Reading Notes for

Category Theory
by Steve Awodey

Zhi Wang

September 22, 2021

Contents

0 Notes and Definitions

0.1 Notes

1 Categories

1.1 Introduction

1.2 Functions of sets

1.3 Definition of a category

1.4 Examples of categories

1

Functions with fibers of at most 2 elements

Assume

$$\forall A, B \in \mathbf{C}\, \forall(f\colon A \to B) \left[\forall b \in B\big(\, \big|f^{-1}(b)\big| \leq 2\big) \iff f \in \mathrm{Hom}_{\mathbf{C}}(A, B)\right].$$

Given that

$$\forall B \in \mathbf{C} \forall b \in B\left[\, \big|1_B^{-1}(b)\big| = 1 \leq 2\right],$$

we know

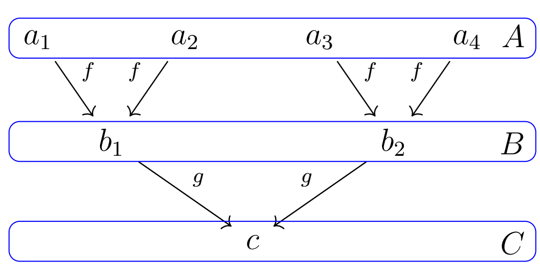
$$\forall B \in \mathbf{C}\left[1_B \in \mathrm{Hom}_{\mathbf{C}}(B, B)\right].$$

Given $\forall A, B, C \in \mathbf{C}, \forall(f\colon A \to B)\,\forall(g\colon B \to C)$

$$\forall b \in B\Big(\, \big|f^{-1}(b)\big| \leq 2\Big) \wedge \forall c \in C\Big(\, \big|g^{-1}(b)\big| \leq 2\Big) \implies \forall c \in C\Big(\, \big|(g \circ f)^{-1}(c)\big| \leq 4\Big)$$

which means that the composite $g \circ f$ does not necessarily exist in $\mathrm{Hom}_{\mathbf{C}}(A, B)$.

Therefore, this is **not** a category.



Functions with finite fiber

Assume

$$\forall A, B \in \mathbf{C}\, \forall (f\colon A \to B) \left[\forall b \in B \big(\left| f^{-1}(b) \right| \in \mathbb{N} \big) \iff f \in \mathrm{Hom}_{\mathbf{C}}(A, B) \right].$$

Given that

$$\forall B \in \mathbf{C} \forall b \in B \left[\left| 1_B^{-1}(b) \right| = 1 \in \mathbb{N} \right],$$

we know

$$\forall B \in \mathbf{C} \left[1_B \in \mathrm{Hom}_{\mathbf{C}}(B, B) \right].$$

Given $\forall A, B, C \in \mathbf{C}, \forall (f\colon A \to B)\, \forall (g\colon B \to C)$

$$\forall b \in B \Big(\left| f^{-1}(b) \right| \in \mathbb{N} \Big) \wedge \forall c \in C \Big(\left| g^{-1}(b) \right| \in \mathbb{N} \Big) \iff \forall c \in C \Big(\left| (g \circ f)^{-1}(c) \right| \in \mathbb{N} \Big)$$

since the multiplication of finite numbers are finite, and converse is true. This means that the composite $g \circ f$ must exist in $\mathrm{Hom}_{\mathbf{C}}(A, B)$.

Therefore, this **is** a category.

Functions with infinite fiber

Since identity map has finite fiber, this is **not** a category.

2

- graphs and graph homomorphisms
- the real numbers R and continuous functions $R \to R$,
- the natural numbers N and all recursive functions $N \to N$, or as in the example of continuous functions, one can take partial recursive functions defined on subsets $U \subseteq N$

11 Category of data types and programs

Let's define a procedure as `Int f(Int a, Int b)  return a * b;`
We have built a morphism/arrow

$$\mathtt{f}\colon \mathtt{Int} \times \mathtt{Int} \to \mathtt{Int}.$$

An example from computer science: Given a functional programming language L, there is an associated category, where the objects are the data types of L, and the arrows are the computable functions of L ("processes," "procedures," "programs"). The composition of two such programs $X \xrightarrow{f} Y \xrightarrow{g} Z$ is given by applying g to the output of f, sometimes also

CATEGORIES 11

written as

$$g \circ f = f; g.$$

The identity is the "do nothing" program.

Categories such as this are basic to the idea of denotational semantics of programming languages. For example, if $\mathbf{C}(L)$ is the category just defined, then the denotational semantics of the language L in a category $\mathbf{D}$ of, say, Scott domains is simply a functor

$$S : \mathbf{C}(L) \to \mathbf{D}$$

since S assigns domains to the types of L and continuous functions to the programs. Both this example and the previous one are related to the notion of "cartesian closed category" that is considered later.

Figure 1

1.5 Isomorphisms

Definition 1.3 while there are "bijective homomorphisms" between non-isomorphic posets.

The inverse must also be homomorphism?

Theorem 1.6 Every category (big or small) is isomorphic to a concrete (small) category. So what make those big?

$$f, h \in \bar{C}, g \circ f, g \circ h \in \bar{D}.$$

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, "g\circ f"] & \\
C \arrow[r, "g"] & D \\
Y \arrow[u, "h"] \arrow[ur, "g\circ h"'] &
\end{tikzcd}
\qquad\qquad
\bar{C} \xrightarrow{\bar{g}} \bar{D}$$

Remark 1.7

What does this mean

as Dedekind cuts or as Cauchy sequences). A better attempt to capture what is intended by the rather vague idea of a "concrete" category is that arbitrary arrows $f : C \to D$ are completely determined by their composites with arrows $x : T \to C$ from some "test object" T, in the sense that $fx = gx$ for all such x implies $f = g$. As we shall see later, this amounts to considering a particular representation of the category, determined by T. A category is then said to be "concrete" when this condition holds for T a "terminal object," in the sense of Section 2.2; but there are also good reasons for considering other objects T, as we see Chapter 2.

Figure 2

If the arrows in the category is too large to form a set, then it might not be isomorphic to a small category?

1.6 Constructions on categories

3 The arrow category

$$\begin{tikzcd}
A \arrow[r, "g_1"] \arrow[d, "f"] & A' \arrow[r, "h_1"] \arrow[d, "f'"] & A'' \arrow[d, "f''"] \\
B \arrow[r, "g_2"] & B' \arrow[r, "h_2"] & B''
\end{tikzcd}$$