Commutative Algebra: Valuation Rings

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Outline

Definition

Let R be an integral domain.
Let K be its field of fractions.

R is a valuation ring if
for any $x \in K$, either $x \in R$ or $x^{-1} \in R$.

Examples

1) $\mathbb{Z}_{(p)} = \left\{ \frac{m}{n} : p \nmid n \right\}$.

Given $x \in \mathrm{Frac}(\mathbb{Z}_{(p)})$, $x = \frac{m}{n}$.

Can choose m, n so that $p | m$ or $p | n$ but not both. $\Rightarrow x \in R$ or $x^{-1} \in R$.

2) $\mathbb{C}[x]_{(x)} = \left\{ \frac{p(x)}{q(x)} : q(0) \neq 0 \right\}$

Given $f \in \mathrm{Frac}(\mathbb{C}[x]_{(x)})$, $f = \frac{p}{q}$.

In lowest terms, $x \nmid p$ or $x \nmid q$.

Non-Examples

1) $\mathbb{Z}$ not a valuation ring.

$\frac{3}{2} = \left(\frac{2}{3}\right)^{-1} \in \mathbb{Q}$, neither in $\mathbb{Z}$.

2) $\mathbb{C}[x,y]_{(x,y)}$ not valuation ring.

$\frac{y}{x} = \left(\frac{x}{y}\right)^{-1} \in \operatorname{Frac}(\mathbb{C}[x,y]_{(x,y)})$

neither is in the ring.

Valuation Rings are Local

Proof: Let $\mathfrak{m}$ be the set of all non-units in R. Claim: $\mathfrak{m}$ ideal.

1) $a \in R$, $x \in \mathfrak{m}$. WTS: $ax \in \mathfrak{m}$.

$x^{-1} \notin R$. Suppose ax is a unit

$(ax)^{-1} \in R \Rightarrow \underbrace{a \, (ax)^{-1}}_{R} = x^{-1} \Rightarrow x^{-1} \in R \Rightarrow \Leftarrow$.

Valuation Rings are Local

2) $x \in \mathfrak{m}$, $y \in \mathfrak{m}$ WTS: $x+y \in \mathfrak{m}$.

$x^{-1} \notin R$, $y^{-1} \notin R$. $xy^{-1} \in \mathrm{Frac}(R)$

either $(xy^{-1}) \in R$ or $x^{-1}y \in R$.

w.l.o.g. $xy^{-1} \in R$ $\quad x+y = \underbrace{(1+xy^{-1})}_{R}\underbrace{y}_{\mathfrak{m}}$

$\Rightarrow x+y \in \mathfrak{m}$.

$\Rightarrow \mathfrak{m}$ maximal ideal.

Rings Containing Valuation Rings

$$R \subset R' \subset K = \mathrm{Frac}(R).$$

Then if R valuation ring,

so is R'.

Valuation Rings are Integrally Closed

Proof Suppose $x \in K$ is integral over R, and $x^{-1} \in R$.

$$(x^{-1})^{n-1}(x^n + a_1 x^{n-1} + \cdots + a_n = 0) \quad a_i \in R.$$

$$x + a_1 + \cdots + a_n (x^{-1})^{n-1} = 0.$$

$$x = -a_1 - a_2 x^{-1} - \cdots - a_n (x^{-1})^{n-1}$$

$$\Rightarrow x \in R.$$

Dominance Order on Subrings with Homomorphisms

Fix a field K, and an algebraically closed field Ω.

Then consider the set

$$\{(A,f) : A \subseteq K \text{ subring}, \ f : A \to \Omega \text{ homomorph.}\}$$

$(A,f) \leq (B,g)$ if $A \subseteq B$ and $g|_A = f$.

Zorn's Lemma $\Rightarrow$ maximal element.

Maximal Element is Local

If (B, g) is a max element of this poset, then $\ker(g)$ is the unique maximal ideal of B.

Proof: Let $\mathfrak{m} = \ker(g)$. $g(B)$ int dom $\Rightarrow$ $\mathfrak{m}$ prime. Take $B_{\mathfrak{m}}$, extend g to $B_{\mathfrak{m}}$ by taking $\bar{g}\left(\frac{x}{y}\right) = \frac{g(x)}{g(y)}$.

$(B, g) \leq (B_{\mathfrak{m}}, \bar{g}) \Rightarrow B_{\mathfrak{m}} = B$.

Maximal Element is a Valuation Ring

Take (B, g) as above, $\mathfrak{m} = \ker(g)$.

Then B is a valuation ring.

<u>Proof</u>: Take $x \in K$. WTS: $x \in B$ or $x^{-1} \in B$.

Lemma: Either $\mathfrak{m}[x] \subsetneq B[x]$ or $\mathfrak{m}[x^{-1}] \subsetneq B[x^{-1}]$.

w.l.o.g $\mathfrak{m}[x] \subsetneq B[x]$. $\exists\, \mathfrak{m}'$ maximal with $\mathfrak{m}[x] \subset \mathfrak{m}'$.

Maximal Element is a Valuation Ring

$$\begin{array}{ccc} B & B[x] & \\ \cup & \cup & \\ \mathfrak{m} & \mathfrak{m}' & \\ & \cup & \\ & \mathfrak{m}[x] & \end{array}$$

$B/\mathfrak{m} \subset B[x]/\mathfrak{m}'$

field extension.

$k \subset k[\bar{x}]$

$\Rightarrow \bar{x}$ alg over k. $\mathfrak{m} = \ker(g)$

$\Rightarrow$ can extend g to $k[\bar{x}]$

$(B, g) < (B[x], \bar{g}) \Rightarrow B[x] = B$

$\Rightarrow x \in B$.

Lemma

Take $x \neq 0 \in K$, $\mathfrak{m} \subseteq B$ maximal.
Either $\mathfrak{m}[x] \neq B[x]$ or $\mathfrak{m}[x^{-1}] \neq B[x^{-1}]$.

Proof Suppose $\mathfrak{m}[x] = B[x]$, $\mathfrak{m}[x^{-1}] = B[x^{-1}]$.

Then,
$$1 = u_0 + u_1 x + \cdots + u_k x^k$$
$$1 = v_0 + v_1 x^{-1} + \cdots + v_\ell (x^{-1})^\ell$$

Suppose $k \geq \ell$.
$$(1 - v_0) x^\ell = v_1 x^{\ell-1} + \cdots + v_\ell .$$

$\forall\, r_0 \in \mathfrak{m}$, $(1 - r_0)$ is a unit.

$\Rightarrow$ $x^{\ell} = w_1 x^{\ell-1} + \dots + w_\ell$.

$x^k = x^{k-\ell} x^{\ell} = w_1 x^{k-1} + \dots + w_\ell x^{k-\ell}$.

Subbing into the polynomial expression for 1 yields a polynomial of strictly lower degree $\Rightarrow\Leftarrow$.

IC $\subseteq$ VR

Let A be an integral domain.
Let B be a valuation ring

$$A \subset B \subset K = \mathrm{Frac}(A).$$

Then integral closure of $A \subseteq B$.

B integrally closed $\Rightarrow$ Any elmt $x \in K$ integral over A is integral over B
$\Rightarrow x \in B \Rightarrow$ int closure of A in B.

$\bigcap$ VR $\subseteq$ IC

Let A be a ring. The integral closure contains the intersection of all valuation rings $B \supseteq A$.

Proof Take x not integral over A.

WTS: $\exists B$ valuation ring, $A \subseteq B$, $x \notin B$.

$x \notin A[x^{-1}]$. $\Rightarrow$ x^{-1} not a unit $\Rightarrow$ $x^{-1} \in m'$

$m' \subseteq A[x^{-1}]$ maximal.

$\bigcap \mathrm{VR} \subseteq \mathrm{IC}$

Take $A[x^{-1}]/\mathfrak{m}' =: k$. Let Ω be the alg. closure of k.

$$\begin{array}{rcl} \pi: & A[x^{-1}] & \longrightarrow \Omega \\ & \uparrow & \\ f: & A & \end{array}$$

$(A, f) < (B, g)$ $\quad B$ valuation ring

$g|_A = f. \Rightarrow g(x^{-1}) = 0. \Rightarrow x \notin B.$

Homomorphisms to Alg Closed Field

Let Ω be an alg closed field, $A \subseteq B$ be integral domains, B f.g. over A and $v \neq 0 \in B$. Then there is $u \neq 0 \in A$ such that $\forall f : A \to \Omega$ with $f(u) \neq 0$, f can be extended to $g : B \to \Omega$ such that $g(v) \neq 0$

Proof: Reduce to $B = A[x]$,

either, x transcendental.

or, x algebraic.

Proof: x transcendental

$B = A[x]$. Take $v \in B$.

$v = a_0 + a_1 x + \cdots + a_n x^n$.

Take $u \in A$ to be a_n. If $f(a_n) \neq 0$,

then, v = nonzero polynomial in x.

Let $g(x)$ = nonroot of this polynomial,

(Ω infinite) $g(v) \neq 0$.

Proof: x algebraic

$B = A[x]$ $\qquad \underline{a_n x^n + \cdots + a_1 x + a_0} = 0$

Given $v \in B$, v^{-1} algebraic over a

$$\underline{a'_m v^{-m} + \cdots + a'_1 v^{-1} + a'_0} = 0.$$

Take $u = a'_m a_n$. $f(u) \neq 0$, then $f: A \to \Omega$ extends to $g: A[u^{-1}] \to \Omega$, setting $g(u^{-1}) = f(u)^{-1}$.

$(A[u^{-1}], g) < (C, h)$, C valuation ring.

Proof: x algebraic

x integral over $A[u^{-1}]$

$\Rightarrow$ x in integral closure of $A[u^{-1}]$.

$\Rightarrow$ $x \in C$. $\Rightarrow$ $B = A[x] \subseteq C$.

$v = p(x) \in C$.

By scaling polynomial, v^{-1} integral over $A[u^{-1}]$ $\Rightarrow$ $v^{-1} \in C$.

$\Rightarrow$ $h(v) \neq 0$. $\Rightarrow$ $h|_B$ extends f s.t. $\underset{0}{\overset{h(v)}{\neq}}$.

Corollary: Hilbert's Nullstellensatz

Let k be a field, B a finitely-generated k-algebra. If B is a field, then it is an algebraic extension of k.

Take $A = k$, $B = B$, $v = 1$, $\Omega = \bar{k}$.

$f : k \hookrightarrow \bar{k}$. $\exists g : B \to \bar{k}$ s.t. $g(1) \neq 0$.

$\Rightarrow \ker(g) \neq B$. B field $\Rightarrow \ker(g) = (0)$.

$\Rightarrow g$ injective $\Rightarrow g$ isomorphism B and a f.g. subfield of $\bar{k} \Rightarrow B$ alg$/k$.