MATH 320: CLASS NOTES

Roots - Chapter 6

This chapter deals with open methods.

1. Newton's Method

2. Secant Method

Analyzing the convergence of the secant method.

Suppose we have a function with a simple root at r and 3-times differentiable in a nbhd.

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \\
x_{n+1} - r &= (x_n - r) - \frac{f(x_n)((x_n - r) - (x_{n-1} - r))}{f(x_n) - f(x_{n-1})} \\
E_{n+1} &= E_n - \frac{f(x_n)(E_n - E_{n-1})}{f(x_n) - f(x_{n-1})} \\
E_{n+1} &= E_n - \frac{(f'(r)E_n + \frac{f''(r)}{2}E_n^2 + O(E_n^3)) \cdot (E_n - E_{n-1})}{f'(r)(E_n - E_{n-1}) + \frac{f''(r)}{2}(E_n^2 - E_{n-1}^2)} + O(E_n^3 - E_{n-1}^3) \\
E_{n+1} &= E_n - \frac{f'(r)(E_n - E_{n-1})(E_n + \frac{f''(r)}{2f'(r)}E_n^2 + O(E_n^3))}{f'(r)(E_n - E_{n-1})(1 + \frac{f''(r)}{2f'(r)}(E_n + E_{n-1}) + O(E_n^2 + E_nE_{n-1} + E_{n-1}^2))} \\
E_{n+1} &= E_n - \frac{(E_n + \frac{f''(r)}{2f'(r)}E_n^2 + O(E_n^3))}{(1 + \frac{f''(r)}{2f'(r)}(E_n + E_{n-1}) + O(E_n^2 + E_nE_{n-1} + E_{n-1}^2))} \\
E_{n+1} &= \frac{E_n(1 + \frac{f''(r)}{2f'(r)}(E_n + E_{n-1}) + O(E_n^2 + E_nE_{n-1} + E_{n-1}^2)) - (E_n + \frac{f''(r)}{2f'(r)}E_n^2 + O(E_n^3))}{(1 + \frac{f''(r)}{2f'(r)}(E_n + E_{n-1}) + O(E_n^2 + E_nE_{n-1} + E_{n-1}^2))} \\
E_{n+1} &= \frac{E_n + \frac{f''(r)}{2f'(r)}(E_n^2 + E_{n-1}E_n) + O(E_n^3 + E_n^2E_{n-1} + E_nE_{n-1}^2)) - (E_n + \frac{f''(r)}{2f'(r)}E_n^2 + O(E_n^3))}{(1 + \frac{f''(r)}{2f'(r)}(E_n + E_{n-1}) + O(E_n^2 + E_nE_{n-1} + E_{n-1}^2))} \\
E_{n+1} &= \frac{\frac{f''(r)}{2f'(r)}E_{n-1}E_n + O(E_n^3 + E_n^2E_{n-1} + E_nE_{n-1}^2)}{(1 + \frac{f''(r)}{2f'(r)}(E_n + E_{n-1}) + O(E_n^2 + E_nE_{n-1} + E_{n-1}^2))} \\
E_{n+1} &= \frac{f''(r)}{2f'(r)}E_{n-1}E_n \left[\frac{1 + O(E_n^2/E_{n-1} + E_n + E_{n-1})}{1 + \frac{f''(r)}{2f'(r)}(E_n + E_{n-1}) + O(E_n^2 + E_nE_{n-1} + E_{n-1}^2)}\right] \\
\implies E_{n+1} &= CE_nE_{n-1}
\end{aligned}
$$

What is the order of convergence? Take q such that

$$\lim_{n\to\infty}\frac{|E_{n+1}|}{|E_n|^q} = C \implies \lim_{n\to\infty}\frac{|C_2\cdot E_n E_{n-1}|}{|E_n|^q} = C$$

$$\implies \lim_{n\to\infty}\frac{|E_{n-1}|}{|E_n|^{q-1}} = C/C_2 \implies \lim_{n\to\infty}\frac{|E_n|^{q-1}}{|E_{n-1}|} = C_2/C$$

$$\implies \lim_{n\to\infty}\frac{|E_n|}{|E_{n-1}|^{1/(q-1)}} = (C_2/C)^{1/(q-1)}$$

But the order of convergence can only take one value, so:

$$1/(q-1) = q \implies q^2 - q - 1 = 0 \implies q = \frac{1}{2}(1 \pm \sqrt{(}5))$$

And the true value is the golden mean.