Commutative Algebra: Cohen-Seidenberg Theorems

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Cohen & Seidenberg

PRIME IDEALS AND INTEGRAL DEPENDENCE

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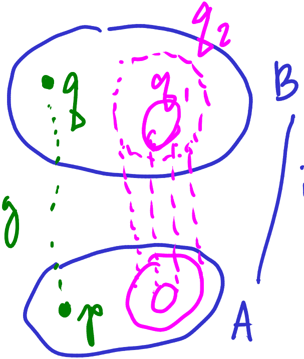
Let $\mathfrak{R}$ and $\mathfrak{S}$ be commutative rings such that $\mathfrak{S}$ contains, and has the same identity element as, $\mathfrak{R}$. If $\mathfrak{p}$ and $\mathfrak{P}$ are prime ideals in $\mathfrak{R}$ and $\mathfrak{S}$ respectively such that $\mathfrak{P} \cap \mathfrak{R} = \mathfrak{p}$ then we shall say that $\mathfrak{P}$ lies over, or contracts to, $\mathfrak{p}$. If over every prime ideal in $\mathfrak{R}$ there lies a prime ideal in $\mathfrak{S}$, we shall say that the "lying-over" theorem holds for the pair of rings $\mathfrak{R}$ and $\mathfrak{S}$.

Outline

Big Picture



Lying over

integral extension

$p_1 \subseteq p_2$

Going Up
q_2 exists

Tools

1) Quotients.

2) Localization.

Going Down
q_1 exists.

Integral Extensions & Fields

Let $A \subseteq B$ be an integral extension of integral domains. Then, B is a field if and only if A is a field.

Proof ($\Leftarrow$) Suppose A field.

$\forall x \in B \quad \exists p(x) = x^n + a_1 x^{n-1} + \dots + a_n = 0 \quad (a_n \neq 0)$

$= \underbrace{-a_n^{-1}}\, x \underbrace{(x^{n-1} + \dots + a_{n-1})}_{x^{-1}!} = 1$

($\Rightarrow$) Suppose B field.

Take $y \in A$. $y^{-1} \in B \quad \exists p(y^{-1}) = (y^{-1})^n + \dots + a_n = 0. \quad (a_n \neq 0)$

$y^{-1} + [a_1 + a_2 y + \dots + a_n y^n] = \underbrace{0}_{y^{-1}} \Rightarrow y^{-1} \in A.$

Integral Extensions & Maximal Ideals

Let $A \subseteq B$ be an integral extension. $\mathfrak{q} \subset B$ prime ideal, $\mathfrak{p} = \mathfrak{q} \cap A$. Then $\mathfrak{q}$ is maximal if and only if $\mathfrak{p}$ is maximal.

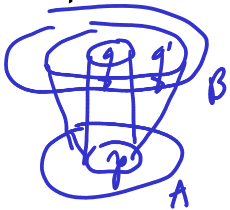
$B/\mathfrak{q}$ is integral extension of $A/\mathfrak{p}$.

By previous result, $B/\mathfrak{q}$ field $\Leftrightarrow$ $A/\mathfrak{p}$ field.

$\Leftrightarrow$ $\mathfrak{q}$ maximal $\Leftrightarrow$ $\mathfrak{p}$ maximal.

Incomparability

Let $A \subseteq B$ be an integral extension. Let $\mathfrak{q}, \mathfrak{q}'$ be prime ideals such that $\mathfrak{q} \subseteq \mathfrak{q}'$ and $\mathfrak{q}^c = \mathfrak{q}'^c = \mathfrak{p}$. Then $\mathfrak{q} = \mathfrak{q}'$.



Proof Localize with $S = A \setminus \mathfrak{p}$.

$S^{-1}B$ integral over $A_\mathfrak{p}$.

$\mathfrak{p}A_\mathfrak{p}$ is maximal. $\mathfrak{q}, \mathfrak{q}'$ preserved by this localization. $S^{-1}\mathfrak{q}$, $S^{-1}\mathfrak{q}'$ must be maximal $\Rightarrow S^{-1}\mathfrak{q} = S^{-1}\mathfrak{q}' \Rightarrow \mathfrak{q} = \mathfrak{q}'$.

The Lying-Over Theorem

Let $A \subseteq B$ be an integral extension. Let $\mathfrak{p} \subset A$ be a prime ideal. Then there exists $\mathfrak{q} \subset B$ prime such that $\mathfrak{q} \cap A = \mathfrak{p}$.

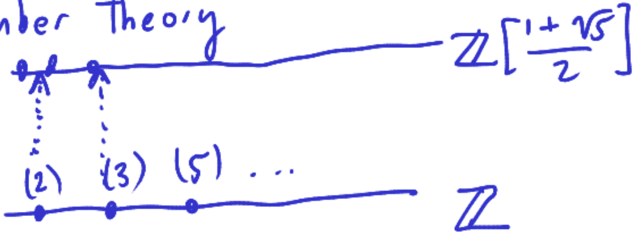
Proof Localize with $S = A \setminus \mathfrak{p}$.

$S^{-1}B$ has a maximal ideal $\mathfrak{m}$. $\mathfrak{m}$ must intersection $A_\mathfrak{p}$ at $\mathfrak{p}A_\mathfrak{p}$. Consider $\mathfrak{m}^c$: this intersects A in $\mathfrak{p}$, and is prime.

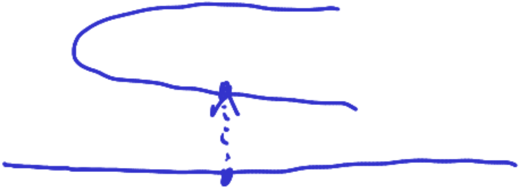
Let $\mathfrak{q} = \mathfrak{m}^c$. $\boxed{\text{Z}}$

Consequences of Lying-Over

① Number Theory

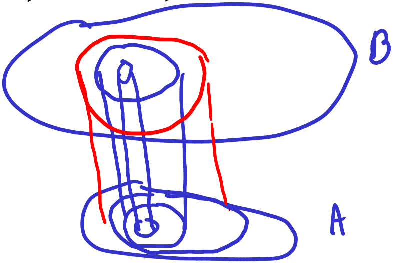


② Geometry: maps from curves to lines.



The Going-Up Theorem

Let $A \subseteq B$ be an integral extension.
Let $\mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n$ be a chain of prime ideals of A.
Let $\mathfrak{q}_1 \subseteq \cdots \subseteq \mathfrak{q}_m$ be a chain of prime ideals of B, with $m < n$ and $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for all $i \leq m$.
Then $\mathfrak{q}_1 \subseteq \cdots \subseteq \mathfrak{q}_m$ can be extended to a chain $\mathfrak{q}_1 \subseteq \cdots \subseteq \mathfrak{q}_n$ with $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for all $i \leq n$.



Proof of Going-Up

Reduce to $n=2$, $m=1$

$\mathfrak{p}_1 \subseteq \mathfrak{p}_2$. $\mathfrak{q}_1$.

Take $B/\mathfrak{q}_1$ integral extension of $A/\mathfrak{p}_1$. $\mathfrak{p}_2 A/\mathfrak{p}_1$ is prime. By Lying Over, $\exists\, \tilde{\mathfrak{q}}_2$ in $B/\mathfrak{q}_1$ prime with $\tilde{\mathfrak{q}}_2 \cap A/\mathfrak{p}_1 = \mathfrak{p}_2 A/\mathfrak{p}_1$.

Corr Thm $\Rightarrow$ $\tilde{\mathfrak{q}}_2$ corresponds to prime ideal $\mathfrak{q}_2$ containing $\mathfrak{q}_1$ with $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$.

Consequences of Going-Up

Krull dimension = length of a longest chain of distinct prime ideals in a ring.

Going-Up ⇒ $A \subset B$ integral extension

krull dim(A) = krull dim(B).

$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \underline{\mathfrak{p}_n} \rightsquigarrow \mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_n . \cancel{\subset \mathfrak{q}_{n+1}}\,?$

maximal

Integral Closure in Localizations

Take $A \subseteq B$ rings, C the integral closure of A in B.
Let S be a multiplicatively closed subset of A.
Then $S^{-1}(C)$ is the int closure of $S^{-1}A$ in $S^{-1}B$.

Proof $\frac{x}{s} \in S^{-1}B$. $p(t) = t^n + \frac{a_1}{s_1}t^{n-1} + \cdots + \frac{a_n}{s_n} = 0$

has $p\left(\frac{x}{s}\right) = 0$. $\frac{x^n}{s^n} + \frac{a_1 x^{n-1}}{s_1 s^{n-1}} + \cdots + \frac{a_n}{s_n} = 0$.

$t = s_1 \cdots s_n$. $t^n x^n + t^{n-1} s_2 \cdots s_n s a_1 x^{n-1} + \cdots + t^{n-1} s_1 \cdots s_{n-1} a_n s^n = 0$.

$\Rightarrow \tilde{p}(tx) = 0 \Rightarrow tx$ integral over A. $\Rightarrow tx \in C$

$\Rightarrow \frac{tx}{st} = \frac{x}{s} \in S^{-1}C$.

"Integrally Closed" is a Local Property

A int closed $\iff$ $A_{\mathfrak{p}}$ int closed $\forall \mathfrak{p}$ prime
$\iff$ $A_{\mathfrak{m}}$ int closed $\forall \mathfrak{m}$ maximal.

$f: A \to$ Integral closure of A. (identity)

f surjective?

$f: A \to C$ surjective
$\iff f_p: A_p \to C_p$ surjective
$\iff f_m: A_m \to C_m$ surjective

surjectivity is a local property (see section on localiz.)

{note: loc. of int cl. = int cl of loc.}

Def: Integral Over $\mathfrak{a} \subseteq A$

If $A \subseteq B$. $x \in B$ satisfies a monic polynomial

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0 \quad a_i \in \mathfrak{a}.$$

Then x integral over $\mathfrak{a}$.

Integral Closure of $\mathfrak{a} \subseteq A$

Lemma 5.14: $C =$ integral closure of A in B. $\mathfrak{a} \subseteq A$ ideal, $\mathfrak{a}^e$ the extension of $\mathfrak{a}$ to C. Then, integral closure of $\mathfrak{a}$ in B is $\mathfrak{r}(\mathfrak{a}^e)$.

set of all elements integral over $\mathfrak{a}$.

Proof ($\subseteq$) $x \in B$ integral over $\mathfrak{a}$

$x^n + a_1 x^{n-1} + \cdots + a_n = 0 \Rightarrow x$ int over $A \Rightarrow x \in C$.

$x^n = -a_1 x^{n-1} - \cdots - a_n \Rightarrow x^n \in \mathfrak{a}^e \Rightarrow x \in \mathfrak{r}(\mathfrak{a}^e)$.

($\supseteq$) $x \in \mathfrak{r}(\mathfrak{a}^e) \Rightarrow \exists n$ s.t. $x^n \in \mathfrak{a}^e \Rightarrow x^n = \sum_{i=1}^{m} a_i x_i$.

$x^n \in A[x_1, \ldots, x_m]$ f.g. module. C-H Thm implies result.

Integral over $\mathfrak{a} \implies$ integral over $r(\mathfrak{a})$

Proposition 5.15 $A \subseteq B$ integral domains, A integrally closed, $x \in B$ integral over $\mathfrak{a} \subseteq A$. Then, x algebraic over $K = \mathrm{Frac}(A)$ with minimal polynomial over K given by $t^n + a_1 t^{n-1} + \cdots + a_n$, then $a_1, \ldots, a_n$ lie in $\mathfrak{r}(\mathfrak{a})$.

Proof:

$$\begin{array}{c} L \\ | \\ K \end{array} \quad x_1, \ldots, x_n \text{ all roots of } p(t).$$

$$p(t) = t^n + \sigma_1(x_1, \ldots, x_n)\, t^{n-1} + \cdots + \sigma_n(x_1, \ldots, x_n)$$

σ_i = i-th elmtry symmetric function.

All x_i's satisfy the same monic polynomial as x $\Rightarrow$ integral over $\mathfrak{a}$.

$\Rightarrow$ elmtry symm functions in the x_i's are also integral over $\mathfrak{a}$.

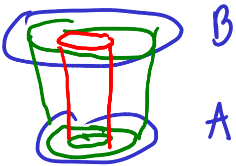
$\Rightarrow$ all coefficients are in $r(\mathfrak{a})$.

The Going-Down Theorem

Let $A \subseteq B$ be an integral extension of integral domain, with A integrally closed.

Let $\mathfrak{p}_1 \supseteq \cdots \supseteq \mathfrak{p}_n$ be a chain of prime ideals of A, and $\mathfrak{q}_1 \supseteq \cdots \supseteq \mathfrak{q}_m$ $(m < n)$ a chain of primes of B s.t. $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for $i \leq m$.

Then, the chain can be extended to $\mathfrak{q}_1 \supseteq \cdots \supseteq \mathfrak{q}_n$ with $\mathfrak{q}_i \cap A = \mathfrak{p}_i$.



Proof: Reduce to $n=2$, $m=1$.

$\mathfrak{p}_1 \supset \mathfrak{p}_2$. $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$.

Take $B_{\mathfrak{q}_1}$. Claim: $\mathfrak{q}_2 = \mathfrak{p}_2 B_{\mathfrak{q}_1}$.

i.e. $\mathfrak{p}_2 B_{\mathfrak{q}_1} \cap A = \mathfrak{p}_2$.

$\frac{y}{s}$, $y \in \mathfrak{p}_2 B$, $s \in B \setminus \mathfrak{q}_1$ $\Rightarrow$ y int over $\mathfrak{p}_2$.

$y^r + a_1 y^{r-1} + \cdots + a_r = 0$ $a_i \in \mathfrak{p}_2$.

Suppose $x \in \mathfrak{p}_2 B_{\mathfrak{q}_1} \cap A$. $\frac{y}{s} = x \Rightarrow s = \frac{y}{x}$

$$\left(\frac{y}{x}\right)^r + \frac{a_1}{x}\left(\frac{y}{x}\right)^{r-1} + \cdots + \frac{a_r}{x^r} = 0.$$

Proof

$\Rightarrow$ s satisfies monic polynomial.

$s \in A \Rightarrow s$ integral over $A = (1)$

$\Rightarrow \frac{a_j}{x^j} \in A$ for all j.

$\Rightarrow$ $\boxed{\text{If } x \notin \mathfrak{p}_2}$, then $\left(\frac{a_j}{x^j}\right) \cdot x^j = a_j \in \mathfrak{p}_2$

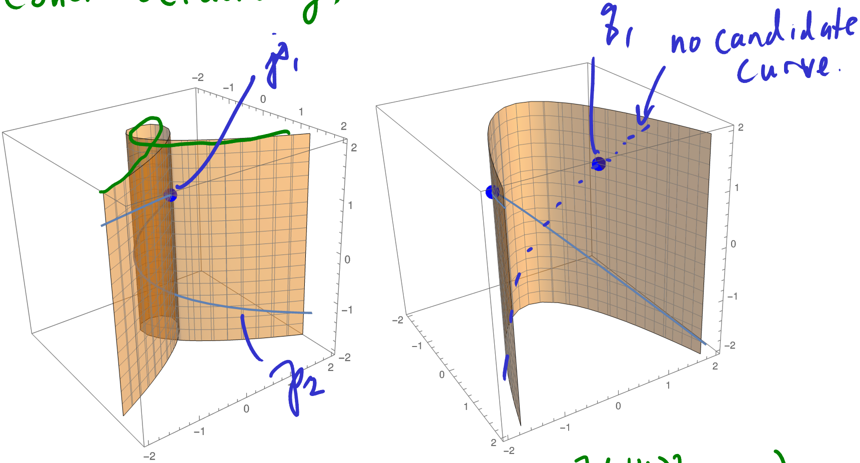
$\Rightarrow \frac{a_j}{x^j} \in \mathfrak{p}_2 \ \forall j. \Rightarrow s$ integral over $\mathfrak{p}_2$.

$\Rightarrow s^r \in B\mathfrak{p}_2 \subseteq B\mathfrak{p}_1 \subseteq \mathfrak{q}_1 \Rightarrow s \in \mathfrak{q}_1 \Rightarrow\Leftarrow$.

$\Rightarrow x \in \mathfrak{p}_2$.

Counter-example: Not integrally closed

(Cohen-Seidenberg)



$$k[x,y,z]/(y^2-x^2-x^3) \subset k\left[x,\frac{y}{x},z\right]/\left(\left(\frac{y}{x}\right)^2-1-x\right)$$