Commutative Algebra: Fractions & Localization

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Outline

Definition (Integral Domain Case)

Guiding example: $\mathbb{Z} \rightsquigarrow \mathbb{Q}$.

Def Let A be a ring. A set $S \subseteq A$ is multiplicatively closed if $1 \in S$ and $x, y \in S \Rightarrow xy \in S$.

Def The localization of A at S, or the ring of fractions with denominators in S as the set $\{ (\frac{a}{s}) : a \in A, s \in S \}$

Definition (Integral Domain Case)

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1 s_2 + a_2 s_1 \quad \text{in } A}{s_1 s_2 \quad \text{in } S}.$$

$$\left(\frac{a_1}{s_1}\right)\left(\frac{a_2}{s_2}\right) = \frac{a_1 a_2 \leftarrow \text{in } A.}{s_1 s_2 \leftarrow \text{in } S}$$

modulo equivalence relation

$$\frac{a}{s} \equiv \frac{b}{t} \quad \text{if} \quad at - bs = 0.$$

Examples: $\mathbb{Z}$

i) $S = \mathbb{Z} \setminus \{0\}$. $S^{-1}\mathbb{Z} = \mathbb{Q}$.

ii) $S = \{n^k : k \geq 0\}$

ex $n = 6$: $S = \{1, 6, 36, 216, \dots\}$

$$S^{-1}\mathbb{Z} = \left\{ \left(\frac{a}{6^k}\right) : a \in \mathbb{Z},\ k \in \mathbb{Z}_{\geq 0} \right\}$$

Problem? $\cancel{\frac{2}{6} = \frac{1}{3}}$ only cancel when it makes sense.

Examples: $\mathbb{Z}$

iii) $S = \mathbb{Z} \setminus (p)$ p prime.

ex $p = 2$. $S =$ odd numbers.

$$S^{-1}\mathbb{Z} = \left\{ \left(\frac{a}{b}\right) : a \in \mathbb{Z},\ b \in \mathbb{Z},\ p \nmid b \right\}$$

ex fractions with odd denominators.

Examples: $k[x]$

i) $S = k[x] \setminus \{0\}$. $S^{-1}k[x] = k(x)$ rational functions

$$S^{-1}k[x] = \left\{ \frac{f(x)}{g(x)} : g \neq 0 \right\}.$$

ii) $S = \{1, x, x^2, \dots\}$

$S^{-1}k[x] = k[x, x^{-1}]$ Laurent polynomial ring.

Examples: $k[x]$

iii) $S = k[x] \setminus (x)$.

$$S^{-1}k[x] = \left\{ \frac{f(x)}{g(x)} : g(x) \text{ not a multiple of } x \right\}$$

= rational functions that are well-behaved at zero.

Definition (General)

Let A be a ring, not necessarily an integral domain.

Consider $\left\{ \left(\frac{a}{s}\right) : a \in A, s \in S \right\}$.

$$\frac{a}{s} \sim \frac{b}{t} \qquad \frac{b}{t} \sim \frac{c}{u} \qquad \frac{a}{s} \sim \frac{c}{u}$$

$$at - bs = 0 \qquad bu - ct = 0$$

$$u(at - bs) = 0 \qquad s(bu - ct) = 0$$

$$atu - cst = 0 \Leftarrow t(au - cs) = 0.$$

Definition (General)

Instead of $\frac{a}{s} \sim \frac{b}{t}$ if $at - bs = 0$,

$\frac{a}{s} \sim \frac{b}{t}$ if there is $u \in S$ s.t.

$u(at - bs) = 0$.

$$S^{-1}A = \left\{ \left(\frac{a}{s}\right) : a \in A, s \in S \right\}$$

modulo $\sim$, $\frac{a}{s} \sim \frac{b}{t}$ if $\exists u \in S$

s.t. $u(at - bs) = 0$.

Universal Property

Let A be a ring, $S \subseteq A$ mult. subset.

$\forall f : A \to B$ ring hom.
with the property
that $f(x)$ is a unit
in B for all $x \in S$.

$\exists!\ g : S^{-1}A \to B$ s.t. the diagram
commutes.

$$\begin{tikzcd}
A \arrow[rr, "i"] \arrow[dr, "f"'] & & S^{-1}A \arrow[dl, "\exists!"] \\
& B &
\end{tikzcd}$$

Localization & Local Rings

Def Local ring is a ring with one maximal ideal.

(maximal ideals of $\mathbb{C}[x,y]$ were $(x-a, y-b)$; $(a,b) \in \mathbb{C}^2$).

If $S = A \setminus \mathfrak{p}$, then $S^{-1}A$ is a local ring with $\mathfrak{p}(S^{-1}A)$.

Notation: $S^{-1}A = A_{\mathfrak{p}}$.

Examples

- $A = \mathbb{Z}$, $\mathfrak{p} = (0)$ or (p)

$$\mathbb{Z}_{(0)} = \mathbb{Q} \qquad \mathbb{Z}_{(p)} = \left\{ \frac{a}{b} : p \nmid b \right\}.$$

- $A = \mathbb{C}[x,y]$, $\mathfrak{p} = (0), (f \text{ irred}), (x-a, y-b)$ $(a,b) \in \mathbb{C}^2$.

$A_{(0)} = \mathbb{C}(x,y)$. $A_{(x-a,y-b)} = \{$ rational fns well defined at (a,b) $\}$

$\{ A_{(f)} =$ rational fns whose denom's are not identically 0 on $f(x,y)=0$.

Ideals of $S^{-1}A$

Bijection : primes of $S^{-1}A$ $\longleftrightarrow$ primes of A that do not meet S.

$\mathfrak{p} \subseteq S^{-1}A$. $\rightsquigarrow$ $f^{-1}(\mathfrak{p}) \subseteq A$.

$f : A \to S^{-1}A$

$x \mapsto \frac{x}{1}$

pre-images of prime ideals are prime.

Ideals of $S^{-1}A$

$$f : A \longrightarrow S^{-1}A$$

$$\mathfrak{p} \subset A \longrightarrow S^{-1}\mathfrak{p} \subset S^{-1}A \quad \text{prime.}$$

int. domain

$\Rightarrow$ $S^{-1}\mathfrak{p}$ prime

$$S^{-1}A / S^{-1}\mathfrak{p} \cong \bar{S}^{-1}(A/\mathfrak{p})$$

$\bar{S}$ = image of S under the quotient.

zero

$\Leftrightarrow$ $S^{-1}\mathfrak{p} = S^{-1}A$

$\Leftrightarrow$ $S \cap \mathfrak{p} \neq \emptyset$.

localization of int. domain is also int. domain or 0.

$\Rightarrow\!\!\!/\ \Leftarrow$

A-module $\to$ $S^{-1}A$-module

Let A be a ring. $S \subseteq A$ mult. subset.

Let M be an A-module.

$$S^{-1}M = \left\{ \frac{m}{s} : m \in M,\ s \in S \right\} / \sim$$

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_2 m_1 + s_1 m_2}{s_1 s_2}.$$

$$\left(\frac{a}{t}\right)\left(\frac{m}{s}\right) = \frac{am}{ts}.$$

$\frac{m_1}{s_1} \equiv \frac{m_2}{s_2}$ if $\exists\, a \in S$ s.t. $a(m_1 s_2 - m_2 s_1) = 0$.