Reading Notes for
Elementary Topology: Problem Textbook

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Contents

Definition (arithmetic progression)**.** An **arithmetic progression** or arithmetic sequence is a sequence of numbers such that the difference between the consecutive terms is constant:

$$a_1, a_1 + d, a_1 + 2d, \dots.$$

Definition (point)**.** Elements in X given a topology (X, Ω) are called **points**.

Definition (particular point topology)**.** Given a discrete topology

$$(X, \mathcal{P}(X)),$$

and a singleton $\{a\}$ disjoint with X, a **particular point topology** or **topology of everywhere dense point** is

$$(X \cup \{a\}, \{\{a\} \cup U \mid U \in \mathcal{P}(X)\} \cup \{\emptyset\}).$$

Remark. Let (Y, Ω) be a particular point topology, then

$$S \in \Omega \iff S = \emptyset \vee a \in S,$$

i.e., a belongs to any nonempty open set.

Definition (Sierpiński space)**.** A particular point topology with two points is called a **Sierpiński space** or topology of connected pair of points. Explicitly, it is

$$(\{a, b\}, \{\emptyset, \{a\}, \{a, b\}\}),$$

where $a \neq b$ and a being the element in all nonempty open sets.

Definition (Cantor set)**.** Let K be the set of real numbers that are sums of series of the form

$$\sum_{k=1}^{\infty} a_k 3^{-k},$$

with $a_k \in \{0, 2\}$. In other words, K is the set of real numbers that are presented as $0.a_1 a_2 \ldots a_k \ldots$ without the digit 1 in the positional system with base 3. K is called the **Cantor set**.

Remark. Then any real number of form $x3^{-j} \in [0, 1]$ $(x \in \mathbb{N}, j \in \mathbb{N})$ must be in K. But K contains irrational numbers as well.

$$K \cap \mathbb{Q} = \{\sum_{p \in P} \frac{2 \cdot 3^p}{3^i} + \sum_{q \in Q} \frac{2 \cdot 3^q}{3^j - 3^i} \mid 0 \leq p < i \leq i + q < j, \}.$$

Definition (family)**.** A **family** of a set S is a subset of $\mathcal{P}(S)$.

Equivalently, a collection F of subsets of a given set S is called a family of subsets of S, or a family of sets over S.

Definition (base)**.** A **base** or basis for the topology Ω of a topological space (X, Ω) is a family B of open subsets of X such that every open set of the topology is equal to a union of some sub-family of B.

Equivalently, B is a base of topology (X, Ω) if

$$S \in \Omega \iff \exists B'[B' \subseteq B \wedge S = \cup B'].$$

Remark. Topologies on a set X with only one base are exactly $\mathcal{S} \in \mathcal{P}(\mathcal{P}(X))$ such that

$$\{ \emptyset, X \} \subseteq \mathcal{S} \wedge \forall \mathcal{T} \subseteq \mathcal{S}[\mathcal{T} \neq \emptyset \implies \exists M \in \mathcal{T} \forall T \in \mathcal{T}(T \subseteq M)],$$

i.e., a topology whose nonempty subset $\mathcal{T} \subseteq \mathcal{S} \subseteq \mathcal{P}(X)$ must have an element containing all other elements in $\mathcal{T}$.

Example: $\mathbb{R}$ whose open sets are intervals $(-1/n, 1/n)$ for $n \in \mathbb{Z}^+$, $\emptyset$, and $\mathbb{R}$.

Example2: For any finite sequence $\emptyset \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq X$, the topology on X consisting of those sets provides another example.

Definition (subbase)**.** Let (X, T) be a topological space. A **subbase** of T is usually defined as a $B \subseteq T$ satisfying one of the two following equivalent conditions:

- T is the smallest topology containing B: any topology T' on X containing B must also contain T.
- All finite intersections of elements of B, together with the set X, forms a basis for T, i.e.,

  $$\Sigma = \{ \bigcap_{i=1}^{n} B_i \mid n \in \mathbb{Z}^+, B_i \in B \} \cup \{ X \}$$

  is a base of T.

For any $S \subseteq \mathcal{P}(X)$, there is a **unique** topology having S as a subbase. In particular, the intersection of all topologies on X containing S satisfies this condition. In general, however, there is no unique subbasis for a given topology. ???

Definition (metric space)**.** See book.

Remark. Consider a metric space

$$(X = \{\, x, y, z \,\}, \rho \colon X \times X \to \mathbb{R}_0^+),$$

where

$$\rho(x, y) = \rho(x, z) = 4, \rho(y, z) = 7.$$

Then

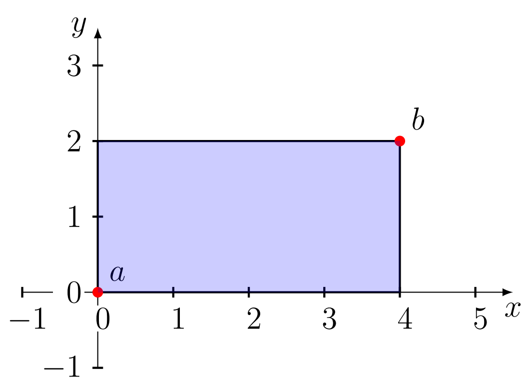
$$\{\, x, y \,\} = B_6(y) \subsetneq B_5(x) = \{\, x, y, z \,\}.$$

Remark. Given $a, b \in \mathbb{R}^2$, the set

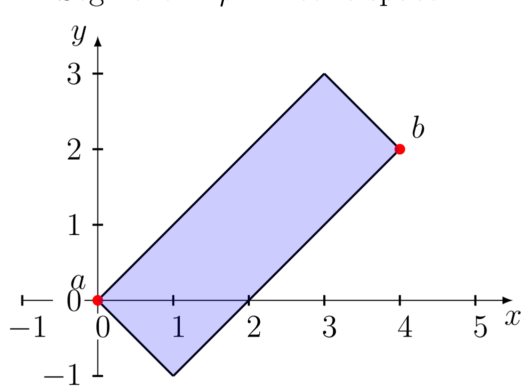
$$\{\, x \in \mathbb{R}^2 \mid \rho(a, x) + \rho(x, b) = \rho(a, b) \,\}$$

is the line segment if ρ is the Euclidean metric.

“Segment in ρ^1 metric space”



“Segment in ρ^∞ metric space”



Definition (dense)**.** Let A and B be two sets in a topological space X. A is **dense** in B if B is a subset of the closure of A. A is **everywhere dense** if the closure of A is X. A set is **nowhere dense** if its exterior is everywhere dense.

Theorem. *A set is everywhere dense iff it intersects any nonempty open set.*

Theorem. *A set is nowhere dense iff its closure has empty interior.*

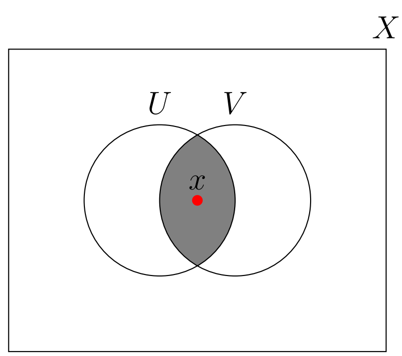
Definition (limit point)**.** A point b is a **limit point** of a set A, if each neighborhood of b intersects $A \setminus \{ b \}$.

Theorem. *Every limit point of a set is its adherent point. The reverse is not true.*

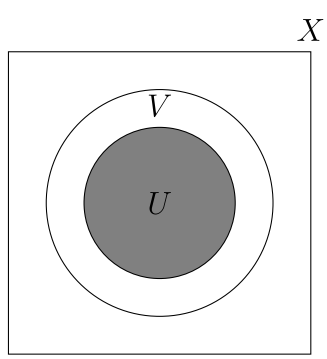
Theorem. *A set is closed iff it contains all of its limit points.*

Definition (isolated point)**.** A point b is an **isolated point** of a set A if $b \in A$ and b has a neighborhood disjoint with $A \setminus \{ b \}$.

U is open in X iff
$\forall x \in U \exists V \subseteq X$ [$x \in V$ and V is open X and $U \cap V$ is open in V].



Therefore we can say U is locally open: a set U is open iff it is open in a neighborhood V of each of its points.



But a locally closed set is not closed.

Definition (locally closed set)**.** Given a topology (X, Ω), below are equivalent statements

- A is **locally closed** in X;
- A is an open subset of its closure;
- A is the intersection of open and closed subsets of X;
- A is closed in the subspace (U, Ω_U), where U is an open set containing A, i.e., $U \in \Omega \wedge A \subseteq U$.

Definition (smallest neighborhood space)**.** A space satisfying one of the below conditions is a **smallest neighborhood space**.

The following statements are equivalent

- each point has a smallest neighborhood,
- the intersection of any collection of open sets is open,
- the union of any collection of closed sets is closed.

Definition (Khalimsky line)**.** The poset topology on $\mathbb{Z}$ with the base

$$\{\,\{\,2k-1, 2k, 2k+1\,\} \mid k \in \mathbb{Z}\,\}$$

is called the **digital line**, or **Khalimsky line**.

Theorem. *In the digital line, each even number is closed and each odd one is open.*

Definition (cyclic order)**.** A finite cyclic group naturally defines a **cyclic order**.

The cyclic order topology determined by the cyclic counterclockwise order of S^1 is the topology generated by the metric $\rho(x, y) = |x - y|$ on $S^1 \subseteq \mathbb{C}$.

Definition (locally continuous)**.** A map f from a topological space X to a topological space Y is said to be **continuous at a point** $a \in X$ if for every neighborhood V of $f(a)$ there exists a neighborhood U of a such that $f(U) \subseteq V$.

p60 isometric embedding, isometry and after skipped