1. A 20 m ladder leans against a wall. The top slides down at a rate of 4 m/s. How fast is the bottom of the ladder moving when it is 16 m from the wall?

Solution: The top of the ladder is descending at a constant, $v_y = \frac{dy}{dt} = -4\ m/s$

(This is not very realistic, of course - this would normally accelerate due to gravity). At the point where the bottom of the ladder is $x = 16m$ from the wall (as required in the question).

At that point, observe the horizontal velocity, $v_x = \frac{dx}{dt} = 3\ m/s$

2. A stone is dropped into a pond, the ripples forming concentric circles which expand. At what rate is the area of one of these circles increasing when the radius is 4 m and increasing at the rate of 0.5 m/s.

Solution: The area of a circle with radius r is $A = \pi r^2$

Differentiate w.r.t. time, and then substitute known values

$$\frac{dA}{dt} = \frac{d}{dt}\left(\pi r^2\right)$$

$$= 2\pi r \frac{dr}{dt}$$

$$= 2\pi (4)(0.5)$$

$$= 4\pi \quad \simeq 12.56\ m^2/s$$

3. For a certain rectangle the length of one side is always three times the length of the other side.

(i) If the shorter side is decreasing at a rate of 2 inches/minute at what rate is the longer side decreasing?

(ii) At what rate is the enclosed area decreasing when the shorter side is 6 inches long and is decreasing at a rate of 2 inches per minute?

Solution: Let us denote the shorter side with x and the longer side with y.

Given y=3x.

(i) Here $\frac{dx}{dt} = -2$ and hence $\frac{dy}{dt} = -6.$

Therefore, the longer side decreasing at a rate of 6 inches per minute.

(ii) Area $A = x\,y$ and hence $\frac{dA}{dt} = -72.$

Therefore, area decreasing at a rate of 72 inches per minute.

4. A thin sheet of ice is in the form of a circle. If the ice is melting in such a way that the area of the sheet is decreasing at a rate of 0.5 m^2/sec at what rate is the radius decreasing when the area of the sheet is 12 m^2?

Answer: $\frac{dr}{dt} = -0.040717$.

Extrema on an Interval :-

Let f be defined on an interval I containing c

① $f(c)$ is the minimum of f on I
if $f(c) \leq f(x)$ for all x in I

② $f(c)$ is the maximum of f on I
if $f(c) \geq f(x)$ for all x in I.

★ The minimum and maximum of a function on an interval are extreme values (or) extrema, of the function on the interval.

★ The minimum and maximum of a function on an interval are also called the absolute minimum and absolute maximum on the interval

Note : The necessary condition for abs. max (or) abs. min is f must be continuous on the closed interval $[a, b]$.

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Ex ①: Find the absolute maximum and absolute minimum values of f on the given interval

(i) $f(x) = 8x^3 - 3x^2 - 9x + 2$; $[-1, 2]$

Sol. $f'(x) = 24x^2 - 6x - 9$

$\Rightarrow 3(8x^2 - 2x - 3) = 0$

$\Rightarrow (4x-3)(2x+1) = 0$

$\Rightarrow x = \frac{3}{4}$; $x = \frac{-1}{2}$

Tabular form:

x	-1	$-1/2$	$3/4$	2
$f(x)$	0	4.75	−3.0625 (Min)	36 (MAX)

Therefore

Absolute max. of f is 36 when $x = 2$

Absolute min of f is −3.0625 when $x = 3/4$.

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(ii) $f(x) = x - 2\sin x \; ; \; [0, 2\pi]$

Sol: $f'(x) = 1 - 2\cos x$

$\Rightarrow 1 - 2\cos x = 0$

$\Rightarrow \cos x = 1/2$

$\Rightarrow x = \pi/3, 5\pi/3$

Tabular form:

x	0	$\pi/3$	$5\pi/3$	2π
$f(x)$	0	-0.6849	6.9668	6.2832

Therefore

Absolute maximum occurs at $\left(\frac{5\pi}{3}, 6.967\right)$

and Absolute minimum occurs at $\left(\frac{\pi}{3}, -0.685\right)$

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(iii) Find the extrema of $f(x) = 2\sin x - \cos 2x$ on the interval $[0, 2\pi]$.

Sol: $f'(x) = 2\cos x + 2\sin 2x$

$\Rightarrow 2\cos x + 2\sin 2x = 0$

$\Rightarrow \cos x + \sin 2x = 0$

$\Rightarrow \cos x + 2\sin x\cos x = 0$

$\Rightarrow \cos x[1 + 2\sin x] = 0$

$\Rightarrow \cos x = 0$ (or) $1 + 2\sin x = 0$

$\Rightarrow x = \pi/2, 3\pi/2$ (or) $\sin x = -1/2$

$x = 7\pi/6, 11\pi/6$

Tabular form:

x	$\pi/2$	$3\pi/2$	$7\pi/6$	$11\pi/6$
$f(x)$	3	-1	-1.5	-1.5

Maximum occurs at $\left(\frac{\pi}{2}, 3\right)$

Minimum occurs at $\left(\frac{7\pi}{6}, -1.5\right)$

and $\left(\frac{11\pi}{6}, -1.5\right)$

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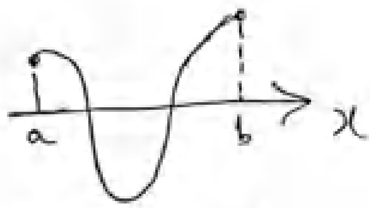
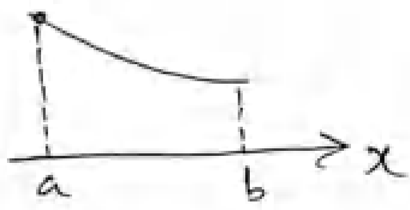
Relative extrema :

1. If there is an open interval containing c on which $f(c)$ is maximum, then $f(c)$ is called a relative maximum of f
2. If there is an open interval containing c on which $f(c)$ is minimum, then $f(c)$ is called a relative minimum of f

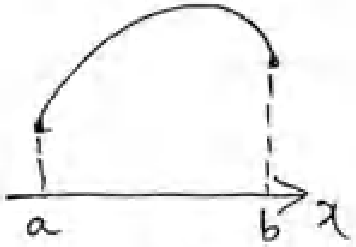
Ex :



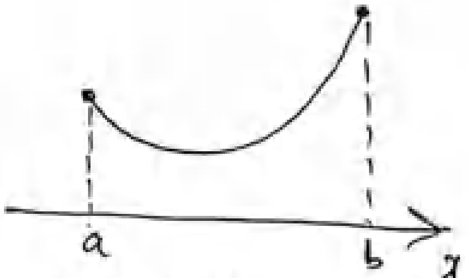
Max & Min at interior points



Max & Min at end points



Max at interior
Min at end points



Min. at interior
Max at end points.

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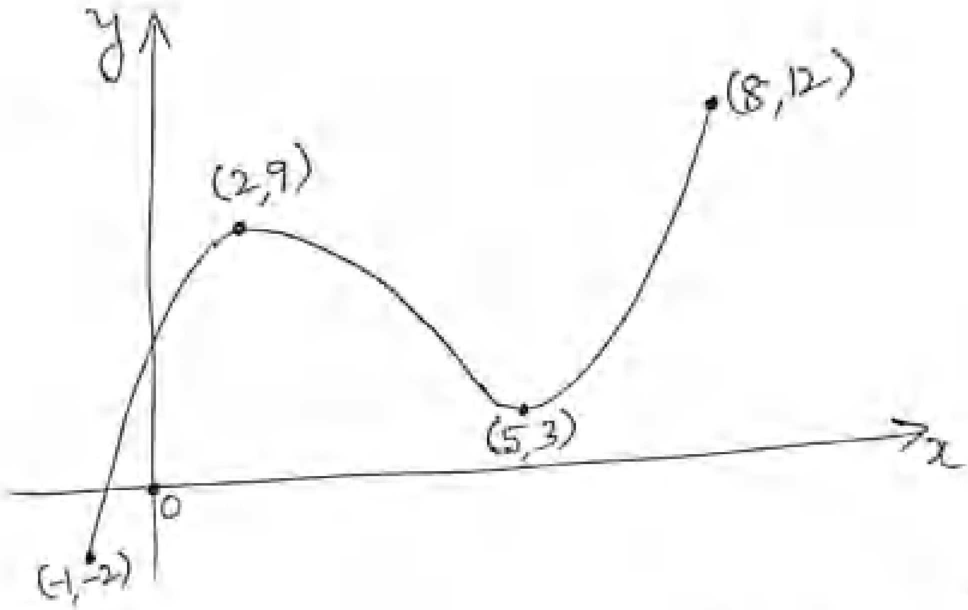
Ex. Let us Consider the following function on the closed interval $[-1, 8]$



From the above figure, we observe that

→ f has an absolute maximum at $(8, 12)$

→ f has an absolute minimum at $(-1, -2)$

→ f has a relative maximum at $(2, 9)$

→ f has a relative minimum at $(5, 3)$

Note : ① Relative max (or) min is also called local max (or) min.

② Existence of max (or) min is same as existence of abs. max (or) abs. min.

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Critical point :

Def.: Let f be defined at c.

If $f'(c)=0$ (or) if f is not differentiable at c

then c is a critical point.

Ex: Find the critical points of the following

(i) $f(x)=5x^3+5x^2+9$

Sol $f'(x)=15x^2+10x$

$\Rightarrow \quad 15x^2+10x=0 \quad \Rightarrow \quad x=0, -2/3$

Critical points : $0, -2/3$

(ii) $f(x) = 8x^{5/4}-8x^{1/4}$

Sol. $f'(x)= 10x^{1/4}-2x^{-3/4}$

$\Rightarrow \quad 10x^{1/4}-2x^{-3/4}=0$

$\Rightarrow \quad 10x^{1/4} = 2x^{-3/4} \quad \Rightarrow \quad x = 1/5.$

$\therefore$ Critical number is $x=1/5$

$f(1/5) = -1.766$

$\therefore$ Critical point is $\left(\frac{1}{5}, -1.766\right)$.

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Rolle's theorem :

Statement : Let $f : [a,b] \to R$ be a function.

If 1. f is continuous in $[a,b]$

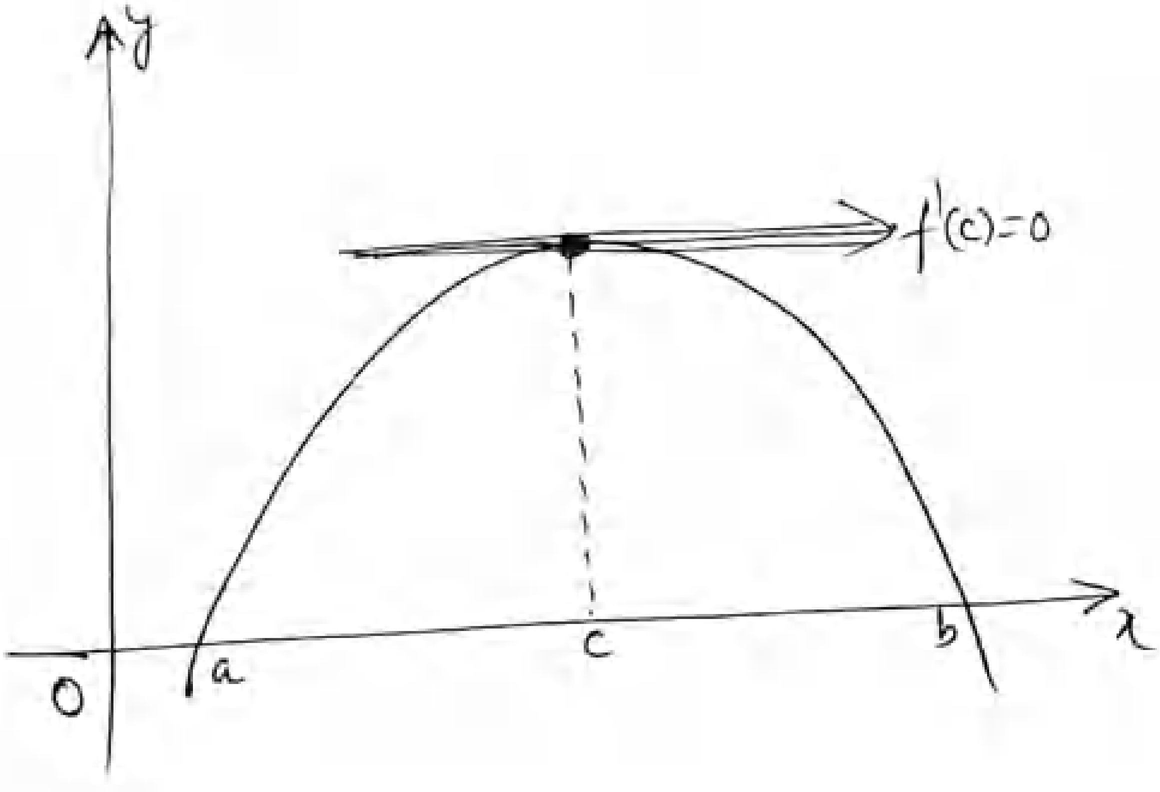
2 f is derivable in (a,b)

and 3. $f(a) = f(b)$,

then there exist some point say c in (a,b) such that

$f'(c) = 0$

The following diagram illustrates this theorem



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Verify Rolle's theorem for the following

① $f(x) = x^2 - x$; $[0, 1]$

② $f(x) = \frac{1}{2+x^2}$; $[-1, 1]$ ([redacted]).

③ $f(x) = 2^x$; $[0, 1]$

④ $f(x) = (x-2)^2 + 4$; $[-2, 2]$

⑤ $f(x) = (x-1)(x-2)(x-3)$; $[1, 3]$

<u>Lagrange's Mean Value theorem</u> :

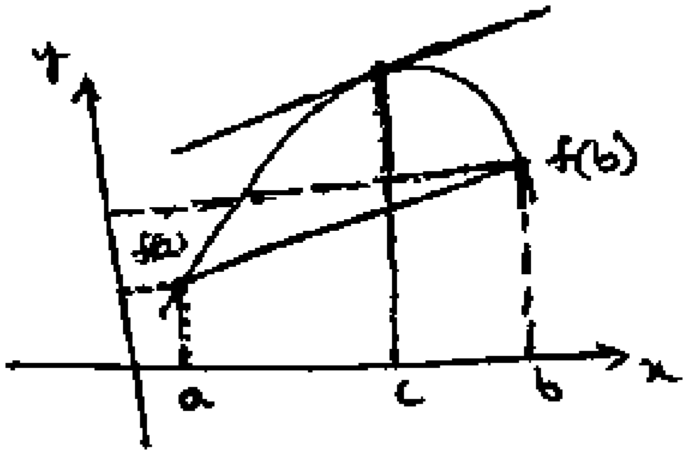
<u>Statement</u> : Let $f : [a, b] \to \mathbb{R}$ be a function.

If (i) f is Continuous on $[a, b]$

(ii) f is derivable on (a, b)

then there exists some point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Check the validity of Lagrange's Mean Value theorem for the following functions. If theorem holds, find a point c

① $f(x) = x^2 - 3x + 5$; $[1, 4]$

② $f(x) = \dfrac{x-1}{x-3}$; $[4, 5]$

③ $f(x) = x^3 + x^2 + 2x - 1$; $[0, 2]$

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Determine the absolute extrema of the following functions

1. $f(x)=8x^3+81x^2-42x-8$ on $[-8,\ 2]$

Answer: Absolute maximum is 1511 at x= −7

and Absolute minimum is −13.3125 at x=1/4

2. $f(x)=3x^4-26x^3+60x^2-11$ on $[1,\ 5]$

Answer: Absolute maximum is 114 at x=5

and Absolute minimum is 26 at x=1.

3. $f(x)=\dfrac{x+4}{2x^2+x+8}$ on $[-10,\ 0]$

Answer: Absolute maximum is 0.5 at x=0

and Absolute minimum is −0.03128 at x=$-4-3\sqrt{2}$.

4. $f(x)=\log_e(x^2+4x+14)$ on $[-4,\ \ 2]$

Answer: Absolute maximum is 3.2581at x=2

and Absolute minimum is 2.3026 at x= −2 .

1. Determine c if $f(x) = x^2 - x$ on $[0,\ 1]$ satisfies Rolle 's Theorem.

Solution:

Clearly f is continuous on $[0,\ 1]$ and differentiable on $(0,\ 1)$.

Also $f(0) = f(1)$.

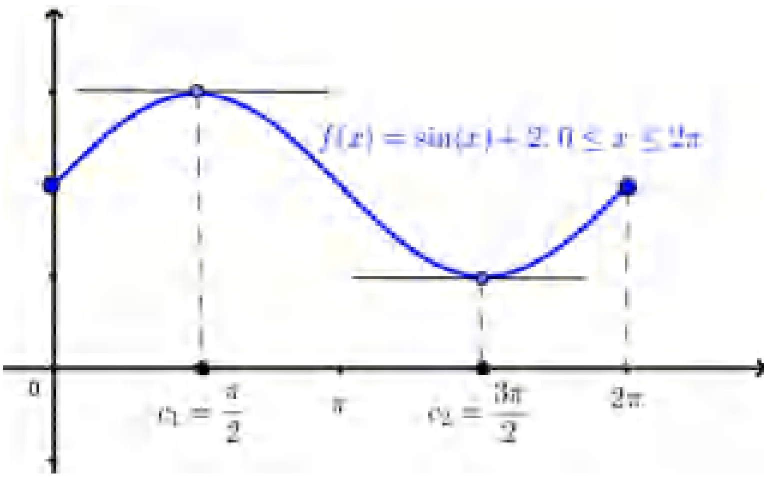
Therefore f satisfies all the conditions of Rolle's theorem.

Hence there exists $c \in (0,\ 1)$ such that $f'(c) = 0$.

Which implies $2c - 1 = 0 \Rightarrow c = \frac{1}{2} \in (0,\ 1)$.

2. Determine all the number(s) c if $f(x) = \sin x + 2$ on $[0,\ 2\pi]$ satisfies Rolle's Theorem.

Answer: $c = \frac{\pi}{2}, \frac{3\pi}{2}$.

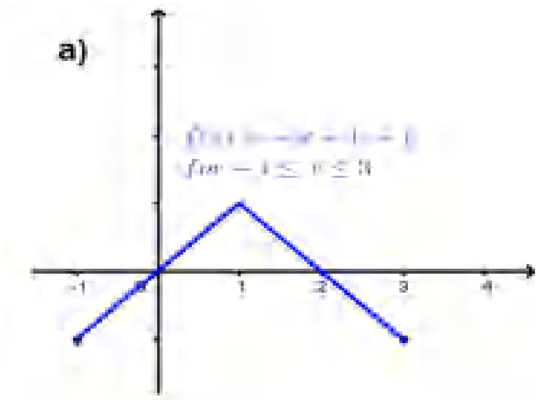


3. Determine all the number(s) c if the following functions satisfies Rolle's Theorem

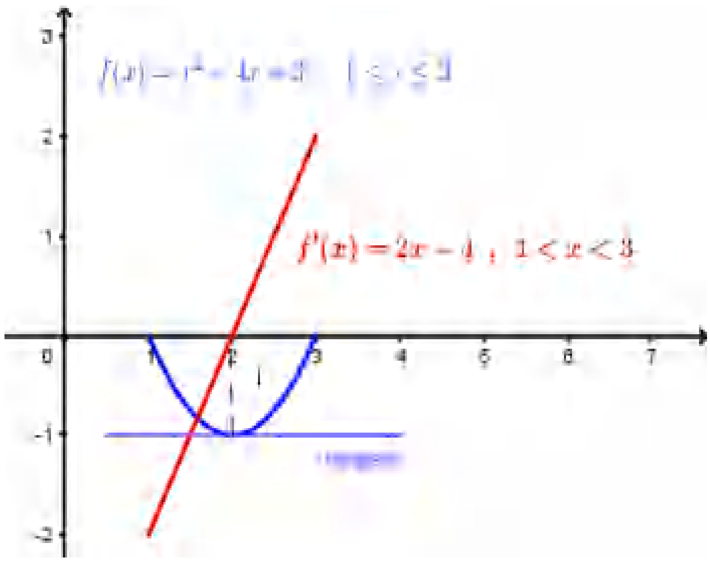
(a) $f(x) = -|x-1| + 1$ on $[-1,\ 3]$

(b) $f(x) = x^2 - 4x + 3$ on $[1,\ 3]$

Hint:



b)



4. Determine all the number(s) c if $A(t) = 8t + e^{-3t}$ on $[-2,\ 3]$ satisfies Lagrange's Mean Value Theorem.

Answer: Cleary $A(t)$ is continuous on $[-2,\ 3]$ and differentiable on $(-2,\ 3)$.

Therefore, $A(t)$ satisfies Lagrange's mean value theorem.

Here, $c = -1.0973$.

5. Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-7,\ 0]$, that $f(-7) = -3$ and that $f'(x) \le 2$. What is the largest possible value for $f(0)$?

Answer: Since $f(x)$ satisfies Lagrange's Mean Value Theorem, there exists $c \in (-7,\ 0)$ such that $f'(c) = \dfrac{f(0) - f(-7)}{0+7} \Rightarrow f(0) = 7f'(c) - 3 \le 7(2) - 3 = 11$.

Therefore, the largest possible value for $f(0)$ is 11.

6. Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-2,\ 5]$, that $f(5) = 14$ and that $f'(x) \le 10$. What is the smallest possible value for $f(-2)$?

1.4: Increasing and Decreasing functions

(i) A function $f(x)$ is increasing over an interval I, if for every $x_1, x_2 \in I$,

$x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$.

Note: If $x_1 \leq x_2 \Rightarrow f(x_1) < f(x_2)$, then $f(x)$ is strictly increasing.

(ii) A function $f(x)$ is decreasing over an interval I, if for every $x_1, x_2 \in I$,

$x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$.

Note: If $x_1 \leq x_2 \Rightarrow f(x_1) > f(x_2)$, then $f(x)$ strictly decreasing.

Using derivatives

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Then

(i) If $f'(x) > 0$ for every $x \in (a, b)$, then $f(x)$ is increasing in the interval (a, b)

(ii) If $f'(x) < 0$ for every $x \in (a, b)$, then $f(x)$ is decreasing on (a, b).

(iii) If $f'(x) = 0$ for every $x \in (a, b)$, then $f(x)$ is constant on (a, b).

Calculation of intervals of increasing (or) decreasing:

Example Problems

① Find the interval of increasing or decreasing of the function $f(x) = x^3 - 3x + 2$.

Sol: Given $f(x) = x^3 - 3x + 2$.

Clearly, $f'(x) = 3x^2 - 3$

Now, $f'(x) = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x = -1, 1$ (critical point)

Sign of $f'(x)$:

$+ + + + + \quad - - - - - - \quad + + + +$

$\quad\quad -1 \quad\quad\quad\quad +1$

It is clear that (i) $f'(x) > 0$ for all $x \in (-\infty, -1)$

(ii) $f'(x) < 0$ for all $x \in (-1, 1)$

and (iii) $f'(x) > 0$ for all $x \in (1, \infty)$.

Therefore, $f(x)$ is increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on $(-1, 1)$.

(2) $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$.

clearly, $f'(x) = 12x^3 - 12x^2 - 24x$

Now, $f'(x) = 0 \Rightarrow x = 0, -1, 2$

(critical points)

sign of $f'(x)$:

So, $f(x)$ is increasing on $(-1, 0) \cup (2, \infty)$ and decreasing on $(-\infty, -1) \cup (0, 2)$

1.5: First Derivative Test

Suppose $f(x)$ is continuous at a critical point x_0.

i) If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then $f(x)$ has local maximum at x_0.

ii) If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0, then $f(x)$ has a local minimum at x_0.

iii) If $f'(x)$ has same sign on an open interval extending left from x_0 and on an open interval extending right from x_0, then $f(x)$ has an inflection point at x_0.

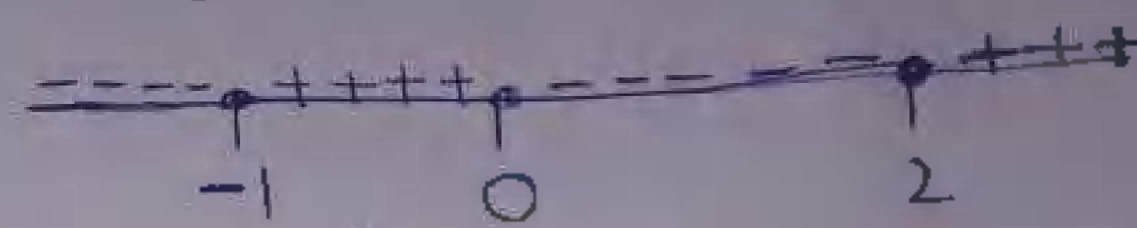
Note: Relative extrema occur where $f'(x)$ changes sign.

Example: ① Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$ on $[-2, 3]$.

Then $f'(x) = 0 \Rightarrow x = -1, 0, 2$

(critical points)

Sign of $f'(x)$



By the First derivative test, f has a relative maximum at $x=0$ and relative minimum at $x=-1$ and $x=2$

② If $f(x)=\sin x+\cos x$ on $[0,2\pi]$, then determine all local extrema for the function.

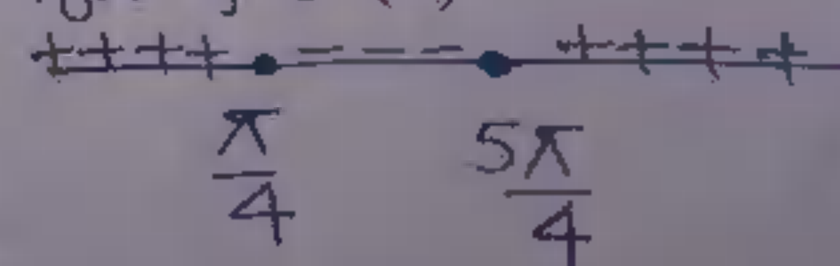
Sol: Suppose $f(x)=\sin x+\cos x$ on $[0,2\pi]$.

Then, $f'(x)=0 \Rightarrow x=\frac{\pi}{4}$ and $\frac{5\pi}{4}$

critical points which are in the interval $[0,2\pi]$

By the First derivative test, f has local minimum at $x=\frac{5\pi}{4}$

Sign of $f'(x)$



and local maximum at $x=\frac{\pi}{4}$.

local maximum is $\sqrt{2}$ and local minimum is $-\sqrt{2}$

(3) A ball is thrown in the air. Its height (meters) at any time t sec is given by

$$h = 3 + 14t - 5t^2.$$

what is its maximum height ?

Sol: Given $h = 3 + 14t - 5t^2$

So, $\frac{dh}{dt} = 14 - 10t$

Now, $\frac{dh}{dt} = 0 \Rightarrow t = 1.4$

Sign of $h'(t)$

$+ + + \quad \bullet \quad - - -$

1.4

Clearly, $h(t)$ has maximum at $t = 1.4$ and the maximum height is 12.8 m.

SECOND DERIVATIVE TEST

Let c be a critical point at which $f'(c)=0$. Suppose $f''(c)$ exists. Then

(i) f has a relative maximum at c if $f''(c)<0$

(ii) f has a relative minimum at c if $f''(c)>0$

and (iii) the test is inconclusive if $f''(c)=0$.

Example Problems

① Use the second derivative test to find the local maximum and local minimum values of the function $f(x)=x^4-2x^2+3$.

Sol: clearly $f'(x)=4x^3-4x$

Now, $f'(x)=0 \Rightarrow 4x(x^2-1)=0$

$\Rightarrow x=0,-1,1$ (critical points)

Here, the second derivative of $f(x)$ is $12x^2-4$

i.e., $f''(x)=12x^2-4$

clearly, (i) $f''(-1)=8>0$, so $f(-1)=2$ is the local minimum value

(ii) $f''(0)=-4<0$, so $f(0)=3$ is the local maximum value.

(iii) $f''(1)=8>0$, so $f(1)=2$ is the local minimum value.

(2) $f(x) = x^3 - 3x^2 + x - 2$

Now,

$f'(x) = 0 \Rightarrow 3x^2 - 6x + 1 = 0$

$\Rightarrow x = \frac{3-\sqrt{6}}{3}, \frac{3+\sqrt{6}}{3}$

clearly, $f''(x) = 6x - 6$

and (i) $f''\left(\frac{3-\sqrt{6}}{3}\right) < 0$, so f has relative maximum at $x = \frac{3-\sqrt{6}}{3}$

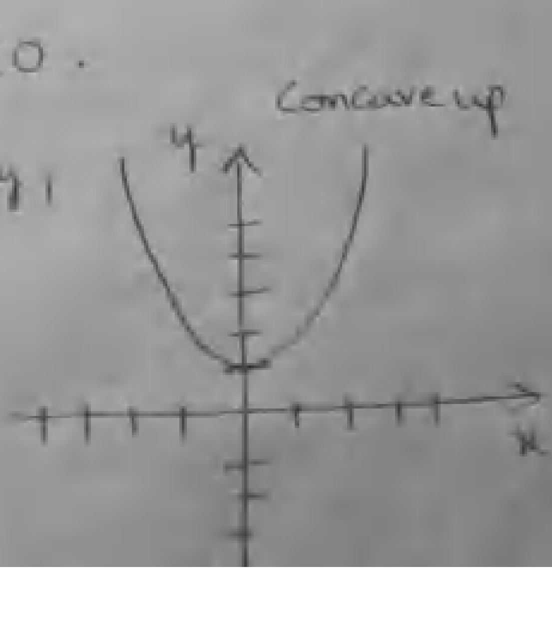
(ii) $f''\left(\frac{3+\sqrt{6}}{3}\right) > 0$, so f has relative minimum at $x = \frac{3+\sqrt{6}}{3}$.

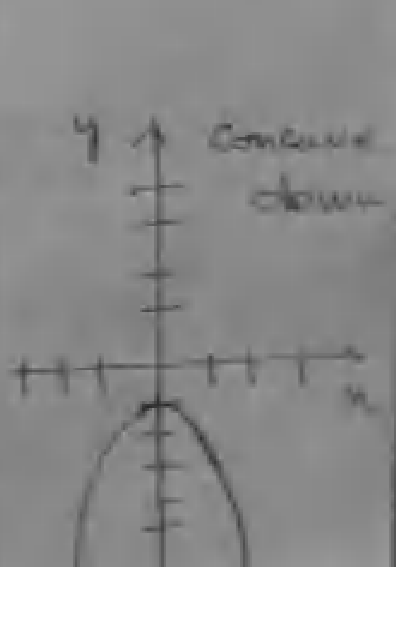
1.6: Concavity

Concavity relates to the rate of change of a function's derivative. A function f is concave up (or upwards) where the ~~[illegible]~~ derivative f' is increasing (or equivalently, f'' is positive). Similarly, f is concave down (or, downwards) where the derivative f' is decreasing (or equivalently, f'' is negative).

OR If the function $f(x)$ is twice differentiable at $x=c$, then the graph of f is concave upwards at $(c, f(c))$ if $f''(c)>0$ and concave downwards at $(c, f(c))$ if $f''(c)<0$.

Graphically,





Example Problems :

① Discuss the concavity of the function $f(x) = x^3 - 3x^2 + x - 2$.

Sol: Given $f(x) = x^3 - 3x^2 + x - 2$.

so, $f'(x) = 3x^2 - 6x + 1$ and $f''(x) = 6x - 6$

Clearly $f''(x) > 0$ for $x > 1$.

and $f''(x) < 0$ for $x < 1$.

Hence, $f(x)$ is concave up on $(1, \infty)$

and concave down on $(-\infty, 1)$.

② Discuss the concavity of $f(x) = 5x^3 + 2x^2 - 3x$.

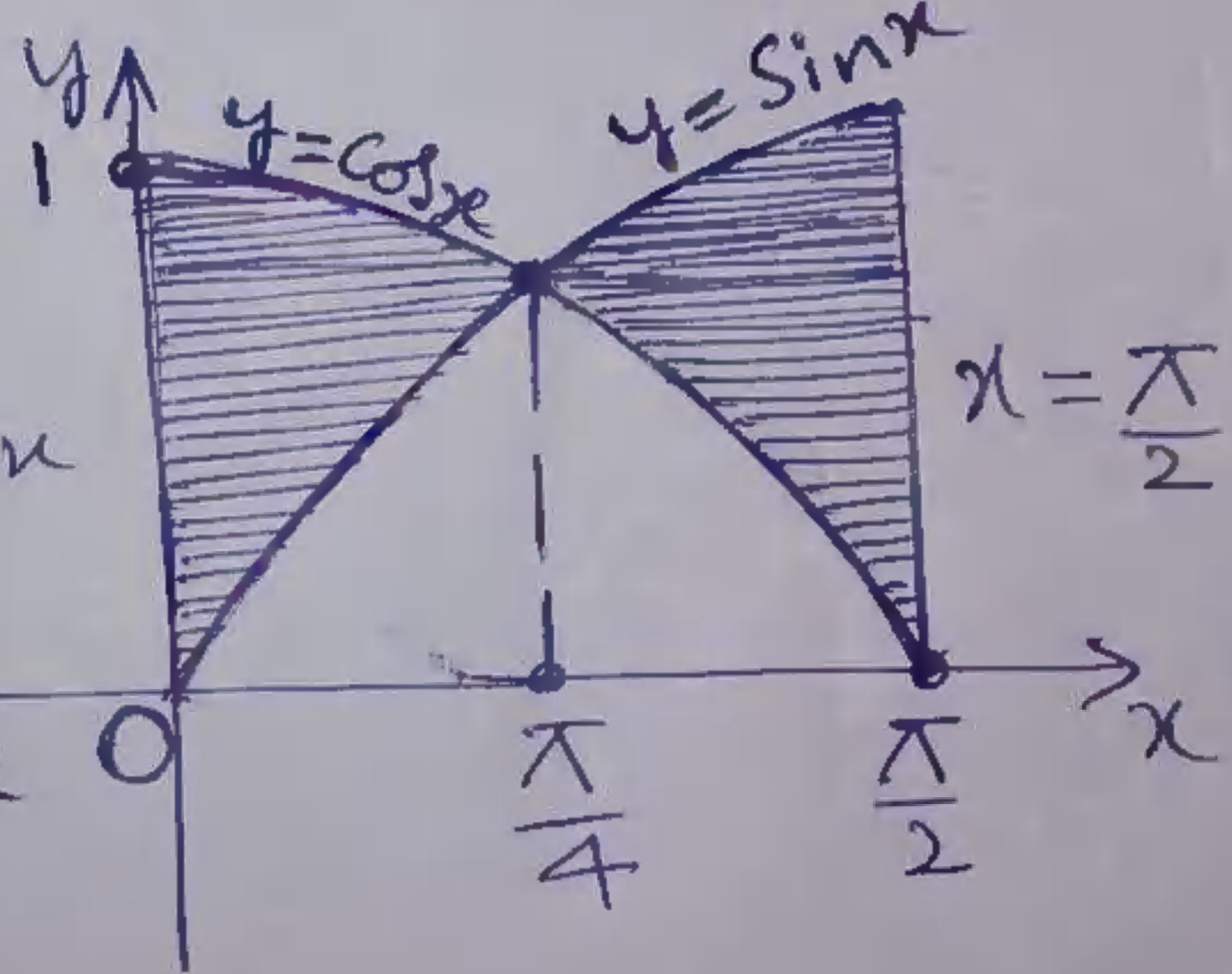
Note: An inflection is where a curve changes from concave upward to concave downward (or, vice versa).

(3). Find the area of the region enclosed by $y = \sin x$, $y = \cos x$, $x = \frac{\pi}{2}$ and the y-axis.

<u>Sol:</u>

Required

$$\text{Area} = \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$



$= 2\sqrt{2} - 2$ sq. units.

(4) Determine the area of the region enclosed by $x = \frac{y^2}{2} - 3$ and $y = x - 1$

<u>Hint:</u> The given curves are intersecting at the points $(-1, -2)$ and $(5, 4)$

Volumes of Solids of revolution:

Volumes using cross-sections:

Definition: The volume of a solid of integrable cross-section area $A(x)$ from $x = a$ to $x = b$ is $V = \int_a^b A(x)\,dx$.

Definition: The volume of a solid of integrable cross-section area $A(y)$ from $y = c$ to $y = d$ is $V = \int_c^d A(y)\,dy$.

To get the cross-sectional area, it is required to cut the object perpen-dicular ($\perp$) to the axis of rotation.

Doing this, the crossection will be either a solid disk if the object is solid or a ring if we have hollowed out a portion of the solid.

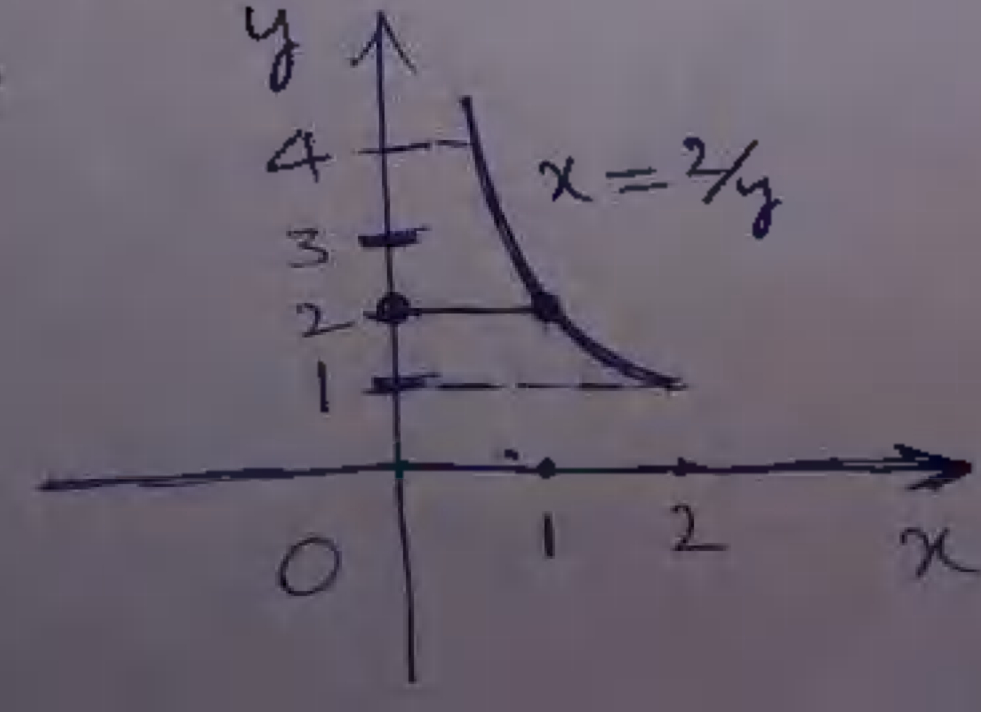
In the case of solid disk, area is $\pi(\text{radius})^2$ where the radius will depend upon the function and the axis of ~~the~~) rotation.

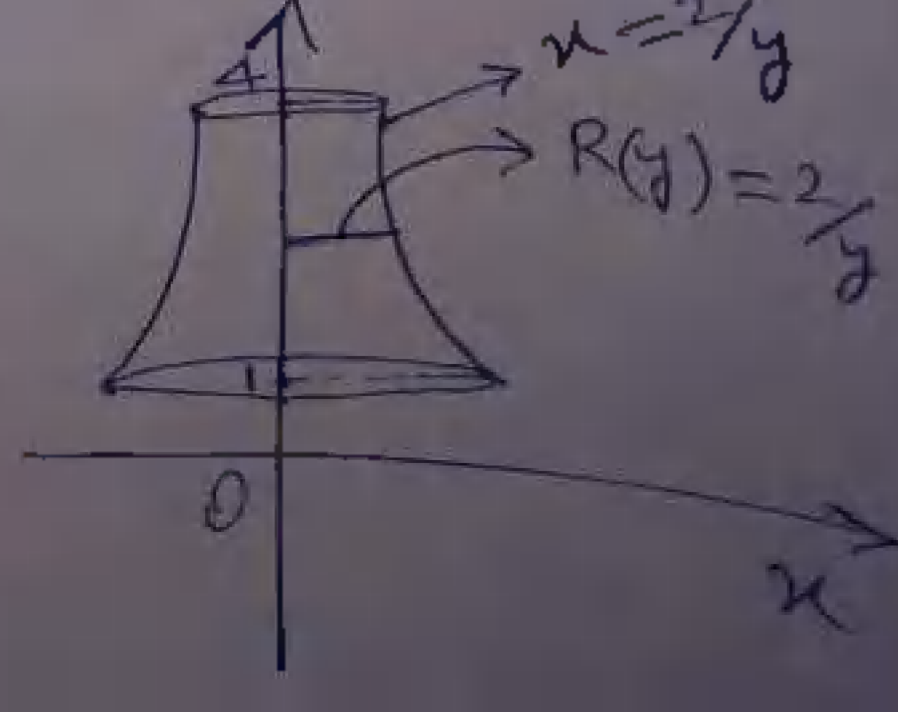
In the case of ring, the area is
$$\pi\left[(\text{outer radius})^2-(\text{Inner radius})^2\right]$$
where again both of the radii depends on the functions given and the axis of rotation.

Example Problems:

① Find the volume of the solid generated by revolving the region between the y-axis and the curve $x=2/y$, $1\le y\le 4$ about the y-axis.

Sol:





Required Volume is $V = \int_1^4 A(y)\, dy$,

where $A(y)$ = the cross-sectional area

$= \pi [R(y)]^2$ (since the cross-section is circle)

$= \pi \left(\frac{2}{y}\right)^2$

$R(y)$ = radius of the circle

= distance between the point $(0, y)$ on y-axis and the point $(2/y, y)$ on $x = 2/y$.

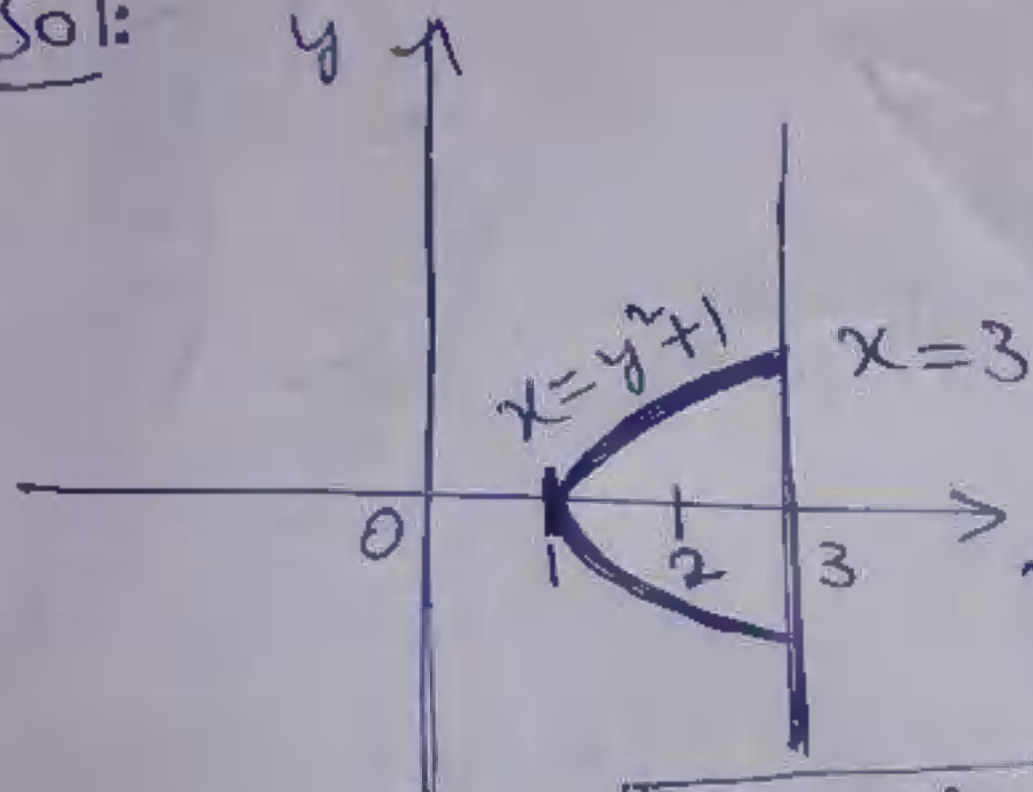
$$= \sqrt{\left(\frac{2}{y} - 0\right)^2 + (y-y)^2} = \frac{2}{y}$$

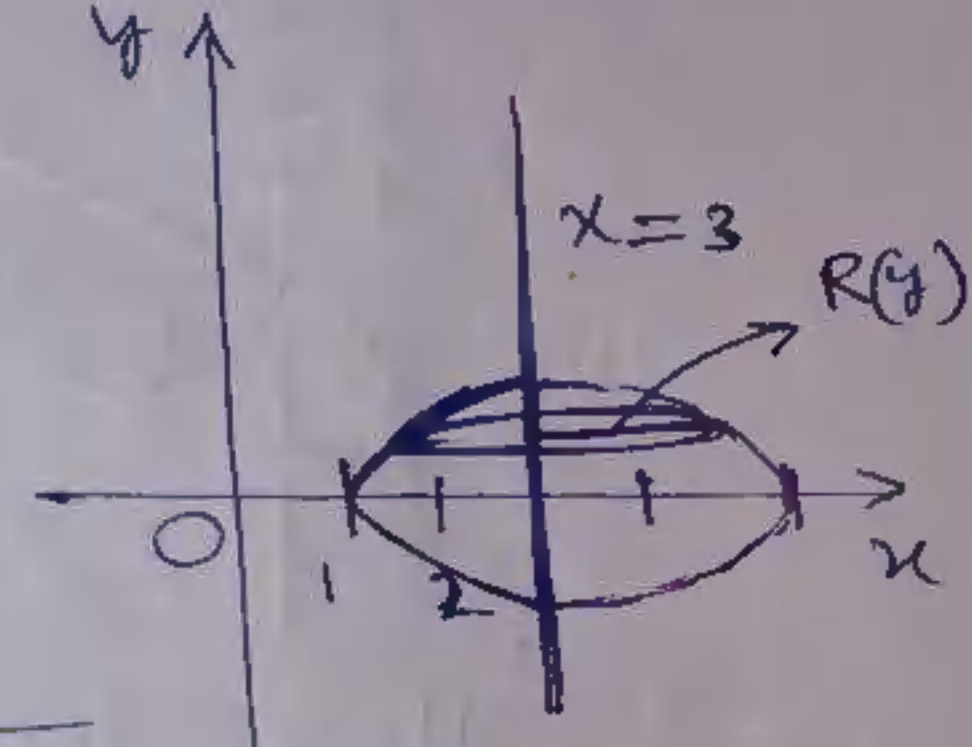
Hence, $\text{Volume} = \pi \int_1^4 \left(\frac{2}{y}\right)^2 dy$

$= 3\pi$ cubic units

(2) Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$

Sol:





Here $R(y) = \sqrt{(y^2+1-3)^2 + (y-y)^2} = y^2 - 2$

given curve $x = y^2 + 1$ and the line $x = 3$ are intersecting at $(3, -\sqrt{2})$ and $(3, \sqrt{2})$

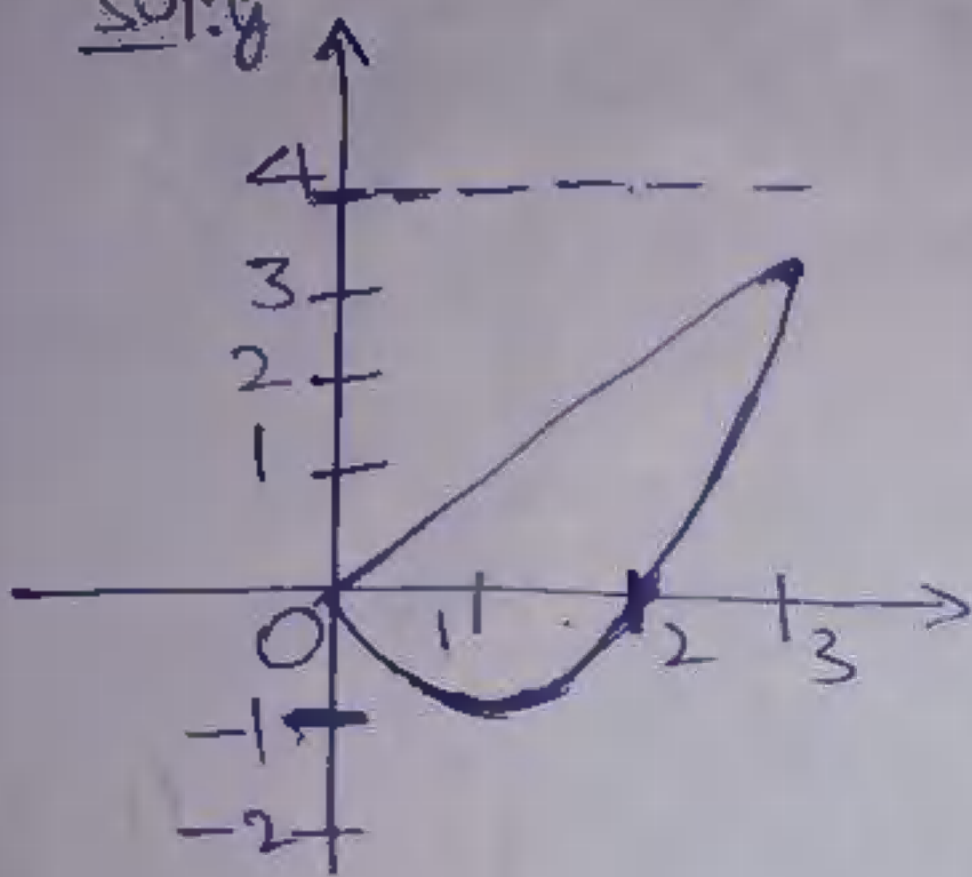
therefore, Volume $= \int_{-\sqrt{2}}^{2} \pi (R(y))^2 \, dy$

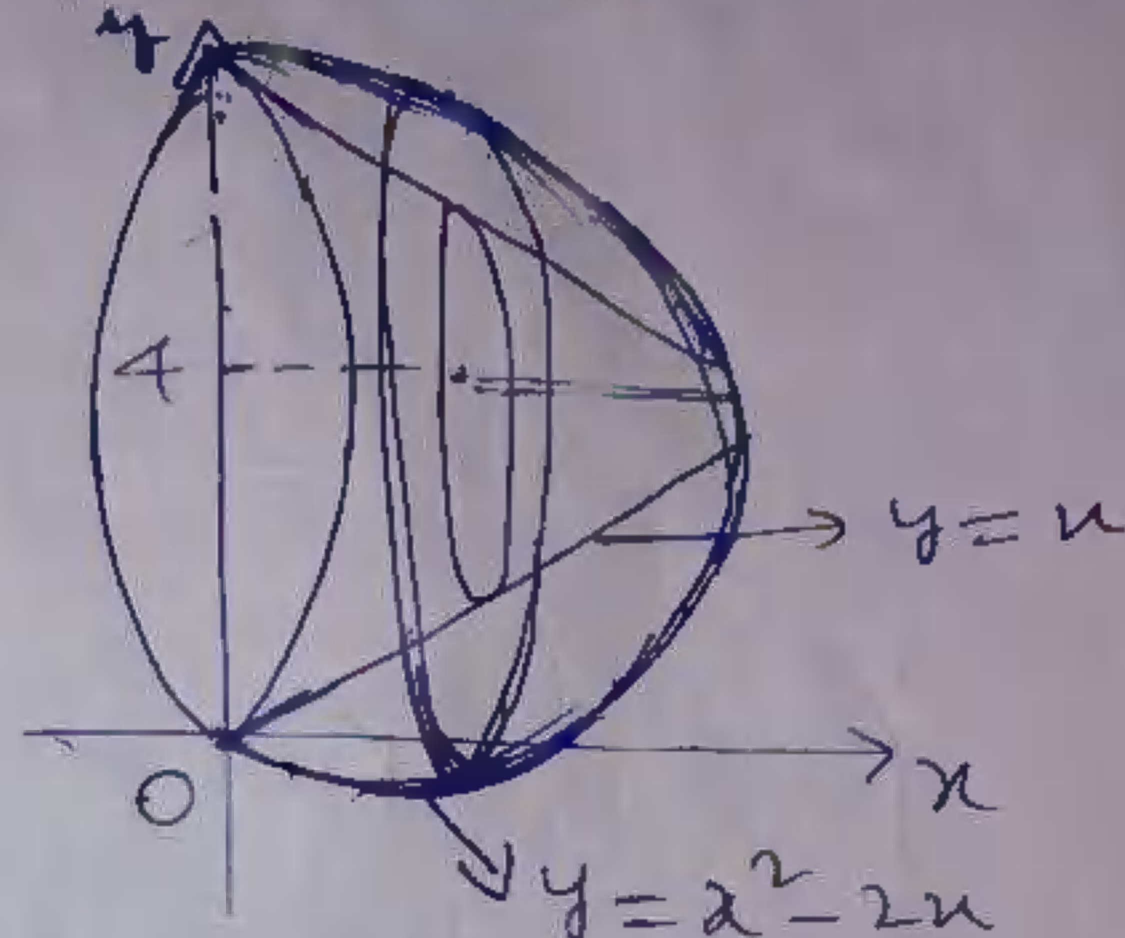
$$= \pi \int_{-\sqrt{2}}^{2} (y^2 - 2)^2 \, dy$$

$$= \frac{64 \pi \sqrt{2}}{15} \text{ cubic units}$$

③ Determine the volume of the solid generated by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

Sol:





Inner radius $r(x) = \sqrt{(x-x)^2 + (4-x)^2}$

$$= 4 - x$$

Outer radius $R(x) = \sqrt{(x-x)^2 + (4-(x^2-2x))^2}$

$$= 4 - (x^2 - 2x)$$

Therefore, Volume $= \int_0^4 \pi \left([4-(x^2-2x)]^2 - [4-x]^2 \right) dx$

$$= \pi \int_0^4 (x^4 - 4x^3 - 5x^2 + 24x)\, dx$$

$$= \frac{153\pi}{6} \text{ cubic units}$$

(4) Determine the volume of the solid obtained by rotating the region bounded by

$y = x^2 - 4x + 5$, $x = 1$, $x = 4$ and the x-axis about the x-axis

<u>Answer:</u> Volume $= 78\pi/5$.

(5) Determine the volume of the solid generated by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$, that lies in the first quadrant about the y-axis.

<u>Answer:</u> $\frac{512\pi}{21}$

Module - 1 Practice Questions

1. For a certain rectangle the length of one side is always three times the length of the other side. If the shorter side is decreasing at a rate of 2 inches per minute, then at what rate is the longer side decreasing.

Answer: 6 inches/minute.

2. Find the critical points of

(i) $f(x) = 8x^3 + 81x^2 - 42x - 8$ Answer: $-7, \frac{1}{4}$

(ii) $f(x) = \frac{x+4}{2x^2+x+8}$ Answer: $\boxed{\frac{-1 \pm \sqrt{63}\, i}{4}}$, $\boxed{-4 \pm 3\sqrt{2}}$

3. A stone is dropped into a pond, the ripples forming concentric circles which expand. At what rate is the area of one of these circles increasing when the radius is 4m and increasing at the rate of 0.5 m/sec.

Answer: 12.56 m^2/sec.

4. Find the local extrema of the function

$f(x) = \frac{x^2}{x^4+16}$. Answer: $f_{max} = f(\pm 2) = \frac{1}{8}$

$f_{min} = f(0) = 0$.

5. obtain the local extrema of $f(x) = x^2 \log_e x$.

Answer: $f_{min} = f\left(\frac{1}{\sqrt{e}}\right) = \frac{-1}{2e}$.

6. Determine all the value(s) of c which satisfy the conclusion of Mean value theorem for $f(x) = 8x + e^{-3}$ on $[-2, 3]$.

Answer: $c = -1.0973 \in (-2, 3)$

7. Verify Rolle's theorem for $f(x) = \sin x + 2$ on $[0, 2\pi]$.

Answer: $c = \frac{\pi}{2}, \frac{3\pi}{2}$

8. Determine the increasing and decreasing interval of the function $f(x) = -x^3 + 3x^2 + 9$

Answer: decreasing in $(-\infty, 0) \cup (2, \infty)$
increasing in $(0, 2)$.

9. Let $f(x) = x^3 - 3x + 1$. Find the inflection points of f and the intervals on which it is concave up or down.

Answer: point of inflection '0'.
concave up on $(0, \infty)$
and concave down on $(-\infty, 0)$.

10. Find the area between the two curves $y = x^2$ and $y = 2x - x^2$

Answer: $\frac{1}{3}$ sq units.

11. Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$ and the x-axis about the x-axis

Answer: Volume $= \frac{78\pi}{5}$ cubic units

12. Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

Answer: volume $= \pi \int_0^3 \left[(-x^2+2x+4)^2 - (4-x)^2\right] dx$

$= \frac{153\pi}{5}$.

Module - 3

Multivariable Calculus

1. Limits and Continuity of a function of two variables

Def: Let f be a function of two variables x and y. The limit of $f(x,y)$ as (x,y) approaches (a,b) is l, written

$\lim\limits_{(x,y)\to(a,b)} f(x,y) = l$, if $\lim\limits_{x\to a}\{\lim\limits_{y\to b} f(x,y)\}$

and $\lim\limits_{y\to b}\{\lim\limits_{x\to a} f(x,y)\}$ exists and

elqual to l.

i.e., $\lim\limits_{x\to a}\{\lim\limits_{y\to b} f(x,y)\} = l = \lim\limits_{y\to b}\{\lim\limits_{x\to a} f(x,y)\}$

Note: At the origin $(0,0)$, we have to verify the limit along the path $y=mx^n$ also if both the limits are equal

Examples

(i) Evaluate $\lim\limits_{(x,y)\to(1,2)} \left(\frac{2x^2y}{x^2+y^2+1}\right)$.

Sol: $\lim\limits_{x\to 1}\left\{\lim\limits_{y\to 2}\left(\frac{2x^2y}{x^2+y^2+1}\right)\right\} = \lim\limits_{x\to 1} \frac{4x^2}{x^2+5}$

$= \frac{2}{3}$

and $\underset{y \to 2}{Lt} \left\{ \underset{x \to 1}{Lt} \left(\frac{2x^2 y}{x^2+y^2+1} \right) \right\} = \underset{y \to 2}{Lt} \frac{2y}{y^2+2} \}$

$= \frac{2}{3}$

Therefore, both $\underset{x \to 1}{Lt} \left\{ \underset{y \to 2}{Lt} \left(\frac{2x^2 y}{x^2+y^2+1} \right) \right\}$ and

$\underset{y \to 2}{Lt} \left\{ \underset{x \to 1}{Lt} \left(\frac{2x^2 y}{x^2+y^2+1} \right) \right\}$ exists and equal

to $\frac{2}{3}$. Hence $\underset{(x,y) \to (1,2)}{Lt} \left(\frac{2x^2 y}{x^2+y^2+1} \right)$

exists and equal to $\frac{2}{3}$

i.e., $\underset{(x,y) \to (1,2)}{Lt} \left(\frac{2x^2 y}{x^2+y^2+1} \right) = \frac{2}{3}$

(ii) Evaluate $\underset{(x,y) \to (0,0)}{Lt} \left(\frac{xy}{x^2+y^2} \right)$

Sol: $\underset{x \to 0}{Lt} \left\{ \underset{y \to 0}{Lt} \left(\frac{xy}{x^2+y^2} \right) \right\} = \underset{x \to 0}{Lt} \; 0 = 0$

and $\underset{y \to 0}{Lt} \left\{ \underset{x \to 0}{Lt} \left(\frac{xy}{x^2+y^2} \right) \right\} = \underset{y \to 0}{Lt} \; 0 = 0$

Along the path $y = mx$,

$\underset{(x,y) \to (0,0)}{Lt} \left(\frac{xy}{x^2+y^2} \right) = \underset{x \to 0}{Lt} \left(\frac{x\, mx}{x^2+m^2x^2} \right) = \underset{x \to 0}{Lt} \frac{m}{m^2+1} = \frac{m}{m^2+1}$

which depends upon the value of m.
Hence the limit does not exist.

(iii) Evaluate
$$\underset{(x,y)\to(1,1)}{Lt}\left(\frac{x(y-1)}{y(x-1)}\right).$$

Sol: $\underset{x\to1}{Lt}\left\{\underset{y\to1}{Lt}\left(\frac{x(y-1)}{y(x-1)}\right)\right\} = \underset{x\to1}{Lt}\ 0 = 0$

and $\underset{y\to1}{Lt}\left\{\underset{x\to1}{Lt}\left(\frac{x(y-1)}{y(x-1)}\right)\right\} = \underset{y\to0}{Lt}\ \frac{1}{0} = \infty.$

Therefore, the limit does not exist.

(iv) Evaluate $\underset{(x,y)\to(0,0)}{Lt}\left(\frac{x-y}{2x+y}\right)$

$\boxed{\text{Ans}}$: does not exist

(v) Evaluate $\underset{(x,y)\to(1,1)}{Lt}\left(\frac{(x-1)^2\log_e x}{(x-1)^2+y^2}\right)$

Ans: 0

(vi) Evaluate $\underset{(x,y)\to(0,0)}{Lt}\left(\frac{xy}{\sqrt{2x^2+y^2}}\right).$

Ans: 0

Continuity:

Def: A function $f(x,y)$ is said to be continuous at the point (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Examples:

(i) Show that the function

$$f(x,y) = \begin{cases} x^2+2y \;; & (x,y)\neq(1,2) \\ 0 \;; & (x,y)=(1,2) \end{cases}$$

is discontinuous at $(1,2)$.

Sol: $\lim_{x\to1}\left\{\lim_{y\to2} f(x,y)\right\} = \lim_{x\to1}\left\{\lim_{y\to2}(x^2+2y)\right\}$

$= \lim_{x\to1}(x^2+4) = 5$

and $\lim_{y\to2}\left\{\lim_{x\to1}(x^2+2y)\right\} = \lim_{y\to2}(1+2y) = 5$

Therefore, $\lim_{(x,y)\to(1,2)} f(x,y) = 5$

But $f(1,2) = 0$

So, $\lim_{(x,y)\to(1,2)} f(x,y) \neq f(1,2)$

Hence f is discontinuous at $(1,2)$

(ii) Investigate the continuity of the function $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

at the origin.

Sol: Along $y = mx$,

$$\lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{xy}{x^2+y^2} \right) = \lim_{x \to 0} \left(\frac{x\,mx}{x^2+m^2x^2} \right)$$

$$= \lim_{x \to 0} \left(\frac{m}{m^2+1} \right)$$

$$= \frac{m}{m^2+1}.$$

which depends on the value of m.

Hence $f(x,y)$ is not continuous at $(0,0)$.

Exercise:

1. verify the continuity of the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

at the origin. Ans: f is continuous at $(0,0)$.

2. verify the continuity of the function

$$f(x,y) = \begin{cases} \dfrac{x^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

at the origin Ans: f is discontinuous at $(0,0)$

③ verify the continuity of the function

$$f(x,y) = \begin{cases} \dfrac{x+y}{x^2+y^2+1}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

at the origin.

Ans: It is continuous at $(0,0)$

④ Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$, if it exists

Ans: does not exist.

⑤ Find $\lim\limits_{(x,y)\to(1,2)} \left(\dfrac{x}{\sqrt{2x+y}}\right)$, if it exists.

Ans: $\frac{1}{2}$

⑥ Find $\lim\limits_{(x,y)\to(0,0)} \left(\dfrac{3x^2y}{x^2+y^2}\right)$, if it exists.

Ans: 0

3.2: Partial Differentiation

Def: Let $z = f(x, y)$. Keeping y constant and varying only x, the partial derivative of z with respect to x is denoted by $\frac{\partial z}{\partial x}$ or, z_x and is defined as

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

Partial derivative of z with respect to y is denoted by $\frac{\partial z}{\partial y}$ or z_y and defined as

$$\frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

Note: (i) $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx}$

(ii) $\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

(iii) $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$

Examples:

1. If $u(x,y) = \log_e\left(\frac{x^2+y^2}{xy}\right)$, then verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Sol: Suppose $u(x,y) = \log_e\left(\frac{x^2+y^2}{xy}\right)$

or, $u(x,y) = \log(x^2+y^2) - \log x - \log y$.

Then, $$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}(2x) - \frac{1}{x}$$

$$= \frac{2x}{x^2+y^2} - \frac{1}{x}$$

So, $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2} - \frac{1}{x}\right)$$

$$= \frac{-4xy}{(x^2+y^2)^2}.$$

And, $$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} \cdot (2y) - \frac{1}{y}$$

$$= \frac{2y}{x^2+y^2} - \frac{1}{y}.$$

So, $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2} - \frac{1}{y}\right)$$

$$= -\frac{4xy}{(x^2+y^2)^2}$$

Therefore, $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

(2) If $u(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$, then find (i) u_x (ii) u_y.

Sol: (i) $u_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$

$$= \frac{x^2}{x^2+y^2} \cdot \left(-\frac{y}{x^2}\right)$$

(3) If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Sol: Suppose $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

Then

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$+ \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{1}{y} + \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right)$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

and $$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \left(\frac{-x}{y^2}\right) + \frac{x^2}{x^2+y^2} \cdot \left(\frac{1}{x}\right)$$

$$= \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

Therefore,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{x}{\sqrt{y^2-x^2}}-\frac{xy}{x^2+y^2}-\frac{x}{\sqrt{y^2-x^2}}$$
$$+\frac{xy}{x^2+y^2}$$

$$=0$$

(4) If $u=\log(x^3+y^3+z^3-3xyz)$,

then S.T. $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=-\frac{9}{(x+y+z)^2}$

Sol: We have

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$$
$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$
$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) \longrightarrow (*)$$

Now,

$$u_x=\frac{\partial u}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3+y^3+z^3-3xyz}$$

and $$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}$$

So, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$= \frac{3}{x+y+z}.$$

From (*), we have

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= 3\left(\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right)$$

$$= 3\left(\frac{-1}{(x+y+z)^2} - \frac{1}{(x+y+z)^2} - \frac{1}{(x+y+z)^2}\right)$$

$$= \frac{-9}{(x+y+z)^2}$$

5. If $u = \log(x^2+y^2)$, then

find $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

Ans: 0

6. If $u = \log(x^3+y^3-x^2y-xy^2)$,

then S.T. $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}$

$$= \frac{-4}{(x+y)^2}$$

Hint:

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u$$

7. If $r^2 = x^2+y^2+z^2$ and $u = r^m$,

then prove that

$$u_{xx} + u_{yy} + u_{zz} = m(m+1)\, r^{m-2}$$

8. If $u(x,y) = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$,

then find $u_{xx} + u_{yy}$.

Ans: 0

9. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$,

then find $\frac{\partial^2 u}{\partial y \partial x}$.

Ans: $\frac{x^2-y^2}{x^2+y^2}$

10. If $u = e^x(x\cos y - y\sin y)$,

then find $u_{xx} + u_{yy}$.

Ans: 0

3.3: Total Differential, Total derivative and Chain rule:

Def: Let $z = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$.

Then (i) The total differential of z is defined as

$$\boxed{dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy}$$

(ii) The total derivative of z with respect to t (or, the total differential coefficient of z w.r.t. t) is denoted by $\frac{dz}{dt}$ and defined as

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

Let $z = f(x, y)$ where $x = \phi(s, t)$ and $y = \psi(s, t)$, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \longrightarrow (1)$$

and
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} \longrightarrow (2)$$

The equations (1) and (2) are known as chain rule for partial differentiation.

Examples:

1. Find the total differential co-efficient of $x^2 y$ with respect to x when x and y are connected by $x^2 + xy + y^2 = 1$

Sol: let $z = x^2 y$.

Then the total derivative of z w.r.t. x is

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$$

$$= (2xy)\,1 + x^2 \frac{dy}{dx} \longrightarrow (*)$$

we have

$$x^2 + xy + y^2 = 1 \longrightarrow (1)$$

Differentiating (1). w.r.t. x, we get

$$2x + \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow (2x+y) + (2y+x)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(2x+y)}{2y+x}$$

Therefore, from ⊛, we get

$$\frac{dz}{dx} = 2xy + x^2\left(-\frac{(2x+y)}{2y+x}\right)$$

$$= 2xy - x^2\frac{(2x+y)}{(2y+x)}$$

(2). If $u = x\log xy$, where $x^3+y^3+3xy=1$, find $\frac{du}{dx}$.

Sol. Suppose $u = x\log(xy)$.

we have

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= (1+\log xy) + \frac{x}{y}\cdot\frac{dy}{dx} \quad \longrightarrow (*)$$

Now, differentiating $x^3+y^3+3xy=1$ w.r.t. x, we get

$$\frac{dy}{dx} = -\left(\frac{x^2+y}{x+y^2}\right)$$

Therefore, from (*), we get

$$\frac{du}{dx} = (1+\log(xy)) - \frac{x}{y}\left(\frac{x^2+y}{x+y^2}\right)$$

(3) Find $\frac{dy}{dx}$ if $x^y + y^x = c$, constant

Ans: $\frac{dy}{dx} = \frac{yx^{y-1} + y^x.\log y}{x^y.\log x + xy^{x-1}}$

(4) If $z = x^2y + xy^2$, then find the total differential of z.

Sol: We have $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

$$dz = 2xy\,dx + 2xy\,dy$$

(5) If $u = f(r, s, t)$, where

$r = x + y$, $s = y + z$ and $t = z +$

find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$.

Sol: Suppose $u = f(r, s, t)$,

where $r = x + y$, $s = y + z$

and $t = z + x$.

By Chain rule, we have

(i) $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x}$$

$$+ \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(1)$$

$$= \frac{\partial u}{\partial r} + \frac{\partial u}{\partial t}.$$

(ii) $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(0)$$

$$= \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

iii)

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(1)$$

$$= \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}.$$

(6) If $u = f(y-z, z-x, x-y)$, find

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

Hint: Let $r = y-z$, $s = z-x$, $t = x-y$.

Then $u = f(r, s, t)$:

Ans: 0

(7) The radius 'r' of ~~the~~ a right circular cone is decreasing at a rate of 3 cm per minute and the height 'h' is increasing at a rate of 2 cm per minute. When $r = 9$ cm and $h = 6$ cm, find the rate of change of its volume.

Sol: Volume of right circular cone is $V = \frac{\pi r^2 h}{3}$

So, $\frac{dv}{dt} = \frac{\partial v}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial v}{\partial h} \cdot \frac{dh}{dt}$

(by Total derivative v w.r.to t.)

$$= \frac{\pi}{3} \left\{ 2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right\}$$

given $r = 9$ cm, $h = 6$ cm, $\frac{dr}{dt} = -3$ cm/min

and $\frac{dh}{dt} = 2$ cm/min.

Therefore, $\frac{dv}{dt} = \frac{\pi}{3} \{ 108(-3) + 81(2) \}$

$= -54\pi$ cm^3/min.

Hence, volume is decreasing at the rate of 54π cm^3/min.

(8) [illegible]

Ans [illegible]

Hint: [illegible]

Volume [illegible]

(9) If [illegible]

(10) If [illegible]

x [illegible]

Prov [illegible]

sol:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x} e^{u} + \frac{\partial z}{\partial y}(-e^{-u})$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v})$$

Therefore,

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^{u}+e^{-v})\frac{\partial z}{\partial x} - (e^{-u}-e^{v})\frac{\partial z}{\partial y}$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

(11) If $z = f(u,v)$ where $u = e^{x}\cos y$ and $v = e^{x}\sin y$, then prove that $\frac{\partial z}{\partial x} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$

3.4 : Jacobian

Def: If u and v are two differentiable functions of two independent variables x and y. Then the Jacobian of u and v with respect to x and y is denoted by $J\left(\frac{u,v}{x,y}\right)$ or, $\frac{\partial(u,v)}{\partial(x,y)}$ and defined as

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Similarly, we can define

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Note: $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$

Problems:

1. If $u = x^2 + y^2$ and $v = xy$, find $\frac{\partial(u,v)}{\partial(x,y)}$

Sol: Suppose $u = x^2 + y^2$ and $v = xy$.

Then, $u_x = \frac{\partial u}{\partial x} = 2x$

$u_y = \frac{\partial u}{\partial y} = 2y$

$v_x = \frac{\partial v}{\partial x} = y$

$v_y = \frac{\partial v}{\partial y} = x.$

Therefore, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$$= \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix}$$

$$= 2x^2 - 2y^2$$

$$= 2(x^2 - y^2)$$

2. If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(r,\theta)}{\partial(x,y)}$.

Sol: Suppose $x = r\cos\theta$ and $y = r\sin\theta$. $\longrightarrow (1)$

we have

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}.$$

From (1), we have

$$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta) \Rightarrow x^2+y^2 = r^2$$

$$\Rightarrow \boxed{r = \sqrt{x^2+y^2}}$$

and

$$\frac{y}{x} = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = \frac{y}{x}$$

$$\Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$$

Now,

$$r_x = \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$

$$r_y = \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$$

3. If $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$,

find $\dfrac{\partial(x, y, z)}{\partial(r, \theta, z)}$.

Hint: $\dfrac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix}$

Ans: $\boxed{r}$

4. In spherical polar coordinates,

$x = r.\sin\theta.\cos\phi$, $y = r\sin\theta.\sin\phi$

and $z = r\cos\theta$, find $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$

Ans: $\boxed{r^2 \sin\theta}$

3.5: Functional Dependence

If $J\left(\frac{u,v}{x,y}\right)=0$, then we say that u and v are functionally dependent.

Otherwise (i.e., $J\left(\frac{u,v}{x,y}\right)\neq 0$), u and v are functionally independent.

Problems: 1. Verify $u=x(1-y)$ and $v=xy$ are functionally dependent or not.

If u and v are functionally dependent, find the relation between u and v.

Sol: Given $u=x(1-y)$ and $v=xy$

So, $u_x = 1-y$ | $v_x = y$

$u_y = -x$ | $v_y = x$

Therefore,
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Therefore,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ 1+y^2 & 1+x^2 \end{vmatrix}$$

$$= 0$$

Hence u and v are functionally dependent.

Now,

$$u = \tan^{-1}x + \tan^{-1}y$$

$$= \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$= \tan^{-1}v$$

$$\boxed{\text{i.e., } u = \tan^{-1}v}$$

3. Prove that $u = x+y+z$,
$V = x^2+y^2+z^2-xy-yz-zx$
and $w = x^3+y^3+z^3-3xyz$ are functionally dependent. Also find the relation between u, v and w.

Sol: $u_x = 1$, $u_y = 1$, $u_z = 1$

$V_x = 2x-y-z$, $V_y = 2y-x-z$

$V_z = 2z-y-x$

$w_x = 3x^2-3yz$, $w_y = 3y^2-3xz$

$w_z = 3z^2-3xy$

Clearly, $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ V_x & V_y & V_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= 0$$

Therefore, u, v and w are functionally dependent.

And,

$$w = x^3+y^3+z^3-3xyz$$

$$= (x^2+y^2+z^2-xy-yz-xy)\,(x+y+z)$$

$$= vu$$

$$\boxed{\text{i.e, } w = uv}$$

4. If $u = \frac{x}{y}$ and $v = \frac{y}{x}$, verify whether u and v are functionally dependent and if so find the relation between u and v.

Sol: Suppose $u = \frac{x}{y}$ and $v = \frac{y}{x}$

Then $u_x = \frac{1}{y}$, $u_y = -\frac{x}{y^2}$

$v_x = \frac{-y}{x^2}$, $v_y = \frac{1}{x}$

Now, $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{y} & \frac{-x}{y^2} \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{1}{xy} - \frac{1}{xy} = 0$$

Therefore, u and v are functionally dependent. And $u = \frac{x}{y}$

$$= \frac{1}{\left(\frac{y}{x}\right)} = \frac{1}{v}$$

Hence $\boxed{uv = 1}$

5. If $u = x^2+y^2+z^2$, $v = xy+yz+zx$ and $w = x+y+z$, verify whether u, v and w are functionally dependent, and if so find the relation between them. Ans: $\boxed{w^2 = u+2v}$

Therefore, $V = [\log u]^2 + 2[\log u]$

8. If $u = \frac{x}{y-z}$, $v = \frac{y}{z-x}$ and $w = \frac{z}{x-y}$,

find the Jacobian of u, v and w with respect to x, y and z (ie., $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Find the relational between u, v and w if they are functionally dependent.

Taylor's series for functions of two variables:

$$f(x,y) = f(a,b) + \left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + (y-b)^3 f_{yyy}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b)\right]$$
$$+ \cdots$$

This is called the Taylor's series expansion of $f(x,y)$ in powers of $(x-a)$ and $(y-b)$ (or, about $x=a$ and $y=b$)

Note ① Taking $x = a+h$ and $y = b+k$, then

$$f(a+h, b+k) = f(a,b) + \left[h f_x(a,b) + k f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + k^2 f_{yy}(a,b) + 2hk f_{xy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + k^3 f_{yyy}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b)\right]$$
$$+ \cdots$$

ere
$x-a=h$
$y-b=k$

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② Above series is called Maclaurin's series if $a=0$ and $b=0$.

Problems:

① Expand $f(x,y)=e^x.\log(1+y)$ in power of x and y upto terms of third degree.

Sol: Given $f(x,y)=e^x\log(1+y)$

here $(a,b)=(0,0)$ (or, $a=0$ and $b=0$)

Now, $f_x(x,y)=e^x\log(1+y) \Rightarrow f_x(0,0)=0$

$f_y=e^x.\frac{1}{1+y} \Rightarrow f_y(0,0)=1$

$f_{xx}(x,y)=e^x\log(1+y) \Rightarrow f_{xx}(0,0)=0$

$f_{xy}(x,y)=e^x.\frac{1}{1+y} \Rightarrow f_{xy}(0,0)=1$

$f_{yy}(x,y)=-e^x.\frac{1}{(1+y)^2} \Rightarrow f_{yy}(0,0)=-1$

$f_{xxx}(x,y)=e^x\log(1+y) \Rightarrow f_{xxx}(0,0)=0$

$f_{xxy}(x,y)=e^x.\frac{1}{1+y} \Rightarrow f_{xxy}(0,0)=1$

$f_{xyy}(x,y)=-e^x.\frac{1}{(1+y)^2} \Rightarrow f_{xyy}(0,0)=-1$

$f_{yyy}(x,y)=2e^x\frac{1}{(1+y)^3} \Rightarrow f_{yyy}(0,0)=2$

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Therefore, by the Taylor's Series, we have

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)]$$
$$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0) + 2xy f_{xy}(0,0)\right]$$
$$+ \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + y^3 f_{yyy}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0)\right]$$
$$+ \cdots$$
$$= y + xy - \frac{1}{2}y^2 + \frac{1}{2}(x^2y - xy^2) + \frac{1}{3}y^3 + \cdots$$

(2) Expand the (~~fn~~) following functions in powers of x and y as far as terms of third degree.

(i) $f(x,y) = e^y \sin x$ (ii) $f(x,y) = e^x \cos y$
(iii) $f(x,y) = e^y \log(1+x)$ (iv) $f(x,y) = \sin x \cos y$

(3) Expand $f(x,y) = \cos xy$ in powers of $(x-1)$ and $\left(y - \frac{\pi}{2}\right)$.

(4) Expand $f(x,y) = x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$

(5) If $f(x,y) = \tan^{-1} xy$, compute $f(0.9, -1.2)$ approximately using Taylor's series

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ol: Given $f(x,y) =$

Now, $f(a,b) = f(1,-1) = -0.7854$

$f_x(x,y) = \frac{y}{1+x^2y^2} \Rightarrow f_x(1,-1) = -\frac{1}{2} = -0.5$

$$f_{xx}(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{y}{1+x^2y^2}\right)$$

$$= -\frac{2xy^3}{(1+x^2y^2)^2}$$

$\Rightarrow f_{xx}(1,-1) = \frac{2}{4} = 0.5$, $f_y = \frac{x}{1+x^2y^2}$

$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{x}{1+x^2y^2}\right)$ | $f_y(1,-1) = 0.5$

$$= -\frac{2x^3y}{(1+x^2y^2)^2}$$

$\Rightarrow f_{yy}(1,-1) = \frac{2}{4} = 0.5$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{(1+x^2y^2)}\right)$$

$$= \frac{1-x^2y^2}{(1+x^2y^2)^2}$$

$\Rightarrow f_{xy}(1,-1) = 0$

Therefore,

$$f(1+(-0.1), -1+(-0.2))$$

$$= -0.7854 + \left[(-0.1)(-0.5) + (-0.2)(0.5)\right]$$

$$+ \frac{1}{2!}\left[(-0.1)^2(0.5) + (-0.2)^2(0.5) + 2(-0.1)(-0.2)(0)\right]$$

$$+ \cdots$$

$$= -0.8029$$

Maxima and Minima for functions of two Variables :

The necessary conditions for $f(x,y)$ to have a maximum or minimum at (a,b) are $f_x(a,b)=0$ and $f_y(a,b)=0$.

<u>Sufficient conditions</u> :

Suppose $f_x(a,b)=0$ and $f_y(a,b)=0$.

Let $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$ and $t = \frac{\partial^2 f}{\partial y^2}$

Then

(i) $f(a,b)$ is a maximum value of $f(x,y)$ if $rt - s^2 > 0$ and $r < 0$ at (a,b).

(ii) $f(a,b)$ is a minimum value of $f(x,y)$ if $rt - s^2 > 0$ and $r > 0$ at (a,b).

(iii) $f(a,b)$ is not an extreme value if $rt - s^2 < 0$ at (a,b). In this case (a,b) is called a saddle point.

(iv) If $rt - s^2 = 0$ at (a,b), then $f(x,y)$ fails to have extreme value and it needs further investigation.

Note: $f(a,b)$ is said to be a stationary value of $f(x,y)$ if $f_x(a,b)=0$ and $f_y(a,b)=0$.
Thus every extreme value is a stationary value.

(1) A rectangular box open at the top is to have volume of 32 cubic feet. Find the dimensions of the box requiring least material for its construction.

Sol: Let the dimensions of the rectangular box be x ft, y ft and z ft.

Given that Volume = 32 cubic ft.

i.e., $xyz = 32 \longrightarrow (1)$

So, $z = \frac{32}{xy}$. Let S be the surface area of the box. Then $S = xy + 2yz + 2zx \longrightarrow (2)$

(Since box is open at top)

Therefore, from (1) and (2), we have

$$S = xy + \frac{64}{x} + \frac{64}{y}.$$

So, $\frac{\partial S}{\partial x} = y - \frac{64}{x^2}$ and $\frac{\partial S}{\partial y} = x - \frac{64}{y^2}$

Now, $\frac{\partial S}{\partial x} = 0 \Rightarrow y - \frac{64}{x^2} = 0 \Rightarrow y = \frac{64}{x^2} \rightarrow (3)$

and $\frac{\partial S}{\partial y} = 0 \Rightarrow x - \frac{64}{y^2} = 0 \Rightarrow x = \frac{64}{y^2} \rightarrow (4)$

Solving (3) and (4), we get

$x = 4$ and $y = 4$.

Now, $r = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}$, $s = \frac{\partial^2 S}{\partial x \partial y} = 1$ and $t = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$

At $(4,4)$, $rt - s^2 = 3 > 0$ and $r = 2 > 0$

$\therefore$ S has minimum at $x = 4$, $y = 4$

and hence from ①, we get $z=2$. Therefore, the required dimensions are 4ft, 4ft and 2 ft to construct the box with least material.

Problems:

1. Find the extremum of the function
$$f(x,y) = x^2+y^2+6x-12.$$

Sol: Given $f(x,y) = x^2+y^2+6x-12$

so, $\frac{\partial f}{\partial x} = 2x+6$ and $\frac{\partial f}{\partial y} = 2y$

Now, $\frac{\partial f}{\partial x} = 0 \Rightarrow 2x+6=0$
$\Rightarrow x+3=0$

and $\frac{\partial f}{\partial y} = 0 \Rightarrow 2y=0$
$\Rightarrow y=0$

Therefore, $(-3,0)$ is an extremum point.

Next, $r = \frac{\partial^2 f}{\partial x^2} = 2$, $s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$

and $t = \frac{\partial^2 f}{\partial y^2} = 2.$

At the point $(-3,0)$, we have $rt-s^2 = 4>0$
$r>0$ and the given function
and hence
$f(x,y)$ is minimum at $(-3,0)$
and the minimum value is $\boxed{-21}$

(2) Obtain the extremum value of the function $f(x,y) = x^4 + y^4 - 2x^2 - 2y^2 + 4xy$

<u>Sol</u>: Here, $\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y$

and $\frac{\partial f}{\partial y} = 4y^3 - 4y + 4x$

Now, $\frac{\partial f}{\partial x} = 0 \Rightarrow x^3 - x + y = 0 \longrightarrow (1)$

and $\frac{\partial f}{\partial y} = 0 \Rightarrow y^3 - y + x = 0 \longrightarrow (2)$

Adding (1) and (2), we get

$$x^3 + y^3 = 0$$

$$\Rightarrow x = -y$$

From (1), we have

$$x^3 - 2x = 0 \Rightarrow x(x^2 - 2) = 0$$

$$\Rightarrow x = 0,\ x = \pm\sqrt{2}$$

$\ldots - 4$.

$\ldots y^2 - 4$.

At(0,0): $rt - s^2 = \ldots 16 + 16 = 0$

So, we cannot say anything about the value of $f(x, y)$ at $(0,0)$.

At$(\sqrt{2}, -\sqrt{2})$:

$r = 20 > 0$ and $rt - s^2 > 0$

Therefore, $f(x,y)$ is minimum at $(\sqrt{2}, -\sqrt{2})$

At$(-\sqrt{2}, \sqrt{2})$: $r = 20 > 0$

[illegible]

Lagrange's method of Undetermined multipliers :

To find the extremum for $f(x,y,z)$ subject to the condition $\phi(x,y,z)=0$

consider the Lagrangean function

$$F(x,y,z)=f(x,y,z)+\lambda\phi(x,y,z)$$

Now, $\frac{\partial F}{\partial x}=0$; $\frac{\partial F}{\partial y}=0$ and $\frac{\partial F}{\partial z}=0$

solving these three equations, we get points at which f has maxima or minima.

① Find the point on the plane $x+2y+3z=4$ that is closed to the origin.

Sol: Let $P(x,y,z)$ be a point on the ~~origin~~ given plane

Then $OP=\sqrt{x^2+y^2+z^2}$.

let $f=x^2+y^2+z^2 \longrightarrow (1)$

we have to minimize f subject to the condition $\phi(x,y,z)=x+2y+3z-4=0$

$\longrightarrow (2)$

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consider Lagrangean function

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z).$$

i.e., $F = x^2+y^2+z^2+\lambda(x+2y+3z-4)$

For f to be minimum,

$$\frac{\partial F}{\partial x}=0 \;;\; \frac{\partial F}{\partial y}=0 \text{ and } \frac{\partial F}{\partial z}=0$$

Now,

$$\frac{\partial F}{\partial x}=0 \Rightarrow 2x+\lambda=0 \Rightarrow x=\frac{-\lambda}{2}$$

$$\frac{\partial F}{\partial y}=0 \Rightarrow 2y+2\lambda=0 \Rightarrow y=-\lambda$$

$$\frac{\partial F}{\partial z}=0 \Rightarrow 2z+3\lambda=0 \Rightarrow z=-\frac{3\lambda}{2}$$

substitute these values in (1), we get

$$-\frac{\lambda}{2}+2(-\lambda)+3\left(-\frac{3\lambda}{2}\right)=4$$

$$\Rightarrow \lambda=-\frac{4}{7}$$

and hence $x=\frac{2}{7}$, $y=\frac{4}{7}$ and $z=\frac{6}{7}$

Therefore, the point on the given plane closest to the origin is $\left(\frac{2}{7},\frac{4}{7},\frac{6}{7}\right)$.

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Problems

① Find the volume of greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

[Hint: Volume $=(2x)(2y)(2z)=8xyz \rightarrow (1)$

and $\phi=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \rightarrow (2)$]

② Find the three positive numbers whose sum is 100 and whose product is maximum.

③ obtain the maximum value of $\cos A\cdot\cos B\cdot\cos C$ in a plane ΔABC.

[Hint: let $f=\cos A\cdot\cos B\cdot\cos C \rightarrow (1)$

In ΔABC, $A+B+C=\pi$

let $\phi=A+B+C-\pi=0 \rightarrow (2)$]

④ Obtain the shortest distance from origin to the surface $xyz^2=2$.

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Problems:

1. obtain the shortest distance from origin to the surface $xyz^2 = 2$.

sol: The distance d from the origin to any point $P(x, y, z)$ on the surface $xyz^2 = 2$ is given by

$$d^2 = x^2 + y^2 + z^2 = f(x, y, z) \text{ say.}$$

let $\phi(x, y, z) \equiv xyz^2 - 2 = 0 \longrightarrow (*)$

let $F = f + \lambda\phi$.

Then $\frac{\partial F}{\partial x} = 2x + \lambda yz^2,$

$\frac{\partial F}{\partial y} = 2y + \lambda xz^2,$

and $\frac{\partial F}{\partial z} = 2z + \lambda 2xyz$

Now, $\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda yz^2 = 0$

$\Rightarrow \frac{x}{yz^2} = \frac{-\lambda}{2} \longrightarrow (1)$

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + \lambda xz^2 = 0$

$\Rightarrow \frac{y}{xz^2} = \frac{-\lambda}{2} \longrightarrow (2)$

and $\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + 2\lambda xyz = 0$

$$\Rightarrow \frac{1}{xy} = -\lambda \longrightarrow (3)$$

From (1) and (2), we have

$$\frac{x}{yz^2} = \frac{y}{xz^2}$$

$$\Rightarrow x^2 = y^2 \Rightarrow x = y$$

From (2) and (3), we have

$$\frac{y}{xz^2} = \frac{1}{2xy}$$

$$\Rightarrow y = \frac{z}{\sqrt{2}}$$

Threfore, from (*), we get

$$x(x)(2x^2) = 2$$

$$\Rightarrow x = 1$$

and hence $y = 1$ and $z = \sqrt{2}$

So, f has minimum at $(1, 1, \sqrt{2})$

Thus the shortest distance from the origin ~~to~~ to the given surface is

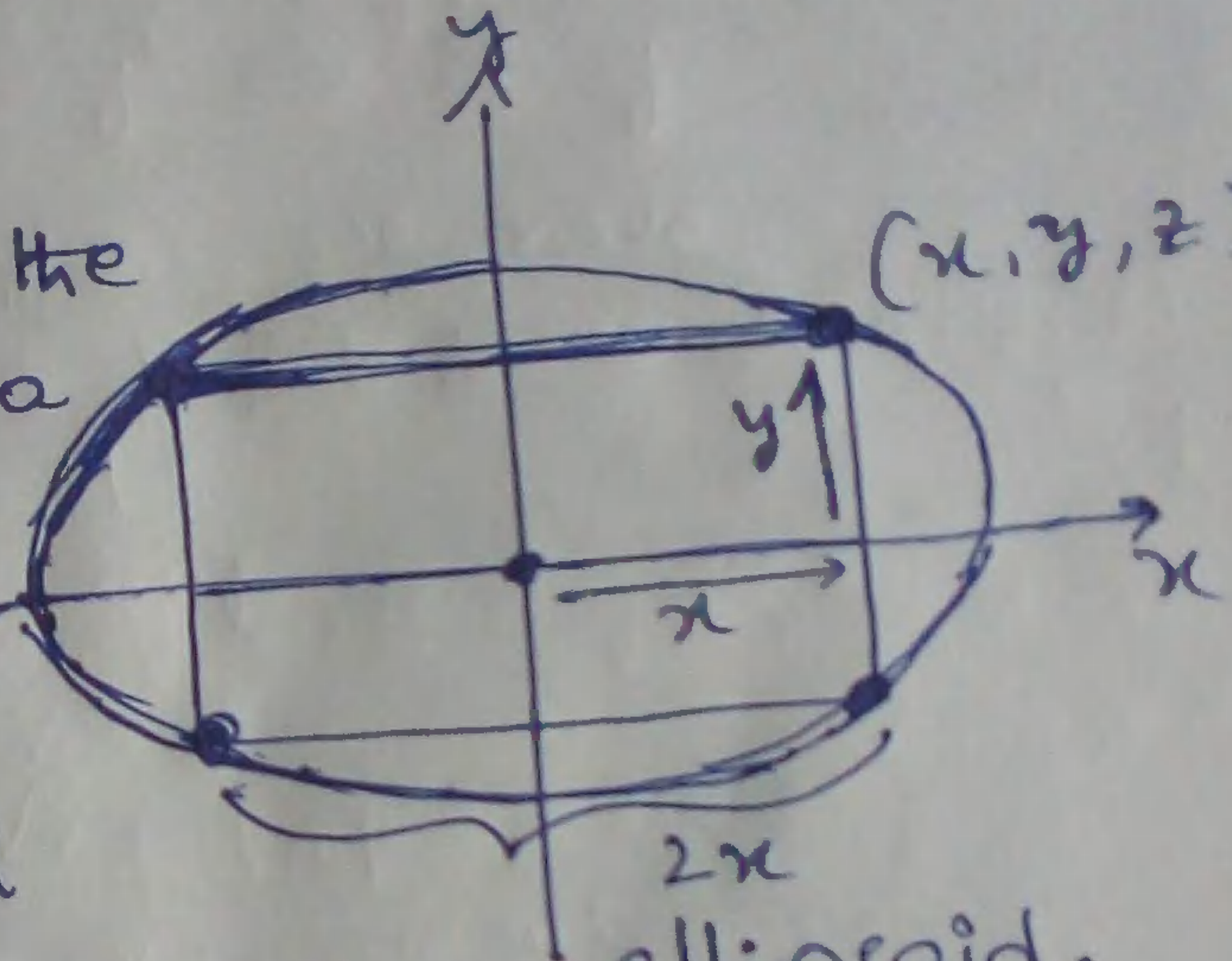
$$d = \sqrt{1^2 + 1^2 + (\sqrt{2})^2} = 2.$$

(2) Obtain the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Sol:

Let (x,y,z) be the coordinates of a corner of a rectangular parallelopiped inscribed in the given ellipsoid.



Then (x,y,z) satisfy the equation of the ellipsoid, so we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \longrightarrow (1)$$

Also, in this case $2x$, $2y$ and $2z$ are the dimensions of the parallelopiped

Therefore, its volume

$$V = 2x.2y.2z = 8xyz = f(x,y,z) \xrightarrow{\text{say}} ②$$

we have to maximize $f(x,y,z)$ subject to the condition ①.

Let $\phi(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \longrightarrow ③$

Let $F = f + \lambda\phi$. Then

$$\frac{\partial F}{\partial x} = 8yz + \lambda\left(\frac{2x}{a^2}\right).$$

$$\frac{\partial F}{\partial y} = 8zx + \lambda\left(\frac{2y}{b^2}\right)$$

and $\frac{\partial F}{\partial z} = 8xy + \lambda\left(\frac{2z}{c^2}\right)$

Now, $\frac{\partial F}{\partial x} = 0 \Rightarrow a^2\left(\frac{yz}{x}\right) = -\frac{\lambda}{4}$

$\frac{\partial F}{\partial y} = 0 \Rightarrow b^2\left(\frac{zx}{y}\right) = \frac{-\lambda}{4}$

$\frac{\partial F}{\partial z} = 0 \Rightarrow c^2\left(\frac{xy}{z}\right) = \frac{-\lambda}{4}$

By solving, we get

$$\lambda = -\frac{3xyz}{2}$$

and hence $x = \frac{a}{\sqrt{3}}$, $y = \frac{b}{\sqrt{3}}$

and $z = \frac{c}{\sqrt{3}}$.

Thus, the maximum volume $= \frac{8abc}{3\sqrt{3}}$.

(3) Find the maximum value of $x^2+y^2+z^2$ under the condition $x+y+z=3$.

Ans: $3a^2$.

Module-5:

Multiple Integrals

1. Evaluation of Double Integrals

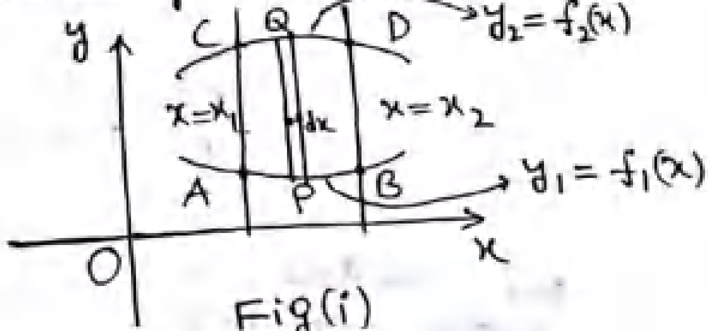
The double integral of $f(x,y)$ over the region A of the xy-plane is written as $\iint_A f(x,y)\,dA$ and expressed as

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dx\,dy.$$

(i) when y_1, y_2 are functions of x and x_1, x_2 are constants; $f(x,y)$ is first integrated w.r.t. y keeping x fixed between limits y_1, y_2 and then the resulting expression is integrated w.r.t. x within the limits x_1, x_2

$$\text{i.e.,}\quad I_1 = \int_{x_1}^{x_2}\left(\int_{y_1}^{y_2} f(x,y)\,dy\right)dx$$

Fig (i) illustrates this process (geometrically illustrated)



Fig(i)

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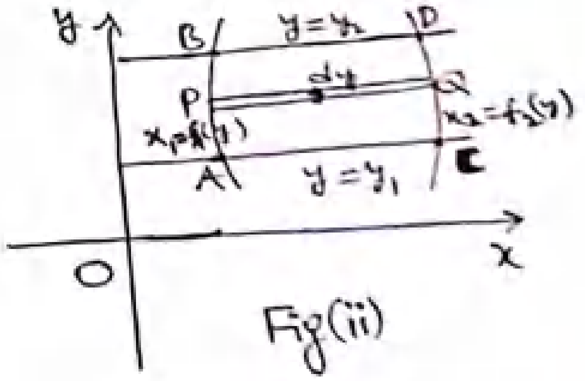
(ii) When x_1, x_2 are functions of y and y_1, y_2 are constants, $f(x,y)$ is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. y between the limits y_1, y_2

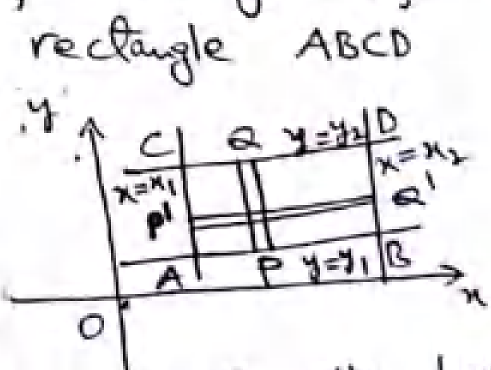
$$\text{i.e.,} \quad I_2 = \int_{y_1}^{y_2} \left(\int_{x_1}^{x_2} f(x,y)\,dx \right) dy$$

Fig(ii) illustrates this process geometrically



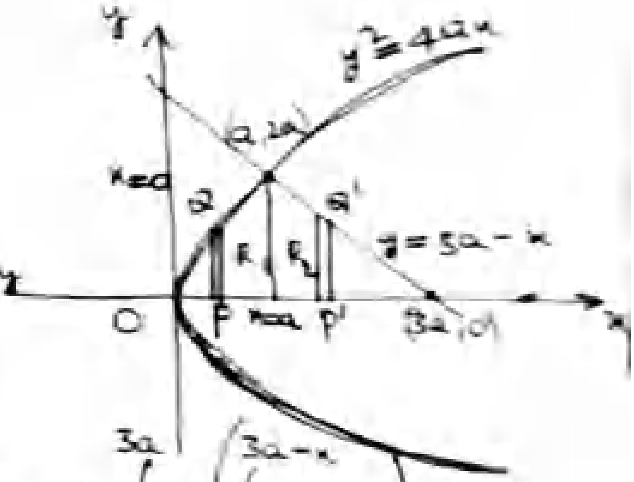
Fig(ii)

(iii) When both pairs of limits are constants, the region of integration is the rectangle ABCD



For constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t. y or vice versa.

1. Using double integral Evaluate the area of the region bounded by the curves $y^2 = 4ax$, $x+y=3a$ and $y=0$



Sol: $$\text{Area} = \iint_R dx\,dy$$

$$= \iint_{R_1} dx\,dy + \iint_{R_2} dx\,dy$$

$$= \int_{x=0}^{a}\left(\int_{y=0}^{2\sqrt{ax}} dy\right)dx + \int_{x=a}^{3a}\left(\int_{y=0}^{3a-x} dy\right)dx$$

$$= \frac{4a^2}{3} + 2a^2 = \frac{10a^2}{3}$$

2. Evaluate $\iint_R xy\,dx\,dy$, where R is the domain bounded by x-axis, $x=2a$ and the curve $x^2=4ay$

3. Evaluate $\iint_R (4xy - y^2)\,dx\,dy$ where **R** is the region (rectangle) bounded by $x=1$, $x=2$, $y=0$ and $y=3$

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Problems

1 Evaluate the following integrals

i) $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$ ii) $\int_0^5 \int_0^{y^2} x(x^2+y^2)\, dx\, dy$

Sol. i) $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx = \int_{x=0}^1 \left[\int_{y=0}^{x^2} e^{y/x}\, dy \right] dx$

$= \int_0^1 \left[x e^{y/x} \right]_0^{x^2} dx$

$= \int_0^1 x(e^x - 1)\, dx$

$= \int_0^1 (xe^x - x)\, dx$

$= \left[e^x(x-1) - \frac{x^2}{2} \right]_0^1$

$= -\frac{1}{2} - (-1) = \frac{1}{2}$

ii) $\int_0^5 \int_0^{y^2} x(x^2+y^2)\, dx\, dy = \int_{y=0}^5 \left[\int_{x=0}^{y^2} (x^3 + xy^2)\, dx \right] dy$

$= \int_0^5 \left[\frac{x^4}{4} + \frac{x^2y^2}{2} \right]_0^{y^2} dy$

$= \int_0^5 \left[\frac{(y^2)^4}{4} + \frac{(y^2)^2 y^2}{2} \right] dx$

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(2)

$$= \int_0^5 \left(\frac{y^8}{4} + \frac{y^6}{2} \right) dy$$

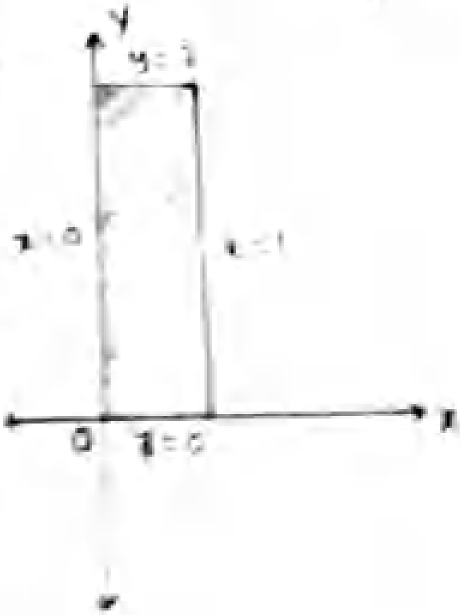
$$= \left(\frac{y^9}{36} + \frac{y^7}{14} \right)_0^5$$

$$= \frac{5^9}{36} + \frac{5^7}{14}$$

$$= \frac{5^7}{4} \left(\frac{25}{9} + \frac{2}{7} \right)$$

2. If A is the area of the rectangular region bounded by the lines $x=0$, $x=1$, $y=0$, $y=2$, evaluate $\int_A (x^2+y^2)\,dA$

Sol. Here, x varies from 0 to 1 and for each x, y varies from 0 to 2.



$$\therefore \int_A (x^2+y^2)\,dA = \int_{x=0}^{1} \left[\int_{y=0}^{2} (x^2+y^2)\,dy \right] dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^2 dx$$

$$= \int_0^1 \left(2x^2 + \frac{8}{3} \right) dx$$

$$= \left(2\frac{x^3}{3} + \frac{8}{3}x \right)_0^1$$

$$= \frac{2}{3} + \frac{8}{3}$$

$$= \frac{10}{3}$$

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3. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$ (3)

Sol. Given Integral $= \int_{x=0}^{1} \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{(1+x^2)+y^2} \right] dx$

$$= \int_0^1 \left[\int_0^{P} \frac{dy}{P^2+y^2} \right] dx \quad \text{where } P = \sqrt{1+x^2}$$

$$= \int_0^1 \left[\frac{1}{P} \operatorname{Tan}^{-1}\left(\frac{y}{P}\right) \right]_0^P dx$$

$$= \int_0^1 \frac{1}{P} \left[\operatorname{Tan}^{-1}(1) - \operatorname{Tan}^{-1}(0) \right] dx$$

$$= \int_0^1 \frac{1}{P} \left[\frac{\pi}{4} - 0 \right] dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}} \quad (\because P = \sqrt{1+x^2}$$

$$= \frac{\pi}{4} \left[\log\left(x + \sqrt{x^2+1}\right) \right]_0^1$$

$$= \frac{\pi}{4} \log(1+\sqrt{2})$$

4. Evaluate $\iint_R y\,dx\,dy$, where R is the region bounded by $y^2 = 4ax$ and $x^2 = 4ay$, $a > 0$.

Sol. Given parabolas $y^2 = 4ax$ and $x^2 = 4ay$ intersect at $(0,0)$ and $(4a, 4a)$. The region bounded by these parabolas is shown in the figure.

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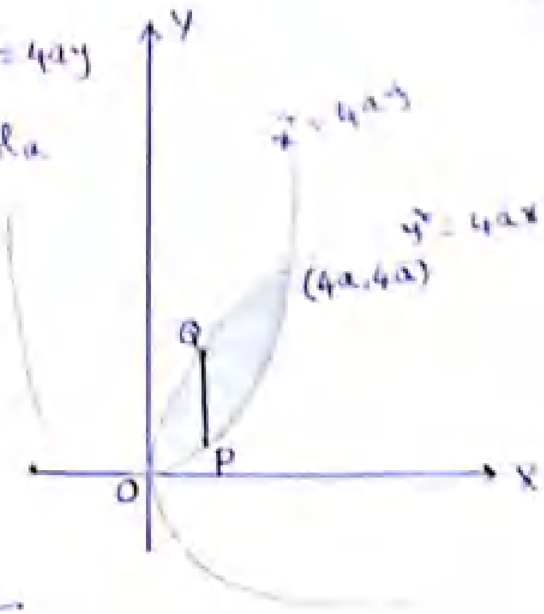
In this region, y increases from a point P on the parabola $x^2 = 4ay$ to a point Q on the parabola $y^2 = 4ax$.



At P, $y = \frac{x^2}{4a}$ and at Q $y = \sqrt{4ax}$.

x increases from 0 to $4a$.

$$\therefore \iint_R y\,dx\,dy = \int_{x=0}^{4a}\left[\int_{y=\frac{x^2}{4a}}^{\sqrt{4ax}} y\,dy\right]dx$$

$$= \int_0^{4a}\left[\frac{y^2}{2}\right]_{\frac{x^2}{4a}}^{\sqrt{4ax}} dx$$

$$= \int_0^{4a} \frac{1}{2}\left[4ax - \frac{x^4}{16a^2}\right]dx$$

$$= \frac{1}{2}\left[2ax^2 - \frac{1}{16a^2}\left(\frac{x^5}{5}\right)\right]_0^{4a}$$

$$= \frac{1}{2}\left[32a^3 - \frac{1}{16a^2}\frac{(4a)^5}{5}\right]$$

$$= \frac{1}{2}\left[32a^3 - \frac{64a^3}{5}\right]$$

$$= \frac{1}{2}\left[\frac{96a^3}{5}\right] = \frac{48}{5}a^3$$

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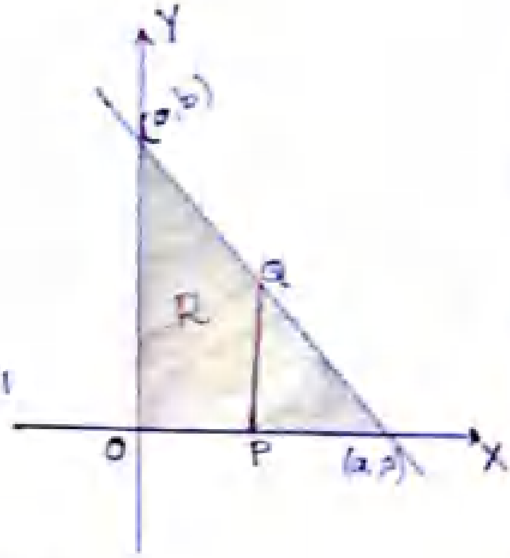
5. Evaluate $\iint xy\,dx\,dy$ over the region in the positive quadrant for which $\frac{x}{a}+\frac{y}{b}\le 1$ (5)

Sol. The shaded region is the region of integration.



In this region, for a fixed x, y varies from 0 to $b\left(1-\frac{x}{a}\right)$ and then x varies from 0 to a. $\left(\because \frac{x}{a}+\frac{y}{b}=1\right)$

$$\iint_R xy\,dx\,dy = \int_{x=0}^{a}\left[\int_{y=0}^{b(1-\frac{x}{a})} y\,dy\right]x\,dx$$

$$= \int_0^a \left[\frac{y^2}{2}\right]_0^{b(1-\frac{x}{a})} x\,dx$$

$$= \frac{1}{2}\int_0^a b^2\left(1-\frac{x}{a}\right)^2 x\,dx$$

$$= \frac{b^2}{2}\int_0^a\left(1-\frac{2x}{a}+\frac{x^2}{a^2}\right)x\,dx$$

$$= \frac{b^2}{2}\int_0^a\left(x-\frac{2x^2}{a}+\frac{x^3}{a^2}\right)dx$$

$$= \frac{b^2}{2}\left[\frac{x^2}{2}-\frac{2x^3}{3a}+\frac{x^4}{4a^2}\right]_0^a$$

$$= \frac{b^2}{2}\left[\frac{a^2}{2}-\frac{2a^3}{3a}+\frac{a^4}{4a^2}\right] = \frac{a^2b^2}{24}$$

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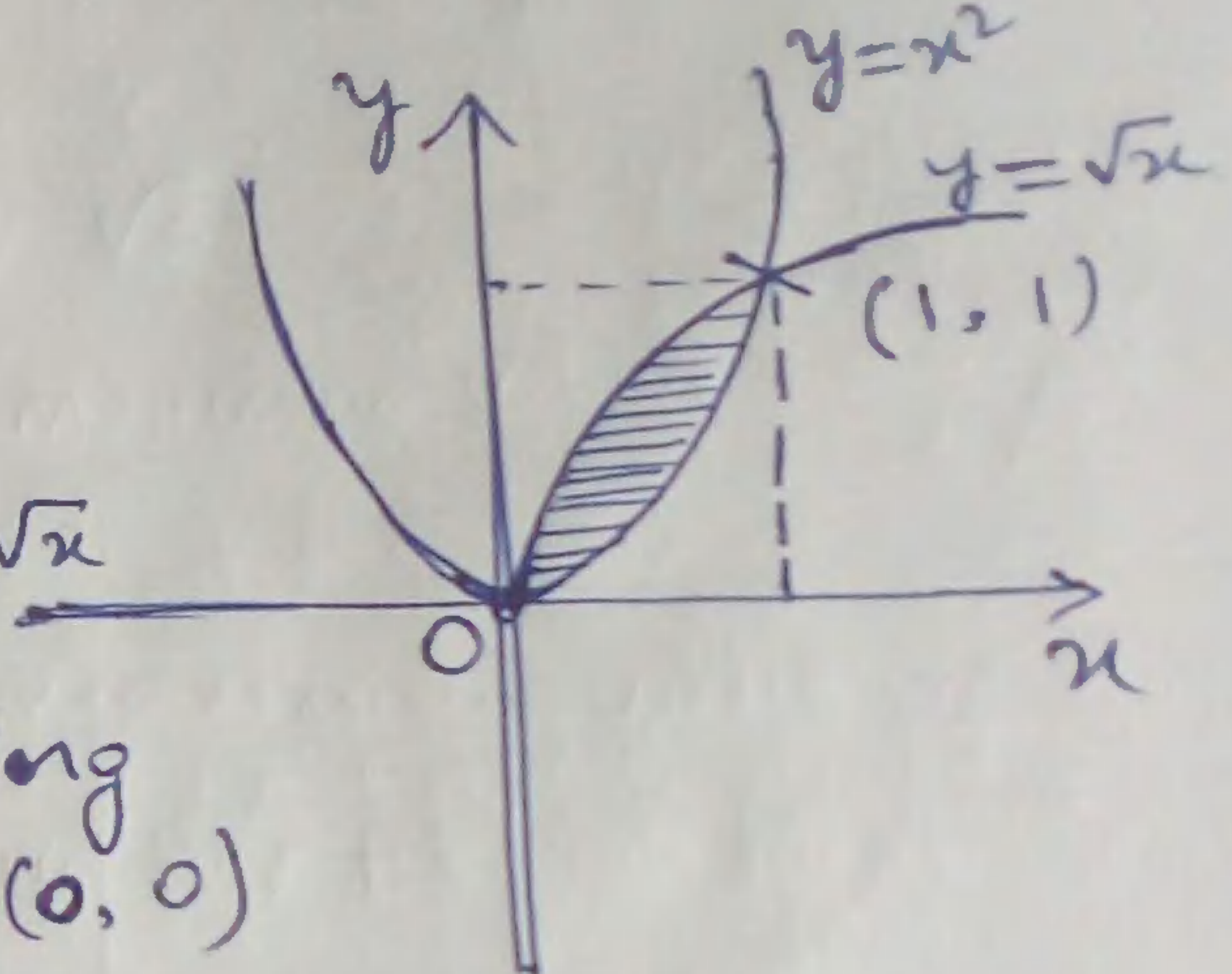
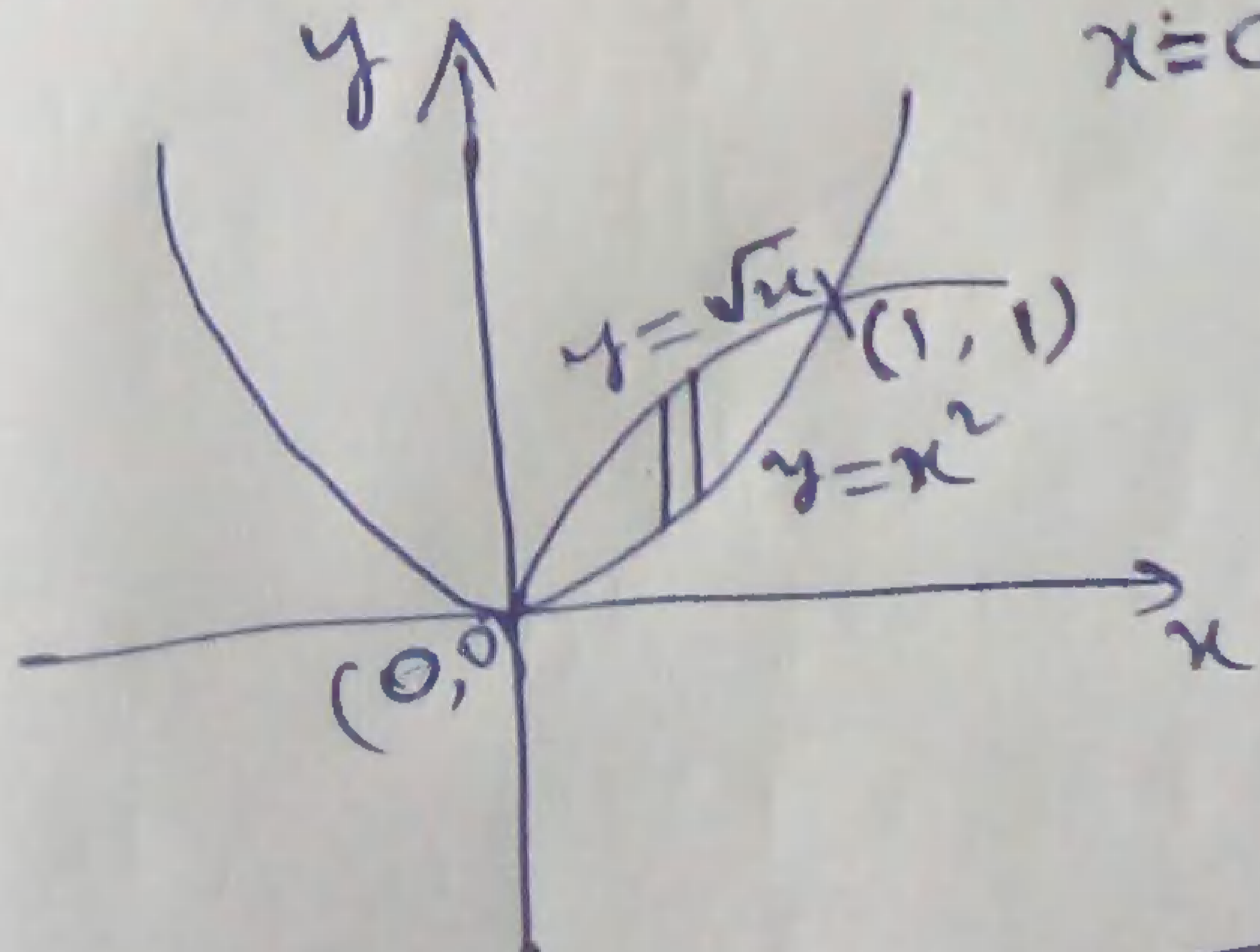
① Obtain the Area of the region bounded by $y = x^2$ and $y = \sqrt{x}$ using double integral.

Sol:



Given curves $y = x^2$ and $y = \sqrt{x}$ are intersecting at the points $(0, 0)$ and $(1, 1)$.

$$\text{Required Area} = \int_{x=0}^{1}\left(\int_{y=x^2}^{\sqrt{x}} dy\right)dx$$



$$= \int_{x=0}^{1} (\sqrt{x} - x^2)\,dx$$

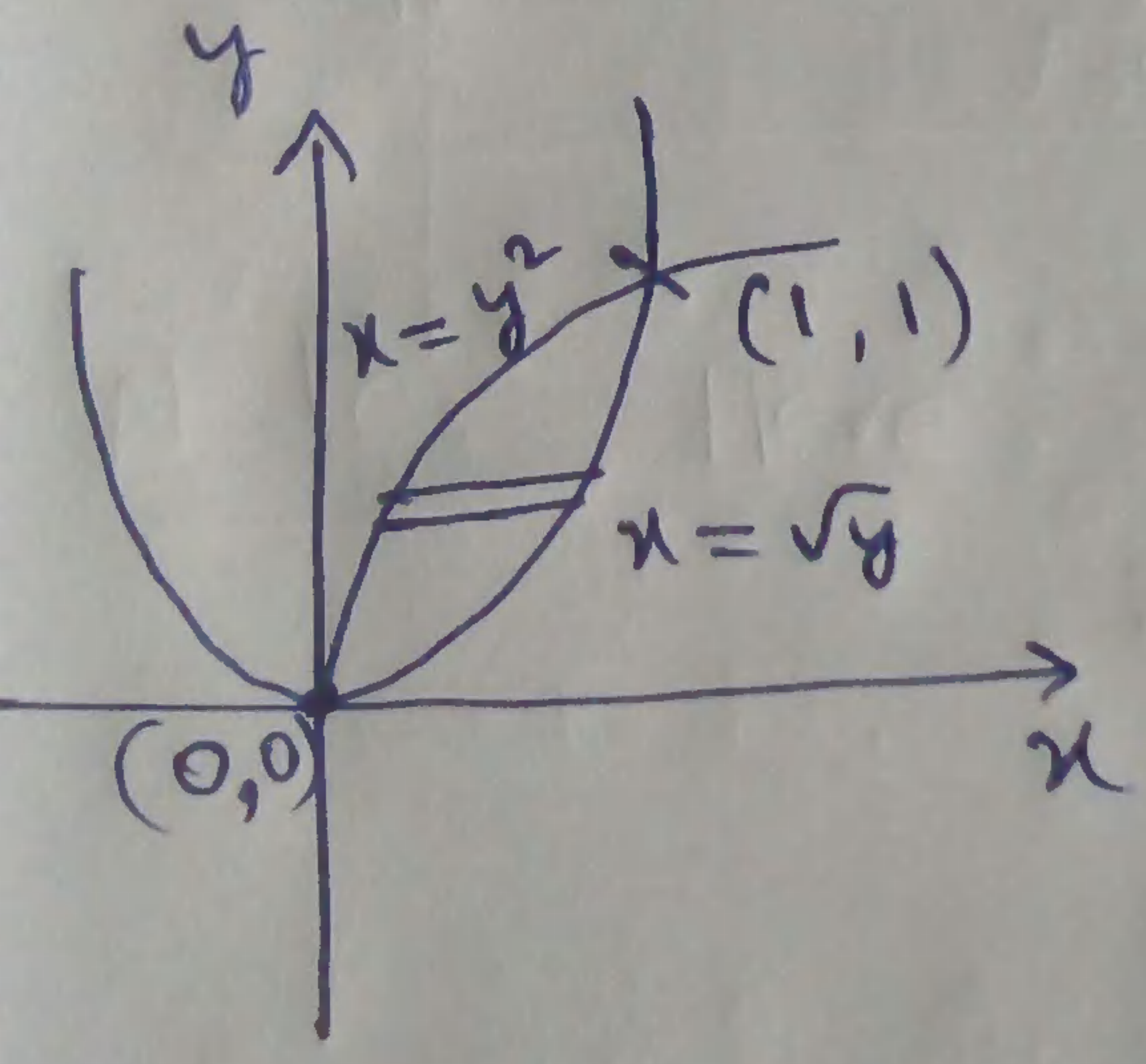
$$= \frac{2}{3}\left(x^{3/2}\right)_0^1 - \left(\frac{x^3}{3}\right)_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq.units}$$

Note: Infact, $\int_{x=0}^{1} (\sqrt{x} - x^2)\, dx$

$$= \int_{x=0}^{1} \left(\int_{y=x^2}^{\sqrt{x}} dy \right) dx$$

OR, Required Area $= \int_{y=0}^{1} \left(\int_{x=y^2}^{\sqrt{y}} dx \right) dy$

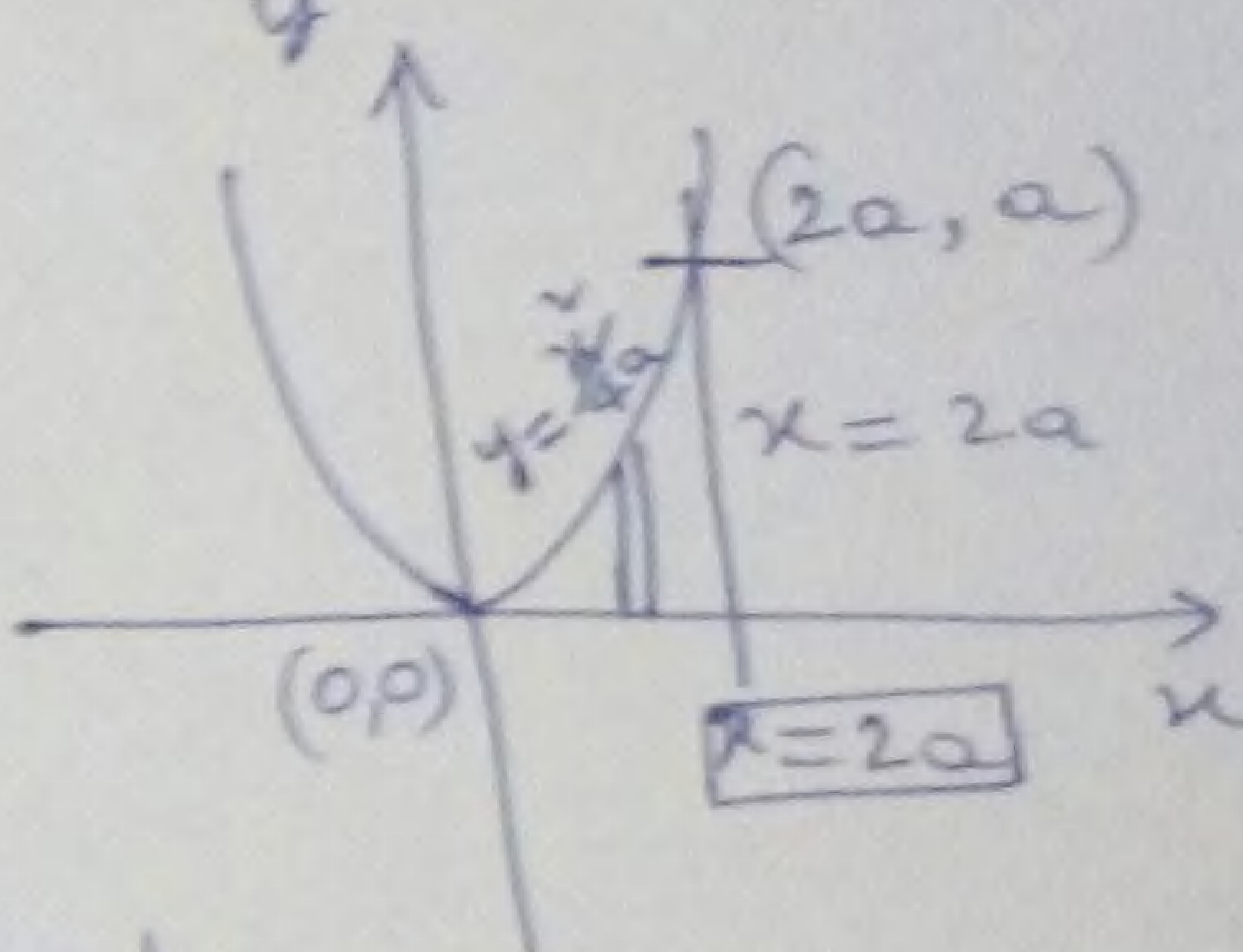


$$= \int_{y=0}^{1} (\sqrt{y} - y^2)\, dy$$

$$= \frac{1}{3} \text{ Squ units.}$$

② Evaluate $\iint_R xy\,dx\,dy$, where R is the region bounded by x-axis, $x = 2a$ and the curve $x^2 = 4ay$.

Sol:



$$\iint_R xy\,dx\,dy = \int_{x=0}^{2a}\left(\int_{y=0}^{y=x^2/4a} xy\,dy\right)dx$$

$$= \int_{x=0}^{2a}\left(x\left[\frac{y^2}{2}\right]_0^{x^2/4a}\right)dx$$

$$= \frac{1}{32a^2}\int_{x=0}^{2a} x^5\,dx = \frac{a^4}{3}$$

Change of order of integration in Double integral.

Problems

1. Change the order of integration and evaluate

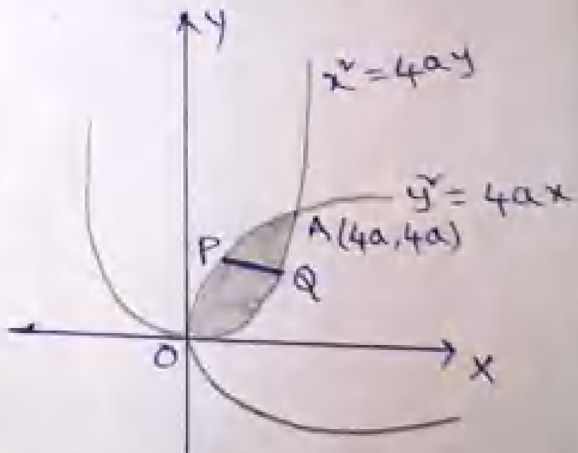
$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}}$$

Sol. In this integral for a fixed x, y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$ and then x varies from 0 to $4a$.

Let us draw the curves $y = \frac{x^2}{4a}$ i.e $x^2 = 4ay$ and $y = 2\sqrt{ax}$ i.e $y^2 = 4ax$.

These two parabolas intersect at $(0,0)$ and $(4a,4a)$

In changing the order of integration, for a fixed y, x varies from $y^2/4a$ to $\sqrt{4ay}$ and then y varies from 0 to $4a$.



$$\int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a}\left[\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\right]dy$$

$$= \int_0^{4a} [x]_{\frac{y^2}{4a}}^{2\sqrt{ay}}\, dy$$

$$= \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy = \left(2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right)_0^{4a}$$

$$= 2\sqrt{a}\,\frac{4a\sqrt{4a}}{3/2} - \frac{64a^3}{12a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

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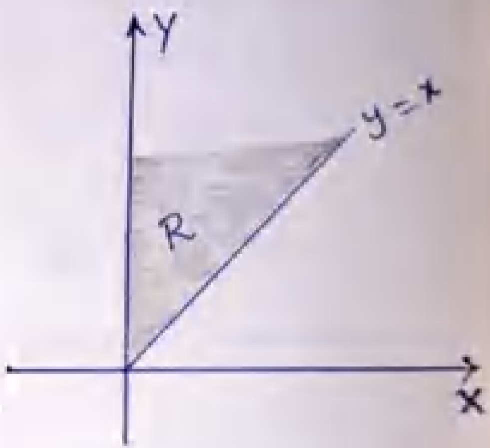
2. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$, by changing the order of integration. (13)

Sol. In the given integral x increases from 0 to ∞ and for each x, y increases from x to ∞. Thus, the lower value of y lies on the line $y=x$.

Therefore, the region of integration is the region in the first quadrant that lies above the line $y=x$.

In changing the order of integration, for a fixed y, x varies from 0 to y and then y varies from 0 to ∞



$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx = \int_{y=0}^{\infty}\left[\int_{x=0}^{y} \frac{e^{-y}}{y}\,dx\right]dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\left[\int_0^y dx\right]dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\cdot y\,dy$$

$$= \int_0^\infty e^{-y}\,dy$$

$$= \left(-e^{-y}\right)_0^\infty$$

$$= -(0-1)$$

$$= 1$$

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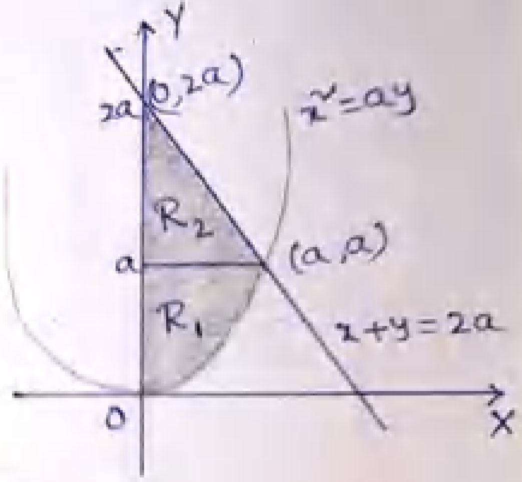
3. Evaluate the following integral by changing the order of integration: $\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$. (14)

Sol. In the given integral, for a fixed x, y varies from $\frac{x^2}{a}$ to $2a-x$ and then x varies from 0 to a.

Let us draw the curves $y=\frac{x^2}{a}$ i.e $x^2=ay$ and $y=2a-x$ i.e the line $x+y=2a$.

The parabola $x^2=ay$ and the line $x+y=2a$ intersect at (a,a).



The shaded region R is the region of integration. Observe that R is made up of two parts R_1 and R_2.

In R_1, for a fixed y, x varies from 0 to $\sqrt{ay}$ and then y varies from 0 to a.

In R_2, for a fixed y, x varies from 0 to $2a-y$ and then y varies from a to $2a$.

$$\therefore \int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx = \int_{y=0}^{a}\left[\int_{x=0}^{\sqrt{ay}} xy\,dx\right]dy + \int_{y=a}^{2a}\left[\int_{x=0}^{2a-y} xy\,dx\right]dy$$

$$= \int_0^a y\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_a^{2a} y\left[\frac{x^2}{2}\right]_0^{2a-y} dy$$

$$= \int_0^a y\left(\frac{ay}{2}\right)dy + \int_a^{2a} y\,\frac{(2a-y)^2}{2}\,dy$$

$$= \frac{a}{2}\int_0^a y^2\,dy + \int_a^{2a} \frac{y}{2}\left(4a^2-4ay+y^2\right)dy$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a + \frac{1}{2}\int_a^{2a}\left(4a^2y-4ay^2+y^3\right)dy$$

$$= \frac{a}{2}\left(\frac{a^3}{3}\right) + \frac{1}{2}\left[2a^2y^2-4a\frac{y^3}{3}+\frac{y^4}{4}\right]_a^{2a}$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[2a^2(4a^2-a^2) - \frac{4a}{3}(8a^3-a^3) + \frac{1}{4}(16a^4-a^4)\right]$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[6a^4 - \frac{28}{3}a^4 + \frac{15}{4}a^4\right]$$

$$= \frac{a^4}{6} + \frac{1}{2}\left(\frac{5a^4}{12}\right)$$

$$= \frac{9a^4}{24}$$

$$= \frac{3}{8}a^4$$

4. Change the order of integration in the integral $\int_0^1 \int_{\sqrt{y}}^{2-y} xy\,dx\,dy$ and hence evaluate it.

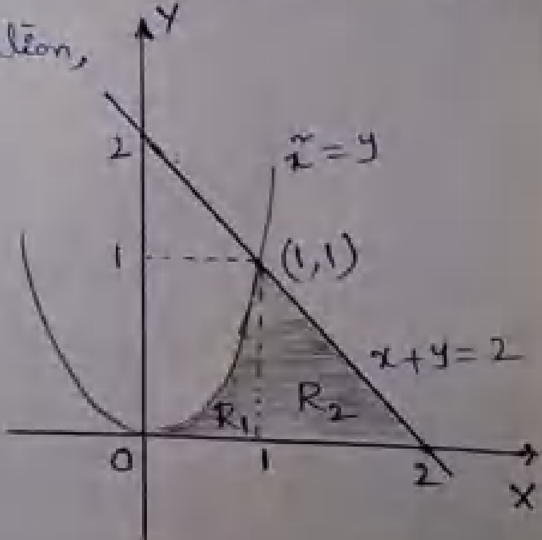
Sol. In the given integral, for a fixed y, x varies from $\sqrt{y}$ to $2-y$ and then y varies from 0 to 1.

Let us draw the curves $x=\sqrt{y}$ (i.e the parabola $x^2=y$) and $x=2-y$ (i.e the line $x+y=2$)

The parabola $x^2=y$ and the line $x+y=2$ intersect at $(1,1)$.

The shaded region R is the region of integration.

observe that R is made up of two parts R_1 and R_2.



To change the order of integration, in R_1, for a fixed x, y varies from 0 to x^2 and then x varies from 0 to 1.

In R_2, for a fixed x, y varies from 0 to $2-x$ and then x varies from 1 to 2.

$$\therefore \int_0^1 \int_{\sqrt{y}}^{2-y} xy\,dx\,dy = \int_{x=0}^{1}\left[\int_{y=0}^{x^2} xy\,dy\right]dx + \int_{x=1}^{2}\left[\int_{y=0}^{2-x} xy\,dy\right]dx$$

17

$$= \int_0^1 x\left[\frac{y^2}{2}\right]_0^{x^2} dx + \int_1^2 x\left[\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= \int_0^1 \frac{x^5}{2}\,dx + \int_1^2 \frac{x}{2}\left[(2-x)^2\right]dx$$

$$= \int_0^1 \frac{x^5}{2}\,dx + \frac{1}{2}\int_1^2 x(4-4x+x^2)\,dx$$

$$= \left(\frac{x^6}{12}\right)_0^1 + \frac{1}{2}\int_1^2 (4x-4x^2+x^3)\,dx$$

$$= \frac{1}{12} + \frac{1}{2}\left[8 - \frac{32}{3} + 4 - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right]$$

$$= \frac{1}{12} + \frac{1}{2}\left[\frac{4}{3} - \frac{11}{12}\right]$$

$$= \frac{1}{12} + \frac{1}{2}\left(\frac{5}{12}\right)$$

$$= \frac{1}{12} + \frac{5}{24}$$

$$= \frac{7}{24}$$

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Problems

① Evaluate the following

(i) $\int_{1}^{4}\int_{0}^{\sqrt{4-x}} xy \, dy \, dx$

(ii) $\int_{0}^{1}\int_{0}^{x^2} e^{y/x} \, dy \, dx$

(iii) $\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} x^3 y \, dx \, dy$

(iv) $\int_{0}^{1}\int_{0}^{1} \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}} \, dx \, dy$

(v) If A is the area of the region bounded by the lines $x=0$, $x=1$, $y=0$, $y=2$, then evaluate $\iint_A (x^2+y^2) \, dx \, dy$.

(vi) If R is the rectangular region with vertices $(0,0), (2,0), (2,3)$, evaluate $\iint_R x^2 y^2 \, dx \, dy$.

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② Evaluate $\iint_R (x+y)^2 \, dx\, dy$, where R is the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

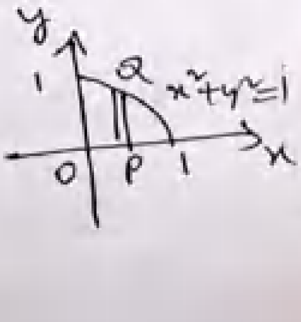
Hint: $\iint_R (x+y)^2 \, dx\, dy = \int_{x=-a}^{a} \left(\int_{y=-b\sqrt{1-\left(\frac{x^2}{a^2}\right)}}^{b\sqrt{1-\left(\frac{x^2}{a^2}\right)}} (x+y)^2 \, dy \right) dx$

$$= \frac{\pi}{4} ab \left(a^2+b^2\right)$$

③ If R is the region bounded by the circle $x^2+y^2=1$ in the first quadrant, evaluate

$$\iint_R \frac{xy}{\sqrt{1-y^2}} \, dx\, dy$$

Hint: $\iint_R \frac{xy}{\sqrt{1-y^2}} \, dx\, dy = \int_{x=0}^{1} \left(\int_{y=0}^{\sqrt{1-x^2}} \frac{xy}{\sqrt{1-y^2}} \, dy \right) dx$



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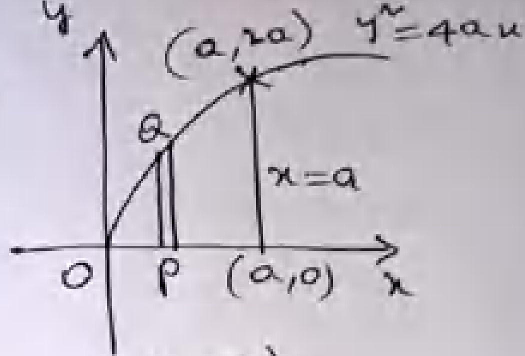
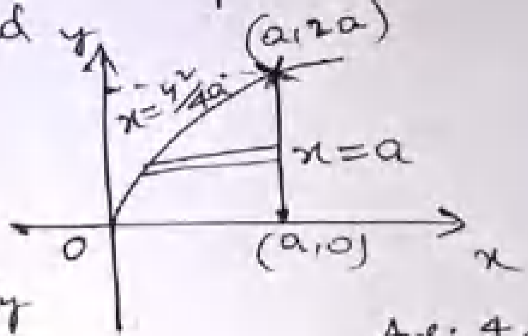
(4) Evaluate $\int_{0}^{a}\int_{0}^{2\sqrt{xa}} x^2\,dy\,dx$ $(a>0)$, by changing the order of integration.

Hint: Given



required



$$\therefore \int_{x=0}^{a}\int_{y=0}^{2\sqrt{ax}} x^2\,dy\,dx$$

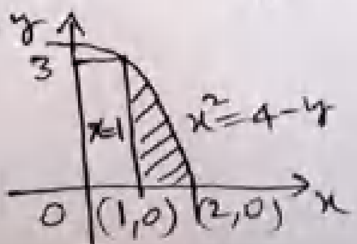
$$= \int_{y=0}^{2a}\left(\int_{x=\frac{y^2}{4a}}^{a} x^2\,dx\right)dy$$

Ans: $\frac{4}{7}a^4$.

(5) Evaluate $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} y^2\,dy\,dx$ by changing the order of integration. $\boxed{\text{Ans: } \frac{\pi}{16}}$

(6) Evaluate $\int_{0}^{3}\int_{1}^{\sqrt{4-y}} (x+y)\,dx\,dy$, by changing the order of integration.

Ans : $\frac{241}{60}$



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(7) Evaluate $\int_0^a \int_{x^2/a}^{2a-x} xy\, dy\, dx$, by changing the order of the integration.

(8) Evaluate $\int_0^\infty \int_0^x x\, e^{-\frac{x^2}{2}}\, dy\, dx$

(9) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2+y^2}\, dy\, dx$ by transforming to polar co-ordinates.

(10) Evaluate $\iint_R \frac{x^2y^2}{x^2+y^2}\, dx\, dy$ over the annular region R between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ with $b>a$, by transforming to polar co-ordinates.

— X —

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Change of Variable in Double integrals

Let $x = f(u,v)$ and $y = g(u,v)$ be the relations between the old variables and the new variables u, v of the new coordinate system.

Then $\iint_R F(x,y)\,dx\,dy = \iint_R F(f,g)\,|J|\,du\,dv$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$, which is called the Jacobian of the coordinate transformation.

Change of variables from cartesian to polar coordinates:

In this case $u = r$, $v = \theta$ and $x = r\cos\theta$, $y = r\sin\theta$

and $J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r(\cos^2\theta + \sin^2\theta) = r$

$\therefore \iint_R F(x,y)\,dx\,dy = \iint_R F(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

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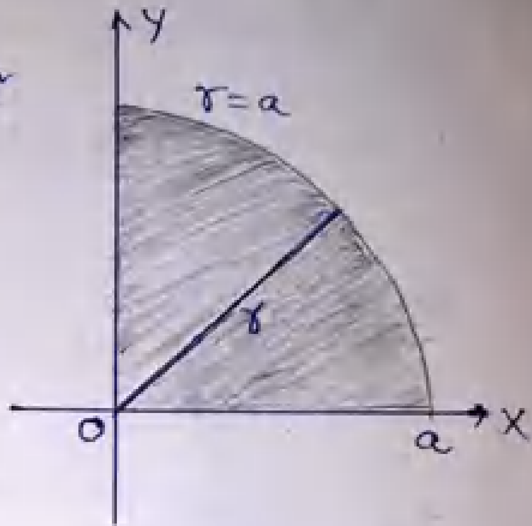
1. Evaluate the double integral $\iint xy\,dx\,dy$ over the positive quadrant bounded by the circle $x^2+y^2=a^2$

Sol. In the positive quadrant bounded by the circle $x^2+y^2=a^2$ the radial distance r varies from 0 to a and the polar angle θ varies from 0 to $\frac{\pi}{2}$



$$\iint xy\,dx\,dy = \int_{r=0}^{a}\int_{\theta=0}^{\frac{\pi}{2}} (r\cos\theta)(r\sin\theta)(r\,dr\,d\theta)$$

$$= \int_0^a r^3\,dr \times \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,d\theta$$

$$= \left(\frac{r^4}{4}\right)_0^a \times \left(\frac{\sin^2\theta}{2}\right)_0^{\frac{\pi}{2}}$$

$$= \frac{a^4}{4} \times \frac{1}{2}$$

$$= \frac{a^4}{8}.$$

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2. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy$ by changing into polar coordinates. (8)

Sol. In the given integral, both x and y increase from 0 to ∞. Therefore the region of integration is the whole of the first quadrant of the XY-plane. In this quadrant r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$.

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} e^{-r^2} r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\int_{r=0}^{\infty} e^{-r^2} r\,dr\right] d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\int_{t=0}^{\infty} \frac{1}{2} e^{-t} dt\right] d\theta \quad (\text{put } r^2 = t$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(-e^{-t}\right)_0^\infty d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{4}$$

<u>Deduction</u>: Using the above result,

$$\frac{\pi}{4} = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy = \int_0^\infty e^{-x^2} dx \times \int_0^\infty e^{-y^2} dy$$

$$= \left[\int_0^\infty e^{-x^2} dx\right]^2$$

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Therefore $\int_0^\infty e^{-x^2}dx = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}.$

3. Evaluate the integral $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\, dy\,dx$ by transforming to polar coordinates.

Sol. In this integral, x increases from 0 to a and for each x, y varies from 0 to $\sqrt{a^2-x^2}$.

Thus, the lower value of y lies on X axis and the upper value of y lies on the curve $y = \sqrt{a^2-x^2}$ i.e $x^2+y^2=a^2$ which is a circle of radius a centred at the origin.

The region of integration is the region in the first quadrant bounded by the circle $x^2+y^2=a^2$. Here θ varies from 0 to $\frac{\pi}{2}$ and r varies from 0 to a.

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\, dx\,dy = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} (r^2\sin^2\theta)\, r\,(r\,dr\,d\theta)$$

$$= \int_{r=0}^{a} r^4\, dr \times \int_{\theta=0}^{\frac{\pi}{2}} \sin^2\theta\, d\theta$$

$$= \left(\frac{r^5}{5}\right)_0^a \times \frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{a^5}{5} \times \frac{\pi}{4}$$

$$= \frac{\pi}{20}a^5.$$

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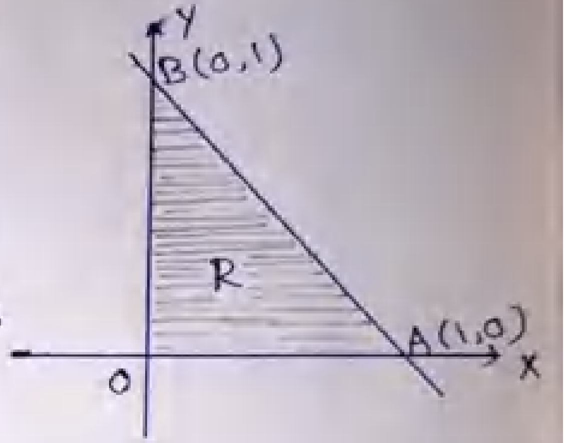
4. By using the transformation $x+y=u$, $y=uv$ (10)

Show that $\int_0^1 \int_0^{1-x} e^{y/x+y} \, dy\,dx = \frac{1}{2}(e-1)$

Sol. The region of integration in the given integral is $y=0$, $y=1-x$, $x=0$ and $x=1$ i.e the triangle OAB.



Given transformation is

$x+y=u \rightarrow 1)$ and $y=uv \rightarrow 2)$

Substitue 2) in 1), we get

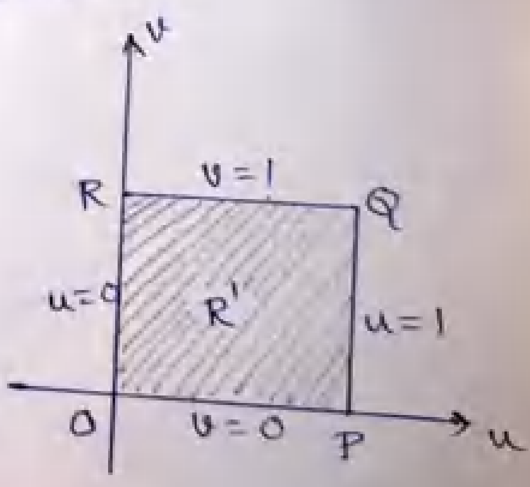
$x+uv=u$ i.e $x=u(1-v) \rightarrow 3$

Now $y=0 \Rightarrow uv=0 \Rightarrow u=0$ or $v=0$

$y=1-x \Rightarrow x+y=1 \Rightarrow u=1$

$x=0 \Rightarrow u=0$ or $v=1$ [using 3)

$\therefore$ The region R is transformed to R' where R' is the square OPQR in uv plane



$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u-uv+uv = u$$

$$\therefore \int_0^1 \int_0^{1-x} e^{y/x+y} \, dx\,dy = \iint_{R'} e^{uv/u} |J| \, du\,dv$$

$$= \int_{v=0}^1 \int_{u=0}^1 e^v u \, du\,dv$$

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$$= \int_0^1 e^v\,dv \int_0^1 u\,du$$ (11)

$$= \left(e^v\right)_0^1 \left(\frac{u^2}{2}\right)_0^1$$

$$= \frac{1}{2}(e-1)$$

5. By changing into polar coordinates evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annular region between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ $(b>a)$.

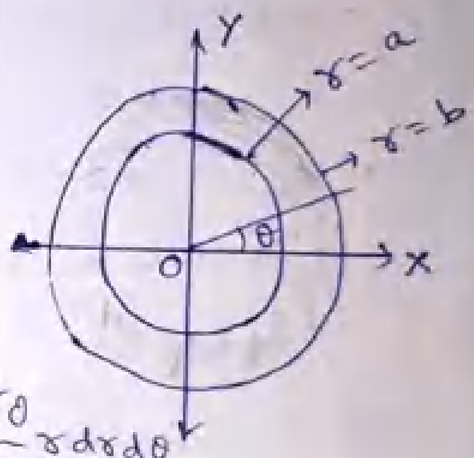
Sol. Changing to polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

Here r varies from a to b

θ varies from 0 to 2π.



$$\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=a}^{b} \frac{r^2\sin^2\theta\, r^2\cos^2\theta}{r^2}\, r\,dr\,d\theta$$

$(\because x^2+y^2=r^2$

$$= \int_{\theta=0}^{2\pi}\int_{r=a}^{b} r^3\cos^2\theta\sin^2\theta\,dr\,d\theta$$

$$= \left(\frac{r^4}{4}\right)_a^b \int_0^{2\pi} \sin^2\theta\cos^2\theta\,d\theta = \frac{b^4-a^4}{4}\int_0^{2\pi} \frac{(\sin 2\theta)^2}{4}\,d\theta$$

$$= \frac{b^4-a^4}{16}\int_0^{2\pi} \sin^2 2\theta\,d\theta = \frac{b^4-a^4}{32}\int_0^{2\pi}(1-\cos 4\theta)\,d\theta$$

$$= \frac{b^4-a^4}{32}\left(\theta - \frac{\sin 4\theta}{4}\right) = \frac{b^4-a^4}{32}\left[2\pi - 0\right]$$

$$= \frac{\pi}{16}\left(b^4-a^4\right)$$

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Triple Integrals

The volume integral of $f(x,y,z)$ over the region R is denoted by $\iiint_R f(x,y,z)\,dx\,dy\,dz$

(or, $\int_V f(x,y,z)\,dv$; here V stands for the volume of R).

Thus, $$\int_V f(x,y,z)\,dv = \iiint_R f(x,y,z)\,dx\,dy\,dz$$

$$= \int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} \int_{z=z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

Note: A volume integral is also called as a Triple Integral.

Problems:

1) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$

Sol: $$\int_{x=0}^{1}\left(\int_{y=0}^{\sqrt{1-x^2}}\left(\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dz\right)dy\right)dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

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$$= \frac{1}{2} \int_{x=0}^{1} \left(\int_{y=0}^{\sqrt{1-x^2}} xy(1-x^2-y^2)\,dy \right) dx$$

$$= \frac{1}{2} \int_{x=0}^{1} x\left\{(1-x^2)\frac{y^2}{2} - \frac{y^4}{4}\right\}_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{8} \int_{x=0}^{1} x(1-2x^2+x^4)\,dx$$

$$= \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{48}$$

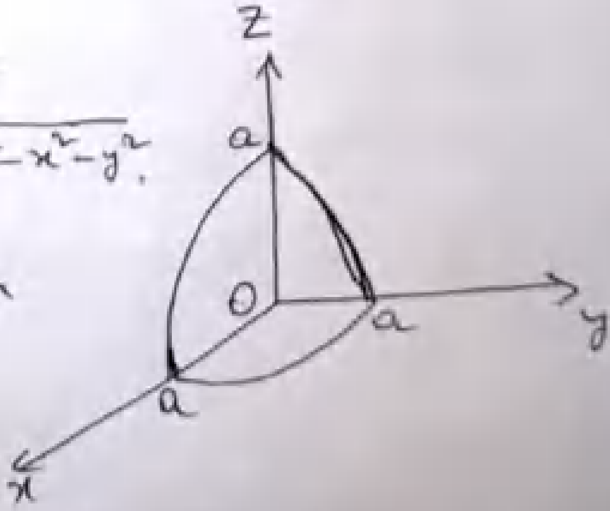
② Evaluate $\iiint_R xy\,dx\,dy\,dz$, where R is the positive octant of the sphere $x^2+y^2+z^2=a^2$

Sol: In the given region,

z varies from 0 to $\sqrt{a^2-x^2-y^2}$.

For $z=0$, y varies from 0 to $\sqrt{a^2-x^2}$.

For $z=0$, $y=0$, x varies from 0 to a.



Therefore, $\iiint_R xy\,dx\,dy\,dz = \int_{x=0}^{a}\left(\int_{y=0}^{\sqrt{a^2-x^2}}\left(\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xy\,dz\right)dy\right)dx$

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$$= \int_{x=0}^{a} x\left(\int_{y=0}^{\sqrt{a^2-x^2}} y\,[z]_0^{\sqrt{a^2-x^2-y^2}}\,dy\right)dx$$

$$= \int_{x=0}^{a} x\left(\int_{y=0}^{\sqrt{a^2-x^2}} y\sqrt{a^2-x^2-y^2}\;dy\right)dx$$

$$= \int_{x=0}^{a} x\left(\int_{t=a^2-x^2}^{0} \left(\frac{-1}{2}\right)\sqrt{t}\,dt\right)dx$$

Taking
$t=(a^2-x^2)-y^2$
then
$dt=-2y\,dy$
and
$y\to 0,\ t\to a^2-x^2$
$y\to\sqrt{a^2-x^2},\ t\to 0$

$$=\frac{1}{2}\int_{x=0}^{a} x\left(\int_{0}^{a^2-x^2}\sqrt{t}\,dt\right)dx$$

$$= \frac{1}{2}\,\frac{2}{3}\int_{x=0}^{a} x\left[t^{3/2}\right]_0^{a^2-x^2}dx$$

$$= \frac{1}{3}\int_{x=0}^{a} x\left(a^2-x^2\right)^{3/2}dx$$

Taking
$a^2-x^2=u$
then
$-2x\,dx=du$
$x\,dx=-\frac{1}{2}du$
and
$x\to 0,\ u\to a^2$
$x\to a,\ u\to 0$

$$= \frac{1}{3}\int_{u=a^2}^{0} u^{3/2}\cdot\left(\frac{-1}{2}\right)du$$

$$= \frac{1}{6}\int_{u=0}^{a^2} u^{3/2}\,du$$

$$= \frac{1}{6}\left[\frac{u^{5/2}}{\left(\frac{5}{2}\right)}\right]_0^{a^2} = \frac{1}{15}\cdot a^5$$

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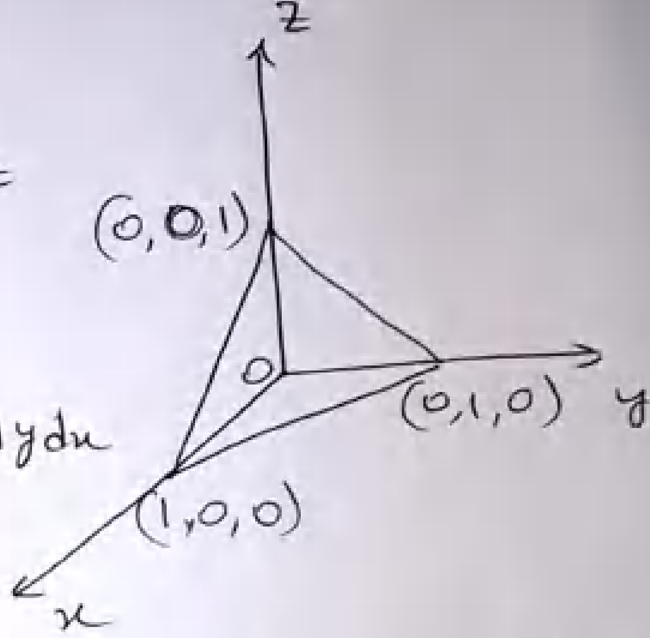
③ Evaluate $\iiint\limits_R (x+y+z)\,dx\,dy\,dz$, where R is the region bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

Sol:



$$\iiint\limits_R (x+y+z)\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} (x+y+z)\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\left(\int_{y=0}^{1-x}\left\{(x+y)\left[z\right]_0^{1-x-y} + \left[\frac{z^2}{2}\right]_0^{1-x-y}\right\}dy\right)dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\left(\int_{y=0}^{1-x}\left\{1-(x+y)^2\right\}dy\right)dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\left[y-\frac{1}{3}(x+y)^3\right]_0^{1-x} dx$$

$$= \frac{1}{6}\cdot\int_{x=0}^{1}(2-3x+x^3)\,dx = \frac{1}{8}.$$

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Problems

① Evaluate the following

(i) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$

(ii) $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy$

(iii) $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$

(iv) $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{x^2+y^2} z^2\, dz\, dy\, dx$

② Evaluate $\iiint_R dx\,dy\,dz$, where R is the finite region of space formed by the planes $x=0, y=0, z=0$ and $2x+3y+4z=12$

③ Evaluate (i) $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

④ Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx$

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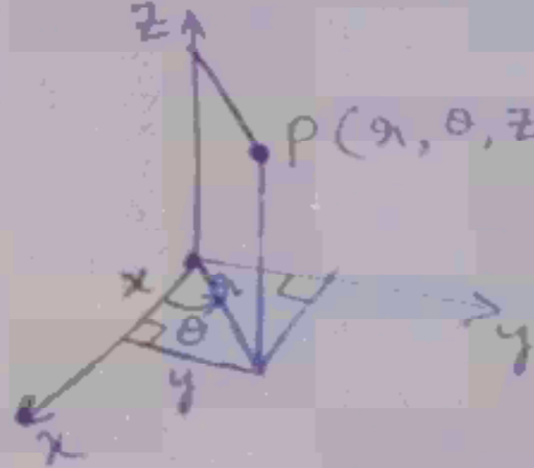
Module 5.5

Cylindrical Polar coordinates :

The relation between cartesian coordinates (x, y, z) and cylindrical coordinates (r, θ, z) for each point P is



$\cos\theta = \frac{x}{r}$

$\Rightarrow \boxed{x = r\cos\theta}$

$\sin\theta = \frac{y}{r}$

$\Rightarrow \boxed{y = r\sin\theta}$

and $z = z$

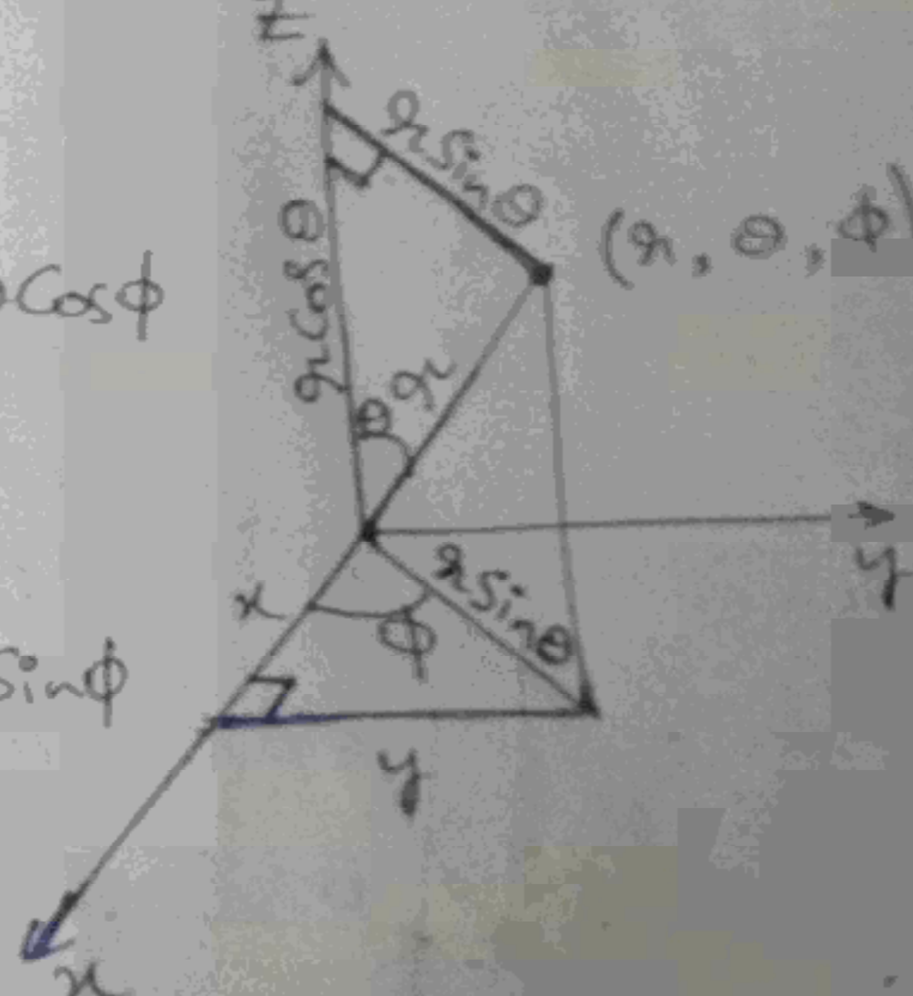
Here $dx\,dy\,dz = |J|\,dr\,d\theta\,dz$

Note: $J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r$

so, $dx\,dy\,dz = r\,dr\,d\theta\,dz$

And $0 \le \theta \le 2\pi$

Spherical Polar Coordinates:



$$\cos\phi = \frac{x}{r\sin\theta}$$

$$\Rightarrow x = r\sin\theta\cos\phi$$

$$\sin\phi = \frac{y}{r\sin\theta}$$

$$\Rightarrow y = r\sin\theta\sin\phi$$

and

$$z = r\cos\theta$$

Here $dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$

And $0 \leq \theta \leq \pi$

$0 \leq \phi \leq 2\pi.$

Problems

1. Using cylindrical polar coordinates, find the volume of the cylinder with base radius 'a' and height h.

Sol: The region of integration is bounded by $x^2+y^2 \le a^2$ and $0 \le z \le h$.

cylindrical polar coordinates are

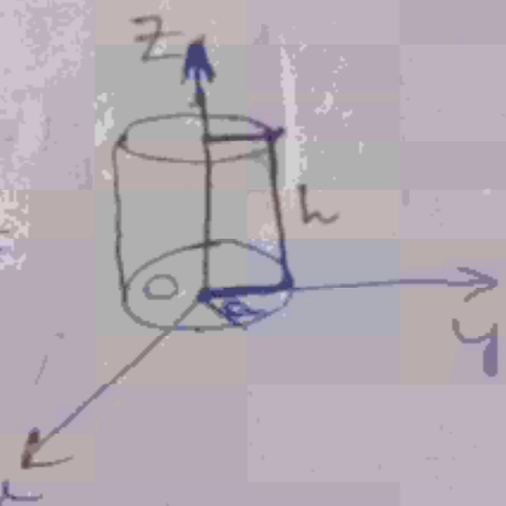
$x = r\cos\theta$, $y = r\sin\theta$ and $z = z$

here r varies from 0 to a

θ varies from 0 to 2π

z varies from 0 to h.

also $dxdydz = r\,dr\,d\theta\,dz$



Therefore, required

Volume $= \iiint dxdydz$

$$= \int_{r=0}^{a}\int_{\theta=0}^{2\pi}\int_{z=0}^{h} r\,dr\,d\theta\,dz$$

$= \pi a^2 h.$

2. Using spherical polar coordinates, find the volume of the sphere $x^2+y^2+z^2=a^2$.

<u>Sol:</u> Volume $= \int\int\int dx\,dy\,dz$

$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{r=0}^{a} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{2\pi a^3}{3}\int_{\theta=0}^{\pi} \sin\theta\, d\theta$$

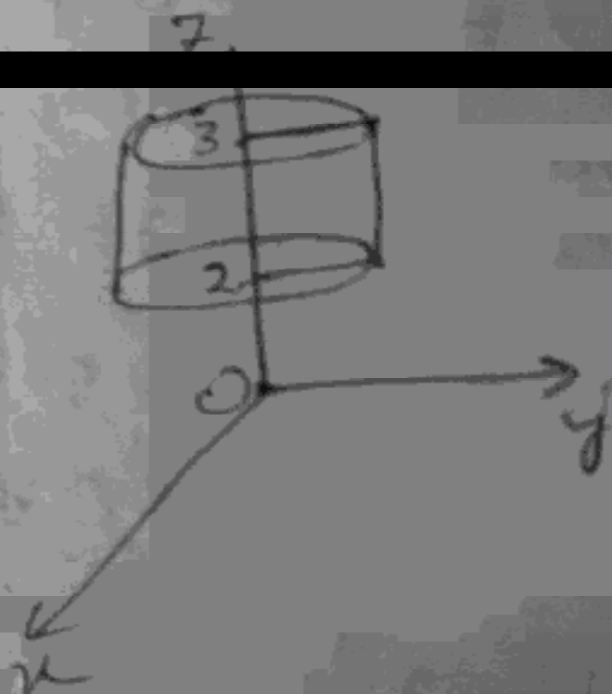
$$= \frac{2\pi a^3}{3}\left[-\cos\theta\right]_0^{\pi}$$

$$= \frac{2\pi a^3}{3}\left\{-((-1)-1)\right\}$$

$$= \frac{4}{3}\pi a^3.$$

3. If R is the region bounded by the cylinder $x^2+y^2=1$ and the planes $z=2$ and $z=3$, then evaluate $\iiint_R dx\,dy\,dz$ using cylindrical polar coordinates.

<u>Sol:</u>



$$\iiint_R dx\,dy\,dz$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=2}^{3} r\,dr\,d\theta\,dz$$

$$= \pi$$

4. Use spherical polar co-ordinates to [illegible] the [illegible]me [illegible]

sphere $x^2+y^2+z^2=1$ and

the cone $z=\sqrt{x^2+y^2}$.

Sol: spherical polar co-ordinates

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$

and $z = r\cos\theta$.

So, $dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$

here $0 \le r \le 1$, $0 \le \phi \le 2\pi$

and $0 \le \theta \le \frac{\pi}{4}$

$$z = \sqrt{x^2+y^2} \Rightarrow z = r\sin\theta$$

$$\Rightarrow r\cos\theta = r\sin\theta$$

$$\Rightarrow \cos\theta = \sin\theta$$

$$\Rightarrow \theta = \frac{\pi}{4}.$$

Or, $x^2+y^2+z^2=1 \Rightarrow z^2+z^2=$

$$\Rightarrow z^2 = \frac{1}{2}$$

$$\Rightarrow z$$

$$\Rightarrow r\cos\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Therefore, required

$$\text{Volume} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{4}} \int_{r=}^{1} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{\pi}{3}\left(2-\sqrt{2}\right)$$

5. Let R be the ice cream cone bounded below by $z = \sqrt{3(x^2+y^2)}$ and above by $x^2+y^2+z^2=4$, then find the volume of R.

Sol: $$\iiint_R dx\,dy\,dz = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/6} \int_{r=0}^{2} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$x^2+y^2+z^2=4 \Rightarrow \frac{z^2}{3}+z^2=4$$

$$\Rightarrow z=\sqrt{3}$$

$$\Rightarrow r\cos\theta=\sqrt{3}$$

$$\Rightarrow \cos\theta=\frac{\sqrt{3}}{2} \quad (\because r=2)$$

$$\Rightarrow \theta=\frac{\pi}{6}$$

Module 4 : Practice Questions

① Evaluate $\iint_R (2yx^2 + 9y^3)\,dA$, where R is the region bounded by $y = \frac{2}{3}x$ and $y = 2\sqrt{x}$.

Answer: $\iint_R (2yx^2+9y^3)\,dA = \int_{x=0}^{9} \int_{y=\frac{2}{3}x}^{2\sqrt{x}} (2yx^2+9y^3)\,dy\,dx$

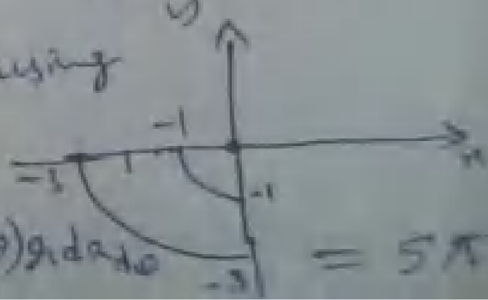
$= \frac{24057}{5}$.

② Evaluate $\iint_R dx\,dy$, where R is the region bounded by $y = 1-x^2$ and $y = x^2-3$

Answer: $\iint_R dx\,dy = \int_{x=-\sqrt{2}}^{\sqrt{2}} \int_{y=x^3-3}^{1-x^2} dy\,dx$

$= \frac{16\sqrt{2}}{3}$

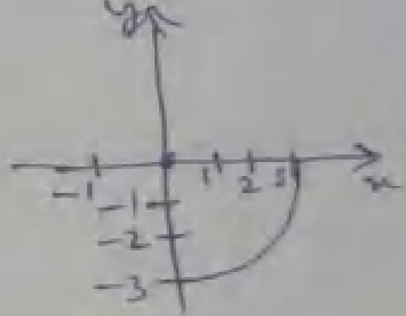
③ Evaluate $\iint_R (y^2+3x)\,dx\,dy$, where R is the region in the third quadrant between $x^2+y^2=1$ and $x^2+y^2=9$ using Polar coordinates.



Answer: $\iint_R (y^2+3x)\,dx\,dy = \int_{\theta=\pi}^{3\pi/2} \int_{r=1}^{3} \left((r\sin\theta)^2 + 3r\cos\theta\right) r\,dr\,d\theta = 5\pi - 26$

(4) Evaluate $\int_0^3 \int_{-\sqrt{9-x^2}}^{0} e^{x^2+y^2}\, dy\, dx$ using polar coordinates.

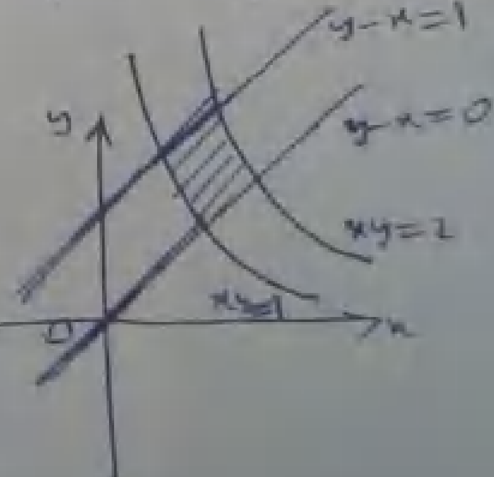
Answer: $\int_{x=0}^{2} \int_{y=-\sqrt{9-x^2}}^{0} e^{x^2+y^2}\, dy\, dx$



$$= \int_{\theta=\frac{3\pi}{2}}^{2\pi} \int_{r=0}^{3} e^{r^2}\, r\, dr\, d\theta$$

$$= \frac{\pi}{4}\left(e^9-1\right).$$

(5) Evaluate $\iint_R (x^2-y^2)\, dx\, dy$, where R is the region bounded by $y-x=1$, $y-x=0$, $xy=2$ and $xy=1$ using change of variables.



Answer: The region can be described simply if we change to coordinates u and v where $u = y-x$ and $v = xy$.

$$\therefore \iint_R (x^2-y^2)\, dx\, dy = \int_{v=1}^{2} \int_{u=0}^{1} (+u)\, du\, dv = +\frac{1}{2}$$

(6) Evaluate $\iint_R e^{x^2-y^2}\,dx\,dy$, where R is the region bounded by $x^2-y^2=1$, $x^2-y^2=4$, $y=0$ and $y=\left(\frac{3}{5}\right)x$ changing into $x^2-y^2=u$ and $x+y=v$.

Answer: $u=x^2-y^2=(x-y)(x+y) \Rightarrow u=(x-y)v \;(\because v=x+y)$

$$\Rightarrow \frac{u}{v}=x-y.$$

So,

$$v+\frac{u}{v}=2x \text{ and } v-\frac{u}{v}=2y.$$

$$y=0 \Rightarrow v-\frac{u}{v}=0 \Rightarrow u=v^2$$

$$y=\frac{3}{5}x \Rightarrow v-\frac{u}{v}=2\left(\frac{3}{5}x\right) \Rightarrow v-\frac{u}{v}=\frac{3}{5}(2x)$$

$$\Rightarrow v-\frac{u}{v}=\frac{3}{5}\left(v+\frac{u}{v}\right)$$

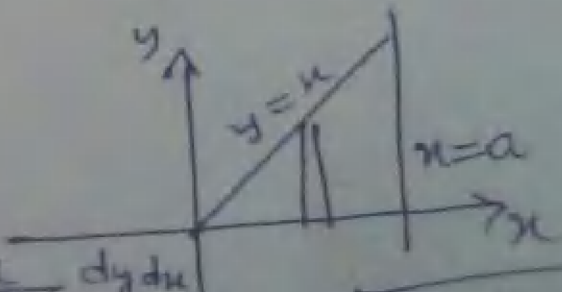
$$\Rightarrow u=\left(\frac{1}{4}\right)v^2$$

And $J=\dfrac{\partial(x,y)}{\partial(u,v)}=\dfrac{1}{2v}$

$$\therefore \iint_R e^{x^2-y^2}\,dx\,dy=\int_{u=1}^{4}\int_{v=\sqrt{u}}^{2\sqrt{u}} e^{u}\,\frac{1}{2u}\,dv\,du$$

$$=\frac{1}{2}\log_e(2)\,(e^4-e).$$

(7) Evaluate $\int_0^a\int_y^a \frac{x}{x^2+y^2}\,dx\,dy$ by changing the order of integration.



Answer: $\int_{y=0}^{a}\int_{x=y}^{a} \frac{x}{x^2+y^2}\,dx\,dy=\int_{x=0}^{a}\int_{y=0}^{x} \frac{x}{x^2+y^2}\,dy\,dx$

$$=\frac{\pi a}{4}.$$

$\boxed{\text{required}}$

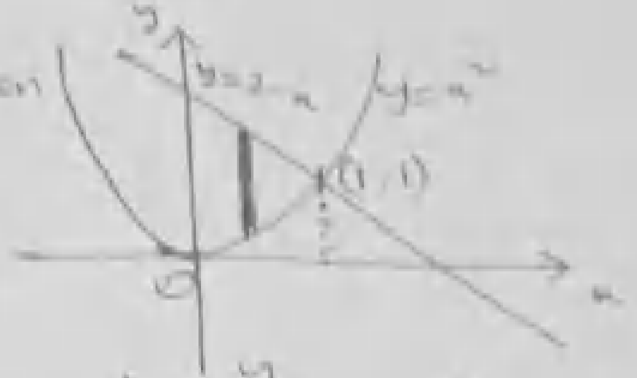
(8) Use change of order of integration, evaluate

$$\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx .$$

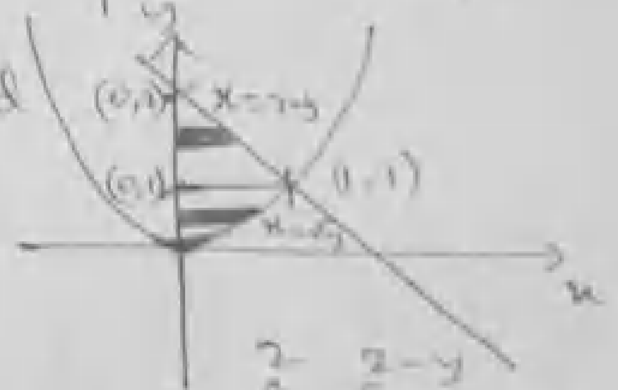
Answer

$y = x^2$
$y = 2 - x$
$x = 0$
$x = 1$

Given



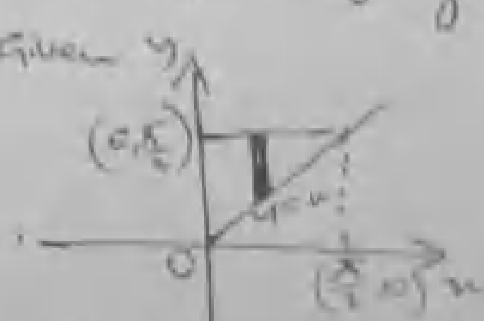
required



$$\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx = \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy \, dx\,dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy \, dx\,dy = \frac{3}{8}$$

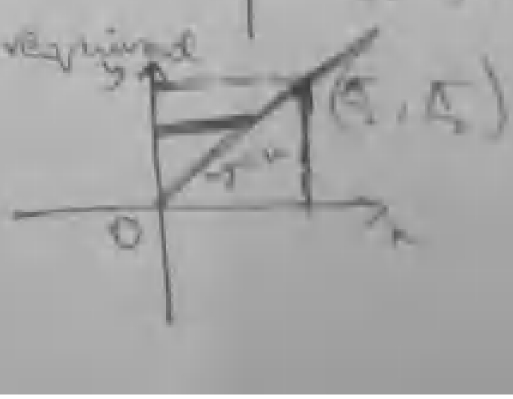
(9) Evaluate $\int_{x=0}^{\frac{\pi}{2}} \int_{y=x}^{\frac{\pi}{2}} \left(\frac{\sin y}{y}\right) dy\,dx$ by changing the order of integration

Given



Answer: $\int_{y=0}^{\frac{\pi}{2}} \int_{x=0}^{y} \left(\frac{\sin y}{y}\right) dx\,dy$

required



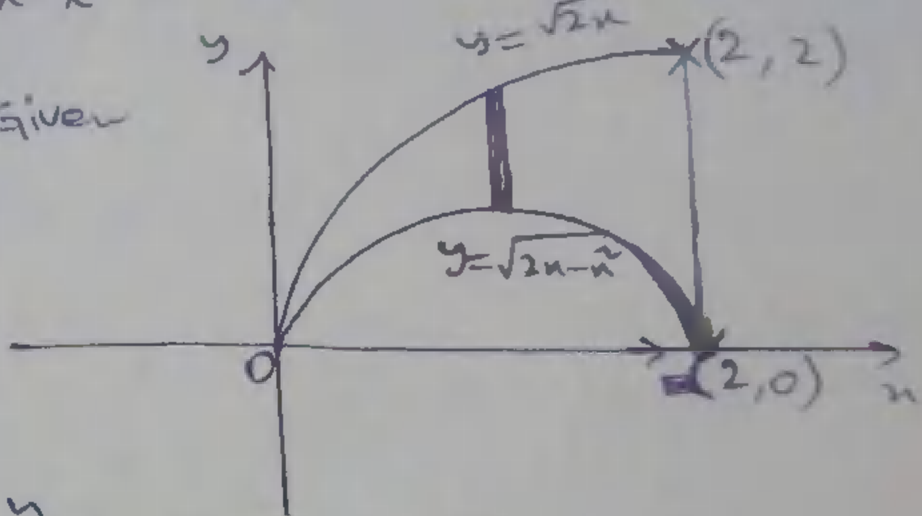
$= 1$

(10) Use change of order of integration, evaluate

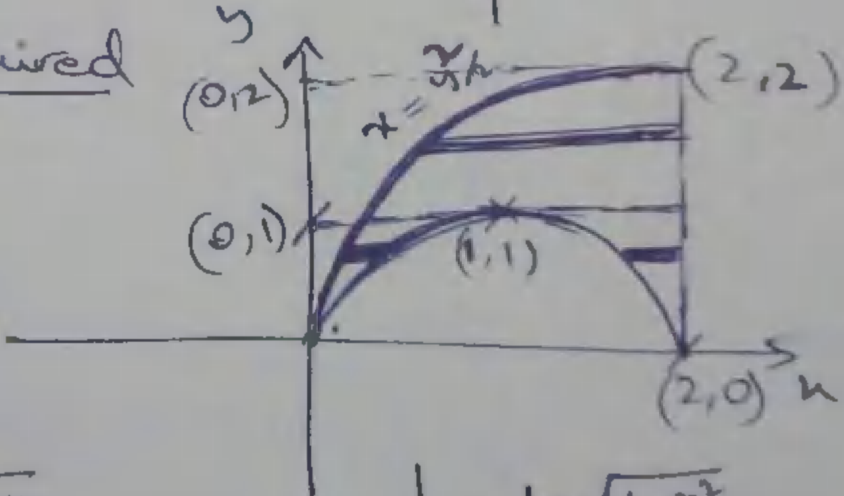
$$\int_{0}^{2} \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} dy\,dx.$$

Answer:

Given



required



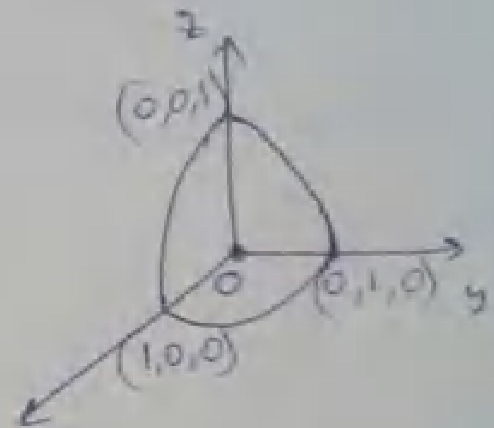
$$\int_{x=0}^{2} \int_{y=\sqrt{2x-x^2}}^{\sqrt{2x}} dy\,dx = \int_{y=0}^{1} \int_{x=y^2/2}^{1-\sqrt{1-y^2}} dx\,dy + \int_{y=0}^{1} \int_{x=1+\sqrt{1-y^2}}^{2} dx\,dy$$

$$+ \int_{y=1}^{2} \int_{x=y^2/2}^{2} dx\,dy$$

$$=$$

(11) Evaluate $\iiint_R xy\,dx\,dy\,dz$, where R is the region bounded by the sphere $x^2+y^2+z^2=1$ in the first octant

Answer:



$$\iiint_R xy\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xy\,dz\,dy\,dx$$

$$= \frac{1}{15}$$

(12) Evaluate $\iiint_R \sqrt{x^2+y^2}\,dx\,dy\,dz$, where R is the region that R lies between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$ and between the xy-plane and the plane $z=x+3$ using cylindrical polar coordinates.

$$\boxed{\begin{matrix} x = r\cos\theta,\ y = r\sin\theta \\ z = z \end{matrix}}$$

Answer: $$\iiint_R \sqrt{x^2+y^2}\,dx\,dy\,dz = \int_{\theta=0}^{2\pi}\int_{r=1}^{2}\int_{z=0}^{r\cos\theta+3} \sqrt{r^2}\, r\,dr\,d\theta\,dz = \boxed{14\pi}$$

$$\boxed{\begin{matrix} xy\text{-plane} \longrightarrow z=0 \\ z=x+3 \longrightarrow z=r\cos\theta+3 \end{matrix}}$$

(13) Evaluate $\iiint_R xyz\,dv$, where R is the region enclosed by the parabolas $z = x^2+y^2$ and $z = 8-(x^2+y^2)$ using cylindrical polar coordinates.

Answer: The parabolas intersect where $z=4$.

$z : r^2$ to $8-r^2$

$r : 0$ to 2

$\theta : 0$ to 2π

$$\iiint_R xyz\,dv = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=r^2}^{8-r^2} (r\cos\theta)(r\sin\theta)\, z\, r\,dz\,dr\,d\theta$$

(14) Evaluate $\iiint_R z\,dv$, where R is the region inside both the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=1$ using spherical polar coordinates

Answer:

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\iiint_R z\,dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/4}\int_{r=0}^{1} (r\cos\theta)\, r^2\sin\theta\,dr\,d\theta\,d\phi.$$

1.8: Beta and Gamma Functions

Definition of Gamma function:

when $n>0$, we define Gamma function as $\overline{|n} = \int_0^\infty e^{-x} \cdot x^{n-1}\,dx$ provided integral exists.

(i) $\overline{|1} = \int_0^\infty e^{-x} \cdot x^{1-1}\,dx = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big]_0^\infty = -(0-1)$
$= 1$

Therefore, $\overline{|1} = 1$

(ii) $\overline{|(n+1)} = \int_0^\infty e^{-x} \cdot x^{(n+1)-1}\,dx$

$$= \int_0^\infty e^{-x} \cdot x^n\,dx$$

$$= x^n(-e^{-x})\Big]_0^\infty - n\int_0^\infty x^{n-1}(-e^{-x})\,dx$$

$$= 0 + n\int_0^\infty e^{-x} \cdot x^{n-1}\,dx$$

$$= n\overline{|n}$$

Therefore, $\overline{|(n+1)} = n\overline{|n}$ $(n>0)$

(iii) $\Gamma(n+1) = n\,\Gamma n$

$$= n(n-1)\,\Gamma(n-1)$$

$$= n(n-1)(n-2)\,\Gamma(n-2)$$

$$\vdots$$

$$= n(n-1)(n-2)\cdots(n-(n-2))(n-(n-1))\,\Gamma(n-(n-1))$$

$$= n(n-1)(n-2)\cdots 2.1\,\Gamma 1$$

$$= 1.2.3\ldots n \qquad (\text{since } \Gamma 1 = 1)$$

$$= n!$$

Therefore, $\Gamma(n+1) = n!$ for $n = 0, 1, 2, 3, \ldots$

(iv) $\Gamma n = \dfrac{\Gamma(n+1)}{n}$ (for $n<0$ and $n \neq -1, -2, -3, \ldots$)

(v) $\Gamma 0, \Gamma(-1), \Gamma(-2), \ldots$ are not defined.

(vi) $\Gamma \frac{1}{2} = \sqrt{\pi}$

(vii) $\Gamma\left(-\frac{1}{2}\right) = \dfrac{\Gamma\left(-\frac{1}{2}+1\right)}{\left(-\frac{1}{2}\right)} = -2\,\Gamma\frac{1}{2} = -2\sqrt{\pi}$

(viii) $\Gamma\frac{7}{2} = \left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right) = \frac{5}{2}\cdot\Gamma\frac{5}{2} = \frac{5}{2}\cdot\frac{3}{2}\Gamma\frac{3}{2}$

$$= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}$$

Therefore, $\Gamma\frac{7}{2} = \dfrac{15}{8}\sqrt{\pi}$

(ix) $\Gamma\left(-\frac{3}{2}\right) = \dfrac{\Gamma\left(-\frac{3}{2}+1\right)}{\left(-\frac{3}{2}\right)} = -\frac{2}{3} \cdot \Gamma\left(-\frac{1}{2}\right)$

$= -\frac{2}{3}\left(-2\sqrt{\pi}\right)$

$= \frac{4}{3}\sqrt{\pi}$

Problems

① Evaluate $\displaystyle\int_0^\infty e^{-ax} \cdot x^{n-1}\, dx$

Sol: Taking $ax = y$, then $a\,dx = dy$

so $dx = \frac{1}{a}\, dy$

And $x \to 0$, $y \to 0$

$x \to \infty$, $y \to \infty$

Therefore, $\displaystyle\int_0^\infty e^{-ax} \cdot x^{n-1}\, dx = \int_0^\infty e^{-y}\left(\frac{y}{a}\right)^{n-1} \cdot \frac{1}{a}\, dy$

$\displaystyle = \frac{1}{a^n}\int_0^\infty e^{-y} \cdot y^{n-1}\, dy$

$= \frac{1}{a^n}\Gamma n$

Note: $\displaystyle \Gamma n = a^n \int_0^\infty e^{-ax}\, x^{n-1}\, dx$

② Evaluate $\int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx$

Sol: Taking $\log\frac{1}{x} = y$, then we get

$\frac{1}{x} = e^{y}$. or, $x = e^{-y}$.

So, $dx = -e^{-y}\,dy$.

And $x \to 0,\ y \to \infty$
$x \to 1,\ y \to 0$

Therefore, $\int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx = \int_\infty^0 y^{n-1}\cdot(-e^{-y})\,dy$

$$= \int_0^\infty e^{-y}\cdot y^{n-1}\,dy$$

$$= \Gamma n.$$

③ Evaluate $\int_0^\infty \sqrt[4]{x}\cdot e^{-\sqrt{x}}\,dx$

Sol: Taking $\sqrt{x} = t$, then $x = t^2$ so that

$dx = 2t\,dt$. And $x \to 0,\ t \to 0$
$x \to \infty,\ t \to \infty$

Therefore, $\int_0^\infty \sqrt[4]{x}\cdot e^{-\sqrt{x}}\,dx = \int_0^\infty x^{1/4}\cdot e^{-\sqrt{x}}\,dx$

$$= \int_0^\infty t^{1/2} \cdot e^{-t}\, 2t\, dt$$

$$= 2\int_0^\infty t^{3/2} \cdot e^{-t}\, dt$$

$$= 2\int_0^\infty e^{-t} \cdot t^{\frac{5}{2}-1}\, dt$$

$$= 2\,\Gamma\left(\tfrac{5}{2}\right)$$

$$= 2 \cdot \tfrac{3}{2} \cdot \Gamma\tfrac{3}{2} = 3\left(\tfrac{1}{2}\right)\Gamma\tfrac{1}{2} = \tfrac{3}{2}\sqrt{\pi}$$

Hence $\int_0^\infty \sqrt[4]{x} \cdot e^{-\sqrt{x}}\, dx = \frac{3}{2}\sqrt{\pi}$.

④ Evaluate $\int_0^\infty \frac{x^a}{a^x}\, dx \;\; (a>0)$

Sol: Taking $a^x = e^t$, we get $x = \frac{t}{\log a}$

so, $dx = \frac{1}{\log a}\, dt$. And $x \to 0$, $t \to 0$
$x \to \infty$, $t \to \infty$

Therefore, $\int_0^\infty \frac{x^a}{a^x}\, dx = \int_0^\infty \left(\frac{t}{\log a}\right)^a \cdot \frac{1}{e^t} \cdot \frac{1}{\log a}\, dt$

$$= \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-t} \cdot t^{(a+1)-1}\, dt$$

$$= \Gamma(a+1) / (\log a)^{a+1}$$

Definition of Beta function:

when $m>0$ and $n>0$, we define Beta function as $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

(i) P.T. $\beta(m,n) = \beta(n,m)$

Sol: we have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Taking $1-x = y$, then $dx = -dy$

And $x \to 0,\ y \to 1$

$x \to 1,\ y \to 0$

Therefore, $\beta(m,n) = -\int_1^0 (1-y)^{m-1}\, y^{n-1}\, dy$

$$= \int_0^1 y^{n-1}.(1-y)^{m-1}dy$$

$$= \beta(n,m)$$

Hence $\beta(m,n) = \beta(n,m)$ ($m>0$ and $n>0$)

(ii) S.T. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta . \cos^{2n-1}\theta\, d\theta$

Sol: we have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Taking $x = \sin^2\theta$, we get $dx = 2\sin\theta\cos\theta\,d\theta$

And $x \to 0,\ \theta \to 0$
$x \to 1,\ \theta \to \pi/2$

Therefore, $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2n-2}\theta \cdot \sin\theta \cdot \cos\theta\,d\theta$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\,d\theta.$$

(iii) P.T. $\beta(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

<u>Sol:</u> We have $\beta(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1}\,dx$

Taking $x = \frac{1}{1+y}$, then $dx = \frac{-1}{(1+y)^2}\,dy$.

And $x \to 0,\ y \to \infty$
$x \to 1,\ y \to 0$

Therefore, $\beta(m,n) = \int_{\infty}^{0} \left(\frac{1}{1+y}\right)^{m-1} \left(1 - \frac{1}{1+y}\right)^{n-1} \cdot \frac{-1}{(1+y)^2}\,dy$

$$= \int_0^{\infty} \left(\frac{1+y-1}{1+y}\right)^{n-1} \cdot \frac{1}{(1+y)^{m+1}}\,dy$$

$$= \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}}\,dy.$$

Hence, $\beta(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

Relation Between Beta and Gamma functions:

$$\beta(m,n) = \frac{\Gamma m \,\Gamma n}{\Gamma(m+n)}, \text{ where } m>0,\ n>0.$$

Sol. we have $\Gamma m = \int_0^\infty e^{-x} . x^{m-1} dx \longrightarrow (1)$

also, we have

$$\frac{\Gamma m}{a^m} = \int_0^\infty e^{-ay} . y^{m-1} dy$$

$$\text{or, } \frac{1}{a^m} = \frac{1}{\Gamma m} \int_0^\infty e^{-ay} y^{m-1} dy \longrightarrow (2)$$

Now,

$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^\infty x^{n-1} \frac{1}{(1+x)^{m+n}} dx$$

$$= \int_0^\infty x^{n-1} \left(\frac{1}{\Gamma(m+n)} \int_0^\infty e^{-(1+x)y} . y^{(m+n)-1} dy \right) dx$$

(from (2))

$$= \frac{1}{\Gamma(m+n)} \int_0^\infty x^{n-1} \left(\int_0^\infty e^{-y} . e^{-xy} . y^{m+n-1} dy \right) dx$$

$$= \frac{1}{\Gamma(m+n)} \int_0^\infty e^{-y} y^{m+n-1} \left(\int_0^\infty e^{-yx} . x^{n-1} dx \right) dy$$

$$= \frac{1}{\Gamma(m+n)} \int_0^\infty e^{-y} \cdot y^{m+n-1} \cdot \frac{\Gamma n}{y^n}\, dy$$

$$= \frac{\Gamma n}{\Gamma(m+n)} \int_0^\infty e^{-y} \cdot y^{m-1}\, dy$$

$$= \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)}.$$

Hence $\beta(m,n) = \dfrac{\Gamma m\, \Gamma n}{\Gamma(m+n)}$ $(m>0, n>0)$

<u>Note:</u> If $0<p<1$, then

$$\Gamma p\, \Gamma(1-p) = \frac{\pi}{\sin p\pi}$$

<u>Problems:</u>

① P.T. $\Gamma\frac{1}{2} = \sqrt{\pi}$

<u>Sol:</u> we have $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

Taking $m=n=\frac{1}{2}$, we get

$$\beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = 2\int_0^{\pi/2} d\theta$$

$$\Rightarrow \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = 2\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{\left(\Gamma\frac{1}{2}\right)^2}{\Gamma 1} = \pi$$

$$\Rightarrow \boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}}$$

(since $\Gamma 1 = 1$)

② Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

Sol: $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^{-1/2}\theta\, d\theta$

$$= \frac{1}{2}\left[2\int_0^{\pi/2} \sin^{2\left(\frac{3}{4}\right)-1}\theta \cdot \cos^{2\left(\frac{1}{4}\right)-1}\theta\, d\theta\right]$$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2}\left(\frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)}\right)$$

$$= \frac{1}{2}\,\frac{\Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right)}{\Gamma 1}$$ (Since $0<p<1$, $\Gamma p\,\Gamma(1-p) = \frac{\pi}{\sin p\pi}$

$$= \frac{1}{2}\,\frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{1}{2}\,\frac{\pi}{\left(\frac{1}{\sqrt{2}}\right)} = \frac{\pi}{\sqrt{2}}$$

Therefore, $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$.

(3) Evaluate $\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta$

Sol: $\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta$

$$= \frac{1}{2}\left(2\int_0^{\pi/2} \sin^{2\left(\frac{p+1}{2}\right)-1}\theta \cdot \cos^{2\left(\frac{q+1}{2}\right)-1}\theta \, d\theta\right)$$

$$= \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}$$

Note: (i) $\int_0^{\pi/2} \sin^2\theta \cdot \cos^3\theta \, d\theta = \frac{1}{2}\frac{\Gamma\frac{3}{2}\cdot\Gamma\frac{4}{2}}{\Gamma\left(\frac{3}{2}+\frac{4}{2}\right)}$

$$= \frac{1}{2}\frac{\Gamma\frac{3}{2}\cdot\Gamma 2}{\Gamma\frac{7}{2}}$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\frac{7}{2}} \quad \left(\text{since } \Gamma 2 = 1! = 1\right)$$

EXERCISE :

① Evaluate $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx$ <u>Answer:</u> $\frac{\sqrt{\pi}}{3}$

② Evaluate $\int_0^\infty x^3 \left(\log \frac{1}{x}\right)^2 dx$ <u>Answer:</u> $\frac{1}{32}$

③ Evaluate $\int_0^\infty e^{-x^2} x^4\, dx$ <u>Answer:</u> $\frac{3}{8}\sqrt{\pi}$

④ Evaluate $\int_0^\infty x^3\, 3^{-x}\, dx$ <u>Answer:</u> $\frac{6}{(\log 3)^4}$

⑤ Evaluate $\int_0^2 (8-x^3)^{1/3}\, dx$ <u>Answer:</u> $\frac{4}{9}\frac{\left(\Gamma\frac{1}{3}\right)^2}{\Gamma\left(\frac{5}{3}\right)}$

⑥ Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\, dx$ <u>Answer:</u> $\sqrt{\pi}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$

⑦ Evaluate $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$ <u>Answer:</u> $\frac{\pi}{\sqrt{2}}$

Module 5 Practice Questions

① Evaluate $\int_0^1 x^3\left(\log_e\left(\frac{1}{x}\right)\right)^2 dx$.

sol: Hint: let $\log_e\left(\frac{1}{x}\right) = y$.

Answer: $\frac{1}{32}$

② Evaluate $\int_0^\infty x^3\, 3^{-x}\, dx$.

Hint: $3^{-x} = e^{-y}$

Answer: $\frac{6}{(\log_e 3)^4}$

③ Evaluate $\int_0^\infty \frac{y}{(1+y^3)^2} dy$. Hint: let $y^3 = x$

Answer: $\frac{2\pi\sqrt{3}}{27}$

④ Evaluate $\int_0^2 (8-x^3)^{1/3} dx$ Hint: $x^3 = 8y$

Answer: $\frac{4}{9}\frac{\left(\Gamma\frac{1}{3}\right)^2}{\Gamma\frac{2}{3}}$

⑤ ~~Prove~~ show that $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx$

Hint: $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \longrightarrow (1)$

Taking $x = \frac{1}{y}$ in the second part of (1)

(6) Evaluate $\int_0^1 \left(\log_e\left(\frac{1}{x}\right)\right)^{n-1} dx \quad (n>0)$

Hint: Taking $\log_e\left(\frac{1}{x}\right) = y$ | Answer: Γn.

or, $\frac{1}{x} = e^y$

(7) Evaluate $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} d\theta$

Answer: $\frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{2}\right) \times \frac{1}{2}\beta\left(\frac{1}{4},\frac{1}{2}\right)$

$$=\left(\frac{1}{2}\frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)}\right)\left(\frac{1}{2}\frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}\right) = \frac{1}{4}\frac{\left(\Gamma\frac{1}{2}\right)^2 \Gamma\frac{1}{4}\,\cancel{\Gamma\frac{3}{4}}}{\left(\Gamma\frac{5}{4}\,\cancel{\Gamma\frac{3}{4}}\right)}$$

$$= \frac{1}{4}\frac{(\sqrt{\pi})^2\,\Gamma\frac{1}{4}}{\left(\frac{5}{4}-1\right)\Gamma\left(\frac{5}{4}-1\right)} = \frac{\pi}{4}\frac{\cancel{\Gamma\frac{1}{4}}}{\frac{1}{4}\cancel{\Gamma\frac{1}{4}}} = \pi.$$

(8) Evaluate $\int_0^{\pi/2} \sin^5\theta \cdot \cos^{7/2}\theta\, d\theta$

Hint: $2m-1=5$ and $2n-1=\frac{7}{2}$

Answer: $\frac{64}{1989}$

(9) When $n \in \mathbb{Z}^+$, show that

$$2^n\,\Gamma\left(n+\tfrac{1}{2}\right) = 1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}$$

Hint: $\Gamma\left(n+\frac{1}{2}\right) = \left(n+\frac{1}{2}-1\right)\Gamma\left(n+\frac{1}{2}-1\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) = \frac{2n-1}{2}\,\Gamma\left(\frac{2n-1}{2}\right)$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\Gamma\left(\frac{2n-3}{2}\right).$$

$$\vdots$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}$$

Hence $2^n\,\Gamma\left(n+\frac{1}{2}\right) = 1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}$.

5.2 Evaluation of Double Integrals

Prove that $\iint_D x^{l-1} y^{m-1}\,dx\,dy = \dfrac{\Gamma l\,\Gamma m}{\Gamma(l+m+1)}\, h^{l+m}$.

where D is the domain $x \geq 0$, $y \geq 0$

and $x+y \leq h$.

Sol: Let $x = hX$ and $y = hY$,

then $dx = h\,dX$ and $dy = h\,dY$.

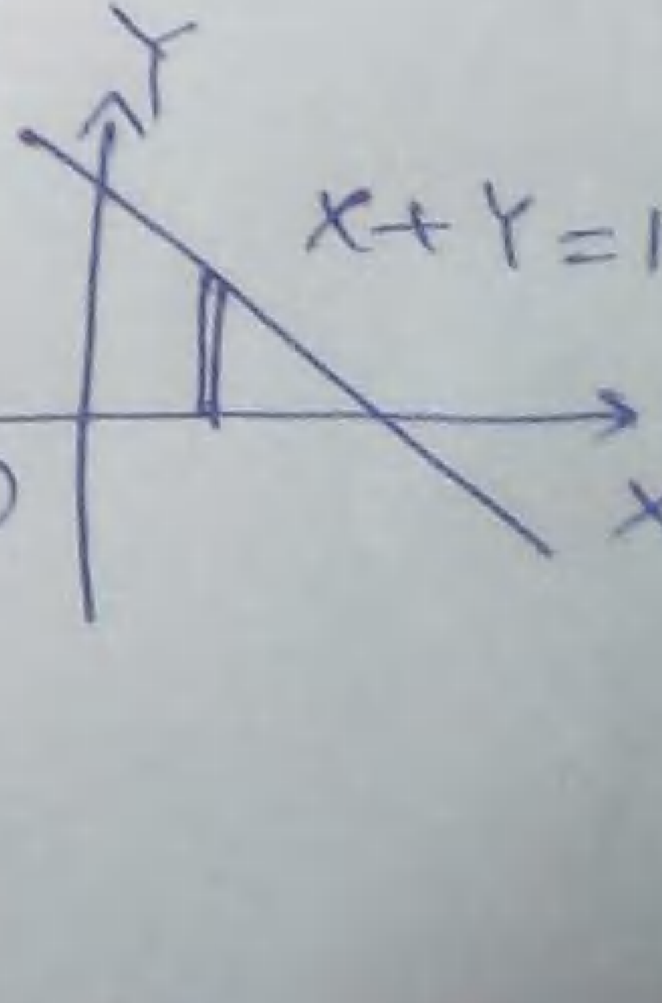
And $dx\,dy = h^2\,dX\,dY$.

Therefore, $\iint_D x^{l-1} y^{m-1}\,dx\,dy$

$$= \iint_{D'} (hX)^{l-1} (hY)^{m-1} h^2\,dX\,dY,$$

where D' is the domain $X \geq 0$, $Y \geq 0$ and $X+Y \leq 1$

$$= h^{l+m} \iint_{D'} X^{l-1} Y^{m-1}\,dX\,dY$$



$$= h^{l+m} \int_{X=0}^{1} \left(\int_{Y=0}^{1-X} X^{l-1} Y^{m-1}\,dY \right) dX$$

$$= h^{l+m} \int_{x=0}^{1} x^{l-1} \left[\frac{Y^m}{m}\right]_0^{1-x} dx$$

$$= \frac{h^{l+m}}{m} \int_{x=0}^{1} x^{l-1} (1-x)^m \, dx$$

$$= \frac{h^{l+m}}{m} \int_{x=0}^{1} x^{l-1} (1-x)^{(m+1)-1} \, dx$$

$$= \frac{h^{l+m}}{m} \, \beta(l, m+1)$$

$$= \frac{h^{l+m}}{m} \frac{\Gamma l \, \Gamma m+1}{\Gamma(l+m+1)} = \frac{h^{l+m}}{\cancel{m}} \frac{\Gamma l \, \cancel{m} \Gamma m}{\Gamma(l+m+1)}$$

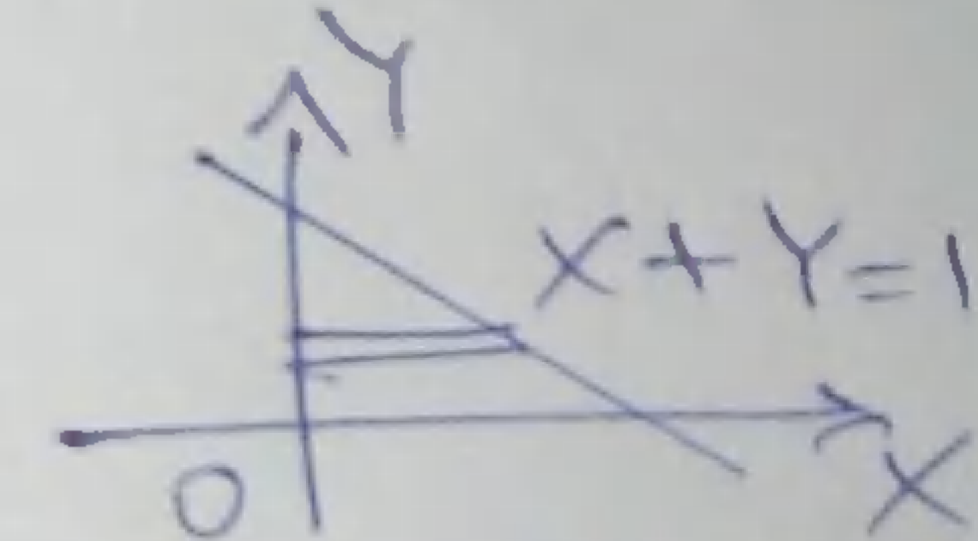
$$= h^{l+m} \frac{\Gamma l \, \Gamma m}{\Gamma(l+m+1)}$$

(i) Evaluate $\iint x^{m-1} y^{n-1}\, dx dy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ interms of Gamma function.

Sol: Let $\frac{x^2}{a^2} = X$ and $\frac{y^2}{b^2} = Y$, then we get

$dx = \frac{a}{2} X^{-\frac{1}{2}} dX$ and $dy = \frac{b}{2} Y^{-\frac{1}{2}} dY$.

And the region of the integration in the XY- plane is given by $X \geq 0$, $Y \geq 0$ and $X+Y \leq 1$.



Therefore $\iint x^{m-1} y^{n-1}\, dx dy$

$$= \int_0^1 \int_0^{1-Y} (a\sqrt{X})^{m-1} (b\sqrt{Y})^{n-1} \frac{a}{2} X^{-\frac{1}{2}} \frac{b}{2} Y^{-\frac{1}{2}} dX dY.$$

$$= \frac{a^m b^n}{4} \int_0^1 \int_0^{1-Y} X^{\frac{m-1}{2}} Y^{\frac{n-1}{2}} X^{-\frac{1}{2}} Y^{-\frac{1}{2}} dX dY$$

$$= \frac{a^m b^n}{4} \int_0^1 \int_0^{1-Y} X^{\frac{m}{2}-1} Y^{\frac{n}{2}-1} dX dY$$

$$= \frac{a^m b^n}{4} \cdot \frac{2}{m} \int_0^1 Y^{\frac{n}{2}-1} (1-Y)^{\frac{m}{2}} dY$$

$$= \frac{a^m b^n}{2m} \int_0^1 Y^{\frac{n}{2}-1} (1-Y)^{\left(\frac{m}{2}+1\right)-1} \, dY$$

$$= \frac{a^m b^n}{2m} \beta\left(\frac{n}{2}, \frac{m+2}{2}\right)$$

$$= \frac{a^m b^n}{2m} \frac{\Gamma\frac{n}{2} \cdot \Gamma\frac{m+2}{2}}{\Gamma\left(\frac{n}{2}+\frac{m}{2}+1\right)}$$

$$= \frac{a^m b^n}{2m} \frac{\Gamma\frac{n}{2} \cdot \frac{m}{2}\Gamma\frac{m}{2}}{\Gamma\left(\frac{m+n}{2}+1\right)}$$

$$= \frac{a^m b^n}{4} \frac{\Gamma\frac{n}{2}\Gamma\frac{m}{2}}{\Gamma\left(\frac{m+n}{2}+1\right)}$$

<u>Dirchlet's</u> <u>Integral</u>:

Prove that $\iiint_V x^{l-1} y^{m-1} z^{n-1} \, dx\,dy\,dz$

$$= \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma(l+m+n+1)},$$

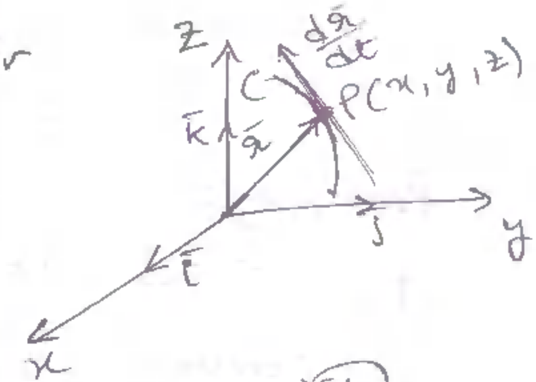
where V is the region $x \geq 0, y \geq 0, z \geq 0$ and $x+y+z \leq 1$.

Module-8:

Vector Differentiation:

As a particle moves along a curve C in space, its position P changes with time t. Accordingly, the co-ordinates (x, y, z) of P are functions of t.

Then the position vector $\bar{r}$ of P becomes



$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

or, $\bar{r} = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k} \longrightarrow (1)$

This is called the ~~curve~~ vector equation of the curve C.

Velocity and Accelaration:

Since equation (1) determines the position of a particle moving along the curve C at any time t, the rate of change of position is given by

$$\frac{d\bar{r}}{dt} = \frac{dx(t)}{dt}\bar{i} + \frac{dy(t)}{dt}\bar{j} + \frac{dz(t)}{dt}\bar{k}$$

This gives the velocity of the particle at time t.

Unit tangent vector to the curve C

is $\dfrac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|}$

and acceleration of the ~~curve~~ particle at time t is

$$\frac{d}{dt}\left(\frac{d\bar{r}}{dt}\right) = \frac{d^2\bar{r}}{dt^2}.$$

Problems:

1. Find the unit tangent vector to the curve $\bar{r}(t) = 4\sin t\,\bar{i} + 4\cos t\,\bar{j} + 3t\,\bar{k}$

2. A particle moving along the curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$. Determine the velocity and acceleration and their magnitudes at any time t.

— x —

Vector point function: Let $S \subseteq R$. If to each t in S there corresponds a unique vector $\bar{u}(t)$ then we say that $\bar{u}$ is a vector function defined on domain S.

Limit of a vector function: Let $\bar{F}(t)$ be a vector function defined in a deleted neighbourhood of t_0 and $\bar{A}$ be a given vector. If to each $\epsilon > 0$ there corresponds a $\delta > 0$ such that,
$0 < |t - t_0| < \delta \Rightarrow |\bar{F}(t) - \bar{A}| < \epsilon$, then $\bar{F}(t)$ is said to tend to vector $\bar{A}$ as $t \to t_0$.

In this case we write $\lim_{t \to t_0} \bar{F}(t) = \bar{A}$.

Continuity of a vector function:

A vector $\bar{F}(t)$ is said to be continuous at $t = t_0$ if $\lim_{t \to t_0} \bar{F}(t) = \bar{F}(t_0)$.

If $\bar{F}$ is continuous at every point in the domain S then we say that $\bar{F}$ is continuous on S.

Derivative of a vector function

Let $\bar{F}$ be a vector function defined on S and $a \in S$. Then if the limit $\lim_{t \to a} \frac{\bar{F}(t) - \bar{F}(a)}{t - a}$ exists then we say that $\bar{F}$ is derivable at a.

Scalar point function: If to each $P(x,y,z)$ of a region R there corresponds a scalar quantity $\phi(x,y,z)$ then ϕ is called scalar point function.

Vector differential operator ∇

The vector differential operator ∇ is defined by

$$\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$$

Gradient of a Scalar point function

Let $\phi(x,y,z)$ be a scalar point function defined in some region of space. Then the vector function $\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$ is called the gradient of ϕ and is denoted by grad ϕ or $\nabla\phi$

i.e $$\text{grad}\,\phi = \nabla\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

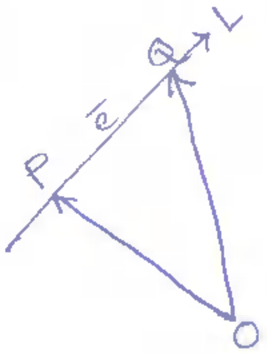
Directional derivative at a point

Let P be a point in space and L be a ray through P in the direction of a given unit vector $\bar{e}$. Let a scalar point function ϕ be defined in a neighbourhood D of P. Let $Q \neq P$ and $Q \in L \cap D$.

If $\lim\limits_{Q \to P} \dfrac{\phi(Q) - \phi(P)}{QP}$ exists, then it is called the directional derivative of scalar point function ϕ at P in the direction of $\bar{e}$.

It is denoted by $\dfrac{\partial \phi}{\partial \bar{e}}$ or $\dfrac{\partial \phi}{\partial s}$ where $s = PQ$

The directional derivative of ϕ in the direction of unit vector $\bar{e}$ is given by $\bar{e} \cdot \nabla \phi$ (or) $\bar{e} \cdot \text{grad}\,\phi$.



Problems:

1) Prove that $\nabla r^n = n r^{n-2} \bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Sol. Given $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Then $r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$

$\therefore r^2 = x^2 + y^2 + z^2$

$\Rightarrow 2r \dfrac{\partial r}{\partial x} = 2x$

$\Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

Now, $\nabla r^n = \bar{i} \dfrac{\partial}{\partial x}(r^n) + \bar{j} \dfrac{\partial}{\partial x}(r^n) + \bar{k} \dfrac{\partial}{\partial x}(r^n)$

$= \sum \bar{i} \dfrac{\partial}{\partial x}(r^n)$

$$= \sum \bar{i}\, n r^{n-1} \frac{\partial r}{\partial x}$$

$$= \sum \bar{i}\, n r^{n-1} \cdot \frac{x}{r}$$

$$= n r^{n-2} \sum x \bar{i}$$

$$= n r^{n-2} (x\bar{i} + y\bar{j} + z\bar{k})$$

$$\therefore \nabla r^n = n r^{n-2} \bar{r}. \quad (\because x\bar{i} + y\bar{j} + z\bar{k} = \bar{r})$$

2) Find a unit normal vector to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1,2,-1)$

Sol. Let $\phi(x,y,z) = x^3 + y^3 + 3xyz - 3$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}(3x^2 + 3yz) + \bar{j}(3y^2 + 3xz) + \bar{k}(3xy)$$

$$\therefore (\nabla\phi)_{at\,(1,2,-1)} = \bar{i}(3-6) + \bar{j}(12-3) + \bar{k}(6)$$

$$= -3\bar{i} + 9\bar{j} + 6\bar{k}$$

$$= 3(-\bar{i} + 3\bar{j} + 2\bar{k})$$

$$\text{unit normal vector} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{3(-\bar{i} + 3\bar{j} + 2\bar{k})}{3\sqrt{1+9+4}}$$

$$= (-\bar{i} + 3\bar{j} + 2\bar{k})/\sqrt{14}$$

3) Find the directional derivative of the function $f(x,y,z) = 2xy + z^2$ at the point $(1,-1,3)$ in the direction of the vector $\bar{i} + 2\bar{j} + 2\bar{k}$.

sol. Given $f(x,y,z) = 2xy + z^2$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= 2y\bar{i} + 2x\bar{j} + 2z\bar{k}$$

$$\therefore (\nabla f)_{at\ (1,-1,3)} = -2\bar{i} + 2\bar{j} + 6\bar{k}$$

The unit vector in the given direction is

$$\bar{e} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1+4+4}} = \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})$$

$\therefore$ The directional derivative of f at $(1,-1,3)$

$$= \bar{e} \cdot \nabla \phi$$

$$= \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k}) \cdot (-2\bar{i} + 2\bar{j} + 6\bar{k})$$

$$= \frac{1}{3}(-2 + 4 + 12)$$

$$= \frac{14}{3}.$$

4) Find the directional derivative of xyz^2+xz at $(1,1,1)$ in the direction of the normal to the surface $3xy^2+y=z$ at $(0,1,1)$.

<u>Sol</u> Let $\phi(x,y,z) = xyz^2+xz$

Then $\nabla\phi = \bar{i}\dfrac{\partial\phi}{\partial x} + \bar{j}\dfrac{\partial\phi}{\partial y} + \bar{k}\dfrac{\partial\phi}{\partial z}$

$= (\cancel{z}+yz^2)\bar{i} + xz^2\bar{j} + (2xyz+x)\bar{k}$

$\therefore (\nabla\phi)_{at(1,1,1)} = 2\bar{i}+\bar{j}+3\bar{k}$

Now, let $f(x,y,z) = 3xy^2+y-z=0$

$\nabla f = \bar{i}\dfrac{\partial f}{\partial x} + \bar{j}\dfrac{\partial f}{\partial y} + \bar{k}\dfrac{\partial f}{\partial z}$

$= 3y^2\bar{i} + (6xy+1)\bar{j} - \bar{k}$

normal to the surface f at $(0,1,1)$ is

$\bar{n} = (\nabla f)_{at(0,1,1)}$

$= 3\bar{i}+\bar{j}-\bar{k}$

$\therefore$ unit normal vector $\bar{e} = \dfrac{\bar{n}}{|\bar{n}|}$

$= \dfrac{3\bar{i}+\bar{j}-\bar{k}}{\sqrt{9+1+1}}$

$= \dfrac{1}{\sqrt{11}}(3\bar{i}+\bar{j}-\bar{k})$

The directional derivative $= \bar{e} \cdot \nabla \phi$

$$= \frac{1}{\sqrt{11}}(3\bar{i}+\bar{j}-\bar{k}) \cdot (2\bar{i}+\bar{j}+3\bar{k})$$

$$= \frac{1}{\sqrt{11}}(6+1-3)$$

$$= \frac{4}{\sqrt{11}}.$$

5) Find the angle between the surfaces at $(2,-1,2)$, the surfaces are $x^2+y^2+z^2=9$, $z=x^2+y^2-3$

Sol. We know that the angle between two surfaces at a point is the angle between their normals at that point.

Let the given surfaces be

$$\phi_1 = x^2+y^2+z^2-9=0$$

$$\phi_2 = x^2+y^2-z-3=0$$

Now $\nabla\phi_1 = \bar{i}\frac{\partial\phi_1}{\partial x} + \bar{j}\frac{\partial\phi_1}{\partial y} + \bar{k}\frac{\partial\phi_1}{\partial z}$

$$= 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

Normal to surface ϕ_1 at $(2,-1,2)$ is

$$\bar{n}_1 = (\nabla\phi_1)_{at\,(2,-1,2)}$$

$$= 4\bar{i} - 2\bar{j} + 4\bar{k}$$

$$\nabla \phi_2 = \bar{i}\,\frac{\partial \phi_2}{\partial x} + \bar{j}\,\frac{\partial \phi_2}{\partial y} + \bar{k}\,\frac{\partial \phi_2}{\partial z}$$

$$= 2x\bar{i} + 2y\bar{j} - \bar{k}$$

Normal to the surface ϕ_2 at $(2,-1,2)$ is

$$\bar{n}_2 = (\nabla \phi_2)_{at\ (2,-1,2)}$$

$$= 4\bar{i} - 2\bar{j} - \bar{k}$$

Let θ be the angle between their normals at $(2,-1,2)$. Then

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1|\,|\bar{n}_2|}$$

$$= \frac{(4\bar{i} - 2\bar{j} + 4\bar{k})\cdot(4\bar{j} - 2\bar{j} - \bar{k})}{\sqrt{16+4+16}\ \sqrt{16+4+1}}$$

$$= \frac{16+4-4}{\sqrt{36}\ \sqrt{21}}$$

$$= \frac{16}{6\sqrt{21}}$$

$$= \frac{8}{3\sqrt{21}}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Divergence of a vector point function

Let $\bar{f}$ be a continuously differentiable vector point function. Then the divergence of $\bar{f}$ is defined as

$$\operatorname{div} \bar{f} = \nabla \cdot \bar{f} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \bar{f}$$

$$= \bar{i} \cdot \frac{\partial \bar{f}}{\partial x} + \bar{j} \cdot \frac{\partial \bar{f}}{\partial y} + \bar{k} \cdot \frac{\partial \bar{f}}{\partial z}$$

$$= \sum \bar{i} \cdot \frac{\partial \bar{f}}{\partial x}$$

It is clear that $\operatorname{div} \bar{f}$ is a scalar function.

If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then we have

$$\operatorname{div} \bar{f} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \left(f_1\bar{i} + f_2\bar{j} + f_3\bar{k}\right)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Note :- If $\operatorname{div} \bar{f} = 0$, then $\bar{f}$ is said to be solenoidal vector.

Physical interpretation : If $\bar{v}$ represents the velocity of a fluid, then $\operatorname{div} \bar{v}$ gives the rate at which the fluid is originating at a point per unit volume. This justifies the name divergence

Problems:

1) Show that $\nabla \cdot \bar{r} = 3$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Sol. Given $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\nabla \cdot \bar{r} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \left(x\bar{i} + y\bar{j} + z\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$= 3.$$

2) If $\bar{F} = (x+3y)\bar{i} + (y-3z)\bar{j} + (x-2z)\bar{k}$, prove that $\bar{F}$ is solenoidal.

Sol. Given $\bar{F} = (x+3y)\bar{i} + (y-3z)\bar{j} + (x-2z)\bar{k}$

$$\therefore \nabla \cdot \bar{F} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-3z) + \frac{\partial}{\partial z}(x-2z)$$

$$= 1 + 1 - 2$$

$$= 0$$

$\therefore$ $\bar{F}$ is solenoidal vector

3) Prove that $\operatorname{div}(r^n \bar{r}) = (n+3) r^n$. Hence show that $\frac{\bar{r}}{r^3}$ is solenoidal.

Sol. Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Then $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$$\therefore r^2 = x^2 + y^2 + z^2$$

$$\Rightarrow 2r \frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}. \text{ Similarly } \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now $r^n \bar{r} = r^n (x\bar{i} + y\bar{j} + z\bar{k})$

$$= (r^n x)\bar{i} + (r^n y)\bar{j} + (r^n z)\bar{k}$$

$$\operatorname{div}(r^n \bar{r}) = \sum \frac{\partial}{\partial x}(r^n x)$$

$$= \sum \left[r^n \cdot 1 + x \cdot n r^{n-1} \frac{\partial r}{\partial x} \right]$$

$$= \sum \left[r^n + x\, n r^{n-1} \frac{x}{r} \right]$$

$$= \sum \left[r^n + n r^{n-2} x^2 \right]$$

$$= \sum r^n + n r^{n-2} \sum x^2$$

$$= 3r^n + n r^{n-2} (x^2 + y^2 + z^2)$$

$$= 3r^n + nr^{n-2} \cdot r^2$$

$$= 3r^n + nr^n$$

$$= (n+3) r^n$$

$\therefore \operatorname{div}(r^n \bar{r}) = (n+3) r^n$

If $n = -3$ then $\operatorname{div}(r^n \bar{r}) = 0$

$\therefore r^n \bar{r}$ is solenoidal if $n = -3$

Then $r^{-3} \bar{r}$ is solenoidal

i.e $\dfrac{\bar{r}}{r^3}$ is Solenoidal.

4) Find $\operatorname{div} \bar{F}$ where $\bar{F} = \operatorname{grad}(x^3 + y^3 + z^3 - 3xyz)$

Sol. Given $\bar{F} = \operatorname{grad}(x^3 + y^3 + z^3 - 3xyz)$

Let $\phi = x^3 + y^3 + z^3 - 3xyz$

Then $\bar{F} = \operatorname{grad} \phi$.

$$= \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$$

$$= (3x^2 - 3yz)\bar{i} + (3y^2 - 3xz)\bar{j} + (3z^2 - 3xy)\bar{k}$$

$$\therefore \operatorname{div} \bar{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

$$= 6x + 6y + 6z$$

$$= 6(x + y + z)$$

Curl of a vector point function

Let $\bar{F}$ be a continuously differentiable vector point function. Then the curl of $\bar{F}$ is defined as $\bar{i} \times \frac{\partial \bar{F}}{\partial x} + \bar{j} \times \frac{\partial \bar{F}}{\partial y} + \bar{k} \times \frac{\partial \bar{F}}{\partial z}$ and is denoted by curl $\bar{F}$ or $\nabla \times \bar{F}$

i.e $\quad \text{curl}\,\bar{F} = \nabla \times \bar{F} = \bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{F}}{\partial y} + \bar{k} \times \frac{\partial \bar{F}}{\partial z}$

$$= \sum \bar{i} \times \frac{\partial \bar{F}}{\partial x}$$

If $\bar{F} = F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k}$ then

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

It is clear that curl $\bar{F}$ is a vector point function.

Irrotational vector

A vector $\bar{F}$ is said to be irrotational if $\nabla \times \bar{F} = \bar{0}$.

If $\bar{F}$ is irrotational then there exists a scalar function $\phi(x,y,z)$ such that $\bar{F} = \nabla\phi$. This ϕ is called scalar potential of $\bar{F}$.

Problems:

1) If $\bar{A} = 3xz^2\bar{i} - yz\bar{j} + (x+2z)\bar{k}$, find curl curl $\bar{A}$

Sol. $$\text{Curl}\,\bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xz^2 & -yz & x+2z \end{vmatrix}$$

$$= \bar{i}(0+y) - \bar{j}(1-6xz) + \bar{k}(0)$$

$$= y\bar{i} + (6xz-1)\bar{j}$$

$$\therefore \text{Curl Curl}\,\bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 6xz-1 & 0 \end{vmatrix}$$

$$= \bar{i}(0-6x) - \bar{j}(0-0) + \bar{k}(6z-1)$$

$$= -6x\bar{i} + (6z-1)\bar{k}$$

1) If $f(r)$ is differentiable, show that $\text{curl}\{\bar{r} f(r)\} = 0$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Sol. $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\therefore r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

$\Rightarrow 2r\dfrac{\partial r}{\partial x} = 2x$

$\Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

Now $\bar{r} f(r) = (x\bar{i} + y\bar{j} + z\bar{k}) f(r)$

$= xf(r)\bar{i} + yf(r)\bar{j} + zf(r)\bar{k}$

$$\therefore \text{curl}\{\bar{r} f(\bar{r})\} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$$= \sum \bar{i}\left[\frac{\partial}{\partial y}(zf(r)) - \frac{\partial}{\partial z}(yf(r))\right]$$

$$= \sum \bar{i}\left[z f'(r)\frac{\partial r}{\partial y} - y f'(r)\frac{\partial r}{\partial z}\right]$$

$$= \sum \bar{i}\left[z f'(r)\frac{y}{r} - y f'(r)\frac{z}{r}\right]$$

$$= \sum \bar{i}\,\frac{f'(r)}{r}\left[zy - yz\right]$$

$$= \bar{0}$$

3. Show that the vector $(x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ is irrotational and find its scalar potential.

Sol. Let $\bar{F} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$

$$\operatorname{curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$= \bar{i}(-x+x) - \bar{j}(-y+y) + \bar{k}(-z+z)$$

$$= \bar{i}(0) - \bar{j}(0) + \bar{k}(0)$$

$$= \bar{0}$$

$\therefore$ F is irrotational vector.

Hence there exists a scalar function ϕ such that $\bar{F} = \nabla\phi$

$$\Rightarrow (x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k} = \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}$$

$$\therefore \frac{\partial\phi}{\partial x} = x^2-yz$$

$$\frac{\partial\phi}{\partial y} = y^2-zx$$

$$\frac{\partial\phi}{\partial z} = z^2-xy$$

we know $d\phi = \frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz$

i.e $d\phi = (x^2-yz)dx+(y^2-zx)dy+(z^2-xy)dz$

$$= x^2dx+y^2dy+z^2dz-(yzdx+zxdy+xydz)$$

$$\Rightarrow d\phi = x^2dx+y^2dy+z^2dz-d(xyz)$$

Integrating, we get

$$\phi = \frac{1}{3}(x^3+y^3+z^3)-xyz+c$$

4) Find the constants a, b, c so that $\bar{F} = (x+2y+az)\bar{i}+(bx-3y-z)\bar{j}+(4x+cy+2z)\bar{k}$ is irrotational. Also find ϕ such that $\bar{F} = \nabla\phi$

Sol. Given $\bar{F} = (x+2y+az)\bar{i}+(bx-3y-z)\bar{j}$
$+(4x+cy+2z)\bar{k}$

$$\operatorname{curl} \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \bar{i}\,(c+1) - \bar{j}\,(4-a) + \bar{k}\,(b-2)$$

Since $\bar{F}$ is irrotational $\operatorname{curl} \bar{F} = \bar{0}$

$$\therefore c+1=0, \quad 4-a=0, \quad b-2=0$$

$$\Rightarrow c=-1, \quad a=4, \quad b=2$$

$\therefore \bar{F} = (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k}$

is irrotational.

Hence there exits a scalar potential ϕ

such that $\bar{F} = \nabla\phi$

$$\therefore (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k}$$

$$= \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\therefore \frac{\partial\phi}{\partial x} = x+2y+4z, \quad \frac{\partial\phi}{\partial y} = 2x-3y-z$$

$$\frac{\partial\phi}{\partial z} = 4x-y+2z$$

Now $\frac{\partial \phi}{\partial x} = x + 2y + 4z$

$$\Rightarrow \phi = \int_x (x + 2y + 4z)\,dx = \frac{x^2}{2} + 2xy + 4xz + f_1(y, z) \longrightarrow 1)$$

$$\frac{\partial \phi}{\partial y} = 2x - 3y - z$$

$$\Rightarrow \phi = 2xy - \frac{3y^2}{2} - yz + f_2(x, z) \longrightarrow 2)$$

and $\frac{\partial \phi}{\partial z} = 4x - y + 2z$

$$\Rightarrow \phi = 4xz - yz + z^2 + f_3(x, y) \longrightarrow 3)$$

from eq: 1), 2), and 3), the Scalar potential satisfying these three equations is

$$\phi = \frac{x^2}{2} + 2xy + 4xz - yz + z^2 - \frac{3y^2}{2} + C$$

i.e $\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4xz + C$

Laplacian operator ∇^2

The Laplacian operator ∇^2 is defined as

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and the equation $\nabla^2 \phi = 0$ i.e $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$

is called the Laplace equation.

pb) Prove that curl grad$\phi = \bar{0}$

Sol. $\text{grad}\,\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$

$$\text{curl}(\text{grad}\,\phi) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \sum \bar{i}\left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right]$$

$$= \sum \bar{i}\,(0)$$

$$= \bar{0}$$

Hence grad ϕ is always irrotational.

Module - 9:

Vector Integration

Line Integral

Any integral which is evaluated along a curve is called a line integral

Let F be a continuous vector point function, defined at each point of a curve C. Then the line integral $\overline{F}$ along C is given by

$$\int_C \overline{F}\cdot d\overline{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz ,$$

where $\overline{F} = F_1\overline{i} + F_2\overline{j} + F_3\overline{k}$ and $d\overline{r} = \overline{i}\,dx + \overline{j}\,dy + \overline{k}\,dz$

Circulation:

If $\overline{v}$ represents the velocity of a fluid particle and C is a closed curve, then the integral $\oint_C \overline{v}\cdot d\overline{r}$ is called the circulation of $\overline{v}$ around C.

Work done by a force:

If $\overline{F}$ represents the force vector acting on a particle moving along an arc AB, then workdone during a small displacement $\delta\overline{r}$ is $\overline{F}\cdot d\overline{r}$

Hence, the total work done by $\overline{F}$ during the displacement from A to B is given by the line integral $\int_A^B \overline{F}\cdot d\overline{r}$

Problems:

1. If $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$, evaluate $\int_C \bar{F} \cdot d\bar{r}$ along the curve C in xy-plane $y = x^3$ from the point $(1,1)$ to $(2,8)$

Sol. Given curve C is $y = x^3$

$$\bar{F} \cdot d\bar{r} = [(5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}] \cdot (dx\,\bar{i} + dy\,\bar{j})$$

$$= (5xy - 6x^2)\,dx + (2y - 4x)\,dy$$

Along $y = x^3$ we have $dy = 3x^2\,dx$

and x varies from 1 to 2.

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_{(1,1)}^{(2,8)} [(5xy - 6x^2)\,dx + (2y - 4x)\,dy]$$

$$= \int_{x=1}^{2} [(5x^4 - 6x^2)\,dx + (2x^3 - 4x)\,3x^2\,dx]$$

$$= \int_1^2 [5x^4 - 6x^2 + 6x^5 - 12x^3]\,dx$$

$$= \left[x^5 - 2x^3 + x^6 - 3x^4\right]_1^2$$

$$= (32 - 16 + 64 - 48) - (1 - 2 + 1 - 3)$$

$$= 35$$

2 Find the workdone in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$

Sol Given $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$

$$\therefore \bar{F}\cdot d\bar{r} = 3x^2dx + (2xz - y)dy + z\,dz$$

The equation of the straight line joining $(0,0,0)$ to $(2,1,3)$ is

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}$$

Then $x = 2t$, $y = t$, $z = 3t$, t varies from 0 to 1

$$\text{workdone by } \bar{F} = \int_{(0,0,0)}^{(2,1,3)} \bar{F}\cdot d\bar{r}$$

$$= \int_{t=0}^{1} [3(2t)^2 2dt + (2\cdot 2t\cdot 3t - t)dt + 3t\cdot 3dt]$$

$$= \int_0^1 (36t^2 + 8t)\,dt$$

$$= (12t^3 + 4t^2)_0^1$$

$$= 12 + 4$$

$$= 16$$

3. If $\bar{F} = (y-2x)\bar{i} + (3x+2y)\bar{j}$, calculate the circulation of $\bar{F}$ about the circle C in the plane $x^2+y^2=4$ oriented in the anticlockwise direction.

Sol The curve is $C: x^2+y^2=4$ in xy plane

Then the parametric equations are

$x = 2\cos t,\ y = 2\sin t$, t varing from 0 to 2π

$\Rightarrow dx = -2\sin t\,dt,\ dy = 2\cos t\,dt$

$$\bar{F}\cdot d\bar{r} = (y-2x)dx + (3x+2y)dy$$

$$= (2\sin t - 4\cos t)(-2\sin t\,dt) + (6\cos t + 4\sin t)(2\cos t\,dt)$$

$$= (-4\sin^2 t + 8\sin t\cos t + 12\cos^2 t + 8\sin t\cos t)dt$$

$$= (16\sin t\cos t + 16\cos^2 t - 4)dt$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi}(16\sin t\cos t + 16\cos^2 t - 4)dt$$

$$= \int_0^{2\pi}\left[8\sin 2t + 16\left(\frac{1+\cos 2t}{2}\right) - 4\right]dt$$

$$= \left[8\left(-\frac{\cos 2t}{2}\right) + 8\left(t + \frac{\sin 2t}{2}\right) - 4t\right]_0^{2\pi}$$

$$= -4 + 8(2\pi + 0) - 8\pi - (-4)$$

$$= 16\pi - 8\pi = 8\pi$$

Surface integrals

Any integral which is to be evaluated over a surface is called a surface integral.

Let $\bar{F}$ be a continuous vector point function. Then the surface integral of $\bar{F}$ over a surface S is given by $\int_S \bar{F}\cdot\bar{n}\,ds$, where $\bar{n}$ is the outward drawn unit normal vector at any point of S.

If $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$, then we have

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint F_1\,dy\,dz + \iint F_2\,dz\,dx + \iint F_3\,dx\,dy$$

Note: Let R_1, R_2, R_3 be the projections of the surface S on xy plane, yz plane, zx planes respectively. Then

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_1} \frac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{k}|}\,dx\,dy$$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_2} \frac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{i}|}\,dy\,dz$$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_3} \frac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{j}|}\,dx\,dz.$$

Problems:

1. Evaluate $\iint \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = (x^2+y^2)\bar{i} - 2x\bar{j} + 2yz\bar{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first octant.

Sol. The normal to the surface S is given by

$\nabla(2x+y+2z) = 2\bar{i}+\bar{j}+\bar{k}$

$\therefore$ unit normal is $\bar{n} = \dfrac{2\bar{i}+\bar{j}+2\bar{k}}{\sqrt{4+1+4}}$

$= \frac{1}{3}(2\bar{i}+\bar{j}+2\bar{k})$

$\Rightarrow \bar{n}\cdot\bar{k} = \frac{2}{3}$

Also $\bar{F}\cdot\bar{n} = [(x^2+y^2)\bar{i} - 2x\bar{j} + 2yz\bar{k}]\cdot\frac{1}{3}(2\bar{i}+\bar{j}+2\bar{k})$

$= \frac{2}{3}(x^2+y^2) - \frac{2}{3}x + \frac{4}{3}yz$

$= \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6-2x-y}{2}\right)$

$= \frac{4y}{3}(3-x)$ $\quad$ ($\because$ since on S, $z = \frac{1}{2}(6-2x-y)$)

$\therefore \iint\limits_S \bar{F}\cdot\bar{n}\,ds = \iint\limits_R \dfrac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{k}|}\,dx\,dy$, where R is projection of S on xy plane

$= \iint\limits_R \frac{4}{3}y(3-x)\cdot\frac{3}{2}\,dx\,dy$

$$= \int_{x=0}^{3} \int_{y=0}^{6-2x} 2y(3-x)\,dy\,dx$$

$$= \int_0^3 2(3-x)\left[\frac{y^2}{2}\right]_0^{6-2x} dx$$

$$= 4\int_0^3 (3-x)^3 dx$$

$$= 4\left[\frac{(3-x)^4}{4(-1)}\right]_0^3$$

$$= -[0-81] = 81$$

2. Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$, where $\bar{F} = z\bar{i} + x\bar{j} + 3y^2 z\bar{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$

Sol. Given that S is the surface of $x^2+y^2=16$

normal to the surface $= \nabla S$

$$= \bar{i}\frac{\partial S}{\partial x} + \bar{j}\frac{\partial S}{\partial y} + \bar{k}\frac{\partial S}{\partial z}$$

$$= 2x\bar{i} + 2y\bar{j}$$

$$= 2(x\bar{i} + y\bar{j})$$

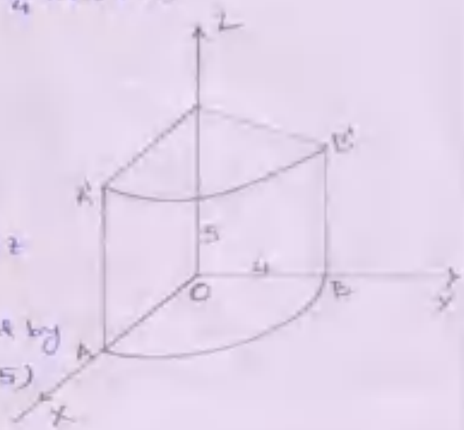
$\therefore$ unit normal $\bar{n} = \dfrac{2(x\bar{i}+y\bar{j})}{2\sqrt{x^2+y^2}}$

i.e $\bar{n} = \dfrac{2(x\bar{i}+y\bar{j})}{2\sqrt{16}} = \dfrac{1}{4}(x\bar{i}+y\bar{j})$

Let R be the projection of S on YZ plane

$\bar{F}\cdot\bar{n} = (z\bar{i} + x\bar{j} + 3y^2 z\bar{k})\cdot\frac{1}{4}(x\bar{i}+y\bar{j})$

$= \frac{1}{4}(xz + xy) = \frac{1}{4}x(y+z)$



and $\bar{n}\cdot\bar{i} = \frac{1}{4}x$

$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \dfrac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{i}|}\,dy\,dz$

$= \int_{z=0}^{5}\int_{y=0}^{4} \frac{1}{4}x(y+z)\cdot\frac{4}{x}\,dy\,dz$

(since the region R is enclosed by $y=0$ to $y=4$ and $z=0$ to $z=5$)

$= \int_{z=0}^{5}\left(\frac{y^2}{2} + yz\right)_{y=0}^{4} dz$

$= \int_0^5 (8+4z)\,dz$

$= (8z + 2z^2)_0^5$

$= 40 + 50$

$= 90$

Volume Integrals

Any integral which is evaluated over a volume is called a volume integral.

If V is the volume bounded by a surface S, then $\iiint_V \bar{F}\,dV$ is called the volume integral of $\bar{F}$

Problems

1. If $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$, evaluate $\int_V \bar{F}\,dV$, where V is the volume of the region bounded by the surfaces $x=0$, $x=1$, $y=0$, $y=6$, $z=x^2$, $z=4$.

Sol: $$\int_V \bar{F}\,dV = \int_{x=0}^{1}\int_{y=0}^{6}\int_{z=x^2}^{4}(2xz\bar{i} - x\bar{j} + y^2\bar{k})\,dx\,dy\,dz$$

$$= \bar{i}\int_0^1\int_0^6\int_{x^2}^4 2xz\,dx\,dy\,dz - \bar{j}\int_0^1\int_0^6\int_{x^2}^4 x\,dx\,dy\,dz + \bar{k}\int_0^1\int_0^6\int_{x^2}^4 y^2\,dx\,dy\,dz \longrightarrow 1)$$

consider, $\int_0^1\int_0^6\int_{x^2}^4 2xz\,dx\,dy\,dz$

$$= \int_0^1\int_0^6 x(z^2)_{x^2}^4\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{6} x(16-x^4)\,dy\,dx.$$

$$= \int_{0}^{1} x(16-x^4)\,[y]_0^6\,dx$$

$$= \int_{0}^{1} 6x(16-x^4)\,dx$$

$$= 6\int_{0}^{1} (16x-x^5)\,dx$$

$$= 6\left[8x^2-\frac{x^6}{6}\right]_0^1$$

$$= 6\left[8-\frac{1}{6}\right] = 47.$$

Now $\displaystyle\int_{x=0}^{1}\int_{y=0}^{6}\int_{z=x^2}^{4} x\,dx\,dy\,dz = \int_{0}^{1}\int_{0}^{6} x\,(z)_{x^2}^{4}\,dy\,dx$

$$= \int_{0}^{1}\int_{0}^{6} x(4-x^2)\,dy\,dx$$

$$= \int_{0}^{1}(4x-x^3)\,[y]_0^6\,dx$$

$$= 6\int_{0}^{1}(4x-x^3)\,dx$$

$$= 6\left[2x^2-\frac{x^4}{4}\right]_0^1$$

$$= 6\left[2-\frac{1}{4}\right]$$

$$= 21/2.$$

Finally, consider $\int_{x=0}^{1}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2\,dx\,dy\,dz$

$$= \int_0^1\int_0^6 y^2 (z)_{x^2}^{4}\,dy\,dx$$

$$= \int_0^1 \left(\frac{y^3}{3}\right)_0^6 (4-x^2)\,dx$$

$$= \frac{6^3}{3}\int_0^1 (4-x^2)\,dx$$

$$= 72\left[4x - \frac{x^3}{3}\right]_0^1$$

$$= 72\left(4-\frac{1}{3}\right)$$

$$= 72\left(\frac{11}{3}\right)$$

$$= 264$$

Substitute these values in eq 1), we get

$$\int_V \bar{F}\,dV = 47\bar{i} - \frac{21}{2}\bar{j} + 264\bar{k}$$

2. If $\bar{F} = (2x^2-3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$, evaluate $\iiint_V \text{curl}\,\bar{F}\,dV$, where V is the closed region bounded by the planes $x=0, y=0, z=0$ and $2x+2y+z=4$.

Sol. Curl $\bar{F}$ = $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2-3z & -2xy & -4x \end{vmatrix}$

$$= \bar{i}(0) - \bar{j}(-4+3) + \bar{k}(-2y-0)$$

$$= \bar{j} - 2y\bar{k}$$

The limits are $z = 0$ to $z = 4-2x-2y$

$y = 0$ to $y = 2-x$

$x = 0$ to $x = 2$

$$\therefore \iiint \nabla\times\bar{F}\, dV = \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} (\bar{j} - 2y\bar{k})\, dx\, dy\, dz$$

$$= \int_0^2 \int_0^{2-x} (\bar{j} - 2y\bar{k}) \left[z\right]_0^{4-2x-2y} dx\, dy$$

$$= \int_0^2 \int_0^{2-x} (\bar{j} - 2y\bar{k})(4-2x-2y)\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} \left[\bar{j}(4-2x-2y) - 2y\bar{k}(4-2x-2y)\right] dy\, dx$$

$$= \int_0^2 \bar{j}\left[4y - 2xy - y^2\right]_{y=0}^{2-x} - 2\bar{k}\left[2y^2 - xy^2 - \frac{2y^3}{3}\right]_{y=0}^{2-x} dx$$

$$= \int_0^2 \left[\bar{j}\left[4(2-x) - 2x(2-x) - (2-x)^2\right]\right.$$

$$-\frac{2}{3}\bar{k}\left[6(2-x)^2-3x(2-x)^2-2(2-x)^3\right]\Big\}\,dx$$

$$=\int_{x=0}^{2}\left[(2-x)^2\bar{j}-\frac{2}{3}(2-x)^3\bar{k}\right]dx$$

$$=\left[\frac{(2-x)^3}{-3}\bar{j}+\frac{2}{3}\frac{(2-x)^4}{4}\bar{k}\right]_{x=0}^{2}$$

$$=\frac{8}{3}\bar{j}-\frac{8}{3}\bar{k}=\frac{8}{3}(\bar{j}-\bar{k})$$

Vector integral Theorems

Gauss Divergence theorem

If $\bar{F}$ is a vector point function having continuous first order partial derivatives over a closed surface S enclosing a volume V, then

$$\iiint_V \nabla\cdot\bar{F}\,dV=\iint_S \bar{F}\cdot\bar{n}\,ds,$$

where $\bar{n}$ is the outward drawn unit normal vector to the surface S.

Problems

1. Verify Gauss divergence theorem for $\overline{F} = (x^3 - yz)\overline{i} - 2x^2y\overline{j} + z\overline{k}$ taken over the surface of the cube bounded by the planes $x = y = z = 2$ and the coordinate planes

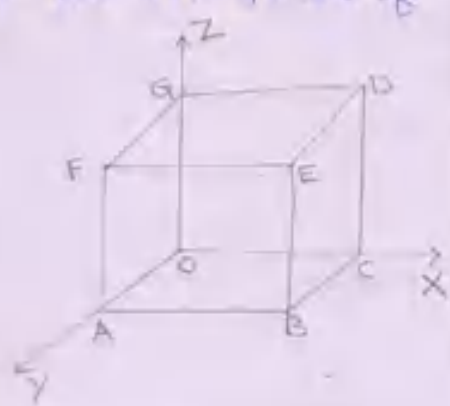
Sol. By Gauss divergence theorem,

$$\int_S \overline{F}\cdot\overline{n}\,ds = \int_V \nabla\cdot\overline{F}\,dV$$

consider the surface integral

$$\int_S \overline{F}\cdot\overline{n}\,ds = \int_{S_1} \overline{F}\cdot\overline{n}\,ds + \int_{S_2} \overline{F}\cdot\overline{n}\,ds + \int_{S_3} \overline{F}\cdot\overline{n}\,ds + \int_{S_4} \overline{F}\cdot\overline{n}\,ds + \int_{S_5} \overline{F}\cdot\overline{n}\,ds + \int_{S_6} \overline{F}\cdot\overline{n}\,ds \longrightarrow (1)$$

where S_1, S_2, S_3 are the six faces of the cube



Over the face S_1 i.e BCDE

$x = 2$

$\bar{n} = \bar{i}$ and $ds = dy\,dz$

$0 \le y \le 2$ and $0 \le z \le 2$

$\bar{F} \cdot \bar{n} = ((x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}) \cdot \bar{i}$

$= x^3 - yz$

$= 8 - yz \quad (\because x = 2$

$\therefore \int_{S_1} \bar{F} \cdot \bar{n}\,ds = \int_{z=0}^{2}\int_{y=0}^{2} (8 - yz)\,dy\,dz$

$= \int_{z=0}^{2} \left(8y - \frac{y^2}{2} z\right)_0^2 dz$

$= \int_0^2 (16 - 2z)\,dz$

$= \left(16z - z^2\right)_0^2$

$= 32 - 4 = 28$

over the face S_2 i.e AOGF

we have $x = 0$

$\bar{n} = -\bar{i}$ and $ds = dy\,dz$

$0 \le y \le 2, \; 0 \le z \le 2$

$\bar{F} \cdot \bar{n} = -(x^3 - yz) = yz \quad (\because x = 0$

$$\therefore \int_{S_2} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{2}\int_{y=0}^{2} yz\,dy\,dz$$

$$= \int_0^2 z\,dz \int_0^2 y\,dy$$

$$= \left(\frac{z^2}{2}\right)_0^2 \left(\frac{y^2}{2}\right)_0^2$$

$$= (2)(2) = 4.$$

over the face S_3 i.e FEDG

$z = 2 \Rightarrow dz = 0$

$\bar{n} = \bar{k}$ and $ds = dx\,dy$

$0 \le x \le 2$ and $0 \le y \le 2$

$\bar{F}\cdot\bar{n} = z = 2$

$$\therefore \int_{S_3} \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{2}\int_{x=0}^{2} 2\,dx\,dy$$

$$= 2\,(x)_0^2\,(y)_0^2$$

$$= 2(2)(2) = 8$$

over the face S_4 i.e OABC

$z = 0$

$\bar{n} = -\bar{k}$ and $ds = dx\,dy$

$0 \le x \le 2,\quad 0 \le y \le 2$

$\bar{F}\cdot\bar{n} = -z = 0 \quad (\because z = 0$

$$\therefore \int_{S_4} \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{2}\int_{x=0}^{2} 0\, dx\,dy = 0$$

Over the face S_5 ie ABEF

we have $y = 2$

$\bar{n} = \bar{j}$ and $ds = dx\,dz$

$0 \le x \le 2$ and $0 \le z \le 2$.

$$\int_{S_5} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{2}\int_{z=0}^{2} (-2x^2y)\,dx\,dz$$

$$= -4\int_{x=0}^{2}\int_{z=0}^{2} x^2\,dx\,dz$$

$$= -4\left(\frac{x^3}{3}\right)_0^2 (z)_0^2$$

$$= -4\left(\frac{8}{3}\right)(2)$$

$$= -\frac{64}{3}$$

Over the face S_6 ie OCDG

$y = 0$

$\bar{n} = -\bar{j}$ and $ds = dx\,dz$

$$\int_{S_6} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{2}\int_{z=0}^{2} 2x^2y\,dx\,dz$$

$$= 0 \quad (\because y = 0$$

$\therefore$ from 1), we get

$$\int_S \bar{F}\cdot\bar{n}\,ds = 28 + 4 + 8 + 0 - \frac{64}{3} + 0 = \frac{56}{3} \longrightarrow 2)$$

Consider the volume integral

$$\int_V \nabla\cdot F\,dV = \int_V (3x^2 - 2x^2 + 1)\,dV$$

$$= \int_{z=0}^{2}\int_{y=0}^{2}\int_{x=0}^{2} (x^2+1)\,dx\,dy\,dz$$

$$= \int_0^2\int_0^2 \left(\frac{x^3}{3} + x\right)_0^2 dy\,dz$$

$$= \int_0^2\int_0^2 \left(\frac{8}{3} + 2\right) dy\,dz$$

$$= \frac{14}{3}\int_0^2 dy \int_0^2 dz$$

$$= \frac{14}{3}(2)(2)$$

$$= \frac{56}{3} \longrightarrow 3)$$

$\therefore$ from 2) and 3), we have

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \nabla\cdot\bar{F}\,dV$$

Hence Gauss divergence theorem is ~~verif~~ verified

2. Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ by Gauss divergence theorem where $\bar{F} = 2x^2y\,\bar{i} - y^2\,\bar{j} + 4xz^2\,\bar{k}$ taken over the region $y^2+z^2=9$, $x=2$ in the first octant.

Sol: By Gauss divergence theorem.

$$\int_S F\cdot\bar{n}\,ds = \int_V \nabla\cdot F\,dv$$

Now $\nabla\cdot F = \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)$

$= 4xy - 2y + 8xz$

The limits are $z = 0$ to $\sqrt{9-y^2}$
$y = 0$ to 3 (∵ In first quadrant)
$x = 0$ to 2

$$\int_V \nabla\cdot F\,dv = \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}}\int_{x=0}^{2}(4xy - 2y + 8xz)\,dx\,dy\,dz$$

$$= \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}}\left[2x^2y - 2xy + 4x^2z\right]_0^2 dz\,dy$$

$$= \int_0^3\int_{z=0}^{\sqrt{9-y^2}}(8y - 4y + 16z)\,dz\,dy$$

$$= \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}}(4y + 16z)\,dz\,dy$$

$$= \int_0^3 \left[4yz + 8z^2\right]_0^{\sqrt{9-y^2}} dy$$

$$= 4\int_0^3 \left[y\sqrt{9-y^2} + 2(9-y^2)\right] dy$$

$$= 4\left[-\frac{1}{2}(9-y^2)^{3/2}\cdot\frac{2}{3} + 2\left(9y - \frac{y^3}{3}\right)\right]_0^3$$

$$= 4\left[0 + 9 + 36 - 0\right]$$

$$= \underline{\underline{180}}$$

Green's theorem in a plane

If R is a closed region in xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where C is traversed in positive (anticlockwise) direction

Problems

1. Verify Green's theorem for $\int_C[(3x^2-8y^2)dx+(4y-6xy)dy]$ where C is the boundary of the region bounded by $x=0, y=0$ and $x+y=1$.

Sol. Given $\int[(3x^2-8y^2)dx+(4y-6xy)]dy = \int_C \bar{F}\cdot d\bar{r}$

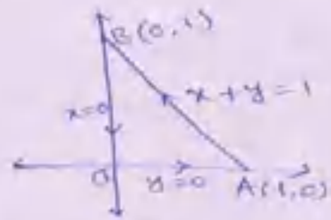
where $\bar{F} = (3x^2-8y^2)\bar{i} + (4y-6xy)\bar{j}$

and $d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where R is the region bounded by $x=0, y=0$ and $x+y=1$



Here $M = 3x^2 - 8y^2$

$N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$, $\frac{\partial N}{\partial x} = -6y$

Now $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (+16y - 6y)\,dx\,dy$

$$= 10\int_{x=0}^{1}\int_{y=0}^{1-x} y\,dy\,dx$$

$$= 10\int_0^1 \left(\frac{y^2}{2}\right)_0^{1-x} dx$$

$$= 5\int_0^1 (1-x)^2 dx$$

$$= 5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= -\frac{5}{3}(0-1)$$

$$= \frac{5}{3} \longrightarrow 1)$$

Now $\int_C M dx + N dy = \int_{OA} M dx + N dy + \int_{AB} M dx + N dy + \int_{BO} M dx + N dy \longrightarrow 2)$

Along OA :-

$y = 0,\ dy = 0$

x varies from 0 to 1

$$\int_{OA} M dx + N dy = \int_{x=0}^{1} 3x^2 dx \quad (\because y = 0 \text{ \& } dy = 0)$$

$$= (x^3)_0^1$$

$$= 1$$

Along AB

we have $x+y=1 \Rightarrow y=1-x$

$\Rightarrow dy = -dx$

x varies from 1 to 0

$$\int_{AB} Mdx + Ndy = \int_{OA} [(3x^2-8y^2)dx + (4y-6xy)dy]$$

$$= \int_{x=1}^{0} [(3x^2 - 8(1-x)^2)dx + (4(1-x) - 6x(1-x))(-dx)]$$

$$= \int_{1}^{0} [3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2]\,dx$$

$$= \int_{1}^{0} (-11x^2 + 26x - 12)\,dx$$

$$= \left(-11\frac{x^3}{3} + 26\frac{x^2}{2} - 12x\right)_1^0$$

$$= 0 - \left(-\frac{11}{3} + 13 - 12\right)$$

$$= \frac{11}{3} - 1$$

$$= \frac{8}{3} \longrightarrow 3)$$

Along BO,

$x=0$, $dx=0$

and y varies from 1 to 0

$$\int_{BO} (Mdx + Ndy) = \int_{y=1}^{0} 4y\,dy$$

$$= -2 \longrightarrow (4)$$

Hence

$$\int_C Mdx + Ndy = 1 + \frac{8}{3} - 2$$

$$= \frac{5}{3} \longrightarrow (5)$$

from 1) and 5), we get

$$\int_C M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Hence, Green's theorem is verified

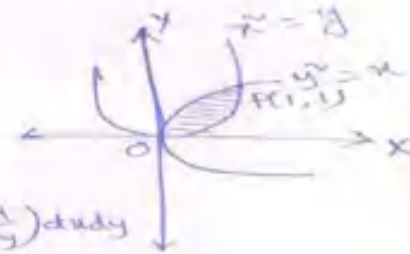
2. Using Green's theorem, evaluate the integral $\int_C (2xy - x^2)\,dx + (x^2+y^2)\,dy$, where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$

Sol: Given that $\int_C (2xy - x^2)\,dx + (x^2+y^2)\,dy$

$$= \int_C M\,dx + N\,dy$$

where $M = 2xy - x^2$, $N = x^2 + y^2$

$$\frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = 2x$$



By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (2x - 2x)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} 0\,dx\,dy$$

$$= 0$$

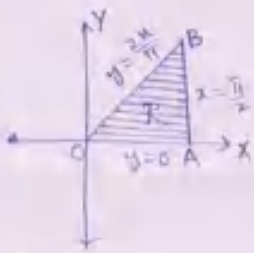
i.e. $\int_C (2xy - x^2)\,dx + (x^2+y^2)\,dy = 0$

3 Evaluate by Green's theorem $\oint_C [(y-\sin x)dx + \cos x\,dy]$ where C is the triangle enclosed by the lines $y=0$, $x=\frac{\pi}{2}$, $y=\frac{2x}{\pi}$

Sol. Given that $\oint_C [(y-\sin x)dx + \cos x\,dy] = \int_C M dx + N dy$

where $M = y - \sin x$, $N = \cos x$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -\sin x$$



By Green's theorem,

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{\frac{\pi}{2}} \int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1)\,dy\,dx$$

$$= -\int_{x=0}^{\frac{\pi}{2}} \int_{y=0}^{\frac{2x}{\pi}} (1+\sin x)\,dy\,dx$$

$$= -\int_0^{\frac{\pi}{2}} (1+\sin x)\,[y]_0^{\frac{2x}{\pi}}\,dx$$

$$= -\int_0^{\frac{\pi}{2}} (1+\sin x)\left(\frac{2x}{\pi}\right) dx$$

$$= -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x(1+\sin x)\,dx$$

$$= -\frac{2}{\pi}\left[x(-\cos x + x) - 1\cdot\left(-\sin x + \frac{x^2}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{2}{\pi}\left[\frac{\pi}{2}\left(0 + \frac{\pi}{2}\right) + 1 - \frac{\pi^2}{8}\right]$$

$$= -\frac{2}{\pi}\left(\frac{\pi^2}{4} - \frac{\pi^2}{8} + 1\right)$$

$$= -\frac{2}{\pi}\left(\frac{\pi^2}{8} + 1\right)$$

$$= -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

$$\therefore \oint_C [(y - \sin x)dx + \cos x\, dy] = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

Stoke's theorem

If S is an open surface bounded by a closed curve C and if $\bar{F}$ is any continuously differentiable vector field, then

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \operatorname{curl}\bar{F}\cdot\bar{n}\, ds$$

where $\bar{n}$ is outward drawn unit normal at any point of S.

Problems :-

1. Verify Stoke's theorem for $\bar{F} = (2x-y)\bar{i} + yz^2\bar{j} - y^2z\bar{k}$ over the upper half of the sphere $x^2+y^2+z^2 = 1$ bounded by the projection of the xy plane.

Sol. Given $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$

The bound C of S is a circle in xy plane i.e $z=0$

By stoke's theorem

$$\int_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \bar{i}(-2yz+2yz) - \bar{j}(0-0) + \bar{k}(0+1)$$

$$= \bar{k}$$

$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \iint_S \bar{k}\cdot\bar{n}\,ds$$

$$= \iint_S dx\,dy$$

$= \pi$ (which is area of the circle $x^2+y^2=1$ i.e $\pi(1^2) = \pi$)

$\longrightarrow 1)$

Now, we have to evaluate the integral $\int_C \bar{F}\cdot d\bar{r}$ where C is the circle $x^2+y^2=1$

Then $x=\cos\theta,\ y=\sin\theta$ $(\because r=1)$

$\Rightarrow dx=-\sin\theta\,d\theta,\ dy=\cos\theta\,d\theta$

θ varies from 0 to 2π over the circle.

$$\int_C \bar{F}\cdot d\bar{r} = \int_C [(2x-y)\,dx - yz^2\,dy - y^2z\,dz]$$

$$= \int_C (2x-y)\,dx, \quad \text{since } z=0,\ dz=0$$

$$= -\int_0^{2\pi}(2\cos\theta-\sin\theta)(-\sin\theta\,d\theta)$$

$$= \int_0^{2\pi}(\sin^2\theta - 2\sin\theta\cos\theta)\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{1-\cos 2\theta}{2} - \sin 2\theta\right]d\theta$$

$$= \left[\frac{1}{2}\left(\theta-\frac{\sin 2\theta}{2}\right)+\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}(2\pi-0)+\frac{1}{2}-0-\frac{1}{2} \quad (\because \cos 4\pi = 1,\ \sin 4\pi = 0)$$

$$= \pi \longrightarrow 2)$$

from 1) & 2): $\int_C \bar{F}\cdot d\bar{r} = \iint_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds$

Hence, ~~Green's~~ stoke's theorem is verified.

2. Verify Stokes theorem for the function $\bar{F} = (x^2 - y^2)\bar{i} + 2xy\bar{j}$, integrated round the rectangle in the plane $z=0$ and bounded by the lines $x=0$, $y=0$, $x=a$ and $y=b$.

Sol. By stoke's theorem

$$\oint_C \bar{F}\cdot d\bar{r} = \int_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds$$

Here $\operatorname{curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$

$$= \bar{i}(0-0) - \bar{j}(0-0) + \bar{k}(2y+2y)$$

$$= 4y\bar{k}$$

$$\iint_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds = \iint_S 4y\bar{k}\cdot\bar{n}\,ds$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} 4y\,dx\,dy$$

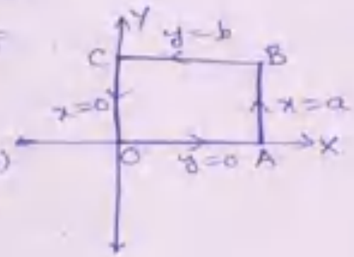
$$= \int_0^a (2y^2)_0^b\,dx$$

$$= 2b^2\int_0^a dx = 2b^2(x)_0^a$$

$$= 2ab^2 \longrightarrow 1)$$

Here C is the rectangle OABC

$$\int_C \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CO} \bar{F}\cdot d\bar{r} \longrightarrow (1)$$



Here $\bar{F}\cdot d\bar{r} = (x^2-y^2)\,dx + 2xy\,dy$

Along OA

$y=0,\ dy=0$

x varies from 0 to a

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_0^a x^2\,dx$$

$$= \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

Along AB

$x=a,\ dx=0$

y varies from 0 to b

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_0^b 2ay\,dy$$

$$= 2a\left(\frac{y^2}{2}\right)_0^b$$

$$= 2a\left(\frac{b^2}{2}\right)$$

$$= ab^2$$

Along BC :

$y = b$, $dy = 0$

x varies from a to 0

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_a^0 (x^2 - b^2)\,dx$$

$$= \left(\frac{x^3}{3} - xb^2\right)_a^0$$

$$= 0 - \left(\frac{a^3}{3} - ab^2\right)$$

$$= -\frac{a^3}{3} + ab^2$$

Along CO :

$x = 0$, $dx = 0$

y varies from b to 0

$$\int_{CO} \bar{F}\cdot d\bar{r} = \int_b^0 0\,dy$$

$$= 0$$

substitute all these in eq 2), we get

$$\oint_C \bar{F}\cdot d\bar{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0$$

$$= 2ab^2 \longrightarrow 3)$$

$\therefore$ from 1) and 3), $\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$

Hence, Stokes theorem is verified.