Commutative Algebra: Operations on Ideals

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Operations on Ideals

Sum, Product, Intersection

Ideal Quotients

Radicals

Extension & Contraction

Definition: Sum, Product, Intersection

Let R be a ring, $\mathfrak{a}, \mathfrak{b} \subseteq R$ ideals.

- Sum : $\mathfrak{a} + \mathfrak{b} = \{ a + b : a \in \mathfrak{a}, b \in \mathfrak{b} \}$.

  $\rightarrow$ ideal! $r(a+b) = ra + rb$.

  $(a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2)$

- Product: $\mathfrak{a}\mathfrak{b} = \langle \{ ab : a \in \mathfrak{a}, b \in \mathfrak{b} \} \rangle$

  $\rightarrow$ not an ideal as a set of products:

  $a_1 b_1 + a_2 b_2 \neq a_3 b_3$.

- Intersection : $\mathfrak{a} \cap \mathfrak{b}$. $\rightarrow$ ideal.

Example: $(10), (12) \subseteq \mathbb{Z}$

Sum: $(10) + (12) = \{10m + 12n : m, n \in \mathbb{Z}\}.$

$$= \{2(5m + 6n) : m, n \in \mathbb{Z}\}.$$

with $m = -1, n = 1, \quad 2 \in (10) + (12).$

$(10) + (12) \subseteq (2)$ since $2 \mid$ every elmt.

$\Rightarrow (10) + (12) = (2).$

Product: $(10)(12) = \{10m \cdot 12n : m, n \in \mathbb{Z}\}$

$$= \{120mn : m, n \in \mathbb{Z}\}.$$

$$= (120).$$

Example: $(10), (12) \subseteq \mathbb{Z}$

Intersection: $(10) \cap (12) = \left\{ n \in \mathbb{Z} : \begin{matrix} 10 \mid n \\ 12 \mid n \end{matrix} \right\}$.

$$= \left\{ n \in \mathbb{Z} \mid 60 \mid n \right\} = (60).$$

Generally...

$$(m) + (n) = (\gcd(m,n)).$$

$$(m)(n) = (mn)$$

$$(m) \cap (n) = (\operatorname{lcm}(m,n)) = \left(\frac{mn}{\gcd(m,n)} \right)$$

Example: $(x^2, y), (x, z) \subseteq \mathbb{C}[x, y, z]$

Sum: $(x^2, y) + (x, z) = (ax^2 + by + cx + dz \mid$
$$a, b, c, d \in \mathbb{C}[x, y, z])$$
$$= (x, y, z).$$

Product: $(x^2, y)(x, z) =$
$$\langle (ax^2 + by)(cx + dz) : a, b, c, d \in \mathbb{C}[x, y, z] \rangle$$
$$= \langle x^3, xy, x^2z, yz \rangle.$$

Example: $(x^2, y), (x, z) \subseteq \mathbb{C}[x, y, z]$

Intersection:

$(x^2, y) \cap (x, z) = \{ p : p \in (x^2, y), p \in (x, z) \}.$

$x^2 | p$ ✓   or $y | p$.

$\Rightarrow x | p$ either $xy | p$ ✓ or $yz | p$ ✓.

$= (x^2, xy, yz)$

Definition: Ideal Quotient

Let R be a ring, $\mathfrak{a}, \mathfrak{b} \subseteq R$ ideals.

$$\mathfrak{a} : \mathfrak{b} = \{x \in R \mid x\mathfrak{b} \subseteq \mathfrak{a}\}.$$

Note: $\mathfrak{a} \subseteq \mathfrak{a} : \mathfrak{b}$ by def of ideal.

Example: $((12) : (10)) \subseteq \mathbb{Z}$

$$(12) : (10) = \{ a \in \mathbb{Z} \mid a(10) \in (12) \}$$

$$\forall m, \exists n, \ 10m \cdot a = 12n \implies 5am = 6n.$$

$$\implies a = 6k.$$

$$\implies (12) : (10) = (6).$$

Generally...

$$(m) : (n) = \left(\frac{m}{\gcd(m,n)} \right).$$

Example: $((xz, yz) : (z)) \subseteq \mathbb{C}[x, y, z]$

$$(xz, yz) : (z) = \{ p : p(z) \subseteq (xz, yz) \}$$
$$= \{ p : \overset{\forall a, \exists b, c}{p \, a \, z} = bxz + cyz \}$$
$$= \{ p : p \, a = bx + cy \}.$$
$$= (x, y).$$

Definition: Annihilator *Ann(x)*

$(0) : (x) \leftarrow Ann(x)$ "annihilator of x".

$$= \{ a \in R : \quad a(x) \subseteq (0) \}$$

$$= \{ a \in R : \quad ax = 0 \}.$$

$$\bigcup_{\substack{x \in R \\ x \neq 0}} Ann(x) = \{ \text{zero divisors of } R \}.$$

Example: $\mathbb{Z}/12\mathbb{Z}$

$\text{Ann}(x)$

$\bar{1}$	$\bar{0}$
$\bar{2}$	$\bar{0}, \bar{6}$
$\bar{3}$	$0, 4, 8$
$\bar{4}$	$0, 3, 6, 9$
$\bar{5}$	0
$\bar{6}$	$0, 2, 4, 6, 8, 10$

$\bar{7}$	0
$\bar{8}$	$0, 3. 6, 9$
$\bar{9}$	$0, 4, 8$
$\overline{10}$	$\bar{0}, \bar{6}$
$\overline{11}$	0

$$\bigcup_{x \neq 0} \text{Ann}(x) = \{0, 2, 3, 4, 6, 8, 9, 10\}$$

Definition: Radical

Let R be a ring, $\mathfrak{a} \subseteq R$ ideal.

$r(\mathfrak{a}) = \sqrt{\mathfrak{a}}$ "radical of $\mathfrak{a}$" or "rad $\mathfrak{a}$"

$$:= \{ x \in R \mid \exists n \in \mathbb{Z}_+, \ x^n \in \mathfrak{a} \}.$$

$\mathfrak{n}$ "nilradical"

$$:= r((0)) = \{ x \in R \mid \exists n \in \mathbb{Z}_+, x^n = 0 \}.$$

$$= \text{set of nilpotents in } R.$$

$\varphi : R \longrightarrow R/\mathfrak{a}.$ $\quad r(\mathfrak{a}) = \varphi^{-1}(\mathfrak{n}(R/\mathfrak{a})).$

Example: $(40) \subseteq \mathbb{Z}$

$$r\big((40)\big) = \{\, n \in \mathbb{Z} \mid \exists k, \ n^k \in (40) \,\}.$$

$$40 = 2^3 \cdot 5. \quad \Rightarrow \quad r\big((40)\big) = (10).$$

Generally,

$$r\left(\left(\prod_{i=1}^{n} p_i^{a_i}\right)\right) = \left(\prod_{i=1}^{n} p_i\right).$$

Example: $(x^2, y^3) \subseteq \mathbb{C}[x, y]$

$$r((x^2, y^3)) = \{f \in \mathbb{C}[x,y] : \exists n, f^n \in (x^2, y^3)\}$$

$$x \in r((x^2, y^3)), \quad y \in r((x^2, y^3))$$

$$(x, y) \subseteq r((x^2, y^3)).$$
$\uparrow$
maximal. $\quad 1 \notin r((x^2, y^3))$

$$\Rightarrow r((x^2, y^3)) \neq \mathbb{C}[x, y]$$

$$\Rightarrow (x, y) = r((x^2, y^3)).$$

Definition: Extension & Contraction

Let $f: A \to B$ be hom of rings.

$a \subseteq A$, $b \subseteq B$ ideals.

- Contraction: $f^{-1}(b) \subseteq A$ ideal. denoted f^c.

- Extension: $f(a) \subseteq B$ <u>not</u> an ideal.

$f(a)B =: a^e \subseteq B$ ideal.

Example: $\mathbb{Z} \to \mathbb{Z}[i]$

$f:$ Integers $\to$ Gaussian integers.

$\qquad n \mapsto n.$

Consider extensions of prime ideals.

1. $p = 2$. $(2)^e = ((1+i))^2$.

$\qquad 2 = 1^2 + 1^2 = (1+i)(1-i)$.

$\qquad (1-i) = -i(1+i)$

2. $p \equiv 1 \mod 4$.

$\qquad (p)^e = $ prod of two <u>distinct</u>

$\qquad\qquad$ prime ideals.

Example: $\mathbb{Z} \to \mathbb{Z}[i]$

3. $p \equiv 3 \bmod 4$.

$(p)^e = (p)$.

Many behaviors for primes under the same homomorphism.