*A FAVORABLE STRATEGY FOR TWENTY-ONE**

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1. *Introduction.*—It has long been an open question as to whether those of the standard gambling games which are not repeated independent trials admit strategies favorable† to the player. There have been numerous implications[2–4] that favorable strategies do not exist. In this note, we settle the issue by showing that there is a markedly favorable mathematical‡ strategy for one of the most widely played games, twenty-one, or blackjack.

2. *Previous Work.*—Our point of departure is the work of Baldwin, Cantey, Maisel, and McDermott,[5, 6] the only serious treatment of blackjack that has been given to date. The reader will find further references and a representative set of rules in their paper. Although there are minor variations in the game, we shall adopt those rules (including insurance[6]).

3. *Method and Results.*—Our calculations are similar to those outlined in Baldwin *et al.*,[5] but there are some very important changes. First, a high-speed computer was programmed to find the player's best possible strategy and the corresponding expectation. The electronic calculator enabled us to dispense with many of the approximations that were needed by Baldwin *et al.* to reduce the calculations to desk computer size. This led to noticeable improvements in results. In particular, the player's expectation for a complete deck was found to be a startling -0.21%. (Baldwin *et al.* give -0.62%). Our second change in approach was to program the computer to do the calculations for arbitrary sets of cards. This made it possible to take into account cards that become visible during play, a feature which is essential for the determination of any winning strategy.§

A standard deck of cards has approximately 3.4×10^7 subsets which are distinguishable under the rules of blackjack. It is thus impractical to compute the optimal strategy for each of these subsets. Instead, we have studied a number of carefully preselected subsets, and from the information gained, several favorable strategies are obtained. Some of our subsets and results are given in Table 1 below.

Let Q(I) be the number of cards of value I. The special subsets in Table 1 differ from a full deck only in that the number of cards of a single value has been altered.

In actual play, these special subsets occur infrequently, and some are even impossible. Even so, they yield a profusion of winning strategies. For example, one

TABLE 1

PLAYER'S EXPECTATION WITH SELECTED SUBSETS

Description of the subset	Player's expectation	Description of the subset	Player's expectation
complete deck	−.0021	Q(7) = 0	.0125
Q(1) = 0	−.0272	Q(8) = 0	.0005
Q(2) = 0	.0142	Q(9) = 0	−.0091
Q(3) = 0	.0189	Q(10) = 12	−.0215
Q(4) = 0	.0236	Q(10) = 20**	.0189
Q(5) = 0	.0329	Q(10) = 24**	.0394
Q(6) = 0	.0187		

of these winning strategies may be obtained by considering the subset Q(5) = 0. Suppose that just before a particular deal the player sees that all fives have been used (so that the unseen cards are a subset of that subset which we describe by Q(5) = 0) and that the unused portion of the deck is ample for that deal. If the player does not take into account the cards other than fives that he has seen on previous deals, then as far as he is concerned, his probabilities for success are the same as for the subset Q(5) = 0. Using Table 1, it follows that the player who adopts the strategy for Q(5) = 0 (see Table 2) when there are no fives remaining has an expectation of 0.0329 at those times.

A winning strategy may now be defined as follows. If Q(5) $\neq$ 0, the player bets the minimum allowed amount, m, merely to remain in the game and follows the complete deck strategy given by Baldwin *et al.* When Q(5) = 0 (and the remainder of the deck will suffice for the next deal), the situation has turned in favor of the player. He now bets a large amount, M, and uses the computed strategy for Q(5) = 0, which is given in Table 2.††

TABLE 2

THE STRATEGY WHEN Q(5) = 0

Pair Splitting
Dealer shows:

Pair	2	3	4	6	7	8	9	10	A
A	X	X	X	X	X	X	X	X	X
10				X					
9	X	X	X	X	X	X	X		
8	X	X	X	X	X	X	X	X	X
7	X	X	X	X	X	X	X		
6	X	X	X	X					
4			X						
3	X	X	X	X	X	X			
2	X	X	X	X	X	X			

Doubling Down
Dealer shows:

Total	2	3	4	6	7	8	9	10	A
20				S					
19		S	S	S					
18	S	S	S	S					
17	S	S	S	S	S				
15			S	S					
14		S	S	S					
13		S	S	S					
11	H	H	H	H	H	H	H	H	H
10	H	H	H	H	H	H	H	H	H
9	H	H	H	H	H				
8			H	H					

Minimum Standing Numbers
Dealer shows:

Total	2	3	4	6	7	8	9	10	A
19							S	S	
18	S	S	S	S	S	S			S
17									H
16				H	H				
15							H	H	
12	H	H	H	H					

Legend:
X: split the pair
S: soft total only
H: hard total only

A disadvantage of this strategy is that the event Q(5) = 0 occurs only in about 3.5 per cent to 10 per cent of the deals (depending on the number of players). A similar remark applies to the other Q(I) = 0 type strategies. However, careful scrutiny has disclosed another strategy which partially overcomes this disadvantage and gives the player a greater expectation as well. This strategy depends on the somewhat surprising fact that all the crucial quantities are almost linearly dependent on the proportion of tens in the deck and nearly independent of the absolute number of tens. The details are too extensive to be given here and will appear elsewhere. The main characteristics are that the player has an advantage almost half the time; his expectation exceeds 0.04 about a tenth of the time, and occasionally (probability 1/5,000–1/10,000) exceeds 0.86.

Since this strategy offers a "spectrum" of favorable expectations, it seems reasonable to let the size of the bets increase with the expectation. The advantages of this procedure have not yet been studied in detail. Nor have we considered in detail the question of letting the player's bet vary with the size of his fortune.

4. *Remarks.*—With only minor modifications, our program can be used by a high-speed computer to play blackjack directly. The computer would play a near-perfect game. If the bets were of constant size, the player would have a decided advantage. If the bet size were varied, the player's advantage would be overwhelming.

The rules variations in Nevada casinos have been tabulated and analyzed. The expectations never vary more than 0.005 from our figures. Consequently our strategies are advantageous regardless of these variations in rules.

The "home" game of blackjack differs from the casino game principally in that the dealer's strategy is fixed in the latter and arbitrary in the former. Using our methods, it is an easy matter to find the optimal player strategy against each possible fixed dealer strategy. The methods of game theory then apply to the case of mixed dealer strategies and yield a complete solution for the home game.

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† A strategy is *favorable* if, for some unifoım bound on the player's bets, his fortune converges with probability 1 to plus infinity.

‡ By saying "mathematical strategy," we mean to exclude such time-honored approaches to winning strategies as physical strategies (defective roulette wheel, defective dice) or the large class of strategems (sleight of hand with the cards, collusion with the dealers, etc.).

§ Further detailed results, together with the lengthy computer routine, the methods used to insure that it is correct, and the discussion of the errors introduced by certain simplifying assumptions, will appear elsewhere.

** Insurance contributes 0.0032 to this value when $Q(10) = 20$ and 0.0073 when $Q(10) = 24$.

†† At the casinos, M/m generally is from 100 to 500. With these values, the overall "expectation" E (i.e., the expected value of the amount won, in units of M, divided by the number of times that M was bet) is greater than 0.03. In fact, $M/m \geqq 15$ insures $E \geqq 0.025$ and if $M/m \geqq 3$, $E > 0$.

[1] Feller, W., *An Introduction to Probability Theory and Its Applications* (New York: John Wiley and Sons, Inc., 1957).

[2] Huff, Darrell, "*The Mathematics of Sex, Gambling, and Insurance,*" *Harper's,* Sept. 1959.

[3] Fox, Philip G. (as told to Stanley Fox), "A Primer for Chumps," *Sat. Eve. Post,* Nov. 21, 1959, pp. 31ff.

[4] Von Mises, Richard, *Probability, Statistics and Truth,* (London: Allen and Unwin, 1957).

[5] Baldwin, R., W. Cantey, H. Maisel, and J. McDermott, "The Optimum Strategy in Blackjack," *J. Am. Stat. Assn.,* **51,** 429–439 (1956).

[6] Baldwin, R., W. Cantey, H. Maisel, and J. McDermott, *Playing Blackjack to Win: A New Strategy for the Game of 21* (New York: M. Barrows, 1957).