Thm $x \not\leq y \Rightarrow y \leq x$

(Proof): $x \not\leq y \iff \big(\underbrace{\exists x_L \in X_L : y \leq x_L}_{(*1)}\big) \vee \big(\underbrace{\exists y_R \in Y_R : y_R \leq x}_{(*2)}\big)$

If $(*1)$ holds, $y \leq x_L \leq x$

If $(*2)$ holds, $y \leq y_R \leq x$

Cor $x < y \iff y \not\leq x$

(Proof): $x < y \iff x \leq y \wedge y \not\leq x$, by def. of "$<$"

Since "$y \not\leq x \Rightarrow x \leq y$", then "$x < y \iff y \not\leq x$"

Thm $x < y \wedge y < z \Rightarrow x < z$

(Proof): Suppose not, i.e.: $x < y \wedge y < z \wedge x \not< z$ (†)

$\iff y \not\leq x \wedge \overbrace{z \not\leq y}^{(*1)} \wedge \overbrace{z \leq x}^{(*2)}$

By the transitive law, $y \not\leq x \Rightarrow y \not\leq s \vee s \not\leq x$ for all $s \in S$

$\Rightarrow \underbrace{y \not\leq z}_{\text{Contradict } (*1)} \vee \underbrace{z \not\leq x}_{\text{Contradict } (*2)}$

Therefore, (†) is wrong. $\Rightarrow$ the thm holds $\square$

Thm For any $x = \{ X_L \mid X_R \} \in \mathbb{S}$, we can remove any member of X_L except the largest number without changing the value of x. Similarly, we can remove any member of X_R except the smallest without changing the value of x

(Proof): Assume that $x = \{ x_1, \cancel{x_2}, \ldots \mid X_R \}$ and that $x_1 < x_2$.

W.T.S.: $\{ x_1, x_2, \cdots \mid X_R \} = \{ x_2, \cdots \mid X_R \}$

$$\Updownarrow$$

$$\{ x_1, x_2, \cdots \mid X_R \} \leq \{ x_2, \cdots \mid X_R \} \quad \wedge \quad \{ x_2, \cdots \mid X_R \} \leq \{ x_1, x_2, \cdots \mid X_R \}$$

$$\Leftrightarrow \begin{cases} \text{(i)} & \neg \exists\, a \in \{ x_1, x_2, \cdots \} : \{ x_2, \cdots \mid X_R \} \leq a \quad \wedge \\ \text{(ii)} & \neg \exists\, b \in X_R : b \leq \{ x_1, x_2, \cdots \mid X_R \} \quad \wedge \\ \text{(iii)} & \neg \exists\, a' \in \{ x_2, \cdots \} : \{ x_1, x_2, \cdots \mid X_R \} \leq a' \quad \wedge \\ \text{(iv)} & \neg \exists\, b' \in X_R : b' \leq \{ x_2, \cdots \mid X_R \} \end{cases}$$

For (i), $\forall a \in \{ x_1, x_2, \cdots \}$, $a = x_1 \quad \vee \quad a \in \{ x_2, \cdots \}$

$$\Rightarrow \quad a < x_2 \in \{ x_2, \cdots \} = \{ x_2, \cdots \mid X_R \}_L \quad \vee \quad a \in \{ x_2, \cdots \} = \{ x_2, \cdots \mid X_R \}_L$$

$$\Rightarrow \quad a < x_2 < \{ x_2, \cdots \mid X_R \} \quad \vee \quad a < \{ x_2, \cdots \mid X_R \}$$

$$\Rightarrow \quad a < \{ x_2, \cdots \mid X_R \} \quad \Rightarrow \quad \text{(i) holds}$$

For (ii) (iii) (iv), they follow immediately from the previous thm: $A < \{ A \mid B \} < B$ □

Cor: If A has a largest member a, then $\{A|B\}=\{a|B\}$

Similarly, if B has a smallest member b, then $\{A|B\}=\{A|b\}$

Thm: If $x=\{X_L \mid X_R\} \in \mathcal{S}$ is greater than all members of A & less than all members of B, then $x=\{X_L, A \mid X_R, B\}$

In other words, $A < x < B \Rightarrow x=\{X_L, A \mid X_R, B\}$

(Proof): W.T.S: $x \leq \{X_L, A \mid X_R, B\} \wedge \{X_L, A \mid X_R, B\} \leq x$

$$\Leftrightarrow \begin{cases} \text{(i)} & \neg \exists x_L \in X_L : \{X_L, A \mid X_R, B\} \leq x_L \quad \wedge \\ \text{(ii)} & \neg \exists \beta \in X_R \cup B : \beta \leq x \quad \wedge \\ \text{(iii)} & \neg \exists \alpha \in X_L \cup A : x \leq \alpha \quad \wedge \\ \text{(iv)} & \neg \exists x_R \in X_R : x_R \leq \{X_L, A \mid X_R, B\}. \end{cases}$$

(i) & (iv) holds following from the previous thm: $A<\{A|B\}<B$

(ii) & (iii) holds following from the previous thm: $A<\{A|B\}<B$

and the def. of A, B in the statement (hypothesis) of the thm $\square$

Thm: If, after day m, the following different surreal numbers exist: $x_1 < x_2 < \cdots < x_n$,

then all new numbers born on day $m+1$ can be represented by: $\{\,|x_1\}, \{x_1|x_2\}, \{x_2|x_3\}, \cdots, \{x_{n-1}|x_n\}, \{x_n|\,\}$

(Proof): W.T.S.:

① $\{\,|x_1\}$ and $\{x_n|\,\}$ give values not known on day m.

② $\{x_i|x_j\}$ gives a value not known on day m if $i+1=j$.

③ $\{x_i|x_j\}$ gives a value already known on day m if $i+1\neq j$

①:

$\{\,|x_1\} < x_1$, where x_1 was the smallest surreal number born on day m, so $\{\,|x_1\}$ must represent a new value

Similarly, for $\{x_n|\,\}$.

②:

$x_i < \{x_i|x_{i+1}\} < x_{i+1}$. Since at day m, we knew no surreal number between x_i and x_{i+1}, then we see $\{x_i|x_{i+1}\}$ gives a new value.

③: Suppose that $i+1<j$

Claim: $\{x_i|x_j\} = x_k$ where $\underbrace{x_k.\text{birthday} = \min\{x_{i+1}.\text{birthday}, \cdots, x_{j-1}.\text{birthday}\}}_{x_k \text{ satisfies:}}$

Proof of claim:

The parents of x_k must either be less than x_{i+1} or greater than x_{j-1} Hence, $(X_k)_L \leq x_i \wedge x_j \leq (X_k)_R$

$\Rightarrow (X_k)_L \leqslant x_i < \{x_i|x_j\} < x_j \leqslant (X_k)_R$

$\Rightarrow \{x_i|x_j\} = \{x_i, (X_k)_L \,|\, x_j, (X_k)_R\}$ —— (*1)

$x_i < \{(X_k)_L \mid (X_k)_R\} < x_j \Rightarrow x_k = \{(X_k)_L \mid (X_k)_R\} = \{x_i, (X_k)_L \mid x_j, (X_k)_R\}$

$\xRightarrow{(*1)} \quad x_k = \{x_i \mid x_j\}$

Therefore, the claim holds. $\square$

Coro If x is the oldest surreal number between a and b,

then $\{a \mid b\} = x$

(Proof): Directly follow from the ~~the~~ claim above $\square$