Reading Notes for
Category Theory
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0.1 Notes

All exercises that are not written here should be considered TODO.

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1 Categories

1.1 Introduction

1.2 Functions of sets

1.3 Definition of a category

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1

Functions with fibers of at most 2 elements Assume

$$\forall A, B \in \mathbf{C}\, \forall (f\colon A \to B) \left[\forall b \in B \Big(\big|f^{-1}(b)\big| \leq 2 \Big) \iff f \in \mathrm{Hom}_{\mathbf{C}}(A, B) \right].$$

Given that

$$\forall B \in \mathbf{C} \forall b \in B \left[\big|1_B^{-1}(b)\big| = 1 \leq 2 \right],$$

we know

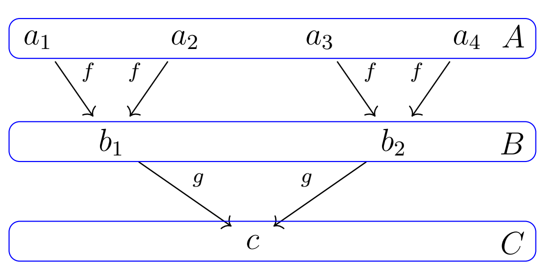
$$\forall B \in \mathbf{C} \left[1_B \in \mathrm{Hom}_{\mathbf{C}}(B, B) \right].$$

Given $\forall A, B, C \in \mathbf{C}, \forall (f\colon A \to B)\, \forall (g\colon B \to C)$

$$\forall b \in B \left(\big|f^{-1}(b)\big| \leq 2 \right) \wedge \forall c \in C \left(\big|g^{-1}(b)\big| \leq 2 \right) \implies \forall c \in C \left(\big|(g \circ f)^{-1}(c)\big| \leq 4 \right)$$

which means that the composite $g \circ f$ does not necessarily exist in $\mathrm{Hom}_{\mathbf{C}}(A, B)$.

Therefore, this is **not** a category.



Functions with finite fiber Assume

$$\forall A, B \in \mathbf{C}\,\forall(f\colon A \to B)\left[\forall b \in B\big(\left|f^{-1}(b)\right| \in \mathbb{N}\big) \iff f \in \mathrm{Hom}_{\mathbf{C}}(A, B)\right].$$

Given that

$$\forall B \in \mathbf{C} \forall b \in B\left[\left|1_B^{-1}(b)\right| = 1 \in \mathbb{N}\right],$$

we know

$$\forall B \in \mathbf{C}\left[1_B \in \mathrm{Hom}_{\mathbf{C}}(B, B)\right].$$

Given $\forall A, B, C \in \mathbf{C}, \forall(f\colon A \to B)\,\forall(g\colon B \to C)$

$$\forall b \in B\Big(\left|f^{-1}(b)\right| \in \mathbb{N}\Big) \wedge \forall c \in C\Big(\left|g^{-1}(b)\right| \in \mathbb{N}\Big) \iff \forall c \in C\Big(\left|(g \circ f)^{-1}(c)\right| \in \mathbb{N}\Big)$$

since the multiplication of finite numbers are finite, and converse is true. This means that the composite $g \circ f$ must exist in $\mathrm{Hom}_{\mathbf{C}}(A, B)$.

Therefore, this **is** a category.

Functions with infinite fiber Since identity map has finite fiber, this is **not** a category.

2

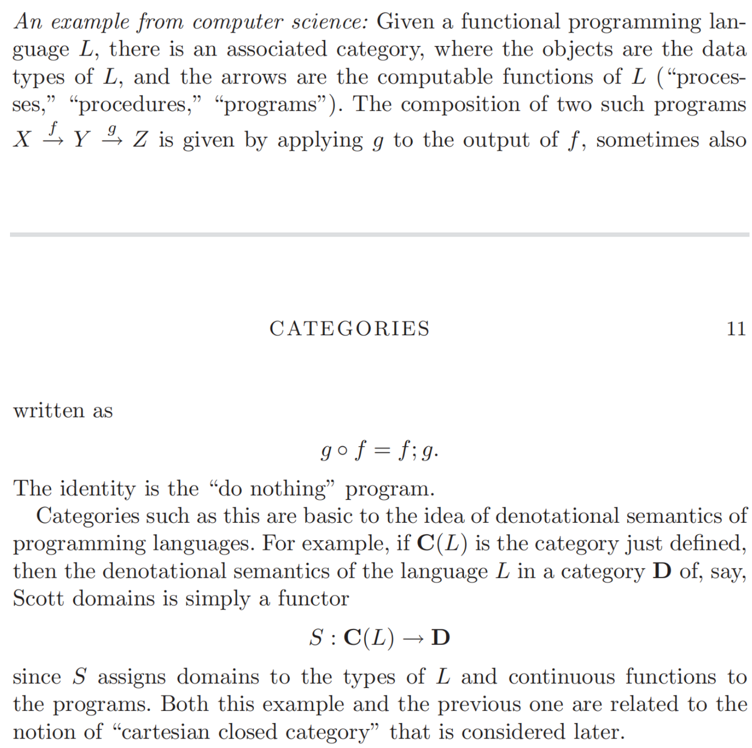
- graphs and graph homomorphisms
- the real numbers R and continuous functions $R \to R$,
- the natural numbers N and all recursive functions $N \to N$, or as in the example of continuous functions, one can take partial recursive functions defined on subsets $U \subseteq N$

11 Category of data types and programs

Let's define a procedure as `Int f(Int a, Int b) return a * b;`

We have built a morphism/arrow

$$\mathtt{f}\colon \mathtt{Int} \times \mathtt{Int} \to \mathtt{Int}.$$



An example from computer science: Given a functional programming language L, there is an associated category, where the objects are the data types of L, and the arrows are the computable functions of L ("processes," "procedures," "programs"). The composition of two such programs $X \xrightarrow{f} Y \xrightarrow{g} Z$ is given by applying g to the output of f, sometimes also

written as

$$g \circ f = f; g.$$

The identity is the "do nothing" program.

Categories such as this are basic to the idea of denotational semantics of programming languages. For example, if $\mathbf{C}(L)$ is the category just defined, then the denotational semantics of the language L in a category $\mathbf{D}$ of, say, Scott domains is simply a functor

$$S : \mathbf{C}(L) \to \mathbf{D}$$

since S assigns domains to the types of L and continuous functions to the programs. Both this example and the previous one are related to the notion of "cartesian closed category" that is considered later.

Figure 1

1.5 Isomorphisms

Definition 1.3 while there are "bijective homomorphisms" between non-isomorphic posets.

The inverse must also be homomorphism?

Theorem 1.6

$$f, h \in \bar{C}, g \circ f, g \circ h \in \bar{D}.$$

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, "g\circ f"] & \\
C \arrow[r, "g"] & D \\
Y \arrow[u, "h"] \arrow[ur, "g\circ h"'] &
\end{tikzcd}
\qquad\qquad
\bar{C} \xrightarrow{\bar{g}} \bar{D}$$

If every category (large or small) was isomorphic to a concrete (small) category, what make those large?

The assumption is wrong. Only small categories are isomorphic to a small category. See this: 1.5.

Also see Warning 1.13 that small does not imply concrete and neither is the converse.

Remark 1.7

Test object What does this mean

> as Dedekind cuts or as Cauchy sequences). A better attempt to capture what is intended by the rather vague idea of a "concrete" category is that arbitrary arrows $f : C \to D$ are completely determined by their composites with arrows $x : T \to C$ from some "test object" T, in the sense that $fx = gx$ for all such x implies $f = g$. As we shall see later, this amounts to considering a particular representation of the category, determined by T. A category is then said to be "concrete" when this condition holds for T a "terminal object," in the sense of Section 2.2; but there are also good reasons for considering other objects T, as we see Chapter 2.

Figure 2

Too many arrows If the arrows in the category is too large to form a set, then it might not be isomorphic to a small category?

1.6 Constructions on categories

3 The arrow category

$$\begin{tikzcd}
A \arrow[r, "g_1"] \arrow[d, "f"] & A' \arrow[r, "h_1"] \arrow[d, "f'"] & A'' \arrow[d, "f''"] \\
B \arrow[r, "g_2"] & B' \arrow[r, "h_2"] & B''
\end{tikzcd}$$

4 Slice category

Definition of $\mathbf{C}/(-)$

$$\mathbf{C}/(-)\colon \mathbf{C} \to \mathbf{Cat}\colon C \to \mathbf{C}/C\colon g \to g_*.$$

Cat to Sets the forgetful functor $U\colon \mathbf{Cat} \to \mathbf{Sets}$ that takes a category to its underlying set of objects.

Do the objects form a set? Guaranteed? Is there a category of categories?

Yes, because **Cat** is defined as the category of *small* categories. There is no category of all categories. Likewise, there is no "set of all sets" or "class of all classes"

But is $\mathbf{C}/C$ a small category? If and only if $\mathbf{C}$ is small?

Principal ideal the slice category P/p is just the "principal ideal" (what does this mean? what is down arrow?) $\downarrow(p)$ of elements $q \in P$ with $q \le p$.

Example 1.8 $1 = \{\,*\,\}$ is mapped to the distinguished point of sets in $\mathbf{Sets}_*$.

1.7 Free categories

Free monoid The Kleene closure of A can be thought as

$$\{\,\text{Empty word}\,\} \sqcup A \sqcup A \times A \sqcup \ldots,$$

where $\times$ is the Cartesian product.

A free monoid can be thought as some monoid isomorphic to the Kleene closure of some set.

A free monoid can be defined as a monoid whose non-identity elements can be uniquely written as a product of its generating elements?

UMP The mapping $i\colon A \to |M(A)|$ must be injection because otherwise there is no monoid homomorphism from $M(A)$ to A^*, the Kleene closure.

Free category Does edges and vertices always form a set?

1.8 Foundations: large, small, and locally small

Definition 1.12

The category of sets is locally small Given two *sets* X and Y, an ordered pair $(x, y) \in X \times Y$ can be viewed as an element in $\mathcal{PP}(X \cup Y)$ (by Kuratowski's definition). Therefore, a function $f\colon X \to Y$ as a subset of $X \times Y$ is an element of $\mathcal{PPP}(X \cup Y)$. In conclusion, the set of all functions from X to Y is a subset of $\mathcal{PPP}(X \cup Y)$, and therefore based on the axiom of replacement and power set, it is a (small) set.

The category of small categories is locally small Since the objects and arrows form two sets, so does their disjoint union.

Therefore, a functor can be viewed as a function between two sets, and therefore **Cat** is locally small.

Non-locally small category Examples

1.9 Exercises

2

a Yes, $\mathbf{Rel} \cong \mathbf{Rel}^{\mathrm{op}}$.

b **Sets** is not isomorphic to its opposite because all terminal objects must be mapped to an initial object as a terminal object in the opposite, than mapped back to a terminal object.

But there are infinite terminal objects in **Sets** (singletons) and only one initial object (the empty set).

c For the whole power set it should be true?

Any subset is mapped to its complement.

This is not necessarily true for a subset of the power set.

3

c There is a bijective monotone function (functor) from a discrete (small) category (which is a poset) to a non-discrete poset category of same objects.

But this is not isomorphism.

5

$$F\colon \mathbf{C}/C \to \mathbf{C}^{\to}$$

is the trivial functor which maps arrows to itself.

6

$$C/\mathbf{C} \cong (\mathbf{C}^{\mathrm{op}}/C)^{\mathrm{op}}.$$

The only difference is the objects are arrows with opposite directions.

$$
\begin{tikzcd}
X \arrow[rr, "a"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\
& C \arrow[dl, "g"] \arrow[dr, "g'"'] & \\
Y \arrow[rr, "b"'] & & Y'
\end{tikzcd}
$$

7 For any positive integer n,

$$F\colon \mathbf{Sets}/n \to \mathbf{Sets}^n,$$

$$(f\colon X \to \{ a_1, \dots, a_n \}) \mapsto (f^{-1}(a_1), \dots, f^{-1}(a_n)).$$

$$(f \to g) \mapsto (f^{-1}(a_1) \to g^{-1}(a_1), \dots, f^{-1}(a_n) \to g^{-1}(a_n)).$$

1. First, the image is contained in the codomain.
2. It is a functor.
3. It is isomorphism?

8 In the preorder category $P(\mathbf{C})$, any information encoded in the arrows is forgotten, only the domain and codomain is kept.

$$
\begin{tikzcd}
A \arrow[loop left, "1_A"] \arrow[loop left, looseness=12, "i_A"'] \arrow[r, bend left, "f_1"] \arrow[r, shift left, "f_2"] & B \arrow[l, shift left, "g"] & C \arrow[l, "h"'] \\
A \arrow[loop left, "1_A"] \arrow[r, shift left, "f'"] & B \arrow[l, shift left, "g'"] & C \arrow[l, "h'"']
\end{tikzcd}
$$

$P(1_A) = P(i_A) = 1_A, P(f_1) = P(f_2) = f'$.

Since any functor preserves domain and codomain, any functor can be mapped as an arrow between two preorder sets.

For example, given two categories $\mathbf{C}, \mathbf{D}$, if the functor F maps *any* arrow in $\mathbf{C}$ $(X \to Y)$ to *an* arrow in $\mathbf{D}$ $(F(X) \to F(Y))$, then the it maps *the* arrow in $P(\mathbf{C})$ $(X \to Y)$ to *the* arrow in $P(\mathbf{D})$ $(F(X) \to F(Y))$.

The identity is preserved since $A \leq A$, the composite is preserved due to the transitivity $A \leq B \wedge B \leq C \implies A \leq C$.

Let I be the inclusion, then $P \circ I = 1$ but $I \circ P \neq 1$.

9

c

$$
\begin{aligned}
a &\to a : 1_a, \\
a &\to b : e, \\
b &\to b : 1_b, \\
b &\to c : f, \\
c &\to c : 1_c, \\
a &\to c : f \circ e, g,
\end{aligned} \tag{1.1}
$$

d The only arrow related to d is 1_d.

$$a \to a : 1_a, (he)^n, (gfe)^n,$$

$$b \to b : 1_b, (eh)^n, (egf)^n,$$

$$c \to c : 1_c, (feg)^n.$$

$$a \to b : e \circ (a \to a); b \to a : (h \vee gf) \circ (b \to b).$$

10 Can you have a directed graph with only 1 vertex, but 6 edges on the same vertex? or only 2 vertices and 3 edges for each direction connecting these?

11

b Assume $\forall A \in \mathbf{Sets}\, \exists! i_A \colon A \to |M(A)|$ as a UMP (existence: 3.4, uniqueness: just pick any one for each A and fix it), then

$$\forall (f \colon A \to B)\, \exists! i_B \circ f\, \exists! M(f)$$

(existential uniqueness of $i_B \circ f$ is given by 2.3) such that the following diagram commutes.

$$
\begin{tikzcd}
\mathbf{Mon} & M(A) \arrow[r, dashed, "\exists! M(f)"] & M(B) \\
\mathbf{Sets} & {|M(A)|} \arrow[r, "|M(f)|"] & {|M(B)|} \\
 & A \arrow[u, "i_A"] \arrow[ur, "i_B \circ f"] \arrow[r, "f"] & B \arrow[u, "i_B"']
\end{tikzcd}
$$

Let $f = 1_A$, and $B = A$. Since $1_{|M(A)|} \circ i_A = i_A \circ 1_A$ and $|1_{M(A)}| = 1_{|M(A)|}$, therefore $M(1_A) = 1_{M(A)}$.

Associativity:

$$\begin{tikzcd}
\textbf{Mon} & M(A) \arrow[r, dashed, "\exists! M(f)"] & M(B) \arrow[r, dashed, "\exists! M(g)"] & M(C) \\
\textbf{Sets} & {|M(A)|} \arrow[r, "{|M(f)|}"] & {|M(B)|} \arrow[r, "{|M(g)|}"] & {|M(C)|} \\
& A \arrow[u, "i_A"] \arrow[ur, "i_B \circ f"'] \arrow[r, "f"'] & B \arrow[u, "i_B"] \arrow[ur, "i_C \circ g"'] \arrow[r, "g"'] & C \arrow[u, "i_C"]
\end{tikzcd}$$

14 TODO

2 Abstract structures

2.1 Epis and monos

Example 2.3

Prove h monic implies $|h|$ monic

$$h\bar{x} \neq h\bar{y} \implies |h|x \neq |h|y,$$

due to UMP of $M(1)$. (If $|h|x = |h|y$, then there exists non-unique maps $h\bar{x}, h\bar{y}$ corresponding it, which contradicts UMP.)

Converse If $f, g : X \to M$ are any distinct homomorphisms, then $|f|, |g| : |X| \to |M|$ are distinct functions. The converse is true, only if there is a monoid homomorphism corresponding to the underlying function. This is to say, function to monoid homomorphism mapping is injective but not surjective.

Example 2.4 Because there is at most 1 arrow between 2 objects!

2.1.1 Sections and retractions

Definition 2.7 A is "smaller" than X.

Functors preserve split epis and split monos, but do not preserve all the epis.

Projective What does projective mean?

Epi into projective object splits Let P be the projective object and $e\colon E \twoheadrightarrow P$ the epi. Since 1_P must exist, therefore by definition of projective object, $\exists m$ such that

$$\begin{tikzcd} & E \arrow[d, two heads, "e"] \\ P \arrow[r, "1_P"'] \arrow[ur, dashed, tail, "\exists m"] & P \end{tikzcd}$$

Therefore e splits and m is mono (by definition of split epi).

More free Projective objects may be thought of as having a more "free" structure, thus permitting "more arrows".

I guess this means: for any epi and any arrow from a projective object sharing the codomain, there must be another arrow from the projective object to the domain of the epi, so there will be "more" arrows than "necessary".

AC vs projective

Proposition 2.1. *If all epis are split, then all objects are projective.*

Proof. Because for any epi $e\colon E \twoheadrightarrow X$ there is a mono $m\colon X \rightarrowtail E$ such that $em = 1_X$, therefore for any $f\colon P \to X$, there must be $\bar{f} = m \circ f$ such that the following diagram commutes.

$$\begin{tikzcd} & E \arrow[dr, two heads, "e"] & \\ P \arrow[r, "f"] \arrow[ur, dashed, "m\circ f"] & X \arrow[u, tail, "m"] \arrow[r, "1_X"] & X \end{tikzcd}$$

∎

It follows that free objects in many (but not all!) categories of algebras then are also projective.

Retract of projective object

Proposition 2.2. *Any retract of a projective object is also projective.*

Proof. Let R be a retract of a projective object P, and let $rs = 1_R$. Then given any epi $e\colon E \twoheadrightarrow X$, and any $f\colon R \to X$, the following diagram must commute.

$$
\begin{tikzcd}
 & & E \arrow[d, two heads, "\forall e"] \\
 & P \arrow[ur, dashed, "\exists\overline{f\circ r}"] \arrow[r, "f\circ r"] \arrow[d, two heads, "r"] & X \\
R \arrow[ur, tail, "s"] \arrow[r, "1_R"] & R \arrow[ur, "\forall f"'] &
\end{tikzcd}
$$

And therefore f lifts across e to $\overline{f \circ r} \circ s$. □

2.2 Initial and terminal objects

Example 2.11

3 Rings In **Rings** (commutative with unit), the ring $\mathbb{Z}$ of integers is initial.

For any finite rings, they might not be homomorphic to other finite rings of different size (e.g., $\mathbb{F}_q$).

$\mathbb{Z}$ is the smallest infinite ring. Since 0 must be mapped to 0 and so is 1, there exists a unique ring homomorphism from $\mathbb{Z}$.

2.3 Generalized elements

Ultrafilter a filter F is an ultrafilter just if for every element $b \in B$, either $b \in F$ or $\neg b \in F$, and not both (exercise!).

Prime ideal Ring homomorphisms $A \to Z$ into the initial ring Z play an analogous and equally important role in algebraic geometry. They correspond to so-called prime ideals, which are the ring-theoretic generalizations of ultrafilters.

Example 2.12

3

$$f = g \iff f1_C = g1_C.$$

Example 2.13 Hom$(X, -)$ is always a functor, and functors always preserve isos.

Example 2.14

Natural number is the revealing object

$$\begin{tikzcd}
\textbf{Mon} & M(1) \arrow[r, dashed, "\exists \bar{x}"] & M \\
\textbf{Sets} & {|M(1)|} \arrow[r, "|\bar{x}|"] & {|M|} \\
& \{*\} \arrow[u, "i"] \arrow[ur, "\forall x"'] &
\end{tikzcd}$$

For any $x \in |M|$, there is a function $x\colon * \mapsto x$. To make the diagram commute, $|\bar{x}|\big(i(*)\big) = x$.

This uniquely corresponds to a monoid homomorphism $\bar{x}\colon i(*) \mapsto x$. Therefore, every element in $|M|$ can be reached by a monoid homomorphism from $M(1)$. And $\mathbb{N}$ is isomorphic to $M(1)$.

Bijection From above, we clearly see that different x correspond to different monoid homomorphism $\bar{x}\colon i(*) \mapsto x$, and exactly one for each. Therefore

$$|\mathrm{Hom}_{\mathbf{Sets}}(1, |M|)| \leq |\mathrm{Hom}_{\mathbf{Mon}}(M(1), |M|)|\,.$$

On the other hand, for any monoid homomorphism $f\colon M(1) \to M$, there is uniquely $x = |f| \circ i$ which makes the diagram commute. Different f must correspond to different x, otherwise it contradicts the UMP axiom.

Universal Property

Bijection in general

Proposition 2.3. *Given a commutative diagram in category* **C**

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

there exists a unique commutative diagram in $\mathbf{C}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dashed, "\exists! h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

where h *is uniquely determined as* $g \circ f$*.*

Proof. Given the definition of commutative diagram, if h exist, it must be $g \circ f$ (uniqueness).

Given the axiom of category, $g \circ f$ must exist if both f and g exist (existence). $\square$

Remark. Therefore, such edge can be arbitrarily added or deleted without loss of generality.

Remark. Note that g is uniquely determined by f and h if and only if f is epi.

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[dr, "h"'] & B \arrow[d, dashed, "\exists g \Longrightarrow \exists! g"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, dashed, "\exists f \Longrightarrow \exists! f"] \arrow[dr, "h"'] & B \arrow[d, tail, "g"] \\ & C \end{tikzcd}$$

Definition (universal property)**.** Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor between categories $\mathbf{C}$ and $\mathbf{D}$, and let $X \in \mathrm{ob}(\mathbf{D}), A, A' \in \mathrm{ob}(\mathbf{C})$.

A **universal morphism from** X **to** F is a unique pair $(A, u\colon X \to F(A))$ in D such that any morphism of the form $f\colon X \to F(A')$ in D, there *exists a unique* morphism $h\colon A \to A'$ in C such that $f = F(h) \circ u$, i.e., the following diagram commutes:

$$\begin{tikzcd} \mathbf{C} & A \arrow[r, dashed, "\exists! h"] & A' \\ \mathbf{D} & F(A) \arrow[r, dashed, "\exists F(h)"] & F(A') \\ & X \arrow[u, "u"] \arrow[ur, "\forall f"'] & \end{tikzcd}$$

A **universal morphism from F to X** is a unique pair $(A, u\colon F(A) \to X)$ in D such that any morphism of the form $f\colon F(A') \to X$ in D, there *exists a unique* morphism $h\colon A' \to A$ in C such that $f = u \circ F(h)$, i.e., the following diagram commutes:

$$\begin{tikzcd}
\mathbf{C} & A & A' \arrow[l, dashed, "\exists! h"'] \\
\mathbf{D} & F(A) \arrow[d, "u"'] & F(A') \arrow[l, dashed, "\exists F(h)"'] \arrow[dl, "\forall f"] \\
& X &
\end{tikzcd}$$

Proposition 2.4. *Given two locally small categories $\mathbf{C}$ and $\mathbf{D}$, if $(A, u\colon X \to F(A))$ is a universal morphism from X to F, then $\forall A' \in \mathrm{ob}(\mathbf{C})$ there is a bijection for hom-sets*

$$\mathrm{Hom}_{\mathbf{C}}(A, A') \cong \mathrm{Hom}_{\mathbf{D}}(X, F(A')).$$

Proof. From the definition of universal morphism, there is a unique $h \in \mathrm{Hom}_{\mathbf{C}}(A, A')$ corresponding each $f \in \mathrm{Hom}_{\mathbf{D}}(X, F(A'))$, and let the mapping be

$$G\colon\ \mathrm{Hom}_{\mathbf{D}}(X, F(A')) \to \mathrm{Hom}_{\mathbf{C}}(A, A')\colon\ f \to h$$

such that the definition diagram commutes.

Conversely, for any $g \in \mathrm{Hom}_{\mathbf{C}}(A, A')$, there exists a unique morphism $F(g) \circ u$ (2.3) such that the definition diagram commutes, therefore

$$G'\colon\ \mathrm{Hom}_{\mathbf{C}}(A, A') \to \mathrm{Hom}_{\mathbf{D}}(X, F(A'))\colon\ g \to F(g) \circ u$$

is another mapping.

Since the diagram commutes, they must be inverse, i.e.,

$$GG' = 1, G'G = 1.$$

$\square$

Remark. Different h corresponds to different f because otherwise it contradicts the universal property.

Different f corresponds to different h because the composite $F(g) \circ u$ is uniquely determined by its components $F(g)$ and u. And there is no other morphism $X \to F(A')$ that makes the diagram commute (2.3).

Corollary 2.5. *Given two locally small categories $\mathbf{C}$ and $\mathbf{D}$, if $(A, u\colon F(A) \to X)$ is a universal morphism from F to X, then $\forall A' \in \mathrm{ob}(\mathbf{C})$ there is a bijection for hom-sets*

$$\mathrm{Hom}_{\mathbf{C}}(A', A) \cong \mathrm{Hom}_{\mathbf{D}}(F(A'), X).$$

Properties of universal morphism See definition here: 2.3.

Proposition 2.6. *Given categories* $\mathbf{C}, \mathbf{D}$*, a functor* $F\colon \mathbf{C} \to \mathbf{D}$*, and a universal morphism from* X *to* F

$$\big(A, u\colon X \to F(A)\big);$$

if

$$\exists f\colon X \rightarrowtail F(B),$$

then u *is monic.*

The dual property claims that given a universal morphism from F *to* X

$$\big(A, u\colon F(A) \to X\big);$$

if

$$\exists f\colon X \twoheadrightarrow F(B),$$

then u *is epic.*

Proof. See 2.8.

If the following diagram commutes, then $i = j$.

$$\begin{tikzcd}
 & F(A) \arrow[r, dashed, "\exists! F(h)"] & F(B) \\
Y \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & X \arrow[u, tail, "u"] \arrow[ur, tail, "f"'] &
\end{tikzcd}$$

$\square$

Remark. There exist non-monic universal morphism from X to F. Let $A = \{\,0\,\}$ be the only object in $\mathbf{C}$, $\mathbf{D}$ be a subcategory of **Sets** with A and $X = \{\,-1, 1\,\}$ as objects and functions as arrows. Define the functor $F\colon \mathbf{C} \to \mathbf{D}$ as $F(A) = A \in \mathbf{D}$. Then a universal morphism from X to F $(A, u\colon X \to A)$ is not monic:

$$\begin{tikzcd}
X \arrow[loop left, "1_X"] \arrow[loop, out=150, in=210, distance=4em, "-1_X"'] \arrow[r, "\exists! u"] & A = F(A) \arrow[loop right, "\exists! 1_A = F(1_A)"]
\end{tikzcd}$$

Corollary 2.7. *Given categories* $\mathbf{C}, \mathbf{D}$*, a functor* $F\colon \mathbf{C} \to \mathbf{D}$*. Let*

$$\big(A, u\colon F(B) \to F(A)\big)$$

be a universal morphism from $F(B)$ to F, then u is monic.

Let

$$\big(A, u\colon F(A) \to F(B)\big)$$

be a universal morphism from F to $F(B)$, then u is epic.

Proof. $\exists 1_{F(B)}\colon F(B) \to F(B)$ that is both epic and monic. And 2.6. $\square$

Corollary 2.8. *Let $U\colon$ **Mon** $\to$ **Sets** be the forgetful functor, then for any universal morphism from a set X to U*

$$\big(A, i\colon X \to U(A)\big),$$

i is monic (injective).

Proof. For any set X, its power set $\mathcal{P}(X)$ is also a set (axiom of power set), as well as a monoid under intersection (or union or symmetric difference).

There is an injection $f\colon X \rightarrowtail \mathcal{P}(X)\colon x \mapsto \{x\}$.
i is monic followed by 2.6. $\square$

Proposition 2.9. *Given categories $\mathbf{C}, \mathbf{D}$, a functor $F\colon \mathbf{C} \to \mathbf{D}$, and a universal morphism from X to F*

$$\big(A, u\colon X \to F(A)\big)$$

or a universal morphism from F to X

$$\big(A, u\colon F(A) \to X\big);$$

then

$$\forall f\colon A \to A[F(f) = 1_{F(A)} \iff f = 1_A].$$

Proof. $\Longleftarrow$: required by the definition of functor.

$\Longrightarrow$: (wlog, consider universal morphism from X to F)

$$\begin{tikzcd}
\mathbf{C} & A \arrow[r, dashed, "\exists! f"] & A \\
\mathbf{D} & F(A) \arrow[r, "F(f)"] & F(A) \\
& X \arrow[u, "u"] \arrow[ur, "u"'] &
\end{tikzcd}$$

Since $f = 1_A$ makes the above diagram commute, based on UMP,

$$\forall g\colon A \to A[F(g) = F(f) = 1_{F(A)} \implies g = f = 1_A].$$

$\square$

Remark. However, it is possible that

$$\exists f\colon A \to A[f \neq 1_A].$$

Proposition 2.10. *Given categories* $\mathbf{C}, \mathbf{D}$*, a functor* $F\colon \mathbf{C} \to \mathbf{D}$*, and a universal morphism from* X *to* F

$$\big(A, u_A\colon X \to F(A)\big);$$

then $\exists u_B$ *such that*

$$\big(B, u_B\colon X \to F(B)\big)$$

is a universal morphism from X *to* F *iff* $A \cong B$ *in* $\mathbf{C}$.

The dual property claims that given a universal morphism from F *to* X

$$\big(A, u_A\colon F(A) \to X\big);$$

then $\exists u_B$ *such that*

$$\big(B, u_B\colon F(B) \to X\big)$$

is a universal morphism from F *to* X *iff* $A \cong B$ *in* $\mathbf{C}$.

Proof. Necessity: UMP determines objects up to isomorphism.

$$\begin{tikzcd}
\mathbf{C} & A \arrow[loop left, "\exists! g\circ f"] \arrow[rr, dashed, shift left, "\exists! f"] & & B \arrow[loop right, "\exists! f\circ g"] \arrow[ll, dashed, shift left, "\exists! g"] \\
\mathbf{D} & F(A) \arrow[loop left, "1_{F(A)}"] \arrow[rr, shift left, "\exists F(f)"] & & F(B) \arrow[loop right, "1_{F(B)}"] \arrow[ll, shift left, "\exists F(g)"] \\
& & X \arrow[ul, "u_A"] \arrow[ur, "u_B"'] &
\end{tikzcd}$$

Per UMP of u_A, $\exists! f$ such that the red diagram commutes:

$$F(f) \circ u_A = u_B.$$

Per UMP of u_B, $\exists! g$ such that the blue diagram commutes:

$$F(g) \circ u_B = u_A.$$

Therefore we have (by plugging in the above equations and applying the axiom of functor):

$$
\begin{aligned}
1_{F(A)} \circ u_A = u_A = F(g) \circ \big(F(f) \circ u_A\big) = F(g \circ f) \circ u_A, \\
1_{F(B)} \circ u_B = u_B = F(f) \circ \big(F(g) \circ u_B\big) = F(f \circ g) \circ u_B,
\end{aligned}
\tag{2.1}
$$

Per UMP of u_A and u_B (2.9), we have

$$g \circ f = 1_A, f \circ g = 1_B.$$

Sufficiency: Given any isomorphism $f\colon A \xrightarrow{\sim} B$ in $\mathbf{C}$ (iso is also epic), the following diagram commutes for any object C in $\mathbf{C}$ and any arrow $c\colon X \to F(C)$ in $\mathbf{D}$.

$$
\begin{tikzcd}
\mathbf{C} & A \arrow[r, two heads, "f"'] \arrow[rr, bend left, "\exists! h"] & B \arrow[r, dashed, "\exists! g"'] & C
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{D} & F(A) \arrow[r, "F(f)"] \arrow[rr, bend left, "F(h)"] & F(B) \arrow[r, dashed, "\exists F(g)"] & F(C) \\
& & X \arrow[ul, "u_A"] \arrow[u, "\exists u_B"'] \arrow[ur, "\forall c"'] &
\end{tikzcd}
$$

where $u_B = F(f) \circ u_A$.

Per UMP of u_A, the red diagram commutes. Since f is epic, g is uniquely determined by f and h. Since f is isomorphism, g exists as $h \circ f^{-1}$. Therefore $\exists! g$ such that the diagram commutes, i.e., (B, u_B) is universal morphism. □

Remark. There might be different isomorphism between A and B, but only one such that the diagram commutes with u_A and u_B.

Generally $F(f \circ g) = 1_{F(B)}$ does not imply $f \circ g = 1_B$. Consider a small category $\mathbf{C}$ containing two objects:

$$A = \{\, 0, 1 \,\}, B = \{\, 0, 1, 2 \,\},$$

and arrows

$$
\begin{aligned}
i_A &\colon A \to B \colon i \mapsto i, \\
f_1 &\colon B \to A \colon i \mapsto i \bmod 2, \\
f_2 &\colon B \to A \colon i \mapsto \min(1, i),
\end{aligned}
$$

and identity arrows and composites. Construct a (preorder) category $\mathbf{D}$ with functor $F\colon \mathbf{C} \to \mathbf{D}$ as

$$\begin{tikzcd}
F(A) \arrow[rr, shift left, "F(i_A)"] & & F(B) \arrow[ll, shift left, "F(f_1)=F(f_2)"] \\
& X \arrow[ul, "u_A"] \arrow[ur] &
\end{tikzcd}$$

Even though (A, u_A) is a UMP, we still have $F(f_1)F(i_A) = 1_{F(B)}$, $f_1 i_A \neq 1_B$.

Initial and terminal objects are defined by universal property Given a category $\mathbf{C}$, define a category $\mathbf{D}$ whose objects are equivalent to objects in $\mathbf{C}$. Define $\mathbf{D}$ such that there is *exactly one* arrow between any two objects in $\mathbf{D}$ (easy to prove $\mathbf{D}$ is a valid category, see 1.9). Define a functor $F\colon \mathbf{C} \to \mathbf{D}$ such that for any arrow $f\colon A \to B$ in $\mathbf{C}$, the corresponding arrow is *the* arrow $F(f)\colon F(A) \to F(B)$ in $\mathbf{D}$ (easy to prove F is a functor if $\mathbf{D}$ is a valid category).

Then an initial object I of $\mathbf{C}$ is defined by a universal morphism from $F(I)$ to F: $(I, 1_{F(I)}\colon F(I) \to F(I))$.

$$\begin{tikzcd}
\mathbf{C} & & I \arrow[r, dashed, "\exists! h"] & A' \\
\mathbf{D} & & F(I) \arrow[loop left, "1_{F(I)}"] \arrow[r, dashed, shift left, "\exists! F(h)"] \arrow[r, shift right, "\forall f"'] & F(A')
\end{tikzcd}$$

In the definition (2.3), $X = F(I), A = I, u = 1_{F(I)}$.

Because for any object A' in $\mathbf{C}$, there is *exactly one* arrow $f\colon X = F(I) \to F(A')$ in $\mathbf{D}$, so there exists *exactly one* arrow from $A = I$ to any object A' in $\mathbf{C}$ (2.4).

A terminal object T of $\mathbf{C}$ is defined by a universal morphism from F to $F(T)$: $(T, 1_{F(T)}\colon F(T) \to F(T))$

$$\begin{tikzcd}
\mathbf{C} & & T & A' \arrow[l, dashed, "\exists! h"'] \\
\mathbf{D} & & F(T) \arrow[loop left, "1_{F(T)}"] & F(A') \arrow[l, dashed, shift right, "\exists! F(h)"'] \arrow[l, shift left, "\forall f"]
\end{tikzcd}$$

In the definition (2.3), $X = F(T), A = T, u = 1_{F(T)}$.

Because for any object A' in $\mathbf{C}$, there is *exactly one* arrow $f\colon F(A') \to X = F(T)$ in $\mathbf{D}$, so there exists *exactly one* arrow from any object A' to $A = T$ in $\mathbf{C}$ (2.5).

2.4 Products

Arrows out of the product To be sure, they are related to the notion of an "exponential" Y^B, via "currying" $\lambda f : A \to Y^B$; we discuss this further in Chapter 6.

Product is defined by universal property

Cumbersome definition Given a category $\mathbf{C}$, and a product diagram in $\mathbf{C}$

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

define a copy $\mathbf{D}$ of $\mathbf{C}$, with one additional object, denoted (A, B). The only arrow from (A, B) to itself is the identity arrow $1_{(A,B)}$. There is no arrow from (A, B) to other different objects in $\mathbf{D}$. For any object C in $\mathbf{C}$, and any arrows $f : C \to A, g : C \to B$ in $\mathbf{C}$, define an arrow $(f, g): C \to (A, B)$ in $\mathbf{D}$. If, for example, there are 3 arrows $C \to A$, and 2 for $C \to B$ in $\mathbf{C}$, then there will be 6 for $C \to (A, B)$ in $\mathbf{D}$.

We now define the composite to make $\mathbf{D}$ a category. For any $h : D \to C$ in $\mathbf{C}$ and any $(f, g): C \to (A, B)$ in $\mathbf{D}$, $(f, g) \circ h := (f \circ h, g \circ h)$ is an arrow from D to (A, B) because both $f \circ h : D \to A$ and $g \circ h : D \to B$ exist in $\mathbf{C}$.

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[dr, dashed, "(f\circ h, g\circ h)"'] & C \arrow[d, "(f,g)"] \\
& (A, B)
\end{tikzcd}$$

Easy to show this definition satisfies the associativity requirement.

Let $F : \mathbf{C} \to \mathbf{D}$ be the trivial functor (since $\mathbf{D}$ is just a little bit more than a copy of $\mathbf{C}$), then a product P is given by the universal morphism from F to (A, B):

$$(P, (p_1, p_2): P \to (A, B)).$$

$$\begin{tikzcd}
P \arrow[d, "(p_1,p_2)"'] & A' \arrow[l, dashed, "\exists! h"'] \arrow[dl, "(x_1,x_2)"] \\
(A, B) &
\end{tikzcd}$$

In the definition, 2.3, $X = (A, B), u = (p_1, p_2), A = P, F(A) = P, F(A') = A', F(h) = h, f = (x_1, x_2)$.

Note that (A, B) is not an object of $\mathbf{C}$, so $F(A') = A' \neq (A, B)$, so any arrow $f\colon A' \to (A, B)$ can be expressed as (x_1, x_2).

By definition of composite in $\mathbf{D}$ and uniqueness of composite (2.3), we have $(x_1, x_2) = (p_1 \circ h, p_2 \circ h)$.

Concise definition Given a category $\mathbf{C}$, define a functor $F\colon \mathbf{C} \to \mathbf{C}\times\mathbf{C}$, where $\mathbf{C} \times \mathbf{C}$ is the product category, as

$$F(A) = (A, A), F(f\colon A \to B) = (f, f)\colon (A, A) \to (B, B).$$

Then a product P is given by the universal morphism from F to (A, B):

$$(P, (p_1, p_2)\colon (P, P) \to (A, B)).$$

$$\begin{tikzcd}
(P,P) \arrow[d, "{(p_1,p_2)}"'] & (A',A') \arrow[l, dashed, "{\exists!(h,h)}"'] \arrow[dl, "{(x_1,x_2)}"] \\
(A,B) &
\end{tikzcd}$$

In the definition, 2.3, $\mathbf{D} = \mathbf{C} \times \mathbf{C}, X = (A, B), u = (p_1, p_2), A = P, F(A) = (P, P), F(A') = (A', A'), F(h) = (h, h), f = (x_1, x_2)$.

By definition of composite in $\mathbf{D}$ and uniqueness of composite (2.3), we have $(x_1, x_2) = (p_1 \circ h, p_2 \circ h)$.

Coproduct is defined by universal property Given a category $\mathbf{C}$, define a functor $F\colon \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ as does 2.4.

Then a coproduct C is given by the universal morphism from (A, B) to F:

$$(C, (i_1, i_2)\colon (A, B) \to (C, C)).$$

$$\begin{tikzcd}
(C,C) \arrow[r, dashed, "{\exists!(h,h)}"] & (A',A') \\
(A,B) \arrow[u, "{(i_1,i_2)}"] \arrow[ur, "{(x_1,x_2)}"'] &
\end{tikzcd}$$

In the definition, 2.3, $\mathbf{D} = \mathbf{C} \times \mathbf{C}, X = (A, B), u = (i_1, i_2), A = C, F(A) = (C, C), F(A') = (A', A'), F(h) = (h, h), f = (x_1, x_2)$.

2.5 Examples of products

3 Check properties

4 Greatest lower bound is just min function in totally ordered set.

6 Lambda calculus type theory

closed terms = no free variables?

$\beta\eta$-equivalence means $\lambda x.x = \lambda y.y$?

Remark 2.18 "Curry–Howard" correspondence Functor from category of proofs to category of types

2.6 Categories with products

Multiplication of arrows $f \times f'$ exists and is uniquely defined because $B \times B'$ is a product and

$$\begin{tikzcd}
& A \times A' \arrow[dl, bend right, "f \circ p_1"'] \arrow[d, dashed, "\exists! f \times f'"] \arrow[dr, bend left, "f' \circ p_2"] & \\
B & B \times B' \arrow[l, "q_1"] \arrow[r, "q_2"'] & B'
\end{tikzcd}$$

Functor from product category to itself First, we need to choose *a* product for each pair of objects because there might be multiple products.

The functor $\times$ maps (A, B) to $A \times B$, and (f, g) to $f \times g$ defined above.

Ternary product Prove the associativity

I-ary product/coproduct

Definition (I-ary product/coproduct)**.** Given any set I, define the I-th power $\mathbf{C}^I$ of a category $\mathbf{C}$ whose object is a family of objects $(C_i)_{i\in I}$ in $\mathbf{C}$ and whose arrow is a family of arrows $(f_i\colon A_i \to B_i)_{i\in I}\colon (A_i)_{i\in I} \to (B_i)_{i\in I}$ in $\mathbf{C}$.

The identity arrow is

$$1_{(A_i)_{i\in I}} \coloneqq (1_{A_i})_{i\in I},$$

and the composite is

$$(f_i\colon B_i \to C_i)_{i\in I} \circ (g_i\colon A_i \to B_i)_{i\in I} \coloneqq (f_i \circ g_i)_{i\in I}.$$

The functor F is defined as $F(A) = (A)_{i\in I}, F(f) = (f)_{i\in I}$.

Then an **I-ary product** P in $\mathbf{C}$ is a universal morphism from F to $(A_i)_{i\in I}$:

$$\Big(P, (p_i)_{i\in I}\colon (P)_{i\in I} \to (A_i)_{i\in I}\Big)$$

such that

$$
\begin{tikzcd}
(P)_{i\in I} \arrow[d, "(p_i)_{i\in I}"'] & (A')_{i\in I} \arrow[l, dashed, "\exists!(h)_{i\in I}"'] \arrow[dl, "\forall(x_i)_{i\in I}"] \\
(A_i)_{i\in I} &
\end{tikzcd}
$$

A J**-ary coproduct** C in $\mathbf{C}$ is a universal morphism from $(A_j)_{j\in J}$ to F:

$$\Big(C, (i_j)_{j\in J} \colon (A_j)_{j\in J} \to (C)_{j\in J}\Big)$$

such that

$$
\begin{tikzcd}
(C)_{j\in J} \arrow[r, dashed, "\exists!(h)_{j\in J}"] & (A')_{j\in J} \\
(A_j)_{j\in J} \arrow[u, "(i_j)_{j\in J}"] \arrow[ur, "\forall(x_j)_{j\in J}"'] &
\end{tikzcd}
$$

2.7 Hom-sets

Representable functor It maps any object B in $\mathbf{C}$ to a set $\mathrm{Hom}(A, B)$, and any arrow $g \colon B \to B'$ to $g_* \colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B') \colon f \mapsto g \circ f$.

Proposition 2.20 This function is a product function of $\mathrm{Hom}(X, p_1)$ and $\mathrm{Hom}(X, p_2)$.

Since it is a function between sets, bijective is equivalent to isomorphic.

Definition 2.21 Let p_1', p_2' be the projection map for $FA \times FB$ in $\mathbf{D}$, then

$$
\begin{tikzcd}
& F(A\times B) \arrow[dl, bend right, "Fp_1"'] \arrow[d, dashed, "\langle Fp_1, Fp_2\rangle"] \arrow[dr, bend left, "Fp_2"] & \\
FA & FA\times FB \arrow[l, "p_1'"] \arrow[r, "p_2'"'] & FB
\end{tikzcd}
$$

If $f = \langle Fp_1, Fp_2\rangle$ is an iso, then $\exists! g \colon FA \times FB \to F(A \times B)$ such that $fg = 1_{FA\times FB}, gf = 1_{F(A\times B)}$. Therefore both f and g are monos and epis.

$$
\begin{tikzcd}
& F(A\times B) \arrow[dl, bend right, "Fp_1"'] \arrow[d, shift left, two heads, tail, "\exists! f"] \arrow[dr, bend left, "Fp_2"] & \\
FA & FA\times FB \arrow[l, "p_1'"] \arrow[r, "p_2'"'] \arrow[u, shift left, "g'"] \arrow[loop below, "1_{FA\times FB}"] & FB
\end{tikzcd}
$$

If there is $g' \colon FA \times FB \to F(A \times B)$ such that the diagram commutes, since f is the only arrow from $F(A \times B)$ to $FA \times FB$ and $1_{FA \times FB}$ from $FA \times FB$ to $FA \times FB$ which make the diagram commute, therefore $f \circ g' = 1_{FA \times FB}$. Since f is epi and mono, $g' = g$. So, g is the only arrow which makes the diagram commute. So $F(A \times B)$ is a product, given the projection maps Fp_1 and Fp_2.

2.8 Exercises

4

b $i = j$ if the following diagram commutes.

$$
\begin{tikzcd}
D \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & A \arrow[r, tail, dashed, "f"] \arrow[dr, tail, "h"'] & B \arrow[d, "g"] \\
& & C
\end{tikzcd}
$$

c $i = j$ if the following diagram commutes.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, two heads, "h"'] & B \arrow[d, dashed, two heads, "g"] & \\
& C \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & D
\end{tikzcd}
$$

7 See 2.1.1.

8 See 2.1.1.

11

Definition of A-monoid Let $U \colon \mathbf{Mon} \to \mathbf{Sets}$ be the forgetful functor and A be any set and M be any monoid. An object in A-**Mon** is a pair $\Big(M, m \in \mathrm{Hom}_{\mathbf{Sets}}\big(A, U(M)\big)\Big)$ and an arrow $h \colon (M, m) \to (N, n)$ in A-**Mon** is an arrow $h \colon M \to N$ in **Mon** such that the following diagram in **Sets** commutes:

$$
\begin{tikzcd}
A \arrow[r, "m"] \arrow[dr, "n"'] & U(M) \arrow[d, "U(h)"] \\
& U(N)
\end{tikzcd}
$$

A-Mon is a category The identity is

$$1_{(M,m)} = 1_M.$$

The composite is

$$h \circ g \colon (M,m) \to (N,n) \to (O,o)$$

such that the following diagram commutes.

$$\begin{tikzcd} & A \arrow[dl, "m"'] \arrow[d, "n"] \arrow[dr, "o"] & \\ U(M) \arrow[r, "U(g)"] & U(N) \arrow[r, "U(h)"] & U(O) \end{tikzcd}$$

Initial objects in A-Mon (M,m) is an initial object in A-**Mon**, iff for any object (N,n) in A-**Mon**, there is an unique arrow $g \in \mathrm{Hom}_{\mathbf{Mon}}(M,N)$ such that the following diagram commutes.

$$\begin{tikzcd} \textbf{Mon} & M \arrow[r, dashed, "\exists! g"] & N \\ \textbf{Sets} & U(M) \arrow[r, "U(g)"] & U(N) \\ & A \arrow[u, "m"] \arrow[ur, "n"'] & \end{tikzcd}$$

This is exactly the definition of free monoid.

14

a See 2.6.

b Given any sets I, X, for any $i \in I$, define an arrow in **Sets**:

$$p_i \colon \mathrm{Hom}_{\mathbf{Sets}}(I,X) \to X \colon f \mapsto f(i).$$

Then let A' be any set, take $x_i \colon A' \to X$ as any function for any $i \in I$. If we want to make the following diagram commute

$$
\begin{tikzcd}
\mathbf{Sets} & \mathrm{Hom}_{\mathbf{Sets}}(I,X) & A' \arrow[l, dashed, "h"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{Sets}^I & \big(\mathrm{Hom}_{\mathbf{Sets}}(I,X)\big)_{i\in I} \arrow[d, "(p_i)_{i\in I}"'] & (A')_{i\in I} \arrow[l, "(h)_{i\in I}"'] \arrow[dl, "(x_i)_{i\in I}"] \\
& (X)_{i\in I} &
\end{tikzcd}
$$

then we need to have the following equivalence for all $i \in I$ (let X^I be $\mathrm{Hom}_{\mathbf{Sets}}(I,X)$):

$$A' \xrightarrow{\quad x_i \quad} X$$

$$a \xmapsto{\quad x_i \quad} x_i(a)$$

$$a \xmapsto{h} h_a \xmapsto{p_i} h_a(i)$$

$$A' \xrightarrow{h} X^I \xrightarrow{p_i} X$$

Therefore

$$\forall i \in I \forall a \in A'[h_a(i) = x_i(a)].$$

$$\because \forall a \in A' \forall i \in I[\exists! x_i(a) \in X],$$

$\therefore \forall a \in A'$, h_a is uniquely and well defined as $i \mapsto x_i(a)$;
also, $h_a \in X^I = \mathrm{Hom}_{\mathbf{Sets}}(I,X)$.

That is to say,

$$\forall a \in A'[\exists! h_a \in X^I],$$

$$\therefore \exists! h \in \mathrm{Hom}_{\mathbf{Sets}}(A', X^I),$$

such that the diagram is commutative.

This is exactly the definition of I-ary product. Therefore

$$\mathrm{Hom}_{\mathbf{Sets}}(I,X) = I^X \cong \prod_{i\in I} X.$$

General Cartesian product of sets Given any set I and any family of sets $(X_i)_{i\in I}$, let

$$P = \{\, f \in \mathrm{Hom}_{\mathbf{Sets}}(I, \bigcup_{i\in I} X_i) \mid \forall i \in I\big(f(i) \in X_i\big) \,\}\,.$$

For any $i \in I$, define an arrow in **Sets**:

$$p_i \colon P \to X_i \colon f \mapsto f(i).$$

Note that $\forall i \in I\, \forall f \in P[\exists! p_i(f) = f(i) \in X_i]$, by definition of P.

Then let A' be any set, take $x_i \colon A' \to X_i$ as any function for any $i \in I$. If we want to make the following diagram commute

$$\begin{tikzcd}
\mathbf{Sets} & P & A' \arrow[l, dashed, "h"'] \\
\mathbf{Sets}^I & \big(P\big)_{i\in I} \arrow[d, "(p_i)_{i\in I}"'] & (A')_{i\in I} \arrow[l, "(h)_{i\in I}"'] \arrow[dl, "(x_i)_{i\in I}"] \\
 & (X_i)_{i\in I} &
\end{tikzcd}$$

then we need to have the following equivalence for all $i \in I$:

$$A' \xrightarrow{\qquad x_i \qquad} X_i$$

$$a \xmapsto{\qquad x_i \qquad} x_i(a)$$

$$a \xmapsto{\ h\ } h_a \xmapsto{\ p_i\ } h_a(i)$$

$$A' \xrightarrow{\ h\ } P \xrightarrow{\ p_i\ } X$$

Therefore

$$\forall i \in I \forall a \in A'[h_a(i) = x_i(a)].$$

By definition of $x_i \colon A' \to X_i$, we have

$$\forall a \in A' \forall i \in I[\exists! x_i(a) \in X_i],$$

$\therefore \forall a \in A'$, h_a is uniquely and well defined as $i \mapsto x_i(a)$;
also, $h_a \in P$ because $\forall i \in I[h_a(i) = x_i(a) \in X_i]$.

That is to say,

$$\forall a \in A'[\exists! h_a \in P],$$

$$\therefore \exists! h \in \mathrm{Hom}_{\mathbf{Sets}}(A', P),$$

such that the diagram is commutative.

This is exactly the definition of I-ary product. Therefore

$$P \cong \prod_{i \in I} X_i.$$

16 Let A be `int` type, and B be `char` type, then $A \to B$ is a function type of signature `char(int)`. The product type $A \times B$ is a struct type {`int, char`}.

TODO

3 Duality

3.1 The duality principle

3.2 Coproducts

Example 3.4

Existence of Kleene closure

Proposition 3.1. *Given a family of sets* $(X_i)_{i \in I}$*, the Cartesian product* $\prod_{i \in I} X_i$ *is a set.*

Proof. See 2.8. It is a result of **Sets** being locally small (1.8). $\square$

Remark. However, the product might be empty without assuming axiom of choice.

Proposition 3.2. *Given a family of sets* $(X_i)_{i \in I}$*, the disjoint union* $\coprod_{i \in I} X_i$ *is a set.*

Proof.

$$\coprod_{i \in I} X_i \coloneqq \{\, (x, i) \in (\bigcup_{i \in I} X_i) \times I \mid x \in X_i, i \in I \,\}$$

is a set due to axiom of specification and axiom of union and 3.1. $\square$

Proposition 3.3.

$$\coprod_{i\in I} X_i \coloneqq \{\, (x,i) \in (\bigcup_{i\in I} X_i) \times I \mid x \in X_i, i \in I \,\}$$

is an I-ary coproduct of a family of sets $(X_i)_{i\in I}$.

Proof. Denote $C = \coprod_{i\in I} X_i$. We have proved this is a set (3.2).

For each $i \in I$, define the inclusion map ι_i as

$$\begin{aligned} \iota_i &\colon X_i \to C, \\ \iota_i(x_i) &\coloneqq (x_i, i). \end{aligned} \tag{3.1}$$

$\because \forall i \in I, x_i \in X_i, \therefore \exists!(x_i, i) \in C.\ \therefore \exists! \iota_i \in \mathrm{Hom}_{\mathbf{Sets}}(X_i, C)$.

By definition of ordered pair, ι_i is an injection for all $i \in I$.

Given any set A, and any family of functions $(\alpha_i \colon X_i \to A)_{i\in I}$, consider the following equivalence for all $i \in I$:

$$X_i \xrightarrow{\qquad\alpha_i\qquad} A$$

$$x_i \xmapsto{\iota_i} (x_i, i) \xmapsto{\gamma} \alpha_i(x_i)$$

$$X_i \overset{\iota_i}{\rightarrowtail} C \xrightarrow{\quad\gamma\quad} A$$

Since (by definition of C and ordered pair)

$$\Big[(x_i, i) \in C \iff (x_i \in X_i \wedge i \in I)\Big] \wedge \Big[(x_i, i) = (y_j, j) \iff (x_i = y_j \wedge i = j)\Big],$$

the images of ι_i for all $i \in I$ form a partition of C. Hence γ is uniquely determined by α_i and ι_i.

$\because \forall i \in I\, \forall x_i \in X_i[\exists! \alpha_i(x_i) \in A], \therefore \exists! \gamma \in \mathrm{Hom}_{\mathbf{Sets}}(C, A)$.

In conclusion, we can build up a commutative diagram as follows.

$$\begin{tikzcd}
(C)_{i\in I} \arrow[r, dashed, "\exists!(\gamma)_{i\in I}"] & (A)_{i\in I} \\
(X_i)_{i\in I} \arrow[u, "(\iota_i)_{i\in I}"] \arrow[ur, "(\alpha_i)_{i\in I}"']
\end{tikzcd}$$

$\square$

Proposition 3.4. *The Kleene closure* $X^* = \bigsqcup_{i \in \mathbb{N}} X^i$ *for any set* X *is a free monoid over* X *under*

$$*\colon \big(\alpha\colon I \to X, I\big), \big(\beta\colon J \to X, J\big) \mapsto \big(([\alpha, \beta]\colon I + J \to X), I + J\big),$$

where $I = \{1, \dots, i\}, J = \{1, \dots, j\}, I + J = \{1, \dots, i + j\}, 0 \cong \varnothing$.

Proof. First, it is a (small) set per 3.1 and 3.2.

Next, define the coproduct diagram in **Sets** as

$$\begin{tikzcd} I \arrow[r, tail, "\iota_I"] & I + J & J \arrow[l, tail, "\iota_{J,I}"'] \end{tikzcd}$$

where $\iota_I\colon i \mapsto i, \iota_{J,I}\colon j \mapsto |I| + j, \iota_\varnothing = \iota_{\varnothing,J}\colon \varnothing \to J, \iota_{J,\varnothing} = 1_J$.

It is easy to prove this is a coproduct diagram by showing the existential uniqueness of $*$ by UMP.

$$\begin{tikzcd} & X & \\ I \arrow[ur, bend left, "\alpha"] \arrow[r, "\iota_I"'] & I + J \arrow[u, dashed, "\exists! {[\alpha,\beta]}"'] & J \arrow[ul, bend right, "\beta"'] \arrow[l, "\iota_{J,I}"] \end{tikzcd}$$

$$[\alpha, \beta](k) = \begin{cases} \alpha(k) & (k \leq |I|) \\ \beta(k - |I|) & (k > |I|) \end{cases}.$$

Note that $I + J \in \mathbb{N}$ because sum of finite numbers is finite.

The identity of the monoid is

$$(\varnothing \to X, \varnothing),$$

which must exist because $\varnothing$ is an initial object in **Sets**.

This is identity because $\varnothing + J = J$ and the inclusion map $\iota_{J,\varnothing} = 1_J$.

$*$ is associative because $+$ is associative (by choosing appropriate inclusion map):

$$\begin{tikzcd} & & I + J + K \arrow[dd, bend right, "\exists! {[\alpha,[\beta,\gamma]]}"'] \arrow[dd, bend left, "\exists! {[[\alpha,\beta],\gamma]}"] & & \\ I \arrow[urr, bend left, "\iota_I"] \arrow[r, "\iota_I"] \arrow[drr, bend right, "\alpha"'] & I + J \arrow[ur, bend left, "\iota_{I+J}"] \arrow[dr, "\exists! {[\alpha,\beta]}"'] & J \arrow[l, "\iota_{J,I}"'] \arrow[r, "\iota_J"] \arrow[d, "\beta"] & J + K \arrow[ul, bend right, "\iota_{J+K,I}"'] \arrow[dl, "\exists! {[\beta,\gamma]}"] & K \arrow[l, "\iota_{K,J}"'] \arrow[ull, bend right, "\iota_{K,I+J}"'] \arrow[dll, bend left, "\gamma"] \\ & & X & & \end{tikzcd}$$

Therefore X^* is a monoid. Let $U\colon \mathbf{Mon} \to \mathbf{Sets}$ be the forgetful functor. Then

$$\bigl(X^*, u\colon x \mapsto (1 \mapsto x, \{\,1\,\})\bigr)$$

is a universal morphism from X to U (given any monoid M and any function $f \in \mathrm{Hom}_{\mathbf{Sets}}(X, U(M))$):

$$\begin{tikzcd}
U(X^*) \arrow[r, dashed, "\exists! U(h)"] & U(M) \\
X \arrow[u, "u"] \arrow[ur, "\forall f"'] &
\end{tikzcd}$$

Because

$$x \xmapsto{u} (1 \mapsto x, \{\,1\,\}) \xmapsto{U(h)} f(x)$$

If $\exists h \in \mathrm{Hom}_{\mathbf{Mon}}(X^*, M)$, then h must be unique, because

$$h(\varnothing \to X, \varnothing) = e_M,$$

where e_M is the identity of M, and for any nonempty I,

$$\begin{aligned} h(i \mapsto x_i, I) &= h([1 \mapsto x_1, 1 \mapsto x_2, \ldots, 1 \mapsto x_{|I|}], \underbrace{1 + \cdots + 1}_{|I| \text{ times}}) \\ &= f(x_1) *_M f(x_2) *_M \cdots *_M f(x_{|I|}). \end{aligned} \tag{3.2}$$

Easy to prove h is a monoid homomorphism as well.

□

Proposition 3.5. *Let $U\colon \mathbf{Mon} \to \mathbf{Sets}$ be the forgetful functor. Given any monoid M and any set $A \cong U(M)$ and a bijection the $f\colon U(M) \xrightarrow{\sim} A$, there exists a unique monoid B such that $U(B) = A$ and f is a monoid isomorphism.*

Proof. If f is a monoid homomorphism, then $\forall a, b \in A$,

$$\begin{aligned} a \cdot_B b &= f(f^{-1}(a)) \cdot_B f(f^{-1}(B)) \\ &= f(f^{-1}(a) \cdot_M f^{-1}(b)) \in B, \end{aligned} \tag{3.3}$$

which uniquely defines a closed multiplication in B.

The identity in B is

$$e_B = f(e_M) \in B$$

where e_M is the identity in M. This uniquely defines an identity element in B.

For any $a \in B$,

$$
\begin{aligned}
e_B \cdot_B a &= f(f^{-1}(e_B) \cdot_M f^{-1}(a)) \\
&= f(e_M \cdot_M f^{-1}(a)) \\
&= f(f^{-1}(a)) \\
&= a.
\end{aligned}
\tag{3.4}
$$

Similarly we have $a \cdot_B e_B = a$. Therefore e_B indeed is the identity element.

Let $a, b, c \in B$, then

$$
\begin{aligned}
(a \cdot_B b) \cdot_B c &= f\Big(f^{-1} \circ f\big(f^{-1}(a) \cdot_M f^{-1}(b)\big) \cdot_M f^{-1}(c)\Big) \\
&= f\Big(\big(f^{-1}(a) \cdot_M f^{-1}(b)\big) \cdot_M f^{-1}(c)\Big) \\
&= f\Big(f^{-1}(a) \cdot_M \big(f^{-1}(b) \cdot_M f^{-1}(c)\big)\Big) \\
&= f\Big(f^{-1}(a) \cdot_M f^{-1} \circ f\big(f^{-1}(b) \cdot_M f^{-1}(c)\big)\Big) \\
&= a \cdot_B (b \cdot_B c).
\end{aligned}
\tag{3.5}
$$

Therefore B is a monoid and f is a monoid homomorphism. Since f is bijective, it is monoid isomorphism. □

Proposition 3.6. *Let $U\colon \mathbf{Mon} \to \mathbf{Sets}$ be the forgetful functor. Given any sets X and A, then $u\colon X \to A$ is a universal morphism from X to U iff $A \cong U(X^*)$ (isomorphic to the underlying set of the Kleene closure) and u is injection.*

Proof. Necessity: u is injection due to 2.8. Isomorphism is needed since UMP determines an object up to isomorphism.

Sufficient condition: since $A \cong U(X^*)$, there exists a monoid isomorphism $f\colon X^* \xrightarrow{\sim} B$ such that $U(B) = A$ and $U(f)\colon U(X^*) \rightarrowtail\!\!\!\!\twoheadrightarrow A$ (3.5). Therefore, the following diagram commutes for any monoid M and any function $g\colon X \to U(M)$ (let i be the UMP defining Kleene closure, see 3.4).

$$
\begin{tikzcd}
U(X^*) \arrow[r, tail, two heads, "U(f)"] \arrow[rr, bend left, "\exists! U(h)"] & U(B) \arrow[r, dashed] & U(M) \\
& X \arrow[ul, "i"] \arrow[u, tail, "u"] \arrow[ur, "\forall g"'] &
\end{tikzcd}
$$

TODO □

Example 3.5

Proposition 3.7.
$$M(A+B) \cong M(A) + M(B).$$

Proof. Given sets A, B and a coproduct diagram $A \overset{i_1}{\to} A+B \overset{i_2}{\leftarrow} B$, let $U\colon \mathbf{Mon} \to \mathbf{Sets}$ the forgetful functor, $M(C)$ be a free monoid over C for any set C, and $\eta_C\colon C \to U \circ M(C)$ be the universal map of $M(C)$ (existence: 3.4). Then the following diagram in **Sets** commutes ($UM = U \circ M$) for any monoid N and any monoid homomorphisms $a\colon M(A) \to N, b\colon M(B) \to N$.

$$\begin{tikzcd}
 & & U(N) & \\
UM(A) \arrow[urr, bend left, "U(a)"] \arrow[r, dashed, red, "\exists! U(f)"'] & UM(A+B) \arrow[ur, dashed, violet, "\exists! U(h)"] & & UM(B) \arrow[ll, dashed, blue, "\exists! U(g)"] \arrow[ul, bend right, "U(b)"'] \\
A \arrow[u, "\eta_A"] \arrow[r, "i_1"'] & A+B \arrow[u, violet, "\eta_{A+B}"] \arrow[uur, dashed, "\exists! [U(a)\circ\eta_A, U(b)\circ\eta_B]"'] & & B \arrow[u, "\eta_B"'] \arrow[ll, "i_2"]
\end{tikzcd}$$

By UMP of free monoid, The red arrow $U(f)$ is uniquely determined by $\eta_{A+B} \circ i_1$, the blue arrow $U(g)$ is uniquely determined by $\eta_{A+B} \circ i_2$.

From the UMP of coproduct, it turns out that $\eta_{A+B} = [U(f) \circ \eta_A, U(g) \circ \eta_B]$.

From UMP of coproduct, the black arrows determine uniquely $[U(a) \circ \eta_A, U(b) \circ \eta_B]\colon A+B \to U(N)$.

From UMP of free monoid, the purple arrow uniquely $h\colon M(A+B) \to N$ such that the diagram commutes.

This is exactly the definition of coproduct in **Mon**, therefore $M(A+B) \cong M(A) + M(B)$. $\square$

Is this a functor? The foregoing example says that the free monoid functor $M : Sets \to Mon$

t the forgetful functor $U : Mon \to Sets$ is representable and so preserves products.

3.3 Equalizers

3.4 Coequalizers

3.5 Exercises