What are connections?

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Vector Fields

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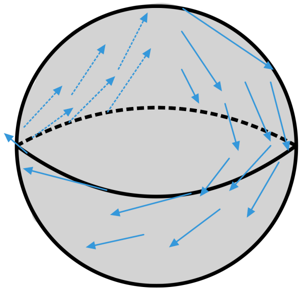

We have S^2 with a collection vectors drawn on it, all tangent to the surface at their base point.

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$$vf$$

For $x_1, \ldots, x_n$ the coordinates of $\mathbb{R}^n$, and partial derivatives $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, we write

$$vf = v^1 \frac{\partial}{\partial x_1} f + \cdots + v^n \frac{\partial}{\partial x_n} f$$

and so we write $v = v^1 \frac{\partial}{\partial x_1} + \cdots + v^n \frac{\partial}{\partial x_n}$

Vector Fields

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the partial derivatives aren't differentiating anything. The right hand side can be fed a function at any time, however.

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But this leaves us with an identification between v, the vector field, and an operator that takes directional derivatives. Despite this seeming abuse of notation, this is a very convenient notion.

Vector Fields

In general, for a manifold M, vector fields are defined as functions

$$v : C^\infty(M) \to C^\infty(M)$$

satifying the following:

$$\begin{aligned} v(f+g) &= v(f) + v(g) \\ v(\alpha f) &= \alpha v(f) \\ v(fg) &= v(f)g + fv(g) \end{aligned}$$

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We shall denote the collection of vector fields on a manifold M by:

$$\mathrm{Vect}(M)$$

Tangent Vectors

Going back to our initial idea of a vector field assigning an arrow to each point of M, we call this kind of arrow a *tangent vector*.

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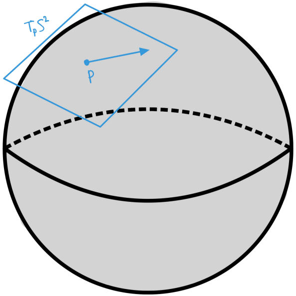

The plane of all vectors tangent to S^2 at the point p is called T_pS^2, *the tangent plane at p*.

Tangent Vectors

We formally define a tangent vector at a point $p \in M$ as a map

$$v_p : C^\infty(M) \to \mathbb{R}$$

by $v_p(f) = v(f)(p)$ *(recall* $v(f)$ *is again a* $C^\infty(M)$ *function).*

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This definition of a tangent vector in fact looks like the arrows in the diagrams since:

$$(v_p + w_p)(f) = v_p(f) + w_p(f)$$

and

$$(\alpha v_p)(f) = \alpha v_p(f)$$

So they indeed form a vector space.

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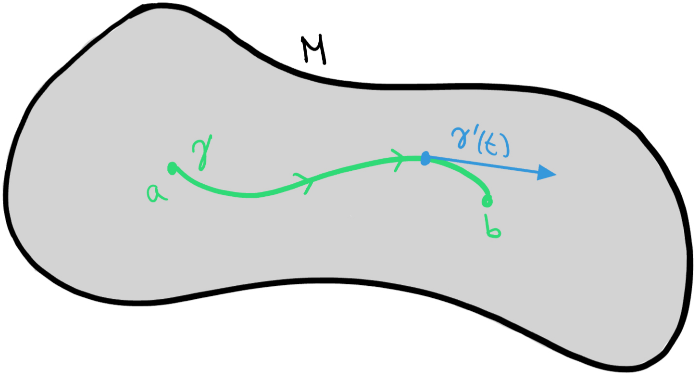

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A curve γ is a function from $\mathbb{R}$ (or some interval) to M that is *smooth*.

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In words, $\gamma'(t)$ differentiates functions in the direction that the curves γ is 'moving' at time t.

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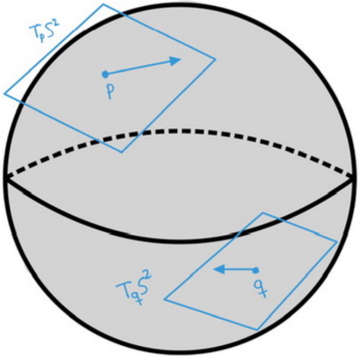

Example

To each point (x, y) in $\mathbb{R}^2$, we will define the basis to the tangent space to be given by:

$$e_1 = \begin{pmatrix} \cos(x) \\ \sin(x) \end{pmatrix}, \qquad e_2 = \begin{pmatrix} -\sin(x) \\ \cos(x) \end{pmatrix}$$

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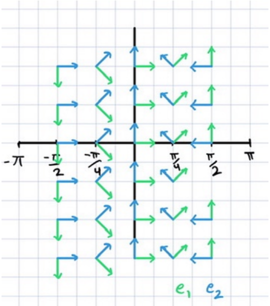

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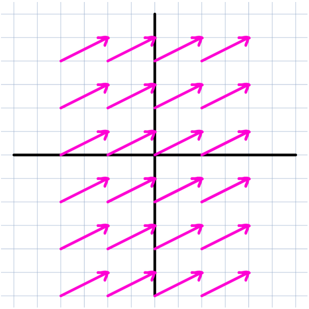

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In our basis e_1, e_2, even though our vector field is constant, the components change as x varies:

$$\begin{aligned} v(x, y) &= v^1 e_1 + v^2 e_2 \\ &= (2\cos x + \sin x) e_1 + (\cos x - 2\sin x) e_2 \end{aligned}$$

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So in this basis, we have:

$$v(x,y) = \begin{pmatrix} 2\cos x + \sin x \\ \cos x - 2\sin x \end{pmatrix}$$

which does not appear constant at all.

Example

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So how do we tell if the vector field is truly changing, or if this is just an artefact of our basis?

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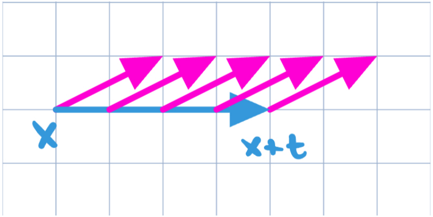

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But this is just the x-partial derivative of v:

$$\begin{aligned} \frac{dv}{dt} &= \frac{d}{dt}(v^1 e_1 + v^2 e_2) \\ &= v^1_x e_1 + v^1 (e_1)_x + v^2_x e_2 + v^2 (e_2)_x \\ &= (v^1_x - v^2) e_1 + (v^1 + v^2_x) e_2 = 0 \end{aligned}$$

Example

$$\frac{dv}{dt} = (v_x^1 - v^2)e_1 + (v^1 + v_x^2)e_2 = 0$$

Notice that in each component e_i, there are two terms:

1. the partial derivative of the component, v_x^i
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2. a correction term, which turns out to be v^2 for e_1 and v^1 for e_2

The notion of the 'true' derivative sees that the vector field is constant, despite the changing components. We wish to arrive at this idea for manifolds more complicated that $\mathbb{R}^n$.

Parallel Transport

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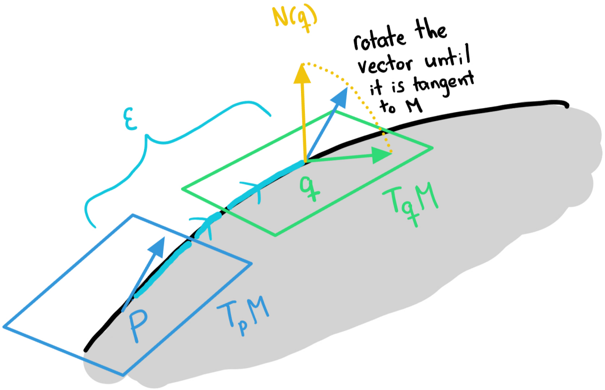
The way we parallel transport a tangent vector $w \in T_pM$ to T_qM for some nearby point q is to simply place w on q. There is no reason for this to be a tangent vector to q, but if we consider M to be living in some larger ambient space, we can 'rotate' w until it lies in T_qM.

Parallel Transport

A picture is worth a thousand words:



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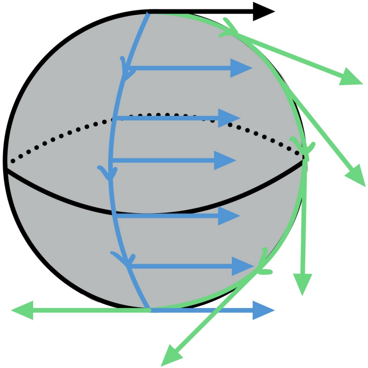

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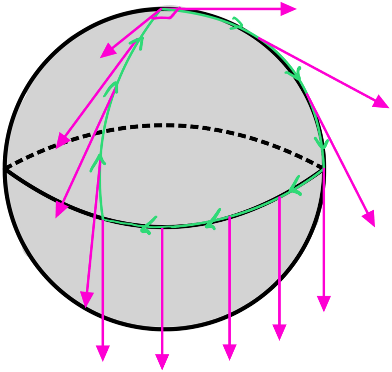

Parallel Transport

Some important properties of parallel transport:

1. If two vectors in the same tangent space are parallel transported along the same curve, then the angle between them remains constant
2. From an initial point and an arbitrary initial tangent vector v (velocity), parallel transport of v along v yields a unique geodesic arising from these conditions.

Covariant Derivatives

Consider a geodesic G on some surface M. Let $G'(t) = v(t)$, the velocity of the curve. Suppose there is a vector field on G, $w(t)$. Parallel transport a tangent vector $w(p) \in T_pM$ along G.

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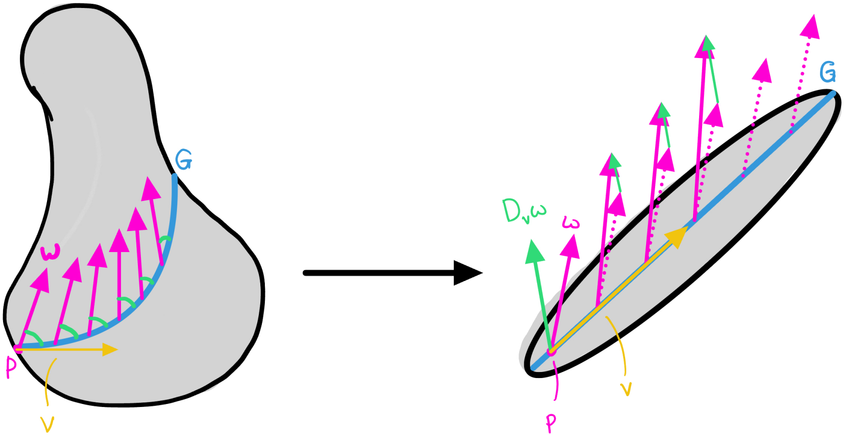
$$D_v w$$

Covariant Derivatives

Once again, pictures are superior:



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$D_v w$ is quantifying the rate at which $w(t)$ is changing as it moves in the direction of v.

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If we are not moving along a geodesic, the picture remains the same, as long as we consider only very short segments.

Covariant Derivatives

One more fact that is obvious from this picture is that $D_v w = 0$ if and only if w is parallel transported along v.

Connections

We can now extend D to be defined on the entirety of a manifold M, so now v and w are vector fields and $D_v w$ is map that assigns to each $x \in M$ the tangent vector corresponding to $D_{v(x)} w(x)$.

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In this way, we may consider D as a map

$$D : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$$

However, the inputs are not symmetric.

Connections

D satisfies the following properties for $v_1, v_2, w_1, w_2 \in \text{Vect}(M)$ and $f \in C^\infty(M)$ and α a scalar:

$$
\begin{aligned}
D_{v_1}(\alpha w_1) &= \alpha D_{v_1} w_1 \\
D_{v_1}(w_1 + w_2) &= D_{v_1} w_1 + D_{v_1} w_2 \\
D_{v_1}(f w_1) &= v(f) w_1 + f D_{v_1} w_1 \qquad (*) \\
D_{v_1 + v_2} w_1 &= D_{v_1} w_1 + D_{v_2} w_1 \\
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\end{aligned}$$

In particular, for the second input D behaves like a derivative ($(*)$ this is known as the *Leibnitz rule*), while D is linear with respect to functions in the first input $(\dagger)$

Connections

So D is collection of all the information about differentiation vector fields and hence parallel transport.

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What about a more general setting? So far we have been exclusively looking at the tangent bundle. This connection is in fact canonical and called the *Levi-Civita connection*.

Connections

In more generality, we don't necessarily have the same notion of angles to measure when vectors are parallel or not, or be able to rotate vectors. We have more general *vector bundles*.

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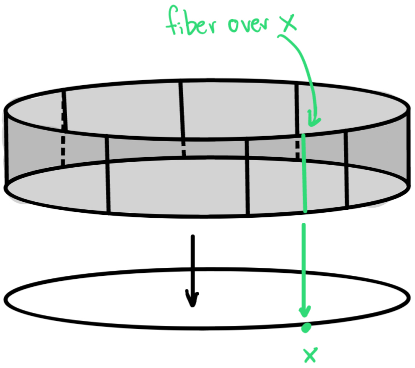
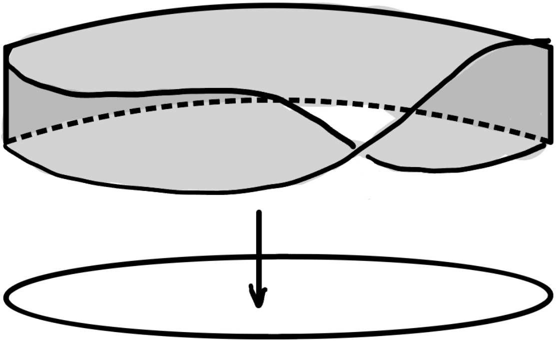


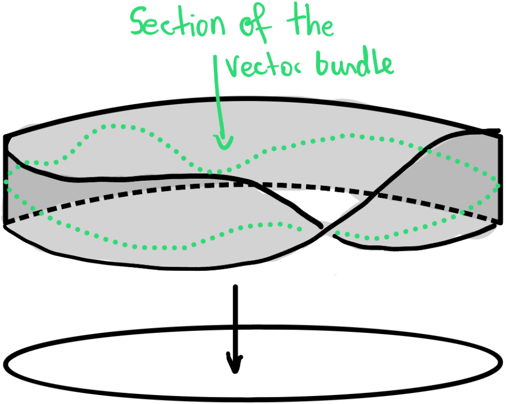
Connections

A non-trivial example of a vector bundle is the mobius bundle:



Connections

A section of vector bundle:



Relating the original example

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Now for a vector bundle E over M, it becomes:

$$D : \mathrm{Vect}(M) \times \Gamma(E) \to \Gamma(E)$$

Relating the original example

We replace vector fields in the second input with sections of our bundle, denote $\Gamma(E)$. So for $v \in \mathrm{Vect}(M)$ and $s \in \Gamma(E)$ we can calculate the covariant derivative:

$$
\begin{aligned}
D_v s &= D_{v^j \partial_j} s \\
&= v^j D_{\partial_j} s \\
&= v^j (D_{\partial_j}(s^i e_i)) \\
&= v^j ((\partial_j s^i) e_i + A_{ji}^k s^i e_k) \\
&= v^j (\partial_j s^i + A_{jk}^i s^k) e_i
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which bears a striking resemblance to the 'true' derivative from earlier:

$$\frac{dv}{dt} = (v^1_x - v^2)e_1 + (v^1 + v^2_x)e_2$$

Thank You!

The end!