Complex Hyperbolic Structures Arising from SU(2,1)-Higgs Bundles

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Setup: The Higgs Bundles 1

An SU(2, 1)-*Higgs bundle* over a Riemann surface X of genus $g \geq 2$ is given by the data of:

- $(\mathcal{U}, \mathcal{W})$, complex vectors bundles over X of rank 2 and 1, respectively, satisfying $\det(\mathcal{U}) = \det(\mathcal{W}^*)$.
- $\beta : \mathcal{U} \to \mathcal{W} \otimes K_X$ and $\gamma : \mathcal{W} \to \mathcal{U} \otimes K_X$.

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Our focus will be on $\mathrm{SU}(2,1)$-Higgs bundles that decompose as line bundles as:

$$\mathcal{V}_1 \xrightarrow{\beta_1} \mathcal{V}_2 \xrightarrow{1} \mathcal{V}_3$$

where $\mathcal{V}_1 \oplus \mathcal{V}_3 = \mathcal{U}$ and $\mathcal{V}_2 = \mathcal{W}$. The map γ decomposes into $\gamma_1 : \mathcal{V}_2 \to \mathcal{V}_1 \otimes K_X$ and $\gamma_3 : \mathcal{V}_2 \to \mathcal{V}_3 \otimes K_X$. We will assume that $\gamma_1 = 0$ and γ_3 is an isomorphism and so will be denoted by 1.

Setup: Higgs Bundles 2

Associated to our SU$(2,1)$-Higgs bundle are the following structures:

1. The holonomy representation $\rho : \pi_1(X) \to \mathrm{SU}(2,1)$. Γ will denote the image of ρ.
2. A signature $(2,1)$ hermitian metric. We will use $\|\cdot\|_x$ to denote the norm induced by the metric, and when the base point x is clear, we will suppress the notation.
3. A flat connection.

Setup: Geometric Structures

For a Lie group G and a homogeneous space X, a (G, X)-structure on a surface M is completely determined by a developing map

$$dev : \tilde{M} \to X$$

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In particular, we will care about $\mathbb{CH}^2$-structures:

$$(\mathrm{SU}(2,1), \mathbb{CH}^2)$$

where $\mathbb{CH}^2$ is defined to be the set of negative lines in $\mathbb{P}(\mathbb{C}^{2,1})$. We will also refer to $\partial\mathbb{CH}^2$-structures, structures where the homogeneous space is the boundary of $\mathbb{CH}^2$.

Main Result

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- There exists a disk bundle $D \to X$ whose total space admits a $\mathbb{CH}^2$ structure, and in particular, the developing map is a bijection.
- There is a circle bundle $B \to X$ whose total space admits a $\partial\mathbb{CH}^2$ structure.
- Furthermore, this is the Guichard-Weinhard geometric structure arising from ρ being Anosov. Since we are in rank 1, being Anosov is equivalent to being convex-cocompact.

The Work of S. Filip

In 2019, Filip studied the case of $\mathrm{SO}(2,3)$ with Higgs bundle of the form:

$$\mathcal{W}_1 \to \mathcal{W}_2 \to \mathcal{W}_3 \to \mathcal{W}_4 \xrightarrow{1} \mathcal{W}_5$$

He showed their corresponding holonomy representations were Anosov and constructed the Guichard-Weinhard geometric structure.

Further, he showed that the of the image of the developing map is the domain of discontinuity coming from the Anosov property of the representation. The complement of the domain of discontinuity was identified with the limit set of Γ.

The methods here were adapted from his work.

The Pullback Bundle 1

We will work in the pullback bundle over the universal cover $\tilde{X}$ of X.

Fix a base point $x_0 \in X$. The fiber $\mathcal{V}(x_0)$ over x_0 will be denoted by V and the decomposition with respect to the hermitian metric will be written $V = V_1 \oplus V_2 \oplus V_3 \cong \mathbb{C}^{2,1}$.

Then the pull-back bundle can be identified with $\tilde{X} \times V$ using the flat connection.

The Pullback Bundle 2

For any vector $u = u_1 + u_2 + u_3 \in V$, we can use the flat connection to identify u with with a flat section of $\tilde{X} \times V$.

For such a section, $u(x)$ will denote the image of the section in the fiber over $x \in \tilde{X}$.

The Key Function

Take $u \in V$ to be a negative vector. Necessarily, $u_2 \neq 0$.

Associated to u, we define a function $f : \tilde{X} \to \mathbb{R}$:

$$f_u(x) := \|u_3(x)\|_x^2$$

so f_u measures the norm of the $\mathcal{V}_3(x)$ component of $u(x)$.

Importantly, $f_u(x)$ has a *unique* minimum and it takes value 0 at the minimum.

Constructing the Bundle

Define

$$B'_x = \{v \in V : \|v\| < 0 \text{ and } v(x) \perp \mathcal{V}_3(x)\}$$

For $v \in B'_x$, $v(x) \in \mathcal{V}_1(x) \oplus \mathcal{V}_2(x)$, so we may identify B'_x with negative vectors inside a copy of $\mathbb{C}^{1,1}$.

Let B_x be the projectivization of B'_x.

Note as well that the projectivization of the negative vectors in V can be identified with $\mathbb{CH}^2$, since $V \cong \mathbb{C}^{2,1}$.

Methods

Let $\tilde{B}$ be the bundle over $\tilde{X}$ where each fiber over x is given by B_x. Then $\tilde{B} \subset \tilde{X} \times \mathbb{CH}^2$.

$\tilde{B}$ is a $\pi_1(X)$-equivariant bundle when the action on the $\mathbb{CH}^2$ is given by ρ. So $\tilde{B}$ descends to a bundle B over X:

$$B := \left(\tilde{B}/\pi_1(X)\right) \to \left(\tilde{X}/\pi_1(X)\right) = X$$

Consider the projection map onto the second factor: $\tilde{B} \to \mathbb{CH}^2$.

Denote this map by

$$dev : \tilde{B} \to \mathbb{CH}^2$$

By construction this map is $\pi_1(X)$-equivariant.

Methods

dev is surjective: Let u be any negative vector in V. Since f_u achieves a minimum at 0 for some $x \in \tilde{X}$, so u is in the image $dev(x)$.

dev is injective: Since f_u has a unique minimum, $f_u(x') \neq 0$ for any $x' \in \tilde{X}$ where $x \neq x'$, so u is not in the image of $dev(x')$ for any other x'.

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The construction of the $\partial\mathbb{CH}^2$ structure is completely analgous, except we consider u as non-zero null vectors. The primary difference is that in this case, f_u may not achieve the minimum, so *dev* will not be a bijection.

Generalizations

Every step works practically the same for $\mathrm{SU}(n, 1)$ and $\mathbb{CH}^n$. The Higgs bundle will decompose as

$$\mathcal{V}_1 \xrightarrow{\beta} \mathcal{V}_2 \xrightarrow{1} \mathcal{V}_3$$

where $\mathcal{V}_1$ is now a rank $n-1$ complex vector bundle.

Also, the case of certain $\mathrm{SO}(2, n)$-Higgs bundles is amenable to these methods. Here the domain of discontuity comes from the representations being Anosov, not just convex cocompact.