Due: October 21, 2016

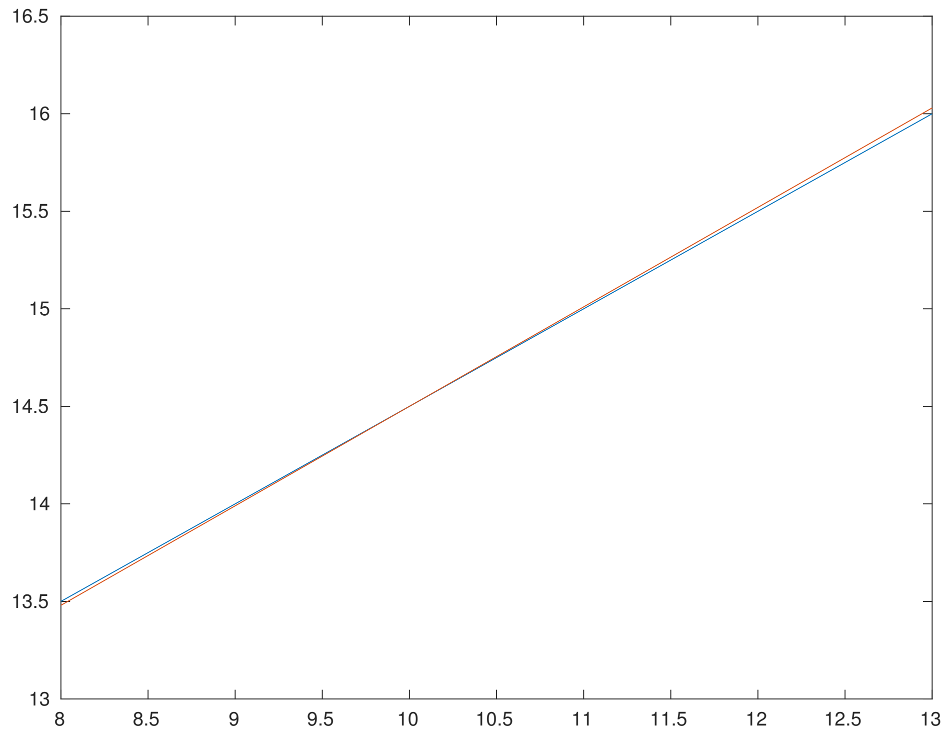
MATH 320: HOMEWORK 5

Please read chapters 9 and 10 in the textbook and complete the following problems.

(1) Problem 9.5

(a) Solve graphically. We use the following code to plot the image below:

```
x1 = 8:.1:13;
x2 = 0.5*x1 + 9.5;
x3 = .51*x1 + 9.4;
plot(x1,x2,x1,x3)
```



From the picture, it seems that $x_1 = 10, x_2 = 14.5$ is the solution.

(b) Compute the determinant. The matrix form of the equation is

$$\begin{pmatrix} 0.5 & -1 \\ 1.02 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -9.5 \\ -18.8 \end{pmatrix}.$$

The determinant of the left-hand matrix is 0.02.

(c) On the basis of (a) and (b), I would guess that the system is ill-conditioned; in particular, the condition number is high. (Indeed, it is 314.5168, using 2-norm.)

(d) Solve by the elimination of unknowns. Start by solving $x_2 = 0.5x_1 + 9.5$, then substitute into

$$1.02x_1 - 2(0.5x_1 + 9.5) = -18.8 \iff 1.02x_1 - 1x_1 - 19 = -18.8$$

$$\iff 0.02x_1 = .2 \iff x_1 = 10 \Rightarrow x_2 = 14.5.$$

The solution is $(10, 14.5)$ as guessed above.

(e) Solve again, but with a_{11} modified slightly to 0.52. This time: $x_2 = 0.52x_1 + 9.5$

$$1.02x_1 - 2(0.52x_1 + 9.5) = -18.8 \iff 1.02x_1 - 1.04x_1 - 19 = -18.8$$
$$\iff -0.02x_1 = .2 \iff x_1 = -10 \Rightarrow x_2 = 4.3.$$

Because the system is so ill-conditioned, even a slight perturbation in the coefficients can lead to a huge shift in the solution. Here, the value of a_{11} changed by .02 but the solution shifted by 20 in the x_1 coordinate, and 10.2 in the x_2 coordinate. Poor condition makes linear systems susceptible to being thrown way off by noise.

(2) Problem 10.12. We have the LU factorization:

$$A = LU = \begin{pmatrix} 1 & & \\ 0.6667 & 1 & \\ -0.3333 & -0.3636 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 & 1 \\ & 7.3333 & -4.6667 \\ & & 3.6364 \end{pmatrix}$$

(a) The determinant of this product can be computed as the product of the two determinants. Since both are triangular, we need only multiply the diagonal elements in the U matrix. $3 \times 7.3333 \times 3.6364 = 80.0004$.

(b) Solving $Ax = b$ can be solved using $LUx = b \iff Ly = b$ and $Ux = y$. So we solve:

$$\begin{pmatrix} 1 & & \\ 0.6667 & 1 & \\ -0.3333 & -0.3636 & 1 \end{pmatrix} y = \begin{pmatrix} -10 \\ 50 \\ -26 \end{pmatrix}$$

Using forward-subsitution, we find $y_1 = -10$, then $-6.667 + y_2 = 50 \Rightarrow y_2 = 56.667$, then $3.333 - 20.6041 + y_3 = -26 \Rightarrow y_3 = -8.7289$.

Then, we use back-substitution to solve the following system:

$$\begin{pmatrix} 3 & -2 & 1 \\ & 7.3333 & -4.6667 \\ & & 3.6364 \end{pmatrix} x = \begin{pmatrix} -10 \\ 56.667 \\ -8.7289 \end{pmatrix}$$

First, we have $x_3 = -2.4004$, then $7.3333 x_2 - 4.6667 \times (-2.4004) = 56.667 \Rightarrow x_2 = 6.200$, and then $3 \times x_1 + (-2) \times (6.2000) + 1 \times (-2.4004) = -10 \Rightarrow x_1 = 1.600$.

The solution therefore is $(1.600, 6.200, -2.400)$.

(3) Problem 10.13

$$A = U^T U = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

We can use MATLAB command `chol(A)` to find the following Cholesky factorization:

$$U = \begin{pmatrix} 1.4142 & -0.7071 & 0 \\ 0 & 1.2247 & -0.8165 \\ 0 & 0 & 1.1547 \end{pmatrix}$$

To find this explicitly, we solve the system of equations:

$$\begin{aligned}
u_{11}^2 &= a_{11} && \Rightarrow u_{11} = \sqrt{2} && \\
u_{11}u_{12} &= a_{12} && \Rightarrow \sqrt{2}u_{12} = -1 && \Rightarrow u_{12} = \frac{-1}{\sqrt{2}} \\
u_{11}u_{13} &= a_{13} && \Rightarrow \sqrt{2}u_{13} = 0 && \Rightarrow u_{13} = 0 \\
u_{12}^2 + u_{22}^2 &= a_{22} && \Rightarrow \frac{1}{2} + u_{22}^2 = 2 && \Rightarrow u_{22} = \sqrt{\frac{3}{2}} \\
u_{12}u_{13} + u_{22}u_{23} &= a_{23} && \Rightarrow 0 + \sqrt{\frac{3}{2}}u_{23} = -1 && \Rightarrow u_{23} = -\sqrt{\frac{2}{3}} \\
u_{13}^2 + u_{23}^2 + u_{33}^2 &= a_{33} && \Rightarrow 0 + \frac{2}{3} + u_{33}^2 = 2 && \Rightarrow u_{33} = \frac{2}{\sqrt{3}}
\end{aligned}$$

(4) Take three points in the plane with distinct x-coordinates so that they are not collinear. The statement "these three points define a parabola" is equivalent to a certain matrix equation involving a 3 x 3 matrix.

(a) What is the matrix equation?
Let the three points be $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3). Then assuming they lie on a parabola $y = ax^2 + bx + c$ means the following matrix equation holds:

$$\begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

(b) Using this equation, write a MATLAB function `parab` which takes as input three points and gives output a string `"y=ax^2 + bx + c"` where a,b, and c are replaced by numbers so that all three points lie on the corresponding parabola.
The MATLAB code used to produce this output is below:

```
function s = parab(x1,y1,x2,y2,x3,y3)
%takes three points in the plane as input
%outputs a string defining the parabola

M = [x1^2 x1 1; x2^2 x2 1; x3^2 x3 1];
V = [y1, y2, y3];
x = (M\V')';
s = sprintf('y = %f x^2 + %f x + %f',...
x(1),x(2),x(3));
```

(c) Evaluate your function given the three points $(0, 1), (2, -3)$, and $(3, 2)$.
The output for this evaluation is

```
 y = 2.333333 x^2 + -6.666667 x + 1.000000
```