Theoretical Background

Namu Kroupa

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1 Matrix equation

To calculate the reflectivity, $R = |b_0|^2$, the matrix equation

$$\mathbf{M}\mathbf{x} = \mathbf{c} \tag{1}$$

must be solved for $\mathbf{x}$, where

$$\mathbf{x} = (b_0, a_1, b_1, a_2, b_2, \ldots, a_M, b_M, a_{M+1})^T \tag{2}$$

and

$$\mathbf{c} = (1, 1, 0, 0, \ldots, 0)^T \tag{3}$$

are $(2M+2)$-dimensional complex vectors.

The matrix $\mathbf{M} \in \mathbb{C}^{(2M+2)\times(2M+2)}$ is

$$\mathbf{M} = \left(\begin{array}{c|c|c|c|c|c|c|c}
\begin{matrix} a \\ b \end{matrix} & \boldsymbol{\beta}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline
\begin{matrix} 0 \\ 0 \end{matrix} & \boldsymbol{\alpha}_1 & \boldsymbol{\beta}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline
\begin{matrix} 0 \\ 0 \end{matrix} & \mathbf{0} & \boldsymbol{\alpha}_2 & \boldsymbol{\beta}_2 & \mathbf{0} & \cdots & \mathbf{0} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline
\begin{matrix} 0 \\ 0 \end{matrix} & \mathbf{0} & \mathbf{0} & \boldsymbol{\alpha}_3 & \boldsymbol{\beta}_3 & \cdots & \mathbf{0} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \boldsymbol{\beta}_{M-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline
\begin{matrix} 0 \\ 0 \end{matrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\alpha}_M & \begin{matrix} c \\ d \end{matrix}
\end{array}\right), \tag{4}$$

(the horizontal lines are for readability) where

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5}$$

To solve Equation 1 efficiently, its "band-storage form" $\mathbf{M}_{\text{band}} \in \mathbb{C}^{5\times(2M+2)}$ is used:

$$\mathbf{M}_{\text{band}} = \begin{pmatrix} 0 & & & & & 0 \\ 0 & & & & & c \\ a & \boldsymbol{c}_0 & \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_{M-1} & d \\ b & & & & & 0 \\ 0 & & & & & 0 \end{pmatrix} \tag{6}$$

where

$$\boldsymbol{c}_i = \begin{pmatrix} 0 & (\boldsymbol{\beta}_i)_{01} \\ (\boldsymbol{\beta}_i)_{00} & (\boldsymbol{\beta}_i)_{11} \\ (\boldsymbol{\beta}_i)_{10} & (\boldsymbol{\alpha}_{i+1})_{01} \\ (\boldsymbol{\alpha}_{i+1})_{00} & (\boldsymbol{\alpha}_{i+1})_{11} \\ (\boldsymbol{\alpha}_{i+1})_{10} & 0 \end{pmatrix} \quad \text{for} \quad i = 0, \ldots, M-1 \tag{7}$$

For s-polarisation,

$$a = -1 \tag{8}$$
$$b = 1 \tag{9}$$
$$c = 1 \tag{10}$$
$$d = (k_x)_{\text{substrate}} \tag{11}$$
$$\boldsymbol{\alpha}_j = \begin{pmatrix} -e^{i\phi_{j-1}} & -1 \\ -e^{i\phi_{j-1}}(k_x)_{j-1} & (k_x)_{j-1} \end{pmatrix} \quad \text{for} \quad j = 1, \ldots, M \tag{12}$$
$$\boldsymbol{\beta}_j = \begin{cases} \begin{pmatrix} 1 & e^{i\phi_0} \\ \frac{(k_x)_0}{(k_x)_{\text{outer}}} & -\frac{(k_x)_0}{(k_x)_{\text{outer}}} e^{i\phi_0} \end{pmatrix} & \text{for} \quad j = 0 \\ \begin{pmatrix} 1 & e^{i\phi_j} \\ (k_x)_j & -(k_x)_j e^{i\phi_j} \end{pmatrix} & \text{for} \quad j = 1, \ldots, M-1 \end{cases}. \tag{13}$$

For p-polarisation,

$$a = -1 \tag{14}$$
$$b = 1 \tag{15}$$
$$c = \frac{(k_x)_{\text{substrate}}}{n_{\text{substrate}}} \tag{16}$$
$$d = n_{\text{substrate}} \tag{17}$$
$$\boldsymbol{\alpha}_j = \begin{pmatrix} -e^{i\phi_{j-1}} \frac{(k_x)_{j-1}}{n_{j-1}} & -\frac{(k_x)_{j-1}}{n_{j-1}} \\ -e^{i\phi_{j-1}} n_{j-1} & n_{j-1} \end{pmatrix} \quad \text{for} \quad j = 1, \ldots, M \tag{18}$$
$$\boldsymbol{\beta}_j = \begin{cases} \begin{pmatrix} \frac{(k_x)_0}{n_0} \frac{n_{\text{outer}}}{(k_x)_{\text{outer}}} & e^{i\phi_0} \frac{(k_x)_0}{n_0} \frac{n_{\text{outer}}}{(k_x)_{\text{outer}}} \\ \frac{n_0}{n_{\text{outer}}} & -e^{i\phi_0} \frac{n_0}{n_{\text{outer}}} \end{pmatrix} & \text{for} \quad j = 0 \\ \begin{pmatrix} \frac{(k_x)_j}{n_j} & e^{i\phi_j} \frac{(k_x)_j}{n_j} \\ n_j & -e^{i\phi_j} n_j \end{pmatrix} & \text{for} \quad j = 1, \ldots, M-1 \end{cases}. \tag{19}$$

2 Fabry-Perot interferometer

As a test case, the Fabry-Perot interferometer is considered, which consists of a single layer of refractive index n_{layer} and thickness d in a medium of refractive index n_{outer} and whose reflectivity at a wavelength λ_{vac} (in vacuum) and incident angle θ_{outer} is given by

$$R = \left| \frac{r(1 - e^{2i\phi_0})}{1 - r^2 e^{2i\phi_0}} \right|^2 \tag{20}$$

where

$$\phi_0 = \frac{2\pi}{\lambda_{\text{vac}}} n_{\text{layer}} d \cos\theta_{\text{layer}} \tag{21}$$
$$r = \begin{cases} \frac{n_{\text{outer}} \cos\theta_{\text{outer}} - n_{\text{layer}} \cos\theta_{\text{layer}}}{n_{\text{outer}} \cos\theta_{\text{outer}} + n_{\text{layer}} \cos\theta_{\text{layer}}} & \text{for s-polarisation,} \\ \frac{n_{\text{layer}} \cos\theta_{\text{outer}} - n_{\text{outer}} \cos\theta_{\text{layer}}}{n_{\text{layer}} \cos\theta_{\text{outer}} + n_{\text{outer}} \cos\theta_{\text{layer}}} & \text{for p-polarisation} \end{cases}. \tag{22}$$

The angle θ_{layer} is the angle of the light ray inside the layer. Its cosine is given by

$$\cos\theta_{\text{layer}} = \sqrt{1 - \left(\frac{n_{\text{outer}} \sin\theta_{\text{outer}}}{n_{\text{layer}}} \right)^2}. \tag{23}$$

In the case of total internal refraction, $\cos\theta_{\text{layer}}$ becomes imaginary and the above equations for R, ϕ_0 and r continue to hold.