Reading Notes for
Quantum Theory, Groups and Representations: An Introduction

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2.2 The group $U(1)$ and its representations

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$\pi_k \colon \mathbb{C}^* \to \mathbb{C}^*$
k must be integer, with the same reason as $U(1)$. If $g, h \in \mathbb{C}^*$ satisfies

$$g = \mathrm{e}^{ia}, h = \mathrm{e}^{ib},$$

where $a, b \in \mathbb{R}$ and $a + b = 2\pi$, then $gh = 1$. Therefore

$$\pi_k(g)\pi_k(h) = \mathrm{e}^{iak}\mathrm{e}^{ibk} = \mathrm{e}^{iak+ibk} = \mathrm{e}^{ik(2\pi)} = \pi_k(1) = 1^k = 1,$$

indicating $k \in \mathbb{Z}$.

2.4 Conservation of charge and $U(1)$ symmetry

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$[U(t), Q] = 0$
The assumption here is "if $[H, Q] = 0$".

3 Two-state Systems and $SU(2)$

4 Linear Algebra Review, Unitary and Orthogonal Groups

4.4 Inner products

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Definition of Inner Product

Definition (nondegenerate bilinear form)**.** A **nondegenerate bilinear form** is a bilinear form $f\colon V \times V \to K$ on a vector space V over a field K, such that $v \mapsto (x \mapsto f(x,v))$ is an isomorphism from V to V^*.

Remark. In finite dimensions, this is equivalent to

$$\forall y \in V[f(x,y) = 0] \implies [x = 0].$$

Definition (sesquilinear form)**.** Over a complex vector space V a map $\varphi\colon V \times V \to \mathbb{C}$ is **sesquilinear** if

$$\begin{aligned} &\varphi(x+y, z+w) = \varphi(x,z) + \varphi(x,w) + \varphi(y,z) + \varphi(y,w), \\ &\varphi(ax, by) = \overline{a}b\,\varphi(x,y), \end{aligned}$$

for all $x, y, z, w \in V$ and all $a, b \in \mathbb{C}$. Here, $\overline{a}$ is the complex conjugate of a scalar a.

Definition (Hermitian form)**.** A sesquilinear form is **Hermitian** if and only if $\langle x, x\rangle$ is real for all x.

4.7 Eigenvalues and eigenvectors

Polynomial and Algebraically Closed Field

Definition (polynomial ring)**.** The **polynomial ring**, $R[x]$, in x over a commutative ring R is the set of functions $f\colon R \to R$, called polynomials in x, of the form

$$f(x) = \sum_{i=0}^{n} a_i x^i,$$

where $a_0, a_1, \ldots, a_n$, the **coefficients** of f, are elements of R.

Proposition 4.1. *If a non-constant polynomial $f(x) \in R[x]$ has a root $\lambda \in R$, then $f(x) = (x - \lambda)g(x)$ where $g(x)$ is a nonzero polynomial in $R[x]$. The degree of $g(x)$ is one less than that of $f(x)$.*

Proof. Since f is non-constant with root $\lambda \in R$ of order n, it can be expressed by

$$f(x) = \sum_{i=0}^{n} a_i x^i,$$

where $n > 0$ and $a_n \neq 0$. Consider $g\colon R \to R$ expressed as (which is undefined if $n = 0$)

$$g(x) = \sum_{i=0}^{n-1} b_i x^i,$$

where

$$\begin{aligned} b_{n-1} &= a_n \neq 0, \\ b_k &= a_{k+1} + \lambda b_{k+1}, \quad \forall 0 \leq k < n-1, \end{aligned} \tag{1}$$

Coefficients b's are in R because all a's and λ are in R. **Therefore $g(x) \in R[x]$. Its degree is $n-1$, and its leading coefficient is nonzero.**

From mathematical induction it can be derived that

$$b_k = \sum_{m=0}^{n-1-k} a_{k+1+m} \lambda^m.$$

The following shows it satisfies $b_k = a_{k+1} + \lambda b_{k+1}$:

$$\begin{aligned} a_{k+1} + \lambda b_{k+1} &= a_{k+1} + \sum_{m=0}^{n-2-k} a_{k+2+m} \lambda^{m+1} \\ &= a_{k+1} + \sum_{m'=1}^{n-1-k} a_{k+1+m'} \lambda^{m'} \\ &= \sum_{m'=0}^{n-1-k} a_{k+1+m'} \lambda^{m'} = b_k, \end{aligned}$$

where $m' = m + 1$.

Therefore,

$$b_0 \lambda = \sum_{m'=1}^{n} a_{m'} \lambda^{m'},$$

where $m' = m + 1$. Since λ is the root of $f(x)$, we have

$$b_0\lambda = -a_0. \tag{2}$$

In conclusion (in case $n = 1$, the last summation evaluates zero), by plugging (1) and (2):

$$\begin{aligned}
(x-\lambda)g(x) &= \sum_{i'=1}^{n} b_{i'-1}x^{i'} - \sum_{i=0}^{n-1} b_i\lambda x^i \\
&= b_{n-1}x^n - b_0\lambda + \sum_{i=1}^{n-1}(b_{i-1} - b_i\lambda)x^i \\
&= a_n x^n + a_0 + \sum_{i=1}^{n-1} a_i x^i \\
&= f(x).
\end{aligned}$$

□

Definition (algebraically closed field)**.** A field F is **algebraically closed** if every non-constant polynomial in $F[x]$ has a root in F.

Proposition 4.2. *Given an algebraically closed field F, every non-constant polynomial in $F[x]$ is a product of first degree polynomials.*

Proof. Every non-constant polynomial $f(x) \in F[x]$ has degree $n > 0$. From the definition of algebraically closed field, $f(x)$ has a root, denoted λ. Therefore, $f(x) = (x - \lambda)g(x)$, where $g(x) \in F[x]$.

This process can be repeated until the remaining polynomial is a nonzero constant. Note that roots of $g(x)$ are also roots of $f(x)$. Since the order of g is one less than that of f, this can be done exactly n times, giving

$$f(x) = k\prod_{i=1}^{n}(x - \lambda_i),$$

where $\lambda_i \in F$ are roots of $f(x)$ and $k \in F$ is the leading coefficient. □

Eigenvalues, Eigenvectors, Schur decomposition, and Jordan form

Definition (algebraic multiplicity)**.** The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

Proposition 4.3. *Given a matrix $A \in \mathrm{M}_n(F)$, where F is an algebraically closed field, the sum of algebraic multiplicities is n.*

Proof. Definition of algebraically closed field. `https://en.wikipedia.org/wiki/Algebraically_closed_field#Every_polynomial_is_a_product_of_first_degree_polynomials` □

Definition (geometric multiplicity)**.** The dimension of the eigenspace associated with an eigenvalue is referred to as the eigenvalue's **geometric multiplicity**.

Theorem 4.4. *Geometric multiplicity is at least 1, at most algebraic multiplicity.*

Corollary 4.5. *There is at least one eigenvalue-eigenvector pair for a square matrix over an algebraically closed field.*

Proof. Worst case scenario: 1 eigenvalue, algebraic multiplicity is the order of the square matrix, geometric multiplicity is 1. □

Corollary 4.6. *Any square matrix over an algebraically closed field is conjugate with an upper triangular matrix.*

Proof. `https://math.stackexchange.com/questions/281833/matrix-similarity-upper-tria`

□

The direct sum of all eigenspaces is not necessarily the original vector space!

Proposition 4.7. *Any square matrix over an algebraically closed field is conjugate with an upper triangular matrix, with its eigenvalues in the main diagonal.*

Proof. In the previous proof, a square matrix A of order n is transformed to

$$B^{-1}AB = \begin{pmatrix} \lambda & C \\ 0 & A' \end{pmatrix} = \begin{pmatrix} \lambda & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{pmatrix},$$

where A' is the $(n-1) \times (n-1)$ submatrix. The characteristic polynomial for A is

$$\det(t\mathbf{I} - A) = \det(B^{-1}(t\mathbf{I} - A)B) = (t - \lambda)\det(t\mathbf{I} - A').$$

Therefore the eigenvalues of A' are also those of A, with the same algebraic multiplicities, except that of λ, which is decreased by 1. $\square$

Corollary 4.8. *The trace of a square matrix is the sum of its eigenvalues.*

Definition (Schur decomposition)**.** Given $A \in \mathrm{M}_n(K)$, where K is an algebraically closed field, the **Schur decomposition** (or Schur's unitary triangularization) of A is

$$A = QUQ^{-1},$$

where Q is a unitary matrix, and U is an upper triangular matrix, called the **Schur form** of A.

Theorem 4.9. *Any matrix $A \in \mathrm{M}_n(K)$, where K is an algebraically closed field, has Schur decomposition.*

The Schur form has the same spectrum, and its eigenvalues are the diagonal entries.

Definition (Jordan block)**.** a **Jordan block** over a ring R (whose identities are the zero 0 and one 1) is a matrix composed of zeroes everywhere except for the diagonal, which is filled with a fixed element $\lambda \in R$, and for the superdiagonal, which is composed of ones.

Definition (Jordan matrix)**.** Any block diagonal matrix whose blocks are Jordan blocks is called a **Jordan matrix**.

Theorem 4.10. *Any $n \times n$ square matrix A whose elements are in an algebraically closed field K is similar to a Jordan matrix J, whose main diagonal entries are eigenvalues of A.*

Remark. Given an eigenvalue λ_i, its geometric multiplicity is the number of Jordan blocks corresponding to λ_i.

The sum of the sizes of all Jordan blocks corresponding to an eigenvalue λ_i is its algebraic multiplicity.

Spectral Theorem

Proposition 4.11. *Any self-adjoint complex matrix is conjugate with a real diagonal matrix through similarity transformation by a unitary matrix.*

Proof. Perform the Schur decomposition for the self-adjoint matrix, denoted H:

$$Q^{-1}HQ = U,$$

where U is upper triangular and Q is unitary. Since H is self-adjoint, we have

$$U^\dagger = Q^\dagger H^\dagger (Q^{-1})^\dagger = Q^{-1}HQ = U.$$

$\square$

Lemma 4.12. *Unitary triangular matrix is diagonal, with diagonal entries being roots of unity.*

Proof. If the unitary matrix is lower triangular, we take its conjugate transpose. The new matrix should be a unitary upper triangular matrix.

Denote the unitary upper triangular matrix as T, then:

$$(T^\dagger T)_{ij} = \sum_{k=1}^{n} \overline{T_{ki}} T_{kj} = \delta_{ij}.$$

Since T is upper triangular, $T_{kj} = 0, \forall k > j$. Therefore,

$$\sum_{k=1}^{j} \overline{T_{ki}} T_{kj} = \delta_{ij}.$$

Especially, for $i = 1, j = 1$,

$$\overline{T_{11}} T_{11} = 1,$$

indicating $T_{11} = \mathrm{e}^{i\theta_1} \neq 0$ where $\theta_1 \in \mathbb{R}$.

For $i \neq 1, j = 1$,

$$\overline{T_{1i}} T_{11} = 0,$$

implying $T_{1i} = 0$ because $T_{11} \neq 0$. $T_{i1} = 0$ for all $i > 1$ because T is upper triangular.

We have proved that $T_{ij} = \mathrm{e}^{i\theta_j}\delta_{ij}$ for $i < 2$ or $j < 2$. If $T_{ij} = \mathrm{e}^{i\theta_j}\delta_{ij}$ for all $i < q$ or $j < q$, then (since $T_{kq} = 0$ for all $k < q$)

$$\delta_{pq} = (T^\dagger T)_{pq} = \sum_{k=1}^{q} \overline{T_{kp}} T_{kq} = \overline{T_{qp}} T_{qq}.$$

When $p = q$, this means $T_{qq} = \mathrm{e}^{i\theta_q} \neq 0$. When $p > q$, since $T_{qq} \neq 0$, we have

$$T_{qp} = \frac{\overline{\delta_{pq}}}{\overline{T_{qq}}} = 0.$$

Simultaneously, T_{pq} because $p > q$ and T is upper triangular. Therefore, $T_{ij} = \mathrm{e}^{i\theta_j}\delta_{ij}$ for all $i = q$ or $j = q$.

Mathematical induction concludes that T is a diagonal matrix with roots of unity as diagonal entries. Therefore its conjugate transpose (lower triangular matrix) has the same property. □

Proposition 4.13. *Any unitary matrix is conjugate with a diagonal matrix of roots of unity through similarity transformation by another unitary matrix.*

Proof. Perform the Schur decomposition for the $n \times n$ unitary matrix denoted U:

$$Q^{-1}UQ = T,$$

where T **is upper triangular** and Q is unitary. Since U is unitary, we have

$$T^\dagger T = Q^\dagger U^\dagger (Q^{-1})^\dagger Q^{-1}UQ = Q^{-1}U^{-1}QQ^{-1}UQ = \mathbf{I}.$$

T is invertible because so are Q and U. Therefore T **is unitary**.

□

Main Text

P45

Spectral theorem

Note that a general (invertible) matrix cannot be diagonalized, but it can be similarity-transformed to upper triangular form, where the main diagonal entries are eigenvalues.

Since similarity transformation does not change the trace of a matrix, the trace is still the sum of eigenvalues.

5 Lie Algebras and Lie Algebra Representations

5.2 Lie algebras of the orthogonal and unitary groups

P52 $U(n)$, different from $O(n)$, is connected because

$$\mathrm{e}^{i\pi} = -1,$$

while this is impossible in $\mathbb{R}$.

There are two components in $O(n)$, one of which containing identity is $SO(n)$.

P53 Prove: Skew-Hermitian matrices in $\mathfrak{u}(n)$ are diagonalizable.

$U(n)$ and $\mathfrak{u}(n)$ matrix are diagonalizable by unitary matrix, so any element Ω in the Lie group $U(n)$ can be written as

$$\Omega = \mathrm{e}^{tX},$$

where $t \in \mathbb{R}$ and X is in the corresponding Lie algebra.

Similarly $SO(n)$ can be written as exponential, but $SL(n)$ is large to be expressed this way. (maybe only Jordan form?)

5.4 Lie algebra representations

P54 Lie algebra representation is algebra homomorphism because it is linear map + preserving Lie bracket.

P57 The adjoint Lie algebra representation is a regular representation.

5.5 Complexification

P61 The complexification of $\mathfrak{su}(2)$ (3-dimension) is $\mathfrak{sl}(2, \mathbb{C})$ (6-real dimension).

The complexification of $\mathfrak{gl}(n, \mathbb{C})$ is a copy of two.

6 The Rotation and Spin Groups in 3 and 4 Dimensions

6.1 The rotation group in three dimensions

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Specialty in 3 dimension

Something very special that happens for orthogonal groups only in dimension $n = 3$ is that the vector representation (the defining representation of $\mathrm{SO}(n)$ matrices on $\mathbb{R}^n$) is isomorphic to the adjoint representation.

This is because any plane is bijectively corresponding to its perpendicular vector in 3 dimension.

Difference between Lie group representation and Lie algebra representation

Note that the adjoint Lie group representation is

$$(Ad, \mathfrak{g})$$

while the adjoint Lie algebra representation is

$$(ad, \mathfrak{g}).$$

Pi vector

Since the vector representation $(\pi_{vector}, \mathbb{R}^3)$ on column vectors and the adjoint representation $(Ad, \mathfrak{so}(3))$ are isomorphic, you can use the same matrix (in the adjoint representation) to represent g (which usually is represented as a column vector).

But the vector and matrix representations are transformed (or acting on other objects) differently (see top of P65).