Solving Poisson's Equation in 2-D

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1 Introduction

Poisson's equation is important partial differential equation used in physics to electrostatic and gravitational fields from their potential fields. Thus, the potential $u(\vec{r})$ must satisfies Poisson's equation,

$$\nabla^2 u(\vec{r}) = \lambda \rho(\vec{r}) \tag{1}$$

where λ is a constant. For instance, with respect to gravitational fields $\lambda = 4\pi^2 G$ [1].

In this paper we will be solving Poisson's equation in 2-D with given boundary conditions for a cases involving both one and two charged wires parallel to the z-axis by first solving equation (1) for $u(\vec{r})$ in a case where the solution is known. This is done to make sure the code for our 2-D Poisson solver is functioning properly. The 2-D Poisson solver works by performing a Fourier Transform to convert the 2-D problem into a 1-D problem and then using a 1-D Poisson solver before transforming it back to a 2-D problem. Therefore, we will begin the paper by solving a 1-D Poisson problem before moving on to explore the potential produced by the electrically charged wires.

2 1-D Poisson Solver

We begin by building and testing our code, Appendix A, for a Poisson problem in one dimension. This is done by discretizing the domain of x into N equal-spaced intervals of δx. For our situation

$$\rho(x) = 1 - 2x^2 \quad , \tag{2}$$

and the boundary conditions are

$$u\left(x_0\right) - \frac{du(x_0)}{dx_0} = 1 \quad , \quad u\left(x_h\right) + \frac{du(x_h)}{dx_h} = 1 \quad . \tag{3}$$

The analytical and numerical solutions shown in Fig. 1 show that the numerical solutions generated by our 1-D Poisson solver are accurate.

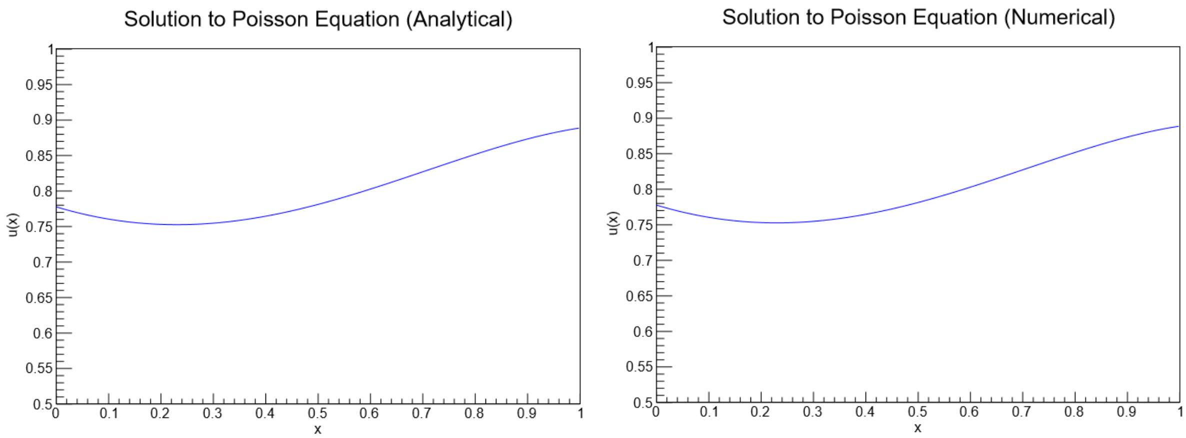

Fig.1 These plots show the solutions, both numerical and analytical , to the 1-D Poisson's equation if given (2) and the boundary conditions (3).

With our 1-D Poisson solver functioning properly we are ready to move on to the 2-D situations.

3 2-D Poisson Solver

We will now explore solutions a 2-D Poisson solver by first solving (1) for a situation in which the solution is known before applying the 2-D Poisson solver to situations involving both 1 and 2 wires that are electrically charged where the solution is not known a priori. Since a reasonable size of mesh for our 2-D Poisson is expensive computationally it is very difficult, if not impossible, to Poisson's equation for 2-D and higher. Therefore, it is useful to first convert the 2-D problem into a 1-D problem by using a Fourier transformation before solving and then converting back into 2-D.

In our first situation in which the analytical solution is known we begin with

$$\rho(x,y) = 6xy(1-y) - 2x^3 \quad . \tag{4}$$

where $0 \le x \le 1$ and $0 \le y \le 1$. The boundary conditions in this situation are $u(0,y)=0$, $u(1,y)=y(1-y)$, and $u(x,0)=u(x,1)=0$. The numerical and analytical solutions are both plotted as a function of x at $y=0.5$ and as a function of y at $x=0.5$, Fig. 2, to ensure that our 2-D Poisson solver is working before moving on to the situations involving the charged wires.

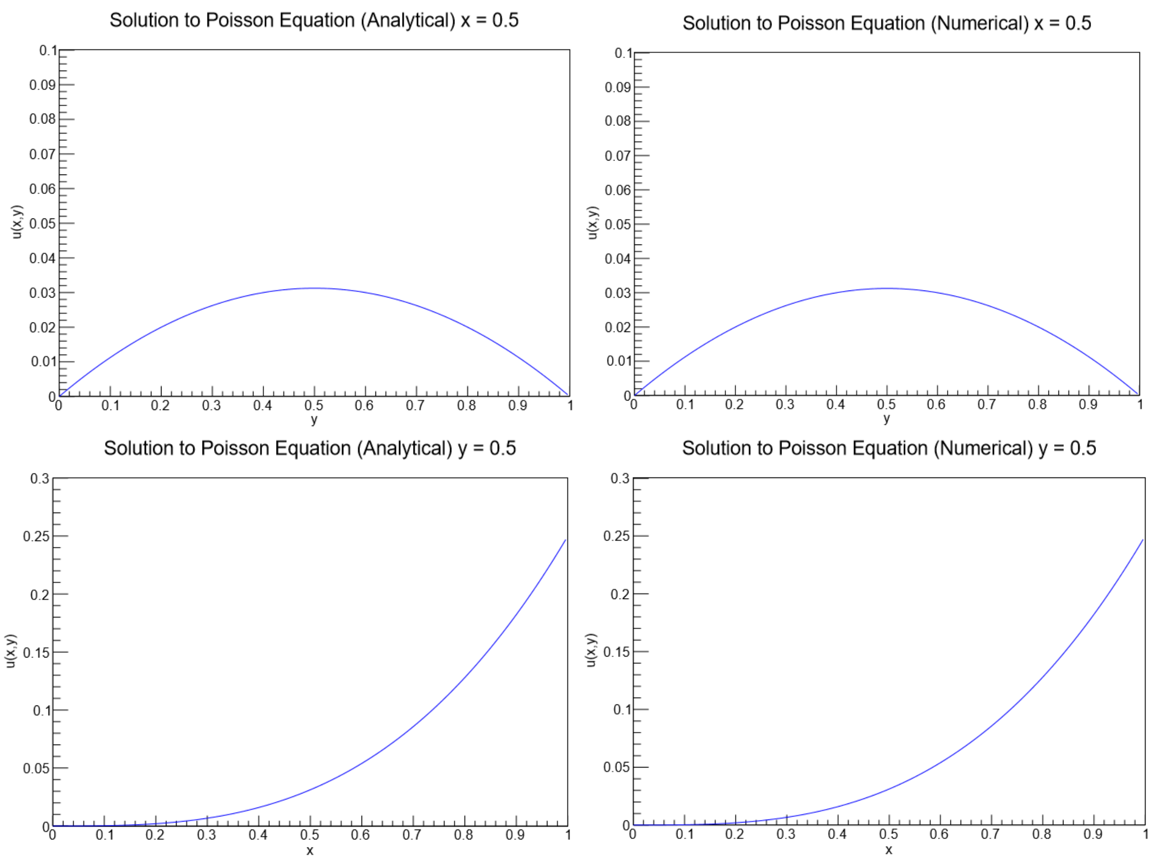

Fig. 2 These plots show the solutions, both numerical and analytical, to the 2-D Poisson's equation if given by equation (4) and its' boundary conditions.

Now that we are confident in functionality of our 2-D Poisson solver we will use it for the situation of a charged wire located at x_0 and y_0 parallel to the z-axis.

We place a charged wire at $x_0 = y_0 = 0.5$ running parallel to the z-axis. The Poisson's equation for the electric potential in this scenario is

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = -\delta(x - x_0)\delta(y - y_0) \quad . \tag{5}$$

The solution to equation (5) is plotted in Fig. 3. The potential is dropping as a function of distance from wire source as expected. We also calculated the electric field E for several distances from the wire, Table 1, which behaves as expected for an infinite wire running along z-axis.

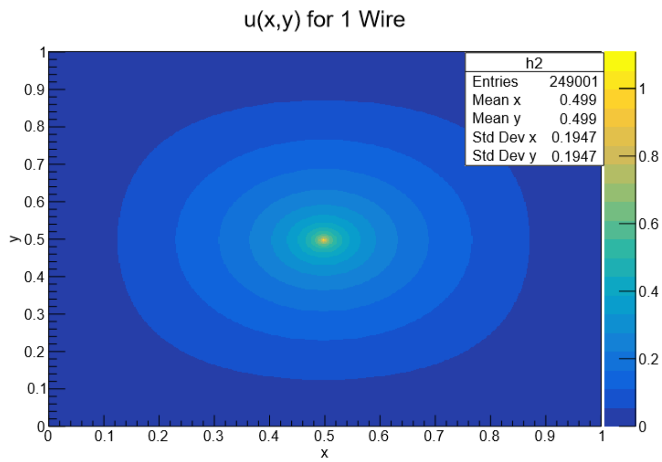

Fig. 3 Solution to the Poisson's equation in 2-D (5).

$\|x-x_0\|$	$\|y-y_0\|$	E
.002	0	112.79
.006	0	31.7642
.1	0	17.6836

Table 1 This table shows the electric field calculated from the potential at the given distances from the wire.

Lastly, in addition to a wire at $x_0 = y_0 = 0.5$ we also place a wire at $x_1 = y_1 = 0.25$ also running parallel to the z-axis. The Poisson's equation for the scaled potential in this case is

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = -q\delta(x)\delta(y)\delta(x-x_0)\delta(y-y_0) \tag{6}$$

where $q = 1$ is the ratio of the currents in the two wires. First we solve $u(x, y)$ numerically with different $(\delta x, \delta y)$, Fig. 4, and different size of mesh, Fig. 5.

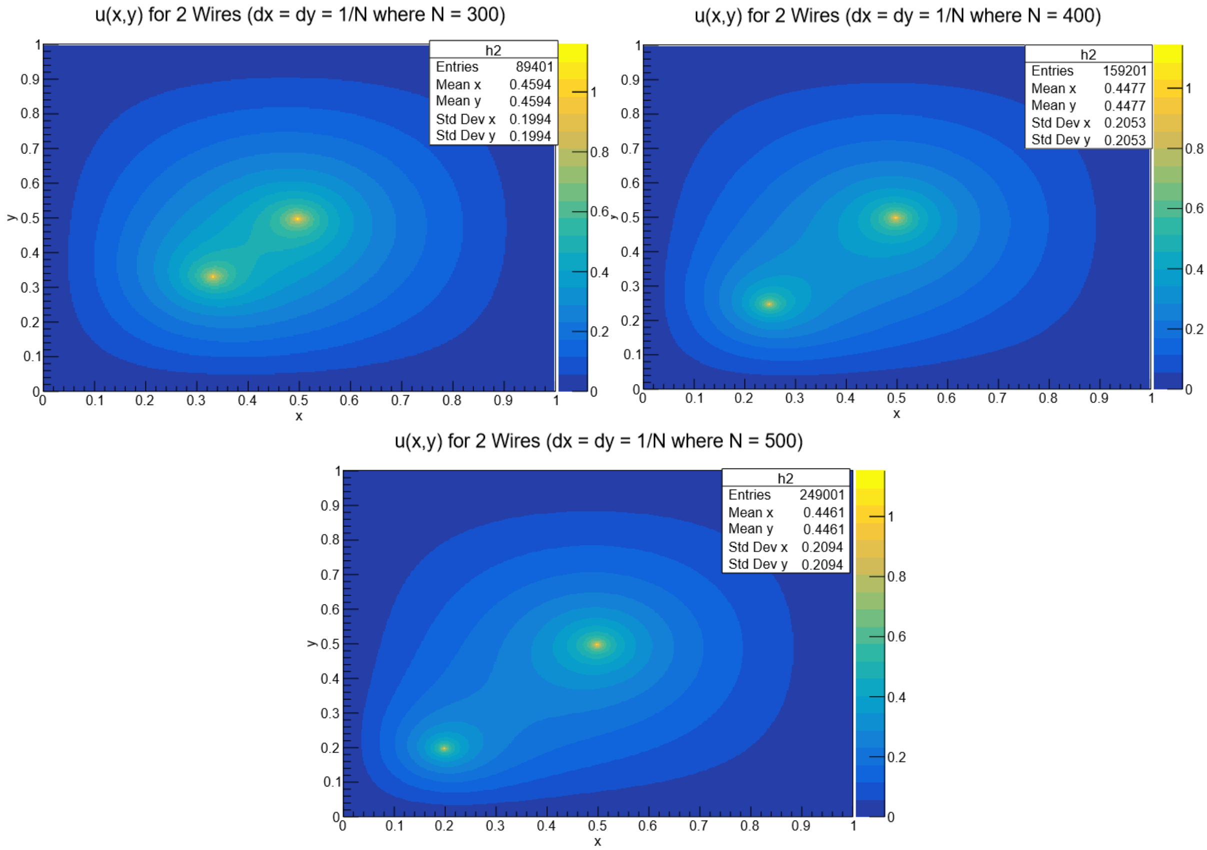

Fig. 4 Solutions to (6) as a function of $(\delta x, \delta y)$ by varying N.

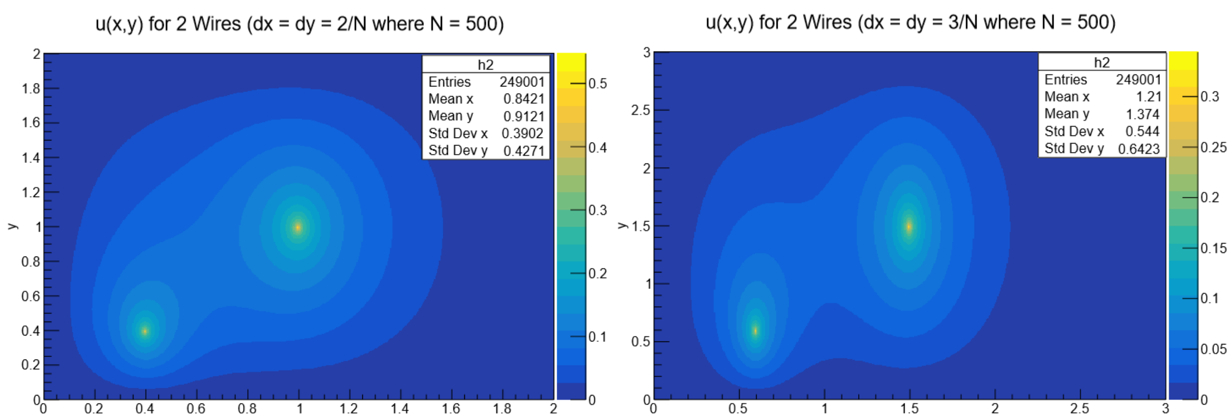

Fig. 4 Solutions to (6) plotted at different sizes of mesh.

4 Conclusion

In this paper we solved the Poisson's equation (1) in 2-D for a situation involving both 1 and 2 wires carrying an electric current. We did this by building a program that solves the2-D Poisson's equation by first converting it to a 1-D problem before solving and then converting back to 2-D. To ensure our program worked correctly we first constructed and studied the 1-D case before moving on to the 2-D cases.

We found that the potential behaved as expected for that produced by in infinitely square wire as seen in Fig. 3, 4, 5.

References

[1] Shi, Jack J., *Lecture Notes For Computational Physics: Chapter 4*. 2019.