

Theoretical Background

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1 Matrix equation

To calculate the reflectivity, $R = |b_0|^2$, the matrix equation

$$\mathbf{M}\mathbf{x} = \mathbf{c} \quad (1)$$

must be solved for \mathbf{x} , where

$$\mathbf{x} = (b_0, a_1, b_1, a_2, b_2, \dots, a_M, b_M, a_{M+1})^T \quad (2)$$

and

$$\mathbf{c} = (1, 1, 0, 0, \dots, 0)^T \quad (3)$$

are $(2M + 2)$ -dimensional complex vectors.

The matrix $\mathbf{M} \in \mathbb{C}^{(2M+2) \times (2M+2)}$ is

$$\mathbf{M} = \begin{pmatrix} a & \beta_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 \\ b & & & & & & & 0 \\ 0 & \alpha_1 & \beta_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 \\ 0 & & & & & & & 0 \\ 0 & \mathbf{0} & \alpha_2 & \beta_2 & \mathbf{0} & \cdots & \mathbf{0} & 0 \\ 0 & & & & & & & 0 \\ 0 & \mathbf{0} & \mathbf{0} & \alpha_3 & \beta_3 & \cdots & \mathbf{0} & 0 \\ 0 & & & & & & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \beta_{N-1} & 0 \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \alpha_N & c \\ 0 & & & & & & & d \end{pmatrix}, \quad (4)$$

(the horizontal lines are for readability) where

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (5)$$

To solve Equation 1 efficiently, its “band-storage form” $\mathbf{M}_{\text{band}} \in \mathbb{C}^{5 \times (2M+2)}$ is used:

$$\mathbf{M}_{\text{band}} = \begin{pmatrix} 0 & & & & & & 0 \\ 0 & & & & & & c \\ a & \mathbf{c}_0 & \mathbf{c}_1 & \cdots & \mathbf{c}_{M-1} & & d \\ b & & & & & & 0 \\ 0 & & & & & & 0 \end{pmatrix} \quad (6)$$

where

$$\mathbf{c}_i = \begin{pmatrix} 0 & (\beta_i)_{01} \\ (\beta_i)_{00} & (\beta_i)_{11} \\ (\beta_i)_{10} & (\alpha_{i+1})_{01} \\ (\alpha_{i+1})_{00} & (\alpha_{i+1})_{11} \\ (\alpha_{i+1})_{10} & 0 \end{pmatrix} \quad \text{for } i = 0, \dots, M-1 \quad (7)$$

For s-polarisation,

$$a = -1 \quad (8)$$

$$b = 1 \quad (9)$$

$$c = 1 \quad (10)$$

$$d = (k_x)_{\text{substrate}} \quad (11)$$

$$\alpha_j = \begin{pmatrix} -e^{i\phi_{j-1}} & -1 \\ -e^{i\phi_{j-1}}(k_x)_{j-1} & (k_x)_{j-1} \end{pmatrix} \quad \text{for } j = 1, \dots, M \quad (12)$$

$$\beta_j = \begin{cases} \begin{pmatrix} 1 & e^{i\phi_0} \\ \frac{(k_x)_0}{(k_x)_{\text{outer}}} & -\frac{(k_x)_0}{(k_x)_{\text{outer}}} e^{i\phi_0} \end{pmatrix} & \text{for } j = 0 \\ \begin{pmatrix} 1 & e^{i\phi_j} \\ (k_x)_j & -(k_x)_j e^{i\phi_j} \end{pmatrix} & \text{for } j = 1, \dots, M-1 \end{cases}. \quad (13)$$

For p-polarisation,

$$a = -1 \quad (14)$$

$$b = 1 \quad (15)$$

$$c = \frac{(k_x)_{\text{substrate}}}{n_{\text{substrate}}} \quad (16)$$

$$d = n_{\text{substrate}} \quad (17)$$

$$\alpha_j = \begin{pmatrix} -e^{i\phi_{j-1}} \frac{(k_x)_{j-1}}{n_{j-1}} & -\frac{(k_x)_{j-1}}{n_{j-1}} \\ -e^{i\phi_{j-1}} n_{j-1} & n_{j-1} \end{pmatrix} \quad \text{for } j = 1, \dots, M \quad (18)$$

$$\beta_j = \begin{cases} \begin{pmatrix} \frac{(k_x)_0}{n_0} \frac{n_{\text{outer}}}{(k_x)_{\text{outer}}} & e^{i\phi_0} \frac{(k_x)_0}{n_0} \frac{n_{\text{outer}}}{(k_x)_{\text{outer}}} \\ \frac{n_0}{n_{\text{outer}}} & -e^{i\phi_0} \frac{n_0}{n_{\text{outer}}} \end{pmatrix} & \text{for } j = 0 \\ \begin{pmatrix} \frac{(k_x)_j}{n_j} & e^{i\phi_j} \frac{(k_x)_j}{n_j} \\ n_j & -e^{i\phi_j} n_j \end{pmatrix} & \text{for } j = 1, \dots, M-1 \end{cases}. \quad (19)$$

2 Fabry-Perot interferometer

As a test case, the Fabry-Perot interferometer is considered, which consists of a single layer of refractive index n_{layer} and thickness d in a medium of refractive index n_{outer} and whose reflectivity at a wavelength λ_{vac} (in vacuum) and incident angle θ_{outer} is given by

$$R = \left| \frac{r(1 - e^{2i\phi_0})}{1 - r^2 e^{2i\phi_0}} \right|^2 \quad (20)$$

where

$$\phi_0 = \frac{2\pi}{\lambda_{\text{vac}}} n_{\text{layer}} d \cos \theta_{\text{layer}} \quad (21)$$

$$r = \frac{n_{\text{outer}} \cos \theta_{\text{outer}} - n_{\text{layer}} \cos \theta_{\text{layer}}}{n_{\text{outer}} \cos \theta_{\text{outer}} + n_{\text{layer}} \cos \theta_{\text{layer}}}. \quad (22)$$

The angle θ_{layer} is the angle of the light ray inside the layer. Its cosine is given by

$$\cos \theta_{\text{layer}} = \sqrt{1 - \left(\frac{n_{\text{outer}} \sin \theta_{\text{outer}}}{n_{\text{layer}}} \right)^2}. \quad (23)$$

In the case of total internal refraction, $\cos \theta_{\text{layer}}$ becomes imaginary and the above equations for R , ϕ_0 and r continue to hold.