The Role of Cohomology in Defining a Symplectic Form on the Character Variety

Author: Zachary Virgilio

University of California, Riverside
zach.virgilio@email.ucr.edu

Introduction

In 1984, William Goldman published his paper "The Symplectic Nature of the Fundamental Groups of Surfaces" where he constructed a symplectic form on the character variety of a surface.

Introduction

In 1984, William Goldman published his paper "The Symplectic Nature of the Fundamental Groups of Surfaces" where he constructed a symplectic form on the character variety of a surface.

In this presentation, we will outline how he did this and highlight the role played by group cohomology.

Background

Let S be a closed oriented topological surface, $\pi = \pi_1 S$ be its fundamental group, and G be a connected Lie group.

$\mathrm{Hom}(\pi, G)$ is the space of representations of $\pi \to G$, endowed with the compact-open topology. G acts on $\mathrm{Hom}(\pi, G)$ via conjugation:

$$g \cdot \rho = g\rho g^{-1} \quad g \in G, \rho \in \mathrm{Hom}(\pi, G)$$

The *character variety* is the quotient space $\mathrm{Hom}(\pi, G)/G$

Goldman showed that if G preserved a nondegenerate bilinear form on its Lie algebra $\mathfrak{g}$, then $\mathrm{Hom}(\pi, G)/G$ had a symplectic structure.

Group Cohomology

We outline the cohomological theory for groups. Let G be a group and M a G-module, that is, there is a left action of G on M such that each $g \in G$ acts as an automorphism of M.

We define the cochain by:

$$C^n(G, M) = \{f : G^n \to M\}$$

(we interpret $G^0 = \mathrm{id}_G$) And the boundary operator $d^{n+1} : C^n(G, M) \to C^{n+1}(G, M)$ by:

$$\begin{aligned} d^{n+1} f(g_1, \ldots, g_{n+1}) = {} & g_1 f(g_2, \ldots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i \cdot g_{i+1}, \ldots, g_{n+1}) \\ & + (-1)^{n+1} f(g_1, \ldots, g_n) \end{aligned}$$

Group Cohomology

With the cochain complex $(C^*(G, M), d^*)$, we define cohomology in the usual manner:

$$H^n(G, M) = \ker d^{n+1} / \operatorname{Im} d^n$$

For $G = \pi$, we have:

$$H^*(\pi, M) = H^*(S, M)$$

So the cohomology of a surface corresponds to the cohomology of its fundamental group.

Tangent vectors as cocycles

In order to define a symplectic form on a surface, we must have a characterization of the tangent space.

Fix $\phi \in \mathrm{Hom}(\pi, G)$. We consider a path ϕ_t in $\mathrm{Hom}(\pi, G)$ such that $\phi_0 = \phi$. Then $\phi_t(x)\phi^{-1}(x)$ is in a neighborhood of the identity for small t, so it is in the image of the exponential map from the Lie algebra:

$$\phi_t(x)\phi^{-1}(x) = e^{tu(x)+\mathcal{O}(t^2)}$$

For $tu(x) + \mathcal{O}(t^2)$ a Taylor expansion of a function from $\pi \to \mathfrak{g}$ and $u(x)$ is the linear component, i.e. $u : \pi \to \mathfrak{g}$.

Rewrite:

$$\phi_t(x) = e^{tu(x)+\mathcal{O}(t^2)}\phi(x)$$

Tangent vectors as cocycles

Since ϕ_t is a homomorphism for all t:

$$\phi_t(xy) = \phi_t(x)\phi_t(y)$$

We differentiate both sides at $t = 0$ to understand the tangent vector:

$$\frac{d}{dt}\Big|_{t=0} e^{tu(xy)+\mathcal{O}(t^2)}\phi(xy) = u(xy)$$

And since:

$$e^{tu(xy)+\mathcal{O}(t^2)} = \phi_t(xy)\phi^{-1}(xy)$$

Tangent vectors as cocylces

So we have:

$$\begin{aligned} u(xy) &= \frac{d}{dt}\Big|_{t=0} e^{tu(xy)+\mathcal{O}(t^2)} \\ &= \frac{d}{dt}\Big|_{t=0} \left(\phi_t(xy)\phi^{-1}(xy)\right) \\ &= \left(\frac{d}{dt}\Big|_{t=0} \phi_t(xy)\right) \phi^{-1}(xy) \end{aligned}$$

Hence:

$$\frac{d}{dt}\Big|_{t=0} \phi_t(xy) = u(xy)\phi(xy)$$

Tangent vectors as cocyles

Now for the other side of the equation

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}(\phi_t(x)\phi_t(y)) &= u(x)\phi(x)\phi(y)+\phi(x)u(y)\phi(y)\\ &= u(x)\phi(xy)+\phi(x)u(y)\phi^{-1}(x)\phi(xy)\\ &= (u(x)+\mathrm{Ad}\,\phi(x)u(y))\,\phi(xy)\end{aligned}$$

Because $\phi_t(xy)=\phi_t(x)\phi_t(y)$, their derivatives must be equal, so we have:

$$u(xy)\phi(xy)=(u(x)+\mathrm{Ad}\,\phi(x)u(y))\,\phi(xy)$$

So the map $u:\pi\to\mathrm{Aut}(\mathfrak{g})$ satisfies:

$$u(xy)=u(x)+\mathrm{Ad}\,\phi(x)u(y)$$

Tangent vectors as cocycles

We have seen that a tangent vector of ϕ is entirely defined by the map $u : \pi \to \mathrm{Aut}(\mathfrak{g})$. Further, $\mathfrak{g}$ is a π module via $\pi \xrightarrow{\phi} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g})$. So we can define the group cohomology $H^1(\pi, \mathfrak{g})$. From our earlier definition, we know that $f \in H^1(\pi, \mathfrak{g})$ is defined by the property that

$$d^2 f(g_1, g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1)$$

We calculate

$$\begin{aligned} d^2 u(x, y) &= \mathrm{Ad}\,\phi(x) u(y) - u(xy) + u(y) \\ &= \mathrm{Ad}\,\phi(x) u(y) - (u(x) - \mathrm{Ad}\,\phi(x) u(y)) + u(x) \\ &= 0 \end{aligned}$$

So u satisfies the cocycle condition and hence u defines an element in $\ker d^2$.

Tangent vectors as cocyles

The converse is immediately clear, any 1-cocycle $u : \pi \to \mathfrak{g}$ must satisfy

$$u(xy) = u(x) + \mathrm{Ad}\,\phi(x)u(y)$$

And so the $\phi_t(x)$ defined by $\phi_t(x) = e^{tu(x)+\mathcal{O}^2(t^2)}\phi(x)$ satisfies $\phi_t(xy) = \phi_t(x)\phi_t(y)$ to the first order, hence it defines a tangent vector.

Thus the tangent space of $\mathrm{Hom}(\pi, G)$ at ϕ is identified with $\ker d^2$.

In fact, when pass to the quotient $\mathrm{Hom}(\pi, G)/G$, tangent vectors are identified with classes in $H^1(\pi, \mathfrak{g})$.

Symplectic forms

A symplectic form ω a section in $\Omega^2(S)$ satisfying closed and nondegeneracy conditions.

In other words, for each $s \in S$, ω_s is a pairing

$$\omega_s : T_sS \times T_sS \to \mathbb{R}$$

satisfying

1. $\omega_s(u, u) = 0$ for all $u \in T_sS$.
2. $\ker \omega_s = \{v \in T_sS : \omega_s(v, u) = 0 \text{ for all } u \in T_sS\}$ is trivial.

And the closed condition means $d\omega = 0$.

A symplectic form on $\mathrm{Hom}(\pi, G)/G$

Since we identify the tangent space of $\mathrm{Hom}(\pi, G)/G$ with $H^1(\pi, \mathfrak{g})$, we need a skew-symmetric pairing of the first cohomology.

A symplectic form on $\mathrm{Hom}(\pi, G)/G$

Since we identify the tangent space of $\mathrm{Hom}(\pi, G)/G$ with $H^1(\pi, \mathfrak{g})$, we need a skew-symmetric pairing of the first cohomology.

In fact, the cup product is exactly what we need:

$$\omega^{(B)} : H^1(\pi, \mathfrak{g}) \times H^1(\pi, \mathfrak{g}) \xrightarrow{\smile} H^2(\pi, \mathbb{R}) - \mathbb{R}$$

Where the coefficient in $H^2(\pi, \mathbb{R})$ is $\mathbb{R}$ because we required the Lie algebra have a nondegenerate bilinear form, B, so we identify $\mathfrak{g}$ with $\mathfrak{g}^*$, and this is equal to $\mathbb{R}$ by Poincarë duality.

A symplectic for on $\mathrm{Hom}(\pi, G)/G$

In fact, checking the alternating and nondegeneracy conditions is quite simple.

Nondegeneracy is a consequence of Poincarë duality on S and alternating follows from the fact the cup product is.

All that remains is to check the closed condition. This is far more difficult and Goldman adapted the technique of Atiyah-Bott of moving to larger space to check the closed condition.