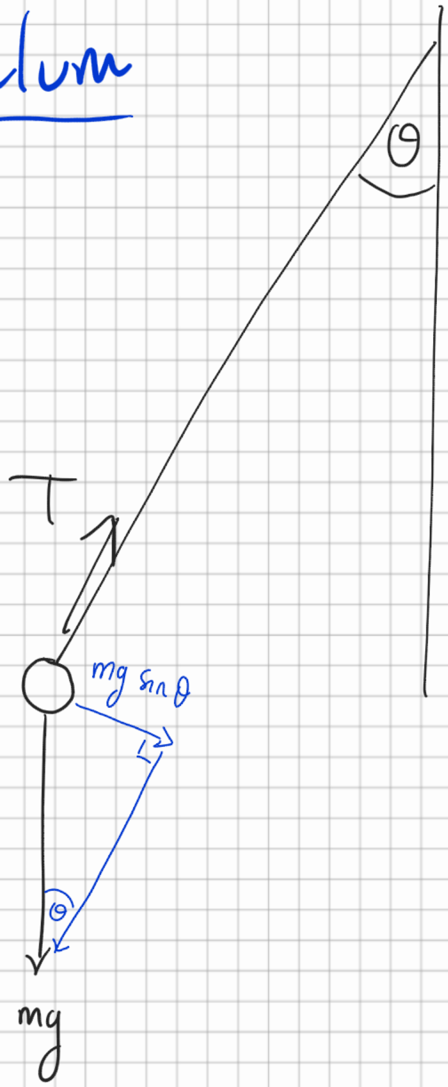
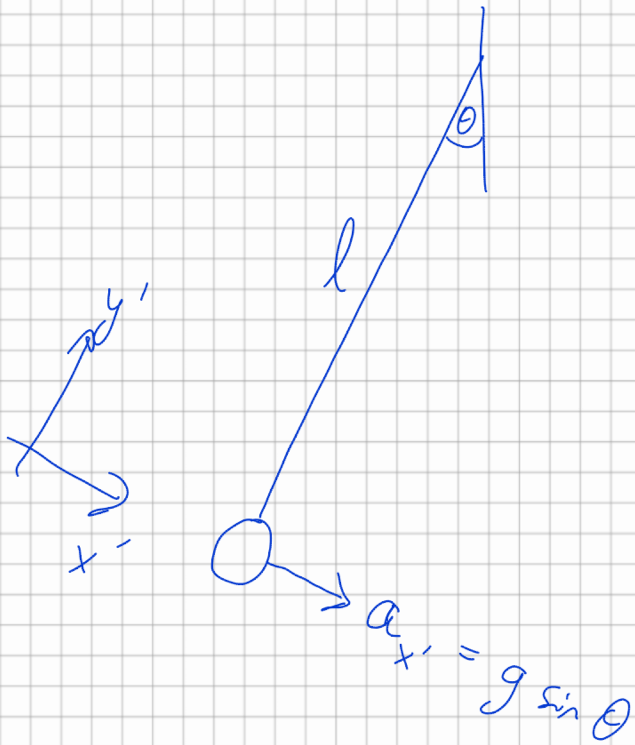
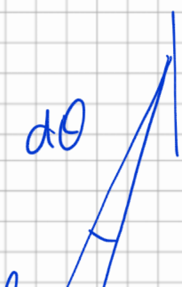
Single Pendulum







$$dx' = l\, d\theta$$

$$-l\ddot{\theta} = \ddot{x}' = g \sin\theta$$

$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$$

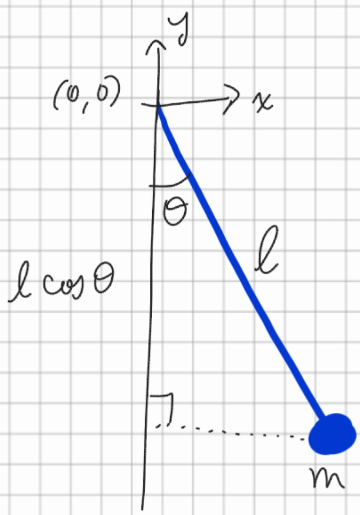
which does not have an analytic solution.

Solution using Euler-Lagrange eq.

Lagrangian $= T - V$

V — Potential energy

T — Kinetic energy



$x = l\sin\theta, \quad y = -l\cos\theta$

$V = mgy = -mgl\cos\theta$

$T = \frac{1}{2}mv^2$

$$v^2 = \dot{x}^2 + \dot{y}^2 = (l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2$$

$$= l^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) = l^2\dot{\theta}^2$$

$\Rightarrow T = \frac{1}{2}ml^2\dot{\theta}^2$

$\Rightarrow L = T - U = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$

Euler-Lagrange equation: $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$

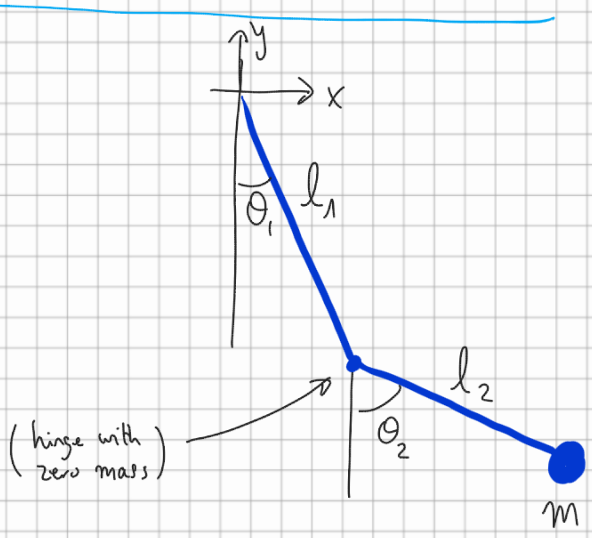
$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = -mgl\sin\theta$

$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0$$

$$\cancel{m} l^2 \ddot{\theta} + \cancel{m} g \cancel{l} \sin\theta = 0$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0} \quad \text{as above.}$$

Double Pendulum



As before, let's write out the Lagrangian.

Coordinates of the mass:

$$x = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

$$y = -l_1 \cos\theta_1 - l_2 \cos\theta_2$$

Velocities:

$$\dot{x} = l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2$$

$$\dot{y} = l_1 \dot{\theta}_1 \sin\theta_1 + l_2 \dot{\theta}_2 \sin\theta_2$$

Kinetic energy:

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2} m \Big(l_1^2 \dot{\theta}_1^2 \cos^2\theta_1 + l_2^2 \dot{\theta}_2^2 \cos^2\theta_2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_1 \cos\theta_2$$
$$+ l_1^2 \dot{\theta}_1^2 \sin^2\theta_1 + l_2^2 \dot{\theta}_2^2 \sin^2\theta_2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_1 \sin\theta_2 \Big)$$

$$= \frac{1}{2} m \Big[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \underbrace{(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)}_{\cos(\theta_1 - \theta_2)} \Big]$$

$$= \frac{1}{2} m \Big[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cdot \cos(\theta_1 - \theta_2) \Big]$$

Potential energy:

$$V = mgy = -mgl_1 \cos\theta_1 - mgl_2 \cos\theta_2 .$$

Lagrangian:

$$L = T - V$$
$$= \frac{1}{2} m \Big[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \Big]$$
$$+ mgl_1 \cos\theta_1 + mg\, l_2 \cos\theta_2 .$$

Euler-Lagrange equations:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = 0 \qquad i = 1, 2$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m l_1^2 \dot{\theta}_1 + m l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \qquad i = 1, 2$$

$$\frac{\partial L}{\partial \dot{\theta}_i} = m l_i^2 \dot{\theta}_i + m l_1 l_2 \dot{\theta}_{3-i} \cos(\theta_1-\theta_2) \qquad i=1,2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = m l_1^2 \ddot{\theta}_1 + m l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) - m l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1-\theta_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = m l_2^2 \ddot{\theta}_2 + m l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - m l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1-\theta_2)$$

$$\frac{\partial L}{\partial \theta_1} = -m l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) - m g l_1 \sin(\theta_1)$$

$$\frac{\partial L}{\partial \theta_2} = + m l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) - m g l_2 \sin(\theta_2)$$

E-L equation for θ_1:

$$\cancel{m} l_1^2 \ddot{\theta}_1 + \cancel{m} l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) - \cancel{m} l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1-\theta_2) + \cancel{m} l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) + \cancel{m} g l_1 \sin(\theta_1) = 0$$

$$l_1^{\cancel{2}} \ddot{\theta}_1 + \cancel{l_1} l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + \cancel{l_1} l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) + g \cancel{l_1} \sin(\theta_1) = 0$$

$$l_1 \ddot{\theta}_1 + l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) + g \sin(\theta_1) = 0$$

E-L equation for θ_2:

$$\cancel{m} l_2^{\cancel{2}} \ddot{\theta}_2 + \cancel{m} l_1 \cancel{l_2} \ddot{\theta}_1 \cos(\theta_1-\theta_2) - \cancel{m} l_1 \cancel{l_2} \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1-\theta_2) - \cancel{m} l_1 \cancel{l_2} \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) + \cancel{m} g \cancel{l_2} \sin(\theta_2) = 0$$

$$l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - l_1 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) + g \sin(\theta_2) = 0$$

Sanity check: $l_1 = 0$

$$\Rightarrow \begin{cases} l_2 \ddot{\theta}_2 + g \sin(\theta_2) = 0 \\ l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) + g \sin(\theta_1) = 0 \end{cases}$$

The first equation is the ordinary eq. of motion for a single pendulum.

The second equation reduces to the first when $\theta_1 = \theta_2$.

Transforming the equations for use with a numeric solver

Original equations:

$$\begin{cases} l_1 \ddot{\theta}_1 + l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + g \sin(\theta_1) = 0 \\ l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + g \sin(\theta_2) = 0 \end{cases}$$

What the solver wants is to

(1) rewrite as a system of 1st-order ODEs,

$$\frac{d}{dt} \vec{y} = f(t, \vec{y})$$

(2) supply the function f.

(1) We do (1) as follows: $y \in \mathbb{R}^4$

$$y_1 = \theta_1, \quad y_2 = \dot{\theta}_1, \quad y_3 = \theta_2, \quad y_4 = \dot{\theta}_2$$

and adding two new ODEs:

$$\frac{d}{dt} y_1 = y_2, \qquad \frac{d}{dt} y_3 = y_4$$

(2) We need to simplify the equations so as to explicitly provide $\frac{d}{dt} y_4 = \cdots, \frac{d}{dt} y_2 = \cdots$

To this end, note that the original functions are linear in $\ddot\theta_1, \ddot\theta_2$. So, it's just a matter of solving a 2×2 system of linear equations:

$$\underbrace{\begin{pmatrix} l_1 & l_2\cos(\theta_1-\theta_2) \\ l_1\cos(\theta_1-\theta_2) & l_2 \end{pmatrix}}_{\hat{=}\, M} \begin{pmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{pmatrix} = f(\theta_1, \dot\theta_1, \theta_2, \dot\theta_2).$$

$$\begin{pmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{pmatrix} = M^{-1} \cdot f(\theta_1, \dot\theta_1, \theta_2, \dot\theta_2)$$

<u>Aside</u>

I'm somewhat worried though that M seems like it can sometimes be singular, e.g. when $l_1 = l_2$ and $\theta_1 = \theta_2$ which gives $M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Specifically, in this case the equations become

$$\begin{cases} l_1\ddot\theta_1 + l_2\ddot\theta_2\cos(\theta_1-\theta_2) + \cancel{l_2\dot\theta_2^2\sin(\theta_1-\theta_2)} + g\sin(\theta_1) = 0 \\ l_2\ddot\theta_2 + l_1\ddot\theta_1\cos(\theta_1-\theta_2) - \cancel{l_1\dot\theta_1^2\sin(\theta_1-\theta_2)} + g\sin(\theta_2) = 0 \end{cases}$$

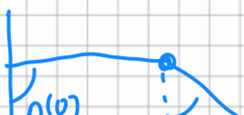
$$l(\ddot\theta_1 + \ddot\theta_2) + g\sin\theta_1 = 0$$

This doesn't seem to make sense... why is there no dependence on $\dot\theta_1, \dot\theta_2$?

It also seems to be degenerate, and so the solver stumbles on this.

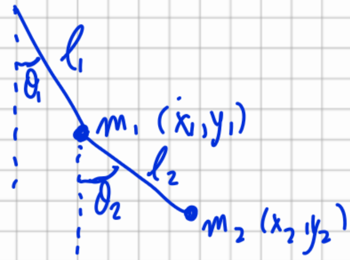
... In practice, I'm overcoming this for now by adding a small value (10^{-3}) to the diagonal.

$\theta^{(0)} = 90°$



$\theta_2^{(0)} = 45^\circ$

Double pendulum with 2 masses



$$x_1 = l_1 \sin\theta_1$$
$$y_1 = -l_1\cos\theta_1$$
$$x_2 = l_1\sin\theta_1 + l_2\sin\theta_2$$
$$y_2 = -l_1\cos\theta_1 - l_2\cos\theta_2$$

Potential energy:
$$\begin{aligned} V &= -m_1 g y_1 - m_2 g y_2 \\ &= -m_1 g l_1\cos\theta_1 - m_2 g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \\ &= -(m_1+m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \end{aligned}$$

Velocities:
$$\left.\begin{aligned} \dot{x}_1 &= l_1\dot\theta_1\cos\theta_1 \\ \dot{y}_1 &= l_1\dot\theta_1\sin\theta_1 \end{aligned}\right\} \Rightarrow v_1^2 = l_1^2\dot\theta_1^2$$
$$\dot{x}_2 = l_1\dot\theta_1\cos\theta_1 + l_2\dot\theta_2\cos\theta_2$$
$$\dot{y}_2 = l_1\dot\theta_1\sin\theta_1 + l_2\dot\theta_2\sin\theta_2$$
$$\Rightarrow v_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = l_1^2\dot\theta_1^2 + l_2^2\dot\theta_2^2 + 2l_1l_2\dot\theta_1\dot\theta_2\underbrace{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)}_{\cos(\theta_1-\theta_2)}$$

Kinetic energy:
$$\begin{aligned} T &= \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 \\ &= \tfrac{1}{2}m_1l_1^2\dot\theta_1^2 + \tfrac{1}{2}m_2l_1^2\dot\theta_1^2 + \tfrac{1}{2}m_2l_2^2\dot\theta_2^2 + m_2l_1l_2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2) \\ &= \tfrac{1}{2}(m_1+m_2)l_1^2\dot\theta_1^2 + \tfrac{1}{2}m_2l_2^2\dot\theta_2^2 + m_2l_1l_2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2) \end{aligned}$$

Lagrangian:
$$\begin{aligned} L &= T - V \\ &= \tfrac{1}{2}(m_1+m_2)l_1^2\dot\theta_1^2 + \tfrac{1}{2}m_2l_2^2\dot\theta_2^2 + m_2l_1l_2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2) + (m_1+m_2)gl_1\cos\theta_1 + m_2gl_2\cos\theta_2 \end{aligned}$$

Euler-Lagrange Equations:
$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta_i} - \frac{\partial L}{\partial\theta_i} = 0 \qquad i=1,2$$

E-L equation for θ_1:
$$\frac{\partial L}{\partial\theta_1} = -m_2l_1l_2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2) - (m_1+m_2)gl_1\sin\theta_1$$

$$\frac{\partial L}{\partial\dot\theta_1} = (m_1+m_2)l_1^2\dot\theta_1 + m_2l_1l_2\dot\theta_2\cos(\theta_1-\theta_2)$$
$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta_1} = (m_1+m_2)l_1^2\ddot\theta_1 + m_2l_1l_2\ddot\theta_2\cos(\theta_1-\theta_2) - m_2l_1l_2\dot\theta_2(\dot\theta_1-\dot\theta_2)\sin(\theta_1-\theta_2)$$

$$(m_1+m_2)l_1^{\cancel{2}}\ddot\theta_1 + m_2\cancel{l_1}l_2\ddot\theta_2\cos(\theta_1-\theta_2) - m_2\cancel{l_1}l_2\dot\theta_2(\cancel{\dot\theta_1}-\dot\theta_2)\sin(\theta_1-\theta_2)$$
$$\cancel{+m_2l_1l_2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2)} + (m_1+m_2)g\cancel{l_1}\sin\theta_1 = 0$$

$$(m_1+m_2)l_1\ddot{\theta}_1 + m_2 l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2 l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) + (m_1+m_2)g\sin\theta_1 = 0$$

E-L equation for θ_2:

$$L = T - V$$

$$= \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + (m_1+m_2)g l_1\cos\theta_1 + m_2 g l_2\cos\theta_2$$

$$\frac{\partial L}{\partial\theta_2} = m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - m_2 g l_2\sin\theta_2$$

$$\frac{\partial L}{\partial\dot{\theta}_2} = m_2 l_2^2\dot{\theta}_2 + m_2 l_1 l_2\dot{\theta}_1\cos(\theta_1-\theta_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}_2} = m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2 l_1 l_2\dot{\theta}_1\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)$$

$$\cancel{m_2} l_2^{\cancel{2}}\ddot{\theta}_2 + \cancel{m_2} l_1\cancel{l_2}\ddot{\theta}_1\cos(\theta_1-\theta_2) - \cancel{m_2} l_1\cancel{l_2}\dot{\theta}_1\sin(\theta_1-\theta_2)(\dot{\theta}_1-\cancel{\dot{\theta}_2})$$
$$-\cancel{m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)} + \cancel{m_2} g\cancel{l_2}\sin\theta_2 = 0$$

$$l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - l_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) + g\sin\theta_2 = 0$$

Sanity check: Converges to the previous solution when $m_1=0$.

Transform to solver form: $\ddot{\theta}_i = f_i(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2)$

$$\begin{cases}(m_1+m_2)l_1\ddot{\theta}_1 + m_2 l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2 l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) + (m_1+m_2)g\sin\theta_1 = 0\\ l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - l_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) + g\sin\theta_2 = 0\end{cases}$$

$$\begin{pmatrix}(m_1+m_2)l_1 & m_2 l_2\cos(\theta_1-\theta_2)\\ m_2 l_1\cos(\theta_1-\theta_2) & m_2 l_2\end{pmatrix}\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} = \begin{pmatrix}m_2 l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) + (m_1+m_2)g\sin\theta_1\\ -m_2 l_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) + m_2 g\sin\theta_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

This matrix is never singular (as long as $m_1, m_2 > 0$), because $M_{11} > M_{21}$ and $M_{12} \leq M_{22}$. Thus we avoid the singularity situation we saw in the previous derivation (which corresponds to $m_1=0$), and which caused numerical issues when solving the equations.