Reading Notes for
The Quantum Theory of Fields

Zhi Wang

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Contents

Notes

Red means question. Blue means important notes.

Volume 1

Chapter 2: Relativistic Quantum Mechanics

2.2 Symmetries

P51 (69)

Eq. (2.2.8)

For *unitary* and *linear* operators, plug Eq. (2.2.6) into Eq. (2.2.2),

$$(U\Phi, U\Psi) = \left(\Phi, U^\dagger U\Psi\right) = (\Phi, \Psi)\,,\ \forall\Phi, \Psi.$$

For *antiunitary* and *antilinear* operators, plug Eq. (2.2.7) into Eq. (2.2.4),

$$(U\Phi, U\Psi) = \left(\Phi, U^\dagger U\Psi\right)^* = (\Phi, \Psi)^*\,,\ \forall\Phi, \Psi.$$

Therefore, $U^\dagger U = 1$. As symmetric transformation should be invertible, $U^\dagger = U^{-1}$.

P54 (72)

Eq. (2.2.17)

Why $t_{bc} = t_{cb}$? (commutativity of derivatives?)

Eq. (2.2.21)

From Eq. (2.2.20)

$$\begin{aligned}
&1 + i\theta^a t_a + i\bar\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + \frac{1}{2}\bar\theta^b\bar\theta^c t_{bc} - \bar\theta^{a'} t_{a'}\theta^{a''} t_{a''} + \cdots \\
&= 1 + i\theta^a t_a + i\bar\theta^a t_a + if^a{}_{bc}\bar\theta^b\theta^c t_a \\
&+ \frac{1}{2}\theta^b\theta^c t_{bc} + \frac{1}{2}\bar\theta^b\bar\theta^c t_{bc} + \frac{1}{2}\bar\theta^b\theta^c t_{bc} + \frac{1}{2}\theta^b\bar\theta^c t_{bc} + \cdots\,.
\end{aligned} \tag{1}$$

Replace a' with b and a'' with c, and exchange b with c for the last term on the right hand side. We can then remove the first five terms on the left hand side and get

$$-\bar\theta^b t_b\theta^c t_c = if^a{}_{bc}\bar\theta^b\theta^c t_a + \frac{1}{2}\bar\theta^b\theta^c t_{bc} + \frac{1}{2}\theta^c\bar\theta^b t_{cb}. \tag{2}$$

Because $t_{bc} = t_{cb}$, and because t_b and θ^c, $\bar{\theta}^b$ and θ^c, θ and $f^a{}_{bc}$ are commutative (why? because θ are just real numbers?), we get the Eq. (2.2.21):

$$t_{bc} = -t_b t_c - i f^a{}_{bc} t_a. \tag{3}$$

Eq. (2.2.23)

Why is $C^a{}_{bc} \equiv -f^a{}_{bc} + f^a{}_{cb}$ a constant? Why is it nonzero but $t_{bc} = t_{cb}$? Aren't derivatives commutative?

P55 (73)

Eq. (2.2.25)

In general the Lie bracket of a connected Lie group is always 0 if and only if the Lie group is *Abelian*. (`https://en.wikipedia.org/wiki/Lie_group#The_Lie_algebra_associated_with_a_Lie_group`)

Eq. (2.2.26)

Plug Eq. (2.2.24) into Eq. (2.2.15) and apply it $N-1$ times ($\theta' = \bar{\theta}' = \frac{\theta}{N}$).

2.3 Quantum Lorentz Transformations

P57 (75)

Eq. (2.3.10)

$$\left(\Lambda^{-1}\right)^\rho{}_\nu = \Lambda_\nu{}^\rho.$$

Definition of groups

$U(\Lambda, a)$ forms the inhomogeneous Lorentz group, or the Poincaré group. $U(\Lambda, 0)$ forms the homogeneous Lorentz group. If $\det \Lambda = 1$, it is proper. If $\Lambda^0{}_0 \geq 1$, it is orthochronous.

2.4 The Poincaré algebra

P59 (77)

Eq. (2.4.2)

Indices lowered and raised:

$$\begin{aligned} \omega_{\sigma\rho} &= \eta_{\mu\sigma}\omega^{\mu}{}_{\rho}, \\ \omega^{\mu}{}_{\rho} &= \eta^{\mu\sigma}\omega_{\sigma\rho}. \end{aligned} \tag{4}$$

When ω is infinitesimal, the matrix Λ is expressed as

$$\Lambda = \begin{pmatrix} 1 & \omega^{0}{}_{1} & \omega^{0}{}_{2} & \omega^{0}{}_{3} \\ \omega^{0}{}_{1} & 1 & \omega^{1}{}_{2} & -\omega^{3}{}_{1} \\ \omega^{0}{}_{2} & -\omega^{1}{}_{2} & 1 & \omega^{2}{}_{3} \\ \omega^{0}{}_{3} & \omega^{3}{}_{1} & -\omega^{2}{}_{3} & 1 \end{pmatrix}. \tag{5}$$

Eq. (2.4.5)

This is because J is just the derivative of U over ω?

P60 (78)

Eq. (2.4.8)

$$\begin{aligned} \left(\Lambda\omega\Lambda^{-1}\right)_{\mu\nu} &= \eta_{\kappa\mu}\Lambda^{\kappa}{}_{\tau}\left(\eta^{\tau\rho}\eta_{\tau\rho}\right)\omega^{\tau}{}_{\sigma}\left(\Lambda^{-1}\right)^{\sigma}{}_{\nu} \\ &= \left(\eta_{\kappa\mu}\Lambda^{\kappa}{}_{\tau}\eta^{\tau\rho}\right)\left(\eta_{\tau\rho}\omega^{\tau}{}_{\sigma}\right)\left(\Lambda^{-1}\right)^{\sigma}{}_{\nu} \\ &= \Lambda_{\mu}{}^{\rho}\omega_{\rho\sigma}\Lambda_{\nu}{}^{\sigma}, \end{aligned} \tag{6}$$

Using Eq. (2.4.2),

$$\begin{aligned} \left(\Lambda\omega\Lambda^{-1}a\right)_{\mu}P^{\mu} &= \Lambda_{\mu}{}^{\rho}\omega_{\rho\sigma}\Lambda_{\nu}{}^{\sigma}a^{\nu}P^{\mu} \\ &= \Lambda_{\nu}{}^{\sigma}\omega_{\sigma\rho}\Lambda_{\mu}{}^{\rho}a^{\mu}P^{\nu} \\ &= -\Lambda_{\nu}{}^{\sigma}\omega_{\rho\sigma}\Lambda_{\mu}{}^{\rho}a^{\mu}P^{\nu}. \end{aligned} \tag{7}$$

Eq. (2.4.9)

$$(\Lambda\epsilon)_{\mu}P^{\mu} = \Lambda_{\mu}{}^{\rho}\epsilon_{\rho}P^{\mu}.$$

For pure translations (with $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$), they tell us that P^{ρ} is translation-invariant, ..., the change of the space-space components of $J^{\rho\sigma}$ under a spatial translation is just the usual change of the angular momentum under a change of the origm relative to which the angular momentum is calculated.

Eq. (2.4.10)

$$
\begin{aligned}
\Lambda_\mu{}^\nu &= \delta_\mu{}^\nu + \omega_\mu{}^\nu, \\
U &= 1 + i\left(\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \epsilon_\rho P^\rho\right) + \cdots, \\
U^{-1} &= 1 - i\left(\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \epsilon_\rho P^\rho\right) + \cdots, \\
UJ^{\rho\sigma}U^{-1} &= J^{\rho\sigma} \\
&\quad + i\left(\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \epsilon_\rho P^\rho\right)J^{\rho\sigma} - i\left(\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \epsilon_\rho P^\rho\right)J^{\rho\sigma} \\
&\quad + \cdots, \\
\text{rhs} &= \left(\delta_\mu{}^\rho + \omega_\mu{}^\rho\right)\left(\delta_\nu{}^\sigma + \omega_\nu{}^\sigma\right)\left(J^{\mu\nu} - \epsilon^\mu P^\nu + \epsilon^\nu P^\mu\right) \\
&= \delta_\mu{}^\rho\delta_\nu{}^\sigma J^{\mu\nu} + \omega_\mu{}^\rho\delta_\nu{}^\sigma J^{\mu\nu} + \delta_\mu{}^\rho\omega_\nu{}^\sigma J^{\mu\nu} \\
&\quad - \delta_\mu{}^\rho\delta_\nu{}^\sigma\epsilon^\mu P^\nu + \delta_\mu{}^\rho\delta_\nu{}^\sigma\epsilon^\nu P^\mu + \cdots \\
&= J^{\rho\sigma} + \omega_\mu{}^\rho J^{\mu\sigma} + \omega_\nu{}^\sigma J^{\rho\nu} - \epsilon^\rho P^\sigma + \epsilon^\sigma P^\rho + \cdots.
\end{aligned} \tag{8}
$$

Eq. (2.4.12)

Switching μ with ν and applying Eq. (2.4.2):

$$
\omega_{\mu\nu}\eta^{\nu\rho}J^{\mu\sigma} = \omega_{\nu\mu}\eta^{\mu\rho}J^{\nu\sigma} = -\omega_{\mu\nu}\eta^{\mu\rho}J^{\nu\sigma}
$$

P61 (79)

Eq. (2.4.16)

Why is P^0 the Hamiltonian H? Why do $\mathbf{P}, \mathbf{J}$ have those physical meanings? Why do they conserve? Why is $\mathbf{K}$ not conserved? Why do you use eigenvalues of conserved operator to label physical states?

Eq. (2.4.18)

In coordinate representation,

$$
\begin{aligned}
R_i &= x^i, \\
P_i &= -i\hbar\frac{\partial}{\partial x^i}.
\end{aligned} \tag{9}
$$

Therefore for any $\psi \in \mathbb{H}$,

$$[R_i, P_j]\,\psi = -x^i i\hbar\frac{\partial\psi}{\partial x^j} + i\hbar\frac{\partial\left(x^i\psi\right)}{\partial x^j} = \delta_{ij} i\hbar\psi. \tag{10}$$

By the definition of the angular momentum,

$$J_i = \epsilon_{ijk} R_j P_k. \tag{11}$$

From $[R_i, R_j] = [P_i, P_j] = 0$, we have,

$$\begin{aligned}
[J_i, J_j] &= \epsilon_{iab} R_a P_b \epsilon_{jcd} R_c P_d - \epsilon_{jcd} R_c P_d \epsilon_{iab} R_a P_b \\
&= \epsilon_{iab}\epsilon_{jcd}\left(R_a\left(R_c P_b - [R_c, P_b]\right) P_d - R_c\left(R_a P_d - [R_a, P_d]\right) P_b\right) \\
&= i\hbar\epsilon_{iab}\epsilon_{jcd}\left(-R_a\delta_{bc} P_d + R_c\delta_{da} P_b\right).
\end{aligned} \tag{12}$$

For the first term, $b = c$, so it must be different from i, j, let it be k. Then replace d with b.

For the second term, $d = a$, so it must be different from i, j, let it be k. Then replace c with a.

$$[J_i, J_j] = i\hbar\left(-\epsilon_{iak}\epsilon_{jkb} R_a P_b + \epsilon_{ikb}\epsilon_{jak} R_a P_b\right).$$

The terms $i = j, a = b$ cancels out, so $i = b, j = a$ for the first term and $i = a, j = b$ for the second term.

$$\begin{aligned}
[J_i, J_j] &= i\hbar\left(-\epsilon_{ijk}\epsilon_{jki} R_j P_i + \epsilon_{ikj}\epsilon_{jik} R_i P_j\right) \\
&= i\hbar\left(\epsilon_{ijk}\epsilon_{kji} R_j P_i + \epsilon_{ijk}\epsilon_{kij} R_i P_j\right) \\
&= i\hbar\epsilon_{ijk}\epsilon_{kab} R_a P_b \\
&= i\hbar\epsilon_{ijk} J_k.
\end{aligned} \tag{13}$$

Eq. (2.4.21)

$$\begin{aligned}
[J_i, P_j] &= \epsilon_{iak} R_a P_k P_j - P_j \epsilon_{iak} R_a P_k \\
&= \epsilon_{iak} R_a P_j P_k - \epsilon_{iak} P_j R_a P_k \\
&= \epsilon_{iak}\left[R_a, P_j\right] P_k \\
&= \epsilon_{iak}\delta_{aj} i\hbar P_k \\
&= i\hbar\epsilon_{ijk} P_k.
\end{aligned} \tag{14}$$

Eq. (2.4.27)

$$\begin{aligned} J_i &= \epsilon_{ijk} J^{jk}, \\ \theta^i &= \frac{1}{2}\epsilon^{ijk}\omega_{jk}, \end{aligned} \tag{15}$$

i.e.,

$$\begin{aligned} \mathbf{J} &= \left\{ J^{23}, J^{31}, J^{12} \right\}, \\ \boldsymbol{\theta} &= \left\{ \omega_{23}, \omega_{31}, \omega_{12} \right\}. \end{aligned} \tag{16}$$

Why is there a negative sign difference from the standard definition? `https://en.wikipedia.org/wiki/Rotation_operator_(quantum_mechanics)#Quantum_mechanical_rotations`

P62 (80)

Eq. (2.4.29)

Given two Hermitian operators A and B, and two real-valued parameters x and y, construct two unitary operators

$$U = \mathrm{e}^{iAx}, V = \mathrm{e}^{iBy}.$$

Then, up to second order:

$$\begin{aligned} U &= 1 + iAx - \frac{1}{2}A^2x^2 + \cdots, \\ V &= 1 + iBy - \frac{1}{2}B^2y^2 + \cdots, \\ \mathrm{e}^{iAx+iBy} &= 1 + iAx + iBy - \frac{1}{2}A^2x^2 - \frac{1}{2}B^2y^2 - \frac{1}{2}AxBy - \frac{1}{2}ByAx + \cdots, \\ UV &= 1 + iAx + iBy - \frac{1}{2}A^2x^2 - \frac{1}{2}B^2y^2 - AxBy + \cdots, \end{aligned} \tag{17}$$

Therefore

$$\begin{aligned} UV &= \left(1 - \frac{1}{2}AxBy + \frac{1}{2}ByAx + \cdots\right)\mathrm{e}^{iAx+iBy} \\ &= \mathrm{e}^{-\frac{1}{2}[A,B]xy}\mathrm{e}^{iAx+iBy}. \end{aligned} \tag{18}$$

In this equation, $A = K_i$, $B = -P_j$, therefore $-\frac{1}{2}[A, B] = \frac{1}{2}[K_i, P_j] = iM\delta_{ij}/2$,

$$\mathrm{e}^{i\mathbf{K}\cdot\mathbf{v}}\mathrm{e}^{-i\mathbf{P}\cdot\mathbf{a}} = \mathrm{e}^{iM\delta_{ij}v_i a_j/2}\mathrm{e}^{i(\mathbf{K}\cdot\mathbf{v}-\mathbf{P}\cdot\mathbf{a})} = \mathrm{e}^{iM\mathbf{a}\cdot\mathbf{v}/2}\mathrm{e}^{i(\mathbf{K}\cdot\mathbf{v}-\mathbf{P}\cdot\mathbf{a})}$$

Conserved quantities

Translocation (space and time) does not change the momentum (and energy). Rotation does not shift the angular momentum.

For a pure translation $T(1, a)$ in the subgroup of the Poincaré group,

$$\langle p|T(1, a)|\psi\rangle = \langle p|\mathrm{e}^{-iP^\mu a_\mu}|\psi\rangle = \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{a}}\langle p|\psi\rangle, \tag{19}$$

which demonstrates that the translation does not change the momentum (and energy) of a system.

Similarly, rotation does not shift the angular momentum:

$$\langle j|\mathrm{e}^{i\mathbf{J}\cdot\boldsymbol{\theta}}|\psi\rangle = \mathrm{e}^{i\mathbf{J}\cdot\boldsymbol{\theta}}\langle j|\psi\rangle.$$

2.5 One-Particle States

P64 (82)

Eq. (2.5.4)

The invariant p^2 is equivalent to $-M^2$ in time-like domain, where M is the rest mass.

Consider the four dimensional vector space over $\mathbb{R}$ to which the four-vector momentum p^μ belongs. The proper orthochronous Lorentz group divides the vector space into equivalent classes:

$$p^\mu = L^\mu{}_\nu(p)\, k^\nu,$$

i.e., for *each* p^μ, there *exists a unique* image k^ν in the quotient space. The mapping is *represented uniquely* by $L(p)$. For *each* element k^ν in the quotient space, there *exists a unique* equivalent class containing four-vector momenta whose mapping towards the quotient space is k^ν. Momenta falling into (belonging to) different equivalent classes are unreachable (not connected) through proper orthochronous Lorentz transformation.

The proper orthochronous Lorentz group divides the four-vector momentum space into three components: the component where the invariant square is positive ($(p^0)^2 < (p^1)^2 + (p^2)^2 + (p^3)^2$, the space-like domain), the time-like component of positive energy ($(p^0)^2 > (p^1)^2 + (p^2)^2 + (p^3)^2$ with $p^0 > 0$), and the time-like component of negative energy ($(p^0)^2 > (p^1)^2 + (p^2)^2 + (p^3)^2$ with $p^0 < 0$).

Eq. (2.5.5)

This is to *define* the σ states in Ψ_p space, with reference to $\Psi_{k,\sigma}$. The assumption is that the σ label is orthogonal to the momenta label, and that the state space of Ψ_k has the same size of that of Ψ_p, i.e., there exists at least one bijection from state space $\{\Psi_{k,\sigma}, \forall\sigma\}$ to state space $\{\Psi_{p,\sigma'}, \forall\sigma'\}$.

This is to choose the way of labeling σ such that the C matrix in Eq. (2.5.3) is an identity matrix multiplied by a constant.

P65 (83)

Eq. (2.5.12)

Why is it δ^3 instead of δ^4? How about the time (0-th) component of the k vector?

Eq. (2.5.13)

Since U is symmetric operation (Eq. (2.2.2)), it does not change the inner product:

$$\langle U\Psi_{k,\sigma'}|U\Psi_{k,\sigma}\rangle = \langle\Psi_{k,\sigma'}|\Psi_{k,\sigma}\rangle = \delta_{\sigma'\sigma}.$$

On the other hand, from Eq. (2.5.8),

$$U\Psi_{k,\sigma} = D_{\rho\sigma}\Psi_{k,\rho}, U\Psi_{k,\sigma'} = D_{\rho'\sigma'}\Psi_{k,\rho'}.$$

Therefore, from Eq. (2.1.2) and Eq. (2.1.3), and plug in Eq. (2.5.12),

$$\delta_{\sigma'\sigma} = D^*_{\rho'\sigma'}D_{\rho\sigma}\langle\Psi_{k,\rho'}|\Psi_{k,\rho}\rangle = D^*_{\rho'\sigma'}D_{\rho\sigma}\delta_{\rho'\rho} = D^*_{\rho\sigma'}D_{\rho\sigma},$$

where the right hand side is just

$$D^*_{\rho\sigma'}D_{\rho\sigma} = \left(D^\dagger\right)_{\sigma'\rho}D_{\rho\sigma} = \left(D^\dagger D\right)_{\sigma'\sigma}.$$

Since symmetric operation is invertible, so is D. Thus, it can be concluded that

$$D^\dagger = D^{-1}.$$

P66 (84)

Table 2.1

The energy p^0 is put last.

If Λ in Eq. (2.5.6) is in the (homogeneous) Lorentz group $O(3,1)$, then the little group for $k^\mu = 0$ should also be $O(3,1)$. Maybe all "S"s in the table should be discarded.

If Λ in Eq. (2.5.6) is in the proper orthochronous (homogeneous) Lorentz group $SO^+(3,1)$, then the little group for $k^\mu = 0$ should also be $SO^+(3,1)$.

(e): $SO^+(2,1)$.

For (c) and (d), consider a infinitesimal $1 + \mathrm{d}\Lambda$ matrix:

$$1 + \mathrm{d}\Lambda = \begin{pmatrix} 1 & {\omega^0}_1 & {\omega^0}_2 & {\omega^0}_3 \\ {\omega^0}_1 & 1 & {\omega^1}_2 & -{\omega^3}_1 \\ {\omega^0}_2 & -{\omega^1}_2 & 1 & {\omega^2}_3 \\ {\omega^0}_3 & {\omega^3}_1 & -{\omega^2}_3 & 1 \end{pmatrix}, \tag{20}$$

acting on

$$\begin{pmatrix} \kappa \\ \kappa \\ 0 \\ 0 \end{pmatrix},$$

then

$$\left(1 + {\omega^0}_1\right)\kappa = \kappa, \left({\omega^0}_2 - {\omega^1}_2\right)\kappa = 0, \left({\omega^0}_3 + {\omega^3}_1\right)\kappa = 0.$$

Therefore, $1 + \mathrm{d}\Lambda$ can be rewritten as

$$1 + \mathrm{d}\Lambda = \begin{pmatrix} 1 & 0 & -a & -b \\ 0 & 1 & -a & -b \\ -a & a & 1 & -\theta \\ -b & b & \theta & 1 \end{pmatrix}, \tag{21}$$

or

$$
\begin{aligned}
\Lambda &= \mathrm{e}^{\mathrm{d}\Lambda} \\
&= \sum_{i=0}^{+\infty} \frac{(\mathrm{d}\Lambda)^i}{i!} \\
&= \begin{pmatrix}
1+(a^2+b^2)\,f(\theta) & -(a^2+b^2)\,f(\theta) & bg(\theta)+ah(\theta) & -ag(\theta)+bh(\theta) \\
(a^2+b^2)\,f(\theta) & 1-(a^2+b^2)\,f(\theta) & bg(\theta)+ah(\theta) & -ag(\theta)+bh(\theta) \\
-bg(\theta)+ah(\theta) & bg(\theta)-ah(\theta) & \cos\theta & -\sin\theta \\
ag(\theta)+bh(\theta) & -ag(\theta)-bh(\theta) & \sin\theta & \cos\theta
\end{pmatrix},
\end{aligned}
\tag{22}
$$

where

$$
\begin{aligned}
f(\theta) &= \sum_{n=0}^{+\infty} \frac{1}{(2n)!} \frac{(-1)^n}{(2n+1)(2n+2)} \theta^{2n} \\
&= \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{(2n)!} \theta^{2n-2}, \\
g(\theta) &= \sum_{n=1}^{+\infty} \frac{(-1)^n}{(2n)!} \theta^{2n-1}, \\
h(\theta) &= \sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{(2n+1)!} \theta^{2n}.
\end{aligned}
\tag{23}
$$

Since

$$
\begin{aligned}
\cos\theta &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} \theta^{2n}, \\
\sin\theta &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} \theta^{2n+1},
\end{aligned}
\tag{24}
$$

we have

$$
\begin{aligned}
f(\theta) &= \frac{1-\cos\theta}{\theta^2}, \\
g(\theta) &= \frac{\cos\theta - 1}{\theta}, \\
h(\theta) &= -\frac{\sin\theta}{\theta}.
\end{aligned}
\tag{25}
$$

In conclusion,

$$\Lambda = \begin{pmatrix} 1+\frac{\left(a^2+b^2\right)(1-\cos\theta)}{\theta^2} & -\frac{\left(a^2+b^2\right)(1-\cos\theta)}{\theta^2} & -\frac{a\sin\theta+b(1-\cos\theta)}{\theta} & -\frac{b\sin\theta-a(1-\cos\theta)}{\theta} \\ \frac{\left(a^2+b^2\right)(1-\cos\theta)}{\theta^2} & 1-\frac{\left(a^2+b^2\right)(1-\cos\theta)}{\theta^2} & -\frac{a\sin\theta+b(1-\cos\theta)}{\theta} & -\frac{b\sin\theta-a(1-\cos\theta)}{\theta} \\ -\frac{a\sin\theta-b(1-\cos\theta)}{\theta} & \frac{a\sin\theta-b(1-\cos\theta)}{\theta} & \cos\theta & -\sin\theta \\ -\frac{b\sin\theta+a(1-\cos\theta)}{\theta} & \frac{b\sin\theta+a(1-\cos\theta)}{\theta} & \sin\theta & \cos\theta \end{pmatrix}, \tag{26}$$

which, when $\theta \to 0$, becomes

$$\lim_{\theta\to 0}\Lambda = \begin{pmatrix} 1+\frac{a^2+b^2}{2} & -\frac{a^2+b^2}{2} & -a & -b \\ \frac{a^2+b^2}{2} & 1-\frac{a^2+b^2}{2} & -a & -b \\ -a & a & 1 & 0 \\ -b & b & 0 & 1 \end{pmatrix}. \tag{27}$$

Easy to see

$$\Lambda(a,b,0)\Lambda(a',b',0) = \Lambda(a+a',b+b',0).$$

$t-x$ and $y-z$ is conserved under $\Lambda(a,b,0)$.

Eq. (2.5.14)

$$\begin{aligned} \langle\Psi_{p',\sigma'}|\Psi_{p,\sigma}\rangle &= \langle\Psi_{p',\sigma'}|N(p)U(L(p))\Psi_{k,\sigma}\rangle && \text{Eq. (2.5.5)} \\ &= N(p)\,\langle\Psi_{p',\sigma'}|U(L(p))\Psi_{k,\sigma}\rangle && \text{Eq. (2.1.2)} \\ &= N(p)\,\langle U(L(p))^\dagger\Psi_{p',\sigma'}|\Psi_{k,\sigma}\rangle && \text{Eq. (2.2.6)} \\ &= N(p)\,\langle U^{-1}(L(p))\Psi_{p',\sigma'}|\Psi_{k,\sigma}\rangle\,, && \text{Eq. (2.2.8)} \end{aligned} \tag{28}$$

$$\begin{aligned} \langle\Psi_{p',\sigma'}|\Psi_{p,\sigma}\rangle &= N(p)\,\langle U(L^{-1}(p))\Psi_{p',\sigma'}|\Psi_{k,\sigma}\rangle && \text{Eq. (2.3.12)} \\ &= N(p)\,\langle\frac{N(p')}{N(k')}\sum_{\sigma''}D_{\sigma''\sigma'}(W(L^{-1}(p),p'))\Psi_{k',\sigma''}|\Psi_{k,\sigma}\rangle && \text{Eq. (2.5.11)} \\ &= \frac{N(p)N^*(p')}{N^*(k')}\sum_{\sigma''}D^*_{\sigma''\sigma'}(W(L^{-1}(p),p'))\,\langle\Psi_{k',\sigma''}|\Psi_{k,\sigma}\rangle && \text{Eq. (2.1.3)} \\ &= \frac{N(p)N^*(p')}{N^*(k')}D^*_{\sigma\sigma'}(W(L^{-1}(p),p'))\delta^3(\mathbf{k}'-\mathbf{k}), && \text{Eq. (2.5.12)} \end{aligned} \tag{29}$$

where $N(k')=1$ due to the definition in Eq. (2.5.5).

P68 (86)

Eq. (2.5.20)

How can you find all representations of SO(3)? `https://math.stackexchange.com/questions/263313/finding-all-irreducible-representations-of-so3`

References

[1] Joe Harris, and William Fulton. *Representation Theory: A First Course.*

[2] Peter Woit *Quantum Theory, Groups and Representations: An Introduction.*

Eq. (2.5.23)

$D^{(j)}(R)$ is the j-th irreducible representation of $R \in \mathrm{SO}(3)$.

"A particle is spin j" = "the wave function describing the particle is the j-th irreducible representation".

It is possible that the wave function is not reducible representation? Then it is not single-particle anymore?

P69 (87)

Mass Zero

W should preserve inner product. It should preserve k as well:

$$(Wt)^\mu k_\mu = (Wt)^\mu (Wk)_\mu = t^\mu k_\mu = -1.$$

P70 (88)

Eq. (2.5.26)

See (27). Exchange 1st and 4th row, 2nd and 3rd row. Then exchange 1st/4th column, 2nd/3rd column. $a \to -\beta, b \to -\alpha$.

Eq. (2.5.28)

This is inconsistent with (26). Because $S(\alpha, \beta)$ is not commutative with $R(\theta)$. So $\exp(\mathrm{d}S + \mathrm{d}R) \neq \exp(\mathrm{d}S)\exp(\mathrm{d}R)$.

(26) is a Lorentz transform. And it preserves inner product between k and t.

To get (26), we need to calculate

$$S\left(-\frac{b\sin\theta + a\,(1-\cos\theta)}{\theta}, -\frac{a\sin\theta - b\,(1-\cos\theta)}{\theta}\right) R\,(\theta)\,.$$

Eq. (2.5.29)–(2.5.31)

Confirmed with Mathematica.

Eq. (2.5.31) shows that the conjugacy class $[S(\alpha, \beta)] \subseteq \{\, S(\alpha', \beta') \mid \alpha' \in \mathbb{R}, \beta' \in \mathbb{R} \,\}$ (because all S's commute).

Actually the conjugacy class is

$$[S(\alpha, \beta)] = \{\, S(\alpha', \beta') \mid \alpha'^2 + \beta'^2 = \alpha^2 + \beta^2 \,\}\,.$$

Test this is isomorphic to ISO(2).

P71 (89)

Eq. (2.5.32)

Consistent with (21).

Diagonalize

Proposition. *If A and B are Hermitian and $[A, B] = 0$, they can be diagonalized simultaneously.*

Proof. Since A is Hermitian, there exists $S^{-1}AS = D$, where D is diagonal. Then the commutativity is expressed as

$$SDS^{-1}B = BSDS^{-1}.$$

Left multiply S^{-1} and right multiply S:

$$DS^{-1}BS = S^{-1}BSD.$$

Let $B' = S^{-1}BS$, then $[D, B'] = 0$.

To write it in a scalar form:

$$(B')_{ij} d_i = (B')_{ij} d_j,$$

where d are diagonal elements in D, i.e., eigenvalues of A.

For this to be true, B' must be block diagonal where the block corresponds to same eigenvalues in D, in which case there is always a way to adjust S matrix (re-diagonalization) such that B' is diagonal. □

P72 (90)

Masslesss Particles

Massless particles are not observed to have any continuous degree of freedom like θ.

Eq. (2.5.39)

Note that $\Psi_{k,\sigma}$ is actually $\Psi_{k,a=0,b=0,\sigma}$. The σ states are diagonalized *within* $a = 0, b = 0$ states. Therefore, this does not contradict with the fact that A, B and J does not commute.

And hence

$$U(S(\alpha, \beta))\Psi_{k,\sigma} = \Psi_{k,\sigma}. \tag{30}$$

Helicity

J_3 is the direction of motion because the four-vector momentum k is chosen to be nonzero in the third spatial axis.

P73 (91)

Lorentz-invariant Helicity

A general Lorentz transform can be decomposed as $L(p)W(\alpha, \beta, \theta)$, the former of which has 3 DOF, the latter has 3.

We have shown from (30) that W preserves helicity (R preserves as well because it is eigenstate).

In short, this is shown in Eq. (2.5.42), Λ should be any proper orthochronous Lorentz transform. (The equation holds because D matrix is diagonal.) The pure translation preserves σ as well because the state is in p-eigenstate. So does space/time inversion.

2.6 Space Inversion and Time-Reversal

P75 (93)

Eq. (2.6.1)

These equations do not hold because there is no connected path for space/time inversion?

P76 (94)

Eq. (2.6.7)

Space inversion is linear and time inversion is antilinear. Both commutes with Hamiltonian.

Volume 2

Volume 3