Equivariant K-stability under finite group action

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ZAG Seminar, July 30, 2020

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- The dimension of X is denoted by n.

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Conjecture

Let X be a \mathbb{Q} -Fano variety and G a reductive group action on X. If X is G-equivariantly K-polystable, then X is K-semistable (resp. K-polystable).

Solution to the equivariant K-stability conjecture:

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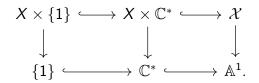
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There is an induced *G*-action on $X \times \mathbb{C}^*$:

$$g(x,t) = (\lambda_t \circ g \circ \lambda_t^{-1}(x), t), g \in G.$$

where λ_t is the 1-parameter subgroup inducing the test configuration.



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where λ_t is the 1-parameter subgroup inducing the test configuration.

The test configuration is called G-equivariant if the above G-action extends to an action on \mathcal{X} commuting with the \mathbb{G}_m -action.

Itersection formula of generalized Futaki invariant

There is a natural $G \times \mathbb{G}_m$ -equivariant compactification $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ of $(\mathcal{X}, \mathcal{L})$ over \mathbb{P}^1 .

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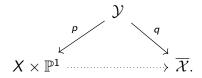
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The **generalized Futaki invariant** of a normal $(\mathcal{X}, \mathcal{L})$ can be computed by

$$\mathsf{Fut}(\mathcal{X},\mathcal{L}) = \frac{1}{(n+1)(-\mathcal{K}_{\mathcal{X}})^n} \left(\frac{n}{r^{n+1}} \overline{\mathcal{L}}^{n+1} + \frac{n+1}{r^n} \overline{\mathcal{L}}^n \cdot \mathcal{K}_{\overline{\mathcal{X}}/\mathbb{P}^1} \right)$$

Norm of a test configuration

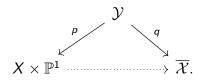
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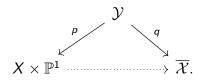


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$$J^{\mathsf{NA}}(\mathcal{X},\mathcal{L}) := rac{p^*(-\mathcal{K}_{X imes\mathbb{P}^1/\mathbb{P}^1})^n\cdot q^*ar{\mathcal{L}}}{(-\mathcal{K}_X)^n} - rac{ar{\mathcal{L}}^{n+1}}{(n+1)(-r\mathcal{K}_X)^n}.$$

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 $J^{NA}(\mathcal{X},\mathcal{L})$ is one of the several equivalent norms of test configurations that can be used to define uniform K-stability.



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- uniformly (*G*-equivariantly) K-stable if there exists $\delta \in (0,1)$, such that $\operatorname{Fut}(\mathcal{X},\mathcal{L}) \geq \delta J^{\operatorname{NA}}(\mathcal{X},\mathcal{L})$ for any (*G*-equivariant) test configuration $(\mathcal{X},\mathcal{L})$.

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Note: If we take $G = \{e\}$, then we recover the usual K-stability notions.



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- Consider $G = \operatorname{Aut}(\mathbb{P}^n) = \operatorname{PGL}(n+1)$. \mathbb{P}^n is uniformly G-equivariantly K-stable, since there is no nontrivial G-equivariant test configuration for \mathbb{P}^n .
- Therefore the equivariant K-stability conjecture is not true for uniform K-stability.

Examples (Tian's criterion)

For a \mathbb{Q} -Fano variety X with a group action G, we can define its G-equivariant alpha invariant

$$\alpha_G(X) = \inf_{m} \left\{ |\operatorname{lct}\left(X, \frac{1}{m}\Sigma\right)| \text{ G-invariant } \Sigma \subset |-mK_X| \right\}.$$

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Theorem (Odaka-Sano)

If $\alpha_G(X) \ge (resp. >) n/(n+1)$, then X is G-equivariantly K-semistable (uniformly G-equivariantly K-stable).



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 V_{22}^* denotes a class of Fano threefolds of degree 22 admitting G-action with $G=\mathbb{C}^*\rtimes\mathbb{Z}/2$. All but two of them have G-equivariant alpha invariants to be 4/5.

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According to the equivariant Tian's criterion, those $X \in V_{22}^*$ with $\alpha_G(X)=4/5$ are uniformly G-equivariantly K-stable and hence K-polystable (Kähler-Einstein) due to the equivariant K-stability conjecture.

Valuative criterion

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where

$$\tau(E) = \sup\{t | \operatorname{vol}_Y(\pi^*(-K_X) - tE) > 0\}$$

is the pseudo-effective threshold of E.



Valuative criterion (Cont'd)

The following two invariants are involved in the valuative criterion:

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Theorem (Fujita, Li)

A Fano variety X is

- K-semistable if and only if $\beta(E) \ge 0$ for every prime divisor E over X;
- ② uniformly K-stable if and only if there exists $\delta \in (0,1)$, such that $\beta(E) \geq \delta j(E)$ for every prime divisor E over X.



Pseudovaluations

Let G < Aut(X) be a group action on X. For any valuation v on X, we define

$$G \cdot v := \inf_{g \in G} g \cdot v,$$

where $g \cdot v$ is the valuation given by $g \cdot v(f) = v(f \circ g)$ for any $f \in \mathbb{C}(X)$.

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For any nonnegative real number x, the ideal sheaf

$$\mathfrak{a}_{\scriptscriptstyle X}(G\cdot v)=\bigcap_{g\in G}\mathfrak{a}_{\scriptscriptstyle X}(g\cdot v)$$

collects regular functions of vanishing order at least x with respect to all $g \cdot v$'s.



Equivariant valuative criterion

Let E be a divisor over X. Define

$$S_X^G(E) := \frac{1}{(-K_X)^n} \int_0^{\tau^G(E)} \operatorname{vol}_X \left(\mathcal{O}_X(-K_X) \otimes \mathfrak{a}_X(G \cdot \operatorname{ord}_E) \right) \, dx,$$

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The equivariant invariants we need for the equivariant valuative criterion are

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Note that the above invariants coincide with the usual ones if E is G-invariant.



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Theorem (Z)

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where R runs through all ramification divisors and e_R is the ramification index of R. For each branch divisor B_i on Y, denote by e_i the ramification index of R with $\sigma(R) = B_i$. Then we have

$$K_X = \sigma^*(K_Y + B),$$

where
$$B = \sum_i \left(1 - \frac{1}{e_i}\right) B_i$$
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Idea of the proof:

- "Only if" part: comparing beta invariants of corresponding prime divisors;
- "If" part: comparing generalized Futaki invariants of corresponding test configurations.



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Consider an étale neighborhood of $[X_0]$ in the corresponding K-moduli stack of the form $[\operatorname{Spec} A/H]$ where $H = \operatorname{Aut}(X_0)$ and $G < \operatorname{Aut}(X) < H$.

A test configuration \mathcal{X} degenerating X to X_0 corresponds to a morphism $\phi: [\mathbb{A}^1/\mathbb{G}_m] \to [\operatorname{Spec} A/H]$ with $\phi(1) = [X]$ and $\phi(0) = [X_0]$.

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Equivalently, we have a 1-parameter subgroup: $\lambda: \mathbb{G}_m \to H$ with

$$\lim_{t\to 0} \lambda(t) \cdot [X] = [X_0].$$



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where $Z_H(G)$ is the centralizer of G inside H.

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Thank you!