Reading Notes for
Metamath Proof Explorer

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Contents

0 Notations

& and $\implies$ is used in statements, while $\wedge$ and $\rightarrow$ is used as logical conjunction.

  Order of operation

1. $\neg$ is evaluated first

2. $\wedge$ and $\vee$ are evaluated next

3. Quantifiers ($\exists$, $\forall$) are evaluated next

4. $\rightarrow$, $\leftrightarrow$ are evaluated last.

1 CLASSICAL FIRST-ORDER LOGIC WITH EQUALITY

In 1.2.5 Logical equivalence, $\neg(\varphi \rightarrow \neg\psi)$ means $\varphi \wedge \psi$.

The 1.2.8 The conditional operator for propositions is equivalent to $\cdot ? \cdot : \cdot$ operator in C programming language.

In 1.2.15 Half adder and full adder in propositional calculus, binary addition is defined.

In 1.4.7 Axiom scheme ax-6 (Existence), this should indicate the existence of a set.

First, we need to define 1.5.3 Axiom scheme ax-12 (Substitution). Then Class Abstraction (df-clab) can be defined.

2 ZF (ZERMELO-FRAENKEL) SET THEORY

2.1 ZF Set Theory - start with the Axiom of Extensionality

2.1.1 Introduce the Axiom of Extensionality

axext2

> **Theorem 19.36v**
>
> It references Theorem 19.36v:
>
> $$\vdash \Big(\exists x(\varphi \rightarrow \psi) \leftrightarrow (\forall x\varphi \rightarrow \psi)\Big).$$
>
> Note: $\forall x\varphi \rightarrow \psi$ means $(\forall x\varphi) \rightarrow \psi$.
>
> To prove this, think φ and ψ as predicates (Boolean-valued functions) of x.
>
> - Left to right:
>
> $\because$ "φ is true for all x" is a stronger condition than "there exists x such that φ is true",
> $\therefore$ "there exists one x such that $\varphi \rightarrow \psi$" indicates ("φ is true for all x" $\rightarrow$ "ψ is true").

- Right to left:

    If "φ is true for all x" will lead to "ψ is true", then there must be one x at which ψ is evaluated true. Therefore at this x, $\varphi \to \psi$.

The set z

Theorem axext2 claims that

$$\vdash \exists z \Big((z \in x \leftrightarrow z \in y) \to x = y \Big).$$

So what is this set z? One of the solutions is

$$z = \begin{cases} \varnothing & (x = y) \\ w & (x \neq y) \end{cases}, \tag{1}$$

where w is any element of $x \triangle y$ (defined in df-symdif), the latter of which must be nonempty if $x \neq y$.

2.1.2 Class abstractions (a.k.a. class builders)

Theorem dfcleq The equality of class is only checked by its containing sets:

$$\vdash \Big(A = B \leftrightarrow \forall x (x \in A \leftrightarrow x \in B) \Big).$$

Definition df-clel The membership of class is checked by set:

$$\vdash \Big(A \in B \leftrightarrow \exists x (x = A \wedge x \in B) \Big).$$

This means, if a class is not a set, then it does not belong to any class.

2.1.5 Restricted quantification

It gives the following definitions:

$$\vdash \Big(\forall x \in A \, \varphi \leftrightarrow \forall x (x \in A \to \varphi) \Big),$$
$$\vdash \Big(\exists x \in A \, \varphi \leftrightarrow \exists x (x \in A \wedge \varphi) \Big). \tag{2}$$
$$\vdash \{\, x \in A \mid \varphi \,\} = \{\, x \mid (x \in A \wedge \varphi) \,\}.$$

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2.1.6 The universal class

V is the class (not set) of all sets.

2.1.9 Proper substitution of classes for sets

Similar to 1.5.3 Axiom scheme ax-12 (Substitution).

2.1.15 "Weak deduction theorem" for set theory

This is similar to 1.2.8, but instead of generating a proposition (well-formed formula), it outputs a class (or set).

2.2 ZF Set Theory - add the Axiom of Replacement

2.2.1 Introduce the Axiom of Replacement

If a map $\varphi' = \forall y \varphi$ maps every set w in V to at most one set z in V, then there exists the **image** y (as a set) of the map φ' from x, i.e., $\varphi' \colon x \to y$ such that each element in y has at least one preimage in x. This is the generalized axiom of replacement:

$$\vdash \Big(\forall w \exists y \forall z \big(\forall y \varphi \to z = y \big) \to \exists y \forall z \big(z \in y \leftrightarrow \exists w (w \in x \wedge \forall y \varphi) \big) \Big).$$

Alternatively (axrep4, 4602*), we rewrite it this way ($\varphi' = \forall y \varphi$ is a well-formed formula not free in y) using Theorem mo2v (2370*):

$$\vdash \exists y \varphi' \implies \vdash \Big(\forall w \exists^* z \varphi' \to \exists y \forall z \big(z \in y \leftrightarrow \exists w (w \in x \wedge \forall \varphi') \big) \Big).$$

The correspondence to the one in the Wikipedia is

Wikipedia	x	y	A	B
Metamath	w	z	x	y

The difference from the one in the Wikipedia is

- In the left half, w (x) is not restricted within x (A) (added in Theorem axrep5).

- In the left half, z (y) does not have to exist.

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- In the right half, y (B) is the image instead of a codomain. (Maybe this is why axiom of separation/specification can be derived?)

- In the right half, "every element in the domain has a mapped value in codomain" becomes "every element in the image has at least a preimage in the domain".

2.2.2 Derive the Axiom of Separation

Other sources   Proving Separation from Replacement
How do the separation axioms follow from the replacement axioms?
Prove the Axiom of Replacement implies the Axiom of Specification.

Derive Axiom schema of specification from Theorem bm1.3ii   Axiom schema of specification *is* the Axiom of Separation.
The assertion of axiom schema of specification is:

Ref	Expression
axspec	$\vdash \left(\forall z \in V\ \exists x\ \forall y \big(y \in x \leftrightarrow (y \in z \wedge \varphi)\big)\right)$

This means a subset x of a set z exists.
Proof of Theorem axspec:

Step	Hyp	Ref	Expression
1		?	$\vdash \left(\forall z \in V\ \forall y(y \in z \wedge \psi \to y \in z)\right)$
2	1	?	$\vdash \left(\forall z \in V\ \exists x\ \forall y(y \in z \wedge \psi \to y \in x)\right)$
3	2	bm1.3ii	$\vdash \left(\forall z \in V\ \exists x\ \forall y(y \in x \leftrightarrow y \in z \wedge \psi)\right)$

The step 1 references the fact that $\vdash (\varphi \wedge \psi \to \varphi)$.
The step 2 references the fact that z exists.
The step 3 references Theorem bm1.3ii:

$$\vdash \exists x \forall y(\varphi \to y \in x) \implies \vdash \exists x \forall y(y \in x \leftrightarrow \varphi)$$

where φ is $y \in z \wedge \psi$.
The axiom of subset/separation/specification is result of the existence of the image of an identity map (restricted by φ).

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2.3 ZF Set Theory - add the Axiom of Power Sets

2.3.2 Derive the Axiom of Pairing

zfpair   Prove axiom of pairing form axiom of power set (no axiom of union). This is because

$$\mathscr{P}\mathscr{P}\varnothing = \{\,\varnothing, \{\,\varnothing\,\}\,\}$$

exists (Theorem pp0ex, 4681).

Then map $\varnothing$ to x and $\{\,\varnothing\,\}$ to y. The image set $\{\,x, y\,\}$ is what we need.

2.3.6 Epsilon and identity relations

E is the membership relationship (Theorem epelg 4845).

$$\vdash \left(B \in \mathrm{V} \to (A \, \mathrm{E} \, B \leftrightarrow A \in B) \right)$$

I is the identity function (Theorem dfid4 4850)

$$\vdash \mathrm{I} = (x \in \mathrm{V} \mapsto x)$$

2.3.7 Partial and complete ordering

These are strict partial/complete(total) ordering.

2.3.8 Founded and well-ordering relations

Set-Like   "R is set-like on A" ($R\,\mathrm{Se}\,A$) means, given any $x \in A$, the class of all $y \in A$ satisfying $y\,R\,x$ is a set (df-se 4893*).

Theorem exse 4897: Any relation on a set is set-like on it.

Theorem epse 4916: The epsilon relation is set-like on any class. (This is the origin of the term "set-like": a set-like relation "acts like" the epsilon relation of sets and their elements.)

$$\vdash \mathrm{E} \, \mathrm{Se} \, A$$

Well-Founded/Ordered   A binary relation R is called well-founded on a class A if every non-empty subset $x \subseteq A$ has a *minimal* element with respect to R.

A binary relation R is called well-ordered on a class A if every non-empty subset $x \subseteq A$ has a *least* element with respect to R (Theorem dfwe2).

Minimal element is not necessarily unique, but a least element, if it exists, is unique and is the only minimal element.

Theorem efrirr 4914: Irreflexivity of the epsilon relation—a class founded by epsilon is not a member of itself.

$$\vdash (\ \mathrm{E\ Fr}\ A \to \neg A \in A)$$

You can always stop the process of finding a containing element?

2.3.9 Relations

Converse   The converse of a binary relation swaps its arguments, i.e.,

$$\vdash A \in \mathrm{V}\ \&\ \vdash B \in \mathrm{V} \implies\ \vdash (A\check{R}B \leftrightarrow BRA)$$

See Theorem brcnv 5120.

Composition   The composition follows the convention (z does not need to be unique).

$$x \xrightarrow{\ B\ } z \xrightarrow{\ A\ } y$$
$$\underbrace{}_{A \circ B}$$

Restriction $A{\restriction}B$ is to restrict the domain of A into B.

Image $A\text{“}B$ is the image of B under A.

Then the axiom of replacement can be expressed as

$$\vdash A = \{\ \langle w, z\rangle \mid \forall y \varphi\ \}\ \&\ \vdash \forall w \exists^* z(wAz) \implies\ A\text{“}x \in \mathrm{V}$$

2.3.10 The Predecessor Class

$\mathrm{Pred}(R, A, X)$ is the class of all elements y of A such that yRX (Theorem elpredg 5502):

$$\vdash \left(\left(X \in B \wedge Y \in A\right) \to \left(Y \in \mathrm{Pred}(R, A, X) \leftrightarrow YRX\right)\right)$$

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2.3.12 Ordinals

Ordinal Number   An ordinal number is a **transitive** class whose nonempty subset must have a least element that belongs to all other elements of the subset. Need proof!
   This need to be checked:

$$\vdash \Big(\mathrm{Ord}\,A \to \forall z\big((z \subseteq A \wedge z \neq \varnothing) \to \exists x \in z \forall y \in z(x \neq y \to x \in y)\big)\Big)$$

But this is not sufficient condition. Consider the following set

$$A = \{\,\varnothing, \{\,\varnothing, \{\,\varnothing\,\}\,\}\,\}.$$

It is not transitive but $\vdash (\,\mathrm{E}\,\mathrm{We}\,A)$.
   The table below might be wrong!

Step	Hyp	Ref	Expression
We		dfwe2	$\vdash \Big(\mathrm{E}\,\mathrm{We}\,A \leftrightarrow \big(\mathrm{E}\,\mathrm{Fr}\,A \wedge \forall x \in A \forall y \in A(x\,\mathrm{E}\,y \vee x = y \vee y\,\mathrm{E}\,x)\big)\Big)$
E1		epel	$\vdash \Big(x \in y \leftrightarrow x\,\mathrm{E}\,y\Big)$
E2		epel	$\vdash \Big(y \in x \leftrightarrow y\,\mathrm{E}\,x\Big)$
We2	We,E1,E2	?	$\vdash \Big(\mathrm{E}\,\mathrm{We}\,A \leftrightarrow \big(\mathrm{E}\,\mathrm{Fr}\,A \wedge \forall x \in A \forall y \in A(x \in y \vee x = y \vee y \in x)\big)\Big)$
Fr		df-fr	$\vdash \Big(\mathrm{E}\,\mathrm{Fr}\,A \leftrightarrow \forall z\big((z \subseteq A \wedge z \neq \varnothing) \to \exists x \in z \forall y \in z \neg y\,\mathrm{E}\,x\big)\Big)$
Ss1		?	$\vdash \Big(z \subseteq A \to (x \in z \to x \in A)\Big)$
Ss2		?	$\vdash \Big(z \subseteq A \to (y \in z \to y \in A)\Big)$
NyEx		?	$\vdash \Big((x\,\mathrm{E}\,y \vee x = y \vee y\,\mathrm{E}\,x) \to \big((\neg y\,\mathrm{E}\,x) \leftrightarrow (x = y \vee x\,\mathrm{E}\,y)\big)\Big)$
WeR	We,Fr,Ss1,Ss2,NyEx,E1	?	$\vdash \Big(\mathrm{E}\,\mathrm{We}\,A \to \forall z\big((z \subseteq A \wedge z \neq \varnothing) \to \exists x \in z \forall y \in z(x = y \vee x \in y)\big)\Big)$
NxEx		?	$\vdash (x \notin x)$
NyEx.1	NxEx,E2	?	$\vdash \Big((x = y) \to (\neg y\,\mathrm{E}\,x)\Big)$
NyEx.2	E2	?	$\vdash \Big((x \in y) \to (\neg y\,\mathrm{E}\,x)\Big)$
FrL	Fr,NyEx.1,NyEx.2	?	$\vdash \Big(\forall z\big((z \subseteq A \wedge z \neq \varnothing) \to \exists x \in z \forall y \in z(x = y \vee x \in y)\big) \to \mathrm{E}\,\mathrm{Fr}\,A\Big)$
W		?	$\vdash \Big(\forall x \in A \forall y \in A \exists w(w = \{x,y\} \wedge w \subseteq A \wedge w \neq \varnothing)\Big)$
W2	W	?	$\vdash \Big(\forall z\big((z \subseteq A \wedge z \neq \varnothing) \to \exists x \in z \forall y \in z(x = y \vee x \in y)\big)$ $\to \forall x \in A \forall y \in A \exists w\big(w = \{x,y\} \wedge \exists u \in w \forall v \in w(u = v \vee u \in v)\big)\Big)$
QED.1	WeR,WeL	?	$\vdash \Big(\mathrm{E}\,\mathrm{We}\,A \leftrightarrow \forall z\big((z \subseteq A \wedge z \neq \varnothing) \to \exists x \in z \forall y \in z(x = y \vee x \in y)\big)\Big)$
QED	QED.1	?	$\vdash \Big(\mathrm{E}\,\mathrm{We}\,A \leftrightarrow \forall z\big((z \subseteq A \wedge z \neq \varnothing) \to \exists x \in z \forall y \in z(x \neq y \to x \in y)\big)\Big)$

   Theorem dfepfr 4918*: An alternate way of saying that the epsilon relation is well-founded.

$$\vdash \Big(\mathrm{E}\,\mathrm{Fr}\,A \leftrightarrow \forall x\big((x \subseteq A \wedge x \neq \varnothing) \to \exists y \in x(x \cap y) = \varnothing\big)\Big)$$

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Theorem wefrc 4927: A nonempty (possibly proper) subclass of a class well-ordered by E has a minimal element.

$$\vdash \Big((\text{ E We } A \wedge B \subseteq A \wedge B \neq \varnothing) \rightarrow \exists x \in B (B \cap x) = \varnothing \Big)$$

Theorem onelss 5573: An element of an ordinal number is a subset of the number.

$$\vdash \Big(A \in \text{On} \rightarrow (B \in A \rightarrow B \subseteq A) \Big)$$

Theorem dford2 8275: Assuming ax-reg 8255, an ordinal is a transitive class on which inclusion satisfies trichotomy.

$$\vdash \Big(\text{Ord } A \leftrightarrow \big(\text{Tr } A \wedge \forall x \in A \forall y \in A (x \in y \vee x = y \vee y \in x) \big) \Big)$$

Theorem ordirr 5549 Epsilon irreflexivity of ordinals: no ordinal class is a member of itself. Theorem 2.2(i) of [BellMachover] p. 469, generalized to classes. We prove this without invoking the Axiom of Regularity.

Theorem nordeq 5550 A member of an ordinal class is not equal to it.

orci/clci sylancb

Limit Ordinal   Theorem dflim3 6814: An alternate definition of a limit ordinal, which is any ordinal that is neither zero nor a successor.

$$\vdash \Big(\text{Lim } A \leftrightarrow \big(\text{Ord } A \wedge \neg (A = \varnothing \vee \exists x \in \text{On } A = \text{suc } x) \big) \Big)$$

2.3.13 Definite description binder (inverted iota)

Define "the unique x such that φ", where φ ordinarily contains x as a free variable. Our definition is meaningful only when there is exactly one x such that φ is true; otherwise, it evaluates to the empty set.

Theorem iota1 5668: Property of iota.

$$\vdash \Big(\exists! x \varphi \rightarrow \big(\varphi \leftrightarrow ((\imath x \varphi) = x) \big) \Big)$$

Theorem iotanul 5669 This theorem is the result if there isn't exactly one x that satisfies φ.

$$\vdash \Big(\neg \exists! x \varphi \rightarrow ((\imath x \varphi) = \varnothing) \Big)$$

2.3.14 Functions

Function   Class F is a function if and only it is a relation where there exists at most one right-hand-side value for any left-hand-side value. (Theorem dffun6 5705*, df-rel)

$$\vdash \left(\mathrm{Fun}\, F \leftrightarrow \left(F \subseteq (\mathrm{V} \times \mathrm{V}) \wedge \forall x \exists^* y\, xFy \right) \right)$$

Theorem funopab 5723*: A class of ordered pairs is a function when there is at most one second member for each pair.

$$\vdash \left(\mathrm{Fun}\, \{\, \langle x, y \rangle \mid \varphi \,\} \leftrightarrow \forall x \exists^* y\, \varphi \right)$$

Theorem funmpt 5726: A function in maps-to notation ("the function defined by the map from x (in A) to $B(x)$", df-mpt) is a function.

$$\vdash \mathrm{Fun}\, (x \in A \mapsto B)$$

Theorem funco 5728 The composition $F \circ G$ of two functions is a function.

Theorem funres 5729 A restriction $F \!\restriction\! A$ of a function is a function.

Theorem funssres 5730 The restriction of a function to the domain of a subclass equals the subclass.

Theorem fun0 5754 The empty set is a function.

Theorem f0 5883 The empty function.

$$\vdash \varnothing : \varnothing \longrightarrow A$$

Function with domain and codomain

$$F : A \longrightarrow B$$

$$F : A \xrightarrow{\text{1-1}} B$$

$$F : A \xrightarrow[\text{onto}]{} B$$

$$F : A \xrightarrow[\text{onto}]{\text{1-1}} B$$

Evaluation The definition of evaluated function (df-fv) does not require Fun F.

$$\vdash (F`A) = (\imath x A F x)$$

It generates empty set if F evaluated at A is not unique or $A \notin \operatorname{dom} F$: Theorem ndmfv 6012, the value of a class outside its domain is the empty set.

$$\vdash \left(\neg A \in \operatorname{dom} F \to (F`A) = \varnothing \right)$$

Theorem fvprc 5981 A function's value at a proper class is the empty set.

$$\vdash \left(\neg A \in \mathrm{V} \to (F`A) = \varnothing \right)$$

Theorem fveq1 5986 Equality theorem for function value.

$$\vdash \left(F = G \to (F`A) = (G`A) \right)$$

Theorem fveq2 5987 Equality theorem for function value.

$$\vdash \left(A = B \to (F`A) = (F`B) \right)$$

Theorem fvex 5997 The value of a class exists.

$$\vdash (F`A) \in \mathrm{V}$$

Isomorphism Isomorphism is a bijection from A onto B with relation R in A preserved in B as S (df-isom).

2.3.15 Cantor's Theorem

Theorem canth 6385 No set is equinumerous to its power set (Cantor's theorem), i.e. no function can map a set onto its power set.

Cantor's Theorem. No set is equinumerous to its power set. Specifically, any set has a cardinality (size) strictly less than the cardinality of its power set. For example, the cardinality of real numbers is the same as the cardinality of the power set of integers, so real numbers cannot be put into a one-to-one correspondence with integers.

Theorem ncanth 6386 Cantor's theorem fails for the universal class (which is not a set but a proper class by vprc 4624). Specifically, the identity function maps the universe onto its power class.

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2.3.16 Restricted iota (description binder)

Definition df-riota 6388 Define restricted description binder. In case there is no unique x such that $(x \in A \land \varphi)$ holds, it evaluates to the empty set.

2.3.17 Operations

Note that the syntax is simply three class symbols in a row surrounded by parentheses.

Difference between operations and relations

- Relations ARB are not parenthesized, while operations are (AFB).

- Relations are *well-formed formulas*, while operations are *classes*.

Theorem funoprabg 6533* "At most one" is a sufficient condition for an operation class abstraction to be a function.

$$\vdash \left(\forall x \forall y \exists^* z \varphi \rightarrow \text{Fun } \{ \langle\langle x, y\rangle, z\rangle \mid \varphi \} \right)$$

2.3.19 Function operation

Given a binary operation R, this maps two functions into a single function:

$$\circ_f R$$

Example:

$$\vdash \left(x \in (\text{dom } f \cap \text{dom } g) \rightarrow \left((f \circ_f Rg) \text{'} x \right) = \left((f\text{'}x) R(g\text{'}x) \right) \right)$$

Given a binary relation R, this is a relation between two functions:

$$\vdash \left((f \circ_r Rg) \leftrightarrow \forall x \in (\text{dom } f \cap \text{dom } g) \, (f\text{'}x) R(g\text{'}x) \right)$$

$$\vdash \left((\text{dom } f \cap \text{dom } g = \varnothing) \rightarrow (f \circ_r Rg) \right)$$

2.3.20 Proper subset relation

This is a relation of sets instead of a wff of classes.

2.4 ZF Set Theory - add the Axiom of Union

2.4.3 Transfinite induction

If $A \in \mathrm{On}$, then $\in$ means $<$, $\cap$ means min of containing elements, $\cup$ means max of containing elements, $\subseteq$ means $\leqslant$.

Theorem tfis 6821* Transfinite Induction Schema. If all ordinal numbers less than a given ordinal number x have a property (induction hypothesis), then all ordinal numbers have the property (conclusion).

$$\vdash \Big(x \in \mathrm{On} \to \big((\forall y \in x [y/x]\varphi) \to \varphi \big) \Big) \implies \vdash \Big(x \in \mathrm{On} \to \varphi \Big)$$

Theorem tfindes 6829* Transfinite Induction with explicit substitution.

1. The first hypothesis is the basis,

2. the second is the induction step for successors,

3. and the third is the induction step for limit ordinals.

$$\vdash [\varnothing/x]\varphi \,\&\, \vdash \Big(x \in \mathrm{On} \to \big(\varphi \to ([\mathrm{suc}\, x/x]\varphi) \big) \Big) \,\&\, \vdash \Big(\mathrm{Lim}\, y \to \big((\forall x \in y\varphi) \to [y/x]\varphi \big) \Big)$$
$$\implies \vdash \Big(x \in \mathrm{On} \to \varphi \Big)$$

2.4.4 The natural numbers (i.e. finite ordinals)

Define the class of natural numbers ω, which are all ordinal numbers that are less than every limit ordinal, i.e., all finite ordinals. (df-om)

Theorem peano2b 6848 A class belongs to omega iff its successor does.

Theorem omsinds 6851* Strong (or "total") induction principle over the finite ordinals.

Allowed substitution hints:

$$\varphi(x), \psi(y), \chi(y), A(y)$$

$$\vdash \Big(x = y \to \big(\varphi \leftrightarrow \psi \big) \Big) \,\&\, \vdash \Big(x = A \to \big(\varphi \leftrightarrow \chi \big) \Big) \,\&\, \vdash \Big(x \in \omega \to \big((\forall y \in x\psi) \to \varphi \big) \Big)$$
$$\implies \vdash \Big(A \in \omega \to \chi \Big)$$

(actually it is $\chi(A)$)

Hypothesis: if $\psi(y)$ holds for all y smaller than x, then $\varphi(x)$ is also true.

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Assertion: $\chi(A)$ is true for all $A \in \omega$.

Note that $\vdash ([\varnothing/x]\varphi)$ must be true, because $\vdash \varnothing \in \omega$ and $\vdash (\forall y \in \varnothing \psi)$, i.e.,

$$\vdash \left(\forall y (y \notin \varnothing \vee \psi) \right)$$

are constantly true.

2.4.6 Finite induction (for finite ordinals)

Theorem findes 6863: Finite induction (mathematical induction) with explicit substitution.

1. The first hypothesis is the basis

2. and the second is the induction step.

$$\vdash [\varnothing/x]\varphi \,\& \vdash \left(x \in \omega \to \left(\varphi \to ([\operatorname{suc} x/x]\varphi) \right) \right) \implies \vdash \left(x \in \omega \to \varphi \right)$$

2.4.7 First and second members of an ordered pair

$$1^{\text{st}}\ 2^{\text{nd}}$$

Theorem 1stnpr 6937 Value of the first-member function at non-pairs.

$$\vdash \left(A \notin (\text{V} \times \text{V}) \to (1^{\text{st}}\,{}^{\backprime}A) = \varnothing \right)$$

Theorem 2ndnpr 6938 Value of the second-member function at non-pairs.

$$\vdash \left(A \notin (\text{V} \times \text{V}) \to (2^{\text{nd}}\,{}^{\backprime}A) = \varnothing \right)$$

Theorem 1st0 6939 The value of the first-member function at the empty set.

$$\vdash (1^{\text{st}}\,{}^{\backprime}\varnothing) = \varnothing$$

Theorem 2nd0 6940 The value of the second-member function at the empty set.

$$\vdash (2^{\text{nd}}\,{}^{\backprime}\varnothing) = \varnothing$$

Theorem op1st 6941 Extract the first member of an ordered pair.

$$\vdash A \in \text{V} \,\& \vdash B \in \text{V} \,\& \implies \vdash (1^{\text{st}}\,{}^{\backprime}\langle A, B \rangle) = A$$

Theorem op2nd 6942 Extract the second member of an ordered pair.

$$\vdash A \in \text{V} \,\& \vdash B \in \text{V} \,\& \implies \vdash (2^{\text{nd}}\,{}^{\backprime}\langle A, B \rangle) = B$$

2.4.8 The support of functions

$$\vdash \left((x \text{ supp } z) = \{\, i \in \text{dom}\, x \mid (x \text{``} \{\, i \,\}) \neq \{\, z \,\} \,\} \right)$$

The reason why $(x \text{``} \{\, i \,\}) \neq \{\, z \,\}$ is used instead of $(x\text{`}i) \neq z$ is that x is allowed to have multiple mapped values for i.

2.4.10 Function transposition

Definition df-tpos 7113* Define the transposition of a function, which is a function $G = \text{tpos } F$ satisfying $G(x, y) = F(y, x)$.

$$\vdash \text{tpos } F = \left(F \circ \left(x \in (\widetilde{\text{dom}\, F} \cup \{\, \varnothing \,\}) \mapsto \cup \{\widetilde{\, x \,}\} \right) \right)$$

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