An Analytic Derivation of Blackjack Win Rates

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We study the win rate for the game of blackjack and derive a stochastic model for the advantage (i.e., expected win) process. Assuming proportional betting, we derive an explicit formula that estimates the win rate as a function of the number of decks used and the proportion of cards dealt before reshuffling. We derive a relationship between the "aggressiveness" of betting and risk. Risk is measured as the probability of wealth dropping below a specified level before it exceeds a higher, specified level.

THE GAME of casino blackjack has engendered much interest since E.O. Thorp's paper in 1960 and his subsequent book, *Beat the Dealer* (1966). Thorp showed that the player's expectation varies according to the undealt cards, and he indicated how to identify situations in which bets with positive expected values could be made.

Using Thorp's ideas, individuals called card counters began to appear in Las Vegas. The better and more disciplined ones had some success (see Anderson 1976). Some players formed teams, both as a way to share risk and information and, more significantly, to facilitate the raising of money from outside sources, since more players meant more hours played and an increased rate of return to investors. The phenomenon of large, well-financed teams earning substantial sums of money is discussed in Uston (1979, 1981) and ECON (1981).

The Nevada casinos reduced their losses to card counters by utilizing various countermeasures, including rule changes, early or selective shuffling, psychic pressure, and expulsions. The Atlantic City, New Jersey casinos, which began operating in 1978, were a particularly fertile ground for card counters. Uston estimates that somewhere between six and two dozen individuals and teams, capitalized at $50,000 or more, operated in Atlantic City at any one time between 1980 and 1982. The Atlantic City casinos' sole permissible response to card counters was to expel them from the blackjack tables. In order to determine the sensitivity of card counter win rate to various changes in playing conditions, the New Jersey State Casino Control Commission ran an experiment during the first 13

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days of December 1979 in which card counters could not be expelled. Also, the Commission retained a consulting firm (ECON) to, among other things, determine a set of rules and playing conditions that would not substantially hurt the general public but would deflate the rapid growth of large, well-financed teams of card counters.

In this paper, we study the effect of various controllable factors on a team's average rate of return, R. The quantity R equals the expected win on a random round, divided by a team's current bankroll. The major determinant of the economic viability of a blackjack team is the rate of return R that can be generated consistent with an acceptable level of risk. A large R, along with an acceptable level of risk, encourages the mutually reinforcing processes of more players on the team and more money for them to play with. For small R, only the well-managed, large and well-capitalized teams can survive.

Since there is no standard set of rules, we initially derive expressions for R and evaluate R for the rules used in Atlantic City prior to the rules change of 1981. Then, we recompute R for the current Atlantic City rules, which are similar to those used in most other casinos.

The vehicle of analysis is a stochastic process model of the advantage process. We will consider a method of betting called proportional betting, a special case of which is similar to the betting scheme proposed by Kelly (1956). Proportional betting is discussed in Friedman (1981) and Wong (1981) and has been commonly used by blackjack teams. We derive the implications of the level of aggressiveness of the betting on risk. Risk will be measured as the probability of the team's bankroll dropping below a specified fraction of its initial bankroll before exceeding a specified multiple of its initial bankroll.

In Section 1, we discuss the theory of card counting. We introduce the stochastic model of the advantage process in Section 2 and then use it to estimate R. Proportional betting is discussed in Section 3. First passage probabilities just alluded to are computed as a function of the aggressiveness of the betting. Section 4 empirically examines, the accuracy of the approximation to the advantage derived in Section 1. In Section 5, we recompute R using the present Atlantic City rules. Section 6 summarizes the results. The rules of blackjack can be found in the Appendix.

1. THE THEORY OF CARD COUNTING

With n cards played out of a d-deck shoe, and a new round about to begin, call the player's expected win on a \$1 bet $A_{n,d}$. This section derives a tractable approximation to $A_{n,d}$. This expectation, also called the advantage, depends upon the playing strategy and upon the set of remaining cards, or equivalently, the set of cards played. We assume that the playing strategy is "good," which means that the playing decisions are based optimally upon the player's hand, the dealer's exposed card, and the true count. For a discussion of playing strategy, see Heath (1975), Griffin (1981), and Epstein (1977).

Let the set of card names be $\{1, 2, \cdots, 10\}$, where an ace is called a one, a jack, queen, or king is called a ten, and all other cards have the same name as their rank. Let X_j be the jth card dealt. Then $A_{n,d}$ is some function of $\{X_1, X_2, \cdots, X_n\}$ and d.

To estimate $A_{n,d}$, we first compute $A_{0,d}$ by enumeration. All possible hands are dealt and after deriving the optimal playing strategy, we compute the expectation $A_{0,d}$. (Details can be found in Heath.) $A_{0,d}$ is called the advantage off the top. A type j card is then removed from the shoe and the process is then repeated. The resulting expectation is the player's advantage if a type j card is initially removed from a d-deck shoe: call it $h(j, d)$. This process is repeated for all $j \in \{1, 2, \cdots, 10\}$.

Let $\tilde{e}_{j,d} = h(j, d) - A_{0,d}$ and $e_{j,d} = [(52d - 1)/52]\tilde{e}_{j,d}$. The quantity $\tilde{e}_{j,d}$ is called the effect of removal of a type j card from a d-deck shoe and $e_{j,d}$ is a normalization of $\tilde{e}_{j,d}$. For convenience, set $e_j = e_{j,6}$, since the dependence of $e_{j,d}$ upon d is very weak. Given the initial Atlantic City rules, $e_1 = -0.00689$, $e_2 = 0.00470$, $e_3 = 0.00561$, $e_4 = 0.00750$, $e_5 = 0.00939$, $e_6 = 0.00585$, $e_7 = 0.00335$, $e_8 = -0.00066$, $e_9 = -0.00293$, $e_{10} = -0.00646$ and $A_{0,6} = 0.00067$. For example, $e_{10} < 0$ since the presence of tens in the deck is favorable to the player. They lead to more blackjacks, better hands after doubling down on 9, 10 or 11 and cause the dealer to exceed 21 more often.

The advantage off the top, $A_{0,d}$, decreases in d but only very slowly for $d \geq 4$. For details, see Griffin (1981, p. 115). Since our major interest is for those cases where $d \geq 4$, we set $c_0 = A_{0,6}$ and ignore the dependence of $A_{0,d}$ upon d. If one wishes to do calculations where d is less than 4, using $A_{0,6}$ in place of $A_{0,d}$ will introduce significant errors.

Let $RC_{n,d} = \sum_{j=1}^{n} e_{X_j}$ be the running count and let

$$TC_{n,d} = \frac{52}{52d - n} RC_{n,d} \tag{1}$$

be the true count. Define

$$\hat{A}_{n,d} = c_0 + TC_{n,d}. \tag{2}$$

Let $\hat{A}_{n,d}$ be an approximation to $A_{n,d}$. The quality of this approximation is empirically considered in Section 4 and a rationale for using it is given in Griffin (1981, Chapter 3).

Let $\sigma^2 = (1/13) \sum_{j=1}^{9} e_j^2 + (4/13)\, e_{10}^2$. This quantity is called the variance of the effect of removal and is a measure of the volatility of $A_{n,d}$. For the rules being considered, $\sigma^2 \approx 3{\cdot}60 \times 10^{-5}$. Finally, note that $\sum_{j=1}^{9} e_j + 4e_{10} \approx 0$.

Card counters behave basically as described in this section. Integer values are assigned to each card type. For example, in the plus-minus count, if $j \in \{2, 3, 4, 5, 6\}$, a $+1$ is assigned, if $j \in \{1, 10\}$, -1 is assigned, and if $j \in \{7, 8, 9\}$, 0 is assigned. The running count is kept (here the e_j's are -1, 0, or $+1$) and the true count is the running count divided by the number of remaining decks. The true count is used for playing decisions and the advantage is assumed to be c_0 plus the appropriate multiple of the true count.

2. STOCHASTIC MODEL OF ADVANTAGE PROCESS

We next discuss the method of betting and then model, $\hat{A}_{n,d}$ using Brownian Bridge and approximate R as a function of the number of decks d and of the proportion ρ of decks dealt out before reshuffling. In this section, we assume $c_0 > 0$, as in the case of the "old" Atlantic City rules. We address the case of $c_0 < 0$ in Section 5 (see the Appendix for rules).

If n cards have been dealt, with a new round about to begin, and the bankroll (present wealth) is W, we initially propose to bet as follows:

$$b = \begin{cases} \dfrac{fW\hat{A}_{n,d}}{\theta^2} & \text{if} \quad \hat{A}_{n,d} > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Here $\theta^2 \approx 1.21$ is the variance of the outcome of a \$1 wager and f is a constant to be chosen. If the player has a log utility function, setting $f = 1$ and betting according to (3) yields bets that are very close to the optimal size. A more complete discussion of (3) is postponed to the next section.

Consider a very closely related betting scheme:

$$\hat{b} = \begin{cases} \dfrac{fW\hat{A}_{n,d}}{\theta^2} & \text{if} \quad RC_{n,d} \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Since $|A_{n,d}| > c_0$ with probability near one, $b = \hat{b}$ with probability near one. When they differ, $\hat{b} = 0$, and b is very small. We find it convenient to use (4) instead of (3) in the calculation of R.

With n cards having been seen, and a new round about to begin, the expected win is $\hat{b}A_{n,d}$. If $f = 1$ and $W = 1$, the unconditional expectation of the win is

$$E\left(\frac{\hat{A}_{n,d}}{\theta^2} \cdot A_{n,d};\, RC_{n,d} \geq 0\right).$$

Define

$$R(d,\,\rho) = \frac{1}{52d\rho} \sum_{n=0}^{52d\rho-1} E\left(\frac{\hat{A}_{n,d}A_{n,d}}{\theta^2};\, RC_{n,d} \geq 0\right). \tag{5}$$

So $R(d,\,\rho)$ is the expected win (with $f = 1$, $W = 1$) on a round starting after n cards have been dealt, where n is chosen randomly on $\{0, 1, 2, \cdots, 52d\rho - 1\}$.

Now, let

$$B(d,\,\rho) = \frac{1}{52d\rho} \sum_{n=0}^{52d\rho-1} E\left(\frac{\hat{A}_{n,d}}{\theta^2};\, RC_{n,d} \geq 0\right). \tag{6}$$

The expression $B(d,\,\rho)$ is the expected bet size (with $f = 1$, $W = 1$) on a round starting after n cards have been dealt, where n is chosen randomly on $\{0, 1, 2, \cdots, 52d\rho - 1\}$.

Finally, let $A(d,\,\rho) = R(d,\,\rho)/B(d,\,\rho)$. Here, $A(d,\,\rho)$ is a measure of win rate discussed in Griffin (1984) and Ethier and Tavaré (1983). It is the ratio of the expected win on a random round divided by the expected bet on a random round and is independent of W and f.

To model $\hat{A}_{n,d}$, note that for $\sum_{j=1}^{9} e_j + 4e_{10} = 0$ and $d = \infty$, $\{RC_{n,\infty},\, n \geq 0\}$ is a random walk with drift 0 and single-step variance σ^2. For $d < \infty$, $RC_{52d,d} = 0$, so the running count is a tied down random walk.

Let $\{\hat{\beta}_t,\, 0 \leq t \leq 1\}$ be a Brownian Bridge Process. Brownian Bridge can be thought of as standard Brownian Motion conditioned to be 0 at $t = 0$ and $t = 1$. In particular,

$$P(\hat{\beta}_t \in dx) = \frac{1}{\sqrt{2t(1-t)}}\, e^{-x^2/2t(1-t)}\, dx \quad \text{for} \quad 0 < t < 1. \tag{7}$$

From Theorem 24.1 of Billingsley (1968),

$$\frac{RC_{[52dt],d}}{\sigma\sqrt{52d}} \Rightarrow \hat{\beta}_t \quad \text{for} \quad 0 \leq t \leq 1 \tag{8}$$

as $d \to \infty$ where the convergence is in $D[0,\,1]$ and $[x]$ denotes the integer part of x.

Using (8), we derive the approximation $\hat{R}$ for R as follows:

Step 1. Replace the right hand side of (5) with its equivalent:

$$\frac{1}{\theta^2\rho} \int_{t=0}^{\rho} E[A_{[52dt],d}\hat{A}_{[52dt],d};\, RC_{[52dt],d} \geq 0]\, dt. \tag{9}$$

Step 2. Replace $A_{[52dt],d}$ in (9) with $\hat{A}_{[52dt],d}$.

Step 3. Following (1) and (2), replace $\hat{A}_{[52dt],d}$ in (9) with $c_0 + 1/d(1-t) \cdot RC_{[52dt],d}$, yielding

$$\frac{1}{\theta^2\rho} \int_{t=0}^{\rho} E\left[\left(c_0 + \frac{1}{d(1-t)}\, RC_{[52dt],d}\right)^2;\, RC_{[52dt],d} \geq 0\right] dt. \tag{10}$$

Step 4. Following (8), replace $RC_{[52dt],d}$ in (10) with $\sigma'\sqrt{d}\hat{\beta}_t$, where $\sigma' = \sigma\sqrt{52}$, and replace the set $\{RC_{[52dt],d} \geq 0\}$ with $\{\hat{\beta}_t \geq 0\}$.

Now, set $\hat{R}(d, \rho)$ equal to the resulting expression. So

$$\hat{R}(d, \rho) = \frac{1}{\theta^2\rho}\int_{t=0}^{\rho} E\left[\left(c_0 + \frac{\sigma'\hat{\beta}_t}{\sqrt{d}(1-t)}\right)^2; \hat{\beta}_t \geq 0\right] dt. \quad (11)$$

Let $\hat{R}(d, \rho)$ be an approximation to $R(d, \rho)$. Following the same sequence of approximations, let $\hat{B}(d, \rho)$ be an approximation to $B(d, \rho)$ with

$$\hat{B}(d, \rho) = \frac{1}{\theta^2\rho}\int_{t=0}^{\rho} E\left[\left(c_0 + \frac{\sigma'\hat{\beta}_t}{\sqrt{d}(1-t)}\right); \hat{\beta}_t \geq 0\right] dt. \quad (12)$$

Finally, set $\hat{A}(d, \rho) = \hat{R}(d, \rho)/\hat{B}(d, \rho)$ and let $\hat{A}(d, \rho)$ be an approximation to $A(d, \rho)$.

THEOREM 1. *For* $0 < \rho < 1$,

$$\hat{R}(d, \rho) = \frac{1}{\theta^2\rho}\left[\frac{\rho c_0^2}{2} + \frac{2c_0\sigma'}{\sqrt{d}}\frac{\alpha(\rho)}{\sqrt{2\pi}} + \frac{\sigma'^2\psi(\rho)}{2d}\right] \quad (13)$$

where $\alpha(\rho) = \sin^{-1}\sqrt{\rho} - \sqrt{\rho(1-\rho)}$, *and* $\psi(\rho) = -1n(1-\rho) - \rho$.

Proof. From (11),

$$\hat{R}(d, \rho) = \frac{1}{\theta^2\rho}\int_{t=0}^{\rho} E\left[\left(c_0^2 + \frac{2c_0\sigma'\beta_t}{\sqrt{d}(1-t)} + \frac{\sigma'^2}{d(1-t)^2}\right); \beta_t \geq 0\right] dt.$$

For $0 < t < 1$,

$$E(c_0^2; \hat{\beta}_t \geq 0) = \frac{c_0^2}{2},$$

$$E(\hat{\beta}_t; \hat{\beta}_t \geq 0) = \frac{\sqrt{t(1-t)}}{\sqrt{2\pi}},$$

$$E(\hat{\beta}_t^2; \hat{\beta}_t \geq 0) = t(1-t)/2.$$

So

$$\hat{R}(d, \rho) = \frac{1}{\theta^2\rho}\int_{t=0}^{\rho}\left(\frac{1}{2}c_0^2 + \frac{2c_0\sigma'}{\sqrt{d}\sqrt{2\pi}}\sqrt{\frac{t}{1-t}} + \frac{\sigma'^2}{2d}\frac{t}{1-t}\right) dt. \quad (14)$$

Now

$$\int_{t=0}^{\rho}\sqrt{\frac{t}{1-t}}\,dt = \sin^{-1}\sqrt{\rho} - \sqrt{\rho(1-\rho)} = \alpha(\rho), \quad (15)$$

$$\int_{t=0}^{\rho} \frac{t}{1-t}\, dt = -\rho - \ln(1-\rho) = \psi(\rho). \tag{16}$$

Substituting (15) and (16) into (14) proves the theorem.

For typical values of ρ and d ($\frac{1}{2} \leq \rho \leq \frac{3}{4}$, $4 \leq d \leq 8$), the last term of (13) dominates.

COROLLARY 1.

$$R(d, \rho) \approx \frac{1}{\theta^2 \rho} \frac{\sigma'^2 \psi(\rho)}{2d} \approx \frac{\sigma'^2}{\theta^2 (2d)} \left[\frac{\rho}{2} + \frac{\rho^2}{3}\right]. \tag{17}$$

Proof. Omitted.

Equation (17) is consistent with the popular blackjack literature, which counsels players to find games with fewer decks and less frequent shuffles (see Thorp 1966, p. 127 and Uston 1981, p. 148). The significance of σ^2 is also highlighted by (17).

THEOREM 2. *For* $0 < \rho < 1$,

$$\hat{B}(d, \rho) = \frac{1}{\theta^2 \rho}\left(\frac{\rho c_0}{2} + \frac{\sigma'}{\sqrt{d}} \frac{\alpha(\rho)}{\sqrt{2\pi}}\right).$$

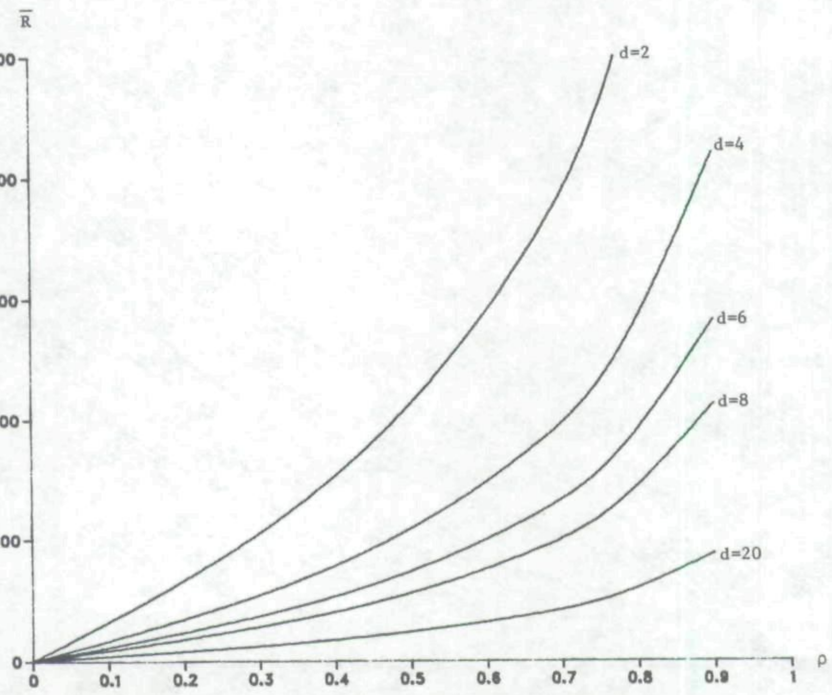

Figure 1. Hourly win rates (old Atlantic City rules).

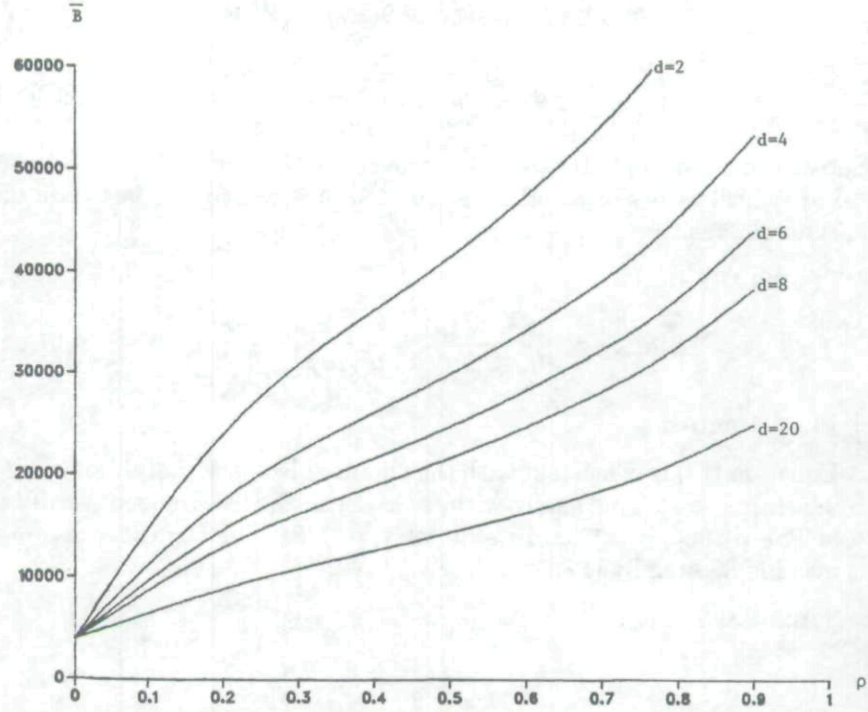

Figure 2. Hourly bet rates (old Atlantic City rules).

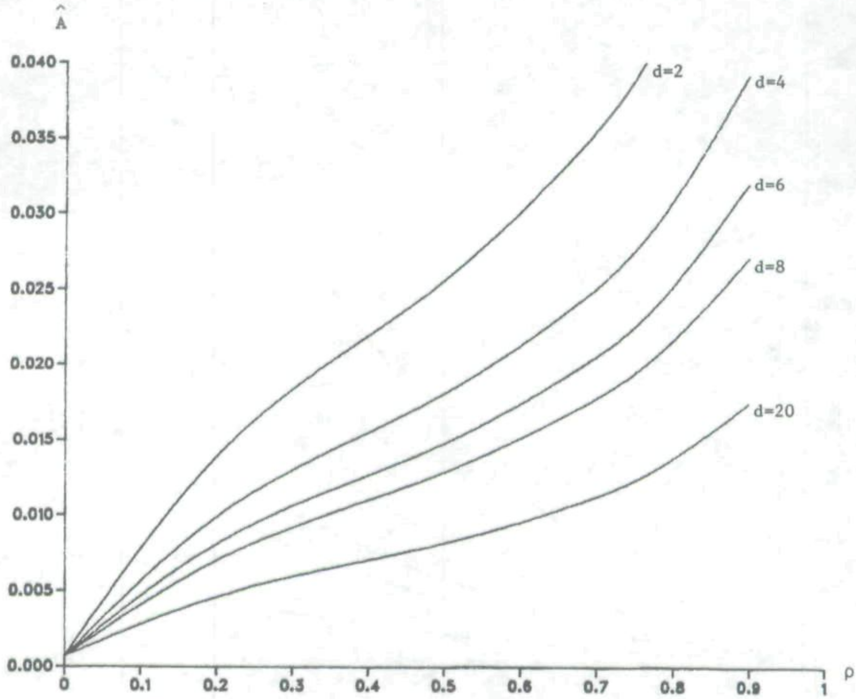

Figure 3. Average advantage (old Atlantic City rules).

Proof. Direct from the proof of Theorem 1.

Rather than graph the values of $\hat{R}$ and $\hat{B}$ for different choices of d and ρ, we define the following closely related quantities and then graph them. Let H be the number of hands played per hour. Let

$$\bar{R}(d, \rho, f, H, W) = fHW\hat{R}(d, \rho),$$

$$\bar{B}(d, \rho, f, H, W) = fHW\hat{B}(d, \rho).$$

Here, $\bar{R}$ and $\bar{B}$ are approximate hourly win and bet rates when the bankroll is W, one bets according to (4), and plays H hands/hour. In Figures 1 and 2, we set $f = 1$, $W = \$100{,}000$ and $H = 70$. The sensitivity of $\bar{R}$ to d and ρ is clear. $\bar{B}$ is less sensitive to d and ρ.

3. PROPORTIONAL BETTING AND FIRST PASSAGE PROBABILITIES

Proportional betting when $f = 1$ has some similarity to a system of betting proposed in Kelly. Kelly's idea is to choose bet sizes that maximize the expected value of the logarithm of the wealth after the wager. Betting in this way implies nice properties related to the exponential rate of growth of the wealth. See Breiman (1961), Ferguson (1965), and Finklestein and Whitley (1981) for a complete discussion.

Consider for the moment a bettor with wealth W and a log utility function. Let Y be the win on a \$1 bet (a \$$r$ wager returns \$$r(1 + Y)$ to the bettor) and let $EY = \mu$ and $\text{Var } Y = \sigma^2$. Suppose that $\mu > 0$, $\sigma^2 >> \mu$ and Y is bounded. Then the optimal wager is approximately $1 \cdot W\mu/\sigma^2$. Replacing the log utility function with some other utility function U satisfying $U' > 0$, $U'' < 0$ gives an optimal wager (subject to mild restrictions on U) of approximately $f \cdot W\mu/\sigma^2$, where $f = -U''(W)/U'(W)$. In this instance, f is not constant.

In theory, a blackjack team could use its utility function to calculate f and then bet according to (3) (or (4)), occasionally changing f as W varies. A much more natural way to proceed in order to choose f is to first require that the probability of wealth exceeding $(1 + \alpha)W_0$ before going below $(1 - \beta)W_0$ is at least p, where W_0 is the initial bankroll and $\alpha > 0$ and $0 < \beta < 1$. Subject to this requirement, a team could choose the highest possible f. In particular, a team might decide to choose the largest f so that the probability of ever going below $\frac{2}{3}W_0$ is at most 0.05. The rationale for proceeding in this way is first, that the investors are interested in such probabilities and, secondly, a team's wealth going below, say $W_0/2$, could cause its players to desert.

In the following discussion, assume that bets are set according to (3). All the arguments can easily be modified without changing the results if, instead, (4) were used.

Specify an $f > 0$ and let W_k be the wealth after k hands. Without loss of generality, set $W_0 = 1$. Choose x and y with $0 < y < 1 < x$.

Let $T_{x,y} = \min\{k > 0\colon W_k \geq x \text{ or } W_k \leq y\}$. Let $P_{x,y}(f) = P(W_{T_{x,y}} \geq x)$.

THEOREM 3. *$P_{x,y}(f) \approx (1 - y^{v^*})/(x^{v^*} - y^{v^*})$, where $v^* = (f - 2)/f$. In particular, for $x = 2$, $y = 1/2$,*

$$P_{2,1/2}(f) \approx \frac{1 - 2^{-v^*}}{2^{v^*} - 2^{-v^*}}. \tag{18}$$

The proof is deferred. $P_{2,1/2}(f)$ is the probability of doubling wealth before halving. Equation 18 appears in Friedman and is drawn in Figure 4.

We now find an appropriate diffusion approximation to $\{W_k, k \geq 0\}$. Call it $\{\omega_t, t \geq 0\}$. Here t has the following interpretation: For $0 \leq t < \rho$, t is the fraction of the first shoe that has been played. For $t = j\rho + t'$, j a positive integer and $0 \leq t' < \rho$, j shoes and the fraction t' of the $(j + 1)$th shoe have been played.

Let $\hat{A}^+_{n,d} = \max(\hat{A}_{n,d}, 0)$. Suppose that at least $k + 1$ rounds are played in the first shoe before reshuffling. Let n_k be the number of cards used in the first k rounds and let Y_{k+1} be the win on a \$1 wager on the

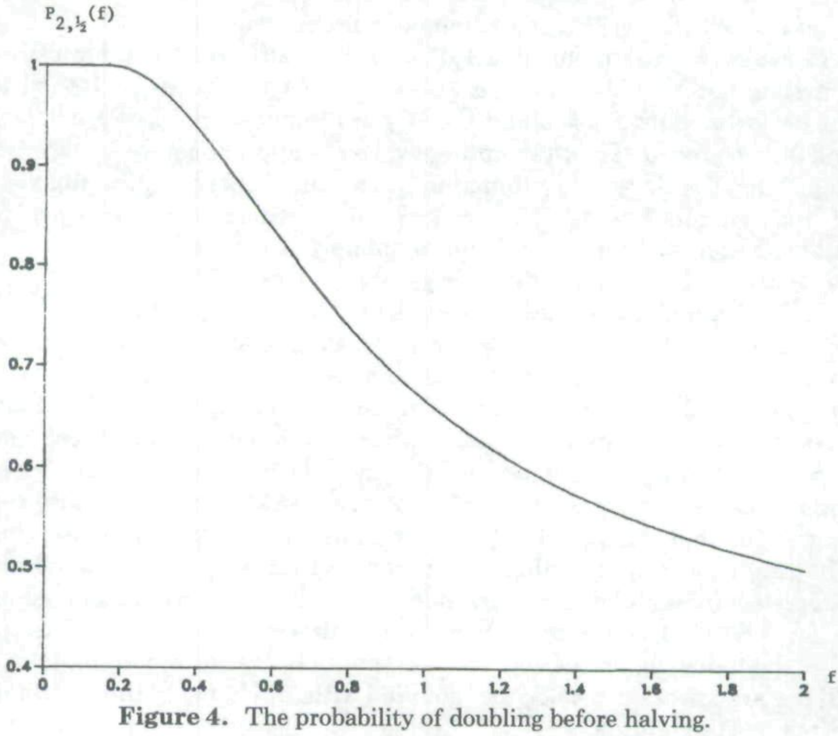

Figure 4. The probability of doubling before halving.

$(k+1)$th round. Recall that the conditional expectation of Y_{k+1} is $A_{n_k,d}$.

Now,

$$W_{k+1} = W_k\left(1 + f\,\frac{Y_{k+1}\hat{A}^+_{n_k,d}}{\theta^2}\right).$$

So, conditioning on the first n_k cards,

$$\begin{aligned} E(W_{k+1} - W_k) &= \frac{W_k f}{\theta^2}\,\hat{A}^+_{n_k,d}A_{n_k,d} \\ &\approx \frac{W_k f}{\theta^2}\,(\hat{A}^+_{n_k,d})^2. \end{aligned} \tag{19}$$

$$\begin{aligned} \mathrm{Var}(W_{k+1} - W_k) &= W_k^2 f^2\,\frac{\theta^2(\hat{A}^+_{n_k,d})^2}{\theta^4} \\ &= \frac{W_k^2 f^2}{\theta^2}\,(\hat{A}^+_{n_k,d})^2. \end{aligned} \tag{20}$$

Let $\lambda = f/\theta^2$. Then, (19) and (20), which are conditional expectations, reduce to:

$$E(W_{k+1} - W_k) \approx \lambda W_k(\hat{A}^+_{n_k,d})^2, \tag{21}$$

$$\mathrm{Var}(W_{k+1} - W_k) = \lambda^2\theta^2 W_k^2(\hat{A}^+_{n_k,d})^2. \tag{22}$$

Let

$$\mu_t = \max\left(c_0 + \frac{\sigma'\hat{\beta}_t}{\sqrt{d}(1-t)},\, 0\right). \tag{23}$$

Compare (9) and (11) for an interpretation of μ_t. In particular, for $n_k = [52dt]$, $\hat{A}^+_{n_k,d}$ and μ_t have an almost identical distribution. Following (21) and (22) and the relationship between μ_t and $\hat{A}^+_{n_k,d}$, define $\{\omega_t,\, 0 \le t \le \rho\}$ as follows:

$$\omega_0 = 1,$$

$$d\omega_t = \lambda\omega_t\mu_t^2\,dt + \lambda\theta\omega_t\mu_t\,db.$$

Here $\{b(t),\ t \ge 0\}$ is standard Brownian motion independent of $\{\mu_t,\, t \ge 0\}$.

To extend ω_t to $t \in [0, \infty]$, let $\{{}^j\omega_t,\, 0 \le t \le \rho\}_{j=1}^{\infty}$ be independent copies of $\{\omega_t,\, 0 \le t \le \rho\}$ and redefine ω_t as follows:

$$\omega_t = \begin{cases} {}^1\omega_t & \text{if } \ t \in [0, \rho) \\ {}^1\omega_\rho\,{}^2\omega_\rho \cdots {}^j\omega_\rho\,{}^{j+1}\omega_{t'} & \text{if } \ t = j\rho + t',\ t' \in [0, \rho). \end{cases}$$

Lemma 1. *For $v^* = 1 - (2/\theta^2\lambda) = (f-2)/f$, $\omega_t^{v^*}$ is a martingale.*

Proof. Choose any υ. Let $u_t = \omega_t^\upsilon$. Then,

$$du_t = \lambda\upsilon\mu_t\omega_t^\upsilon\theta\, db + \upsilon\omega_t^\upsilon\lambda\mu_t^2(1 + (½)(\upsilon - 1)\theta^2\lambda)\, dt.$$

For $\upsilon = 1 - (2/\theta^2\lambda)$, $1 + (½)(\upsilon - 1)\theta^2\lambda = 0$. Hence, $du_t = \lambda\upsilon\mu_t\omega_t^\upsilon\theta\, db$ if $\upsilon = \upsilon^*$, in which case, it is a martingale.

Proof of Theorem 3. Define $T'_{x,y} = \inf\{t > 0 : \omega_t = x \text{ or } \omega_t = y\}$. Because $P(\int_0^\rho \mu_s ds > \varepsilon) > \delta$, for some $\varepsilon > 0$, $\delta > 0$, we can easily see that $P(T'_{x,y} < \infty) = 1$.

Apply the optional stopping theorem to the martingale $\omega_t^{\upsilon^*}$. So

$$P(\omega_{T'_{x,y}} = x)x^{\upsilon^*} + (1 - P(w_{T'_{x,y}} = x))y^{\upsilon^*} = \omega_0^{\upsilon^*} = 1.$$

Solving,

$$P(\omega_{T'_{x,y}} = x) = \frac{1 - y^{\upsilon^*}}{x^{\upsilon^*} - y^{\upsilon^*}}.$$

So

$$P_{x,y}(f) = P(W_{T_{x,y}} \geq x) \approx \frac{1 - y^{\upsilon^*}}{x^{\upsilon^*} - y^{\upsilon^*}}.$$

The accuracy of the diffusion approximation depends crucially on the distribution of $f\mu_t$. If $f\mu_t$ stays small, the approximation is a good one. An analysis by simulation of $A_{n,d}$ is presented in Section 4.

Another source of error in the diffusion approximation is that $A_{n,d} \neq \hat{A}_{n,d}$. The difference between $A_{n,d}$ and $\hat{A}_{n,d}$ is also discussed in Section 4.

4. EMPIRICAL ISSUES

We now discuss how well $A_{n,d}$ is approximated by $\hat{A}_{n,d}$. One hundred six-deck shoes were randomly generated and m_i cards were dealt out of the ith shoe. The m_i were chosen independently and uniformly on $\{0, 1, 2, \cdots, 233\}$. We then computed $\hat{A}_{m_i,6}$ and $A_{m_i,6}$. Let $\hat{a}_i$ and a_i be, respectively, those values. The empirical results are:

$$\frac{1}{100} \textstyle\sum_{i=1}^{100} \hat{a}_i = 0.00214, \tag{24}$$

$$\frac{1}{100} \textstyle\sum_{i=1}^{100} a_i = 0.00330, \tag{25}$$

$$\left[\frac{1}{100} \textstyle\sum_{i=1}^{100} (a_i - \hat{a}_i)^2\right]^{1/2} = 0.0022, \tag{26}$$

$$\max_{1\leq i\leq 100} a_i = 0.0735.$$

The discrepancy between (24) and (25) is expected because, as the absolute value of the true count deviates from zero, the "accuracy" of the playing decisions increases, which induces a nonlinear effect. Finally, (26) should not be viewed as a test statistic but rather as a measure of the "closeness" of $\hat{A}_{n,d}$ and $A_{n,d}$.

5. RECOMPUTATION OF $\hat{R}$ UNDER PRESENT ATLANTIC CITY RULES

Take the set of rules to be those currently used in Atlantic City. The new values of the various parameters are:

$$c_0 = -0.0057$$

$$\begin{aligned} e_1 &= -0.0059 & e_6 &= 0.0046 \\ e_2 &= 0.0038 & e_7 &= 0.0028 \\ e_3 &= 0.0045 & e_8 &= -0.0002 \\ e_4 &= 0.0062 & e_9 &= -0.0020 \\ e_5 &= 0.0068 & e_{10} &= -0.0051 \end{aligned}$$

$$\sigma^2 = \frac{1}{13}\left(\textstyle\sum_{i=1}^{9} e_i^2 + 4e_{10}^2\right) \approx 2.35 \times 10^{-5}.$$

Since c_0 is now negative, (4) makes little sense, since the running count $RC_{n,d}$ can be positive while the estimated advantage $\hat{A}_{n,d}$ can be negative. Therefore, (11) and (12) are no longer appropriate.

Noting that $\{c_0 + [\sigma'\hat{\beta}_t/\sqrt{d}(1-t)] \geq 0\} = \{\hat{\beta}_t \geq k(1-t)\}$, where $k = -c_0\sqrt{d}/\sigma'$, we redefine $\hat{R}$ and $\hat{B}$ as follows:

$$\hat{R}(d, \rho) = \frac{1}{\theta^2\rho}\int_{t=0}^{\rho} E\left[\left(c_0 + \frac{\sigma'\hat{\beta}_t}{\sqrt{d}(1-t)}\right)^2; \hat{\beta}_t \geq k(1-t)\right] dt, \quad (27)$$

$$\hat{B}(d, \rho) = \frac{1}{\theta^2\rho}\int_{t=0}^{\rho} E\left[\left(c_0 + \frac{\sigma'\hat{\beta}_t}{\sqrt{d}(1-t)}\right); \hat{\beta}_t \geq k(1-t)\right] dt. \quad (28)$$

Note that if Steps 1–4 are repeated, with the sole difference that (3) is used instead of (4), the results would be (27) and (28) instead of (11) and (12). Let $\overline{\phi}(s) = (1/\sqrt{2\pi})\int_{u=s}^{\infty} e^{-u^2/2}\,du$.

THEOREM 4.

$$\hat{R}(d, \rho) = \frac{1}{\theta^2\rho}\left[c_0^2 I_1 + \frac{2c_0\sigma' I_2}{\sqrt{d}} + \frac{\sigma'^2}{d} I_3\right],$$

$$\hat{B}(d, \rho) = \frac{1}{\sigma^2\rho}\left[c_0 I_1 + \frac{\sigma' I_2}{\sqrt{d}}\right],$$

where

$$I_1 = \int_{t=0}^{\rho} \bar{\phi}\left(k \sqrt{\frac{(1-t)}{t}}\right) dt,$$

$$I_2 = \frac{1}{\sqrt{2\pi}} \int_{t=0}^{\rho} \sqrt{\frac{t}{(1-t)}}\, e^{-k^2[(1-t)/2t]}\, dt,$$

$$I_3 = \frac{1}{\sqrt{2\pi}} \int_{t=0}^{\rho} k \sqrt{\frac{t}{(1-t)}}\, e^{-k^2[(1-t)/2t]}\, dt + \int_{t=0}^{\rho} \frac{t}{(1-t)}\, \bar{\phi}\left(k \sqrt{\frac{(1-t)}{t}}\right) dt.$$

Proof. Similar to the proof of Theorem 1.

As in Section 2, set

$$\bar{R}(d, \rho, f, H, W) = fHW\hat{R}(d, \rho),$$
$$\bar{B}(d, \rho, f, H, W) = fHW\hat{B}(d, \rho),$$
$$\hat{A}(d, \rho) = \hat{R}(d, \rho)/\hat{B}(d, \rho).$$

Figures 5 through 7 are analogous to Figures 1 through 3.

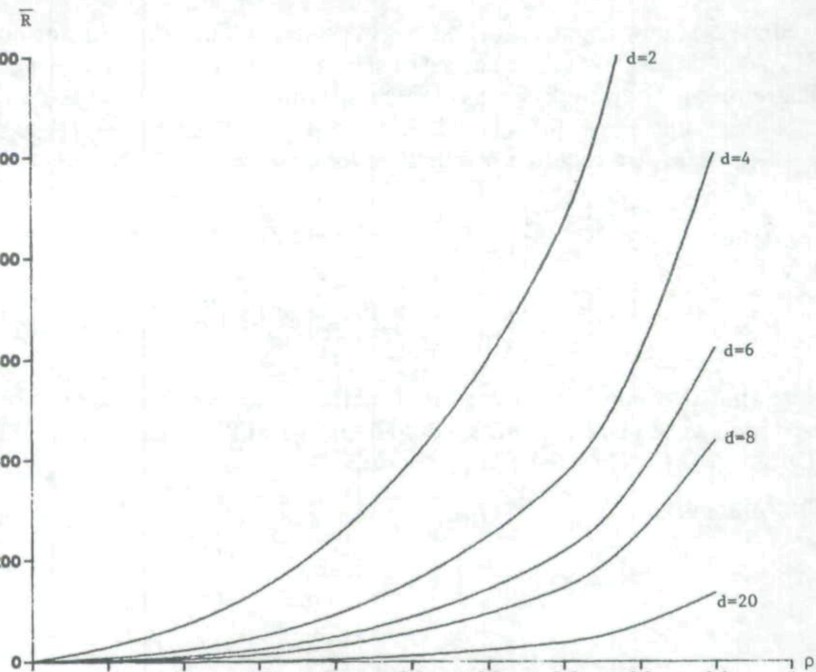

Figure 5. Hourly win rates (new Atlantic City rules).

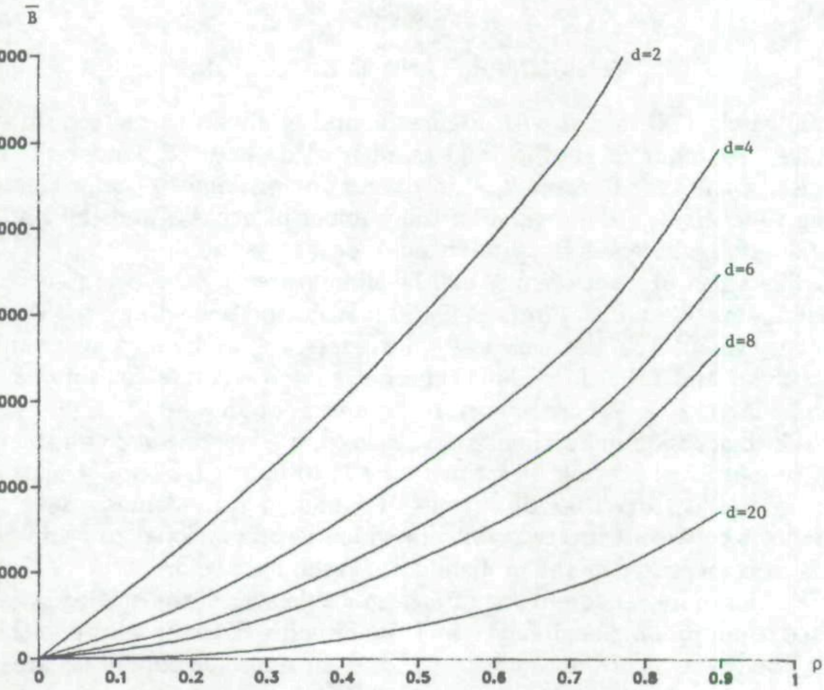

Figure 6. Hourly bet rates (new Atlantic City rules).

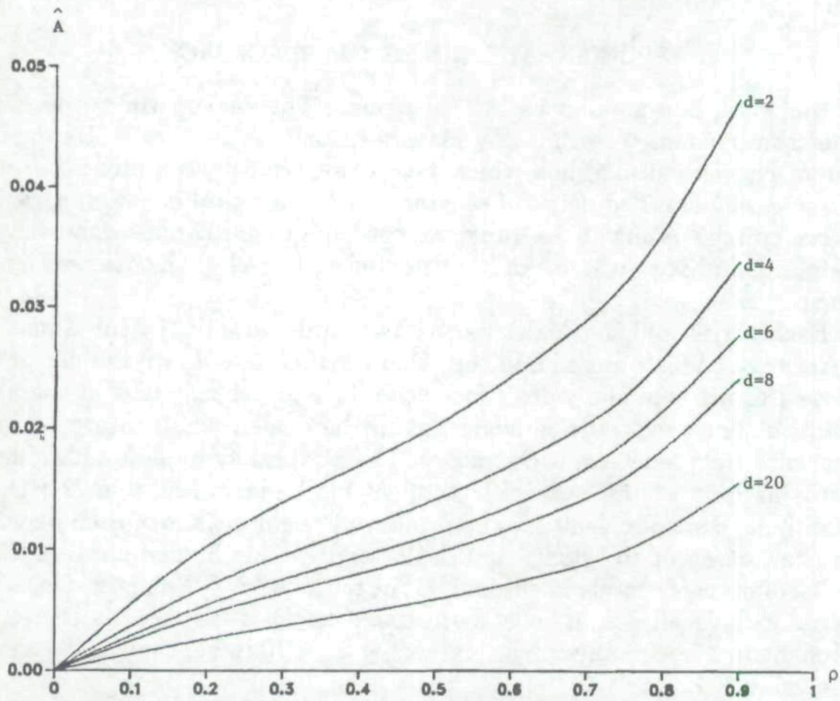

Figure 7. Average advantage (new Atlantic City rules).

6. SUMMARY AND CONCLUSIONS

Formula (17), along with Figures 1 and 5, shows the effects of the rules, frequency of shuffles and number of decks used. Under the old rules, where $c_0 > 0$ is small, $\hat{R}$ is almost proportional to the product of the violatility σ^2, the inverse of the number of decks d and the sum of $\rho/2 + \rho^2/3$ where ρ is the proportion of decks used.

The effect of a set of rules can be summarized by the two associated parameters c_0 and σ^2. For $c_0 < 0$ and d large and ρ small, we see that $\hat{R}$ is very small. As d decrease and ρ increases, we see by a comparison of Figures 1 and 5 that formula (17) becomes more accurate. So, for d small and ρ large, σ^2 is a more important parameter than c_0.

The discussion in Section 3 provides a compelling rationale for the use of proportional betting. If a constant f is to be used, Figure 4 or some variant of it (for different choices of x and y) fully demonstrates the tradeoff between expected win rate which is proportional to f and risk which is measured as the probability of reaching x before y.

Furthermore, Theorem 3 is often applicable even if the betting system used is not proportional. Given an initial wealth W at the start of a shoe, let f be the ratio of the product of θ^2 and the expected sum of squares of wagers over the shoe to the product of W and the expected win over the shoe. If f is nearly constant as a function of W, (18) still holds.

APPENDIX: THE RULES OF BLACKJACK

Blackjack is a popular casino card game. The dealer, who works for the casino, competes against N players (usually N is 1 to 7). He deals out of a device called a shoe, which, before use, typically contains between 4 and 8 well-shuffled decks of playing cards. Each card has a numerical value corresponding to its rank, except for picture cards, which have value 10, and aces, which can be evaluated as 1 or 11 at the discretion of the player.

Each player and the dealer receive two cards initially. Of the dealer's two cards, one is exposed and the other remains face down; the player's two cards are commonly dealt face down (the player may look at them), although their exposure is irrelevant to the dealer, who conforms to a mechanistic, predetermined strategy. The player may request additional cards as long as the numerical total of his hand is less than 21; the additional cards are dealt face up. Following the decisions of each player to draw cards or to "stick," the dealer exposes his hidden card. If the value of his cards totals 17 through 21, he must "stick." When the dealer's cards include an ace, it must be counted as 11 if his total is thereby brought to 17 or more without exceeding 21; otherwise, the ace is given the value of 1.

If the player "busts" (exceeds 21), he automatically loses his bet. If he does not exceed 21, but the dealer subsequently busts, the player automatically wins. If neither player nor dealer "busts," the hand closer to 21 is the winning one. In the case where the dealer and the player conclude with hands of the same total, the play is a tie (or "standoff" or "push") and the bet is canceled. A win for the player receives a payoff at even odds.

An ace and a face card or 10 comprising the first two cards dealt to either player or dealer constitute a two-card total of 21, called a "natural" or "blackjack." Such a combination wins over all others. If the player receives a "blackjack" and the dealer does not, the payoff to the player is 1.5 times his wager. The dealer wins only the amount wagered when he has a "Blackjack" and the player does not. "Blackjack" occurring in both the player and dealer hands constitutes a tie.

All used cards are placed in a discard rack. When a prespecified number of cards have been used, the next round is preceded by a reshuffling of all the cards.

A set of additional options are often, but not always, available to the player. Generally, a player, having received two cards, may double his bet and then will receive exactly one card. This choice is called doubling down.

In the initial Atlantic City rules, a player could, upon seeing his first two cards and the dealer's exposed card, surrender half his bet to the dealer and remove the remainder of his bet. This "early surrender" option is extremely valuable and its nonavailability since 1981 is the only difference between the initial and present Atlantic City rules. The Atlantic City rules are now typical of rules elsewhere.

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