Name:____________________

MATH 320: QUIZ 5

(1) (6 points) Define the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & 0 \\ -2 & 0 & 2 \end{bmatrix}.$$

(a) Compute an LU Decomposition using "Naive Gaussian elimination." What are L and U?

(b) Solve the equation $Ax = (1, 2, 1)^T$ for x by first solving $Ld = b$ and then solving $Ux = d$ (using L and U from the last step). What are d and x?

a) Naive Gaussian Elimination:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & 0 \\ -2 & 0 & 2 \end{bmatrix} \quad \begin{array}{l} -2R_1 + R_2 \Rightarrow f_{21} = 2 \\ R_1 + R_2 \Rightarrow f_{31} = -1 \end{array} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{1}{3} & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -6 \\ 0 & -1 & 5 \end{bmatrix} \quad \frac{1}{3}R_2 + R_3 \Rightarrow f_{32} = -\frac{1}{3} \qquad U = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -6 \\ 0 & 0 & 3 \end{bmatrix}$$

b) $Ld = b \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{1}{3} & 1 \end{bmatrix} d = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow d_1 = 1,$

$d_2 + 2d_1 = 2 \Rightarrow d_2 = 0$

$d_3 - d_1 - \frac{1}{3}d_2 = 1 \Rightarrow d_3 = 2$

$Ux = d \Leftrightarrow \begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -6 \\ 0 & 0 & 3 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \Rightarrow 3x_3 = 2 \Rightarrow x_3 = \frac{2}{3}$

$3x_2 - 6x_3 = 0 \Rightarrow x_2 = \frac{4}{3}$

$2x_1 - x_2 + 3x_3 = 1 \Rightarrow x_1 = \frac{1}{6}$

$d = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad x = \begin{pmatrix} 1/6 \\ 4/3 \\ 2/3 \end{pmatrix}$

(c) Compute the LU Decomposition using "Gaussian elimination with partial pivoting." What are L, U, and P?

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & 0 \\ -2 & 0 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 4 & 1 & 0 \\ 2 & -1 & 3 \\ -2 & 0 & 2 \end{bmatrix} \begin{matrix} -\frac{1}{2}R_1 + R_2 \Rightarrow f_{21} = \frac{1}{2} \\ \frac{1}{2}R_1 + R_3 \Rightarrow f_{31} = -\frac{1}{2} \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 4 & 1 & 0 \\ 0 & -\frac{3}{2} & 3 \\ 0 & \frac{1}{2} & 2 \end{bmatrix} \frac{1}{3}R_2 + R_3 \Rightarrow f_{32} = -\frac{1}{3}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{3} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 4 & 1 & 0 \\ 0 & -\frac{3}{2} & 3 \\ 0 & 0 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(because of initial row switch)

(2) (4 points) Write a MATLAB function that computes a Cholesky decomposition for a 3×3 matrix A (assuming the input is symmetric and positive semi-definite). Precise wording is not as important as conceptual accuracy. Don't use the `chol` function!

```
function  U = cholesky (A)
  % input: 3x3 symm. pos. semi-def. matrix A
  % output: 3x3 upper-triangular matrix U
            such that U^T U = A

  U = zeros(3);      % initialize the matrix.
  U(1,1) = A(1,1)^(1/2);
  U(1,2) = A(1,2)/U(1,1);
  U(1,3) = A(1,3)/U(1,1);
  U(2,2) = (A(2,2) - U(1,2)^2)^(1/2);
  U(2,3) = (A(2,3) - U(1,2) * U(1,3)) / U(2,2);
  U(3,3) = (A(3,3) - U(1,3)^2 - U(2,3)^2)^(1/2);

end
```

Reference: $U^T U = \begin{bmatrix} u_{11} & & \\ u_{12} & u_{22} & \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ & u_{22} & u_{23} \\ & & u_{33} \end{bmatrix} = \begin{bmatrix} u_{11}^2 & u_{11}u_{12} & u_{11}u_{13} \\ & u_{12}^2 + u_{22}^2 & u_{12}u_{13} + u_{22}u_{23} \\ & & u_{13}^2 + u_{23}^2 + u_{33}^2 \end{bmatrix}$