-\pi i) + \int_{0}^{\infty} f(x)\,dx = 0$$

Let $z = re^{i\theta}$. Then

$$\int_{C_r} f(z)\,dz = \int_{\pi}^{0} \frac{e^{ire^{i\theta}}}{re^{i\theta}}\, rie^{i\theta} d\theta$$

$$= -i\int_{0}^{\pi} e^{ir(\cos\theta + i\sin\theta)}\,d\theta$$

Taking $r \to 0$, we get

$$\int_{C_r} f(z)\,dz = -i\int_{0}^{\pi} d\theta$$

$$= -\pi i$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \pi i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \pi i$$

Equating the imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$$

Since $\frac{\sin x}{x}$ is even, we have

$$2\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \pi.$$

Hence $$\boxed{\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}}$$

(2) Evaluate $\int_{-\infty}^{\infty} \frac{\cos 2x}{x} \, dx$ <u>Answer:</u> 0

(3) Evaluate $\int_{-\infty}^{\infty} \frac{\sin 2x}{x} \, dx$ <u>Answer:</u> π

Fall Sem. 2023-24

Tutorial– 3

Programme Name & Branch: B. Tech.

Course Name & Code: Complex Variables and Linear Algebra (BMAT201L)

Practice all the questions:

1. Obtain the Taylor's series expansion of the function $f(z)=\dfrac{1}{z^2-z-6}$ about $z=1$.

   Answer: $f(z)=\dfrac{-1}{6}+\dfrac{1}{5}\left(\dfrac{1}{2^2}-\dfrac{1}{3^2}\right)(z-1)-\left(\dfrac{1}{2^3}+\dfrac{1}{3^3}\right)(z-1)^2+\left(\dfrac{1}{2^4}-\dfrac{1}{3^4}\right)(z-1)^3+\ldots$

2. Obtain the Laurent series expansion of the function $f(z)=\dfrac{z}{z^2-4z+3}$ in the region $1<|z|<3$.

   Answer: $f(z)=\dfrac{-1}{6}\left(1+\dfrac{z}{3}+\dfrac{z^2}{3^2}+\ldots\right)-\dfrac{1}{z}\left(1+\dfrac{1}{z}+\dfrac{1}{z^2}+\ldots\right)$

3. Obtain the Laurent series of the function $f(z)=\dfrac{z}{(z-1)(z-3)}$ about $z=3$.

4. Obtain the Laurent series of the function $f(z)=\dfrac{z^2-1}{z^2+5z+6}$ in the region $2<|z|<3$.

5. Discuss the singularities of (i) $f(z)=\dfrac{1-\cos z}{z}$ (ii) $f(z)=e^{1/z}$

6. Find the poles and the corresponding residues at each pole of the function $f(z)=\dfrac{z^2+2z}{(z+1)^2(z^2+4)}$.

   Answer: Here $z=-1$ is a pole of order two and $z=-2i,\ z=2i$ are simple poles. And the corresponding Residues are $\dfrac{2}{25},\ \dfrac{1+7i}{2},\ \dfrac{1-7i}{2}$ respectively.

7. Find the poles and the corresponding residues of the function $f(z)=\dfrac{z^2+1}{(z-2)(z+1)^3}$

8. Evaluate $\displaystyle\int_C \dfrac{e^{2z}}{(z-1)^2(z-2)}\,dz$ where C is $|z|=\dfrac{1}{2}$.

9. Evaluate $\int_C \frac{z^3 e^{-z}}{(z-1)^2}\,dz$ where C is $|z-1| = \frac{1}{2}$.

Answer: By Cauchy's integral formula, $\int_C \frac{z^3 e^{-z}}{(z-1)^2}\,dz = \frac{\pi i}{e}$

10. Evaluate $\int_C \frac{z^2-1}{z^2+1}\,dz$ where C is $|z-i| = 1$.

Answer: By Cauchy's integral formula, $\int_C \frac{z^2-1}{z^2+1}\,dz = -2\pi.$

11. Evaluate $\int_c \frac{z^2+2z}{(z+1)^2(z^2+4)}\,dz$ where C is $|z| = \frac{3}{2}$ using Residue theorem.

Answer: Here $z = -1$ is a pole of order two which is inside of C and $z = -2i,\ z = 2i$ are simple poles which are outside of C. Therefore $\int_c \frac{z^2+2z}{(z+1)^2(z^2+4)}\,dz = 2\pi i(\mathrm{Res.} f(z))_{z=-1} = 2\pi i\left(\frac{2}{25}\right)$

12. Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$ where C is $|z| = \frac{3}{2}$ using Residue theorem.

Answer: $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = 2\pi i$.

Module 7: Matrices and System of Equations

7.1 Eigen values and eigen vectors

Def: Let A be a square matrix of order n. Let X be a non-zero ~~var~~ ($n \times 1$) vector such that $AX = \lambda X$, where λ is a real or complex scalar. Then λ is called an eigen value of A corresponding to X and X is called an eigen vector of A corresponding to λ.

Note: ① Eigen value is also called as characteristic value or characteristic root.
② Eigen vector is also called as characteristic vector.

Characteristic equation of a square matrix:

Let A be a square matrix of order n. Then $|A - \lambda I| = 0$, where I is an identity matrix of order n, is called the characteristic equation of A.

Note: ① $|A - \lambda I|$ is called the characteristic polynomial of A.
② The roots of the characteristic equation

of A are the eigen values of A.

<u>Properties of the eigenvalues:</u>

Let A be a square matrix of order n. Then

1. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of A,
   then (i) $\lambda_1 + \lambda_2 + \cdots + \lambda_n =$ trace of A
   (ii) $\lambda_1 \cdot \lambda_2 \cdots \lambda_n = |A|$
   (iii) $\lambda_1, \lambda_2, \ldots, \lambda_n$ are also the eigen values of A^T
   (iv) For any positive integer k, $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ are the eigenvalues of A^k.
2. If A is non-singular, then all the eigenvalues of A are non-zero.
3. If A is non-singular and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of A, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ are the eigen values of A^{-1}.
4. If A is a diagonal or a triangular matrix, then the principal diagonal elements of A are the eigen values of A.
5. If λ is an eigen value of A and K is constant, then $K\lambda$ is an eigenvalue of KA.
6. All the eigenvalues of a real symmetric matrix are real.

7. The eigen values of an orthogonal matrix are of unit modulus.

Examples

1. Find the eigen values and the corresponding eigen values of (i) $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$

Sol: (i) Given $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.

The characteristic equation of A is

$|A - \lambda I| = 0$ i.e., $\begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (4-\lambda)(2-\lambda) - 3 = 0$

$\Rightarrow \lambda^2 - 6\lambda + 5 = 0$

$\Rightarrow (\lambda - 1)(\lambda - 5) = 0$

$\Rightarrow \lambda = 1, 5$ are the eigen values of A.

Now,

(i) The eigen vectors corresponding to $\lambda = 1$ are given by $(A - \lambda I)X = 0$

i.e., $\begin{bmatrix} 4-1 & 3 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or, $\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \xrightarrow{3R_2 - R_1} \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

which gives $x + y = 0$

let $y = k$, we get $x = -k, y = k$

$\therefore X = K\begin{bmatrix}-1\\1\end{bmatrix}.$

(ii) The eigen vectors corresponding to $\lambda = 5$ are given by $(A-\lambda I)X = 0$

i.e., $\begin{bmatrix}4-5 & 3\\ 1 & 2-5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

or, $\begin{bmatrix}-1 & 3\\ 1 & -3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\xrightarrow{R_2+R_1} \begin{bmatrix}-1 & 3\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

which gives $x - 3y = 0$

let $y = K$, an arbitrary constant, then we get $x = 3K$, $y = K$.

$\therefore X = K\begin{bmatrix}3\\1\end{bmatrix}$

(ii) The characteristic equation of A is

$|A - \lambda I| = 0$ i.e., $\begin{vmatrix}1-\lambda & 0 & 2\\ 0 & 2-\lambda & 3\\ 0 & 0 & 5-\lambda\end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(2-\lambda)(5-\lambda) = 0$

$\Rightarrow \lambda = 1, 2, 5$ are the eigen values of A.

(i) The eigen vectors corresponding to $\lambda = 1$ are given by $(A - \lambda I)X = 0$

i.e., $\begin{bmatrix} 1-1 & 0 & 2 \\ 0 & 2-1 & 3 \\ 0 & 0 & 5-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or, $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{R_3-2R_1} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

which gives

$z=0$ and $y+3z=0$

So, $y=0$ and hence $0.x=0$

$\therefore X=\begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix}=K\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(ii) The eigen vectors corresponding to $\lambda=2$ are given by $(A-\lambda I)X=0$

i.e., $\begin{bmatrix} 1-2 & 0 & 2 \\ 0 & 2-2 & 3 \\ 0 & 0 & 5-2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or, $\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\xrightarrow{R_3-R_2} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

which gives $x-2z=0$ and $3z=0$

$\Rightarrow x=0$

Now, $-1.x+0.y+2.z=0$

$\Rightarrow 0.y=0$

$\therefore X=\begin{bmatrix} 0 \\ K \\ 0 \end{bmatrix}=K\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(iii) The eigen vectors corresponding to $\lambda = 5$ are given by $(A - \lambda I)X = 0$

i.e., $$\begin{bmatrix} 1-5 & 0 & 2 \\ 0 & 2-5 & 3 \\ 0 & 0 & 5-5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, $$\begin{bmatrix} -4 & 0 & 2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives $2x - z = 0$

and $y - z = 0$

Let $z = k$, an arbitrary constant, then

we get $x = \frac{k}{2}$, $y = k$ and $z = k$

$$\therefore X = k \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

(2) Find the eigen values and the corresponding eigen vectors of (i) $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$\boxed{\lambda = 2, 2, 8}$

(ii) $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$

$\boxed{\lambda = -1, 1, 4}$

③ Find the eigenvalues and the corresponding eigenvectors of (i) $A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

④ If 3 and 15 are two of the eigenvalues of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, find $|A|$ without expanding determinant.

<u>Note</u>: If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all distinct eigen values of a matrix A of order n, then the corresponding eigenvectors $X_1, X_2, \ldots, X_n$ are linearly independent.

<u>Cayley-Hamilton Theorem:</u>

Every square matrix satisfies its characteristic equation.

① verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and hence find A^{-1} if exists. Also find $A^5 - 4A^4 + 3A^3 - 2A^2 + A + I$.

Sol: The characteristic equation of A is $|A-\lambda I|=0$

i.e., $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2-1=0$

$\Rightarrow \boxed{\lambda^2-4\lambda+3=0}$

$\Rightarrow (\lambda-1)(\lambda-3)=0$

$\Rightarrow \lambda = 1, 3$

$\therefore \lambda^2-4\lambda+3=0$
is the characteristic equation of A.

Verification:: To show $A^2-4A+3I=0$

Now, $A^2-4A+3I = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - 4\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} - \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Hence Cayley-Hamilton theorem is verified.

Since $|A| = 3 \neq 0$, A^{-1} exists.

we have $A^2-4A+3I=0$

$\Rightarrow A^{-1}(A^2-4A+3I)=0$

$\Rightarrow A-4I+3A^{-1}=0$

$\Rightarrow A^{-1} = \frac{1}{3}(4I-A) = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

To find $A^5 - 4A^4 + 3A^3 - 2A^2 + A + I$.

$$A^5 - 4A^4 + 3A^3 - 2A^2 + A + I$$

$$= A^3(A^2 - 4A + 3I) - 2A^2 + A + I$$

$$= A^3(0) - 2A^2 + A + I$$

$$= -2A^2 + A + I$$

$$= -2\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -7 \\ -7 & -7 \end{bmatrix}$$

(2) verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ and hence find A^{-1} if exists.

7.2 System of Equations

Def: Consider the following system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

The above system of equations can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

i.e., $AX = B$, where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

We can solve the above system of equations using the following methods.

(i) Gaussian elimination method,

and (ii) Gauss Jordan method.

Examples:

① Solve $2x-y+z=1$
$x+2y-z=4$
$3x+y+2z=7$

using (i) Gauss elimination method,

and (ii) Gauss Jordan method.

Sol: The given system can be written as

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

i.e., $AX = B$, where $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & 1 & 2 \end{bmatrix}$,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

Augmented matrix is $[A \; B]$

is
$$\left[\begin{array}{ccc:c} 2 & -1 & 1 & 1 \\ 1 & 2 & -1 & 4 \\ 3 & 1 & 2 & 7 \end{array}\right]$$

(i) Gauss elimination method

$$\left[\begin{array}{ccc:c} 2 & -1 & 1 & 1 \\ 1 & 2 & -1 & 4 \\ 3 & 1 & 2 & 7 \end{array}\right] \xrightarrow[2R_3-3R_1]{2R_2-R_1} \left[\begin{array}{ccc:c} 2 & -1 & 1 & 1 \\ 0 & 5 & -3 & 7 \\ 0 & 5 & 1 & 11 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \begin{bmatrix} 2 & -1 & 1 & 1 \\ 0 & 5 & -3 & 7 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3\left(\frac{1}{4}\right)} \left[\begin{array}{ccc:c} 2 & -1 & 1 & 1 \\ 0 & 5 & -3 & 7 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, we get

$$2x - y + z = 1$$
$$5y - 3z = 7$$
$$z = 1$$

and hence $x = 1$, $y = 2$ and $z = 1$

(ii) Gauss Jordan method

$$\left[\begin{array}{ccc:c} 2 & -1 & 1 & 1 \\ 1 & 2 & -1 & 4 \\ 3 & 1 & 2 & 7 \end{array}\right] \xrightarrow[2R_3 - 3R_1]{2R_2 - R_1} \left[\begin{array}{ccc:c} 2 & -1 & 1 & 1 \\ 0 & 5 & -3 & 7 \\ 0 & 5 & 1 & 11 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc:c} 2 & -1 & 1 & 1 \\ 0 & 5 & -3 & 7 \\ 0 & 0 & 4 & 4 \end{array}\right]$$

$$\xrightarrow{R_3\left(\frac{1}{4}\right)} \left[\begin{array}{ccc:c} 2 & -1 & 1 & 1 \\ 0 & 5 & -3 & 7 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 - R_3]{R_2 + 3R_3} \left[\begin{array}{ccc:c} 2 & -1 & 0 & 0 \\ 0 & 5 & 0 & 10 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{5R_1 + R_2} \left[\begin{array}{ccc:c} 10 & 0 & 0 & 10 \\ 0 & 5 & 0 & 10 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2\left(\frac{1}{5}\right)]{R_1\left(\frac{1}{10}\right)} \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, we get

$x = 1$, $y = 2$ and $z = 1$.

(2) Solve

$$x + y + 2z - w = 3$$
$$2x - 3y + z + w = 1$$
$$3x + 2y - z + 2w = 6$$
$$x + 3y - 3z - 4w = -3$$

using (i) Gauss elimination and (ii) Gauss Jordan methods.

Answer: $x = 1$, $y = 1$, $z = 1$ and $w = 1$

(3) Solve

$$2x - 3y + 5z = 9$$
$$x + 2y - z = 1$$
$$3x - y + 2z = 6$$

using (i) Gauss elimination and (ii) Gauss Jordan methods.

Answer: $x = 1$, $y = 1$ and $z = 2$.

Fall Sem. 2023-24

Tutorial- 4

Programme Name & Branch: B. Tech.

Course Name & Code: Complex Variables and Linear Algebra (BMAT201L)

Practice all the questions

1. Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$ using Residue theorem. Answer: $\frac{\pi}{6}$

2. Evaluate $\int_0^{\infty} \frac{2x^2-1}{x^4+5x^2+4}\, dx$ using Residue theorem. Answer: $\frac{\pi}{4}$

3. Evaluate $\int_0^{\infty} \frac{x\sin x}{1+x^2}\, dx$ using Residue theorem. Answer: $\frac{\pi i}{2e}$

4. Evaluate $\int_{-\infty}^{\infty} \frac{\cos 2x}{x}\, dx$. Answer: 0

5. Find the eigenvalues and the corresponding eigenvectors of $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

   Answer: eigenvalues are $\lambda = 0, 3, 15$ and the corresponding eigenvectors are

   $X_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \; X_2 = \begin{pmatrix} -4 \\ -2 \\ 4 \end{pmatrix}, \; X_3 = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}.$

6. Verify Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ and hence find A^4 .

7. Solve $x+2y-5z=-2,\;\; 2x-y+3z=4,\;\; x+3y-z=3$ using Gauss elimination and Gauss Jordan methods. Answer: x=1, y=1, z=1

Module 4: Vector spaces

4.1

Def. of a Group: Let G be a non-empty set together with one binary operation '$\circ$'. Then $(G, \circ)$ is called a group if it satisfies the following:

(i) Associativity: $a \circ (b \circ c) = (a \circ b) \circ c$ for $a, b, c \in G$,

(ii) Existence of identity: For all $a \in G$, there exists an element $e \in G$ such that $a \circ e = e \circ a = a$.
Here e is called an identity element of G with respect to '$\circ$'.

(iii) Existence of inverse: For each $a \in G$, there exists an element $b \in G$ such that $a \circ b = b \circ a = e$.
Here b is called an inverse of a in G with respect to '$\circ$'.

Note: A mapping $\circ : G \times G \to G$ is called a binary operation on G.

E.g. (i) $(\mathbb{N}, +)$ is not a group.

for, $0 \notin \mathbb{N}$ (0 is an additive identity.)

(ii) $(\mathbb{Z}, +)$ is a group.

(iii) $(\mathbb{Z}, \cdot)$ is not a group.

for, $2 \in \mathbb{Z}$ but $\frac{1}{2} \notin \mathbb{Z}$

i.e., multiplicative inverse of 2 is not in $\mathbb{Z}$..

(iv) $(\mathbb{R}, +)$ is a group.

(v) $(\mathbb{R}, \cdot)$ is not a group.

for, $0 \in \mathbb{R}$ has no multiplicative inverse.

(vi) $(\mathbb{R}^*, \cdot)$ is a group, where $\mathbb{R}^* = \mathbb{R} - \{0\}$.

Abelian Group: A group $(G, \circ)$ is said to be abelian if $a \circ b = b \circ a$ for all $a, b \in G$.

e.g. (i) $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$, $(\mathbb{Q}, +)$ are all abelian groups.

(ii) $(\mathbb{R}^*, \cdot)$ is an abelian group.

Def. of a Ring: Let R be a non-empty set togther with two binary operations '$+$' and '$\cdot$'. Then $(R,+,\cdot)$ is called a ring if

(i) $(R,+)$ is an abelian group

(ii) $(R,\cdot)$ is a semi group

(i.e., $a\cdot(b\cdot c)=(a\cdot b)\cdot c$ for all $a,b,c\in R$)

(iii) Distributive laws hold good in R.

i.e., $a\cdot(b+c)=a\cdot b+a\cdot c$

and $(a+b)\cdot c=a\cdot c+b\cdot c$

for all $a,b,c\in R$.

e.g. (i) $(\mathbb{Z},+,\cdot)$ is a ring

(ii) $(\mathbb{R},+,\cdot)$ is a ring

(iii) $(\mathbb{Q},+,\cdot)$, $(\mathbb{C},+,\cdot)$ are all rings.

Definition of a Field: Let F be a non-empty set with two binary operations '$+$' and '$\cdot$'. Then $(F,+,\cdot)$ is called a field if

(i) $(F,+)$ is an abelian group

(ii) $(F^*,\cdot)$ is an abelian group ($F^*=F-\{0\}$)

(iii) Distributive laws are hold good in F

i.e., $a\cdot(b+c)=a\cdot b+a\cdot c$
and $(a+b)\cdot c=a\cdot c+b\cdot c$ for all $a,b,c\in F$.

e.g. (i) $(\mathbb{Z},+,\cdot)$ is not a field.
(ii) $(\mathbb{Q},+,\cdot)$ is a field
(iii) $(\mathbb{R},+,\cdot)$ and $(\mathbb{C},+,\cdot)$ are fields.

Vector Space: Let V be a non-empty set whose elements are called vectors and F be a field whose elements are called scalars. Then V (i.e., $(V,+,\cdot)$) is said to be a vector space over F if

(i) $(V,+)$ is an abelian group.
(ii) $a\in F,\ \alpha\in V \Rightarrow a\cdot\alpha\in V$
(iii) $a\cdot(\alpha+\beta)=a\cdot\alpha+a\cdot\beta$ for $a\in F$ and $\alpha,\beta\in V$
(iv) $(a+b)\alpha=a\cdot\alpha+b\cdot\alpha$ for $a,b\in F$ and $\alpha\in V$
(v) $a\cdot(b\alpha)=(a\cdot b)\alpha$ for $a,b\in F$ and $\alpha\in V$

and (vi) $1\cdot\alpha=\alpha$ for $1\in F$ and $\alpha\in V$.

Note:
If $(V,+,\cdot)$ is a vector space over a field F, then we say $V(F)$ is a vector space (or, a linear space)

e.g. (i) $\mathbb{R}$ is a vector space over a field $(\mathbb{Q}, +, \cdot)$
i.e., $\mathbb{R}(\mathbb{Q})$ is a vector space

(ii) $\mathbb{C}(\mathbb{R})$ is a vector space

(iii) $\mathbb{R}(\mathbb{C})$ is not a vector space.
for, $1 \in \mathbb{R}$ and $i \in \mathbb{C} \Rightarrow 1 \cdot i = i \notin \mathbb{R}$

(iv) $\mathbb{R}^n(\mathbb{R})$ is a vector space $(n = 1, 2, 3, \cdots)$

(v) $(M_{2\times 2}, +, \cdot)$ is a vector space over a field $(\mathbb{R}, +, \cdot)$ (here $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$)

Subspace: Let $V(F)$ be a vector space. A non-empty subset W of V is said to be a subspace of V over a field F if W is itself a vector space over F under the same operations as defined on V.

Example ① clearly
$W = \{ (a, b, c) \mid a, b, c \in \mathbb{R} \}$ is a subspace of a vector space $\mathbb{R}^3(\mathbb{R})$.

② $W = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$ is a subspace of a vector space $M_{2\times 2}(\mathbb{R})$.

* The necessary and sufficient condition for a non-empty subset W of a vector space $V(F)$ to be a subspace of V is

$$a, b \in F \text{ and } \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W.$$

Note: (1) The intersection of any two subspaces of a vector space is a subspace.

(2) The intersection of an arbitrary collection of subspaces of a vector space is also a subspace

(3) The union of two subspace of a vector space is not necessarily a subspace.

(4) The union of two subspaces W_1 and W_2 of a vector space $V(F)$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Def.: Let W_1 and W_2 be two subspaces of a vector space $V(F)$, then $W_1 + W_2 = \{\alpha + \beta \mid \alpha \in W_1, \beta \in W_2\}$ is called the linear sum of W_1 and W_2.

Note: $W_1 + W_2$ is a subspace of $V(F)$.

Def.: Let W_1 and W_2 be two subspaces of a vector space $V(F)$. Then V is said to be a direct sum of W_1 and W_2 if each element of V can be uniquely expressed as ~~a~~ the sum of an an element of W_1 and an element of W_1.

If V is a direct sum of W_1 and W_2, then we write $V = W_1 \oplus W_2$

* The necessary and sufficient condition for a vector space V to be the direct sum of its subspaces W_1 and W_2 are

(i.) $V = W_1 + W_2$ (ii) $W_1 \cap W_2 = \{0\}$.

Linear combination of vectors

Def.: Let $V(F)$ be a vector space. An element $\alpha \in V$ is said to be a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$ if there exists $a_1, a_2, \ldots, a_n \in F$ such that $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$.

Example: ① The vector $\alpha = (-7, -6)$ is a linear combination of the vectors $\alpha_1 = (-2, 3)$ and $\alpha_2 = (1, 4)$ ([illegible])

since $(-7, -6) = 2(-2, 3) - 3(1, 4)$

i.e., $\alpha = 2\alpha_1 + (-3)\alpha_2$

Linear Span: Let $V(F)$ be a vector space and S be any non-empty subset of V. Then the linear span of S (or, simply span of S) is denoted by $L(S)$ (or, span(S)) and defined as the set of all finite linear combinations of elements of S.

i.e., $L(S) = \left\{ \sum_{i=1}^{n} a_i \alpha_i \;\middle|\; a_i \in F,\ \alpha_i \in S \text{ for } i=1,2,\dots,n \right\}$.

Clearly $S \subseteq L(S)$.

Note: ① span(S) is the smallest subspace of V that contains S.

Examples: ① $L(\{(1,0,0),(0,1,0),(0,0,1)\}) = \mathbb{R}^3(\mathbb{R})$.

for, for $(x,y,z) \in \mathbb{R}^3$, $(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$.

② $L\left(\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}\right) = M_{2\times 2}(\mathbb{R})$.

for, for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

③ Show that the set $S=\{(2,5),(1,3)\}$ is a spanning set for $\mathbb{R}^2(\mathbb{R})$

sol: Let $(a,b)\in\mathbb{R}^2$. Suppose $(a,b)=r(2,5)+s(1,3)$

Then $2r+s=a$
$5r+3s=b$

i.e., $\begin{bmatrix}2&1\\5&3\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$

$\xrightarrow{2R_2-5R_1}\begin{bmatrix}2&1\\0&1\end{bmatrix}\begin{bmatrix}r\\s\end{bmatrix}=\begin{bmatrix}a\\2b-5a\end{bmatrix}$

which gives $2r+s=a$
and $s=2b-5a$

$\Rightarrow r=\dfrac{6a-2b}{2}\Rightarrow r=3a-b$

$\therefore (a,b)=(3a-b)(2,5)+(2b-5a)(1,3)$.

and hence every element of $\mathbb{R}^2$ can be expressed as a linear combination of elements of S. Thus $L(S)=\mathbb{R}^2$

Linearly dependent and independent set of vectors.

Def: Let $V(F)$ be a vector space over a field F. Then a finite set $\{\alpha_1,\alpha_2,\dots,\alpha_n\}$ of vectors of V is said to be (i) linearly dependent if there exists scalars $a_1,a_2,\dots,a_n\in F$

not all are zero such that

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0$$

(ii) linearly independent if $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0$ where $a_1, a_2, \cdots a_n \in F$ implies $a_1 = 0, a_2 = 0, \cdots, a_n = 0$

Example: ① The set of vectors $(2,1,4), (1,-1,2), (3,1,-2)$ i.e., $\{(2,1,4),(1,-1,2),(3,1,-2)\}$ is a linearly independent subset of $\mathbb{R}^3(\mathbb{R})$.

Suppose $a, b, c \in \mathbb{R}$ such that

$$a(2,1,4) + b(1,-1,2) + c(3,1,-2) = 0$$

$$\Rightarrow 2a + b + 3c = 0 \; ; \; a - b + c = 0 \; ; \; 4a + 2b - 2c = 0$$

i.e., $\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\xrightarrow[R_3 - 2R_1]{R_2 - 2R_1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -4 & -1 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Therefore, we get $a = 0, b = 0, c = 0$

Hence the vectors $(2,1,4), (1,-1,2), (3,1,-2)$ are linearly independent.

② Show that the set $S = \{(1,2,1), (3,1,5), (3,-4,7)\}$ is ~~a~~ linearly dependent subset of $\mathbb{R}^3(\mathbb{R})$.

Note: If every finite subset of an infinite set S of vectors is linearly independent, then S is linearly independent.

(2) If $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a linearly independent subset of a vector space $V(F)$, then every element $\alpha \in L(S)$ has a unique representation in the form $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$, where $a_1, a_2, \ldots, a_n \in F$.

Basis of a Vector Space:

Definition: Let $V(F)$ be a vector space and S any an non-empty subset of V. Then S is said to be basis of V if (i) S is Linearly independent
(ii) $L(S) = V$

Examples: (1) $S = \{(1,0,0)\ (0,1,0), (0,0,1)\}$ is linearly independent and $L(S) = \mathbb{R}^3(\mathbb{R})$.

$\therefore$ S is a basis of $\mathbb{R}^3(\mathbb{R})$.

This basis is called **the standard** basis of $\mathbb{R}^3(\mathbb{R})$.

(2) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

is the standard basis of $M_{2\times 2}(\mathbb{R})$

③ verify the vectors $(1,2,1)$, $(3,1,5)$, $(3,-4,7)$ form a basis of $\mathbb{R}^3(\mathbb{R})$ or not.

<u>Sol:</u> Let $a, b, c \in \mathbb{R}$ such that

$$a(1,2,1)+b(3,1,5)+c(3,-4,7)=0$$

Then
$$a+3b+3c=0$$
$$2a+b-4c=0$$
$$a+5b+7c=0$$

i.e.,
$$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_3-R_1]{R_2-2R_1}\begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & -10 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{5R_3+2R_2}\begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & -10 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, a, b, c are all not zero ~~and hence~~.

Hence, the vectors $(1,2,1)$, $(3,1,5)$, $(3,-4,7)$ are linarly dependent.

Thus, the vectors $(1,2,1)$, $(3,1,5)$, $(3,-4,7)$ do not form a basis of $\mathbb{R}^3(\mathbb{R})$.

<u>Dimension of a vector space:</u>

Let $V(F)$ be a vector space. The the number of elements of a basis of V is called the dimension of V over F and we write $\dim_F(V)$ or simply $\dim(V)$.

4.2 Finite dimensional vector space

Def: Let V be a vector space over a field F. Then we say that V is finite dimensional if it is spanned by a finite set of vectors.

Def: The dimension of a finite dimensional vector space V is the number of elements in a basis of V.

Note: ① For convenience, we will define the dimension of the trivial vector space $\{0\}$ to be zero, even though $\{0\}$ doesn't have a basis.

② Every basis of an n-dimensional vector space contains exactly n vectors.

③ Let $V(F)$ be an n-dimensional vector space and $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a subset of V. Then

(i) If S is linearly independent, then S is a basis of V.

(ii) If $L(S) = V$, then S is a basis of V.

Examples: ① The dimension of a vector space $\mathbb{R}(\mathbb{R})$ is one.

(2) The dimension of a vector space $\mathbb{C}(\mathbb{R})$ is two.

(3) For any positive integer n, let P_n denote the vector space of polynomials with real co-efficients of degree at most n. Then the set $S = \{1, x, x^2, x^3\}$ is a basis for P_3. Hence the dimension of P_3 is 4.

Note: (1) Let $V(F)$ be a vector space. The set $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of vectors in V form a basis of V if and only if every vector α in V can be expressed uniquely as a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

(2) Let $V(F)$ be an n-dimensional vector space. Then any subset of V contains more than n elements is linearly dependent.

(3) The set of vectors consists zero vector is linearly dependent.

(4) If $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$, then the set $\{\alpha, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ is Linearly dependent.

Problems:

① Find a basis and dimension of the subspace

$$W = \{(x, y, z, w) \in \mathbb{R}^4 \mid x - y + z - w = 0\}$$ of

a vector space $\mathbb{R}^4(\mathbb{R})$.

Sol: Any vector $\alpha = (x, y, z, w)$ in W satisfies $x - y + z - w = 0$, or equivalently

$$x = y - z + w.$$

Therefore, we have

$$(x, y, z, w) = (y - z + w, y, z, w)$$

$$= y(1, 1, 0, 0) + z(-1, 0, 1, 0) + w(1, 0, 0, 1)$$

and hence; the set $S = \{(1, 1, 0, 0), (-1, 0, 1, 0), (1, 0, 0, 1)\}$

is a spanning set for the subspace W.

i.e., $L(S) = W$.

Suppose $a, b, c \in \mathbb{R}$ such that

$$a(1, 1, 0, 0) + b(-1, 0, 1, 0) + c(1, 0, 0, 1) = 0$$

then $a - b + c = 0$; $a = 0$, $b = 0$, $c = 0$

$$\Rightarrow a = 0, b = 0, c = 0.$$

Therefore, $S = \{(1,1,0,0), (-1,0,1,0), (1,0,0,1)\}$ is linearly independent.

Hence $S = \{(1,1,0,0), (-1,0,1,0), (1,0,0,1)\}$ is a basis for the subspace W.

Thus $\dim W = 3$.

(2) Find the basis and dimension of the subspace $W = \{(x, y, z) \mid x - y = 0\}$ of a vector space $\mathbb{R}^3(\mathbb{R})$.

<u>Answer:</u> Basis is $\{(1,1,0), (0,0,1)\}$ and dimension 2.

(3) Show that the set $S = \{(2,1,4), (1,-1,2), (3,1,-2)\}$ form a basis for a vector space $\mathbb{R}^3(\mathbb{R})$.

(4) Is the set $S = \{(1,2,1), (2,1,0), (0,0,1)\}$ form a basis for $\mathbb{R}^3(\mathbb{R})$.

$\boxed{4.3}$ Row Space, Column Space and Nullspace.

<u>Terminology:</u> Let A be the 2×3 matrix $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}$.

Then (i) The row vectors of A are ~~(1,2,-1),(0,1,3)~~
$[1 \;\; 2 \;\; -1]$ and $[0 \;\; 1 \;\; 3]$ in $\mathbb{R}^3$

(ii) The column vectors of A are
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ in $\mathbb{R}^2$.

<u>Definition:</u> Let A be a $m\times n$ matrix. Then

(i) The row space of A is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A.

(ii) The column space of A is the subspace of $\mathbb{R}^m$ spanned by the column vectors of A.

<u>Note:</u> If a matrix A is row-equivalent to a matrix B in row-echelon form, then the non-zero row vectors of B form a basis for the row space of A.

<u>Example:</u> Finding a basis for Row space.

(1) Let $A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$, then find a basis for row space of A.

Sol: $A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} \xrightarrow[R_5 - 2R_1]{R_4 - R_1} \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & -4 & -1 & 0 \\ 0 & -1 & -2 & -2 & -3 \end{bmatrix}$

$$\xrightarrow[R_5 + R_2]{R_4 + 2R_2} \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$$

$$\xrightarrow[R_5 + R_3]{R_4 - R_3} \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(Echelon form of A)

Therefore $\alpha_1 = (1, 1, 4, 1, 2)$, $\alpha_2 = (0, 1, 2, 1, 1)$, $\alpha_3 = (0, 0, 0, 1, 2)$ form a basis for the row space of A. Hence the dimension of the row space (of A) is 3.

<u>Example</u>: Finding a basis for the column space of A.

<u>Note</u>: A basis for the row space of A^T is the same as finding a basis for the column space of A.

① Let $A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$, then find a basis for the column space of A.

Sol: $A^T = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & -1 & 1 \\ 4 & 2 & 0 & 0 & 6 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 2 & 1 \end{bmatrix}$

$$\xrightarrow[R_4 - R_1;\ R_5 - 2R_1]{R_2 - R_1;\ R_3 - 4R_1} \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 2 & 0 & -4 & -2 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 1 & 2 & 0 & -3 \end{bmatrix}$$

$$\xrightarrow[R_4 - R_2;\ R_5 - R_2]{R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 2 & -2 \end{bmatrix}$$

$$\xrightarrow{R_{3,4}} \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & -2 \end{bmatrix}$$

$$\xrightarrow{R_5 - 2R_3} \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(Echelon form of A^T)

Therefore, $\alpha_1=(1,0,0,1,2)$, $\alpha_2=(0,1,0,-2,-1)$, $\alpha_3=(0,0,1,1,1)$ form a basis for the column space of A. Hence, the dimension of the column space is 3.

Note: If A is an $m\times n$ matrix, then the row space and the column space of A have the same dimension.

Definition: The dimension of the row (or, column) space of a matrix A is called the rank of A. It is denoted by $\text{rank}(A)$ (or, $\rho(A)$).

Definition: If A is a $m\times n$ matrix, then the set of all solutions of the homogeneous system of linear equations $Ax=0$ is a subspace of $\mathbb{R}^n$ called the nullspace of A and denoted by $N(A)$.

i.e, $N(A)=\{x\in\mathbb{R}^n \mid Ax=0\}$.

The dimension $N(A)$ is called the nullity of A.

Example: Finding a basis for the nullspace of A.

① let $A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$, then find a basis and dimension for $N(A)$.

Sol: we need to solve the system $Ax = 0$

i.e., $\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\xrightarrow[R_5 - 2R_1]{R_4 - R_1} \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & -4 & -1 & 0 \\ 0 & -1 & -2 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_5 + R_2]{R_4 + 2R_1} \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_5 + R_3]{R_4 - R_3} \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_1-R_2}\begin{bmatrix}1&0&2&0&1\\0&1&2&1&1\\0&0&0&1&2\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

$$\xrightarrow{R_2-R_3}\begin{bmatrix}1&0&2&0&1\\0&1&2&0&-1\\0&0&0&1&2\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

Therefore, we get

$$x_1+2x_3+x_5=0$$
$$x_2+2x_3-x_5=0$$
$$x_4+2x_5=0.$$

Let $x_5=K_1$ and $x_3=K_2$, then

$$x_1=-K_1-2K_2$$
$$x_2=K_1-2K_2$$
$$x_3=K_2$$
$$x_4=-2K_1$$
$$x_5=K_1$$

$$\therefore\ x=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}-K_1-2K_2\\K_1-2K_2\\K_2\\-2K_1\\K_1\end{bmatrix}=K_1\begin{bmatrix}-1\\1\\0\\-2\\1\end{bmatrix}+K_2\begin{bmatrix}-2\\-2\\1\\0\\0\end{bmatrix}$$

Hence, the set $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ form a basis for $N(A)$.

Thus $\dim(N(A)) = 2$. i.e., $\text{nullity}(A) = 2$.

<u>Note:</u> If A is a $m \times n$ matrix of rank r, then the dimension of the solution space of $Ax = 0$ is $n - r$.

i.e., $\dim(N(A)) = n - \text{rank}(A)$

or, $\boxed{\text{rank}(A) + \text{nullity}(A) = n}$

(number of ~~rows~~ columns of A.)

<u>Problems</u>

① Let $A = \begin{bmatrix} 2 & 4 & -3 & -6 \\ 7 & 14 & -6 & -3 \\ -2 & -4 & 1 & -2 \\ 2 & 4 & -2 & -2 \end{bmatrix}$.

Find (i) basis for row space of A

(ii) basis for column space of A

(iii) basis for nullspace of A

(iv) $\text{rank}(A)$ (v) $\text{nullity}(A)$.

<u>Answer:</u> (i) $\{(1, 2, -1, -1), (0, 0, 1, 4)\}$

(ii) $\{(2, 7, -2, 2), (-3, -6, 1, -2)\}$

(iii) $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$ (iv) 2 (v) 2.

Fall Sem. 2023-24

Tutorial- 5

Programme Name & Branch: B. Tech.

Course Name & Code: Complex Variables and Linear Algebra (BMAT201L)

Practice all the questions

1. Is the set $S=\{(1,\ 1,\ 2),\ (1,\ 2,\ 5),\ (5,\ 3,\ 4)\}$ of vectors form a basis for $\mathrm{R}^3(\mathrm{R})$?

Answer: The set $S=\{(2,\ 1,\ 4),\ (1,\ -1,\ 2),\ (3,\ 1,\ -2)\}$ is linearly dependent and hence it is not a basis for $\mathrm{R}^3(\mathrm{R})$.

2. Is the set $S=\{(1,\ 2,\ 1),\ (3,\ 1,\ 5),\ (3,\ -4,\ 7)\}$ forms a basis of a vector space $\mathrm{R}^3(\mathrm{R})$.

Answer: Yes.

3. If W is the subspace of a vector space $\mathrm{R}^4(\mathrm{R})$ generated by the vectors $(1,\ -2,\ 5,\ -3)$, $(2,\ 3,\ 1,\ -4),\ (3,\ 8,\ -3,\ -5)$, then find a basis and dimension of W.

Answer: $\begin{pmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{pmatrix} \xrightarrow[\mathrm{R}_3-3R_1]{\mathrm{R}_2-2R_1} \begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{pmatrix} \xrightarrow{\mathrm{R}_3-2R_2} \begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Therefore, the two non-zero row vectors $(1,\ -2,\ 5,\ -3),\ (0,\ 7,\ -9,\ 2)$ forms a basis of W and hence $\dim \mathrm{W}=2$.

4. Find the basis and dimension of the row space of $\mathrm{A}=\begin{pmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{pmatrix}$.

Also find the nullity (A).

Answer: The set $\{(1,\ -2,\ 5,\ -3),\ (0,\ 7,\ -9,\ 2)\}$ form a basis for the row space of A and hence the dimension of the row space of A is 2 (i.e., the rank of A is 2). Also nullity (A) = $\mathrm{n}-\mathrm{rank}(A)$ = 2.

Module 5 – Linear Transformations

5.1

Linear Transformation

Definition: Let $U(F)$ and $V(F)$ be two vector spaces. Then the function $T: U \longrightarrow V$ is called a linear transformation from U into V if

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \text{ for all } a, b \in F \text{ and } \alpha, \beta \in U.$$

Note: ① If T is a linear transformation from U into itself (i.e., $T: U \to U$), then T is called a Linear operator on U.

② If $T: U \longrightarrow F$ is a linear transformation, then T is called a linear functional on U.

③ An n dimensional vector space V over a field F is denoted by $V_n(F)$.

Properties of Linear Transformation

Let $T: U(F) \longrightarrow V(F)$ be a linear transformation from the vector space $U(F)$ to the vector space $V(F)$. Then

(i) $T(\hat{0}) = \bar{0}$ where $\hat{0} \in U$ and $\bar{0} \in V$

(ii) $T(-\alpha) = -T(\alpha)$ for all $\alpha \in U$

(iii) $T(\alpha - \beta) = T(\alpha) - T(\beta)$ for all $\alpha, \beta \in U$

(iv) $T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) = a_1T(\alpha_1) + a_2T(\alpha_2) + \cdots + a_nT(\alpha_n)$

for $a_1, a_2, \ldots, a_n \in F$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in U$.

Problems

① The mapping $T: V_3(\mathbb{R}) \longrightarrow V_2(\mathbb{R})$ is defined by $T(x, y, z) = (x-y, x-z)$ for $(x, y, z) \in V_3$. Is T a linear transformation?

Sol: Let $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$ be any two vectors in V_3.

For $a, b \in \mathbb{R}$,

$$T(a\alpha + b\beta) = T(a(x_1, y_1, z_1) + b(x_2, y_2, z_2))$$

$$= T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$

$$= ((ax_1 + bx_2) - (ay_1 + by_2), (ax_1 + bx_2) - (az_1 + bz_2))$$

$$= (a(x_1 - y_1) + b(x_2 - y_2), a(x_1 - z_1) + b(x_2 - z_2))$$

$$= a(x_1 - y_1, x_1 - z_1) + b(x_2 - y_2, x_2 - z_2)$$

$$= aT(x_1, y_1, z_1) + bT(x_2, y_2, z_2)$$

$$= aT(\alpha) + bT(\beta).$$

Therefore, T is a linear transformation.

2. Is the mapping $T: \mathbb{R}^3(\mathbb{R}) \longrightarrow \mathbb{R}^2(\mathbb{R})$ defined by $T(x, y, z) = (|x|, 0)$ a linear transformation?

Sol: Let $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2) \in \mathbb{R}^3$.

For $a, b \in \mathbb{R}$, $T(a\alpha + b\beta) = T(a(x_1, y_1, z_1) + b(x_2, y_2, z_2))$

$= T(ax_1+bx_2, ay_1+by_2, az_1+bz_2)$

$= (|ax_1+bx_2|, 0)$

and $aT(\alpha)+bT(\beta) = aT(x_1, y_1, z_1) + bT(x_2, y_2, z_2)$

$= a(|x_1|, 0) + b(|x_2|, 0)$

$= (a|x_1| + b|x_2|, 0)$

Clearly $T(a\alpha+b\beta) \neq aT(\alpha)+bT(\beta)$

(since $|ax_1+bx_2| \leq |a||x_1|+|b||x_2|$)

Hence T is not a linear transformation.

3. Is $T: \mathbb{R}^3(\mathbb{R}) \longrightarrow \mathbb{R}^3(\mathbb{R})$ defined by

$T(x, y, z) = (x+1, y, z)$ a linear transformation

Answer: not a L.T.

Note: Let $U(F)$ and $V(F)$ be two Vector Spaces and $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ a basis of U. Let $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be a set of vectors in V. Then there exists a unique linear transformation $T: U \longrightarrow V$ such that $T(\alpha_i) = \beta_i$ for $i=1,2,\ldots,n$

4. Fin a linear Transformation $T : \mathbb{R}^2(\mathbb{R}) \to \mathbb{R}^2(\mathbb{R})$ is defined by $T(2,3) = (4,5)$ and $T(1,0) = (0,0)$.

Sol: Let $S = \{(2,3), (1,0)\}$. First we have to show that S is a basis of $\mathbb{R}^2$.

Let $(x,y) \in \mathbb{R}^2$. For $a, b \in \mathbb{R}$,

$$(x,y) = a(2,3) + b(1,0)$$

$$= (2a+b, 3a)$$

$$\Rightarrow 2a+b = x \text{ and } 3a = y$$

$$\Rightarrow a = \frac{y}{3} \text{ and } b = \frac{3x-2y}{3}$$

$$\therefore (x,y) = \frac{y}{3}(2,3) + \left(\frac{3x-2y}{3}\right)(1,0)$$

and hence $L(S) = \mathbb{R}^2$. Clealy S is L.I.

Therefore, S is a basis of $\mathbb{R}^2$.

Now, $T(x,y) = T\left(\frac{y}{3}(2,3) + \left(\frac{3x-2y}{3}\right)(1,0)\right)$

$$= \frac{y}{3}T(2,3) + \frac{3x-2y}{3}T(1,0)$$

$$= \frac{y}{3}(4,5) + \frac{3x-2y}{3}(0,0)$$

$$= \left(\frac{4y}{3}, \frac{5y}{3}\right)$$

which is the rearuired linear transformation

5. Find a linear transformation $T: \mathbb{R}^3(\mathbb{R}) \longrightarrow \mathbb{R}(\mathbb{R})$ defined by $T(1,1,1)=3$, $T(0,1,-2)=1$, $T(0,0,1)=-2$.

Answer: $T(x,y,z) = 8x-3y-2z$.

6. Find a linear transformation $T: V_2(\mathbb{R}) \longrightarrow V_2(\mathbb{R})$ defined by $T(1,2)=(3,0)$ and $T(2,1)=(1,2)$.

Answer: $T(x,y) = \left(\frac{5y-x}{3}, \frac{4x-2y}{3}\right)$.

Range space and Null space of a Linear Transformation

Definition: Let $U(F)$ and $V(F)$ be vectorspaces and $T: U \longrightarrow V$ a linear transformation. Then the range of T is denoted by $R(T)$ and defined as

$$R(T) = \{T(\alpha) \mid \alpha \in U\}$$

Clearly $R(T)$ is a subspace of $V(F)$.

Definition: Let $U(F)$ and $V(F)$ be vector spaces and $T: U \longrightarrow V$ a linear transformation. Then the Null space of T is denoted by $N(T)$ and defined as $N(T) = \{\alpha \in U \mid T(\alpha) = \hat{0}\}$, $\hat{0}$ is the zero vector in V.

Clearly $N(T)$ is a subspace of $U(F)$.

The Null space $N(T)$ is also called as the Kernel of T.

Dimension of Range space and Null space

Definition: Let $T: U(F) \longrightarrow V(F)$ be a linear transformation where U is a finite dimensional vector space. Then

(i) The dimension of the rang space $R(T)$ is defined as rank of T and is denoted by $\rho(T)$. i.e., $\rho(T) = \dim R(T)$.

(ii) The dimension of the Null space $N(T)$ is defined as nullity of T and is denoted by $\nu(T)$. i.e., $\nu(T) = \dim N(T)$.

Note: $\boxed{\text{rank}(T) + \text{nullity}(T) = \dim U}$.

Problems:

① Let $T: \mathbb{R}^3(\mathbb{R}) \longrightarrow \mathbb{R}^2(\mathbb{R})$ be a linear transformation defined by $T(x, y, z) = (x+y, y+z)$. Then find the dimension of range space $R(T)$ and the null space $N(T)$.

Sol: Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ be the starand basis of $\mathbb{R}^3$.

Then $T(1,0,0)=(1,0)$
$T(0,1,0)=(1,1)$
$T(0,0,1)=(0,1)$

Let $S_1=\{(1,0),(1,1),(0,1)\}$. Then $S_1 \subseteq R(T)$.

Now, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_3-R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Therefore, $\{(1,0),(0,1)\}$ forms a basis of $R(T)$.
Hence $\dim R(T) = 2$.

For $\alpha \in N(T)$, $T(\alpha)=\hat{0} \Rightarrow T(x,y,z)=(0,0)$
$\Rightarrow (x+y,\ y+z)=(0,0)$
$\Rightarrow x+y=0$
and $y+z=0$

Let $z=k$, then we get $y=-k$ and hence
$x=k,\ y=-k,\ z=k$.

$\therefore \left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$ form a basis for $N(T)$ and hence

$\dim N(T) = \text{nullity}(T) = 1$

Therefore, $\dim R(T) + \dim N(T) = 2+1$

i.e., $\boxed{\text{rank}(T) + \text{nullity} = 3 = \dim \mathbb{R}^3}$

2. Let $T: \mathbb{R}^4(\mathbb{R}) \longrightarrow \mathbb{R}^3(\mathbb{R})$ be a linear transformation defined by $T(a,b,c,d)=(a-b+c+d,\ a+2c-d,\ a+b+3c-3d)$ for $a,b,c,d \in \mathbb{R}$, then find basis and dimension of $R(T)$ and $N(T)$. Answer: (i) $\{(1,1,1),(0,1,2)\}$ is a basis of $R(T)$
$\dim R(T)=2$, $\dim N(T)=\text{nullity}(T)=2$ (ii) $\{(-2,-1,1,0),(1,2,0,1)\}$ is a basis of N(T)

Invertible Linear Transformation

Non-Singular Transformation:

Definition: A linear transformation $T: U(F) \to V(F)$ is said to be non-singular if the null space of T consists zero vector alone. [i.e., $N(T) = \{\hat{0}\}$, where $\hat{0}$ is the zero vector in $U(F)$.]

i.e, $\alpha \in U$ and $T(\alpha) = \bar{0} \Rightarrow \alpha = \hat{0}$.

Note: (1) Let $U(F)$ and $V(F)$ be two finite dimensional vector spaces. The linear transformation $T: U \to V$ is one-one onto iff (if and only if) T is non-singular. So, $T: U \to V$ is invertible if and only if T is non-singular.

(2) A linear transformation $T: U(F) \to V(F)$ is said to be invertible if there exists a linear transformation $S: V(F) \to U(F)$ such that $T \circ S = S \circ T = I$, identity transformation.

Here S is the inverse of T.

(3) Let $U(F)$ and $V(F)$ be two vector spaces. Then the set $L(U, V)$ of all linear transformations from $U(F)$ into $V(F)$ is a vector space over a field F. This is called a vector space of linear transformations.

(4) If $\dim U(F) = m$ and $\dim V(F) = n$, then $\dim(L(U,V)) = mn$.

Problems

(1) If a linear transformation $T: \mathbb{R}^3(\mathbb{R}) \longrightarrow \mathbb{R}^3(\mathbb{R})$ defined by $T(x,y,z) = (2x, (4x-y), (2x+3y-z))$ is invertible, then find T^{-1}.

Sol: Let $T(x,y,z) = (a,b,c)$

Then $T^{-1}(a,b,c) = (x,y,z)$ (Since T is invertible)

Now, $T(x,y,z) = (a,b,c)$

$\Rightarrow (2x, 4x-y, 2x+3y-z) = (a,b,c)$

$\Rightarrow 2x = a,\ 4x-y = b,\ 2x+3y-z = c$

$\Rightarrow x = \frac{a}{2},\ y = 2a-b,\ z = 7a-3b-c$

$\therefore T^{-1}(a,b,c) = \left(\frac{a}{2}, 2a-b, 7a-3b-c\right)$

(2) Show that a linear transformation $T: \mathbb{R}^3(\mathbb{R}) \longrightarrow \mathbb{R}^3(\mathbb{R})$ defined by $T(x,y,z) = (x+y+z, y+z, z)$ is invertible and hence find T^{-1}.

Sol: we have T is invertible if T is non-singular

For $\alpha = (x, y, z) \in \mathbb{R}^3$, $T(\alpha) = \hat{0}$

$$\Rightarrow T(x, y, z) = (0, 0, 0)$$
$$\Rightarrow (x+y+z, y+z, z) = (0, 0, 0)$$
$$\Rightarrow x+y+z = 0, \; y+z = 0, \; z = 0$$
$$\Rightarrow x = 0, \; y = 0, \; z = 0$$
$$\Rightarrow \alpha = \hat{0}$$

Therefore, $N(T) = \{\hat{0}\}$ and hence T is non-singular. Thus T is invertible.

Let $T(x, y, z) = (a, b, c)$. Then $T^{-1}(a, b, c) = (x, y, z)$

Now, $T(x, y, z) = (a, b, c) \Rightarrow (x+y+z, y+z, z) = (a, b, c)$

$$\Rightarrow x+y+z = a, \; y+z = b, \; z = c$$
$$\Rightarrow x = a-b, \; y = b-c, \; z = c.$$

$\therefore T^{-1}(a, b, c) = (a-b, b-c, c)$

3. Show that a linear transformation $T: \mathbb{R}^3(\mathbb{R}) \to \mathbb{R}^3(\mathbb{R})$ defined by $T(x, y, z) = (3x, x-y, 2x+y+z)$ is invertible and hence find T^{-1}.

4. Show that a linear transformation $T: \mathbb{R}^2(\mathbb{R}) \to \mathbb{R}^2(\mathbb{R})$ defined by $T(x, y) = (x+y, x-y)$ is invertible and ~~(and)~~ hence find T^{-1}.

5.2 Matrix of Linear Transformation

Let $U(F)$ and $V(F)$ be two finite dimensional vector spaces such that $\dim U = n$ and $\dim V = m$. Let $T: U \to V$ be a linear transformation. Let $B_1 = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be the ordered basis of U and $B_2 = \{\beta_1, \beta_2, \dots, \beta_m\}$ be the ordered basis of V.

For every $\alpha \in U$, $T(\alpha) \in V$ and $T(\alpha)$ can be expressed as a linear combination of the elements of the basis B_2. So,

$$T(\alpha_1) = a_{11}\beta_1 + a_{21}\beta_2 + \dots + a_{m1}\beta_m$$

$$T(\alpha_2) = a_{12}\beta_1 + a_{22}\beta_2 + \dots + a_{m2}\beta_m$$

$$\vdots$$

$$T(\alpha_n) = a_{1n}\beta_1 + a_{2n}\beta_2 + \dots + a_{mn}\beta_m$$

writing the co-ordinates of $T(\alpha_1), T(\alpha_2), \dots, T(\alpha_n)$ successively as columns of a matrix, we get

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

This matrix represented as $[a_{ij}]_{m \times n}$ is called the matrix of the linear transformation T with respect to the bases B_1 and B_2. we can denote this by $[T : B_1, B_2]$ or $[T] = [a_{ij}]_{m\times n}$.

Here $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$ are co-ordinate vectors of $T(\alpha_1), T(\alpha_2), \dots, T(\alpha_n)$ with respect to a basis B_2 respectively.

i.e., $\left[T(\alpha_1)\right]_{B_2} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \left[T(\alpha_2)\right]_{B_2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, [T(\alpha_n)]_{B_2} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$

Note: (i) Let $T : V(F) \longrightarrow V(F)$ be a linear operator such that $\dim V = n$. If $B_1 = B_2 = B$ (say), then the above said matrix is called the matrix of T relative to the ordered basis B of V. It is denoted by $[T : B]$ or $[T]_B$.

(ii) Let $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis of an n-dimensional vector space $V(F)$. Then for any $\alpha \in V$, $\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$, where $a_1, a_2, \dots, a_n \in F$.

Therefore, the co-ordinate vector of α relative to the basis B is $[\alpha]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$.

Problems

1. Let $T: \mathbb{R}^2(\mathbb{R}) \longrightarrow \mathbb{R}^3(\mathbb{R})$ be a linear transformation defined by $T(x,y) = (x+y,\ 2x-y,\ 7y)$. Find the matrix of linear transformation $[T : B_1, B_2]$ where B_1 is the standard basis of $\mathbb{R}^2$ and B_2 is the standard basis of $\mathbb{R}^3$.

Sol: We have $B_1 = \{(1,0),(0,1)\}$ is the standard basis of $\mathbb{R}^2$ and $B_2 = \{(1,0,0),(0,1,0),(0,0,1)\}$ is the standard basis of $\mathbb{R}^3$.

Now,
$$T(1,0) = (1,2,0)$$
$$= 1(1,0,0) + 2(0,1,0) + 0(0,0,1)$$
$$T(0,1) = (1,-1,7)$$
$$= 1(1,0,0) - 1(0,1,0) + 7(0,0,1).$$

Therefore,
$$[T : B_1, B_2] = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}.$$

2. Let $T: \mathbb{R}^3(\mathbb{R}) \longrightarrow \mathbb{R}^2(\mathbb{R})$ be a linear transformation defined by $T(x,y,z) = (3x+2y-4z,\ x-5y+3z)$. Find the matrix of T relative to the bases

$B_1 = \{(1,1,1),(1,1,0),(1,0,0)\}$ and $B_2 = \{(1,3),(1,5)\}$.

Sol: Let $(a,b) \in \mathbb{R}^2$ be such that
$$(a,b) = p(1,3) + q(2,5)$$
then
$$(a,b) = (p+2q,\ 3p+5q)$$
$$\Rightarrow a = p+2q,\quad b = 3p+5q$$
$$\Rightarrow p = -5a+2b,\quad q = 3a-b$$

$\therefore \; (a,b) = (-5a+2b)(1,3) + (3a-b)(2,5).$

Now, $T(1,1,1) = (1,-1)$

$= -7(1,3) + 4(2,5)$

$T(1,1,0) = (5,-4)$

$= -33(1,3) + 19(2,5)$

$T(1,0,0) = (3,1)$

$= -13(1,3) + 8(2,5).$

Therefore, the matrix of T relative to B_1 and B_2 is $[T : B_1, B_2] = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$

* Suppose, $\alpha = (1,1,1)$, then $[T(\alpha)]_{B_2} = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$ is the co-ordinate vector of $T(\alpha)$ relative to B_2

3. Let $T : \mathbb{R}^2(\mathbb{R}) \longrightarrow \mathbb{R}^2(\mathbb{R})$ be a linear operator defined by $T(x,y) = (4x-2y, 2x+y)$. Find the matrix of T w.r.t. the basis $B = \{(1,1), (-1,0)\}$.

Sol: let $(a,b) \in \mathbb{R}^2$ be such that

$$(a,b) = p(1,1) + q(-1,0)$$

then $(a,b) = \cancel{(p-q)+p}\ (p-q,\ p)$

$$\Rightarrow a = p-q, \quad b = p$$

$$\Rightarrow p = b, \quad q = b-a$$

$$\therefore (a,b) = b(1,1) + (b-a)(-1,0)$$

Now, $T(1,1) = (2,3)$

$$= 3(1,1) + 1(1,0)$$

$$T(-1,0) = (-4,-2)$$

$$= -2(1,1) + 2(-1,0)$$

Therefore, the matrix of T w.r.t. the basis

$B = \{(1,1),\ (-1,0)\}$ is

$$[T:B] = [T]_B = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$$

* Here $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ are the co-ordinate

vectors

5.3 Change of Basis

Let $V(F)$ be an n-dimensional vector space.
Let $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $B' = \{\beta_1, \beta_2, \dots, \beta_n\}$ be two arbitrary bases for V. Let us suppose

$$\beta_1 = a_{11}\alpha_1 + a_{21}\alpha_2 + \dots + a_{n1}\alpha_n$$
$$\beta_2 = a_{12}\alpha_1 + a_{22}\alpha_2 + \dots + a_{n2}\alpha_n$$
$$\vdots$$
$$\beta_n = a_{1n}\alpha_1 + a_{2n}\alpha_2 + \dots + a_{nn}\alpha_n$$

Then the matrix of transformation from B to B' is

$$P = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}.$$

Here, the matrix P is called the transition matrix from B to B'.
(or, the change of basis matrix)

Note: Since $B' = \{\beta_1, \beta_2, \dots, \beta_n\}$ is L.I, the matrix P is invertible and P^{-1} is called the transition matrix from B' to B.

Examples

1. Let $B=\{(1,0),(0,1)\}$ and $B'=\{(1,3),(2,5)\}$ be the bases of $\mathbb{R}^2(\mathbb{R})$. Find the transition matrix from B to B'. Also find the transition matrix from B' to B.

Sol: (i) clearly, $(1,3) = 1(1,0) + 3(0,1)$

$$(2,5) = 2(1,0) + 5(0,1)$$

Therefore, the transition matrix from B to B' is $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

(ii) For any $(x,y) \in \mathbb{R}^2$,

$$(x,y) = a(1,3) + b(2,5)$$

$$\Rightarrow x = a+2b, \quad y = 3a+5b$$

$$\Rightarrow a = 2y-5x; \ b = 3x-y.$$

$$\therefore (x,y) = (2y-5x)(1,3) + (3x-y)(2,5)$$

Now,

$$(1,0) = -5(1,3) + 3(2,5)$$

$$(0,1) = (+2)(1,3) - 1(2,5)$$

Therefore, the transition matrix from B' to B is $\begin{bmatrix} -5 & 2 \end{bmatrix}$.

(2) Find the transition matrix from $B = \{(1,2),(3,4)\}$ to $B' = \{(7,3),(4,2)\}$, where B and B' are bases of a vector space $\mathbb{R}^2(\mathbb{R})$.

(3) Find the transition matrix from $B = \{(9,2),(4,-3)\}$ to $B' = \{(2,1),(-3,1)\}$, where B and B' are the bases of $\mathbb{R}^2(\mathbb{R})$.

(4) Find the transition matrix from $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ to $B' = \{(1,1,0),(0,1,2),(1,1,1)\}$, where B and B' are the bases of $\mathbb{R}^3(\mathbb{R})$.

Similarity of Matrices

Definition: Let A and B be square matrices of order n over a field F. Then B is said to be similar to A if there exists an invertible matrix C of order n with elements in F such that $B = C^{-1}AC$.

Fall Sem. 2023-24

Tutorial– 6

Programme Name & Branch: B. Tech.

Course Name & Code: Complex Variables and Linear Algebra (BMAT201L)

Practice all the questions

1. Is a mapping $T:\mathbb{R}^3(\mathbb{R})\to\mathbb{R}^3(\mathbb{R})$ defined by $\mathrm{T(x,\ y,\ z)=(x+y-z,\ }2x-y+z,\ -x+y+2z)$ a linear transformation?

   Answer: Yes

2. Which of the following functions are linear transformations from $\mathrm{R}^2(\mathrm{R})$ into $\mathrm{R}^2(\mathrm{R})$.

   (i) $\mathrm{T(x,\ y)=(x^2,\ y)}$ (ii) $\mathrm{T(x,\ y)=(x,\ }|y|)$ (iii) $\mathrm{T(x,\ y)=(x-y,\ 0)}$

   Answer: (i) not a linear transformation (ii) not a linear transformation (iii) a linear transformation

3. Find the linear transformation $\mathrm{T{:}R^2(R)\to R^2(R)}$ such that $\mathrm{T(2,\ 3)=(4,\ 5)}$ and $\mathrm{T(1,\ 0)=(0,\ 0)}$.

Answer: $\mathrm{T(x,\ y)}=\left(\dfrac{4\mathrm{y}}{3},\ \dfrac{5\mathrm{y}}{3}\right)$.

4. Let $\mathrm{T{:}R^3(R)\to R^3(R)}$ be a linear transformation defined by

   $\mathrm{T(x,\ y,\ z)=(2x,\ 4x-y,\ }2x+3y-z)$ for $\mathrm{x,\ y,\ z\in R}$. Show that T is invertible and find $\mathrm{T^{-1}}$.

Answer: $\mathrm{T}^{-1}(a,\ b,\ c)=\left(\dfrac{a}{2},\ 2a-b,\ 7a-3b-c\right)$.

5. Let $\mathrm{T{:}R^2(R)\to R^3(R)}$ be a linear transformation defined by $\mathrm{T(x,\ y)=(x+y,\ }x-\mathrm{y},\ y)$. Find the rank and nullity of T.

Answer: rank of T = 2 and nullity of T = 0.

6. Let $\mathrm{T{:}R^3(R)\to R^3(R)}$ be a linear transformation defined by

   $\mathrm{T(x,\ y,\ z)=(y+z,\ }x-\mathrm{z},\ -x-y)$. Find the matrix of the linear transformation T with respect to the basis $\mathrm{B}=\{(0,\ 1,\ -1),\ (1,\ 0,\ -1),\ (1,\ -1,\ 0)\}$.

Answer: $[\mathrm{T}]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

7. Let $\mathrm{T}:\mathrm{R}^3(\mathrm{R}) \to \mathrm{R}^2(\mathrm{R})$ be a linear transformation defined by

$\mathrm{T}(x,\ y,\ z)=(2x+y-z,\ 3x-2y+4z)$. Find the matrix of the linear transformation T with respect to the bases $\mathrm{B}_1=\{(1,\ 1, 1),\ (1,\ 1,\ 0),\ (1,\ 0,\ 0)\}$ and $\mathrm{B}_2=\{(1,\ 3),\ (1,\ 4)\}$.

Answer: $[\mathrm{T}:\ \mathrm{B}_1, \mathrm{B}_2] = \begin{pmatrix} 3 & 11 & 5 \\ -1 & -8 & -3 \end{pmatrix}$.

8. Let $\mathrm{B}=\{(2,\ 3),\ (3,\ 5)\}$ and $\mathrm{B}^1=\{(4,\ 3),\ (1,\ 2)\}$ be the bases of a vector space $\mathrm{R}^2(\mathrm{R})$. Find the transition matrix (or, change of basis matrix) from B to B^1.

Answer: $\mathrm{P}=\begin{pmatrix} 11 & -1 \\ -6 & 1 \end{pmatrix}$.

Module 6: Inner Product Spaces

Definition: Let $V(F)$ be a vector space where F is a field of real or complex numbers. An inner product on V is a function $f: V \times V \longrightarrow F$ such that

(i) $f(\alpha, \beta) = \overline{f(\beta, \alpha)}$, where $\overline{f(\beta, \alpha)}$ is the complex conjugate of $f(\beta, \alpha)$, for all $\alpha, \beta \in V$

(ii) $f(\alpha, \alpha) >$ for $\alpha \neq \bar{0}$ and $f(\alpha, \alpha) = 0$ for $\alpha = \bar{0}$

and (iii) $f(a\alpha + b\beta, r) = af(\alpha, r) + bf(\beta, r)$ for all $\alpha, \beta, r \in V$ and $a, b \in F$.

The vector space $V(F)$ in which an inner product f defined as above is called an inner product space and it is denoted by (V, f).

Notation: In general $f(\alpha, \beta)$ is denoted by $\langle \alpha, \beta \rangle$ or (α, β) or $\langle \alpha | \beta \rangle$. Here after we use $\langle \alpha, \beta \rangle$ for $f(\alpha, \beta)$.

Therefore, the above conditions of the inner product are written as follows:

(i) $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$

(ii) $\langle \alpha, \alpha \rangle$ for $\alpha \neq \bar{0}$ and $\langle \alpha, \alpha \rangle = 0$ for $\alpha = \bar{0}$.

and (iii) $\langle a\alpha + b\beta, r \rangle = a \langle \alpha, r \rangle + b \langle \beta, r \rangle$ for all $\alpha, \beta, r \in V$, $\bar{0} \in V$ and $a, b, 0 \in F$.

Example:

① If $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$ are the elements of a vector space $\mathbb{R}^3(\mathbb{R})$, then

$\langle \alpha, \beta \rangle = x_1 x_2 + y_1 y_2 + z_1 z_2$ is an inner product on $\mathbb{R}^3$.

Note: The above product is called the dot product of α and β and is denoted by $\alpha \cdot \beta$.

i.e., $\langle \alpha, \beta \rangle = \alpha \cdot \beta$.

This is called the standard inner product on $\mathbb{R}^3$.

② For $A, B \in M_{n \times n}(\mathbb{R})$, define $\langle A, B \rangle = \mathrm{Tr}(B^T A)$ is an inner product on $M_{n \times n}(\mathbb{R})$.

Note: $\mathrm{Tr}(B^T A) =$ Trace of $(B^T A)$

Norm or Length of a vector:

Definition: Let $V(F)$ be an inner product space. The norm (or, length) of a vector $\alpha \in V$ is denoted by $\|\alpha\|$ and defined as the positive square root of $\langle \alpha, \alpha \rangle$.

i.e., $\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle}$

Note: ① For $\alpha \in V$, $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$

② In an inner product space $V(F)$,

(i) $\|a\alpha\| = |a| \, \|\alpha\|$ for $\alpha \in V$ and $a \in F$.

(ii) $|\langle \alpha, \beta \rangle| \leq \|\alpha\| \, \|\beta\|$ for $\alpha, \beta \in V$.

(iii) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ for $\alpha, \beta \in V$

Definition: Let $V(F)$ be an inner product space. A vector $\alpha \in V$ is said to be a unit vector if $\|\alpha\| = 1$.

Definition: Let $V(F)$ be an inner product space and $\alpha, \beta \in V$. α is said to be orthogonal to β if $\langle \alpha, \beta \rangle = 0$.

Orthogonal and Orthonormal sets

Definition: Let $V(F)$ be an inner product space. A non-empty sub set S of V is said to be

(i) an orthogonal if for every pair of distinct vectors of S are orthogonal.

~~i.e., for every $\alpha_i, \alpha_j \in S$, $\langle \alpha_i, \alpha_j \rangle = 0$ (for $i \neq j$).~~

i.e., for every α_i, α_j $(i \neq j)$ in S, $\langle \alpha_i, \alpha_j \rangle = 0$.

(ii) an orthonormal if S is orthogonal and for every $\alpha \in S$, $\|\alpha\| = 1$.

Examples:

① $S = \left\{ \left(\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right), \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right) \right\}$ is an orthonormal set in $\mathbb{R}^3(\mathbb{R})$.

② $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is an orthonormal set in $\mathbb{R}^3(\mathbb{R})$

③ $S = \{(1,-2,-2), (2,-1,2), (2,2,-1)\}$ is an orthogonal set in $\mathbb{R}^3(\mathbb{R})$ but it is not an orthonormal set.

Gram-Schmidt orthogonalization Process

Let $V(F)$ be an inner product space and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis for V. Let $\beta_1 = \alpha_1$,

$$\beta_2 = \alpha_2 - \frac{\langle \alpha_2, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\langle \alpha_3, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} \beta_1 - \frac{\langle \alpha_3, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle} \beta_2$$

$$\vdots$$

$$\beta_n = \alpha_n - \frac{\langle \alpha_n, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} \beta_1 - \cdots - \frac{\langle \alpha_n, \beta_{n-1} \rangle}{\langle \beta_{n-1}, \beta_{n-1} \rangle} \beta_{n-1}$$

Then $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is an orthogonal basis for V.

And hence $\left\{ \frac{\beta_1}{\|\beta_1\|}, \frac{\beta_2}{\|\beta_2\|}, \ldots, \frac{\beta_n}{\|\beta_n\|} \right\}$ is an orthonormal basis for V.

<u>Note:</u> Every finite dimensional inner product space has an orthogonal basis and hence it has an orthonormal basis.

<u>Problems:</u>

① Given $\{(2,1,3), (1,2,3), (1,1,1)\}$ is a basis of $\mathbb{R}^3(\mathbb{R})$. Obtain an orthogonal basis of $\mathbb{R}^3(\mathbb{R})$ using Gram Schmidt orthogal process.

<u>Sol:</u> Let $\alpha_1 = (2,1,3)$, $\alpha_2 = (1,2,3)$, $\alpha_3 = (1,1,1)$.

By Gram Schmidt orthogonalization process an orthogonal basis $\{\beta_1, \beta_2, \beta_3\}$ is given by

$$\beta_1 = \alpha_1 = (2,1,3).$$

$$\beta_2 = \alpha_2 - \frac{\langle \alpha_2, \beta_1 \rangle \beta_1}{\langle \beta_1, \beta_1 \rangle} = (1,2,3) - \frac{\langle (1,2,3), (2,1,3) \rangle (2,1,3)}{\langle (2,1,3), (2,1,3) \rangle}$$

$$= (1,2,3) - \frac{13}{14}(2,1,3)$$

$$= \frac{1}{14}(-12, 15, 3)$$

$$\beta_3 = \alpha_3 - \frac{\langle \alpha_3, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}\beta_1 - \frac{\langle \alpha_3, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle}\beta_2$$

$$= (1,1,1) - \frac{\langle (1,1,1),(2,1,3) \rangle}{14}(2,1,3)$$

$$- \frac{\langle (1,1,1), \frac{1}{14}(-12,15,3) \rangle}{\left(\frac{378}{196}\right)} \frac{1}{14}(-12,15,3)$$

$$= (1,1,1) - \frac{6}{14}(2,1,3) - \frac{\frac{6}{14} \cdot \frac{1}{14}(-12,15,3)}{\left(\frac{378}{196}\right)}$$

$$= (1,1,1) - \frac{6}{14}(2,1,3) - \frac{6}{378}(-12,15,3)$$

$$= (1,1,1) - \frac{6}{14}(2,1,3) - \frac{1}{378}(-72,90,18)$$

$$= \left(\frac{1}{3}, \frac{1}{3}, \frac{-1}{3}\right)$$

Therefore, $\left\{(2,1,3), \left(\frac{-12}{14}, \frac{15}{14}, \frac{3}{14}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{-1}{3}\right)\right\}$

is an orthogonal basis of $\mathbb{R}^3(\mathbb{R})$.

Hence $\left\{\left(\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right), \left(\frac{-12}{\sqrt{378}}, \frac{15}{\sqrt{378}}, \frac{3}{\sqrt{378}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)\right\}$

is an orthonormal basis of $\mathbb{R}^3(\mathbb{R})$.

(2) Use Gram Schmidt process to obtain an orthogonal basis of $\mathbb{R}^3(\mathbb{R})$, given that $\{(1,0,1), (1,0,-1), (0,3,4)\}$ is a basis of $\mathbb{R}^3(\mathbb{R})$ and hence find an orthonormal basis of $\mathbb{R}^3(\mathbb{R})$

Answer: $\left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right), (0,1,0)\right\}$

is an orthonormal basis of $\mathbb{R}^3(\mathbb{R})$

(3) Prove that $\left\{\left(\frac{3}{5}, 0, \frac{4}{5}\right), \left(-\frac{4}{5}, 0, \frac{3}{5}\right), (0,1,0)\right\}$

is an orthonormal set in $\mathbb{R}^3$.

Fall Sem. 2023-24

Tutorial– 7

Programme Name & Branch: B. Tech.

Course Name & Code: Complex Variables and Linear Algebra (BMAT201L)

Practice all the questions

1. If $\alpha=(1,\ 3,\ -1),\ \beta=(-1,\ 1,\ 2)$ are two vectors in an inner product space $R^3(R)$, find (i) $\|\alpha\|$ (ii) $\langle\alpha,\ \beta\rangle$ (iii) angle between α and β.

Answer: (i) $\|\alpha\|=\sqrt{\langle\alpha,\ \alpha\rangle}=\sqrt{11}$ (ii) $\langle\alpha,\ \beta\rangle=1(-1)+3(1)+(-1)2=0$

(iii) $\cos\theta=\dfrac{\langle\alpha,\ \beta\rangle}{\|\alpha\|\,\|\beta\|}=\dfrac{0}{\sqrt{11}\sqrt{6}}=0\Rightarrow\theta=\dfrac{\pi}{2}$.

2. Is $S=\{(1,\ -1,1),\ (0,\ 1,\ 1),\ (2,\ 1,\ -1)\}$ an orthogonal set in an inner product space $R^3(R)$.

Answer: Clearly every pair of distinct vector are orthogonal. Hence S is an orthogonal set in $R^3(R)$.

3. Is $S=\left\{\left(\dfrac{1}{\sqrt{2}},\ \dfrac{1}{\sqrt{2}}\right),\left(\dfrac{1}{\sqrt{2}},\ \dfrac{-1}{\sqrt{2}}\right)\right\}$ an orthonormal set in an inner product space $R^2(R)$.

Answer: Cleary S is an orthogonal set and each vector in S is a unit vector. Hence S is an orthonormal set in $R^2(R)$.

4. Using Gram Schmidt process finds an orthogonal basis for $R^3(R)$, given that $\{(1,\ 2,\ 2),\ (-1,\ 0,\ 2),\ (0,\ 0,\ 1)\}$ is a basis for R^3. Also find an orthonormal basis for $R^3(R)$.

Answer: $\{(1,\ 2,\ 2),\ (-4/3,\ -2/3,\ 4/3),\ (2/9,\ -2/9,\ 1/9)\}$ is an orthogonal basis for $R^3(R)$ and hence its orthonormal basis is $\left\{\left(\dfrac{1}{3},\ \dfrac{2}{3},\ \dfrac{2}{3}\right),\left(-\dfrac{2}{3},\ -\dfrac{1}{3},\ \dfrac{2}{3}\right),\left(\dfrac{2}{3},\ -\dfrac{2}{3},\ \dfrac{1}{3}\right)\right\}$.

Name: Reg. No:

Fall Semester 2023-24, Quiz–2, Slot: C1+TC1+TCC1

BMAT 201L - Complex Variables and Linear Algebra.

Answer all the questions. Each question carries ONE mark.

1. The characteristic equation of $A=\begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}$ is -------------

2. Let $A=\begin{pmatrix} 1 & 2 & 4 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{pmatrix}$. If $A^3 = aA^2 - bA + cI$, then the values of a= -----, b= ---- and c= ----

3. The eigenvector corresponding to an eigenvalue $\lambda=3$ of $A=\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ is -----

4. Let $A=\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Then the eigenvalues A^5 are -----------

5. Which one of the following set of vectors is linearly independent sub set of a vector space $\mathbb{R}^2$ over a field of real numbers $\mathbb{R}$.

   (a) $\{(1, 1, 1), (1, 2, 0), (0, 0, 0)\}$ (b) $\{(1, 1, 1), (1, 2, 0), (0, -1, 2)\}$

   (c) $\{(1, 1, 1), (1, 2, 0), (0, -1, 1)\}$ (d) $\{(0, -2, 4), (1, 2, 0), (0, -1, 2)\}$

6. Which one of the following set of vectors is a basis for a vector space $\mathbb{R}^3$ over a field of real numbers $\mathbb{R}$.

   (a) $B=\{(1, 1, 1), (0, 1, 1), (0, 0, 0)\}$
   (b) $B=\{(1, -1, 1), (0, 1, 0), (1, 0, 1)\}$
   (c) $B=\{(1, 0, 0), (0, 0, 1), (0, 1, -1)\}$

7. The basis of the Null space of $A=\begin{pmatrix} -1 & 2 & 3 \\ -2 & 4 & 6 \end{pmatrix}$ is -------

8. The dimension of the row space of $A=\begin{pmatrix} -1 & -2 & -3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix}$ is -------

(a) 1 (b) 3 (c) 0 (d) 2

9. If $T: R^2(R) \to R^2(R)$ is an invertible linear operator defined by

$T(x, \ y) = (x + y, \ \ x - 2y)$, then $T^{-1}(a, \ \ b)$= ------------

10. Let $T: R^2(R) \to R^2(R)$ be a linear transformation defined by

$T(x, \ y) = (x + y, \ \ x - y)$, then N(T)= --------

KEY:

1. $\lambda^3 - 6\lambda^2 + 8\lambda = 0$ 2. 11, 34, 24 3. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ 4. 1, 243 5. b 6. c

7. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$ 8. d 9. $\left(\dfrac{2a+b}{3}, \ \dfrac{a-b}{3} \right)$ 10. $\{\hat{0}\}$

SEMESTER: FALL 2023-24

COURSE NAME: Complex Variables and Linear Algebra. **COURSE CODE: B**MAT201L

SLOT: C1+TC1+TCC1

Course Outcome (CO): 1, 2, 3, 4, 5

At the end of this course the students should be able to

1. Construct an analytic function and find complex potential of fluid flow and electric field.
2. Find the image of the straight line by elementary transformations and to express analytic function in power series.
3. Evaluate real integral using the techniques of contour integration.
4. Use the power of inner product and norm for analysis.
5. Use matrices and transformations for solving engineering problems.

Assessment Methods, Rubrics and Other Guidelines

Theory Component (Class Number: VL2023240102031)

Assessment Type	Due Date	Weightage	Remarks
Quiz 1	NA	10	Before CAT1
DA 1	13-10-2023	10	Before CAT 2
Quiz 2	NA	10	Before FAT
CAT 1	As per the announcement by the University	15	
CAT 2	As per the announcement by the University	15	
FAT	As per the announcement by the University	40	

Fall Sem. 2023-24

DA1

Programme Name & Branch: B. Tech.

Course Name & Code: Complex Variables and Linear Algebra (BMAT201L)

Maximum Marks: 10

Answer all the questions

1. Evaluate $\int_0^{\pi} \frac{1}{4+\sin^2\theta}\, d\theta$ using residue theorem.

   Answer: $\int_0^{\pi} \frac{1}{4+\sin^2\theta}\, d\theta = \int_0^{\pi} \frac{2}{9-\cos 2\theta} d\theta = \int_0^{2\pi} \frac{1}{9-\cos\phi} d\phi = \frac{-2}{i}\int_C \frac{1}{z^2-18z+1} dz = \frac{\pi}{2\sqrt{5}}$, C is the unit circle $|z|=1$.

2. Let $A = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}$. Use Cayley-Hamilton theorem to find A^{-1} and A^4.

   Answer: The characteristic equation of A is $\lambda^3 - 16\lambda^2 + 81\lambda - 126 = 0$. By Cayley-Hamilton theorem, we have $A^3 - 16A^2 + 81A - 126I = 0$. Therefore $A^{-1} = \begin{pmatrix} 5/21 & -2/21 & 0 \\ -2/21 & 5/21 & 0 \\ 23/126 & -13/63 & 1/6 \end{pmatrix}$

   and $A^4 = \begin{pmatrix} 1241 & 1160 & 0 \\ 1160 & 1241 & 0 \\ -865 & 1970 & 1296 \end{pmatrix}$

3. Solve the system of equations $2x+3y-z=4;\quad x-y+5z=5;\quad 3x+4y+z=8$ using Gauss Jordan method.

   Answer: $x=1,\ y=1,\ z=1.$

4. a) Is the set $S = \{(1,\ 1,\ 2),\ (1,\ 2,\ 5),\ (5,\ 3,\ 4)\}$ form a basis for $\mathrm{R}^3(\mathrm{R})$?

   Answer: Not a basis

   b) Let $\mathrm{W}=\{(\mathrm{x},\ \mathrm{y},\ \mathrm{z},\ \mathrm{w}) \mid x = y+z-w\}$ be the subspace of a vector space $\mathrm{R}^4(\mathrm{R})$. Find the basis and the dimension of W.

Answer: $\{(1,\ 1,\ 0,\ 0),\ (1,\ 0,\ 1,\ 0),\ (-1,\ 0,\ 0,\ 1)\}$ is a basis of W and hence dim $W = 3$.

5. Find the dimension of the null space of $A = \begin{pmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{pmatrix}$.

Answer: $\left\{ \begin{pmatrix} -17/7 \\ 9/7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 17/7 \\ -2/7 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis of the null space $N(A)$ and hence the dim $N(A) = 2$.

Name: Reg. No:

Quiz–I, Fall Semester 2023-2024, **Slot: C1+TC1+TCC1**

BMAT201L - Complex variables and Linear Algebra.

Answer all the questions. Each question carries ONE mark.

1. Which one of the following is analytic?
   (a) $f(z)=x^2-y^2+i2xy$ (b) $f(z)=e^x(\cos y-i\sin y)$ (c) $f(z)=x^2+ixy$ (d) $f(z)=\overline{z}$
2. If the real part of the analytic function $f(z)$ is $\dfrac{x}{x^2+y^2}$, then $f(z)=\ldots\ldots\ldots$
   (a) $\dfrac{-1}{z}+c$ (b) $\dfrac{1}{2z}+c$ (c) $\dfrac{1}{z}+c$ (d) $\dfrac{2}{z}+c$
3. The image of the circle $|z|=2$ in the z-plane under the transformation $w=z+3+2i$ is -----
4. The image of the line $y=2x$ under the transformation $w=z^2$ is --------------
5. The image of the line $y-x+1=0$ in the z-plane under the transformation $w=\dfrac{1}{z}$ is ---------
6. The image of the line $x=2$ in the z-plane under the transformation $w=e^z$ is --------------
7. The bilinear transformation which maps points $i,\ 0,\ 1$ into the points $1-i,\ \infty,\ 0$
   (a) $w=\dfrac{i(z-1)}{z+1}$ (b) $w=\dfrac{i(z+1)}{z}$ (c) $w=\dfrac{-i(z+1)}{z}$ (d) $w=\dfrac{i(1-z)}{z}$
8. The Laurent series expansion of $f(z)=\dfrac{1}{(z-1)(z-2)}$ in the region $1<|z|<2$ is ……
9. The residue of $f(z)=\dfrac{z^2}{(z-1)^2(z+2)}$ at the pole $z=1$ is ……………..
10. The value of $\displaystyle\int_c \left(\frac{z^2-1}{z^2+1}\right)dz$, where c is $|z-i|=1$, is ……………

Key: 1. $f(z) = x^2 - y^2 + i2xy$

2. $\dfrac{1}{z} + c$

3. $u^2 + v^2 - 6u - 4v = -9$

4. $4u = -3v$

5. $u^2 + v^2 - u - v = 0$

6. $u^2 + v^2 = e^4$

7. $w = \dfrac{i(1-z)}{z}$

8. $f(z) = -\dfrac{1}{z}\left(1 + \dfrac{1}{z} + \dfrac{1}{z^2} + \ldots\right) - \dfrac{1}{2}\left(1 + \dfrac{z}{2} + \dfrac{z^2}{4} + \ldots\right)$

9. 5/9

10. -2π

VIT

Vellore Institute of Technology

(Deemed to be University under section 3 of UGC Act, 1956)

SCHOOL OF ADVANCED SCIENCES

Fall Semester 2023-2024

Continuous Assessment Test – II

Programme Name & Branch: B.Tech. **Slot: C1+TC1+TCC1**

Course Name & code: Complex Variables and Linear Algebra – BMAT201L

Class Number (s): Common QP for the mentioned slot

Exam Duration: 90 Min. **Maximum Marks: 50**

General instruction(s): Answer ALL the Questions

Q.No.	Question	Max Marks	CO	BL
1.	Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3}\,dx$ using Cauchy's residue theorem.	10	CO3	BL2
2. (a)	Find the eigenvalue corresponding to an eigenvector X=$\begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$ of a matrix A=$\begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}$	5	CO5	BL3
2. (b)	Let A=$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$. Use Cayley-Hamilton theorem to find the values of a and b such that $A^3 = a\,A + b\,I$, where I=$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	5	CO5	BL3
3.	Using Gauss Jordan, Solve the linear system. $x + 2y - z - w = -4$; $3x - y + 2z + 2w = 10$; $x + y - 3z + w = -5$; $2x - 3y + 4z - 7w = 6$	10	CO5	BL3
4.(a)	Let $R^3(R)$ be a Vector Space and $S = \{(1,1,0), (1,0,2), (1,1,1)\}$ be a linear independent subset of R^3. Express a vector $(1,2,-1)$ as a Linear combination of the elements of S.	5	CO4	BL4
4.(b)	Let $W = \{(x,y,z,w) \in R^4 \mid x = 2y,\ z = 3w - y\}$ be a subspace of a vector space $R^4(R)$. Find the basis and dimension of W.	5	CO4	BL5
5.	Find the dimension of the row space of $A = \begin{bmatrix} 2 & 1 & 0 & 4 & 3 \\ 1 & 0 & 5 & 2 & 1 \\ 3 & 2 & 4 & 7 & 0 \\ 2 & 6 & 1 & 0 & 1 \end{bmatrix}$	10	CO4	BL2

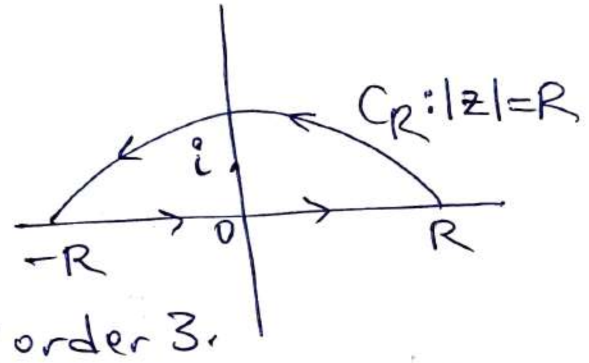
FS(23-24) - BMAT201L - C1+TC1+TCC1

CAT2 - KEY:

1. consider $\int_C f(z)\,dz = \int_C \frac{z^2}{(1+z^2)^3}\,dz$, where

C is the closed contour consisting the upper half of the circle $C_R : |z| = R$ and the ~~red~~ real line from $-R$ to R.

Here $f(z) = \frac{z^2}{(z^2+1)^3}$.



clearly,

$z = i$ and $z = -i$

are the poles of $f(z)$ of order 3.

And $z = i$ is inside of C and $z = -i$ is outside of C.

Now, $[\text{Res.} f(z)]_{z=i} = \frac{1}{2!} \lim_{z \to i} \frac{d^2}{dz^2} (z-i)^3 f(z)$

$$= \frac{1}{2} \lim_{z \to i} \frac{d^2}{dz^2} \frac{z^2}{(z+i)^3}$$

$$= \frac{1}{2}\left(\frac{4}{32i}\right) = \frac{1}{16i}$$

Therefore, By Cauchy's Residue theorem, we have

$$\int_C f(z)\,dz = 2\pi i \left(\frac{1}{16i}\right)$$

$$\Rightarrow \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = \frac{\pi}{8}$$

Taking $R \to \infty$, we get

$$\int_{-\infty}^{\infty} f(x)\,dx + 0 = \frac{\pi}{8}$$

Hence $$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3}\,dx = \frac{\pi}{8}$$

(2) (a)

$$AX = \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 25 \\ -10 \\ 20 \end{bmatrix} = 5 \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$$

$$= \lambda X$$

Therefore, the required eigenvalue is $\lambda = 5$.

(b) Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$.

The characteristic equation of A is $|A - \lambda I| = 0$. i.e., $\begin{vmatrix} 1-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^2 - 4\lambda + 5 = 0$$

By the Cayley-Hamilton theorem, we have

$$A^2 - 4A + 5I = 0$$

or, $A^2 = 4A - 5I$

$$\Rightarrow A^3 = 4A^2 - 5A$$

$$\Rightarrow A^3 = 4(4A - 5I) - 5A$$

$$\Rightarrow A^3 = 16A - 5A - 20I$$

$$\Rightarrow A^3 = 11A - 20I$$

Therefore, $a = 11$ and $b = -20$.

(3) The given system of equations can be written as

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 3 & -1 & 2 & 2 \\ 1 & 1 & -3 & 1 \\ 2 & -3 & 4 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -4 \\ 10 \\ -5 \\ 6 \end{bmatrix}$$

ie., $AX = B$

Now,

$$\begin{bmatrix} A & B \end{bmatrix} = \left[\begin{array}{cccc:c} 1 & 2 & -1 & -1 & -4 \\ 3 & -1 & 2 & 2 & 10 \\ 1 & 1 & -3 & 1 & -5 \\ 2 & -3 & 4 & -7 & 6 \end{array}\right]$$

can be reduced as $\left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$

using Gauss Jordan method.

$\therefore x = 1$, $y = -1$; $z = 2$ and $w = 1$.

4 a For any $(x, y, z) \in \mathbb{R}^3$,

$$(x, y, z) = a(1,1,0) + b(1,0,2) + c(1,1,1).$$

i.e., $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y-x \\ z \end{bmatrix}$$

$$\xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y-x \\ z+2y-2x \end{bmatrix}$$

which implies that

$$a = 2x - y - z\ ; \quad b = x - y \text{ and } c = z + 2y - 2x$$

$$\therefore (x, y, z) = (2x-y-z)(1,1,0) + (x-y)(1,0,2) + (z+2y-2x)(1,1,1)$$

Hence,

$$(1, 2, -1) = 1(1,1,0) - 1(1,0,2) + 1(1,1,1)$$

(4)(b) For any $(x, y, z, w) \in W$,

$$(x, y, z, w) = (2y, y, 3w - y, w)$$
$$= y(2, 1, -1, 0)$$
$$+ w(0, 0, 3, 1)$$

Therefore, $\{(2,1,-1,0), (0,0,3,1)\}$ forms a basis of W and hence $\dim W = 2$.

(5) $$A = \begin{bmatrix} 2 & 1 & 0 & 4 & 3 \\ 1 & 0 & 5 & 2 & 1 \\ 3 & 2 & 4 & 7 & 0 \\ 2 & 6 & 1 & 0 & 1 \end{bmatrix}$$

By applying row operations, we can reduce A as $\begin{bmatrix} 2 & 1 & 0 & 4 & 3 \\ 0 & -1 & 10 & 0 & -1 \\ 0 & 0 & 9 & 1 & -1 \\ 0 & 0 & 0 & -87 & 194 \end{bmatrix}$

Therefore, the number of non-zero rows is 4 and hence the dimension of the row space of A is $\underline{\underline{4}}$.

Worked examples — Conformal mappings and bilinear transformations

Example 1

Suppose we wish to find a bilinear transformation which maps the circle $|z - i| = 1$ to the circle $|w| = 2$. Since $|w/2| = 1$, the linear transformation $w = f(z) = 2z - 2i$, which magnifies the first circle, and translates its centre, is a suitable choice. (Note that there is no unique choice of bilinear transformation satisfying the given criteria.) Since $f(i) = 0$, f maps the inside of the first circle to the inside of the second.

Suppose now we wish to find a bilinear transformation g which maps the inside of the first circle to the outside of the second circle. Let $g(z) = (\alpha z + \beta)/(\gamma z + \delta)$. We choose $g(i) = \infty$, so that $g(z) = (\alpha z + \beta)/(z - i)$ without the loss of generality. Three points on the first circle are $0, 1 + i$ and $2i$, and $g(0) = i\beta, g(1 + i) = \alpha(1 + i) + \beta$ and $g(2i) = 2\alpha - i\beta$. All three of these points must lie on $|w| = 2$, so the simplest choice is $\alpha = 0$ and $\beta = 2$. Then $g(z) = 2/(z - i)$.

Suppose now we wish to find a bilinear transformation h which maps the circle $|z - i| = 1$ to the real line. Since $0, 1 + i$ and $2i$ lie on the given circle and the given line passes through $0, 1$ and ∞, we simply choose h so that $h(0) = 0, h(1 + i) = 1$ and $h(2i) = \infty$ say. Then $h(z) = z/(iz + 2)$. Note that $h(i) = i$, so h maps the region given by $|z - 1| < 1$ to the upper half-plane. This can be shown formally by letting $z = x + iy$. Then

$$h(z) = \frac{2x - i[x^2 + (y - 1)^2 - 1]}{x^2 + (y - 2)^2}$$

and $|z - i| < 1 \quad \Rightarrow \quad x^2 + (y - 1)^2 < 1$.

Example 2 Find a conformal map of the unit disk $|z| < 1$ onto the right half-plane $\mathrm{Re}\, w > 0$.

Solution

We are naturally led to look for a bilinear transformation that maps the circle $|z| = 1$ onto the imaginary axis. The transformation must therefore have a pole on the circle, according to our earlier remarks. Moreover, the origin $w = 0$ must also lie on the image of the circle. As a first step, let's look at

$$w = f_1(z) = \frac{z + 1}{z - 1}, \qquad (i)$$

which maps 1 to ∞ and -1 to 0.

From the geometric properties of bilinear transformations, we can conclude that (i) maps $|z| = 1$ onto *some* straight line through the origin. To see *which* straight line, we plug in $z = i$ and find that the point

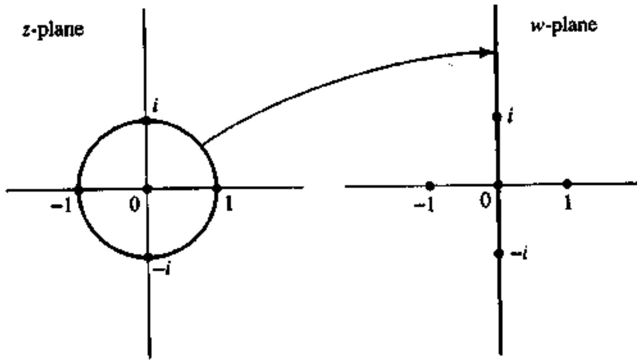
$$w = \frac{i + 1}{i - 1} = -i$$

also lies on the line. Hence the image of the circle under f_1 must be the imaginary axis.

To see which half-plane is the image of the interior of the circle, we check the point $z = 0$. It is mapped by (i) to the point $w = -1$ in the *left* half-plane. This not what we want, but it can be corrected by a final rotation of π, yielding

$$w = f(z) = -\frac{z + 1}{z - 1} = \frac{1 + z}{1 - z} \qquad (ii)$$

as an answer to the problem. (Of course, any subsequent vertical translation or magnification can be permitted.)



Example 3 Find the image of the *interior* of the circle $C : |z-2| = 2$ under the bilinear transformation

$$w = f(z) = \frac{z}{2z-8}.$$

Solution

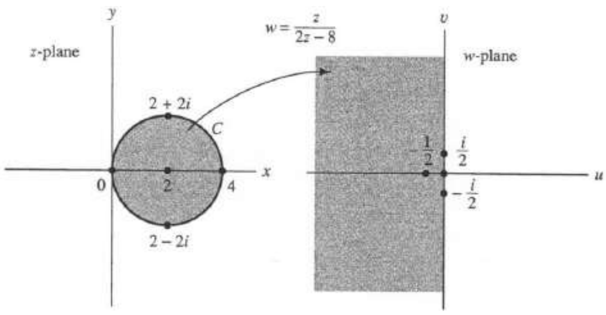
First we find the image of the circle C. Since f has a pole at $z = 4$ and this point lies on C, the image has to be a straight line. To specify this line all we need is to determine two of its finite points. The points $z = 0$ and $z = 2 + 2i$ which lie on C have, as their images,

$$w = f(0) = 0 \quad \text{and} \quad w = f(2+2i) = \frac{2+2i}{2(2+2i)-i} = -\frac{i}{2}.$$

Thus the image of C is the imaginary axis in the w-plane. From connectivity, we know that the interior of C is therefore mapped either onto the right half-plane $\operatorname{Re} w > 0$ or onto the left half-plane $\operatorname{Re} w < 0$. Since $z = 2$ lies inside C and

$$w = f(2) = \frac{2}{4-8} = -\frac{1}{2}$$

lies in the left half-plane, we conclude that the image of the interior of C is the left half-plane.



Example 4 – Find a bilinear transformation that maps the region $D_1 : |z| > 1$ onto the region $D_2 : \operatorname{Re} w < 0$.

Solution

We shall take both D_1 and D_2 to be left regions. This is accomplished for D_1 by choosing any three points on the circle $|z| = 1$ that give it a negative (clockwise) orientation, say

$$z_1 = 1, \quad z_2 = -i, \quad z_3 = -1.$$

Similarly the three points

$$w_1 = 0, \quad w_2 = i, \quad w_3 = \infty$$

on the imaginary axis make D_2 it left region. Hence a solution to the problem is given by the transformation that takes

$$1 \text{ to } 0, \quad -i \text{ to } i, \quad -1 \text{ to } \infty.$$

This we obtain by setting

$$(w, 0, i, \infty) = (z, 1, -i, -1),$$

that is,

$$\frac{w-0}{i-0} = \frac{(z-1)(-i+1)}{(z+1)(-i-1)},$$

which yields

$$w = \frac{(z-1)(1+i)}{(z+1)(-i-1)} = \frac{1-z}{1+z}.$$

Example 5 – Mapping the unit disk onto an infinite horizontal strip

We will describe a sequence of analytic and one-to-one mappings that takes the unit circle onto an infinite horizontal strip. The first linear fractional transformation, $w_1 = -i\phi(z)$, is obtained by multiplying by $-i$ the linear fractional transformation $\phi(z)$, where $\phi(z) = i\dfrac{1-z}{1+z}$ maps the unit disk onto the upper half-plane, and multiplication by $-i$ rotates by the angle $-\dfrac{\pi}{2}$, the effect of $-i\phi(z)$ is to map the unit disk onto the right half-pane.

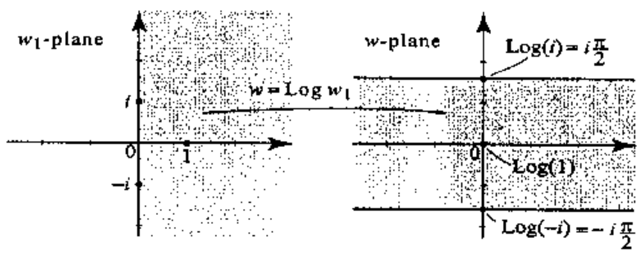


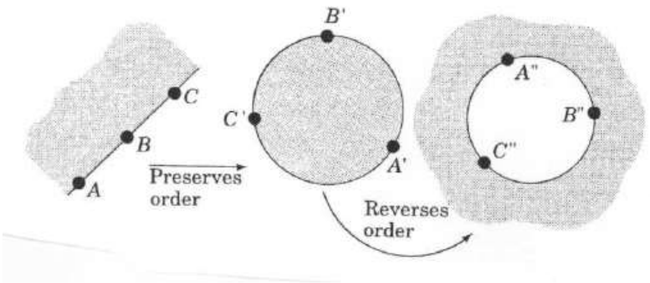
Figure The principal branch of the logarithm, $\operatorname{Log} z$, maps the right half-plane onto an infinite horizontal strip.

In the figure, $\operatorname{Log} w_1 = \ln|w_1| + i\operatorname{Arg} w_1$ is the principal branch of the logarithm. As w_1 varies in the right half-plane, $\operatorname{Arg} w_1$ varies between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$, which explains the location of the horizontal boundary of the infinite strip. The desired mapping is

$$w = f(z) = \operatorname{Log}(-i\phi(z)) = \operatorname{Log}\frac{1-z}{1+z}.$$

Example 6 Find a linear fractional transformation that maps the *interior* of the circle $|z-i|=2$ onto the *exterior* of the circle $|w-1|=3$.

We need only three points on the first circle in clockwise order and three on the second circle in counterclockwise order. Let $z_1=-i, z_2=-2+i, z_3=3i$ and $w_1=4, w_2=1+3i, w_3=-2$. We have



$$\frac{(w-4)(3+3i)}{(w+2)(-3+3i)}=\frac{(z+i)(-2-2i)}{(z-3i)(-2+2i)},$$

which, when solved for w, defines the mapping

$$w=\frac{z-7i}{z-i}.$$

The center of the circle, $|z-i|=2$, is $z=i$. Our transformation maps this point to $w=\infty$, which is clearly in the exterior of the circle. $|w-1|=3$.

Example 7 Find a linear fractional transformation that maps the half-plane defined by $\operatorname{Im}(z)>\operatorname{Re}(z)$ onto the interior of the circle $|w-1|=3$.

We shall regard the specified half-plane as the "interior" of the "circle" through ∞ defined by the line $\operatorname{Im}(z)=\operatorname{Re}(z)$. As noted earlier, it is usually convenient to use ∞ when possible as a point on a line. Then three points in "clockwise" order are $z_1=\infty, z_2=0$, and $z_3=-1-i$. Three points on the circle, $|w-1|=3$, in clockwise order are $w_1=1+3i, w_2=4, w_3=-2$. The unique linear fractional transformation mapping these points in order is defined by

$$w=\frac{\alpha z+\beta}{z+\gamma},$$

where from the images of ∞ and 0, we must have

$$\alpha=1+3i$$
$$\frac{\beta}{\gamma}=4,$$

and from the image of $z_3=-1-i$, we must have

$$\frac{\alpha(-1-i)+\beta}{(-1-i)+\gamma}=-2.$$

By substituting $\beta=4\gamma$ into the preceding equation, we have

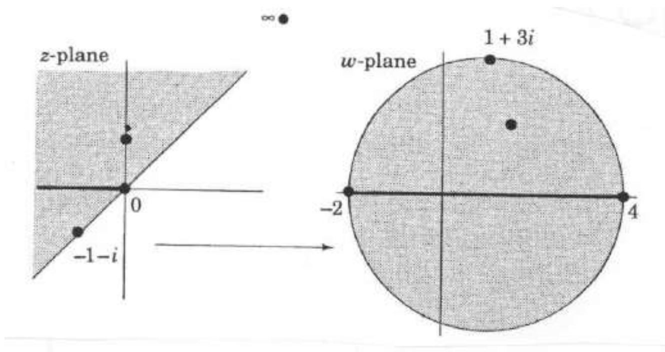
$$\beta=4i \quad \text{and} \quad \gamma=i,$$

so that

$$w = \frac{(1+3i)z+4i}{z+i}$$

is the required linear fractional transformation.

Now $z_0 = i$ is in the half-plane defined by $\operatorname{Im}(z) > \operatorname{Re}(z)$. Its image under the transformation is $w_0 = \frac{5}{2} + \frac{3}{2}i$ and as $w_0 - 1 = \frac{3}{2} + \frac{3}{2}i$ has a modulus of $\frac{3}{2}\sqrt{2} \approx 2.12132 < 3$, w_0 is in the interior of $|w-1| = 3$.



SCHOOL OF ADVANCED SCIENCES

FALL Semester 2023-2024
Continuous Assessment Test – I

Programme Name & Branch : B.Tech.,
Slot: C1+TC1+TCC1
Course Name & code: Complex Variables and Linear Algebra & BMAT201L
Exam Duration: 90 Min. **Maximum Marks: 50**

General instruction(s): **Answer all questions**

Q.No.	Question	Max Marks	CO	BL
1.	Show, by considering the function $f(z)$ defined by $f(z)=\dfrac{2xy(x+iy)}{x^2+y^2}$ for $z\neq 0$ and $f(0)=0$, that the C-R equations are not the sufficient conditions for a function to be analytic.	10M	CO1	BL4
2.	Show that $v(x,y)=x^3-3x^2y+2x+1+y^3-3xy^2$ is a harmonic function and obtain an analytic function for which $v(x,y)$ is the imaginary part. Also find the conjugate harmonic of $v(x,y)$.	10M	CO1	BL5
3.	Find the Bilinear transformation which maps $z_1=-2i,\ z_2=i,\ z_3=\infty$ onto $w_1=0,\ w_2=-3,\ w_3=\dfrac{1}{3}$ respectively. Also find (i) the invariant points of the transformation. (ii) the image of $\lvert z\rvert<1$ in the w-plane.	10M	CO2	BL5
4.	Find the Laurent's series for the function $f(z)=\dfrac{1}{(z+1)(z+2)^2}$ in the following regions: i) $\lvert z-1\rvert<2$ ii)) $2<\lvert z-1\rvert<3$	10M	CO3	BL3
5.	a) Find the location and nature of singularity of $f(z)=\dfrac{1-e^z}{z^3}$. b) Using Cauchy integral formula, evaluate $\displaystyle\int_C \frac{z+1}{z^3-2z^2}\,dz$, where C is a circle $\lvert z-2-i\rvert=2$.	4M+6M	CO3	BL3

BMAT201L CAT Answer key

(1) $u = \dfrac{2x^2y}{x^2+y^2}$; $v = \dfrac{2xy^2}{x^2+y^2}$.

$u_x\big|_{(0,0)} = \lim\limits_{x\to 0} \dfrac{u(x,0)-u(0,0)}{x} = 0.$

$v_y\big|_{(0,0)} = \lim\limits_{y\to 0} \dfrac{u(0,y)-v(0,0)}{y} = 0.$

$u_y\big|_{(0,0)} = \lim\limits_{y\to 0} \dfrac{u(0,y)-u(0,0)}{y} = 0.$

$v_x\big|_{(0,0)} = \lim\limits_{x\to 0} \dfrac{v(x,0)-v(0,0)}{x} = 0.$

$\therefore$ C-R equations are true at (0,0).

consider $f'(z) = \lim\limits_{z\to 0} \dfrac{f(z)-f(0)}{z}$

$= \lim\limits_{\substack{y\to mx\\ x\to 0}} \dfrac{2xy\,\cancel{(x+iy)}}{x^2+y^2\,\cancel{(x+iy)}}$

$= \lim\limits_{x\to 0} \dfrac{2m}{1+m^2}$ depends on m, not unique

$\therefore f'(z)$ does not exist at (0,0)

$\Rightarrow f(z)$ is not analytic at (0,0).

(2) $v_x = 3x^2-6xy+2-3y^2$ || $v_y = -3x^2+3y^2-6xy$

$v_{xx} = 6x-6y$ || $v_{yy} = 6y-6x$.

$\Rightarrow v_{xx}+v_{yy} = 0 \Rightarrow v$ is harmonic

$f'(z) = u_x + i v_x$ $\quad$ $\boxed{\text{Replace } x \text{ by } z \text{ \& } y \text{ by } 0.}$

$= -3z^2 + i(3z^2+2)$

$\Rightarrow f(z) = (i-1)z^3 + 2iz + C \longrightarrow$ Analytic fn.

(C → real constant)

Replace z by $x+iy$, we have

$$u(x,y) = -3x^2y + y^3 - x^3 + 3xy^2 - 2y + c$$

3) $$\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}.$$

(ie) $$'' = \frac{(z_1-z_2)}{(z_3-z_2)} \quad (\text{as } z_4 \to \infty)$$

(ie) $$\frac{10w}{3w-1} = -iz+1.$$

(ie) B.T ⟵ $$\boxed{w = \frac{iz-2}{3iz+4}} \text{ (or) } w = \frac{z+2i}{3z-4i}$$

Put $w=z$ to get invariant pts, $z = \frac{1}{6}\left\{(4i+1) \pm \sqrt{-15+32i}\right\}$

Image of $|z|<1$.

$$z = \frac{-4w-2}{3iw-i}$$

(ie) $$\left|\frac{4w+2}{3w-1}\right| < 1 \Rightarrow u^2+v^2+\frac{22}{7}u+\frac{3}{7} < 0.$$

(ie) The interior part of the unit circle in z-plane is mapped onto the interior of the circle $u^2+v^2+\frac{22}{7}u+\frac{3}{7}=0$ in w-plane.

(3)

4) Put $z-1=u \Rightarrow z=u+1$

$$\therefore \frac{1}{(z+1)(z+2)^2} = \frac{1}{(u+2)(u+3)^2} = \frac{A}{u+2}+\frac{B}{u+3}+\frac{C}{(u+3)^2}$$

$A=1,\ B=C=-1.$

(i) $|u|<2 \Rightarrow \left|\frac{u}{2}\right|<1$ & $\left|\frac{u}{3}\right|<1.$

$\therefore \frac{1}{u+2}-\frac{1}{u+3}-\frac{1}{(u+3)^2}$ takes the form (1)

$$\frac{1}{2}\left(1+\frac{u}{2}\right)^{-1}-\frac{1}{3}\left(1+\frac{u}{3}\right)^{-1}-\frac{1}{9}\left(1+\frac{u}{3}\right)^{-2}$$

$$=\frac{1}{2}\left\{1-\frac{u}{2}+\left(\frac{u}{2}\right)^2-\cdots\right\}-\frac{1}{3}\left\{1-\frac{u}{3}+\left(\frac{u}{3}\right)^2-\cdots\right\}$$
$$-\frac{1}{9}\left\{1-2\left(\frac{u}{3}\right)+3\left(\frac{u}{3}\right)^2-\cdots\right\}$$

Replace u by $(z-1)$, gives the series.

(ii) $2<|u|<3$

$\left|\frac{u}{3}\right|<1$ & $\left|\frac{2}{u}\right|<1.$

$$\therefore (1) \Rightarrow \frac{1}{u}\left(1+\frac{2}{u}\right)^{-1}-\frac{1}{3}\left(1+\frac{u}{3}\right)^{-1}-\frac{1}{9}\left(1+\frac{u}{3}\right)^{-2}$$

$$=\left(\frac{1}{u}-\frac{2}{u^2}+\frac{2^2}{u^3}-\cdots\right)-\frac{1}{3}\left(1-\frac{u}{3}+\left(\frac{u}{3}\right)^2-\cdots\right\}$$
$$-\frac{1}{9}\left\{1-2\left(\frac{u}{3}\right)+3\left(\frac{u}{3}\right)^2-\cdots\right\}$$

replace u by $(z-1)$, gives the required series.

(5) (AM) a) $z=0$ is a singularity.

$$\frac{\cancel{1}-\left\{\cancel{1}+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right\}}{z^3}$$

$$= -\frac{1}{z^2 . 1!}-\frac{1}{z . 2!}-\frac{1}{3!}-\frac{z}{4!}-\frac{z^2}{5!}\cdots$$

(-ve) powers of $(z-0)$ after the 2^{nd} term are missing

$\Rightarrow$ $z=0$ is a pole of order 2.

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5) The poles are given by

$$z^2(z-2)=0. \Rightarrow z=0,2.$$

out of these poles only $z=2$ lies inside C.

$$\therefore \int_C \frac{z+1}{z^3-2z^2}\,dz = \underbrace{\int_C \frac{\left(\frac{z+1}{z^2}\right)}{z-2}\,.dz}_{\text{of the form}}$$

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i . f(a) \quad \text{— (1)}$$

Where

$$f(a) = f(z)\Big|_{z=a} = \frac{z+1}{z^2}\Big|_{z=2} = \frac{3}{4}$$

$$\therefore (1) \Rightarrow 2\pi i\left(3/4\right) = \frac{3}{2}\pi i$$