Blackjack in Holland Casino's: basic, optimal and winning strategies.

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Abstract

This paper considers the cardgame Blackjack according to the rules of Holland Casino's in the Netherlands. Expected gains of strategies are derived with simulation and also with analytic tools. New efficiency concepts based on the gains of the basic and the optimal strategy are introduced. A general method for approximating expected gains for strategies based on card counting systems is developed. In particular it is shown how Thorp's Ten Count system and the High Low system should be used in order to get positive expected gains. This implies that in Holland Casino's it is possible to beat the dealer in practice.

Keywords: blackjack, Holland Casino's, cardgames, basic strategy, optimality, card counting, Ten Count system, High-Low system

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1 Introduction

The cardgame of Blackjack (also known as Twenty One) is still today one of the most popular casino games. It has engendered much interest since by clever play it is possible for players to get an advantage over the house. This discovery was revealed in the sixties with the publication of the paper Thorp (1960) and the subsequent famous book Thorp (1966) entitled "Beat the Dealer". Thorp showed that the player's expectation varies according to the undealt cards, and he indicated how to identify situations with a positive expectation. By raising the bet in such games an overall positive expected result can be obtained. Such winning strategies will beat the dealer in the long run.

However, casino's took their counter measures and changed the rules in order to get the advantage back. Today, these rules vary strongly between and even within casino's. For most of the variations it is still possible to obtain a serious advantage for professional hard working card counters. Although this mere fact seems to disturb casino boards terribly, the game is still attractive to exploit because most players are really amateurs and lose a lot of money. Another reason is that a winning strategy for one version of the game is a losing one for another variation.

There is a tremendous literature available on BJ (Blackjack). A lot of books are filled with strategy tables to use. Some of them are unreliable because they are based on rough approximating probability calculations; even the class of game variations for which they are supposed to be appropriate is not clearly indicated. The serious ultimate guide for references is Dalton (1993). We mention here the easily available and reliable mathematical books and papers Baldwin et. al. (1956), Epstein (1977), Gottlieb (1985), Griffin (1988), Chambliss and Roginski (1990), Yakowitz and Kollier (1992), and the appendix of Black (1993). All these publications deal exclusively with the American way of playing: with a dealer's hole card. In Europe in most casino's the game is played without a hole card.

The goal of this paper is to give a profound analysis of Blackjack as it is played in Holland Casino's in the Netherlands (Amsterdam, Breda, Eindhoven, Groningen, Nijmegen, Rotterdam, Scheveningen, Valkenburg, Zandvoort and Schiphol Airport). These BJHC-rules are exactly the same in all cities. They are typical Dutch in so far that the precise combination of the variations does not appear elsewhere. (We will describe the rules exactly in section 2.) Shortly, the differences with the Las Vegas Strip standard (see Dalton (1993), p. 65) are: all cards dealt face up and no hole card (the European way), a six-deck shoe with a cut card between 1/2 and 2/3, restrictions for doubling on card combinations, doubling after splitting, unrestricted repeated splitting (for splitted aces one card only), and a three-sevens bonus. As in Las Vegas there is no (early or late) surrender. Experiments are going on in Amsterdam and Zandvoort with card shuffling machines, leading to games that are more or less played with complete shoes only.

A keystone for professional playing is the so called *basic strategy*. This strategy for BJHC was published first in Van der Genugten (1993). Thereafter this strategy was revealed (and derived independently) by two Dutch professional Blackjack players Wind and Wind (1994).

In this paper we will analyse strategies for the BJHC-game and the concepts on which they are based. Utmost care is taken to give a clear definition of them since in literature this is often a source of confusion. Results are obtained with a special purpose computer package. It consists of 5 interrelated computer programs written in Turbo-Pascal:

- BJ1SIM: a highly flexible simulation program for obtaining expected gains for (combinations of) arbitrary strategies,
- BJ1ISTRT, BJ1FSTRT: two analytic programs which calculate expected gains for a given stock and a given strategy; I (nfinite) indicates drawing with replacement and F(inite) stands for drawing without replacement,
- BJ1IGAME, BJ1FGAME: two analytic programs which calculate optimal expected gains and corresponding optimal decision tables; here I and F have the same meaning as before.

Much of the material in this paper is, with minor changes, applicable to rules in other European casino's. For rules outside Europe differences are somewhat bigger due to the presence of the hole-card.

The paper is organized as follows. In section 2 we give a description of the rules of BJHC.

In section 3 we discuss the two components of strategies: the betfunction and the playing strategy. We formulate precisely the concept of optimality. This leads to a sound definition of the basic strategy. To fix the ideas we have also included its decision table in this section.

In section 4 we consider the expected gains of strategies. By means of BJ1SIM we can give these gains for some naive strategies and the basic strategy. Also some rough estimates are given for the optimal strategy. We conclude this section by introducing efficiency concepts for arbitrary strategies.

The steady-state analysis in section 5 makes clear which tools are needed for computer calculations for expected gains. These tools in the form of computer programs are described in the following two sections.

Section 6 describes the programs BJ1IGAME (for the construction of the optimal strategy) and BJ1ISTRT (for given arbitrary strategies) under the assumptions that cards are drawn with replacement. In particular it contains the construction of the basic strategy.

Section 7 describes the corresponding programs BJ1FGAME and BJ1FSTRT for the practical situation that cards are drawn without replacement. As far as we know the coding system needed for doing actual calculations has never appeared in literature before.

In section 8 a method is given for estimating the expected gains for arbitrary strategies, in particular for the basic strategy and the optimal strategy. These estimation method only gives crude estimates due to the fact that the sample size is rather small but yet too large to prevent replacement of approximations with BJ1IGAME by the (more ore less) exact values with BJ1FGAME.

In section 9 we follow another approach by means of (linear and non-linear) approximations of expected gains by card fractions. Here we describe the general setup and its relations to card counting systems for betting. Analytic results can be obtained by approximating the distribution of the running count by that of the Brownian bridge.

In section 10 we consider card counting systems more in detail. We restrict ourselves to a discussion of TTC (Thorp's Ten Count) and HiLo (High-Low).

Finally, in section 11 we describe how the card counting systems of section 10 can be used for playing decisions. Since optimal betting often involves maximal bets, high budgets are needed. Therefore we consider also some other betting concepts more suitable for low budget players. For readers only interested in practical strategies which beat the dealer this is the most interesting section.

2 BJHC-rules

In this section we will give a description of the BJHC-rules together with some notation to be used in the following. Game constants for which we will consider alternatives are presented as variables together with their standard values.

BJHC is a card game that is played with 2-7 players; mostly the number of players is $a = 7$. The dealer, who is a member of the house, deals the cards out of a device called a *shoe*. A *complete* shoe consists of $n = 6$ decks of playing cards of size 52 (therefore in total $k = 52n = 312$ cards).

Cards are always dealt face up. So, at least in theory, every player can know the composition of the shoe at any stage of the game by observing the dealt cards.

Face cards have the value 10 (T); non-face cards have their indicated value. An A (ace) is counted as 1 or 11 depending on the other cards in the hand. If the sum of a hand with at least one ace counted as 11 would exceed 21, then all aces are counted as 1, otherwise one ace is counted as 11. A hand or sum is called *soft* if it contains an ace counted as 11; otherwise it is called *hard*. The main goal of players is to get hands with a sum as close as possible to but never exceeding 21 by *drawing* (asking the dealer for cards one after another) or *standing* (requesting no more cards) at the right moment. He *busts* (loses) if his (hard) sum exceeds 21. After all players the dealer draws cards too. He has no choice at all: he draws on sums ≤ 16, stands on sums ≥ 17 and ≤ 21 (hard or soft) and busts (loses) on a (hard) sum > 21. If a player and the dealer both stand, then the game is lost for the one holding the smallest sum. The combination (A, T) is called *"Blackjack"* and beats any other sum of 21. Equals sums give a *draw*.

We code cards by their value and the ace by 1. In general the card distribution in the shoe at a certain stage of the game is random and will be denoted by $C = (C(1), \ldots, C(10))$. Realizations will be denoted correspondingly with $c = (c(1), \ldots, c(10))$.

The playing stock C_1 for the first *game* is the (non-random) complete shoe $c_0 = (kp_1, kp_2, \ldots, kp_{10}) = (4n, 4n, \ldots, 4n, 9n) = (24, 24, \ldots, 24, 96)$, where $p_1 = \cdots = p_9 = 1/13, p_{10} = 4/13$ are the cards fractions in one deck. The remaining cards in the shoe after the first game become the (random) playing stock C_2 of the second game and so on. Used cards are placed into a discard rack. If during (or at the end of) a game the size $\sum C(i)$ of the current stock C in the shoe decreases to a level equal to or less than $k(1-\lambda)$, then after this game the cards are reshuffled and the next game starts again with a complete shoe. In practice the fraction is marked by positioning a cut card in the shoe at about a played fraction $\lambda = 2/3$ corresponding to a level of 104 remaining cards. However, in BJHC dealers are allowed to lower the cut card position to $\lambda = 1/2$. This appears to be a disadvantage for the players and is only done when professional card counters join the game. We call a *rowgame* a whole sequence of games, from a complete shoe up to the game in which cut card falls or is reached.

At this moment the HC's in Amsterdam and Zandvoort are experimenting with card shuffling machines. After each game cards are automatically reshuffled. In this case a rowgame consists of exactly one game. If this reshuffling would be completely random, this would correspond to BJ with a fraction $\lambda = 0$. (In practice there is a slight correlation between successive drawings.)

Outside the Netherlands there are still casino's which offer Blackjack without a cut card. This corresponds to a fraction $\lambda = 1$. For that case, and also for other high values of λ, the shoe will get empty during a game. Then the cards in the discard rack are reshuffled and placed into the shoe for playing the remaining part of the game. In this paper we assume that then the next game is started with a reshuffled complete shoe. In BJHC the discard rack is never used for this purpose because the cut card position λ is too small. However, for a general description and analysis it is worthful to consider the whole range $\lambda \in [0, 1]$.

We describe in detail one game together with the decision points of the players.

The game starts with the *betting* of the players. The minimum and maximum bet can vary with the table. Today in BJHC the possible combinations (in Dutch guilders) are (10, 500), (20, 1000), (40, 1500) and in the "cercle privé" (100, 2500) (the combination (5, 500) in Scheveningen no longer exists). Fixing the minimum bet at the unit amount $B_{\text{min}} = 1$, the possible values of the maximum bets are $B_{\text{max}} = 50$, 37.5 and 25. Bets must be in the range $[1, B_{\text{max}}]$.

After the player's betting round the dealer gives one card to each of the players and to himself (the *dealercard*). Then a second card is dealt to each of the players to make it a pair (not yet the dealer). So at this stage all hands of players contain two cards.

If the dealercard is an A, every player may ask for *insurance* (IS) against a possible dealer's "Blackjack" later on. This is a side bet with an amount $\frac{1}{2}\times$ his original bet.

A player with the card combination "Blackjack" has to stand.

Next, players without "Blackjack", continue playing their hand, one after another, from player 1 to a.

If both cards of a hand have the same value, a player may *split* (SP) those cards and continue separately with two hands containing one card. To the additional hand a new bet equal to the original bet must be added. The first step in playing a splitted hand is that the dealer adds one new card to make it a pair. Repeated splitting is allowed without any restriction. However, with a no further splitted hand of two aces standing is obligatory. Splitted pairs cannot count as "Blackjack".

If a pair (splitted or not) has a hard sum 9, 10 or 11 or a soft sum 19, 20 or 21 (not Blackjack), *doubling down* (DD) is permitted. Then the player doubles his original bet, draws exactly one card and has to stand thereafter. A soft sum becomes hard because every ace in this hand gets automatically the value 1.

Finally, if a hand is not doubled, the player can *draw* or *stand* (D/S) as long as he did not stand or bust. Standing on a (hard or soft) 21 is obligatory. A non-splitted hand of three sevens gets a bonus of $1\times$ the original bet.

After all players have played their hands the dealer draws cards for himself according to the fixed rule already indicated.

A winning player gains an amount $1\times$ his original bet and even $1\frac{1}{2}\times$ if he wins with "Blackjack". A losing player loses his bet. In case of a draw a player gains nor loses: his bet is returned.

If at least one player has taken insurance against a dealer's ace then, even in the case that no player stands, the dealer must draw at least one card to see if he gets "Blackjack". If he has "Blackjack" then the player gains $2\times$ his insurance, otherwise he loses this insurance. In practice, if a player insures his own "Blackjack", he always gains $1\times$ his bet. Therefore, the dealer gives him immediately this gain and removes the player's cards from the table. This particular form of insurance is called *evenmoney*. (Of course, for evenmoney alone the dealer would not draw a card.)

In the following we consider the number of decks n, the cut card position λ, the number of players a and the maximum bet $B_{\max}$ as parameters. For the standard values $n = 6, \lambda = 2/3, a = 7$ the time needed for one game is about 1 minute. Reshuffling takes 2 minutes. Since one rowgame contains approximately 10 games, this gives 12 minutes per rowgame or 5 rowgames per hour. So a professional player can play 10000 rowgames (or 100000 games) yearly if he works hard for 2000 hours per year. This should be kept in mind in judging expected gains per (row)game of strategies. For theoretical purposes concerning approximations we will also consider games in which every card is drawn with replacement. We refer to these games by the parameter values $n = \infty$ and $\lambda = 0$. This implies that rowgames coincide with games.

3 Strategies and optimality

Consider a game with fixed parameters n, a, λ and $B_{\max}$. A strategy (H_ν, S_ν) for a player ν consists of two parts: a *betting* strategy H_ν which prescribes the betsize at the start of each new game and a *playing* strategy S_ν which prescribes the playing decisions IS, SP, DD, D/S at any stage of the game.

We restrict the class of all possible strategies of a player ν in the following way. His betsize at the start of a game shall only depend on the stock at that moment; therefore it can be characterized by a betfunction $H_\nu(c) \in [1, B_{\max}], c \in \mathcal{C}$, with $\mathcal{C} = \{c_0\} \cup \{c :$

$\sum c(i) > k(1-\lambda)\}$ the class of possible stocks which can be encountered with betting.

The playing decisions of the player ν at a certain stage of the game shall only depend on the current or past stocks in that game and the exposed cards on the table at that stage. So a playing strategy S_ν is a function which specifies the playing decisions for every possible sequence of stocks and table cards during a game. More precisely, let $d_0(c)$ denotes the sequence of the $2a+1$ cards dealt by the dealer ($\nu = 0$) before the playing round starts, $d_\nu(c)$ (for $\nu = 0, \ldots, a$) the whole sequence of cards used by the players $0, \ldots, \nu$ and, more specific, $d_{\nu j} = (d_{\nu-1}(c), x_{1\nu}, \ldots, x_{j\nu})$ the sequence up to the stage in which player ν already got additionally j cards $x_{1\nu}, \ldots, x_{j\nu}$. Then $S_\nu(d_{\nu j})$ prescribes the relevant playing decision at any stage $d_{\nu j}$. This constitutes a class $\mathcal{S}_\nu$ of playing strategies. The stocks during successive games only depend on the playing strategies $S_\nu \in \mathcal{S}_\nu$ for $\nu = 1, \ldots, a$ of the players and not on their betfunctions. The restriction to such playing strategies gives no loss of generality at all.

Denote by $G_1(c)$ the (random) gain of a player ν for a game with starting stock $c \in \mathcal{C}$ and minimum bet $B_{\min} = 1$. Then the (random) gain $G(c)$ of this player using the betfunction $H(c)$ is given by $G(c) = H(c)G_1(c), c \in C$.

For given playing strategies $S_1, \ldots, S_a$ the probability distribution $\mathcal{L}(G(c))$ is fixed. Given these strategies we call the betfunction H_ν of player ν *optimal* if it maximizes $E(G(c))$ for every $c \in \mathcal{C}$. Clearly, H_ν is optimal for

$$H_\nu(c) = \begin{cases} 1 & \text{if} \quad E(G_1(c)) \leq 0 \\ B_{\max} & \text{if} \quad E(G_1(c)) > 0. \end{cases}$$

For fixed $S_j, j \neq \nu$, the distribution $\mathcal{L}(G_1(c))$ only depends on the choice $S_\nu \in \mathcal{S}_\nu$. Given the S_j with $j \neq \nu$ we call the playing strategy S_ν optimal if $S_\nu(d_{\nu j})$ maximizes $E(G_1(c)|d_{\nu j})$ for every stage $d_{\nu j}$ of the game that can be reached by player ν and for every stock $c \in \mathcal{C}$.

Optimality for player ν depends on the playing strategies S_j of other players as well. In analyzing strategies for player ν we must make a specific choice for the playing strategies of the other players. A reasonable and pragmatic approach is to consider possible improvements of player ν amid other players of moderate skill playing independently of each other and following a simple so called *basic* strategy. Although in practice moderate players do not quite reach the level of this strategy, we choose it as a well defined reference point (see e.g. Bond (1974), Keren and Wagenaar (1985), Wagenaar (1988), Chau and Phillips (1995)).

We define the *basic strategy* S_{bas} as the playing strategy which would be optimal under the *theoretical* assumption that *all* cards are drawn *with* replacement (i.e. the game with $n = \infty$ and $\lambda = 0$). Clearly, under this assumption $E(G_1(c_0)|d_{\nu j})$ will only depend on $d_{\nu j}$ through the dealercard and the cards in the hand(s) of player ν and not of the playing strategies S_j of the other players $j \neq \nu$. Therefore S_{bas} is the same for all players and can be tabulated as a function of the dealercard and characteristics of the player's hand. We describe its construction in section 6. Table 1 gives the result.

So from now on while evaluating numerically the quality of the strategy of a particular player we assume that the other players follow the basic strategy S_{bas}. Therefore the optimal playing strategy S_{opt} will only depend on the number of decks n, the number of players a, the cut card position λ and the particular player ν. We denote by H_{bas}, H_{opt} the optimal betfunctions belonging to S_{bas}, S_{opt}, respectively. These functions depend on $B_{\max}$ too.

4 Expected gains and efficiency

Consider fixed parameters n, a, λ and $B_{\max}$. For a fixed choice of *playing* strategies for each player, we consider the expected gain of a particular player with strategy (H, S).

The random sequence of all successive stocks by dealing one card after another during the m^{th} game starting with stock C_m and ending with C_{m+1} determines the gain G_m of the m^{th} game. Then the average gain $\mu_G = \mu_G(H, S)$ per game in the long run is given by

$$\mu_G = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} G_i, \quad a.s. \tag{1}$$

(The average bet $\mu_B = \mu_B(H, S)$ per game in the log run is defined similarly). Let GR_m be the sum of all gains in the m^{th} rowgame and N_m the number of games in this rowgame. Then the average gain μ_{GR} and number of games μ_N per rowgame is given by

$$(\mu_{GR}, \mu_N) = \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} (GR_m, N_m), \quad a.s. \tag{2}$$

Clearly,

$$\mu_G = \mu_{GR}/\mu_N. \tag{3}$$

Table 1. Basic Strategy S_{bas} of BJHC

INSURANCE: never IS

SPLITTING (Split = X; No Split = –)

Dealercard Pair	A	2	3	4	5	6	7	8	9	T
AA	–	X	X	X	X	X	X	X	X	X
22	–	X	X	X	X	X	X	–	–	–
33	–	X	X	X	X	X	X	–	–	–
44	–	–	–	–	X	X	–	–	–	–
55	–	–	–	–	–	–	–	–	–	–
66	–	X	X	X	X	X	–	–	–	–
77	–	X	X	X	X	X	X	–	–	–
88	–	X	X	X	X	X	X	X	X	–
99	–	X	X	X	X	X	–	X	X	–
TT	–	–	–	–	–	–	–	–	–	–

DOUBLE DOWN (DDown = X; No DDown = –)

Dealercard Sum	A	2	3	4	5	6	7	8	9	T
Hard 9	–	–	X	X	X	X	–	–	–	–
Hard 10	–	X	X	X	X	X	X	X	X	–
Hard 11	–	X	X	X	X	X	X	X	X	–
Soft 19-21	–	–	–	–	–	–	–	–	–	–

DRAW/STAND (Draw = X; Stand = –)

Dealercard Sum	A	2	3	4	5	6	7	8	9	T
Hard $\leq$ 11	X	X	X	X	X	X	X	X	X	X
Hard 12	X	X	X	–	–	–	X	X	X	X
Hard 13	X	–	–	–	–	–	X	X	X	X
Hard 14	X	–	–	–	–	–	X	X	X	X
Hard 15	X	–	–	–	–	–	X	X	X	X
Hard 16	X	–	–	–	–	–	X	X	X	X
Hard $\geq$ 17	–	–	–	–	–	–	–	–	–	–
Soft $\leq$ 17	X	X	X	X	X	X	X	X	X	X
Soft 18	X	–	–	–	–	–	–	–	X	X
Soft $\geq$ 19	–	–	–	–	–	–	–	–	–	–

The simulation program BJ1SIM estimates for any given playing strategy and any simulation run of M rowgames the values of μ_{GR}, μ_N and μ_G. The reliability of the simulation depends strongly on the quality of the random generator. The simulation program is written in Turbo-Pascal. Its builtin random generator is based on a linear congruent method with a cyclus that would give reliable results for a length of about 300-400 rowgames with 7 players. Therefore, as an alternative, an algorithm of Bays and Durham is used exactly in the way as indicated in Press et al. (1989), section 7.1, p. 215; see also Van der Genugten (1993), section 2.2.7, p. 106 for more details. For a reasonable accuracy a simulation lengh of about M = 50,000,000 rowgames is needed. On a PC-Pentium 90 such a simulation run requires 4 days.

In order to give an idea about the losses that are suffered with simple naive strategies we performed a simulation for BJHC with $a = 7$ players, giving player ν the naive strategy "stand if sum $\geq \nu + 11$ and draw otherwise". The strategy "stand for sum ≥ 12" means "never bust" and "stand for sum ≥ 17" is called "mimic the dealer". The betsize is 1.

Table 2. Sim. gains of naive playing strategies $(n = 6, \lambda = 2/3, a = 7, H \equiv 1)$
playing strategy: never IS, SP or DD; S if sum $\geq \nu + 11$ and D otherwise
(M = 50,200,000 rowgames $-$ μ_N = 10.13 games)

Pν	μ_{GR}	$\pm$95% CI	$\mu_G(1, S_{\nu+11})$
D	4.622	0.004	0.4562
P5	−0.524	0.001	−0.0517
P4	−0.527	0.001	−0.0521
P6	−0.571	0.001	−0.0564
P3	−0.586	0.001	−0.0579
P2	−0.688	0.001	−0.0679
P1	−0.814	0.001	−0.0804
P7	−0.911	0.001	−0.0899

In the first column the player D refers to the dealer and Pν to player ν. The third column contains the half length of a 95% confidence interval for μ_{GR}. We see that these simple strategies lead to a disaster. Even the relatively best player P5 standing on 16 suffers a loss of more than 5%. This is much more than a pure chance game as Roulette would cost! Certainly such players are welcome at the Blackjack tables in HC.

All players can do much better with a little bit more effort by following the basic strategy. Since the use of pencil and paper is strictly forbidden in BJHC, they just have to learn table 1 by heart. A simulation result with BJ1SIM is given in table 3.

Table 3. Sim. gains of S_{bas} $(n = 6, \lambda = 2/3, a = 7, H \equiv 1)$
playing strategy: see table 1
(M = 50,000,000 rowgames $-$ μ_N = 9.86 games)

P	μ_{GR}	$\pm$95% CI	$\mu_G(1, S_{bas})$
D	0.3726	0.005	0.0378
P5	-0.053	0.001	-0.0054
P2	-0.053	0.001	-0.0054
P6	-0.053	0.001	-0.0054
P4	-0.053	0.001	-0.0054
P7	-0.054	0.001	-0.0054
P3	-0.054	0.001	-0.0054
P1	-0.054	0.001	-0.0055

We see that $\mu_G(1, S_{bas}) = -0.0054$ is almost the same for all players and therefore independent of the position at the table. Although the value is still negative it is much better than the values of μ_G for the naive strategies in table 2.

The gain $\mu_G(1, S_{bas})$ for the basic strategy does hardly depend on the number of players. Table 4 gives the simulation result for 1 instead of 7 players.

Table 4. Sim. gains of S_{bas} $(n = 6, \lambda = 2/3, a = 1, H \equiv 1)$
(M = 1,000,000,000 rowgames $-$ μ_N = 39.5 games)

	μ_{GR}	$\pm$95% CI	$\mu_G(1, S_{bas})$
P1	-0.217	0.001	-0.0050

This value differs slighly from $\mu_G(1, S_{bas}) = -0.0054$ for $n = 6, \lambda = 2/3, a = 7$. Roughly spoken, the basic strategy with bet 1 gives a loss of 0.0050 to 0.0055 for all players and is independent of the number of players a.

Rather crude estimates of μ_G can be given for the optimal betfunctions H_{bas}, H_{opt} in combination with the playing strategies S_{bas}, S_{opt}. Table 5 gives some results for a particular player, thereby assuming that the other players play the basic strategy. In the following sections we will discuss the accuracy of these figures in detail.

Table 5. Estimated gains $(n = 6, \lambda = 2/3, B_{\max} = 50)$

Strategy	μ_G
$(1, S_{bas})$	-0.005
$(1, S_{opt})$	-0.004
(H_{bas}, S_{bas})	$+0.08$
(H_{opt}, S_{opt})	$+0.11$

The table shows that there exist strategies (H, S) with positive expected gains. Using such strategies will beat the dealer in the long run.

Consider for fixed playing strategies of the other players the strategy (H, S) of a particular player. We define the *total efficiency* $\mathrm{TE}(H, S)$, the *betting efficiency* $\mathrm{BE}(H, S)$ and the (playing) *strategy efficiency* $\mathrm{SE}(H, S) = \mathrm{SE}(S)$ by, respectively,

$$\begin{aligned}
\mathrm{TE}(H, S) &= \frac{\mu_G(H, S) - \mu_G(1, S_{bas})}{\mu_G(H_{opt}, S_{opt}) - \mu_G(1, S_{bas})} \\
\mathrm{BE}(H, S) &= \frac{\mu_G(H, S) - \mu_G(1, S)}{\mu_G(H_{opt}, S_{opt}) - \mu_G(1, S)} \\
\mathrm{SE}(S) &= \frac{\mu_G(1, S) - \mu_G(1, S_{bas})}{\mu_G(1, S_{opt}) - \mu_G(1, S_{bas})}.
\end{aligned}$$

Clearly,

$$\mathrm{TE}(H, S) = \mathrm{BE}(H, S) + \mathrm{TM}.\mathrm{SE}(S).(1 - \mathrm{BE}(H, S)),$$

where TM is the table multiplier (not depending on S) defined by

$$\mathrm{TM} = \frac{\mu_G(1, S_{opt}) - \mu_G(1, S_{bas})}{\mu_G(H_{opt}, S_{opt}) - \mu_G(1, S_{bas})}.$$

For obtaining a high betting efficiency of the strategy (H, S) we see that much effort should be put into the approximation H of the optimal betfunction H_{bas} in a simple playable way; the improvement of the playing strategy S from S_{bas} towards S_{opt} is less important. This is even more true when the table multiplier TM is small. Then the total efficiency TE of (H, S) is almost completely determined by its betting efficiency BE. So the improvement of S towards S_{opt} for influencing SE is of minor importance.

For $n = 6$ and $B_{\max} \in [25, 50], \lambda \in [1/2, 2/3]$ the table multipliers TM of BJHC are in the range $0.01 < \text{TM} < 0.03$ and therefore very small. (The figures in table 5 are in agreement with this.) In fact the large number of decks $n = 6$ has for a great deal reduced the effect of skill to betting.

5 Steady-state analysis

Consider a fixed choice of playing strategies. The random sequence $C_1, C_2, \ldots$ of starting stocks form an ergodic Markov chain with state space $\mathcal{C}$ and initial value $C_1 = c_0$. Denote by

$$\pi(c) = \lim_{m\to\infty} P\{C_m = c\}, \quad c \in \mathcal{C} \tag{4}$$

its limit distribution (independent of c_0). We can express the average gains in the long run as expectations of gains in only one game if we start this game with a random stock $C_1 = C$ with $\mathcal{L}(C) = \pi$ (the steady state). Then for $G_1 = G_1(C)$ and $G = G(C) = H(C)G_1(C)$ we have according to the LLN for Markov chains:

$$\mu_{G_1} = E(G_1) = \sum_{c\in C} \pi(c)E(G_1(c)) \tag{5}$$

and more general,

$$\mu_G = E(G) = \sum_{c\in C} \pi(c)H(c)E(G_1(c)). \tag{6}$$

So, at least in theory, we can use (6) for calculating the expected gain μ_G of any betfunction H by determining $\pi(c)$ and $E(G_1(c)), c \in \mathcal{C}$.

In evaluating numerically the strategy of a particular player we take for π the limit probabilities for the assumed standard case that all players follow S_{bas}. So we neglect the effect that π will change when the particular player deviates from S_{bas}. In practice this effect is small and good approximations will be obtained. Neglecting this effect, we see from (5) that the playing strategy S_{opt} of a player as defined in section 3 maximizes his μ_{G_1}. The corresponding betfunction H_{opt} (depending on S_{opt}) maximizes his μ_G in (6). The same holds for the optimal betfunction H_{bas} corresponding to S_{bas}.

For BJ with $a = 1$ player we can calculate $E(G_1(c)|d_{1j})$ for every $c \in \mathcal{C}$ and for every card sequence d_{1j} of the player. This can be done not only for a given playing strategy but also for the optimal strategy. We distinguish the cases $n = \infty$ (drawing with replacement) and $n < \infty$ (drawing without replacement).

For $n = \infty$ the calculations are relatively simple because the card fractions in the stock remain unchanged. The computer programs BJ1IGAME and BJ1ISTRT solve the problems for a given stock c in about 0.5 sec. on a PC-Pentium 90. Details are described in section 6.

For $n < \infty$ the calculations are very complicated since all possible stock developments from a given stock c have to be taken into account. Yet, by a special coding system for such developments we were able to solve the problem. The details of the programs BJ1FGAME and BJ1FSTRT are described in section 7. However, the needed computer time for a given stock c of moderate size with $n = 6$ decks is about 5 days on the PC-Pentium 90 (and on a VAX mainframe still 19 hours). For many $c \in \mathcal{C}$ the differences between the values of $E(G_1(c))$ for $n = \infty$ and moderate finite n are small. This will be discussed in section 7. Therefore in applying (6) we take $n = \infty$ for approximations with values of λ not too close to 1.

In BJ with a number of players $a > 1$, for a particular player ν there is also information contained in $d_{\nu-1}(c)$ and conditioning should be performed for the whole sequence $d_{\nu j}$. This is simply impossible to do. However, the differences with $a = 1$ player may expected to be small. Therefore we will use the obtained results for one player also as approximations for the general case of a particular player among the other players.

With these approximations a straightforward computation of μ_G by (6) is still impossible, even if we use BJ1IGAME, BJ1ISTRT instead of BJ1FGAME, BJ1FSTRT for the calculations. The problem is the large number of stocks in $\mathcal{C}$ (about $(4n+1)^9(16n+1)$; for $n = 6$ resulting in 3.7×10^{14}). Therefore we follow an approach which mixes simulation and analysis by conditioning to the fraction t of played cards. This kind of estimation is described in section 8.

6 Expected gains for infinite decks

In this section we assume that cards are drawn with replacement. Given a stock $c \in \mathcal{C}$ we describe the program BJ1IGAME which maximizes $E(G_1(c)|d_{1j})$ for any sequence $d_{1j} = (d_0(c), x_{11}, \ldots, x_{1j})$, where $d_0(c)$ contains the dealercard and the hands of two cards of all players and where $x_{11}, \ldots, x_{1j}$ denotes the cards of the player thereafter. (Since all cards are drawn with replacement the stock at stage d_{1j} is still c). The program registrates also the corresponding optimal decision table and intermediate results.

We omit the description of the modification from BJ1IGAME to BJ1ISTRT for a given strategy instead of the optimal one.

Table 6 gives the result for the starting stock c_0 for $n = 6$. The unconditional mean becomes $E(G_1(c_0)) = -0.0061$. Table 7 gives the result for a stock c obtained from c_0 if 10 cards of each card value $2, 3, \ldots, 6$ are removed.

Table 6 is in fact an extension of table 1 containing S_{bas} since it optimizes decisions for the starting stock c_0. The main part has an entry for each dealercard $1, \ldots, 10$. Each hand has three columns:

Dec = coded optimal decision,

Opt = expected gain for the optimal decision,

Dif = difference with the expected gain of the second best decision.

(Note that during the game "splitting" comes before "double down", and "double down" before "draw/stand".) For example, with a hand (5,5) against a dealercard 8 we see under "splitting" Dec = 2. So we should not split but doubledown. In this case the expected gain is Opt = 0.287. Splitting would give a difference Dif = 0.631 compared with the optimal decision, leading to an expected gain of $0.287 - 0.631 = -0.344$. The sum of (5,5) is H(ard)10. Under "doubledown" for H10 we get the same value Opt = 0.287. The second best decision (draw or stand) has a difference Dif = 0.089, leading to an expected gain $0.287 - 0.089 = 0.198$ for "not doubledown". Under "draw/stand" we find that this value corresponds to Dec = 1 (drawing). The difference is 0.708 leading to an expected gain of $0.198 - 0.708 = -0.510$ for standing.

Under "splitting" and "draw/stand" the code 777 refers to the situation that the extra bonus for three sevens can be obtained and 77 or H14 to the situation that this is not the case.

Table 7 has been added to show that for $c \neq c_0$ the optimal decisions can be quite different from those of S_{bas}. It contains some very striking optimal decisions. Under "splitting" we see that the decisions for a dealercard 8 and a hand (7, 7) depend on the extra bonus for three sevens. For a dealercard 4–6 even a hand (T, T) should be splitted. Under "double down" we see that for a dealercard 5–6 we should not stand on S21 but double down. Under "draw/stand" we see again the influence of the bonus of three sevens on the optimal decisions.

We describe the algorithms leading to the results above. These algorithms have been implemented in BJ1IGAME. The algorithm for *insurance* is very easy. Let $f_{10} = c(10)/\Sigma c(i)$ be the fraction of tens in the current stock c. This equals the probability that the dealer gets BJ. Therefore the expected gain with insurance is $-\frac{1}{2} + \frac{3}{2} f_{10}$. So we should insure if $f_{10} > \frac{1}{3}$.

The algorithms for *splitting, doubledown* and *draw/stand* work backwards.

Table 6. Optimal decisions and expected gains for the starting stock using BJ1IGAME (with replacement)

Stock:312   24 24 24 24 24 24 24 24 24 96

GAME VALUE: -0.006144

(Decisions: 0=Stand 1=Draw 2=DoubleDown 3=Split)

INSURANCE
Decision: No - Opt: 0.000 - Dif: 0.038

Dealer:		1	2	3	4	5	6	7	8	9	T
SPLITTING											
A A	Dec	1	3	3	3	3	3	3	3	3	3
	Opt	-0.322	0.609	0.658	0.707	0.757	0.817	0.633	0.507	0.368	0.119
	Dif	0.176	0.528	0.554	0.581	0.600	0.631	0.468	0.412	0.368	0.260
2 2	Dec	1	3	3	3	3	3	3	1	1	1
	Opt	-0.483	-0.084	-0.015	0.060	0.153	0.225	0.007	-0.159	-0.241	-0.344
	Dif	0.414	0.031	0.067	0.109	0.165	0.214	0.096	0.015	0.124	0.257
3 3	Dec	1	3	3	3	3	3	3	1	1	1
	Opt	-0.518	-0.138	-0.056	0.031	0.125	0.195	-0.052	-0.217	-0.293	-0.389
	Dif	0.413	0.003	0.051	0.103	0.160	0.208	0.099	0.012	0.123	0.255
4 4	Dec	1	1	1	1	3	3	1	1	1	1
	Opt	-0.444	-0.022	0.008	0.039	0.076	0.140	0.082	-0.060	-0.210	-0.307
	Dif	0.522	0.145	0.099	0.050	0.005	0.025	0.212	0.227	0.256	0.381
5 5	Dec	1	2	2	2	2	2	2	2	2	1
	Opt	-0.251	0.359	0.409	0.461	0.513	0.576	0.392	0.287	0.144	-0.054
	Dif	0.750	0.552	0.526	0.497	0.461	0.464	0.584	0.631	0.663	0.679
6 6	Dec	1	3	3	3	3	3	1	1	1	1
	Opt	-0.550	-0.212	-0.124	-0.031	0.066	0.132	-0.213	-0.272	-0.340	-0.429
	Dif	0.486	0.041	0.110	0.180	0.233	0.286	0.044	0.131	0.230	0.349
7 7	Dec	1	3	3	3	3	3	3	1	1	1
	Opt	-0.612	-0.131	-0.048	0.040	0.131	0.232	-0.049	-0.372	-0.431	-0.507
	Dif	0.432	0.162	0.204	0.251	0.298	0.386	0.273	0.017	0.125	0.236
777	Dec	1	3	3	3	3	3	3	1	1	1
	Opt	-0.535	-0.131	-0.048	0.040	0.131	0.232	-0.049	-0.295	-0.354	-0.430
	Dif	0.509	0.085	0.127	0.174	0.221	0.309	0.196	0.094	0.202	0.312
8 8	Dec	1	3	3	3	3	3	3	3	3	1
	Opt	-0.666	0.076	0.149	0.223	0.300	0.413	0.325	-0.020	-0.387	-0.575
	Dif	0.222	0.369	0.401	0.434	0.467	0.566	0.740	0.438	0.123	0.039
9 9	Dec	0	3	3	3	3	3	0	3	3	0
	Opt	-0.377	0.196	0.259	0.324	0.393	0.472	0.400	0.235	-0.077	-0.242
	Dif	0.329	0.074	0.111	0.148	0.194	0.189	0.030	0.129	0.106	0.195
T T	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	0.146	0.640	0.650	0.661	0.670	0.704	0.773	0.792	0.758	0.435
	Dif	0.649	0.275	0.238	0.200	0.158	0.128	0.259	0.396	0.525	0.542
DOUBLE DOWN											
H 9	Dec	1	1	2	2	2	2	1	1	1	1
	Opt	-0.353	0.074	0.121	0.182	0.243	0.317	0.172	0.098	-0.052	-0.218
	Dif	0.562	0.013	0.020	0.053	0.085	0.121	0.068	0.125	0.249	0.367
H10	Dec	1	2	2	2	2	2	2	2	2	1
	Opt	-0.251	0.359	0.409	0.461	0.513	0.576	0.392	0.287	0.144	-0.054
	Dif	0.374	0.176	0.203	0.230	0.256	0.288	0.136	0.089	0.028	0.108
H11	Dec	1	2	2	2	2	2	2	2	2	1
	Opt	-0.209	0.471	0.518	0.566	0.615	0.667	0.463	0.351	0.228	0.033
	Dif	0.331	0.232	0.257	0.283	0.307	0.334	0.171	0.121	0.070	0.021
S19	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	-0.115	0.386	0.404	0.423	0.440	0.496	0.616	0.594	0.288	-0.019
	Dif	0.800	0.325	0.284	0.241	0.196	0.179	0.512	0.620	0.589	0.566
S20	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	0.146	0.640	0.650	0.661	0.670	0.704	0.773	0.792	0.758	0.435
	Dif	0.771	0.281	0.241	0.200	0.158	0.128	0.381	0.505	0.614	0.597
S21	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	0.331	0.882	0.885	0.889	0.892	0.903	0.926	0.931	0.939	0.812
	Dif	0.871	0.411	0.368	0.323	0.277	0.235	0.463	0.580	0.711	0.800
DRAW/STAND											
H 3	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.465	-0.101	-0.069	-0.036	0.000	0.024	-0.057	-0.131	-0.215	-0.322
	Dif	0.304	0.192	0.183	0.175	0.167	0.178	0.418	0.380	0.328	0.254
H 4	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.483	-0.115	-0.083	-0.049	-0.012	0.011	-0.088	-0.159	-0.241	-0.344
	Dif	0.287	0.178	0.170	0.162	0.155	0.165	0.387	0.351	0.302	0.232
H 5	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.501	-0.128	-0.095	-0.061	-0.024	-0.001	-0.119	-0.188	-0.267	-0.366
	Dif	0.269	0.165	0.157	0.150	0.143	0.153	0.356	0.322	0.277	0.210
H 6	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.518	-0.141	-0.107	-0.073	-0.035	-0.013	-0.152	-0.217	-0.293	-0.389
	Dif	0.251	0.152	0.145	0.138	0.132	0.141	0.323	0.293	0.251	0.187

Dealer:		1	2	3	4	5	6	7	8	9	T
H 7	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.522	-0.109	-0.077	-0.043	-0.007	0.029	-0.069	-0.211	-0.285	-0.371
	Dif	0.247	0.184	0.176	0.168	0.160	0.183	0.407	0.300	0.258	0.204
H 8	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.444	-0.022	0.008	0.039	0.071	0.115	0.082	-0.060	-0.210	-0.307
	Dif	0.325	0.271	0.260	0.250	0.238	0.269	0.558	0.451	0.333	0.269
H 9	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.353	0.074	0.101	0.129	0.158	0.196	0.172	0.098	-0.052	-0.218
	Dif	0.416	0.367	0.354	0.340	0.325	0.350	0.647	0.609	0.491	0.358
H10	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.251	0.182	0.206	0.230	0.256	0.288	0.257	0.198	0.117	-0.054
	Dif	0.518	0.475	0.458	0.442	0.423	0.441	0.732	0.708	0.660	0.522
H11	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.209	0.238	0.260	0.283	0.307	0.334	0.292	0.230	0.158	0.033
	Dif	0.561	0.531	0.513	0.494	0.475	0.487	0.768	0.740	0.701	0.609
H12	Dec	1	1	1	0	0	0	1	1	1	1
	Opt	-0.550	-0.253	-0.234	-0.211	-0.167	-0.154	-0.213	-0.272	-0.340	-0.429
	Dif	0.219	0.039	0.019	0.002	0.026	0.017	0.263	0.239	0.203	0.147
H13	Dec	1	0	0	0	0	0	1	1	1	1
	Opt	-0.582	-0.293	-0.252	-0.211	-0.167	-0.154	-0.269	-0.324	-0.387	-0.469
	Dif	0.187	0.015	0.039	0.063	0.090	0.082	0.206	0.187	0.156	0.106
H14	Dec	1	0	0	0	0	0	1	1	1	1
	Opt	-0.612	-0.293	-0.252	-0.211	-0.167	-0.154	-0.321	-0.372	-0.431	-0.507
	Dif	0.157	0.069	0.096	0.124	0.154	0.147	0.154	0.139	0.112	0.068
777	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.535	-0.216	-0.175	-0.134	-0.090	-0.077	-0.244	-0.295	-0.354	-0.430
	Dif	0.234	0.077	0.077	0.077	0.077	0.077	0.231	0.216	0.189	0.145
H15	Dec	1	0	0	0	0	0	1	1	1	1
	Opt	-0.640	-0.293	-0.252	-0.211	-0.167	-0.154	-0.370	-0.417	-0.472	-0.543
	Dif	0.129	0.124	0.154	0.185	0.218	0.212	0.106	0.094	0.072	0.033
H16	Dec	1	0	0	0	0	0	1	1	1	1
	Opt	-0.666	-0.293	-0.252	-0.211	-0.167	-0.154	-0.415	-0.458	-0.509	-0.575
	Dif	0.104	0.178	0.212	0.245	0.282	0.277	0.061	0.052	0.034	0.001
H17	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	-0.639	-0.153	-0.117	-0.081	-0.045	0.012	-0.107	-0.382	-0.423	-0.464
	Dif	0.055	0.383	0.414	0.446	0.478	0.520	0.377	0.124	0.131	0.152
H18	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	-0.377	0.122	0.148	0.176	0.200	0.283	0.400	0.106	-0.183	-0.242
	Dif	0.364	0.744	0.768	0.793	0.815	0.891	0.991	0.697	0.433	0.433
H19	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	-0.115	0.386	0.404	0.423	0.440	0.496	0.616	0.594	0.288	-0.019
	Dif	0.694	1.115	1.132	1.150	1.166	1.219	1.331	1.308	1.003	0.732
H20	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	0.146	0.640	0.650	0.661	0.670	0.704	0.773	0.792	0.758	0.435
	Dif	1.044	1.495	1.505	1.516	1.525	1.558	1.625	1.643	1.609	1.296
S12	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.322	0.082	0.104	0.127	0.156	0.186	0.165	0.095	0.000	-0.142
	Dif	0.448	0.375	0.356	0.338	0.324	0.340	0.641	0.606	0.543	0.434
S13	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.347	0.047	0.074	0.102	0.133	0.162	0.122	0.054	-0.038	-0.174
	Dif	0.422	0.339	0.326	0.314	0.301	0.315	0.598	0.565	0.505	0.402
S14	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.373	0.022	0.051	0.080	0.112	0.139	0.080	0.013	-0.075	-0.206
	Dif	0.397	0.315	0.303	0.291	0.279	0.293	0.555	0.524	0.468	0.370
S15	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.398	-0.000	0.029	0.059	0.092	0.118	0.037	-0.027	-0.112	-0.237
	Dif	0.372	0.293	0.281	0.270	0.259	0.272	0.512	0.483	0.431	0.339
S16	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.422	-0.021	0.009	0.040	0.073	0.099	-0.005	-0.067	-0.149	-0.268
	Dif	0.347	0.272	0.261	0.251	0.241	0.253	0.470	0.444	0.395	0.307
S17	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.432	-0.000	0.029	0.059	0.091	0.128	0.054	-0.073	-0.150	-0.259
	Dif	0.207	0.152	0.146	0.140	0.136	0.116	0.161	0.309	0.273	0.206
S18	Dec	1	0	0	0	0	0	0	0	1	1
	Opt	-0.372	0.122	0.148	0.176	0.200	0.283	0.400	0.106	-0.101	-0.210
	Dif	0.005	0.059	0.058	0.057	0.052	0.093	0.229	0.066	0.082	0.032
S19	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	-0.115	0.386	0.404	0.423	0.440	0.496	0.616	0.594	0.288	-0.019
	Dif	0.196	0.262	0.255	0.248	0.237	0.256	0.395	0.442	0.280	0.140
S20	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	0.146	0.640	0.650	0.661	0.670	0.704	0.773	0.792	0.758	0.435
	Dif	0.397	0.457	0.444	0.431	0.414	0.416	0.516	0.594	0.642	0.489
GAIN FOR SUM=21											
	BJ	1.038	1.500	1.500	1.500	1.500	1.500	1.500	1.500	1.500	1.385
	NoBJ	0.331	0.882	0.885	0.889	0.892	0.903	0.926	0.931	0.939	0.812

Table 7. Optimal decisions and expected gains for a modified stock using BJ1IGAME (with replacement)

```
Stock:262   24 14 14 14 14 14 24 24 24 96

GAME VALUE:   0.055378

(Decisions: 0=Stand 1=Draw 2=DoubleDown 3=Split)

INSURANCE
Decision: Yes       -  Opt:   0.050       -   Dif:   0.050

Dealer:      1      2      3      4      5      6      7      8      9      T

SPLITTING
A A Dec      1      3      3      3      3      3      3      3      3      3
    Opt -0.399  0.884  0.968  1.076  1.204  1.248  1.060  0.895  0.672  0.281
    Dif  0.033  0.797  0.819  0.855  0.907  0.949  0.893  0.796  0.690  0.478
2 2 Dec      1      3      3      3      3      3      3      3      1      1
    Opt -0.578  0.080  0.261  0.472  0.687  0.689  0.205 -0.029 -0.313 -0.432
    Dif  0.427  0.167  0.272  0.399  0.530  0.545  0.349  0.192  0.002  0.235
3 3 Dec      1      3      3      3      3      3      3      3      1      1
    Opt -0.635  0.021  0.201  0.408  0.618  0.610  0.043 -0.171 -0.393 -0.501
    Dif  0.436  0.141  0.242  0.363  0.488  0.495  0.257  0.129  0.029  0.244
4 4 Dec      1      1      3      3      3      3      1      1      1      1
    Opt -0.524  0.015  0.097  0.304  0.512  0.488  0.158 -0.043 -0.257 -0.386
    Dif  0.633  0.089  0.011  0.136  0.263  0.205  0.326  0.314  0.318  0.473
5 5 Dec      1      2      2      2      2      2      2      2      2      1
    Opt -0.274  0.545  0.632  0.756  0.894  0.944  0.749  0.587  0.359 -0.034
    Dif  0.971  0.701  0.617  0.547  0.493  0.568  1.052  1.084  1.070  0.935
6 6 Dec      1      3      3      3      3      3      1      1      1      1
    Opt -0.602 -0.158  0.024  0.230  0.435  0.411 -0.215 -0.270 -0.350 -0.468
    Dif  0.668  0.069  0.165  0.278  0.393  0.402  0.120  0.267  0.398  0.531
7 7 Dec      1      3      3      3      3      3      3      3      1      1
    Opt -0.710 -0.054  0.135  0.348  0.562  0.610  0.072 -0.445 -0.528 -0.605
    Dif  0.512  0.173  0.275  0.396  0.520  0.601  0.493  0.026  0.164  0.337
777 Dec      1      3      3      3      3      3      3      1      1      1
    Opt -0.619 -0.054  0.135  0.348  0.562  0.610  0.072 -0.380 -0.437 -0.514
    Dif  0.604  0.081  0.184  0.305  0.428  0.509  0.402  0.065  0.256  0.429
8 8 Dec      0      3      3      3      3      3      3      3      3      0
    Opt -0.765  0.167  0.334  0.533  0.747  0.841  0.582  0.082 -0.454 -0.661
    Dif  0.283  0.394  0.475  0.581  0.705  0.832  1.025  0.598  0.134  0.109
9 9 Dec      0      3      3      3      3      3      3      3      3      0
    Opt -0.466  0.339  0.475  0.669  0.864  0.921  0.623  0.449 -0.006 -0.332
    Dif  0.349  0.233  0.297  0.412  0.528  0.493  0.114  0.313  0.230  0.194
T T Dec      0      0      3      3      3      3      0      0      0      0
    Opt  0.045  0.608  0.668  0.968  1.325  1.405  0.818  0.839  0.787  0.384
    Dif  0.594  0.063  0.052  0.313  0.612  0.651  0.029  0.205  0.380  0.452

DOUBLE DOWN
H 9 Dec      1      2      2      2      2      2      2      2      1      1
    Opt -0.407  0.199  0.321  0.477  0.632  0.687  0.431  0.222 -0.034 -0.264
    Dif  0.501  0.069  0.132  0.212  0.292  0.317  0.149  0.041  0.172  0.384
H10 Dec      1      2      2      2      2      2      2      2      2      1
    Opt -0.274  0.545  0.632  0.756  0.894  0.944  0.749  0.587  0.359 -0.034
    Dif  0.309  0.272  0.316  0.378  0.447  0.472  0.355  0.271  0.155  0.075
H11 Dec      1      2      2      2      2      2      2      2      2      2
    Opt -0.223  0.681  0.765  0.870  0.992  1.021  0.785  0.636  0.441  0.108
    Dif  0.270  0.340  0.382  0.435  0.496  0.511  0.364  0.286  0.187  0.026
S19 Dec      0      0      0      2      2      2      0      0      0      0
    Opt -0.215  0.344  0.385  0.477  0.632  0.687  0.699  0.661  0.274 -0.113
    Dif  0.693  0.145  0.065  0.021  0.108  0.098  0.268  0.439  0.480  0.535
S20 Dec      0      0      2      2      2      2      0      0      0      0
    Opt  0.045  0.608  0.632  0.756  0.894  0.944  0.818  0.839  0.787  0.384
    Dif  0.629  0.063  0.017  0.101  0.181  0.191  0.069  0.252  0.428  0.493
S21 Dec      0      0      0      0      2      2      0      0      0      0
    Opt  0.222  0.870  0.871  0.878  0.992  1.021  0.939  0.946  0.954  0.794
    Dif  0.716  0.190  0.106  0.007  0.087  0.104  0.154  0.309  0.513  0.686

DRAW/STAND
H 3 Dec      1      1      1      1      1      1      1      1      1      1
    Opt -0.535 -0.057  0.016  0.099  0.181  0.171 -0.066 -0.146 -0.242 -0.375
    Dif  0.230  0.170  0.157  0.147  0.139  0.162  0.378  0.370  0.346  0.286
H 4 Dec      1      1      1      1      1      1      1      1      1      1
    Opt -0.578 -0.086 -0.011  0.073  0.157  0.144 -0.144 -0.221 -0.313 -0.432
    Dif  0.187  0.141  0.130  0.121  0.115  0.135  0.300  0.294  0.275  0.229
H 5 Dec      1      1      1      1      1      1      1      1      1      1
    Opt -0.623 -0.115 -0.037  0.049  0.134  0.119 -0.197 -0.280 -0.374 -0.486
    Dif  0.143  0.113  0.104  0.097  0.092  0.109  0.246  0.235  0.214  0.175
H 6 Dec      1      1      1      1      1      1      1      1      1      1
    Opt -0.635 -0.119 -0.041  0.045  0.131  0.115 -0.214 -0.300 -0.393 -0.501
    Dif  0.130  0.108  0.100  0.093  0.089  0.106  0.230  0.215  0.195  0.160
```

Dealer:		1	2	3	4	5	6	7	8	9	T
H 7	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.611	-0.079	-0.002	0.083	0.167	0.180	-0.056	-0.257	-0.365	-0.473
	Dif	0.154	0.148	0.138	0.131	0.125	0.171	0.388	0.259	0.223	0.188
H 8	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.524	0.015	0.086	0.168	0.249	0.283	0.158	-0.043	-0.257	-0.386
	Dif	0.241	0.242	0.227	0.216	0.207	0.274	0.602	0.472	0.330	0.274
H 9	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.407	0.130	0.188	0.264	0.340	0.370	0.282	0.180	-0.034	-0.264
	Dif	0.358	0.357	0.329	0.312	0.298	0.361	0.725	0.696	0.554	0.397
H10	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.274	0.272	0.316	0.378	0.447	0.472	0.395	0.317	0.203	-0.034
	Dif	0.491	0.499	0.457	0.426	0.405	0.463	0.839	0.833	0.791	0.627
H11	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.223	0.340	0.382	0.435	0.496	0.511	0.421	0.351	0.254	0.082
	Dif	0.543	0.567	0.523	0.483	0.454	0.502	0.865	0.866	0.841	0.743
H12	Dec	1	0	0	0	0	0	1	1	1	1
	Opt	-0.602	-0.227	-0.141	-0.048	0.042	0.009	-0.215	-0.270	-0.350	-0.468
	Dif	0.163	0.030	0.082	0.132	0.176	0.134	0.229	0.245	0.237	0.193
H13	Dec	1	0	0	0	0	0	1	1	1	1
	Opt	-0.653	-0.227	-0.141	-0.048	0.042	0.009	-0.320	-0.371	-0.433	-0.529
	Dif	0.112	0.123	0.181	0.238	0.290	0.248	0.123	0.145	0.155	0.131
H14	Dec	1	0	0	0	0	0	1	1	1	1
	Opt	-0.710	-0.227	-0.141	-0.048	0.042	0.009	-0.422	-0.471	-0.528	-0.605
	Dif	0.055	0.225	0.288	0.352	0.411	0.369	0.022	0.044	0.059	0.055
777	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.619	-0.136	-0.049	0.044	0.134	0.101	-0.330	-0.380	-0.437	-0.514
	Dif	0.147	0.092	0.092	0.092	0.092	0.092	0.114	0.136	0.151	0.147
H15	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	-0.765	-0.227	-0.141	-0.048	0.042	0.009	-0.444	-0.516	-0.588	-0.661
	Dif	0.004	0.338	0.406	0.475	0.540	0.496	0.081	0.056	0.038	0.031
H16	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	-0.765	-0.227	-0.141	-0.048	0.042	0.009	-0.444	-0.516	-0.588	-0.661
	Dif	0.013	0.375	0.447	0.522	0.591	0.544	0.095	0.077	0.055	0.045
H17	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	-0.677	-0.115	-0.033	0.055	0.142	0.179	-0.034	-0.389	-0.469	-0.550
	Dif	0.111	0.526	0.599	0.673	0.745	0.767	0.536	0.216	0.193	0.171
H18	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	-0.466	0.106	0.178	0.257	0.335	0.428	0.509	0.136	-0.235	-0.332
	Dif	0.341	0.797	0.864	0.935	1.002	1.086	1.152	0.782	0.448	0.417
H19	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	-0.215	0.344	0.385	0.455	0.524	0.589	0.699	0.661	0.274	-0.113
	Dif	0.624	1.097	1.137	1.203	1.265	1.326	1.429	1.389	1.006	0.664
H20	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	0.045	0.608	0.616	0.655	0.713	0.753	0.818	0.839	0.787	0.384
	Dif	0.933	1.436	1.444	1.483	1.539	1.578	1.640	1.661	1.608	1.219
S12	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.399	0.087	0.148	0.222	0.297	0.300	0.168	0.098	-0.018	-0.198
	Dif	0.366	0.314	0.289	0.269	0.255	0.291	0.612	0.614	0.569	0.463
S13	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.443	0.049	0.113	0.188	0.265	0.263	0.075	0.007	-0.090	-0.249
	Dif	0.322	0.276	0.254	0.236	0.223	0.254	0.519	0.523	0.498	0.412
S14	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.502	0.003	0.071	0.149	0.228	0.221	-0.024	-0.094	-0.188	-0.326
	Dif	0.264	0.230	0.212	0.197	0.186	0.212	0.420	0.422	0.399	0.335
S15	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.566	-0.049	0.023	0.105	0.187	0.177	-0.102	-0.181	-0.281	-0.410
	Dif	0.200	0.178	0.164	0.153	0.145	0.168	0.342	0.335	0.307	0.250
S16	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.581	-0.057	0.016	0.099	0.181	0.171	-0.123	-0.205	-0.304	-0.429
	Dif	0.184	0.170	0.157	0.147	0.139	0.162	0.321	0.310	0.284	0.232
S17	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	-0.560	-0.022	0.050	0.132	0.213	0.229	0.017	-0.171	-0.279	-0.404
	Dif	0.117	0.093	0.084	0.077	0.071	0.050	0.051	0.218	0.190	0.146
S18	Dec	0	0	0	0	0	0	0	0	1	1
	Opt	-0.466	0.106	0.178	0.257	0.335	0.428	0.509	0.136	-0.189	-0.326
	Dif	0.018	0.045	0.049	0.051	0.051	0.108	0.300	0.120	0.046	0.005
S19	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	-0.215	0.344	0.385	0.455	0.524	0.589	0.699	0.661	0.274	-0.113
	Dif	0.168	0.186	0.171	0.168	0.162	0.197	0.390	0.449	0.270	0.113
S20	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	0.045	0.608	0.616	0.655	0.713	0.753	0.818	0.839	0.787	0.384
	Dif	0.320	0.335	0.299	0.277	0.266	0.281	0.423	0.522	0.584	0.418

GAIN FOR SUM=21

BJ	0.950	1.500	1.500	1.500	1.500	1.500	1.500	1.500	1.500	1.363
NoBJ	0.222	0.870	0.871	0.878	0.904	0.918	0.939	0.946	0.954	0.794

The first step is to calculate the expected gains for *standing* for every dealercard and every hard and soft sum. This can be programmed in a simple way using recursion by traversing all possible hands of the dealer (see e.g. Van der Genugten (1993), §3.2.1 and §3.2.3: the procedures given here can be easily combined when only the expected gain is needed). However, computation is not very efficient since many hands are permutations of each other and have the same probability. Therefore we made a database containing for each dealercard all (in some way) ordered dealer hands ending with a sum between 17 and 21 together with its frequency. Table 8 gives as an example all ordered rows for a dealercard $D = 10$ leading to a sum $S = 17$. The last column indicates the frequency of the permutations.

Table 8. Frequencies of permutations (D = 10, S = 17)

Card										Freq.
A	2	3	4	5	6	7	8	9	T	
2	1	1	0	0	0	0	0	0	0	6
5	1	0	0	0	0	0	0	0	0	1
1	0	2	0	0	0	0	0	0	0	2
4	0	1	0	0	0	0	0	0	0	1
1	3	0	0	0	0	0	0	0	0	3
1	0	0	0	0	1	0	0	0	0	1
3	2	0	0	0	0	0	0	0	0	4
0	1	0	0	1	0	0	0	0	0	2
0	0	1	1	0	0	0	0	0	0	2
0	0	0	0	0	0	1	0	0	0	1
3	0	0	1	0	0	0	0	0	0	1
1	1	0	1	0	0	0	0	0	0	4
0	2	1	0	0	0	0	0	0	0	3
2	0	0	0	1	0	0	0	0	0	1

So from the first row of table 8 we see that there are 6 sequences containing 2 Aces, 1 with cardvalue 2 and 1 with cardvalue 3 leading to a sum 17 with a dealercard T.

Table 9 shows the reduction obtained in this way.

Table 9. Reduction by reordering

Dealer card	Ordered	Non-ordered	Dealer card	Ordered	Non-ordered
A	782	4720	6	334	1273
2	1014	10350	7	255	796
3	788	6149	8	186	478
4	591	3641	9	140	319
5	451	2223	T	93	160
			Total	4634	30109

This suggests a reduction of computer time of $30109/4634 \approx 6$. In reality the factor between non-recursion and recursion is about 3. This reduction is very important since the algorithm is used again and again.

The second step is to calculate the maximal expected gain for *draw/stand* and the corresponding decisions. Since cards are drawn with replacement the stock does not change during drawing. Therefore the calculations can be done backwards, starting with a hand H21. For H20 we calculate the expected gain for drawing by conditioning to the drawn card using the result for H21. By comparing this with the expected gain for standing on H20 we get the optimal decision and the corresponding expected gain. In this way we can continue down to H12. From this point on we calculate backwards the results for the pairs (H11, S21), (H10, S20) down to (H2, S12).

The third step is to calculate the maximal expected gain for *doubledown*. The calculations are rather straightforward by conditioning to the card drawn after doubling.

The fourth step is to calculate the maximal expected gain for *splitting*. The fact that repeated splitting is allowed causes a big problem. It is simply impossible to do the calculations by conditioning to all outcomes of splitted pairs, even if we would take a reasonable upperbound for the number of splittings. Therefore, not in the simulations but only in the calculations, we act as if the splitting rules are defined slightly different.

We assume that the splitting part is completed before the specific hands are played out. At a certain splitting stage (T, Sp) the player has a total of T hands from which exactly Sp hands of one card can still receive a splitting card (the splitting hands); the other $T - Sp$ hands of two cards have already received a non-splitting card (the non-splitting hands). Splitting one of the Sp hands gives the new stage $(T + 1, Sp + 1)$, no splitting results in $(T, Sp - 1)$. We act as if non-splitting cards are only inspected by the dealer. Therefore these cards are unknown to the player and do not influence decisions. Splitting starts with stage (2, 2) and stops with $Sp = 0$ or if $T =$ MaxSplit, an upperbound needed for backward computer calculations. In the rare case that the upperbound (MaxSplit, Sp) is attained, the splitting stage ends by splitting the last hand, generating Sp splitting hands of one card. (For drawing with replacement we act as if the splitting stage stops also if only one card in the stock is left.) This concludes the splitting part.

Before continuing, all non-splitting cards of the $T - Sp$ non-splitting hands are taken back now by the dealer (only important for drawing without replacement) and added to the reshuffled stock. After this, one of the T hands is chosen at random and played out. Multiplying its gain and bet by T will give the final result for all hands together (so the other $T-1$ hands are not played at all). If the selected hand is one of the Sp splitting hands, then the first card given by the dealer can be any card including a card with the value of the splitting card. However, if it is one of the $T-Sp$ non-splitting hands, then the first card of the dealer must be a card with a value different from the splitting card; otherwise it is returned to the stock and the dealing procedure is repeated until such a card is received.

Intuitively, it will be clear that this modification of the splitting rules has almost no influence on $E(G_1(c))$, especially if cards are drawn with replacement. Although rather cumbersome, conditioning to all possible branches of stages (T, Sp) can be performed.

For drawing with replacement repeated splitting as far as possible is optimal if s-plitting is better than any other decision (for drawing without replacement this is not necessarily true). In all computations for drawing with replacement we took MaxSplit = 10 (and MaxSplit = 6 for drawing without replacement). These values are large enough for having almost no influence on the expected value of the game.

The fifth and final step is to calculate the expected value $E(G_1(c))$ of the game by conditioning to the dealercard and the hand (first two cards) of a player. In this step the possiblity of "Blackjack" and insurance has to be taken into account.

We checked the optimal calculated value of $E(G_1(c_0))$ =-0.00614 (see table 19) by simulating the basic strategy for $n = \infty, \lambda = 0, a = 1$. For a simulation run of $M =$ 1,000,000,000 (row)games we found -0.00615 ± 0.00007 (95%$-$CI), a very satisfactory result. Furthermore, we could not find a more simple splitting modification with an acceptable accuracy. Other alternatives proposed in literature (see e.g. Griffin (1988), Ch. 11, p. 155 and Van der Genugten (1993), §3.2.5, p. 161) appeared to be not accurate enough.

7 Expected gains for finite decks

In this section we assume that cards are drawn without replacement. As in section 6 we describe the program BJ1FGAME which maximizes $E(G_1(c)|d_{1j})$ for any sequence $d_{1j} = (d_0(c), x_{11}, \ldots, x_{1j})$ for $a = 1$ player. Intermediate results will only be given for the starting stock c_0 for a game with $n = 6$ decks. (Again we omit the description of the modifications from BJ1IGAME to BJ1FSTRT for a given strategy instead of the optimal one.) Table 10 gives the results for $E(G_1(c_0)|d_0(c))$, the conditional expectations given the dealercard and the players hand of two cards. For the unconditional mean we find $E(G_1(c_0)) = -0.0052$.

For the infinite case $n = \infty$ we found the value -0.0061 (see section 6). The difference is mainly due to the different procedures in drawing cards. Intermediate results specify the expected gains given the dealercard and the hand of two cards of the player. So these 3 cards are removed from the starting stock before the calculations for splitting, double down and draw/stand begin. In the table the hand of the player is specified by its (hard or soft) sum and one of its cards C1. Under NS (non-split) we tabulated the expected gain given its non-splittable sum. It is the weighted mean (with appropriate probabilities) of the expected gains with the same sum. A similar table of expected gains for the starting stock playing S_{bas} using BJ1FSTRT (without replacement) has been omitted. Results come very close to that of table 10. So we use this table also for the basic strategy.

It is interesting to compare these results with the corresponding infinite case. Therefore we have added table 11 which has precisely the same structure. In fact this table summarizes table 6 in the appropriate way.

Comparison of tables 10 and 11 shows that the optimal decisions for $n = 6$ and $n = \infty$ coincide for allmost all player's hands. There are some exceptions, e.g. for $n = 6$ we should draw for (10, 2) and stand for all other hands against a dealercard 4. The effect on the expected gain is small.

The final question is whether the analytic results obtained by the approximating rules for splitting correspond to simulation results. For this we performed two simulations.

Table 12 gives the simulation results for drawing with replacement ($n = \infty$) and one player ($a = 1$) playing S_{bas}. The overall expected gain is found to be -0.0061, in complete agreement with the analytic results. There is also a good correspondence for the intermediate results (compare with table 11). Table 13 gives the simulation results for drawing without replacement ($n = 6$) and one player ($a = 1$), playing again S_{bas}. This table can be compared with table 10 because the difference between S_{opt} and S_{bas} will be very small. The overall expected gain of S_{bas} is found to be -0.0053, slightly less than the analytic result of -0.0052 for S_{opt}. Again there is a good correspondence for the intermediate results.

Table 10. Optimal decisions and expected gains for the starting stock using BJ1FGAME (without replacement)

Stock: 312 24 24 24 24 24 24 24 24 24 96

GAME VALUE: -0.005208

(Decisions: 0=stand 1=draw 2=ddown 3=split)

Sum	C1	1	2	3	4	5	6	7	8	9	T
H 4	2	-0.4860 1	-0.0774 3	-0.0056 3	0.0702 3	0.1742 3	0.2363 3	0.0091 3	-0.1568 1	-0.2379 1	-0.3425 1
H 5	3	-0.5044 1	-0.1286 1	-0.0954 1	-0.0581 1	-0.0170 1	0.0022 1	-0.1195 1	-0.1872 1	-0.2660 1	-0.3661 1
	NS	-0.5044	-0.1286	-0.0954	-0.0581	-0.0170	0.0022	-0.1195	-0.1872	-0.2660	-0.3661
H 6	4	-0.5238 1	-0.1420 1	-0.1068 1	-0.0700 1	-0.0282 1	-0.0086 1	-0.1537 1	-0.2197 1	-0.2945 1	-0.3904 1
	3	-0.5238 1	-0.1331 3	-0.0489 3	0.0506 3	0.1507 3	0.2090 3	-0.0504 3	-0.2192 1	-0.2953 1	-0.3901 1
	NS	-0.5238	-0.1420	-0.1068	-0.0700	-0.0282	-0.0086	-0.1537	-0.2197	-0.2945	-0.3904
H 7	5	-0.5269 1	-0.1077 1	-0.0740 1	-0.0377 1	0.0026 1	0.0357 1	-0.0687 1	-0.2117 1	-0.2854 1	-0.3724 1
	4	-0.5284 1	-0.1107 1	-0.0768 1	-0.0388 1	0.0014 1	0.0340 1	-0.0691 1	-0.2133 1	-0.2885 1	-0.3742 1
	NS	-0.5276	-0.1093	-0.0754	-0.0383	0.0020	0.0348	-0.0689	-0.2125	-0.2869	-0.3733
H 8	6	-0.4497 1	-0.0202 1	0.0107 1	0.0454 1	0.0801 1	0.1181 1	0.0839 1	-0.0592 1	-0.2099 1	-0.3079 1
	5	-0.4487 1	-0.0205 1	0.0103 1	0.0464 1	0.0816 1	0.1227 1	0.0838 1	-0.0595 1	-0.2114 1	-0.3079 1
	4	-0.4477 1	-0.0198 1	0.0116 1	0.0480 1	0.1048 3	0.1566 3	0.0866 1	-0.0591 1	-0.2098 1	-0.3064 1
	NS	-0.4492	-0.0204	0.0105	0.0459	0.0809	0.1204	0.0839	-0.0593	-0.2106	-0.3079
H 9	7	-0.3577 1	0.0761 1	0.1325 2	0.1962 2	0.2573 2	0.3242 2	0.1743 1	0.0997 1	-0.0522 1	-0.2189 1
	6	-0.3575 1	0.0775 1	0.1319 2	0.1989 2	0.2661 2	0.3273 2	0.1761 1	0.1012 1	-0.0520 1	-0.2171 1
	5	-0.3564 1	0.0779 1	0.1333 2	0.1998 2	0.2694 2	0.3368 2	0.1763 1	0.0999 1	-0.0522 1	-0.2166 1
	NS	-0.3572	0.0772	0.1326	0.1983	0.2642	0.3294	0.1756	0.1003	-0.0521	-0.2175
H10	8	-0.2540 1	0.3689 2	0.4209 2	0.4683 2	0.5252 2	0.5821 2	0.3963 2	0.2876 2	0.1487 2	-0.0535 1
	7	-0.2540 1	0.3697 2	0.4225 2	0.4784 2	0.5288 2	0.5857 2	0.4014 2	0.2926 2	0.1458 2	-0.0536 1
	6	-0.2546 1	0.3709 2	0.4242 2	0.4804 2	0.5381 2	0.5898 2	0.4060 2	0.2916 2	0.1478 2	-0.0530 1
	5	-0.2536 1	0.3732 2	0.4257 2	0.4811 2	0.5405 2	0.5989 2	0.4047 2	0.2930 2	0.1492 2	-0.0526 1
	NS	-0.2542	0.3699	0.4225	0.4756	0.5307	0.5858	0.4013	0.2906	0.1474	-0.0534
H11	9	-0.2092 1	0.4789 2	0.5215 2	0.5715 2	0.6248 2	0.6716 2	0.4619 2	0.3470 2	0.2255 2	0.0303 1
	8	-0.2092 1	0.4804 2	0.5294 2	0.5756 2	0.6281 2	0.6747 2	0.4645 2	0.3474 2	0.2257 2	0.0310 1
	7	-0.2091 1	0.4821 2	0.5326 2	0.5846 2	0.6318 2	0.6780 2	0.4676 2	0.3491 2	0.2272 2	0.0311 1
	6	-0.2089 1	0.4861 2	0.5355 2	0.5875 2	0.6413 2	0.6827 2	0.4697 2	0.3535 2	0.2300 2	0.0317 1
	NS	-0.2091	0.4819	0.5297	0.5797	0.6314	0.6767	0.4659	0.3493	0.2271	0.0310
H12	10	-0.5500 1	-0.2519 1	-0.2314 1	-0.2104 1	-0.1636 0	-0.1547 0	-0.2126 1	-0.2720 1	-0.3407 1	-0.4242 1
	9	-0.5549 1	-0.2555 1	-0.2370 1	-0.2080 0	-0.1625 0	-0.1535 0	-0.2182 1	-0.2786 1	-0.3482 1	-0.4303 1
	8	-0.5550 1	-0.2544 1	-0.2330 1	-0.2068 0	-0.1614 0	-0.1526 0	-0.2179 1	-0.2791 1	-0.3487 1	-0.4299 1
	7	-0.5551 1	-0.2533 1	-0.2319 1	-0.2025 0	-0.1602 0	-0.1516 0	-0.2196 1	-0.2792 1	-0.3478 1	-0.4297 1
	6	-0.5564 1	-0.1959 3	-0.1032 3	-0.0056 3	0.0958 3	0.1362 3	-0.2206 1	-0.2793 1	-0.3472 1	-0.4300 1
	NS	-0.5521	-0.2530	-0.2324	-0.2084	-0.1627	-0.1538	-0.2152	-0.2750	-0.3439	-0.4267

Sum	C1	1	2	3	4	5	6	7	8	9	T
H13	10	-0.5819 1	-0.2958 0	-0.2542 0	-0.2088 0	-0.1632 0	-0.1543 0	-0.2693 1	-0.3244 1	-0.3827 1	-0.4653 1
	9	-0.5827 1	-0.2916 0	-0.2511 0	-0.2076 0	-0.1621 0	-0.1532 0	-0.2698 1	-0.3260 1	-0.3837 1	-0.4660 1
	8	-0.5881 1	-0.2886 0	-0.2468 0	-0.2064 0	-0.1611 0	-0.1522 0	-0.2781 1	-0.3335 1	-0.3913 1	-0.4730 1
	7	-0.5895 1	-0.2886 0	-0.2466 0	-0.2030 0	-0.1609 0	-0.1555 0	-0.2784 1	-0.3347 1	-0.3921 1	-0.4736 1
	NS	-0.5840	-0.2931	-0.2516	-0.2075	-0.1624	-0.1540	-0.2719	-0.3274	-0.3854	-0.4677
H14	10	-0.6127 1	-0.2955 0	-0.2519 0	-0.2084 0	-0.1628 0	-0.1540 0	-0.3246 1	-0.3696 1	-0.4282 1	-0.5048 1
	9	-0.6140 1	-0.2892 0	-0.2507 0	-0.2072 0	-0.1618 0	-0.1529 0	-0.3254 1	-0.3716 1	-0.4302 1	-0.5058 1
	8	-0.6155 1	-0.2890 0	-0.2473 0	-0.2069 0	-0.1618 0	-0.1561 0	-0.3255 1	-0.3715 1	-0.4318 1	-0.5076 1
	7	-0.5488 1	-0.1186 3	-0.0326 3	0.0607 3	0.1430 3	0.2315 3	-0.0475 3	-0.3065 1	-0.3667 1	-0.4437 1
	NS	-0.6134	-0.2934	-0.2509	-0.2079	-0.1625	-0.1541	-0.3249	-0.3702	-0.4291	-0.5054
H15	10	-0.6419 1	-0.2931 0	-0.2515 0	-0.2080 0	-0.1624 0	-0.1537 0	-0.3689 1	-0.4169 1	-0.4722 1	-0.5424 1
	9	-0.6433 1	-0.2897 0	-0.2512 0	-0.2077 0	-0.1625 0	-0.1567 0	-0.3687 1	-0.4174 1	-0.4728 1	-0.5438 1
	8	-0.6389 1	-0.2896 0	-0.2477 0	-0.2076 0	-0.1657 0	-0.1569 0	-0.3625 1	-0.4109 1	-0.4670 1	-0.5386 1
	NS	-0.6416	-0.2920	-0.2508	-0.2079	-0.1630	-0.1547	-0.3679	-0.4160	-0.4714	-0.5420
H16	10	-0.6647 1	-0.2936 0	-0.2520 0	-0.2084 0	-0.1631 0	-0.1575 0	-0.4086 1	-0.4531 1	-0.5045 1	-0.5708 1
	9	-0.6647 1	-0.2902 0	-0.2517 0	-0.2084 0	-0.1664 0	-0.1575 0	-0.4084 1	-0.4536 1	-0.5050 1	-0.5715 1
	8	-0.6647 1	0.0838 3	0.1566 3	0.2249 3	0.3054 3	0.4076 3	0.3214 3	-0.0279 3	-0.3893 3	-0.5714 1
	NS	-0.6647	-0.2929	-0.2519	-0.2084	-0.1638	-0.1575	-0.4086	-0.4532	-0.5046	-0.5710
H17	10	-0.6373 0	-0.1538 0	-0.1175 0	-0.0780 0	-0.0446 0	0.0083 0	-0.1089 0	-0.3842 0	-0.4222 0	-0.4636 0
	9	-0.6373 0	-0.1504 0	-0.1174 0	-0.0811 0	-0.0448 0	0.0083 0	-0.1097 0	-0.3875 0	-0.4213 0	-0.4605 0
	NS	-0.6373	-0.1531	-0.1175	-0.0786	-0.0447	0.0083	-0.1091	-0.3849	-0.4220	-0.4630
H18	10	-0.3748 0	0.1212 0	0.1479 0	0.1741 0	0.2001 0	0.2810 0	0.3977 0	0.1041 0	-0.1852 0	-0.2387 0
	9	-0.3748 0	0.1984 3	0.2525 3	0.3207 3	0.3942 3	0.4642 3	0.3996 0	0.2301 3	-0.0812 3	-0.2355 0
	NS	-0.3748	0.1212	0.1479	0.1741	0.2001	0.2810	0.3977	0.1041	-0.1852	-0.2387
H19	10	-0.1118 0	0.3862 0	0.4012 0	0.4200 0	0.4408 0	0.4941 0	0.6150 0	0.5911 0	0.2839 0	-0.0136 0
	NS	-0.1118	0.3862	0.4012	0.4200	0.4408	0.4941	0.6150	0.5911	0.2839	-0.0136
H20	10	0.1512 0	0.6379 0	0.6480 0	0.6585 0	0.6709 0	0.7028 0	0.7720 0	0.7904 0	0.7561 0	0.4380 0

S12	-0.3260 1	0.6143 3	0.6626 3	0.7131 3	0.7653 3	0.8189 3	0.6324 3	0.5037 3	0.3666 3	0.1275 3
S13	-0.3516 1	0.0455 1	0.0737 1	0.1037 1	0.1373 1	0.1627 1	0.1202 1	0.0518 1	-0.0341 1	-0.1691 1
S14	-0.3766 1	0.0215 1	0.0499 1	0.0817 1	0.1157 1	0.1402 1	0.0765 1	0.0165 1	-0.0727 1	-0.2013 1
S15	-0.4032 1	-0.0019 1	0.0281 1	0.0594 1	0.0944 1	0.1184 1	0.0364 1	-0.0284 1	-0.1124 1	-0.2348 1
S16	-0.4294 1	-0.0228 1	0.0071 1	0.0394 1	0.0746 1	0.1015 1	-0.0079 1	-0.0694 1	-0.1514 1	-0.2689 1
S17	-0.4368 1	0.0007 1	0.0301 1	0.0620 1	0.0989 1	0.1293 1	0.0547 1	-0.0717 1	-0.1473 1	-0.2555 1
S18	-0.3764 1	0.1240 0	0.1511 0	0.1802 0	0.2031 0	0.2805 0	0.4019 0	0.1081 0	-0.0985 1	-0.2067 1
S19	-0.1179 0	0.3886 0	0.4066 0	0.4222 0	0.4429 0	0.4939 0	0.6157 0	0.5959 0	0.2876 0	-0.0158 0
S20	0.1440 0	0.6424 0	0.6494 0	0.6598 0	0.6722 0	0.7026 0	0.7733 0	0.7907 0	0.7595 0	0.4388 0
S21	1.0388 0	1.5000 0	1.5000 0	1.5000 0	1.5000 0	1.5000 0	1.5000 0	1.5000 0	1.5000 0	1.3883 0

Table 11. Optimal decisions and expected gains for the starting stock using BJ1IGAME (with replacement)

Stock: 312 24 24 24 24 24 24 24 24 24 96

GAME VALUE: -0.006144

(Decisions: 0=stand 1=draw 2=ddown 3=split)

Sum	C1	1	2	3	4	5	6	7	8	9	T
H 4	2	-0.4829 1	-0.0842 3	-0.0153 3	0.0597 3	0.1526 3	0.2249 3	0.0073 3	-0.1593 1	-0.2407 1	-0.3439 1
H 5	3	-0.5006 1	-0.1282 1	-0.0953 1	-0.0615 1	-0.0240 1	-0.0012 1	-0.1194 1	-0.1881 1	-0.2666 1	-0.3662 1
	NS	-0.5006	-0.1282	-0.0953	-0.0615	-0.0240	-0.0012	-0.1194	-0.1881	-0.2666	-0.3662
H 6	4	-0.5183 1	-0.1408 1	-0.1073 1	-0.0729 1	-0.0349 1	-0.0130 1	-0.1519 1	-0.2172 1	-0.2926 1	-0.3887 1
	3	-0.5183 1	-0.1377 3	-0.0560 3	0.0305 3	0.1247 3	0.1952 3	-0.0525 3	-0.2172 1	-0.2926 1	-0.3887 1
	NS	-0.5183	-0.1408	-0.1073	-0.0729	-0.0349	-0.0130	-0.1519	-0.2172	-0.2926	-0.3887
H 7	5	-0.5224 1	-0.1092 1	-0.0766 1	-0.0430 1	-0.0073 1	0.0292 1	-0.0688 1	-0.2106 1	-0.2854 1	-0.3714 1
	4	-0.5224 1	-0.1092 1	-0.0766 1	-0.0430 1	-0.0073 1	0.0292 1	-0.0688 1	-0.2106 1	-0.2854 1	-0.3714 1
	NS	-0.5224	-0.1092	-0.0766	-0.0430	-0.0073	0.0292	-0.0688	-0.2106	-0.2854	-0.3714
H 8	6	-0.4441 1	-0.0218 1	0.0080 1	0.0388 1	0.0708 1	0.1150 1	0.0822 1	-0.0599 1	-0.2102 1	-0.3071 1
	5	-0.4441 1	-0.0218 1	0.0080 1	0.0388 1	0.0708 1	0.1150 1	0.0822 1	-0.0599 1	-0.2102 1	-0.3071 1
	4	-0.4441 1	-0.0218 1	0.0080 1	0.0388 1	0.0758 3	0.1404 3	0.0822 1	-0.0599 1	-0.2102 1	-0.3071 1
	NS	-0.4441	-0.0218	0.0080	0.0388	0.0708	0.1150	0.0822	-0.0599	-0.2102	-0.3071
H 9	7	-0.3532 1	0.0744 1	0.1208 2	0.1819 2	0.2431 2	0.3171 2	0.1719 1	0.0984 1	-0.0522 1	-0.2181 1
	6	-0.3532 1	0.0744 1	0.1208 2	0.1819 2	0.2431 2	0.3171 2	0.1719 1	0.0984 1	-0.0522 1	-0.2181 1
	5	-0.3532 1	0.0744 1	0.1208 2	0.1819 2	0.2431 2	0.3171 2	0.1719 1	0.0984 1	-0.0522 1	-0.2181 1
	NS	-0.3532	0.0744	0.1208	0.1819	0.2431	0.3171	0.1719	0.0984	-0.0522	-0.2181
H10	8	-0.2513 1	0.3589 2	0.4093 2	0.4609 2	0.5125 2	0.5756 2	0.3924 2	0.2866 2	0.1443 2	-0.0536 1
	7	-0.2513 1	0.3589 2	0.4093 2	0.4609 2	0.5125 2	0.5756 2	0.3924 2	0.2866 2	0.1443 2	-0.0536 1
	6	-0.2513 1	0.3589 2	0.4093 2	0.4609 2	0.5125 2	0.5756 2	0.3924 2	0.2866 2	0.1443 2	-0.0536 1
	5	-0.2513 1	0.3589 2	0.4093 2	0.4609 2	0.5125 2	0.5756 2	0.3924 2	0.2866 2	0.1443 2	-0.0536 1
	NS	-0.2513	0.3589	0.4093	0.4609	0.5125	0.5756	0.3924	0.2866	0.1443	-0.0536
H11	9	-0.2087 1	0.4706 2	0.5178 2	0.5660 2	0.6147 2	0.6674 2	0.4629 2	0.3507 2	0.2278 2	0.0334 1
	8	-0.2087 1	0.4706 2	0.5178 2	0.5660 2	0.6147 2	0.6674 2	0.4629 2	0.3507 2	0.2278 2	0.0334 1
	7	-0.2087 1	0.4706 2	0.5178 2	0.5660 2	0.6147 2	0.6674 2	0.4629 2	0.3507 2	0.2278 2	0.0334 1
	6	-0.2087 1	0.4706 2	0.5178 2	0.5660 2	0.6147 2	0.6674 2	0.4629 2	0.3507 2	0.2278 2	0.0334 1
	NS	-0.2087	0.4706	0.5178	0.5660	0.6147	0.6674	0.4629	0.3507	0.2278	0.0334
H12	10	-0.5504 1	-0.2534 1	-0.2337 1	-0.2111 0	-0.1672 0	-0.1537 0	-0.2128 1	-0.2716 1	-0.3400 1	-0.4287 1
	9	-0.5504 1	-0.2534 1	-0.2337 1	-0.2111 0	-0.1672 0	-0.1537 0	-0.2128 1	-0.2716 1	-0.3400 1	-0.4287 1
	8	-0.5504 1	-0.2534 1	-0.2337 1	-0.2111 0	-0.1672 0	-0.1537 0	-0.2128 1	-0.2716 1	-0.3400 1	-0.4287 1
	7	-0.5504 1	-0.2534 1	-0.2337 1	-0.2111 0	-0.1672 0	-0.1537 0	-0.2128 1	-0.2716 1	-0.3400 1	-0.4287 1
	6	-0.5504 1	-0.2123 3	-0.1238 3	-0.0310 3	0.0658 3	0.1322 3	-0.2128 1	-0.2716 1	-0.3400 1	-0.4287 1
	NS	-0.5504	-0.2534	-0.2337	-0.2111	-0.1672	-0.1537	-0.2128	-0.2716	-0.3400	-0.4287

Sum	C1	1	2	3	4	5	6	7	8	9	T
H13	10	-0.5825 1	-0.2928 0	-0.2523 0	-0.2111 0	-0.1672 0	-0.1537 0	-0.2691 1	-0.3236 1	-0.3872 1	-0.4695 1
	9	-0.5825 1	-0.2928 0	-0.2523 0	-0.2111 0	-0.1672 0	-0.1537 0	-0.2691 1	-0.3236 1	-0.3872 1	-0.4695 1
	8	-0.5825 1	-0.2928 0	-0.2523 0	-0.2111 0	-0.1672 0	-0.1537 0	-0.2691 1	-0.3236 1	-0.3872 1	-0.4695 1
	7	-0.5825 1	-0.2928 0	-0.2523 0	-0.2111 0	-0.1672 0	-0.1537 0	-0.2691 1	-0.3236 1	-0.3872 1	-0.4695 1
	NS	-0.5825	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.2691	-0.3236	-0.3872	-0.4695
H14	10	-0.6123 1	-0.2928 0	-0.2523 0	-0.2111 0	-0.1672 0	-0.1537 0	-0.3213 1	-0.3719 1	-0.4309 1	-0.5074 1
	9	-0.6123 1	-0.2928 0	-0.2523 0	-0.2111 0	-0.1672 0	-0.1537 0	-0.3213 1	-0.3719 1	-0.4309 1	-0.5074 1
	8	-0.6123 1	-0.2928 0	-0.2523 0	-0.2111 0	-0.1672 0	-0.1537 0	-0.3213 1	-0.3719 1	-0.4309 1	-0.5074 1
	7	-0.5354 1	-0.1305 3	-0.0478 3	0.0397 3	0.1311 3	0.2319 3	-0.0485 3	-0.2950 1	-0.3540 1	-0.4304 1
	NS	-0.6123	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.3213	-0.3719	-0.4309	-0.5074
H15	10	-0.6400 1	-0.2928 0	-0.2523 0	-0.2111 0	-0.1672 0	-0.1537 0	-0.3698 1	-0.4168 1	-0.4716 1	-0.5425 1
	9	-0.6400 1	-0.2928 0	-0.2523 0	-0.2111 0	-0.1672 0	-0.1537 0	-0.3698 1	-0.4168 1	-0.4716 1	-0.5425 1
	8	-0.6400 1	-0.2928 0	-0.2523 0	-0.2111 0	-0.1672 0	-0.1537 0	-0.3698 1	-0.4168 1	-0.4716 1	-0.5425 1
	NS	-0.6400	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.3698	-0.4168	-0.4716	-0.5425
H16	10	-0.6657 1	-0.2928 0	-0.2523 0	-0.2111 0	-0.1672 0	-0.1537 0	-0.4148 1	-0.4584 1	-0.5093 1	-0.5752 1
	9	-0.6657 1	-0.2928 0	-0.2523 0	-0.2111 0	-0.1672 0	-0.1537 0	-0.4148 1	-0.4584 1	-0.5093 1	-0.5752 1
	8	-0.6657 1	0.0760 3	0.1485 3	0.2234 3	0.3002 3	0.4127 3	0.3254 3	-0.0202 3	-0.3865 3	-0.5752 1
	NS	-0.6657	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4148	-0.4584	-0.5093	-0.5752
H17	10	-0.6386 0	-0.1530 0	-0.1172 0	-0.0806 0	-0.0449 0	0.0117 0	-0.1068 0	-0.3820 0	-0.4232 0	-0.4644 0
	9	-0.6386 0	-0.1530 0	-0.1172 0	-0.0806 0	-0.0449 0	0.0117 0	-0.1068 0	-0.3820 0	-0.4232 0	-0.4644 0
	NS	-0.6386	-0.1530	-0.1172	-0.0806	-0.0449	0.0117	-0.1068	-0.3820	-0.4232	-0.4644
H18	10	-0.3771 0	0.1217 0	0.1483 0	0.1759 0	0.1996 0	0.2834 0	0.3996 0	0.1060 0	-0.1832 0	-0.2415 0
	9	-0.3771 0	0.1961 3	0.2592 3	0.3243 3	0.3931 3	0.4725 3	0.3996 0	0.2352 3	-0.0774 3	-0.2415 0
	NS	-0.3771	0.1217	0.1483	0.1759	0.1996	0.2834	0.3996	0.1060	-0.1832	-0.2415
H19	10	-0.1155 0	0.3863 0	0.4044 0	0.4232 0	0.4395 0	0.4960 0	0.6160 0	0.5939 0	0.2876 0	-0.0187 0
	NS	-0.1155	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	-0.0187
H20	10	0.1461 0	0.6400 0	0.6503 0	0.6610 0	0.6704 0	0.7040 0	0.7732 0	0.7918 0	0.7584 0	0.4350 0

S12	-0.3219 1	0.6094 3	0.6578 3	0.7073 3	0.7569 3	0.8167 3	0.6335 3	0.5073 3	0.3680 3	0.1189 3
S13	-0.3474 1	0.0466 1	0.0741 1	0.1025 1	0.1334 1	0.1617 1	0.1224 1	0.0541 1	-0.0377 1	-0.1737 1
S14	-0.3727 1	0.0224 1	0.0508 1	0.0801 1	0.1119 1	0.1392 1	0.0795 1	0.0133 1	-0.0752 1	-0.2057 1
S15	-0.3977 1	-0.0001 1	0.0292 1	0.0593 1	0.0920 1	0.1182 1	0.0370 1	-0.0271 1	-0.1122 1	-0.2373 1
S16	-0.4224 1	-0.0210 1	0.0091 1	0.0400 1	0.0734 1	0.0988 1	-0.0049 1	-0.0668 1	-0.1486 1	-0.2684 1
S17	-0.4320 1	-0.0005 1	0.0290 1	0.0593 1	0.0912 1	0.1281 1	0.0538 1	-0.0729 1	-0.1498 1	-0.2586 1
S18	-0.3720 1	0.1217 0	0.1483 0	0.1759 0	0.1996 0	0.2834 0	0.3996 0	0.1060 0	-0.1007 1	-0.2097 1
S19	-0.1155 0	0.3863 0	0.4044 0	0.4232 0	0.4395 0	0.4960 0	0.6160 0	0.5939 0	0.2876 0	-0.0187 0
S20	0.1461 0	0.6400 0	0.6503 0	0.6610 0	0.6704 0	0.7040 0	0.7732 0	0.7918 0	0.7584 0	0.4350 0
S21	1.0385 0	1.5000 0	1.5000 0	1.5000 0	1.5000 0	1.5000 0	1.5000 0	1.5000 0	1.5000 0	1.3846 0

Table 12. Simulated expected gains of S_{bas} using BJ1SIM (with replacement: $n = \infty$ and $a = 1$), (M = 1,000,000,000 rowgames)

Stock: 312 24 24 24 24 24 24 24 24 24 96

Game value: -0.006150

Sum	C1	1	2	3	4	5	6	7	8	9	T
H 4	2	-0.4839	-0.0818	-0.0148	0.0645	0.1479	0.2261	0.0063	-0.1585	-0.2405	-0.3436
H 5	3	-0.5002	-0.1283	-0.0945	-0.0626	-0.0239	-0.0011	-0.1199	-0.1884	-0.2670	-0.3666
	NS	-0.5002	-0.1283	-0.0945	-0.0626	-0.0239	-0.0011	-0.1199	-0.1884	-0.2670	-0.3666
H 6	4	-0.5182	-0.1416	-0.1080	-0.0719	-0.0350	-0.0138	-0.1522	-0.2171	-0.2928	-0.3887
	3	-0.5182	-0.1329	-0.0578	0.0318	0.1296	0.1913	-0.0575	-0.2187	-0.2912	-0.3885
	NS	-0.5182	-0.1416	-0.1080	-0.0719	-0.0350	-0.0138	-0.1522	-0.2171	-0.2928	-0.3887
H 7	5	-0.5230	-0.1101	-0.0792	-0.0437	-0.0078	0.0290	-0.0686	-0.2105	-0.2871	-0.3719
	4	-0.5232	-0.1097	-0.0752	-0.0445	-0.0075	0.0287	-0.0682	-0.2074	-0.2868	-0.3709
	NS	-0.5231	-0.1099	-0.0772	-0.0441	-0.0076	0.0289	-0.0684	-0.2089	-0.2869	-0.3714
H 8	6	-0.4449	-0.0228	0.0088	0.0401	0.0706	0.1155	0.0834	-0.0597	-0.2114	-0.3070
	5	-0.4432	-0.0216	0.0085	0.0397	0.0717	0.1154	0.0830	-0.0608	-0.2100	-0.3078
	4	-0.4423	-0.0219	0.0079	0.0374	0.0774	0.1434	0.0807	-0.0581	-0.2104	-0.3060
	NS	-0.4440	-0.0222	0.0086	0.0399	0.0711	0.1154	0.0832	-0.0602	-0.2107	-0.3074
H 9	7	-0.3516	0.0746	0.1175	0.1791	0.2479	0.3178	0.1708	0.0995	-0.0521	-0.2178
	6	-0.3528	0.0743	0.1224	0.1808	0.2449	0.3188	0.1698	0.0985	-0.0503	-0.2180
	5	-0.3521	0.0751	0.1208	0.1822	0.2412	0.3182	0.1728	0.0981	-0.0531	-0.2183
	NS	-0.3522	0.0747	0.1202	0.1807	0.2447	0.3183	0.1711	0.0987	-0.0519	-0.2180
H10	8	-0.2511	0.3614	0.4098	0.4591	0.5099	0.5745	0.3938	0.2872	0.1421	-0.0536
	7	-0.2494	0.3610	0.4107	0.4620	0.5151	0.5751	0.3924	0.2851	0.1426	-0.0533
	6	-0.2511	0.3592	0.4128	0.4611	0.5122	0.5768	0.3916	0.2855	0.1442	-0.0539
	5	-0.2521	0.3627	0.4108	0.4632	0.5110	0.5771	0.3905	0.2875	0.1380	-0.0541
	NS	-0.2506	0.3605	0.4111	0.4607	0.5124	0.5754	0.3926	0.2859	0.1430	-0.0536
H11	9	-0.2111	0.4731	0.5195	0.5658	0.6117	0.6695	0.4622	0.3496	0.2281	0.0331
	8	-0.2085	0.4715	0.5165	0.5652	0.6146	0.6714	0.4635	0.3519	0.2299	0.0330
	7	-0.2102	0.4720	0.5150	0.5686	0.6180	0.6639	0.4644	0.3555	0.2256	0.0325
	6	-0.2085	0.4701	0.5200	0.5647	0.6135	0.6682	0.4620	0.3510	0.2307	0.0329
	NS	-0.2096	0.4717	0.5177	0.5661	0.6145	0.6682	0.4630	0.3520	0.2286	0.0328
H12	10	-0.5505	-0.2526	-0.2336	-0.2108	-0.1668	-0.1536	-0.2131	-0.2721	-0.3396	-0.4288
	9	-0.5501	-0.2545	-0.2325	-0.2126	-0.1679	-0.1537	-0.2157	-0.2715	-0.3415	-0.4292
	8	-0.5513	-0.2528	-0.2356	-0.2113	-0.1674	-0.1531	-0.2135	-0.2715	-0.3396	-0.4289
	7	-0.5498	-0.2524	-0.2348	-0.2125	-0.1666	-0.1514	-0.2134	-0.2739	-0.3386	-0.4288
	6	-0.5489	-0.2104	-0.1219	-0.0243	0.0681	0.1310	-0.2135	-0.2727	-0.3400	-0.4294
	NS	-0.5505	-0.2529	-0.2339	-0.2114	-0.1670	-0.1533	-0.2136	-0.2722	-0.3397	-0.4289

Sum	C1	1	2	3	4	5	6	7	8	9	T
H13	10	-0.5828	-0.2936	-0.2526	-0.2114	-0.1669	-0.1538	-0.2688	-0.3232	-0.3868	-0.4695
	9	-0.5820	-0.2920	-0.2521	-0.2099	-0.1659	-0.1542	-0.2689	-0.3241	-0.3872	-0.4699
	8	-0.5810	-0.2923	-0.2535	-0.2123	-0.1686	-0.1525	-0.2699	-0.3240	-0.3863	-0.4697
	7	-0.5828	-0.2927	-0.2533	-0.2114	-0.1674	-0.1526	-0.2687	-0.3254	-0.3866	-0.4694
	NS	-0.5824	-0.2931	-0.2527	-0.2113	-0.1671	-0.1535	-0.2690	-0.3238	-0.3868	-0.4696
H14	10	-0.6117	-0.2922	-0.2522	-0.2113	-0.1680	-0.1543	-0.3214	-0.3721	-0.4313	-0.5073
	9	-0.6109	-0.2928	-0.2499	-0.2118	-0.1673	-0.1559	-0.3211	-0.3709	-0.4311	-0.5065
	8	-0.6117	-0.2924	-0.2533	-0.2127	-0.1659	-0.1540	-0.3201	-0.3721	-0.4314	-0.5072
	7	-0.5363	-0.1283	-0.0453	0.0393	0.1319	0.2339	-0.0456	-0.2958	-0.3530	-0.4306
	NS	-0.6116	-0.2924	-0.2520	-0.2116	-0.1675	-0.1545	-0.3212	-0.3719	-0.4313	-0.5072
H15	10	-0.6398	-0.2929	-0.2529	-0.2113	-0.1676	-0.1540	-0.3700	-0.4170	-0.4722	-0.5428
	9	-0.6407	-0.2930	-0.2538	-0.2125	-0.1687	-0.1523	-0.3688	-0.4169	-0.4724	-0.5431
	8	-0.6401	-0.2922	-0.2517	-0.2096	-0.1670	-0.1532	-0.3691	-0.4155	-0.4724	-0.5427
	NS	-0.6400	-0.2928	-0.2528	-0.2112	-0.1677	-0.1536	-0.3696	-0.4168	-0.4723	-0.5428
H16	10	-0.6654	-0.2917	-0.2528	-0.2113	-0.1675	-0.1534	-0.4147	-0.4592	-0.5094	-0.5755
	9	-0.6652	-0.2924	-0.2519	-0.2111	-0.1670	-0.1548	-0.4165	-0.4583	-0.5087	-0.5754
	8	-0.6655	0.0756	0.1488	0.2234	0.3059	0.4129	0.3244	-0.0141	-0.3885	-0.5740
	NS	-0.6654	-0.2918	-0.2526	-0.2112	-0.1674	-0.1537	-0.4150	-0.4590	-0.5092	-0.5755
H17	10	-0.6392	-0.1533	-0.1169	-0.0804	-0.0452	0.0120	-0.1069	-0.3827	-0.4234	-0.4645
	9	-0.6390	-0.1546	-0.1183	-0.0816	-0.0431	0.0127	-0.1060	-0.3817	-0.4243	-0.4643
	NS	-0.6392	-0.1535	-0.1172	-0.0806	-0.0448	0.0121	-0.1067	-0.3825	-0.4235	-0.4645
H18	10	-0.3772	0.1228	0.1490	0.1752	0.2001	0.2842	0.3981	0.1063	-0.1830	-0.2416
	9	-0.3764	0.2001	0.2574	0.3185	0.3930	0.4677	0.3993	0.2314	-0.0811	-0.2417
	NS	-0.3772	0.1228	0.1490	0.1752	0.2001	0.2842	0.3981	0.1063	-0.1830	-0.2416
H19	10	-0.1153	0.3860	0.4042	0.4230	0.4389	0.4973	0.6155	0.5942	0.2871	-0.0188
	NS	-0.1153	0.3860	0.4042	0.4230	0.4389	0.4973	0.6155	0.5942	0.2871	-0.0188
H20	10	0.1458	0.6397	0.6500	0.6612	0.6702	0.7037	0.7736	0.7920	0.7583	0.4350
S12		-0.3219	0.6093	0.6619	0.7036	0.7557	0.8206	0.6350	0.5094	0.3725	0.1183
S13		-0.3453	0.0477	0.0736	0.1031	0.1348	0.1614	0.1215	0.0544	-0.0375	-0.1735
S14		-0.3724	0.0215	0.0516	0.0806	0.1129	0.1386	0.0790	0.0136	-0.0753	-0.2065
S15		-0.3962	0.0010	0.0290	0.0587	0.0924	0.1200	0.0372	-0.0269	-0.1129	-0.2365
S16		-0.4214	-0.0207	0.0088	0.0400	0.0742	0.1009	-0.0039	-0.0674	-0.1497	-0.2686
S17		-0.4316	0.0023	0.0276	0.0600	0.0916	0.1269	0.0539	-0.0722	-0.1494	-0.2582
S18		-0.3714	0.1209	0.1477	0.1767	0.1982	0.2841	0.3987	0.1073	-0.0983	-0.2096
S19		-0.1161	0.3875	0.4049	0.4233	0.4406	0.4957	0.6165	0.5938	0.2876	-0.0189
S20		0.1486	0.6392	0.6520	0.6608	0.6695	0.7044	0.7738	0.7912	0.7582	0.4351
S21		1.0378	1.5000	1.5000	1.5000	1.5000	1.5000	1.5000	1.5000	1.5000	1.3846

Table 13. Simulated expected gains of S_{bas} using BJ1SIM (without replacement; $n = 6$ and $a = 1$), (M = 1,000,000,000 rowgames)

Stock: 312 24 24 24 24 24 24 24 24 24 96

Game value: -0.005297

Sum	C1	1	2	3	4	5	6	7	8	9	T
H 4	2	-0.4863	-0.0823	-0.0108	0.0742	0.1755	0.2390	0.0122	-0.1564	-0.2385	-0.3426
H 5	3	-0.5049	-0.1287	-0.0931	-0.0610	-0.0168	0.0027	-0.1211	-0.1872	-0.2671	-0.3654
	NS	-0.5049	-0.1287	-0.0931	-0.0610	-0.0168	0.0027	-0.1211	-0.1872	-0.2671	-0.3654
H 6	4	-0.5241	-0.1417	-0.1071	-0.0711	-0.0278	-0.0093	-0.1528	-0.2178	-0.2935	-0.3912
	3	-0.5250	-0.1354	-0.0490	0.0483	0.1485	0.2050	-0.0536	-0.2179	-0.2946	-0.3901
	NS	-0.5241	-0.1417	-0.1071	-0.0711	-0.0278	-0.0093	-0.1528	-0.2178	-0.2935	-0.3912
H 7	5	-0.5275	-0.1078	-0.0755	-0.0388	0.0024	0.0377	-0.0680	-0.2114	-0.2853	-0.3724
	4	-0.5284	-0.1107	-0.0748	-0.0382	0.0016	0.0360	-0.0691	-0.2135	-0.2877	-0.3744
	NS	-0.5279	-0.1093	-0.0752	-0.0385	0.0020	0.0369	-0.0686	-0.2125	-0.2865	-0.3734
H 8	6	-0.4505	-0.0190	0.0108	0.0459	0.0803	0.1167	0.0844	-0.0588	-0.2103	-0.3083
	5	-0.4494	-0.0212	0.0106	0.0461	0.0820	0.1220	0.0831	-0.0602	-0.2120	-0.3088
	4	-0.4481	-0.0177	0.0115	0.0511	0.1055	0.1555	0.0878	-0.0573	-0.2096	-0.3063
	NS	-0.4500	-0.0201	0.0107	0.0460	0.0811	0.1194	0.0838	-0.0595	-0.2111	-0.3086
H 9	7	-0.3583	0.0767	0.1344	0.1954	0.2546	0.3236	0.1735	0.1017	-0.0521	-0.2187
	6	-0.3576	0.0774	0.1328	0.1988	0.2641	0.3284	0.1765	0.1019	-0.0510	-0.2176
	5	-0.3564	0.0789	0.1311	0.1982	0.2707	0.3364	0.1757	0.0992	-0.0523	-0.2178
	NS	-0.3574	0.0777	0.1328	0.1975	0.2630	0.3295	0.1752	0.1009	-0.0518	-0.2180
H10	8	-0.2540	0.3721	0.4195	0.4682	0.5272	0.5808	0.3943	0.2907	0.1483	-0.0533
	7	-0.2555	0.3696	0.4233	0.4768	0.5282	0.5844	0.4019	0.2910	0.1466	-0.0534
	6	-0.2531	0.3709	0.4224	0.4803	0.5410	0.5859	0.4058	0.2896	0.1496	-0.0535
	5	-0.2543	0.3683	0.4228	0.4857	0.5387	0.5994	0.4049	0.2893	0.1511	-0.0538
	NS	-0.2542	0.3708	0.4217	0.4750	0.5321	0.5837	0.4007	0.2904	0.1482	-0.0534
H11	9	-0.2105	0.4754	0.5228	0.5757	0.6251	0.6709	0.4600	0.3454	0.2244	0.0304
	8	-0.2086	0.4801	0.5287	0.5759	0.6248	0.6750	0.4630	0.3475	0.2241	0.0301
	7	-0.2086	0.4821	0.5294	0.5821	0.6309	0.6755	0.4687	0.3483	0.2263	0.0308
	6	-0.2094	0.4856	0.5359	0.5850	0.6400	0.6849	0.4671	0.3570	0.2289	0.0311
	NS	-0.2093	0.4808	0.5292	0.5797	0.6301	0.6765	0.4646	0.3496	0.2260	0.0306
H12	10	-0.5504	-0.2516	-0.2313	-0.2119	-0.1637	-0.1551	-0.2128	-0.2721	-0.3411	-0.4241
	9	-0.5561	-0.2545	-0.2362	-0.2089	-0.1636	-0.1541	-0.2181	-0.2787	-0.3499	-0.4298
	8	-0.5568	-0.2541	-0.2342	-0.2071	-0.1620	-0.1542	-0.2170	-0.2783	-0.3483	-0.4304
	7	-0.5563	-0.2519	-0.2327	-0.2023	-0.1612	-0.1511	-0.2193	-0.2781	-0.3483	-0.4304
	6	-0.5566	-0.1916	-0.1015	-0.0121	0.0884	0.1383	-0.2198	-0.2813	-0.3454	-0.4299
	NS	-0.5530	-0.2524	-0.2326	-0.2094	-0.1631	-0.1543	-0.2151	-0.2747	-0.3444	-0.4267

Sum	C1	1	2	3	4	5	6	7	8	9	T
H13	10	-0.5824	-0.2957	-0.2549	-0.2089	-0.1632	-0.1544	-0.2691	-0.3247	-0.3829	-0.4650
	9	-0.5816	-0.2921	-0.2506	-0.2066	-0.1625	-0.1536	-0.2690	-0.3263	-0.3842	-0.4658
	8	-0.5881	-0.2893	-0.2470	-0.2073	-0.1618	-0.1529	-0.2759	-0.3341	-0.3904	-0.4732
	7	-0.5899	-0.2898	-0.2465	-0.2024	-0.1605	-0.1568	-0.2762	-0.3353	-0.3914	-0.4741
	NS	-0.5842	-0.2934	-0.2519	-0.2074	-0.1625	-0.1544	-0.2710	-0.3278	-0.3854	-0.4676
H14	10	-0.6127	-0.2954	-0.2519	-0.2078	-0.1629	-0.1549	-0.3253	-0.3698	-0.4287	-0.5051
	9	-0.6131	-0.2887	-0.2507	-0.2086	-0.1626	-0.1533	-0.3267	-0.3709	-0.4315	-0.5066
	8	-0.6144	-0.2901	-0.2483	-0.2074	-0.1636	-0.1556	-0.3255	-0.3719	-0.4325	-0.5073
	7	-0.5486	-0.1135	-0.0315	0.0657	0.1417	0.2334	-0.0467	-0.3069	-0.3648	-0.4437
	NS	-0.6131	-0.2934	-0.2511	-0.2078	-0.1630	-0.1547	-0.3256	-0.3703	-0.4298	-0.5057
H15	10	-0.6422	-0.2942	-0.2511	-0.2082	-0.1628	-0.1530	-0.3690	-0.4165	-0.4728	-0.5426
	9	-0.6423	-0.2887	-0.2497	-0.2065	-0.1631	-0.1575	-0.3688	-0.4167	-0.4724	-0.5441
	8	-0.6383	-0.2901	-0.2494	-0.2087	-0.1659	-0.1569	-0.3629	-0.4119	-0.4669	-0.5385
	NS	-0.6416	-0.2926	-0.2506	-0.2080	-0.1634	-0.1544	-0.3680	-0.4158	-0.4717	-0.5422
H16	10	-0.6653	-0.2942	-0.2519	-0.2087	-0.1628	-0.1569	-0.4092	-0.4532	-0.5044	-0.5708
	9	-0.6643	-0.2908	-0.2506	-0.2097	-0.1666	-0.1574	-0.4079	-0.4548	-0.5058	-0.5717
	8	-0.6662	0.0869	0.1549	0.2273	0.3081	0.4046	0.3203	-0.0309	-0.3899	-0.5727
	NS	-0.6651	-0.2936	-0.2517	-0.2089	-0.1636	-0.1570	-0.4090	-0.4536	-0.5047	-0.5710
H17	10	-0.6377	-0.1544	-0.1171	-0.0786	-0.0451	0.0083	-0.1086	-0.3840	-0.4222	-0.4632
	9	-0.6371	-0.1504	-0.1164	-0.0816	-0.0436	0.0089	-0.1101	-0.3877	-0.4203	-0.4603
	NS	-0.6376	-0.1536	-0.1170	-0.0792	-0.0448	0.0085	-0.1089	-0.3847	-0.4218	-0.4626
H18	10	-0.3757	0.1211	0.1477	0.1746	0.1998	0.2815	0.3974	0.1038	-0.1849	-0.2384
	9	-0.3743	0.1985	0.2607	0.3200	0.3937	0.4586	0.3997	0.2332	-0.0894	-0.2347
	NS	-0.3757	0.1211	0.1477	0.1746	0.1998	0.2815	0.3974	0.1038	-0.1849	-0.2384
H19	10	-0.1121	0.3863	0.4006	0.4197	0.4410	0.4949	0.6145	0.5914	0.2835	-0.0139
	NS	-0.1121	0.3863	0.4006	0.4197	0.4410	0.4949	0.6145	0.5914	0.2835	-0.0139
H20	10	0.1512	0.6379	0.6481	0.6582	0.6713	0.7030	0.7719	0.7902	0.7563	0.4380
S12		-0.3245	0.6164	0.6633	0.7120	0.7664	0.8208	0.6291	0.5051	0.3655	0.1272
S13		-0.3519	0.0465	0.0725	0.1042	0.1377	0.1639	0.1218	0.0528	-0.0342	-0.1693
S14		-0.3771	0.0233	0.0512	0.0842	0.1151	0.1413	0.0776	0.0170	-0.0729	-0.2023
S15		-0.4028	-0.0010	0.0298	0.0588	0.0947	0.1187	0.0383	-0.0286	-0.1122	-0.2355
S16		-0.4297	-0.0231	0.0071	0.0397	0.0737	0.1002	-0.0094	-0.0680	-0.1515	-0.2699
S17		-0.4359	-0.0002	0.0310	0.0615	0.1009	0.1306	0.0542	-0.0722	-0.1450	-0.2558
S18		-0.3762	0.1241	0.1518	0.1804	0.2049	0.2815	0.4019	0.1087	-0.0983	-0.2057
S19		-0.1177	0.3885	0.4056	0.4223	0.4426	0.4946	0.6151	0.5964	0.2881	-0.0151
S20		0.1443	0.6424	0.6494	0.6619	0.6728	0.7029	0.7736	0.7900	0.7597	0.4385
S21		1.0383	1.5000	1.5000	1.5000	1.5000	1.5000	1.5000	1.5000	1.5000	1.3882

The comparisons also show that the splitting rules for analysis give good approximations for these rules in reality.

We conclude the exposition of the results for finite decks with table 14 containing maximal expected gains for varying n.

Table 14. Expected gains $E(G_1(c_0))$ for S_{opt} $(a = 1)$

n	$E(G_1(c_0))$	n	$E(G_1(c_0))$	n	$E(G_1(c_0))$	n	$E(G_1(c_0))$
∞	-0.0061	20	-0.0059	4	-0.0047	1	-0.0029
100	-0.0061	6	-0.0052	3	-0.0043	$1/2$	$+0.0071$
50	-0.0060	5	-0.0050	2	-0.0033		

The table shows that S_{opt} (or S_{bas}) gives a higher expected gain when the number of decks n decreases, although for $n \geq 1$ the value is still negative. For a stock around the cut card position $\lambda = 2/3$ with a starting stock $n = 6$ and with unchanged card proportions we get from the table under $6(1 - \lambda) = 2$ a value -0.0033, a difference of $-0.0052 + 0.0033 = 0.0019$. Roughly spoken, this bias can be expected in our calculations if we replace BJ1FGAME, BJ1FSTRT by BJ1IGAME, BJ1ISTRT in order to speed up the calculations. This will be done in the following sections. Multiplication with $B_{\max} = 50$ gives a value 0.01, a very rough estimation of the order of the bias in the expected gain of betting strategies like H_{opt} or H_{bas}.

We describe now the algorithms contained in BJ1FGAME for the case of finite decks by indicating the differences with the infinite case as discussed in section 6.

The algorithm for insurance needs no modification. In the algorithm for standing only the probabilities for the possible card drawings have to be changed.

The big problem is the algorithm for Draw/Stand. Calculations must be done backwards for H21 down to (H2, S12). However, now we have also to keep track of all changes in the stock during card drawing. It is rather easy to give a recursive algorithm for this (see e.g. Van Der Genugten (1993), §3.3.2 and p. 220, 221). However, this only gives results within a reasonable time for sums not too far from 21 and is simply unusable for small sums. The reason is that during recursion the same results, which are very time consuming to obtain, are calculated again and again. Therefore we have developed a non-recursive algorithm which works also backwards and in which each result is only calculated once.

This algorithm works with a special coding system for all sums and stocks between H21 and (H2, S12) using the usual binary representations of (non-negative) integers.

It is constructed in such a way that not only coding but also decoding is very fast. Since it has not appeared in literature before we give a rather detailed description by means of a simple example.

We consider the choice between draw or stand for a hard sum H17 with a dealercard T and the starting stock c_0 (for $n = 6$). The following scheme gives the higher sums H18−H21 together with all card sequences involved and their relative numbers for retrieval:

	H21						H20					H19				H18	
H17	8	1	:	1	1	1	4	1	:	1	1	2	1	:	1	1	1
	9	1	:	1	2		5	1	:	2		3	2				
	10	1	:	2	1		6	2	:	1							
	11	1	:	3			7	3									
	12	2	:	1	1												
	13	2	:	2													
	14	3	:	1													
	15	4															

The general construction for lower sums will be clear from this. Card sequences for higher sums are obtained from card sequences for lower sums in such a way that all sequences are ordered from low to high cards (e.g. H18 generates H19, H19 and H18 generate H20, etc.). For H17−H21 we need two arrays S(Stand) and HO(Hard Optimal) of length $2^{21-17} = 16$ to keep track of the expected gains for standing and optimal decisions (between drawing and standing).

At the highest level $21 - 17 = 4$ we calculate the expected gains for standing in the indicated order. Since standing is obligatory on H21 this is also the expected gain for the optimal decisions:

H21 (level 4)									
1	1	1	1	–	H0(1 1 1 1)	=	S(1 1 1 1)	=	0.834
1	1	2		–	H0(1 1 2)	=	S(1 1 2)	=	0.822
1	2	1		–	H0(1 2 1)	=	S(1 2 1)	=	S(1 1 2)
1	3			–	H0(1 3)	=	S(1 3)	=	0.816
2	1	1		–	H0(2 1 1)	=	S(2 1 1)	=	S(1 1 2)
2	2			–	H0(2 2)	=	S(2 2)	=	0.810
3	1			–	H0(3 1)	=	S(3 1)	=	S(1 3)
4				–	H0(4)	=	S(4)	=	0.810

Note that expected gains for standing are only calculated for non-decreasing card sequences.

In the following steps we calculate also the expected gains for drawing by conditioning to the drawn card and make comparisons:

H20 (level 3)

1 1 1 Card 1 → H0(1 1 1 1)
Card 2 − 10 → − 1

$$\left.\begin{array}{l} D = -0.875 \\ S(1\ 1\ 1) = 0.450 \end{array}\right\} \Rightarrow H0(1\ 1\ 1) = 0.450$$

1 2 Card 1 → H0(1 2 1)
Card 2 − 10 → − 1

$$\left.\begin{array}{l} D = -0.864 \\ S(1\ 2) = 0.440 \end{array}\right\} \Rightarrow H0(1\ 2) = 0.440$$

2 1 Card 1 → H0(2 1 1)
Card 2 − 10 → − 1

$$\left.\begin{array}{l} D = -0.864 \\ S(2\ 1) = S(1\ 2) \end{array}\right\} \Rightarrow H0(2\ 1) = 0.440$$

3 Card 1 → H0(3 1)
Card 2 − 10 → − 1

$$\left.\begin{array}{l} D = -0.859 \\ S(3) = 0.435 \end{array}\right\} \Rightarrow H0(3) = 0.435$$

H19 (level 2)

1 1 Card 1 → H0(1 1 1)
Card 2 → H0(1 1 2)
Card 3 − 10 → − 1

$$\left.\begin{array}{l} D = -0.755 \\ S(1\ 1) = -0.010 \end{array}\right\} \Rightarrow H0(1\ 1) = -0.010$$

2 Card 1 → H0(2 1)
Card 2 → H0(2 2)
Card 3 − 10 → − 1

$$\left.\begin{array}{l} D = -0.754 \\ S(2) = -0.018 \end{array}\right\} \Rightarrow H0(2) = -0.237$$

H18 (level 1)

1 Card 1 → H0(1 1)
Card 2 → H0(1 2)
Card 3 → H0(1 3)
Card 4 − 10 → − 1

$$\left.\begin{array}{l} D = -0.674 \\ S(1) = -0.237 \end{array}\right\} \Rightarrow H0(1) = -0.237$$

H17 (level 0)

Card 1 $\rightarrow$ H0(1)
Card 2 $\rightarrow$ H0(2)
Card 3 $\rightarrow$ H0(3)
Card 4 $\rightarrow$ H0(4)
Card 5 $-$ 10 $\rightarrow$ $-$ 1
———————+

$$\left.\begin{array}{l}\mathrm{D} = -0.616 \\ \mathrm{S} = -0.464\end{array}\right\} \Rightarrow \mathrm{H} = -0.464 \ (\mathrm{diff} = 0.152)$$

The resulting value H $= -0.464$ and the difference diff $= 0.152$ can be found in table 19 under DRAW/STAND for H17 and dealercard T. Note that no element in H0 and S is calculated twice during the backwards procedure from H21 to H17. The values of H0 and S with a high index Nr follow from those with lower indices.

The coding and decoding of the relative positions in H and S is indicated by means of the following table:

Sequence				Code				Nr - 2^{4-1}
1	1	1	1	0	0	0	0	0
1	1	2		0	0	1	0	1
1	2	1		0	1	0	0	2
1	3			0	1	1	0	3
2	1	1		1	0	0	0	4
2	2			1	0	1	0	5
3	1			1	1	0	0	6
4				1	1	1	0	7

The card sequence generates the code by writing successively $j-1$ ones and 1 zero for each card j, finally followed by dropping the last zero. By interpreting this code as an integer in binary form and by adding the absolute position 2^{4-1} we get the index number Nr. This coding process is easily reversed.

In general we need also an array S0 (Soft Optimal) for the soft sums. This leads to 3 arrays HO, SO and S of length $2^{21-1} \approx 10^6$. For low hard sums decoding a given number can lead to cards $j > 10$. Of course such array elements remain unused. In Turbo Pascal long integers must be used, leading to 3 arrays of 6 Mb. This can only be implemented efficiently by means of files on a ram disk of about 20 Mb. Today such PC's are no exception any longer.

The algorithm for splitting uses the modified rules which have already been described in section 6 (including the details for drawing without replacement). For the whole algorithm the splitting part takes 80-90% of the needed computer time, caused by the fact that repeated splitting is allowed.

8 Estimation of expected gains

Consider a fixed choice of playing strategies and one particular player. We continue the steady-state analysis in section 5 by conditioning to the fraction t of played cards.

Since every rowgame starts anew with the complete shoe c_0 and cards are randomly dealt one after another, the conditional distribution of C given its card total $k - kt$ (with $t = 0, 1/k, \ldots, \lambda - 1/k$ the fraction of dealt cards) becomes:

$$\pi(c|t) = P(C = c|C \in \mathcal{C}_t) = \binom{kp_1}{c(1)} \cdots \binom{kp_{10}}{c(10)} \Big/ \binom{k}{kt}, \tag{7}$$

where $\mathcal{C}_t = \{c : \sum c(j) = k - kt\}$ is the set of stocks containing $k - kt$ cards.

So we can write

$$\pi(c) = \sum_{t=0}^{\lambda-1/k} p(t)\pi(c|t) \tag{8}$$

with

$$p(t) = P\{C \in \mathcal{C}_t\} \tag{9}$$

By the LLN for Markov chains we have

$$p(t) = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} I(C_i \in \mathcal{C}_t), \quad a.s. \tag{10}$$

So $p(t)$ is the fraction of all stocks at the start of games with card total $k - kt$ in the long run. Note that $\pi(c|t)$ does not depend on the playing strategies but that $p(t)$ does.

By conditioning with respect to the event $\{C \in \mathcal{C}_t\}$ that C contains a played fraction t, we get from (6), (8), (9):

$$\mu_G = \sum_{t=0}^{\lambda-1/k} p(t)E_t(G) \tag{11}$$

with

$$E_t(G) = E_t(H(C)G_1(C)) = \sum_{c\in\mathcal{C}_t} \pi(c|t)H(c)E(G_1(c)). \tag{12}$$

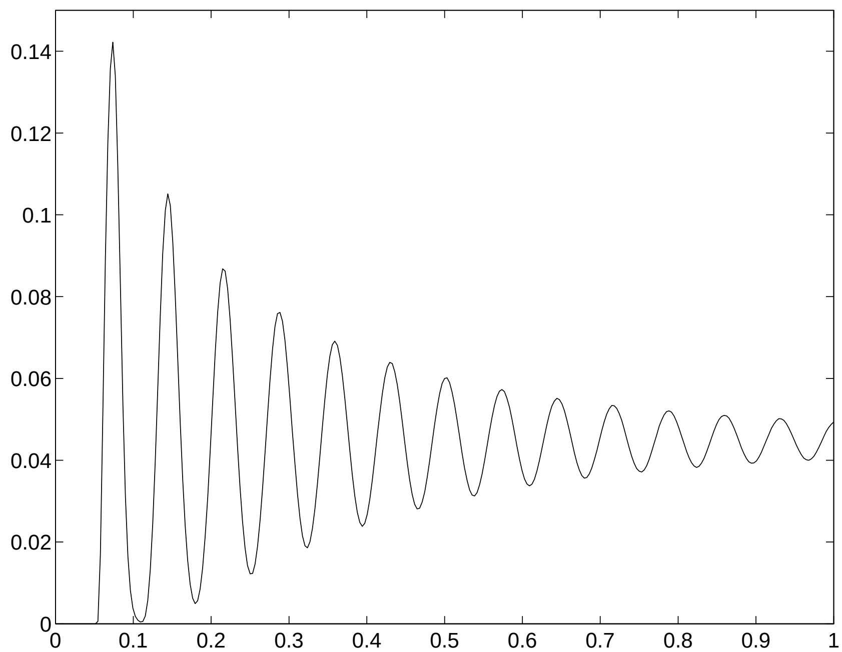
Here E_t denotes the conditional expection given $\{C \in \mathcal{C}_t\}$. In particular, for the optimal betfunction H corresponding to the chosen playing strategy we get

$$E_t(G) = E_t(G_1(C)) + (B_{\max} - 1)E_t(G_1(C))I(G_1(C) > 0)) \tag{13}$$

Note that substitution of $B_{\max} = 1$ gives the result for the unit betfunction $H \equiv 1$.

By giving all players the basic strategy S_{bas}, the probabilities $p(t)$ can be determined from (10) by simulation using BJ1SIM. Figure 15 gives a graphical presentation of $p(t)/p(0)$ for $0 < t < 1$ for $n = 6, \lambda = 1, a = 7$ based on a simulation of $M =$ 40,000,000 rowgames. Notice the oscillating pattern due to the fact that every rowgame starts anew with the same starting stock c_0. From the $p(t)$ for $\lambda = 1$ we easily get the $p(t)$ for arbitrary λ by truncation and rescaling.

Figure 15. Graphical representation of $p(t)/p(0)$ **for** $0 < t < 1(n = 6, \lambda = 1, a = 7, S_{bas};\ \ p(0) = 0.0690)$, (M = 40,000,000 rowgames)



As mentioned at the end of section 5 we cannot calculate (11) from (12) due to the fact that the calculation of $E(G_1(c))$ for all c is too time consuming. For optimal betting we used simulation of a restricted size to get for all t an approximation of the conditional expectations $E_t(G_1(C))$ and $E_t(G_1(C)I(G_1(C) > 0))$ appearing in (13). We did this for the basic strategy S_{bas} as well as the optimal playing strategy S_{opt}. For both we took the same stock sequence of 286,024 games obtained by simulating the basic strategy for $a = 7$ players using BJ1SIM. For each starting stock c in this sequence we calculated t and $E(G_1(c))$ using BJ1IGAME and BJ1ISTRT. (This took about $2 \times 5 = 10$ days computing time on a Pentium-90.) By taking means over the same t-values we got rough estimates for $E_t(G_1(C))$ and $E_t(G_1(C))I(G_1(C) > 0))$. Substitution of these estimates into (13) and using (11) with the values of $p(t)$ already obtained by simulation gives an estimate of the expected gain μ_G.

Figure 16 Est. expected gain $\mu_G(H_{bas}, S_{bas})$ **for** $1/6 < \lambda < 5/6$

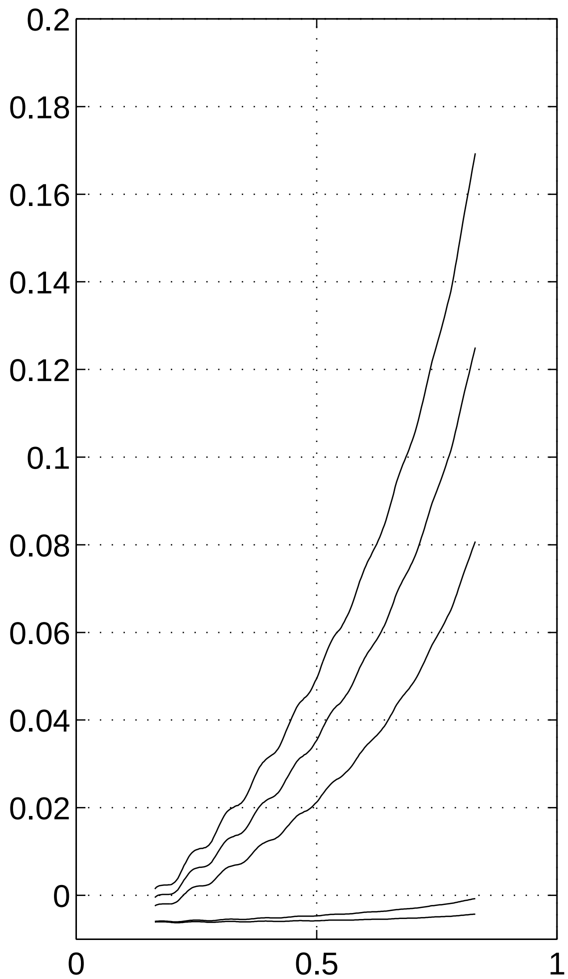


Figure 17 Est. expected gain $\mu_G(H_{opt}, S_{opt})$ **for** $1/6 < \lambda < 5/6$

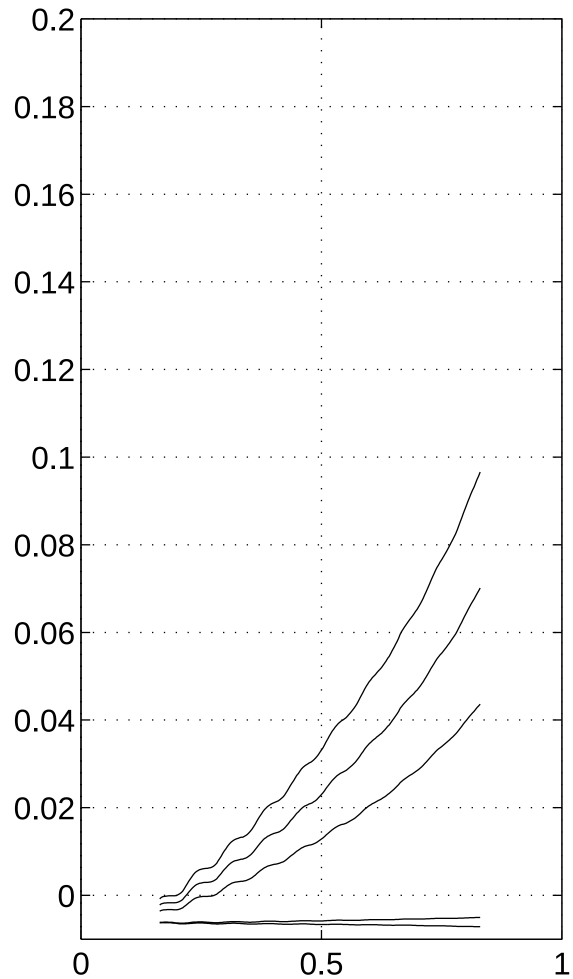


Figure 16 gives the estimates (for $n = 6$) for (H_{bas}, S_{bas}) and the values $B_{\max} = 1$, 2, 25, 37.5 and 50 ($1/6 \leq \lambda \leq 5/6$). We see that for $B_{\max} = 1$ and 2 the values of μ_G

are negative. Even with optimal betting no positive gain can be obtained with this low values of $B_{\max}$. For the usual values $B_{\max} = 25$, 37.5 and 50 the value of μ_G is always positive for $\lambda \geq 1/3$. So in the usual range $1/2 \leq \lambda \leq 2/3$ players can obtain a positive gain by applying (H_{bas}, S_{bas}). Of course, this has only theoretical value since the betting strategy H_{bas} cannot be tabulated.

Figure 17 gives the same result for the optimal strategy (H_{opt}, S_{opt}). The conclusions concerning positive gains are about the same. However, the positive gains are higher than for the basic strategy.

In particular table 18 gives the expected gains for the most important values $\lambda = 1/2, \lambda = 2/3$ and $B_{\max} = 25$, 37.5 and 50.

Table 18. Est. expected gains $\mu_G(H_{bas}, S_{bas})$ and $\mu_G(H_{opt}, S_{opt})$ using BJ1IGAME, BJ1ISTRT $(n = 6, a = 7)$

	$\mu_G(H_{bas}, S_{bas})$		$\mu_G(H_{opt}, S_{opt})$	
$B_{\max}$	$\lambda = 1/2$	$\lambda = 2/3$	$\lambda = 1/2$	$\lambda = 2/3$
1	−0.0067	−0.0068	−0.0058	−0.0053
2	−0.0059	−0.0054	−0.0047	−0.0033
25	0.013	0.026	0.021	0.043
37.5	0.023	0.043	0.035	0.068
50	0.033	0.060	0.049	0.093

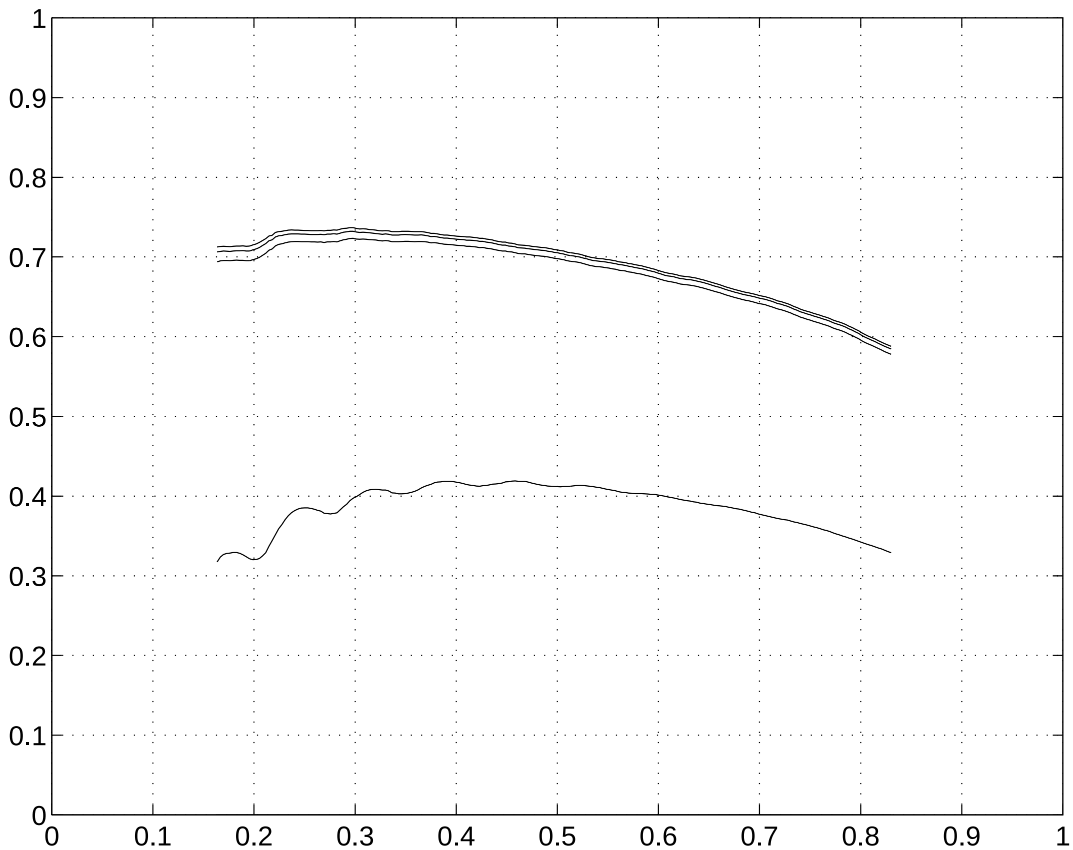
The fact that BJ1IGAME, BJ1ISTRT have been used instead of BJ1FGAME, BJ1FSTRT causes a negative bias in table 18 (see section 7). E.g. the value $\mu_G(H_{opt}, S_{opt}) = 0.093$ for $\lambda = 2/3, B_{\max} = 50$ is too small. Furthermore the small sequence of 286,024 games will also lead to inaccuracy (a good estimator for the variance has not been found). We estimate the correction for $\lambda = 2/3, B_{\max} = 50$ on about 0.02 (see section 9). The corrected values have been published in table 5.

From the estimates of the expected gains efficiencies can be calculated. Since all efficiencies are relative values no corrections are needed. Figure 19 shows the BE (betting efficiency) of (H_{bas}, S_{bas}) for $1/6 \leq \lambda \leq 5/6$ and $B_{\max} = 1$, 2, 25, 37.5 and 50.

For $B_{\max}$

$= 1$ the efficiency is 0 by definition. For $B_{\max} = 2$ the betting efficiency is very low (between 0.3 and 0.4 for varying values of λ). For the usual BJHC-values $B_{\max} = 25$, 37.5 and 50 the efficiency is more or less constant and only depends on λ. It decreases from 0.7 for $\lambda = 0.3$ to 0.6 for $\lambda = 0.8$. This again shows that for S_{bas} it is much more important to use a good betfunction than improving S_{bas} itself.

Figure 19. BE(H_{bas}, S_{bas}) **for** $B_{\max}$ =1, 2, 25, 37.5, 50 **and for** $1/6 < \lambda < 5/6$ $(n = 6, a = 7)$



9 Card counting systems for betting

A card counting system is a vector $\psi_1 \in \mathbb{R}^{10}$ with the interpretation that card j gets the score $\psi_{j1}, j = 1, \ldots, 10$. During a rowgame a player cumulates the scores of all dealt cards using his counting system ψ_1. This sum of scores is called the *running count*. He makes his betting and playing decisions according to the *true count*, by definition the running count divided by the number of remaining cards in the shoe. Sometimes a player uses even a second counting system $\psi_2 \in \mathbb{R}^{10}$ simultaneously for special decisions. This is called *side* counting.

For our general framework we consider q-dimensional card systems $\Psi = \{\psi_{gj}\} = [\psi'_1 \ldots \psi'_q] \in \mathbb{R}^{10\times q}$.

In this section we focus on the construction of card counting systems for betting for a particular player given the playing strategies of all players.

The first step in the analysis is to construct a card counting system $\Phi = \{\phi_{hj}\} = [\phi'_1 \ldots \phi'_p] \in \mathbb{R}^{10\times p}$ with which a good approximation of the optimal betfunction can be obtained. Such a system will be rather theoretical because its scores will not be nice figures and therefore too complicated to use in practice.

Therefore the second step in the analysis is to replace the theoretical system Φ by a practical system Ψ having nice figures.

The basic idea behind the system Φ is that the conditional gain distribution $\mathcal{L}_t(G_1(C))$ given $\mathcal{C}_t$ will almost not vary among all stocks C with the same vector $F(t) = (F_1(t), \ldots, F_{10}(t) \in \mathbb{R}^{10}$ of played card fractions $F_j(t)$, defined by

$$F_j(t) = \frac{c_0(j) - C(j)}{k - kt}, j = 1, \ldots, 10. \tag{14}$$

So we can try the approximation

$$\mathcal{L}_t(G_1(C)) \approx \mathcal{L}(\bar{G}_1(U(t))) \tag{15}$$

with $\bar{G}_1$ some (possibly non-linear) deterministic function of the *theoretical true count*

$$U(t) = \Phi' F(t) \tag{16}$$

of p linear combinations of the $F_j(t)$ with the counting system $\Phi = \{\varphi_{hj}\} = [\varphi'_1, \ldots, \varphi'_p]' \in \mathbb{R}^{10\times p}$ as coefficients. The components $U_h(t)$ in $U(t) = [U_1(t), \ldots, U_p(t)]'$ are given by

$$U_h(t) = \varphi'_h F(t) = \sum_{j=1}^{10} \varphi_{hj} F_j(t), \quad h = 1, \ldots, p. \tag{17}$$

The corresponding *theoretical running* count $V(t) = [V_1(t), \ldots, V_p(t)]'$ is given by

$$V(t) = k(1-t)U(t). \tag{18}$$

This can be seen as follows. Let $X_1, X_2, \ldots$ denote the successive cards starting with the complete shoe c_0. Then with (14), (16) and (18):

$$V(t) = \sum_{j=1}^{10} \varphi_j(c_0(j) - C(j)) = \sum_{j=1}^{10} \varphi_j \sum_{i=1}^{kt} I(X_i = j) = \sum_{i=1}^{kt} \varphi_{X_i}. \tag{19}$$

Indeed, $V(t)$ is obtained by summing up the card values φ_{X_i} of any new dealt card X_i. The true count $U(t)$ is determined by dividing the running count $V(t)$ by the size $k(1-t)$ of the current stock. (In practice often one deck of 52 cards is taken as count unit. Then $F(t)$ and $U(t)$ should be multiplied by 52 and the factor in (18) becomes $n(1-t)$.)

In order to get an idea about the kind of approximations involved we calculated $E(G_1(c))$ for c obtained from the starting stock $c_0 = (24 \quad 24 \ldots 24 \quad 96)$ by deleting successively all cards with a particular card value. Figure 20 gives the result for each card j as a function of the played fraction $F_j(t)$ for S_{bas}. Figure 21 gives the same result for S_{opt}. (The results are approximations obtained by using the programs BJ1ISTRT, BJ1IGAME instead of BJ1FSTRT, BJ1FGAME.)

The *low* cards 2-6 exhibit an increasing pattern of $E(G_1(c))$ with $F_j(t)$. So playing stocks containing few low cards are a disadvantage for players. For the *high* cards 7-10 and A=1 we see that card 7 gives a slightly positive pattern, cards 8 is neutral, card 9 is slightly negative and the cards T=10 and A=1 are strongly negative. So stocks rich with tens or aces are advantegeous for players.

The effect of deleting cards is almost linear (with the exception of card T for high fractions: there is also a striking difference between the basic and the optimal strategy). In particular, using an one-dimensional linear approximation $\bar{G}_1$ with positive slope, we see that $\phi_{j1} > 0$ for $j = 2, \ldots, 7$, $\phi_{81} \approx 0$ and $\phi_{j1} < 0$ for $j = 1, 9$ and 10. The figures 20 and 21 make also clear that high quality approximations may not be expected with linear approximations.

For a given playing strategy the construction of a theoretical card counting system Φ is done by calculating $E(G_1(c))$ for c in a sufficiently large and widely spreadout set

$\mathcal{C}_{fit}$ of stocks. By choosing a suitable parametric form for $\bar{G}_1$, its parameters and Φ can be estimated. In section 10 we will investigate such linear approximations in detail. Quadratic approximations for tens have also been worked out. Results are not presented here. The obtained improvements are small in relation to the increase of complexity. Further research is needed here.

Now suppose that a particular player uses a betfunction H based on the theoretical true count, say

$$H(c) = \bar{H}(U(t)), c \in \mathcal{C}. \tag{20}$$

Then for the gain $G = G(C) = H(C)G_1(C)$ we get with (15):

$$\mathcal{L}_t(G) = \mathcal{L}_t(\bar{H}(U(t))G_1(C)) \approx \mathcal{L}(\bar{H}(U(t)\bar{G}_1(U(t)). \tag{21}$$

Figure 20. $E(G_1(c))$ **for card fractions** $F_j(t)$ **w.r.t** S_{bas} **using BJ1ISTRT.**

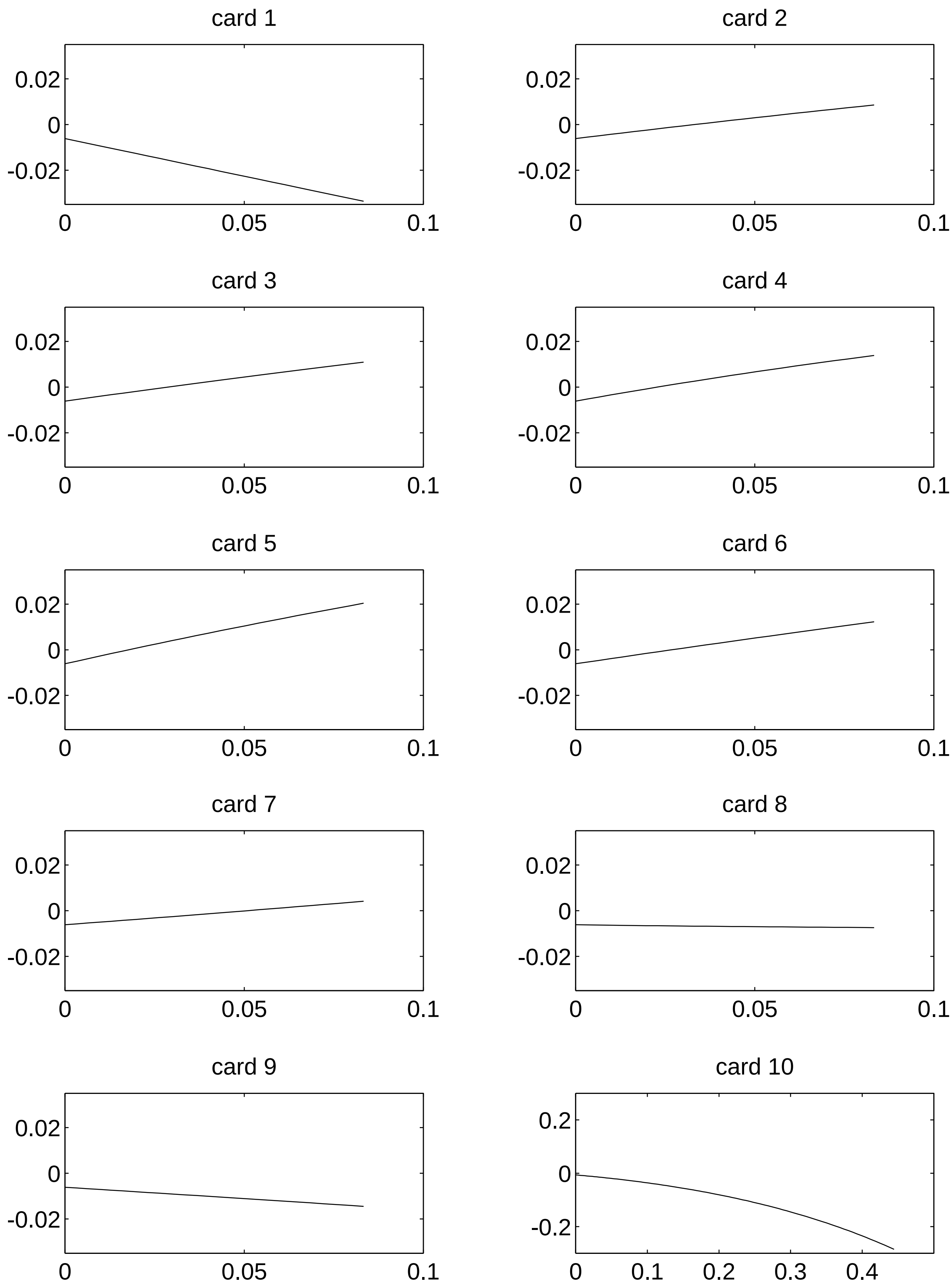
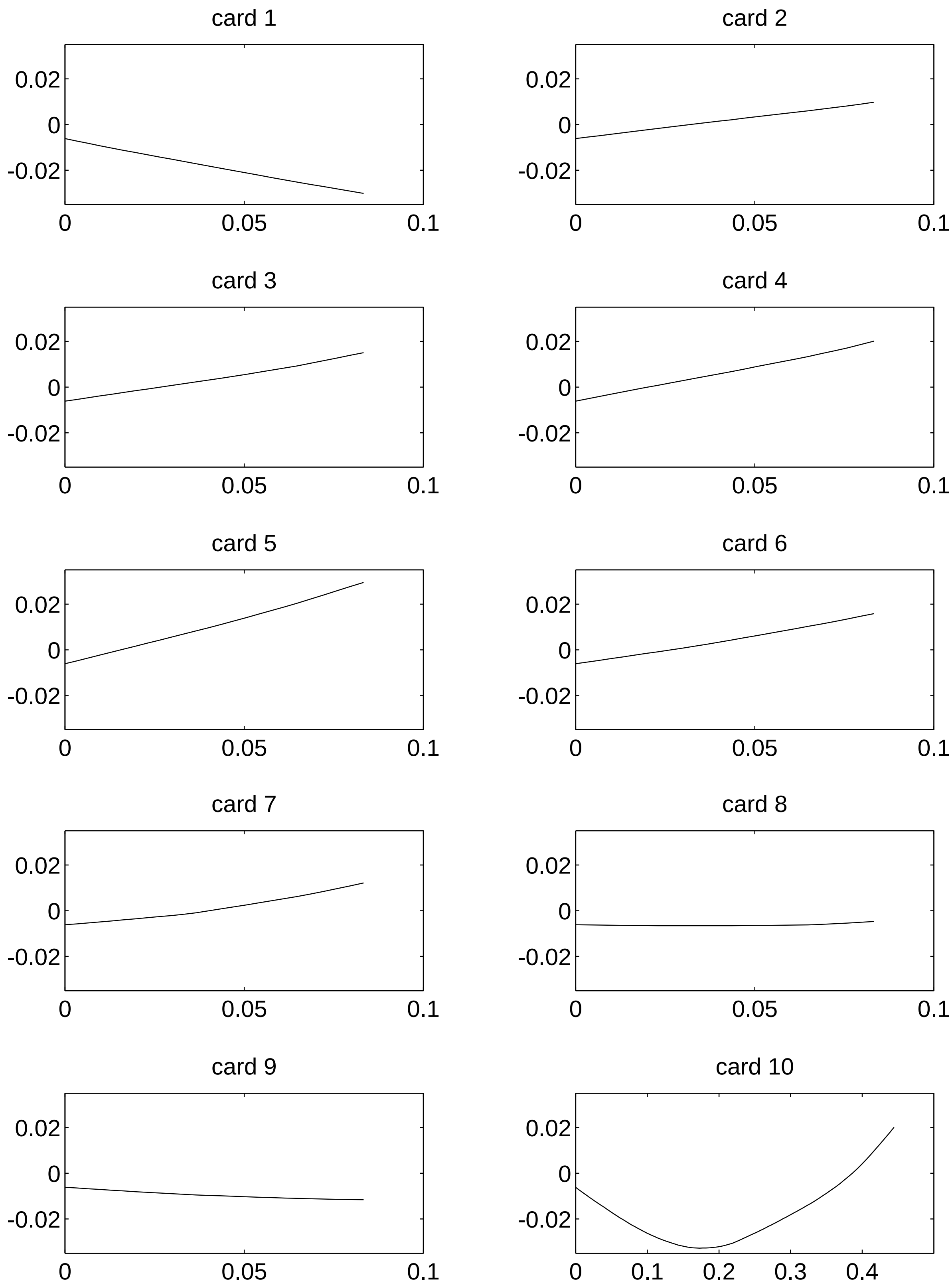


Figure 21. $E(G_1(c))$ **for card fractions** $F_j(t)$ **w.r.t.** S_{opt} **(using BJ1IGAME).**



So the optimal betfunction maximizing $E_t(G)$ can be found approximately by maximizing the expectation of the distribution on the right hand side of (21). This leads to $\bar{H}$ given by

$$\bar{H}(U(t)) = \begin{cases} 1 & \text{if} \quad \bar{G}_1(U(t)) \leq 0 \\ B_{\max} & \text{if} \quad \bar{G}_1(U(t)) > 0. \end{cases} \tag{22}$$

For the corresponding maximal expected gain given $\mathcal{C}_t$ we have

$$\begin{aligned} E_t(G) &\approx E\{\bar{H}(U(t))\bar{G}_1(U(t))\} = \\ &= E\{\bar{G}_1(U(t))\} + (B_{\max} - 1)E\{\bar{G}_1(U(t))I(\bar{G}_1(U(t)) > 0)\}. \end{aligned} \tag{23}$$

In relation to Φ we choose a simple approximating system $\Psi = \{\psi_{gj}\} = [\psi_1', \ldots, \psi_{10}']' \in \mathbb{R}^{10\times q}$ using very nice figures $(0, \pm 1, \pm 2$, etc.). This leads to a q-dimensional card counting system with *practical true* count $T(t)$ and *running* count $S(t)$ given by

$$T(t) = \Psi' F(t) \tag{24}$$

$$S(t) = k(1-t)T(t). \tag{25}$$

For a given betfunction $\bar{H}$ based on $U(t)$ we choose a betfunction $\widehat{H}$ based on $T(t)$ by considering the LS-approximation $\widehat{\Phi}$ of Φ by Ψ. So,

$$\widehat{\Phi} = 1_{10} a_0' + \Psi A \tag{26}$$

with

$$A = \Sigma_{\psi\psi}^{-1}\Sigma_{\psi\varphi}, \quad a_0 = \mu_\varphi - A'\mu_\psi \tag{27}$$

$$\mu_\varphi = \sum_{j=1}^{10} p_j \varphi_j, \quad \mu_\psi = \sum_{j=1}^{10} p_j \psi_j \tag{28}$$

$$\Sigma_{\psi\varphi} = \sum_{j=1}^{10} p_j(\psi_j - \mu_\psi)(\varphi_j - \mu_\varphi)' \tag{29}$$

and $\Sigma_{\varphi\varphi}, \Sigma_{\varphi\psi}$ and $\Sigma_{\psi\psi}$ defined similarly. We write $\Sigma_\varphi = (diag(\Sigma_{\varphi\varphi}))^{1/2}, \Sigma_\psi = (diag(\Sigma_{\psi\psi}))^{1/2}$.

With (16), (25), and (26) this leads to the corresponding estimate $\widehat{U}(t)$ of $U(t)$:

$$\widehat{U}(t) = \widehat{\Phi}' F(t) = a_0 + A'\Psi' F(t)$$

or, with (27)

$$\widehat{U}(t)=\mu_\varphi+\Sigma_{\varphi\psi}\Sigma_{\psi\psi}^{-1}(T(t)-\mu_\psi). \tag{30}$$

According to (15) we use the less precise approximation

$$\mathcal{L}_t(G_1(C))\approx\mathcal{L}(\bar{G}_1(\widehat{U}(t))). \tag{31}$$

Now, suppose that a particular player uses his practical count system Ψ instead of the theoretical system Φ. Then according to (20) he will use the betfunction

$$H(c)=\bar{H}(\widehat{U}(t))=\widehat{H}(T(t)). \tag{32}$$

Then for the corresponding gain G we get with (31), (32):

$$\mathcal{L}_t(G)=\mathcal{L}_t(\bar{H}(\widehat{U}(t))G_1(C))\approx\mathcal{L}(\bar{H}(\widehat{U}(t))\bar{G}_1(\widehat{U}(t))). \tag{33}$$

The analogue of (22) leads to the betfunction $\widehat{H}$, given by

$$\widehat{H}(T(t))=H(\widehat{U}(t))=\begin{cases}0 & \text{if } \ \bar{G}_1(\widehat{U}(t))\le 0\\ B_{\max} & \text{if } \ \bar{G}_1(\widehat{U}(t))>0.\end{cases} \tag{34}$$

For the corresponding expected gain given $\mathcal{C}_t$ we have (compare (23)):

$$E_t(G)\approx E\{\bar{G}_1(\widehat{U}(t))\}+(B_{\max}-1)E\{\bar{G}_1(\widehat{U}(t))I(\bar{G}_1(\widehat{U}(t))>0)\}. \tag{35}$$

Of course, if Ψ differs much from Φ the expected gain $E_t(G)$ is no longer a good approximation of the maximal expected gain.

Card counting systems based on the practical true count $T(t)$ can be analyzed quite easily for many-deck games, since $(T(t),0<t<1)$ converges for $n\to\infty$ after suitable standardisation to the q-product of the Brownian bridge. Using this we get for a n-deck the following approximation (see Appendix A for details):

$$T(t)\overset{\mathcal{L}}{\approx}\frac{t}{1-t}\mu_\psi+\sqrt{\frac{t}{1-t}}\frac{1}{\sqrt{k-1}}\Sigma_\psi Z_q, \tag{36}$$

where Z_q has q independent $N(0,1)$ (standard normal) components. Note that the second term in (36) is $0(1/\sqrt{n})$ since $k=52n$.

Combination of (30) and (36) leads to

$$\widehat{U}(t)\overset{\mathcal{L}}{\approx}\left(\mu_\varphi-\frac{1-2t}{1-t}\Sigma_{\varphi\psi}\Sigma_{\psi\psi}^{-1}\mu_\psi\right)+\sqrt{\frac{t}{1-t}}\frac{1}{\sqrt{k-1}}\Sigma_\varphi R_{\varphi\psi}Z_q \tag{37}$$

with correlationmatrix $R_{\varphi\psi}$ defined by

$$R_{\varphi\psi}=\Sigma_\varphi^{-1}\Sigma_{\varphi\psi}\Sigma_\psi^{-1}. \tag{38}$$

For $\Psi=\Phi$ we get from (37) an approximation of $\mathcal{L}(U(t))$ of the form (36) (take $\psi=\varphi$ and use $R_{\varphi\varphi}=1$).

10 Centered linear card counting systems

We will only consider *centered* linear card counting systems Φ and Ψ. So we assume $\mu_\varphi = 0$, $\mu_\psi = 0$. Then from (30) we get

$$\widehat{U}(t) = \Sigma_{\varphi\psi}\Sigma_{\psi\psi}^{-1}T(t). \tag{39}$$

For linear $\bar{G}_1$ it suffices to take $p = q = 1$. So

$$\bar{G}_1(\widehat{U}(t)) = g_0 + \widehat{U}(t) \tag{40}$$

with $g_0 = E(G_1(c_0))$.

Then the betfunction (34) and the corresponding expected gain conditional to $\mathcal{C}_t$ of (35) can be rewritten. Substitution of

$$t_0 = -g_0\Sigma_{\psi\psi}/\Sigma_{\varphi\psi} \tag{41}$$

leads with (39), (40) to

$$\widehat{H}(T(t)) = \begin{cases} 1 & \text{if} \quad T(t) \le t_0 \\ B_{\max} & \text{if} \quad T(t) > t_0 \end{cases} \tag{42}$$

and

$$E_t(G) \approx (g_0 + E(\widehat{U}(t))) + (B_{\max} - 1)[g_0 P(\widehat{U}(t) > -g_0) + E(\widehat{U}(t)I(\widehat{U}(t) > -g_0)]$$

With (37) for $\mu_\varphi = \mu_\psi = 0$ this gives for $Z = Z_1$:

$$E_t(G) \approx g_0 + (B_{\max} - 1)[g_0 P(Z > -g_0/g_1(t)) + g_1(t)E(Z\ I(Z > -g_0/g_1(t)))]$$

with

$$g_1(t) = \sqrt{\frac{t}{1-t}}\frac{1}{\sqrt{k-1}}\Sigma_\varphi R_{\varphi\psi}. \tag{43}$$

Define

$$Z_k(t) = \int_t^\infty u^k z(u)du, \quad k = 0, 1, \ldots \tag{44}$$

with z the density of $N(0,1)$ (note that $Z_1(t) = z(t)$). Then, finally

$$E_t(G) \approx g_0 + (B_{\max} - 1)[g_0 Z_0(-g_0/g_1(t)) + g_1(t)Z_1(-g_0/g_1(t))]. \tag{45}$$

In order to get reasonable approximations for the BJHC-range $\lambda \in [\frac{1}{2}, \frac{1}{3}]$ we took for $\mathcal{C}_{fit}$ the set

$$\mathcal{C}_{fit} = \{c : \Sigma c(j) \in [104, \;\; 312], \;\; c(j) \bmod 8 = 0 \text{ for all } j, \;\; G_c^2 < \chi^2_{9;0.0001}\}, \tag{46}$$

where G_c^2 is the χ^2-distance

$$G_c^2 = \sum_{j=1}^{10} (c(j) - p_j \Sigma c(i))^2 / (p_j \Sigma c(i)).$$

It appears that $\#\mathcal{C}_{fit}$= 28183. For any $c \in \mathcal{C}_{fit}$ we calculated $E(G_1(c))$ for S_{bas} and S_{opt} with BJ1ISTRT and BJ1IGAME. From (15), (16) and (40) we see $E(G_1(c)) \approx E(\bar{G}_1(\Phi' F(t)) = g_0 + \Phi' F(t)$. We calculated the LS-solution of (g_0, Φ) under the linear restriction $\mu_\varphi = 0$. This leads to corresponding linear approximations $\widehat{H}_{bas}$ $\widehat{H}_{opt}$ of H_{bas}, H_{opt}. The result is contained in table 22 (of course $g_0 = E(G_1(c_0)) = -0.0061$).

Table 22. Φ-values of the betfunctions $\widehat{H}_{bas}$ and $\widehat{H}_{opt}$

card	$\widehat{H}_{bas}$	$\widehat{H}_{opt}$
1	−0.3411	−0.3285
2	+0.1861	+0.2125
3	+0.2153	+0.2620
4	+0.2708	+0.3414
5	+0.3451	+0.4397
6	+0.2253	+0.2362
7	+0.1154	+0.1657
8	−0.0254	+0.0006
9	−0.1006	−0.0825
10	−0.2227	−0.3118

The signs of the coefficients in table 22 agree with the slope directions in figures 19 and 20.

Table 23 compares for $\lambda = 1/2$ and $2/3$ and various values of $B_{\max}$ the estimated values of the expected gains according to table 18 and the calculated approximated values by linear card counting with coefficients in table 22. Table 24 gives the same figures for the optimal playing strategy.

Table 23. Expected gains μ_G for $S_{bas}(n = 6)$

	$B_{\max}$	1	2	25	37.5	50
$\lambda = \frac{1}{2}$	H_{bas} (est.)	-0.0067	-0.0059	0.013	0.023	0.033
	$\widehat{H}_{bas}$ (lin.)	-0.0061	-0.0052	0.016	0.027	0.038
$\lambda = \frac{2}{3}$	H_{bas} (est.)	-0.0068	-0.0054	0.026	0.043	0.060
	$\widehat{H}_{bas}$ (lin.)	-0.0061	-0.0046	0.030	0.049	0.068

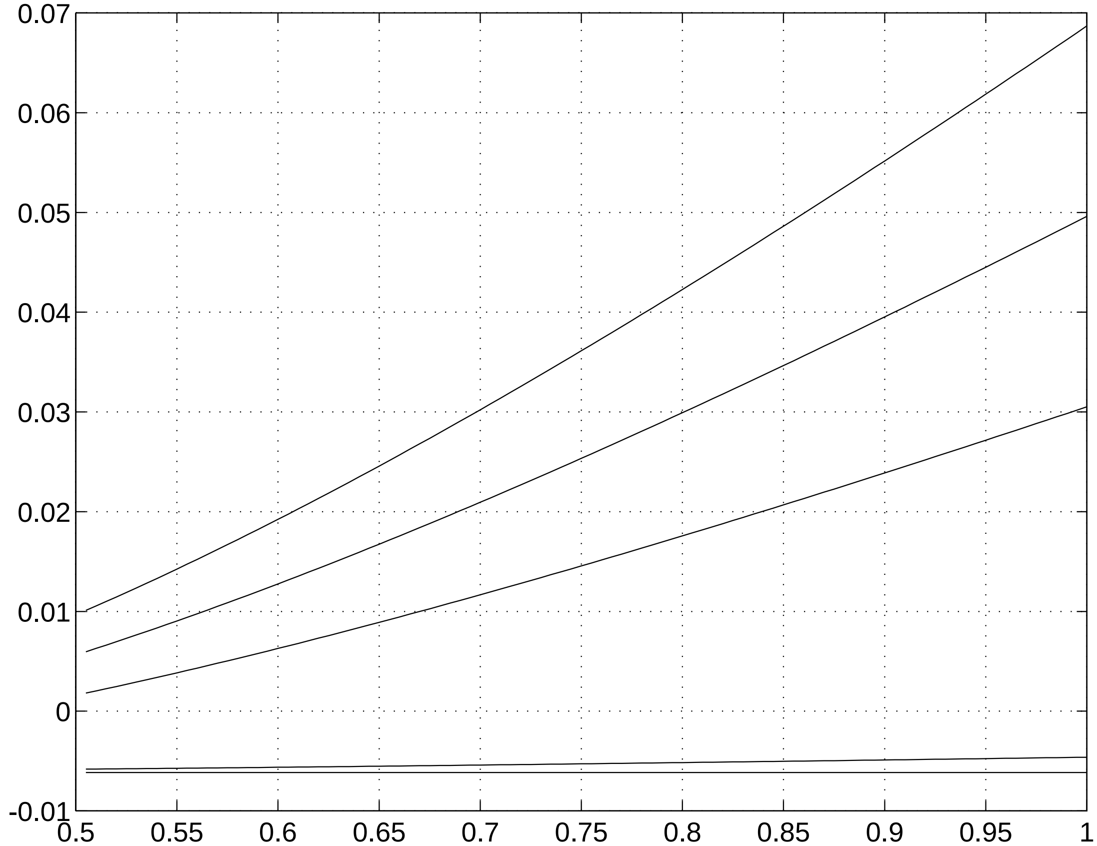
Table 24. Expected gains μ_G for $S_{opt}(n = 6)$

	$B_{\max}$	1	2	25	37.5	50
$\lambda = \frac{1}{2}$	H_{opt} (est.)	-0.0058	-0.0047	0.021	0.035	0.049
	$\widehat{H}_{opt}$ (lin.)	-0.0061	-0.0048	0.027	0.044	0.062
$\lambda = \frac{2}{3}$	H_{opt} (est.)	-0.0053	-0.0033	0.043	0.068	0.093
	$\widehat{H}_{opt}$ (lin.)	-0.0061	-0.0039	0.047	0.075	0.102

The quality of the approximations in table 23 and 24 is moderate but sufficient to rely on for further analysis. Improvement can only be obtained by using non-linear approximations. They still contain the bias caused by the replacement of BJ1FGAME, BJ1FSTRT by BJ1IGAME, BJ1IGAME. In table 24 we find $\mu_G(\widehat{H}_{opt}, S_{opt}) = 0.102$ for $B_{\max} = 50, \lambda = 2/3$. Adding the bias of 0.01 this leads to the value 0.11 in table 5.

Figure 25. Est. μ_G for S_{bas} as function of $R_{\varphi\psi}(B_{\max} = 1, 2, 25, 37.5$ and $50; \lambda = 1/2)$

Figure 26. Est. μ_G for S_{bas} as function of $R_{\varphi\psi}(B_{\max} = 1, 2, 25, 37.5$ and $50; \lambda = 2/3)$



For the construction of approximations Ψ of Φ we only consider S_{bas}. From (43) and (45) it follows that the estimated expected gain μ_G of a particular Ψ-approximation only depends on the correlationcoefficient $R_{\varphi\psi}$. Figure 25 gives the estimated μ_G as a function of $R_{\varphi\psi} \in [0.5, 1]$ for $\lambda = 1/2$ and for $B_{\max} = 1, 2, 25, 37.5$ and 50. Figure 26 does the same for $\lambda = 2/3$.

The figures show clearly the strong increase of the expected gain by using Ψ-approximations of Φ with increasing correlationcoefficient $R_{\varphi\psi}$. We consider here only two famous practical (centered) card counting systems: TTC (*Thorp's Ten Count*) and HiLo (*High-Low*). Table 27 gives the definition of both systems.

Table 27. Ψ-values of TTC and HiLo

card	1	2	3	4	5	6	7	8	9	10
TTC	4	4	4	4	4	4	4	4	4	−9
HiLo	−1	1	1	1	1	1	0	0	0	−1

The TTC-system is the most simple card counting system. Its true count $T = T(t)$ can easily expressed as a function of the so-called T-ratio $TR = TR(t)$ of a stock: the number of non-tens divided by the number of tens. Clearly, for a given stock $c = (c(1), \dots, c(10))$ with $\sum c(i) = k(1-t)$ we get for the true count

$$\begin{aligned} T &= \{4\textstyle\sum_{i=1}^{9}(p_i k - c(i)) - 9(p_{10}k - c(10))\}/\{k(1-t)\} = \\ &= 13c(10)/\textstyle\sum c(i) - 4 = 13/(1+TR) - 4 \end{aligned}$$

or

$$T = \frac{9 - 4TR}{1 + TR}, \quad TR = \frac{9 - T}{4 + T}.$$

In relation to the Φ-values of the basic strategy, table 28 follows with (41) and table 22, 27.

Table 28. (Φ, Ψ)-values of counting systems with respect to S_{bas}

	$R_{\varphi\psi}$	t_0	
TTC	0.66	+0.248	$TR_0 = 2.06$
HiLo	0.96	+0.0248	$52t_0 = +1.3$

With tables 23, 24, 28 and figures 25, 26 this leads to the following expected gains in table 29.

Table 29. Est. μ_G of counting systems for S_{bas} $(n = 6)$

	$B_{\max}$	1	2	25	37.5	50
$\lambda = \frac{1}{2}$	$\widehat{H}_{TTC}$	−0.0061	−0.0058	0.0015	0.0055	0.010
	$\widehat{H}_{HiLo}$	−0.0061	−0.0053	0.014	0.024	0.034
	$\widehat{H}_{bas}$	−0.0061	−0.0052	0.016	0.027	0.038
$\lambda = \frac{2}{3}$	$\widehat{H}_{TTC}$	−0.0061	−0.0055	0.0092	0.017	0.025
	$\widehat{H}_{HiLo}$	−0.0061	−0.0047	0.028	0.046	0.063
	$\widehat{H}_{bas}$	−0.0061	−0.0046	0.030	0.049	0.068

Clearly, the performance of the HiLo-system is much better than that of the TTC-system. However, TCC is much easier to use in practice.

As a final check we performed a simulation for $\lambda = 2/3,\ B_{\max} = 50$ with the Ψ-values of TTC and HiLo and the underlying Φ. Table 30 gives the results.

Table 30. Sim. gains of counting systems for S_{bas} $(n = 6, \lambda = 2/3,\ B_{\max} = 50)$
(M = 50,000,000 rowgames - $\mu_N = 9.86$)

betfunction	μ_{GR}	$\pm$ 95%	μ_G
$\widehat{H}_{TTC}$ $(TR_0 = 2.06)$	0.294	± 0.02	0.0299
$\widehat{H}_{HiLo}$ $(52t_0 = 1.3)$	0.687	± 0.02	0.0698
$\widehat{H}_{bas}$ $(t_0 = -0.0061)$	0.733	± 0.02	0.0744

Comparing table 30 with $B_{\max} = 50, \lambda = 2/3$ in table 29, we see again the bias in the estimates.

11 Card counting systems for playing

Card counting systems are also used for playing decisions. For BJHC the number of decks $n = 6$ is large and therefore playing decisions different from the basic strategy can only increase the expected gain by a small amount.

We only consider playing decisions for the TTC and HiLo systems. A more systematic approach using linear approximations based on expected differences in gains for individual decisions will be left for further research.

Table 31 gives the TTC-playing strategy S_{TTC}. This table has been constructed in the following way (see also Van Der Genugten (1993), p. 143). In the starting stock $c_0 = (c(1), \ldots, c(10)) = (24, \ldots, 24, 96)$ the number of tens $c(10)$ has been varied in the range 1−196, thereby traversing the whole range of interesting T-ratio's. For each stock the optimal decision table has been calculated. (This is the same method as used in Thorp (1966).) In fact table 31 summarizes the whole range. It may be argued that in this way the table only optimizes decisions for $c(1) = \cdots = c(9)$. This is certainly true. However, the method works and can be performed in a simple way.

The table shows clearly the non-linear effect on the expected gains of the decisions as well.

Table 32 gives the HiLo-playing strategy S_{HiLo}. This table has been copied from Wind & Wind (1994), p. 59-61. The precise construction is not identicated there but is based on linear approximations of expected gains for the various decisions. Note that the running count HL is given by the fraction of decks (instead of cards).

Table 31. Playing strategy S_{TTC} (T = Ten Ratio)

INSURANCE: insure if $T \leq 2$

SPLITTING (Split if $T \leq$, if underlined then split it $T \geq$)

Dealercard Pair	A	2	3	4	5	6	7	8	9	T
AA	1.4	4.2	4.5	4.7	5.0	5.3	4.3	3.9	3.7	3.2
22	0	3.0	3.7	∞	∞	∞	∞	$\underline{2.6}$	$\underline{8.0}$	0
33	0	2.3	∞	∞	∞	∞	∞	$\underline{2.4}$	$\underline{5.5}$	0
44	0	1.4	1.6	1.9	2.3	2.4	$\underline{6.2}$	$\underline{7.0}$	0	0
55	0	0	0	0	0	0	0	0	0	0
66	0	2.4	2.7	3.0	3.3	3.7	0	0	0	0
77	0	3.6	4.1	4.8	5.8	∞	∞	2.1	0	0
777	0	3.0	3.3	3.7	4.2	5.3	4.9	0	0	0
88	0	∞	∞	∞	∞	∞	∞	∞	∞	$\underline{2.8}$
99	0	2.6	2.9	3.1	3.5	3.4	1.8	4.1	4.1	0
TT	0	1.4	1.5	1.6	1.7	1.8	0	0	0	0

DOUBLE DOWN (DDown if $T \leq$)

Dealercard Sum	A	2	3	4	5	6	7	8	9	T
H9	0	2.2	2.4	2.6	2.9	3.2	1.8	1.5	0	0
H10	0	3.6	3.9	4.2	4.8	5.1	3.5	3.0	2.5	0
H11	0	3.9	4.1	4.5	5.0	5.4	3.9	3.3	2.8	2.1
S19	0	1.2	1.3	1.4	1.5	1.6	0	0	0	0
S20	0	1.4	1.5	1.6	1.7	1.8	0	0	0	0
S21	0	0	0	0	1.0	1.0	0	0	0	0

DRAW/STAND (Draw if $T \geq$)

Dealercard Sum	A	2	3	4	5	6	7	8	9	T
H12	0	2.0	2.2	2.3	2.4	2.3	0	0	0	1.1
H13	1.2	2.3	2.5	2.7	2.9	2.8	0	0	0	1.3
H14	1.3	2.7	3.0	3.2	3.5	3.3	0	0	1.2	1.6
777	0	2.2	2.4	2.5	2.7	2.7	0	0	0	0
H15	1.4	3.2	3.5	3.9	4.3	4.1	0	0	1.4	1.9
H16	1.5	3.8	4.2	4.7	5.4	5.0	0	1.4	1.8	2.2
H17	3.0	∞	∞	∞	∞	∞	∞	∞	∞	∞
H18	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
S17	0	0	0	0	0	0	0	0	0	0
S18	2.1	5.1	5.5	6.2	6.6	∞	∞	5.1	0	0
S19	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

Table 32 Playing strategy S_{HiLo} (HL = 52 $\times$ true count)

INSURANCE: insure if $HL \geq 3$

SPLITTING (Split if $HL >$)

Dealercard Pair	A	2	3	4	5	6	7	8	9	T
AA	$+\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	-9
22	$+\infty$	$-3^1/_2$	-6	-8	$-9^1/_2$	$-\infty$	$-\infty$	$+4^1/_2$	$+\infty$	$+\infty$
33	$+\infty$	0	-5	-8	$-9^1/_2$	$-\infty$	$-\infty$	$+5^1/_2$	$+\infty$	$+\infty$
44	$+\infty$	$+\infty$	$+8$	$+3$	$-^1/_2$	$-1^1/_2$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
55	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
66	$+\infty$	-2	$-4^1/_2$	$-6^1/_2$	$-8^1/_2$	$-\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
7(7)7	$+\infty$	-7	$-8^1/_2$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$+\infty$	$+\infty$	$+\infty$
88	$+\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$+\infty$
99	$+\infty$	-3	$-4^1/_2$	$-5^1/_2$	-7	$-7^1/_2$	$+3$	-9	$-\infty$	$+\infty$
TT	$+\infty$	$+\infty$	$+8^1/_2$	$+6^1/_2$	$+5$	$+4^1/_2$	$+\infty$	$+\infty$	$+\infty$	$+\infty$

DOUBLE DOWN (DDown if $HL >$)

Dealercard Sum	A	2	3	4	5	6	7	8	9	T
H9	$+\infty$	$+1$	-1	-3	$-4^1/_2$	$-^1/_2$	$+3^1/_2$	$+7^1/_2$	$+\infty$	$+\infty$
H10	$+\infty$	-9	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-6^1/_2$	$-4^1/_2$	$-1^1/_2$	$+\infty$
H11	$+\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-7^1/_2$	-5	$+3^1/_2$
S19	$+\infty$	$+\infty$	$+\infty$	$+9$	$+6^1/_2$	$+7$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
S20	$+\infty$	$+\infty$	$+8^1/_2$	$+6^1/_2$	$+5$	$+4^1/_2$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
S21	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+8$	$+8$	$+\infty$	$+\infty$	$+\infty$	$+\infty$

DRAW/STAND (Draw if $HL \leq$)

Dealercard Sum	A	2	3	4	5	6	7	8	9	T
H12	$+\infty$	$+3$	$+1^1/_2$	0	$-1^1/_2$	-1	$+\infty$	$+\infty$	$+\infty$	$+\infty$
H13	$+\infty$	-1	$-2^1/_2$	-4	$-5^1/_2$	$-5^1/_2$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
H14(777)	$+\infty$	-4	-5	$-6^1/_2$	-8	$-8^1/_2$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
H15	$+9^1/_2$	-6	-7	$-8^1/_2$	$-9^1/_2$	$-\infty$	$+\infty$	$+\infty$	$+8$	$+4^1/_2$
H16	$+8^1/_2$	$-9^1/_2$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$+8$	$+7$	$+4^1/_2$	0
H17	$-6^1/_2$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
H18	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
S17	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
S18	$+1$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$+\infty$	$+\infty$
S19	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$

We studied the effect of S_{bas}, S_{TTC} and S_{HiLo} in combination with various betfunctions. Based on table 28 and some further simulations with S_{TTC} and S_{HiLo} we took finally the (modified) simple bounds:

for $\hat{H}_{TTC}$: $t_0 = +0.194$ $(TR_0 = 2.1)$

for $\hat{H}_{HiLo}$: $t_0 = +0.0288$ $(52t_0 = 1.5)$

These betfunctions always take the maximum bet $B_{\max}$ for appropriate values of T and HL respectively. This leads to a large variance in the gain. So this can only be played in practice by high budget players with a large starting capital. For low budget players with a low (or moderate) capital other betfunctions come into view. Therefore we consider also two low budget betfunctions $\tilde{H}_{TTC}$ and $\tilde{H}_{HiLo}$ specified in table 33.

Table 33. Betting of low and high budget players ($B_{\max}$ = 50)

	TTC		HiLo		
Class	$\hat{H}_{TTC}$ (High)	$\tilde{H}_{TTC}$ (Low)	Class	$\hat{H}_{HiLo}$ (High)	$\tilde{H}_{HiLo}$ (Low)
$T > 2.3$	1	1	HL< 1/2	1	1
$2.1 < T \leq 2.3$	1	1	$1/2 \leq HL < 1$	1	1
$2.0 < T \leq 2.1$	50	2	$1 \leq HL < 2$	50	5
$1.9 < T \leq 2.0$	50	6	$2 \leq HL < 3$	50	10
$1.8 < T \leq 1.9$	50	9	$3 \leq HL < 4$	50	15
$1.7 < T \leq 1.8$	50	13	$4 \leq HL < 5$	50	20
$T \leq 1.7$	50	21	HL ≥ 5	50	25

The betting strategy $\tilde{H}_{TTC}$ aims to minimize the probability of ruin starting with a moderate capital. This probability can be found approximately by finding the negative root of $M(t) = 1$, where M is the moment generating function of the gain of one rowgame. For details we refer to Van der Genugten (1993), §3.1.6, p. 150-151.

The betting strategy $\tilde{H}_{HiLo}$ is proposed by Wind and Wind (1994) §6.3, table 24, p. 50. It is a conservative interpretation of the Kelly-principle to choose the bet in such a way that the expected growth of one's capital is maximized.

We made several simulation runs to obtain the performance of these strategies. Table 34 gives the results.

Table 34. Simulated gains $(n = 6, a = 7, \lambda = 2/3, B_{\max} = 50)$
(M = 50,000,000 rowgames $-\mu_N = 9.98$ games)

Strategy	μ_{GR}	σ_{GR}	$\pm 95\% CI$	μ_G	μ_B
$(\widehat{H}_{HiLo}, S_{HiLo})$	+0.918	87.7	0.027	+0.0931	14.7
$(\widehat{H}_{HiLo}, S_{bas})$	+0.702	86.5	0.027	+0.0711	14.4
$(\widehat{H}_{TTC}, S_{TTC})$	+0.569	75.5	0.023	+0.0576	11.2
$(\widehat{H}_{TTC}, S_{bas})$	+0.321	73.8	0.023	+0.0326	10.8
$(\tilde{H}_{HiLo}, S_{HiLo})$	+0.240	20.9	0.006	+0.0243	3.60
$(\tilde{H}_{HiLo}, S_{bas})$	+0.171	20.1	0.006	+0.0174	3.51
$(\tilde{H}_{TTC}, S_{TTC})$	+0.062	11.2	0.003	+0.0063	2.02
$(\tilde{H}_{TTC}, S_{bas})$	+0.009	10.7	0.003	+0.0009	1.97
$(H \equiv 1, S_{TTC})$	−0.0425	3.51	0.0011	−0.0043	1.11
$(H \equiv 1, S_{HiLo})$	−0.0456	3.50	0.0011	−0.0046	1.10
$(H \equiv 1, S_{bas})$	−0.0524	3.50	0.0011	−0.0053	1.10

In this table we have also given the standard deviation σ_{GR} of one rowgame and the mean bet μ_B of one game.

We see that the HiLo-system is better (but also more complicated) than the TTC-system. There is a large difference between the high-budget systems $\widehat{H}$ and the low-budget systems $\tilde{H}$.

Appendix

For $n = 1, 2, \dots$ consider a stock of k_n cards with $k_n \to \infty$. Let $x_{ni} \in \mathbb{R}^n$ denote the score vector belonging to card i such that

$$\sum_{i=1}^{k_n} x_{ni} = 0, \sum_{i=1}^{k_n} x_{ni}x'_{ni} = I_q, \max_{1\le i\le k_n} |x_{n_i}| \to 0.$$

Consider drawing without replacement from this stock. Let X_{ni} be the score of the i^{th} drawn card and set

$$Z_n(t) = \sum_{i=1}^{[k_n t]} X_{ni}, \quad 0 \le t \le 1.$$

Then

$$Z_n(t) \Rightarrow W_q(t), \;\; 0 \le t \le 1, \;\; n \to \infty,$$

where $(W_q(t), 0 \le t \le 1)$ denotes the q-product of the Brownian bridge. (See Billengsly (1968), theorem 24.1 for $q = 1$; the general case easily follows by considering linear combinations).

We apply Billingsly's theorem for deriving (36). Set $k_n = 52n$ and let

$$x_{ni} = \frac{1}{\sqrt{k_n}}\Sigma_\psi^{-1}(\psi_j - \mu_\psi), \;\; i \in K_j, \; j = 1, \dots, 10,$$

where K_j denotes the set of all cards with card value j. Then

$$\sum_{i=1}^{k_n} x_{ni} = \tfrac{1}{\sqrt{k_n}}\Sigma_\psi^{-1}\sum_{j=1}^{10}(k_n p_j)(\psi_j - \mu_\psi) = 0$$

$$\sum_{i=1}^{k_n} x_{ni}x'_{ni} = \tfrac{1}{k_n}\Sigma_\psi^{-1}(\sum_{j=1}^{10}(k_n p_j)(\psi_j - \mu_\psi)(\psi_j - \mu_\psi)')\Sigma_\psi^{-1} = I_q$$

$$\max_{1\le i\le k_n} |x_{ni}| \le \tfrac{1}{\sqrt{k_n}}||\Sigma_\psi^{-1}||_\infty \max_{1\le j\le 10} |\psi_j - \mu_\psi| \to 0.$$

Therefore, $Z_n(t) \Rightarrow W_q(t), \; 0 \le t \le 1$. In particular, for fixed t:

$$W_q(t) \overset{\mathcal{L}}{=} (1-t)B_q(\frac{t}{1-t}) \overset{\mathcal{L}}{=} \sqrt{t(1-t)}Z_q,$$

where B_q denotes Brownian motion. Substitution leads to

$$Z_n(t) = \frac{1}{\sqrt{k_n}}\Sigma_\psi^{-1}(S(t) - [k_n t]\mu_\psi)$$

or

$$S(t) = \sqrt{k_n}\Sigma_\psi Z_n(t) + [k_n t]\mu_\psi \overset{\mathcal{L}}{\approx} \sqrt{k_n}\Sigma_\psi\sqrt{t(1-t)}Z_q + k_n t\mu_\psi.$$

Division by $k_n(1-t)$ leads with (25) to (36) with the factor k_n instead of $k_n - 1$. The replacement with $k_n - 1$ has been made to get a full agreement with the variance of the hypergeometric distribution from the trivial counting system by considering only one card value.

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