Introduction to Surreal Numbers

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Outline

Definition of Surreal Numbers

Definition (Surreal Number)

A surreal number $s \in S$ is a pair of two sets of previously created surreal numbers, denoted by $s = \{S_L | S_R\}$, such that

$$\neg(\exists s_l \in S_L, \exists s_r \in S_R \text{ such that } s_r \leq s_l).$$

We only consider the case when $|S_L|$ and $|S_R|$ are finite in this talk.

Definition ($\leq$ Relation)

Here the ordering $\leq$ of surreal numbers is defined recursively as follows – for $x = \{X_L | X_R\}$ and $y = \{Y_L | Y_R\}$, $x \leq y$ if and only if:

1. There is no $x_L \in X_L$ such that $y \leq x_L$.
2. There is no $y_R \in Y_R$ such that $y_R \leq x$.

Well-definedness of Surreal Numbers

If S_L or S_R is empty, the condition is vacuously true, since statements like "$\forall x \in \emptyset$" are always true.

Definition (Birthday)

We define the birthday of a surreal number as the ordinal day on which it is first created:

- $0 = \{|\}$ is born on day 1.
- For any surreal number $x = \{X_L | X_R\}$,

$$\text{birthday}(x) := \max\left(\max_{x_l \in X_L} \text{birthday}(x_l), \max_{x_r \in X_R} \text{birthday}(x_r)\right) + 1$$

The birthday reflects the recursive depth of construction. Then $\leq$ is well-defined, since the integer $\text{birthday}(x) + \text{birthday}(y)$ is strictly decreasing in the recursive definition of $\leq$.

Ordering Relations

Definition (Ordering Relations)

The other ordering relations in S are defined as:

- $x < y \iff x \leq y$ and $y \nleq x$
- $x = y \iff x \leq y$ and $y \leq x$

Some Examples

- **Zero:** $0 \equiv \langle | \rangle := \{\emptyset | \emptyset\}$.
- **One:** $1 \equiv \langle 0 | \rangle$.
- **Minus One:** $-1 \equiv \langle | 0 \rangle$.

These satisfy all consistency conditions.

Ordering: $0 \leq 0$

Theorem

$0 \leq 0$

Proof.

We verify the two defining conditions for $0 \leq 0$, where $0 = \{\emptyset | \emptyset\}$.

1. There is no $x \in \emptyset$ such that $0 \leq x$. This is vacuously true because the left set is empty.
2. There is no $y \in \emptyset$ such that $y \leq 0$. This is also vacuously true because the right set is empty.

Since both conditions hold, we conclude that $0 \leq 0$. □

Corollary

The expression $\{0|0\}$ is not a surreal number.

Ordering: $0 < 1$

Theorem

$0 < 1$.

Proof.

(1) $0 \leq 1$**:** We check the two conditions for $0 \leq 1$, where $0 = \{\emptyset|\emptyset\}$ and $1 = \{0|\emptyset\}$.

1. There is no $x \in \emptyset$ such that $1 \leq x$. This is vacuously true.
2. There is no $y \in \emptyset$ such that $y \leq 0$. This is also vacuously true.

Both conditions hold, so $0 \leq 1$.

(2) $1 \not\leq 0$**:** $1 = \{0|\emptyset\}$, so its only left option is $1_L = 0$, and we already know $0 \leq 0$. Thus $1 \not\leq 0$. Hence $0 < 1$. □

Ordering: $0 < \frac{1}{2}, \frac{1}{2} < 1$

Define $\frac{1}{2} \equiv \langle 0|1 \rangle$, clearly $0 < 1$ so the form is legal.

Theorem

$0 < \frac{1}{2}$, $\frac{1}{2} < 1$.

Proof.

(1) $0 < \frac{1}{2}$ We show $0 \leq \frac{1}{2}$. Since $0 = \{\emptyset|\emptyset\}$, condition (1) is vacuously true. The right set of $\frac{1}{2}$ is $\{1\}$, and $1 \not\leq 0$, so condition (2) also holds. Thus $0 \leq \frac{1}{2}$. To show $\frac{1}{2} \not\leq 0$: The left set of $\frac{1}{2}$ is $\{0\}$, and $0 \leq 0$ is true, so condition (1) fails. Hence $\frac{1}{2} \not\leq 0$. Thus $0 < \frac{1}{2}$. $\square$

Proof (continued)

Proof.

(2) $\frac{1}{2} < 1$ We show $\frac{1}{2} \leq 1$. For $x \in \{0\}$, $1 \not\leq 0$, so condition (1) holds. The right set of 1 is empty, so condition (2) is vacuous. Thus $\frac{1}{2} \leq 1$. To show $1 \not\leq \frac{1}{2}$: The left set of 1 is $\{0\}$, and $\frac{1}{2} \not\leq 0$, so condition (1) fails. Hence $\frac{1}{2} < 1$. □

Reflexivity of $\leq$

Theorem (Reflexivity of $\leq$)

If x is a surreal number, then $x \leq x$.

Proof.

We prove the statement by induction on the birthday of x. Recall that

$$x \leq x \iff \neg(\exists x_L \in X_L : x \leq x_L) \land \neg(\exists x_R \in X_R : x_R \leq x),$$

where $x = \{X_L | X_R\}$.
Base case. For $x = 0 = \{\emptyset|\emptyset\}$, both conditions hold vacuously; hence $0 \leq 0$.
Inductive step. Assume the statement holds for all y whose birthday is strictly less than x, e.g., for every $y = x_L \in X_L$ or $y = x_R \in X_R$. $\square$

Proof (continued)

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Suppose, for contradiction, that $x \not\leq x$. By definition, this means that at least one of the following holds:

$$(\exists x_L \in X_L : x \leq x_L) \text{ or } (\exists x_R \in X_R : x_R \leq x).$$

Case 1. Assume $\exists x_L \in X_L$ such that $x \leq x_L$. Fix such an x_L. Then the definition of $x \leq x_L$ gives:

$$\neg(\exists x_L' \in X_L : x_L \leq x_L') \wedge \neg(\exists x_{LR} \in (x_L)_R : x_{LR} \leq x). \quad (*)$$

But the first part of $(*)$ contradicts the inductive hypothesis $x_L \leq x_L$.
Case 2. Assume $\exists x_R \in X_R$ such that $x_R \leq x$. Then, by the same reasoning as in Case 1, we obtain a contradiction with the inductive hypothesis $x_R \leq x_R$. Since both cases lead to contradictions, the assumption $x \not\leq x$ is impossible. Thus $x \leq x$ for all surreal numbers x. □

Transitivity of $\leq$

Theorem (Transitivity of $\leq$)

Let x, y, z be surreal numbers. If $x \leq y$ and $y \leq z$, then $x \leq z$.

Proof.

Define the proposition

$$p(x, y, z) : (x \leq y) \wedge (y \leq z) \rightarrow (x \leq z).$$

We argue by induction on the total birthday $\mathrm{bd}(x) + \mathrm{bd}(y) + \mathrm{bd}(z)$, i.e. $p(x', y', z')$ holds whenever

$$\mathrm{bd}(x') + \mathrm{bd}(y') + \mathrm{bd}(z') < \mathrm{bd}(x) + \mathrm{bd}(y) + \mathrm{bd}(z).$$

Assume, for contradiction, that $x \not\leq z$. By definition, we must have either

$$(\exists x_L \in X_L : z \leq x_L) \text{ or } (\exists z_R \in Z_R : z_R \leq x).$$

Proof (continued)

Proof (continued).

Case 1. Suppose there exists $x_L \in X_L$ such that $z \leq x_L$. Since $y \leq z$ and

$$\mathrm{bd}(y) + \mathrm{bd}(z) + \mathrm{bd}(x_L) < \mathrm{bd}(x) + \mathrm{bd}(y) + \mathrm{bd}(z)$$

one can apply $p(y, z, x_L)$ and get

$$y \leq x_L.$$

However, we also have $x \leq y$, which by definition says

$$y \not\leq x_L$$

for all $x_L \in X_L$. This gives a contradiction. $\square$

Proof (continued).

Case 2. Suppose there exists $z_R \in Z_R$ such that $z_R \leq x$. Since $x \leq y$, we can once again get $z_R \leq y$ by induction hypothesis $p(z_R, x, y)$. But we also have $y \leq z$, which by definition says

$$z_R \not\leq y$$

for all $z_R \in Z_R$. This again gives a contradiction. □

So far we didn't use the property that for $s \in \mathbb{S}$, $(s_r \in S_R) \not\leq (s_L \in S_L)$. In Lean formalization, we therefore used the more generalized structure

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inductive Game where
| mk :  List Game → List Game → Game
```

to prove these results. For extra properties of $\mathbb{S}$, we introduced `Surreal := {g:Game // IsSurreal g}` to prove them.