Commentative Algebra: Tensor Products

Outline

Tensor Product as Quotient

Let A be a ring, M, N be A-modules.

- Take A-linear combinations of pairs in $M \times N$. Typical element:

$$\sum_{i \in I} a_i (m_i, n_i). \quad =: A^{(M \times N)}$$

- Let D be the submodule of $A^{(M \times N)}$ generated by: 1) $a(m,n) - (am, n)$.
2) $a(m,n) - (m, an)$. 3) $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$
4) $(m, n_1 + n_2) - (m, n_1) - (m, n_2)$.

Tensor Product as Quotient

$$M \otimes N := A^{(M \times N)} / D.$$

$\forall (x, y) \in A^{(M \times N)} \rightsquigarrow$ image in $M \otimes N$ is written as $x \otimes y$.

$$ax \otimes y \doteq x \otimes (ay) = a(x \otimes y)$$

$$x_1 + x_2 \otimes y = x_1 \otimes y + x_2 \otimes y$$

$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2.$$

Tensor Product by Universal Property

For M, N A-modules, there exists an A-module $M \otimes N$ and an A-bilinear map $g: M \times N \to M \otimes N$ with following property:

$$\begin{tikzcd}
M \times N \arrow[d, "g"'] \arrow[dr, "f"] & \\
M \otimes N \arrow[r, dotted, "\exists! f'"] & P
\end{tikzcd}$$

For any A-module P, A-bilinear map $f: M \times N \to P$, there is a unique A-mod homom $f': M \otimes N \to P$ s.t. $f' \circ g = f$.

Tensor Product by Universal Property

Suppose $M \square N$ satisfies the same property:

the property forces a unique A-module isomorphism.

$$
\begin{tikzcd}
M \times N \arrow[d, "g"] \arrow[dr, "f"] & \\
M \otimes N \arrow[r, dotted, "\exists! f'"] & P
\end{tikzcd}
$$

$M \square N$

$$
\begin{tikzcd}
M \times N \arrow[d, "g'"'] \arrow[dr, bend left, "g"] & \\
M \square N \arrow[r, "\exists!"] & M \otimes N
\end{tikzcd}
$$

Multi-linear Tensor Product

Differences:

① A-linear in each factor.

$$\begin{tikzcd} M_1 \times \cdots \times M_r \arrow[d, "g"] \arrow[dr, "f"] & \\ M_1 \otimes \cdots \otimes M_r \arrow[r, dotted, "\exists! f'"] & P \end{tikzcd}$$

② In the existence proof, the free module and its submodule look diff.

Some Quick Isomorphisms (Prop 2.14)

$$(M \otimes N) \cong (N \otimes M)$$

$$x \otimes y \mapsto y \otimes x$$

$$(M \otimes N) \otimes P \cong M \otimes (N \otimes P) \cong M \otimes N \otimes P$$

$$(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z) \mapsto (x \otimes y \otimes z)$$

$$(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$$

$$(x, y) \otimes z \mapsto x \otimes z, \; y \otimes z$$

$$A \otimes M \cong M$$

$$a \otimes m \mapsto am.$$

$$M \otimes (N \otimes P) \underset{g}{\overset{f}{\rightleftarrows}} M \otimes N \otimes P$$

Fix $x \in M$. $f_x : N \times P \to M \otimes N \otimes P$

$(y, z) \mapsto x \otimes y \otimes z$

This is A-bilinear $\Rightarrow \exists!\, f'_x : N \otimes P \to M \otimes N \otimes P$.

Now let $f : M \times (N \otimes P) \to M \otimes N \otimes P$

$(x, t) \mapsto f'_x(t)$

This is A-bilinear $\Rightarrow \exists!\, f' : M \otimes (N \otimes P) \to M \otimes N \otimes P$

$x \otimes (y \otimes z) \mapsto x \otimes y \otimes z$

$g : M \times N \times P \to M \otimes (N \otimes P)$

$(x, y, z) \mapsto x \otimes (y \otimes z)$

A-trilinear $\Rightarrow \exists!\, g' : M \otimes N \otimes P \to M \otimes (N \otimes P)$

$$(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$$

Exercise!

Restriction & Extension of Scalars

Let A, B be rings. Take $f : A \to B$ a ring homomorphism.

1) If M is a B-module, it has an A-module structure via $ax = f(a)x$. $a \in A, x \in M$.

2) If M is an A-module take $B \otimes_A M$ ← this has a B-module structure. $b(b' \otimes_A x) = bb' \otimes_A x$.

$b \otimes ax = a(b \otimes x) = (f(a)b) \otimes x$.

Claim: $- \otimes_A M$ is a Functor A-Mod$\to$ A-Mod

Fix an A-module M.

Objects: $N \longrightarrow N \otimes_A M$

Morphisms: $f : N \to P$ A-module homomorphism

$f' : N \otimes_A M \longrightarrow P \otimes_A M$

$n \otimes m \longmapsto f(n) \otimes m.$

In other words $f' = f \otimes 1_M.$

"Defn": Adjoint Functors

Kinda like an inverse functor.

$F : \mathcal{C} \to \mathcal{D}$. $G : \mathcal{D} \to \mathcal{C}$

For $X \in \mathrm{Obj}(\mathcal{C})$, $Y \in \mathrm{Obj}(\mathcal{D})$

$$\mathrm{Hom}_{\mathcal{C}}(X, GY) \leftrightarrow \mathrm{Hom}_{\mathcal{D}}(FX, Y).$$

in a natural way.

– $\otimes M$ and $\mathrm{Hom}(M,\cdot)$ are Adjoints

Given N, P in category of A-modules

there is a natural bijection

$$\mathrm{Hom}(N \otimes M, P) \longleftrightarrow \mathrm{Hom}(N, \mathrm{Hom}(M, P))$$

Given $f : N \otimes M \to P \rightsquigarrow f' : N \to \mathrm{Hom}(M, P)$.

Fix $n \in N$. What results is $f_n : M \to P$.

Given $f' : N \to \mathrm{Hom}(M, P) \overset{?}{\rightsquigarrow} f : N \otimes M \to P$

$f(n \otimes m) = f'(n)(m) \in P$.

$- \otimes M$ is a right-exact functor

Since $- \otimes M$ is a left adjoint, it is right exact, i.e. given

$$N' \to N \to N'' \to 0 \quad \text{exact}$$

$$\Rightarrow N' \otimes M \to N \otimes M \to N'' \otimes M \to 0 \quad \text{exact.}$$

$- \otimes M$ preserves cokernels.

– $\otimes M$ is a right-exact functor

$N' \to N \to N'' \to 0$ exact

$0 \to \mathrm{Hom}(N'', P) \to \mathrm{Hom}(N, P) \to \mathrm{Hom}(N', P)$ exact.

$0 \to \mathrm{Hom}(N'', \mathrm{Hom}(M, P')) \to \mathrm{Hom}(N, \mathrm{Hom}(M, P'))$
$\to \mathrm{Hom}(N', \mathrm{Hom}(M, P'))$ exact.

$\Rightarrow 0 \to \mathrm{Hom}(N'' \otimes M, P') \to \mathrm{Hom}(N \otimes M, P')$
$\to \mathrm{Hom}(N' \otimes M, P')$ exact.

$\Rightarrow N' \otimes M \to N \otimes M \to N'' \otimes M \to 0$ exact.

– $\otimes M$ is NOT left-exact

$A = \mathbb{Z}$. Category: $\mathbb{Z}$-mod.

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

$\downarrow \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$

$$0 \to \underbrace{\mathbb{Z} \otimes \mathbb{Z}_2}_{\text{not exact}} \xrightarrow{(\times 2 \otimes \mathrm{Id})} \mathbb{Z} \otimes \mathbb{Z}_2 \to \mathbb{Z}_2 \otimes \mathbb{Z}_2 \to 0.$$

$\mathrm{Im}(0 \to \mathbb{Z} \otimes \mathbb{Z}_2) = 0. \neq \mathrm{Ker}(\mathbb{Z} \otimes \mathbb{Z}_2 \to \mathbb{Z} \otimes \mathbb{Z}_2)$

$n \otimes x \mapsto 2n \otimes x = n \otimes 2x = 0$

Defn: Flat Module

TFAE conditions characterizing *flatness*.
M is a flat module if

1. $- \otimes M$ is an exact functor.
2. $- \otimes M$ preserves short exact sequences.
3. $f : N' \to N$ injective $\implies$
   $f \otimes 1 : N' \otimes M \to N \otimes M$ injective.
4. $f : N' \to N$ injective for N', N finitely generated $\implies$
   $f \otimes 1 : N' \otimes M \to N \otimes M$ injective.

(1) $\Leftarrow$ (2) break down long to short.

(2) $\Leftarrow$ (3) $-\otimes M$ preserves cokernels
(3) means it preserves kernels.