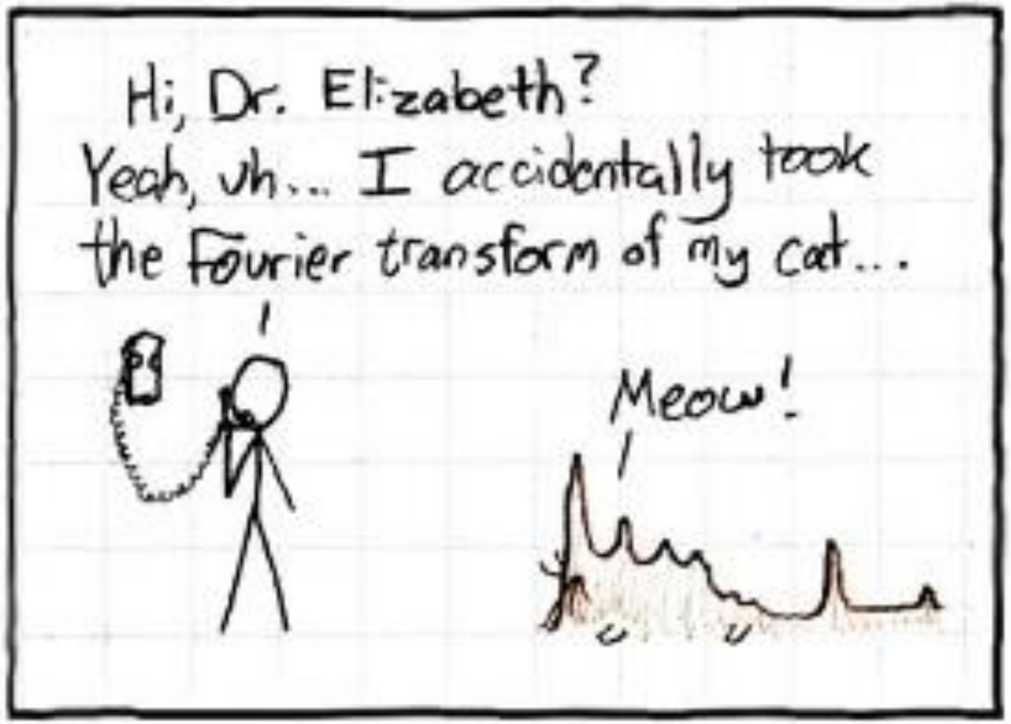
Frequency domain



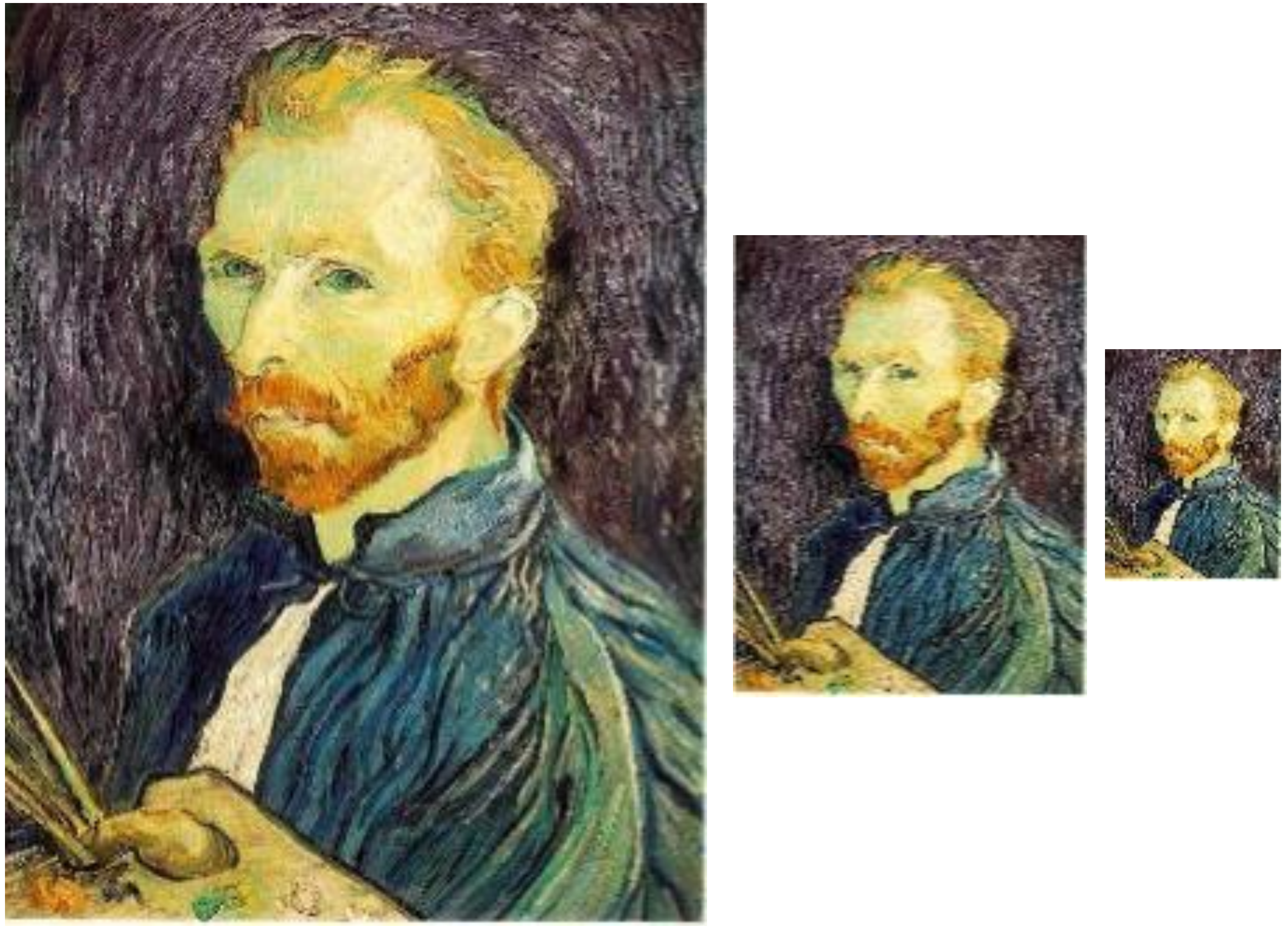
ICS 483 Computer Vision
Fall 2025, Lecture 4

Announcement

- Quiz 1 is due tonight 11:55pm. 10% pts off for each late day, and no pts after two days.
- Assignment 1: You can finish the convolution and filter part for now. Please start early! Watch the 20-min Hough Transform lecture from FPCV (First principle of computer vision by Shree Nayar) for the Hough Trans
- If you are have not received the participation verification (by submitting the in-class quiz and completing the Python tutorial), please talk to me after class.

Recap

- **Image downsampling.**
- Aliasing.
- Gaussian image pyramid.
- Laplacian image pyramid.
- Fourier series.
- Frequency domain.
- Fourier transform.
- Frequency-domain filtering.
- Revisiting sampling.



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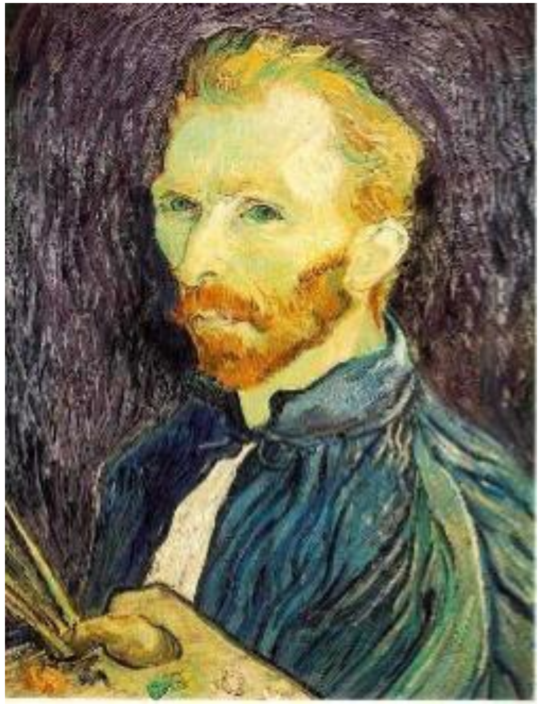

1/2

1/4 (2x zoom)

1/8 (4x zoom)

Recap

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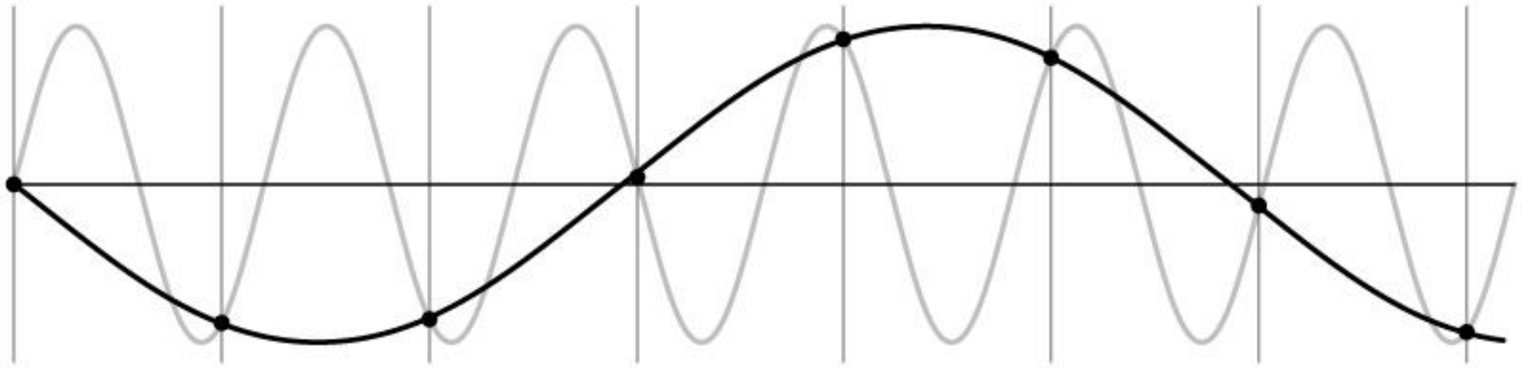

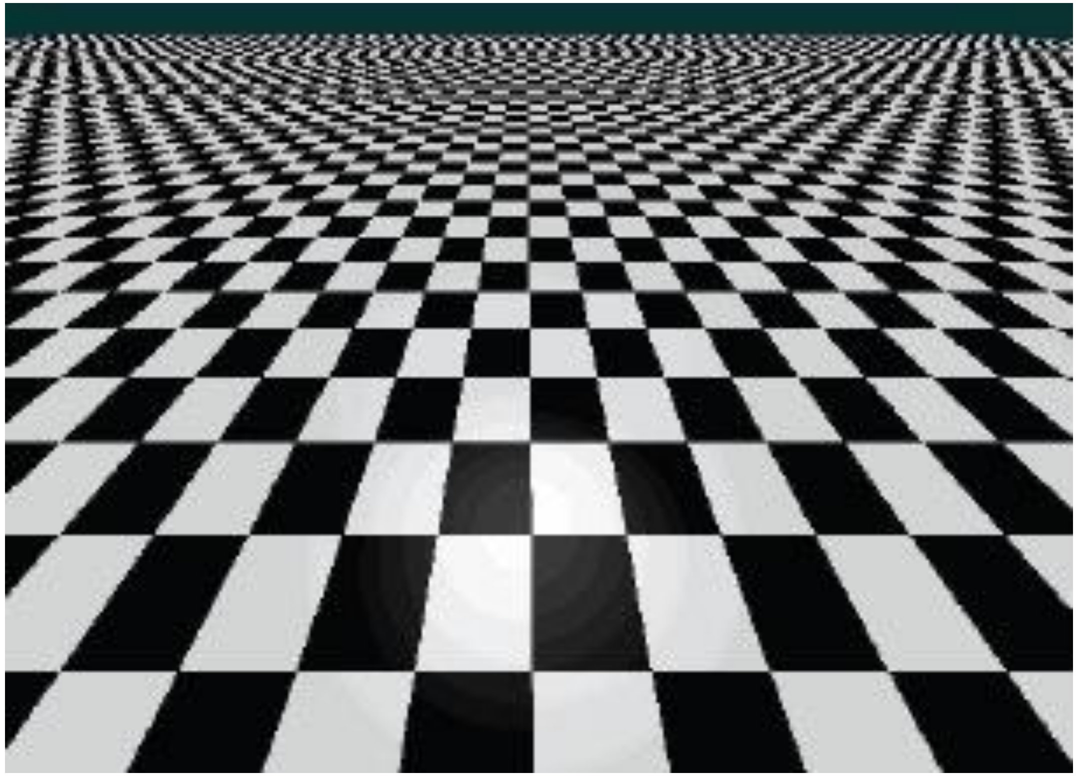

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spatial

temporal

Recap

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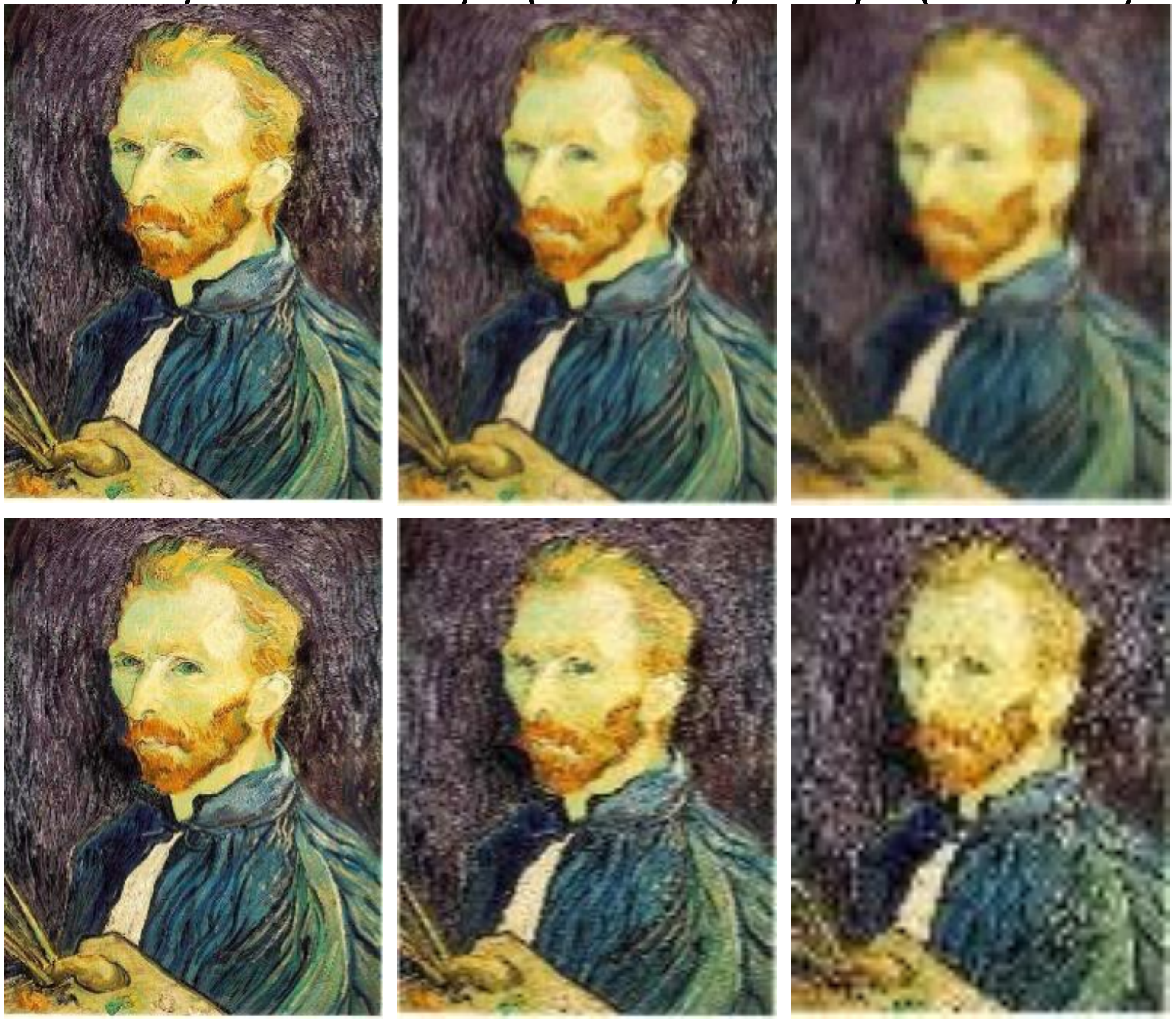
1/2 | 1/4 (2x zoom) | 1/8 (4x zoom)

Gaussian smoothed

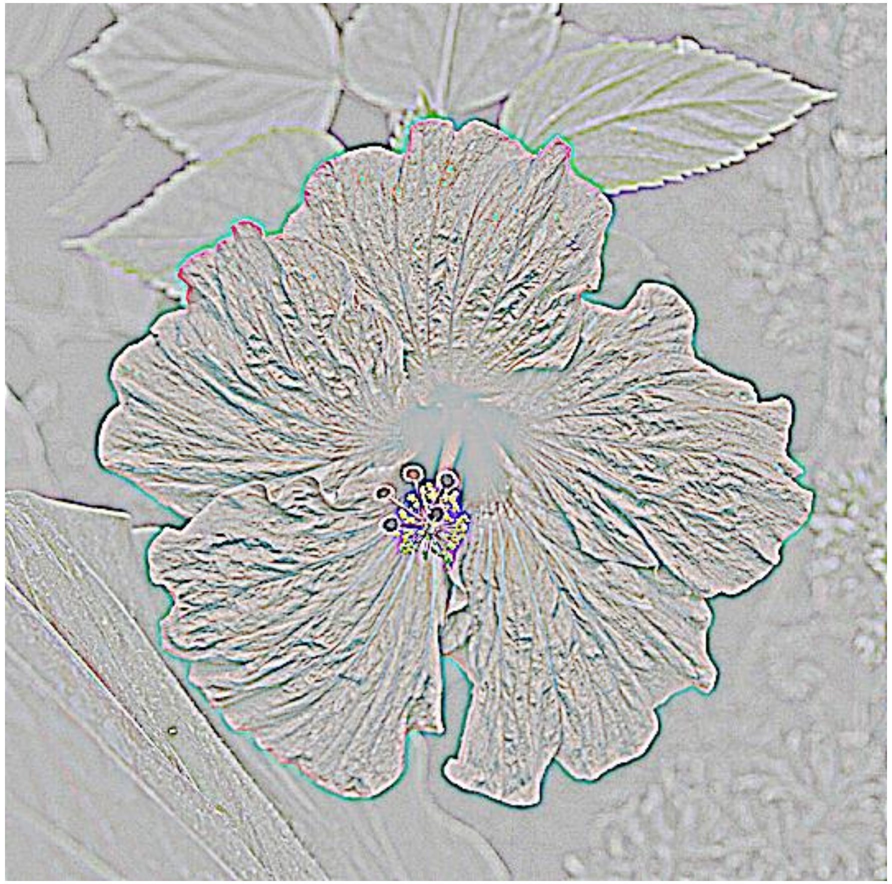
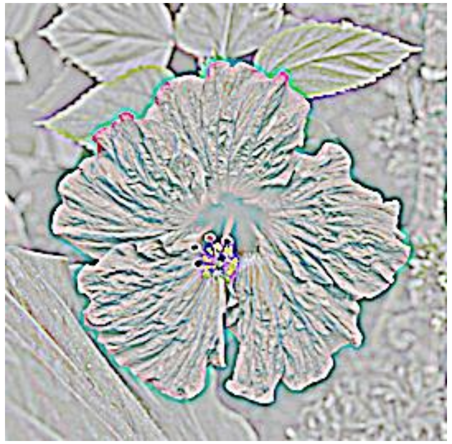
naive



Gaussian vs Laplacian Pyramid



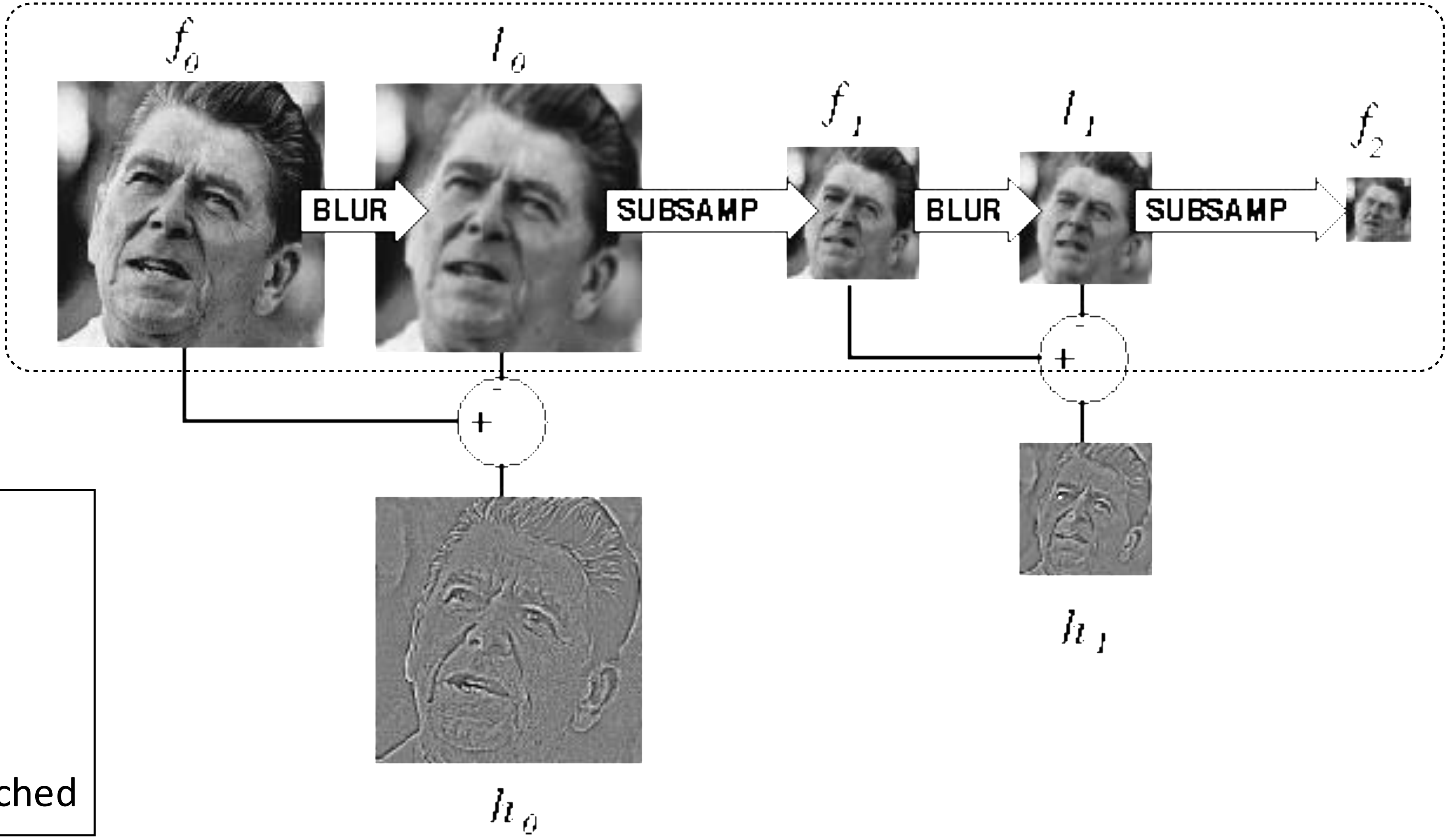

Shown in opposite order for space.

Which one takes more space to store?

Constructing a Laplacian pyramid

It's a Gaussian pyramid.



Algorithm

repeat:

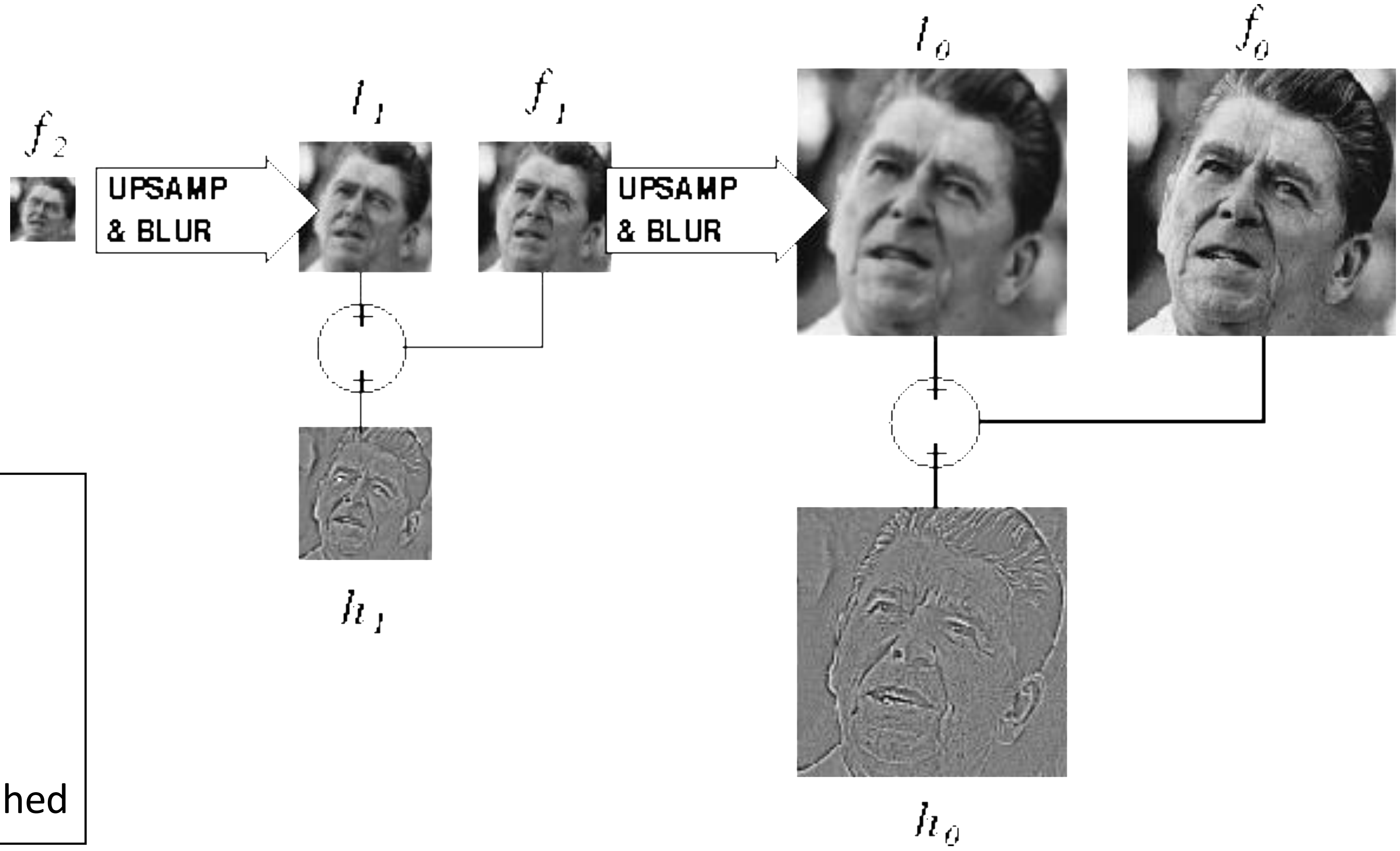
filter

compute residual

subsample

until min resolution reached

Reconstructing the original image



Algorithm

repeat:

upsample

sum with residual

until orig resolution reached

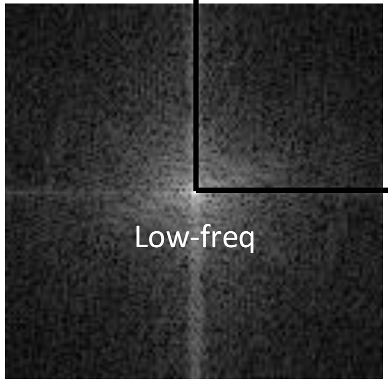
Frequency Domain

What are “bass” and ”treble” in images?

More filtering examples

original image

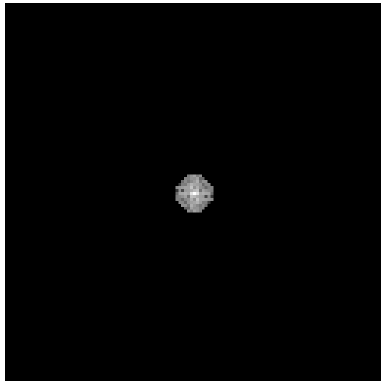
frequency magnitude



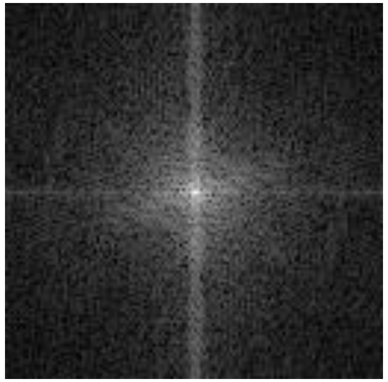
More filtering examples

original image

low-pass filter



frequency magnitude



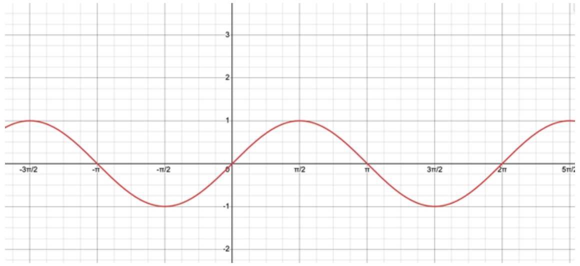
Fourier series

Basic building block

$$A \sin(\omega x + \phi)$$

Fourier's claim: Add enough of these to get <u>any</u> *periodic* signal you want!

Basic building block



[Visualizer]

$$A\sin(\omega x + \phi)$$

amplitude

sinusoid

angular frequency

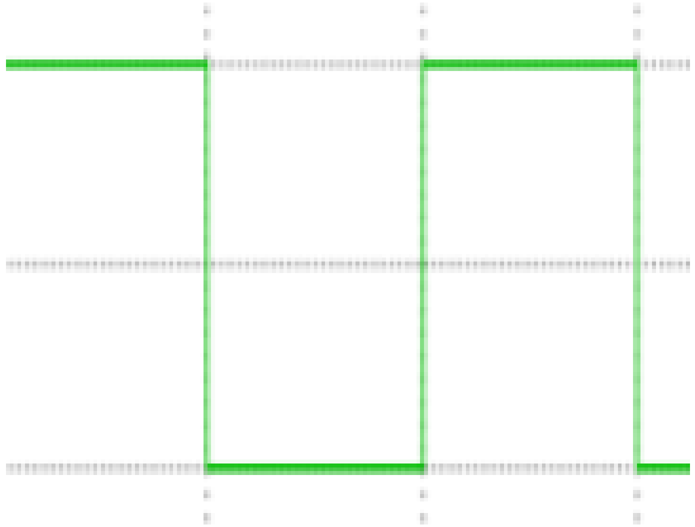
variable

phase

Fourier's claim: Add enough of these to get <u>any</u> *periodic* signal you want!

Examples

How would you generate this function?

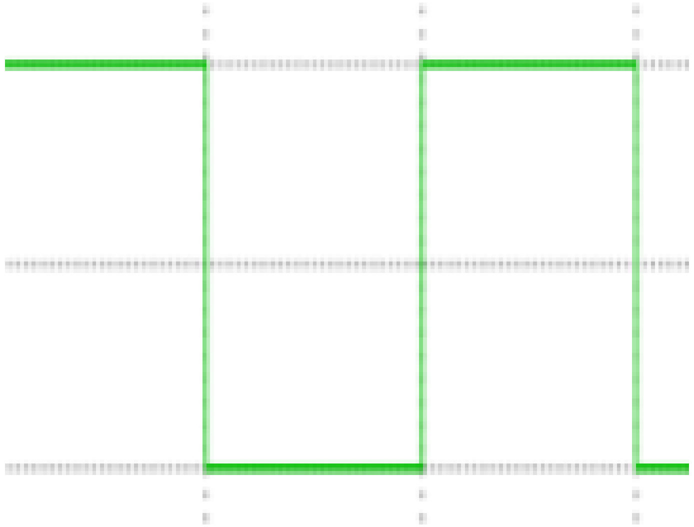


square wave

= ? + ?

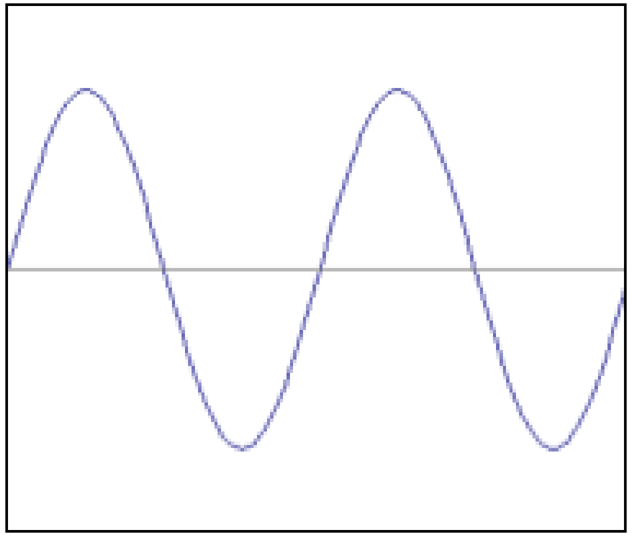
Examples

How would you generate this function?

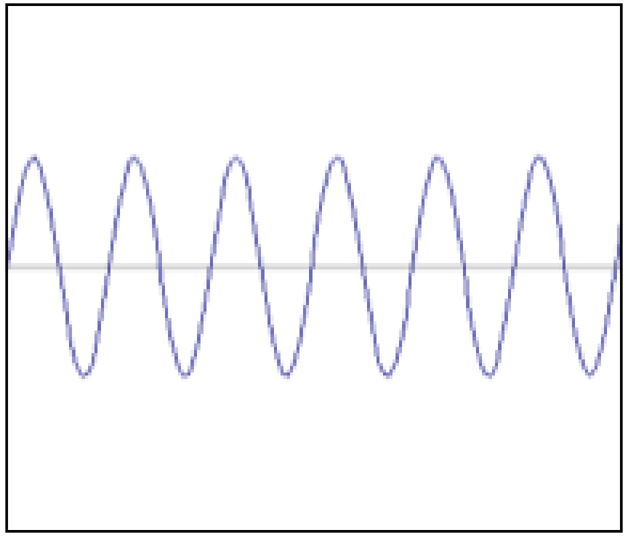


square wave

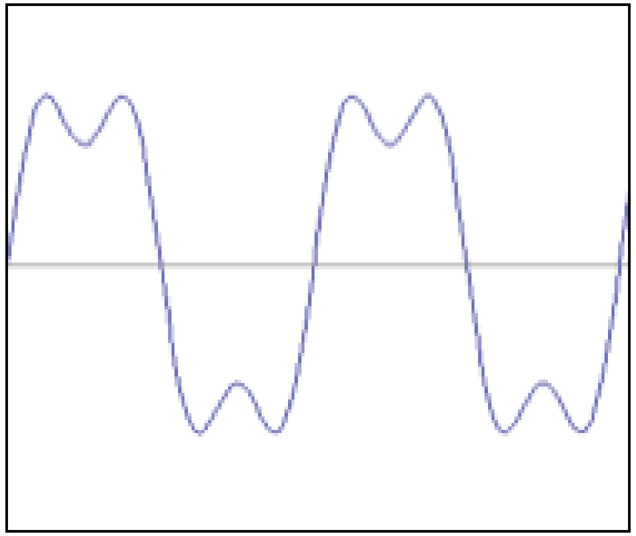
$\approx$



$+$



$=$



Examples

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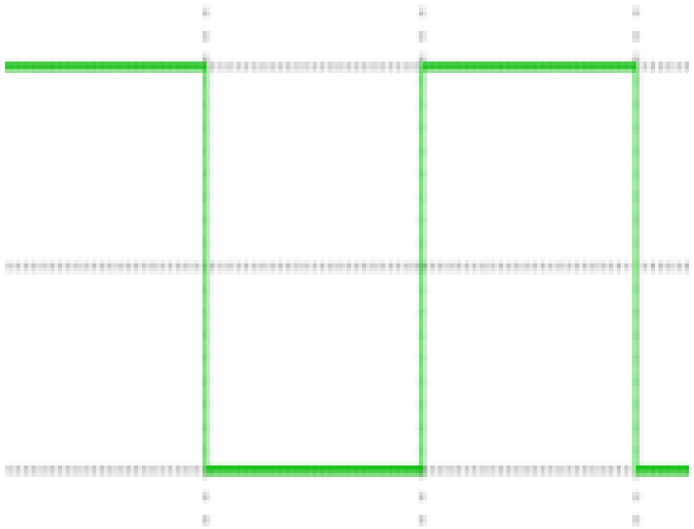
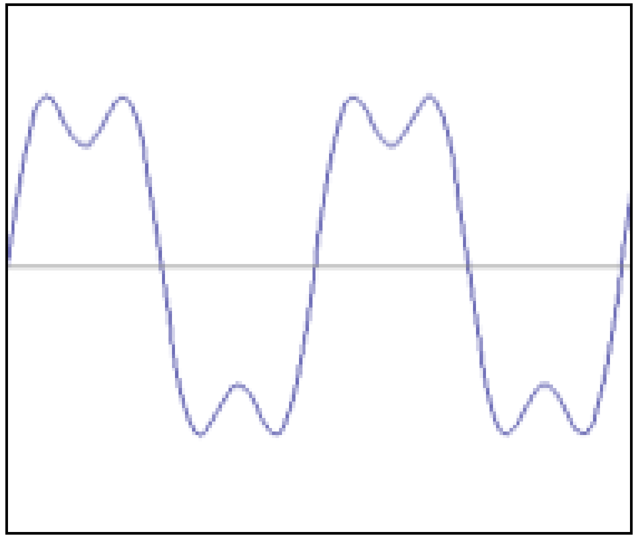
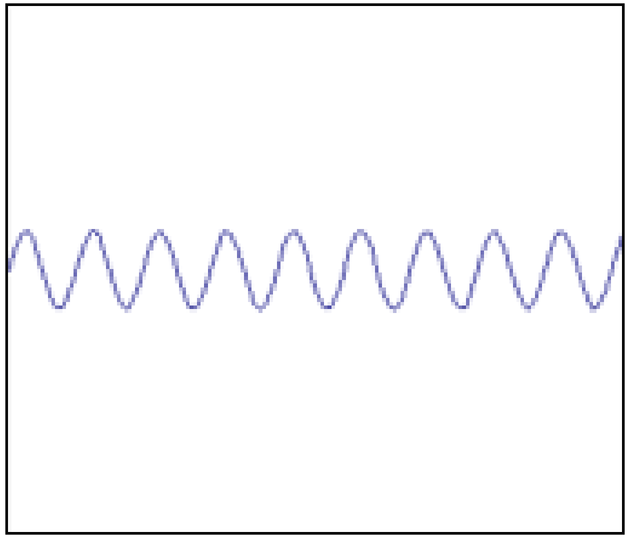


square wave

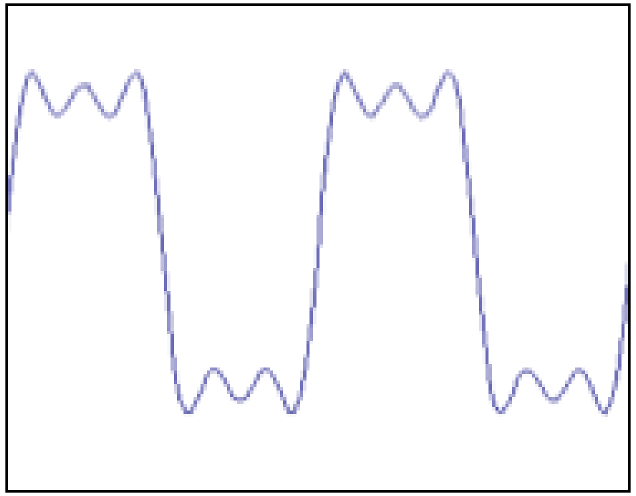
≈



+



=



Examples

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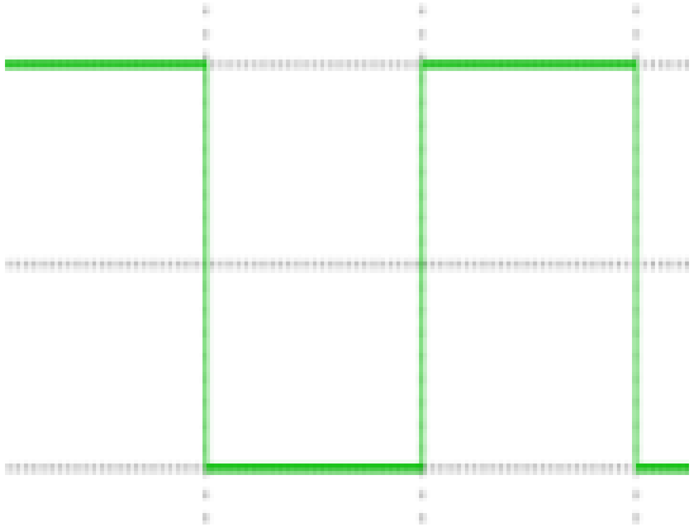
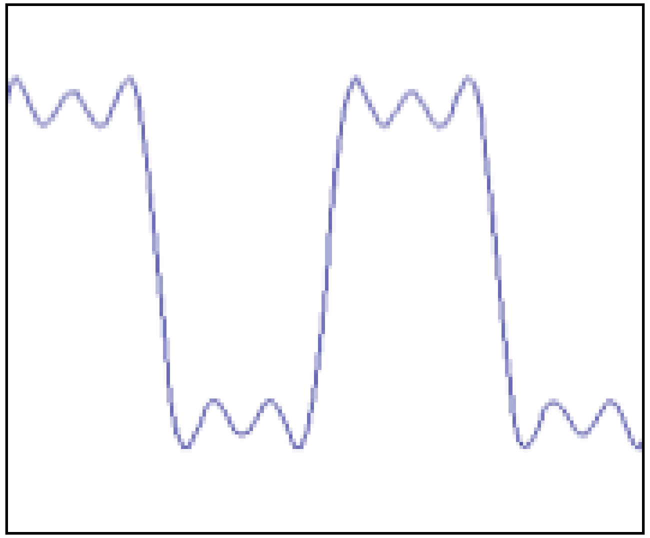


square wave

≈



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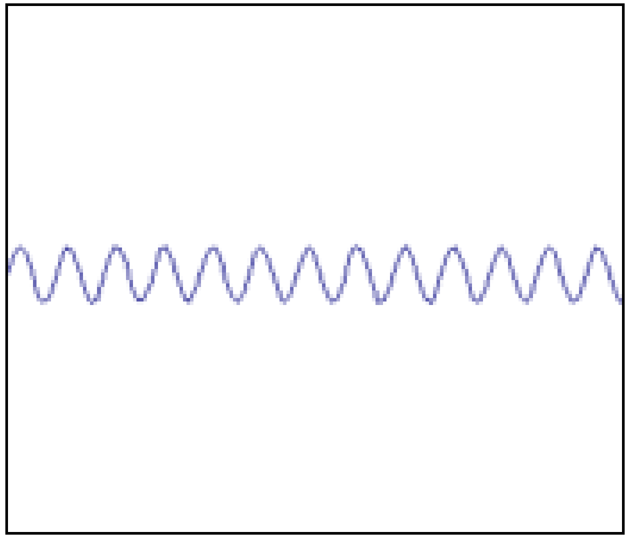
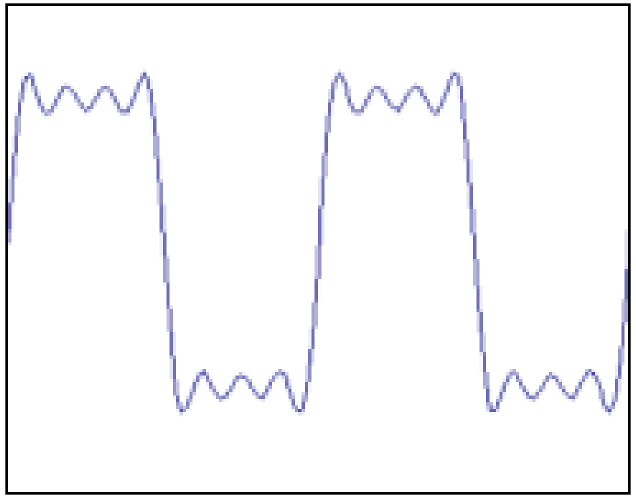


=



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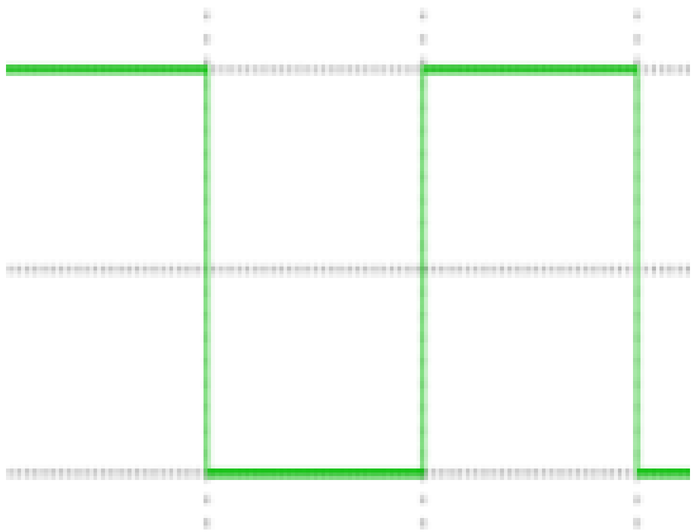
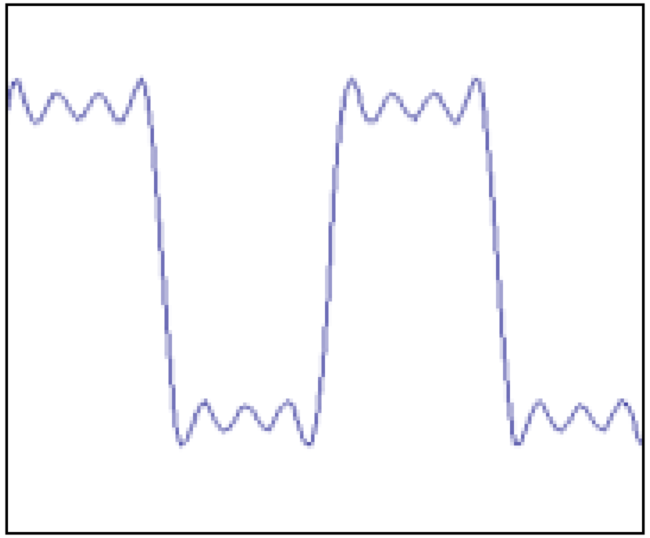


square wave

≈



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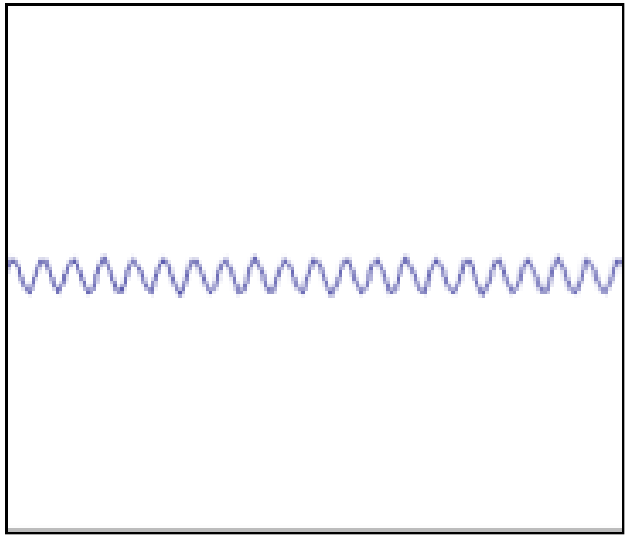
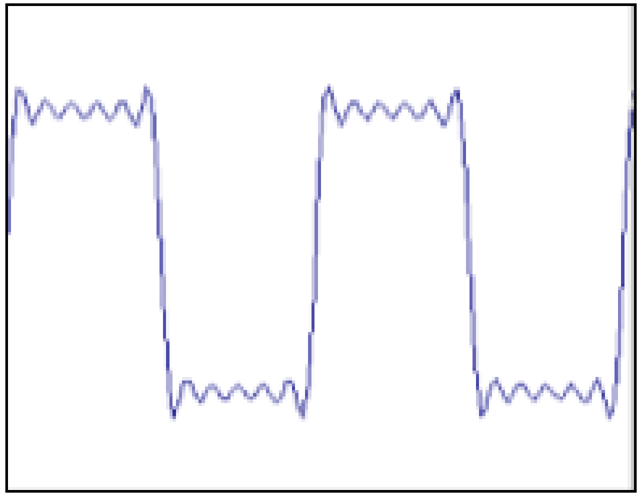


=



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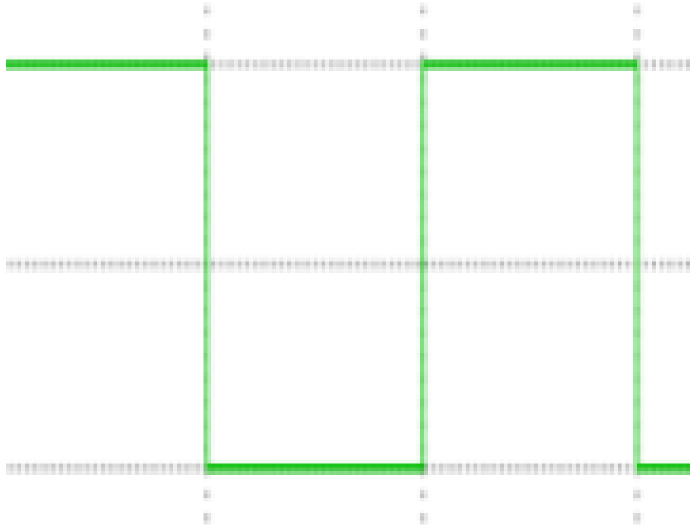
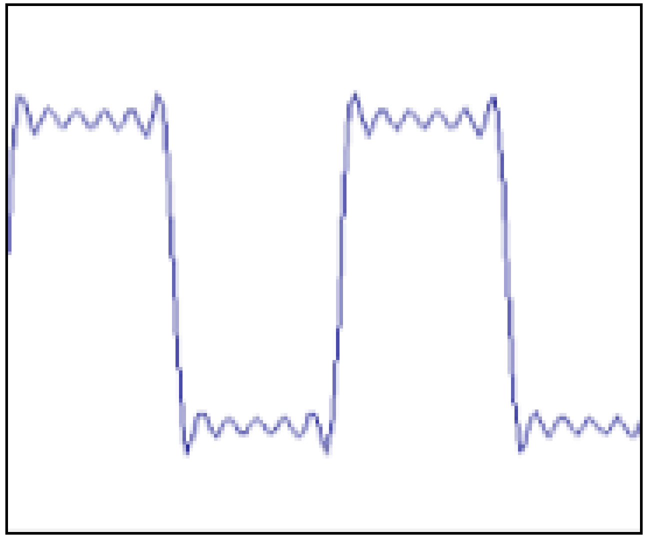
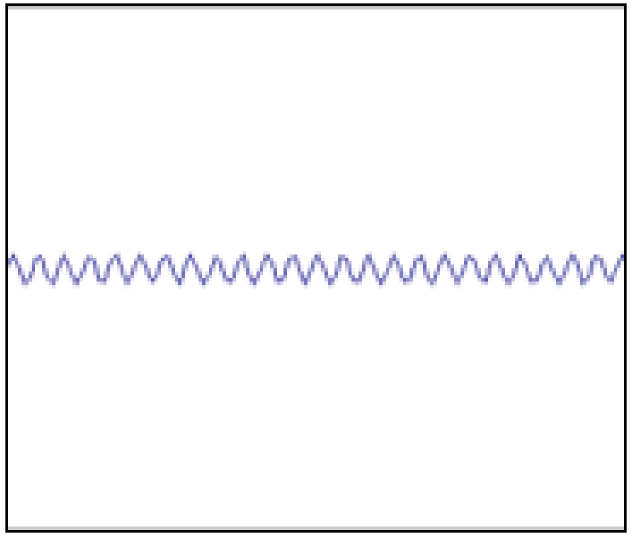


square wave

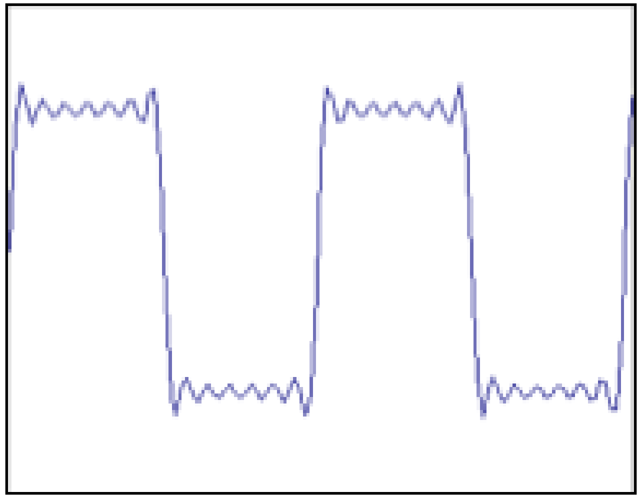
$\approx$



$+$

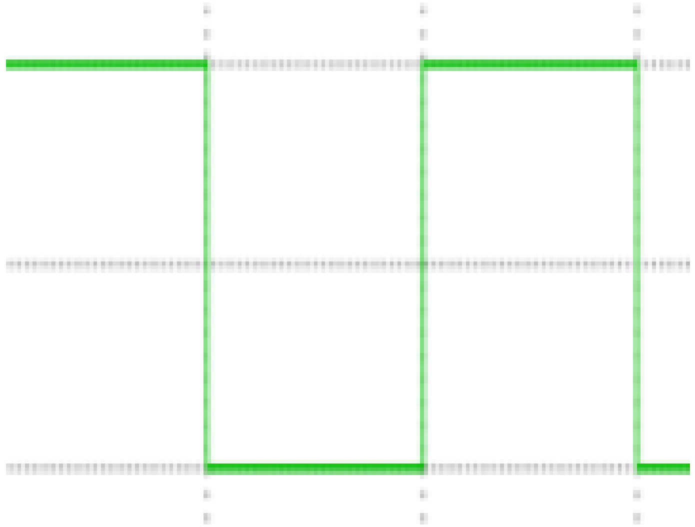


$=$



How would you express this mathematically?

Examples



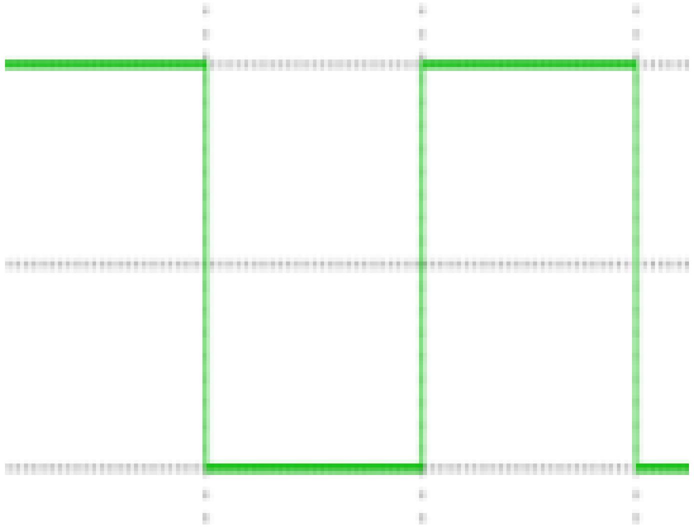
$$= \quad A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kx)$$

square wave

infinite sum of sine waves

How would could you visualize this in the frequency domain?

Examples



square wave

=

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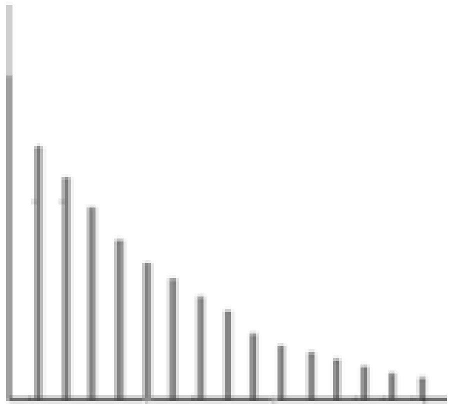
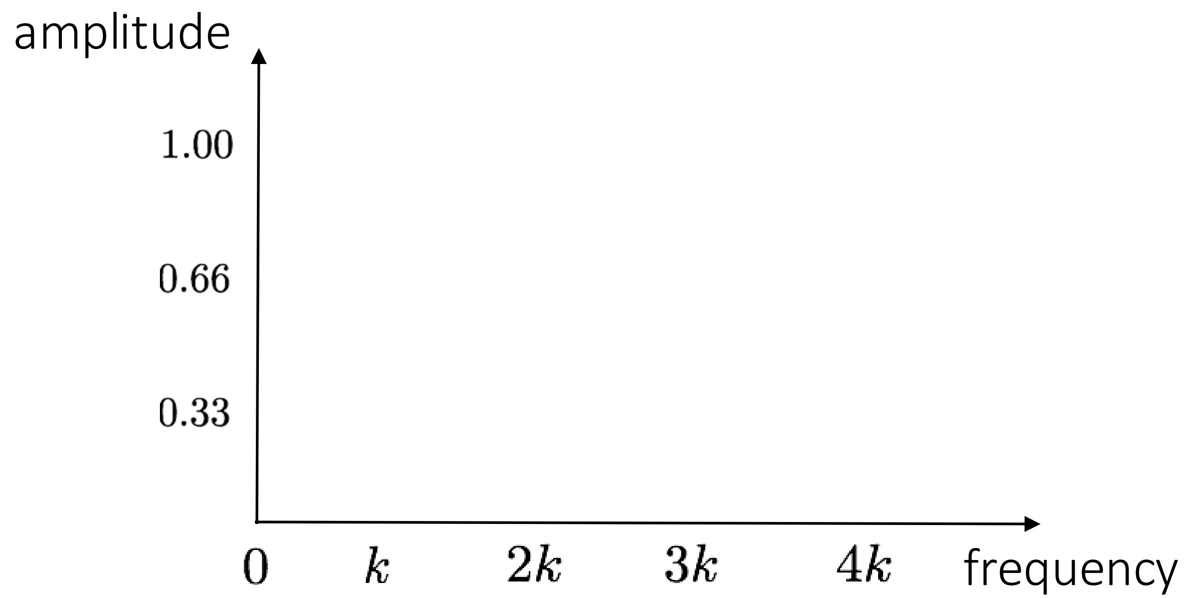
infinite sum of sine waves

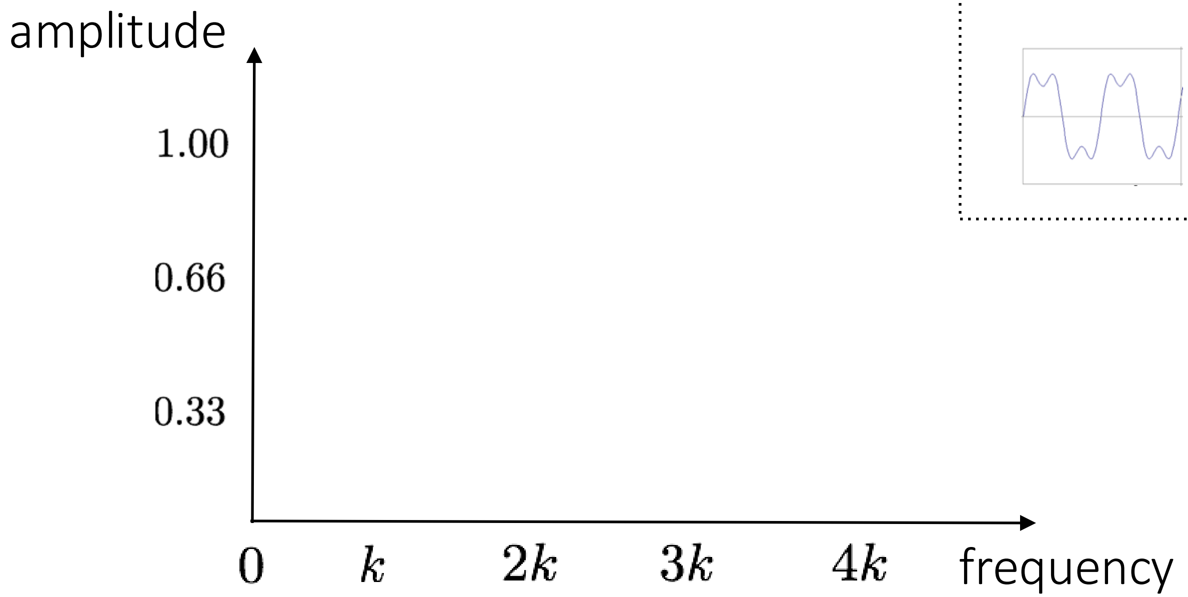


Frequency domain

Visualizing the frequency spectrum

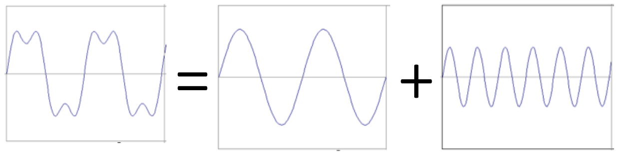


Visualizing the frequency spectrum



Recall the temporal domain visualization

$$f(x) = \sin(2\pi kx) + \frac{1}{3}\sin(2\pi 3kx)$$



Visualizing the frequency spectrum

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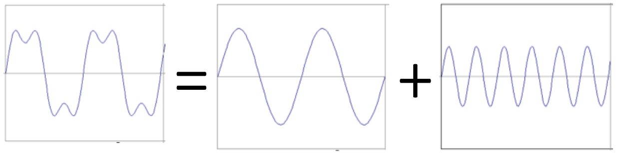

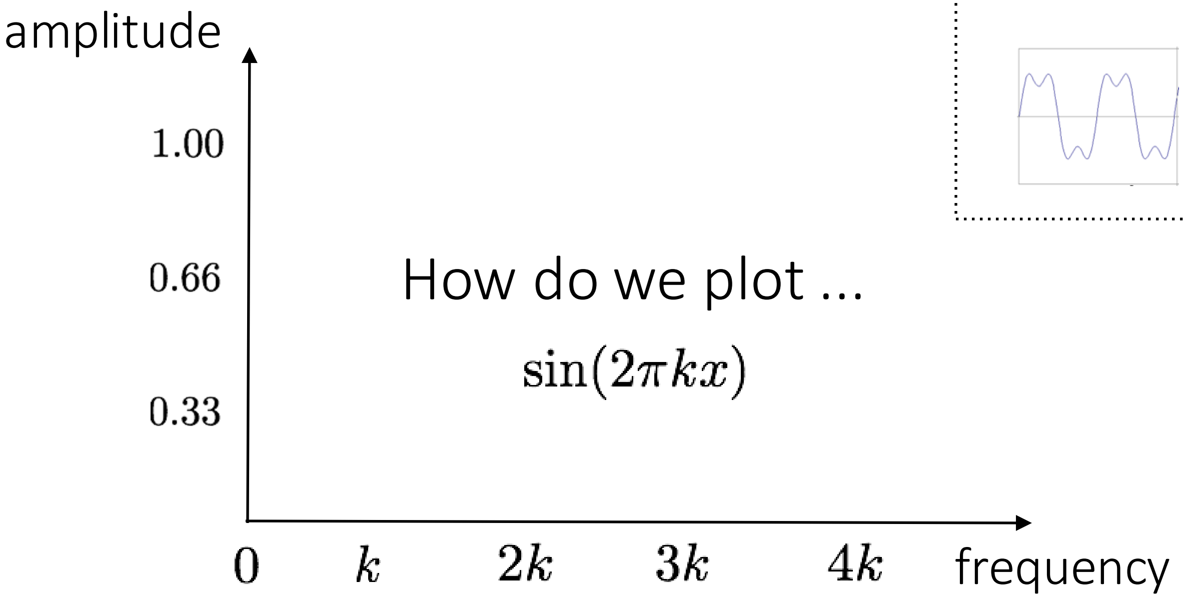

How do we plot ...

$$\sin(2\pi kx)$$

Visualizing the frequency spectrum

Recall the temporal domain visualization

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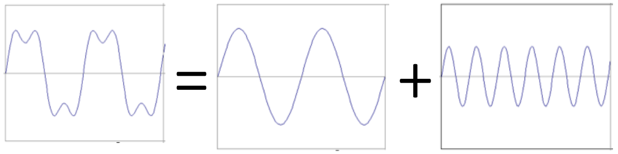
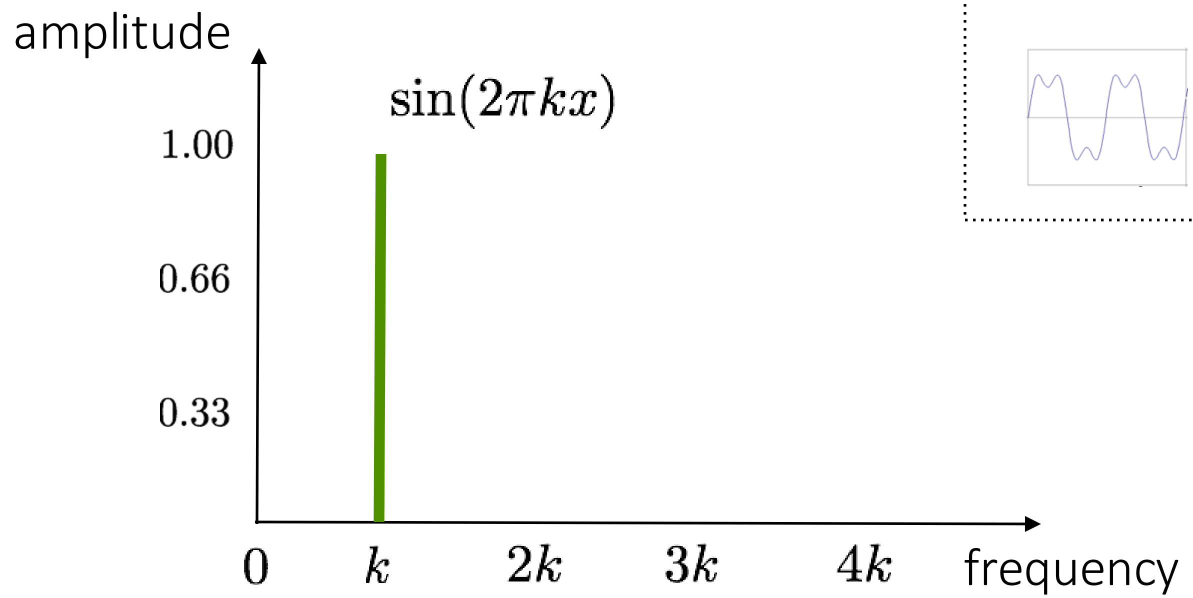
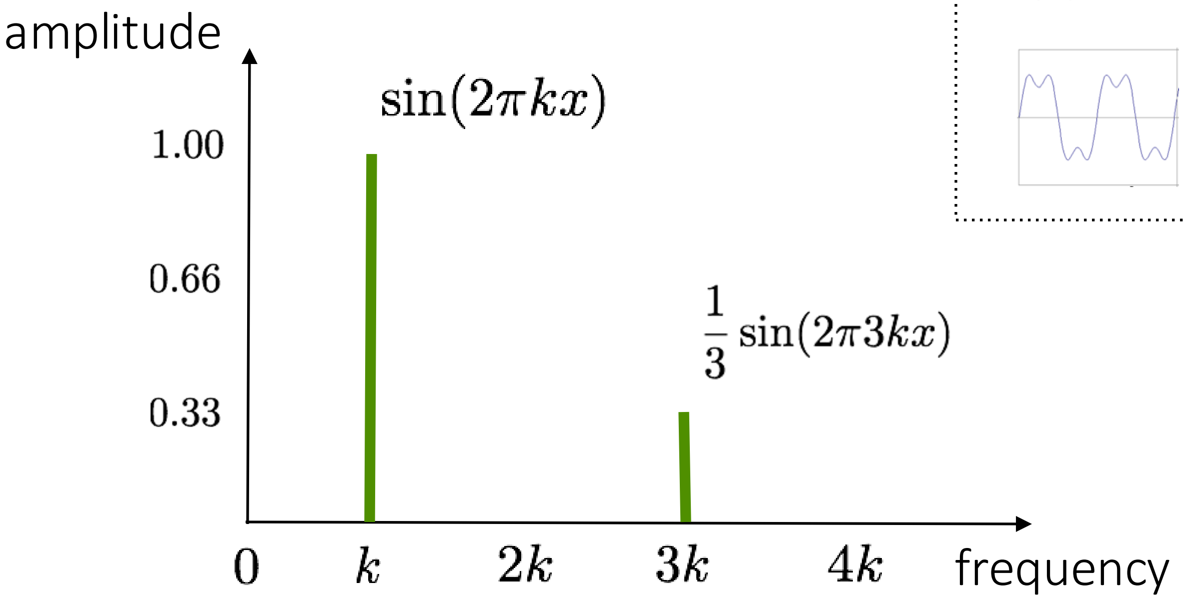




Visualizing the frequency spectrum



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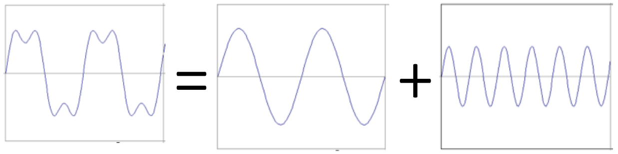

Visualizing the frequency spectrum

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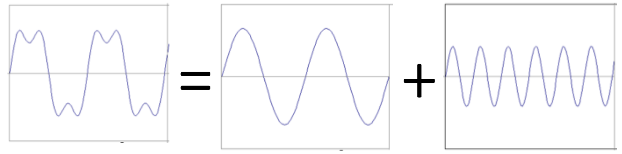
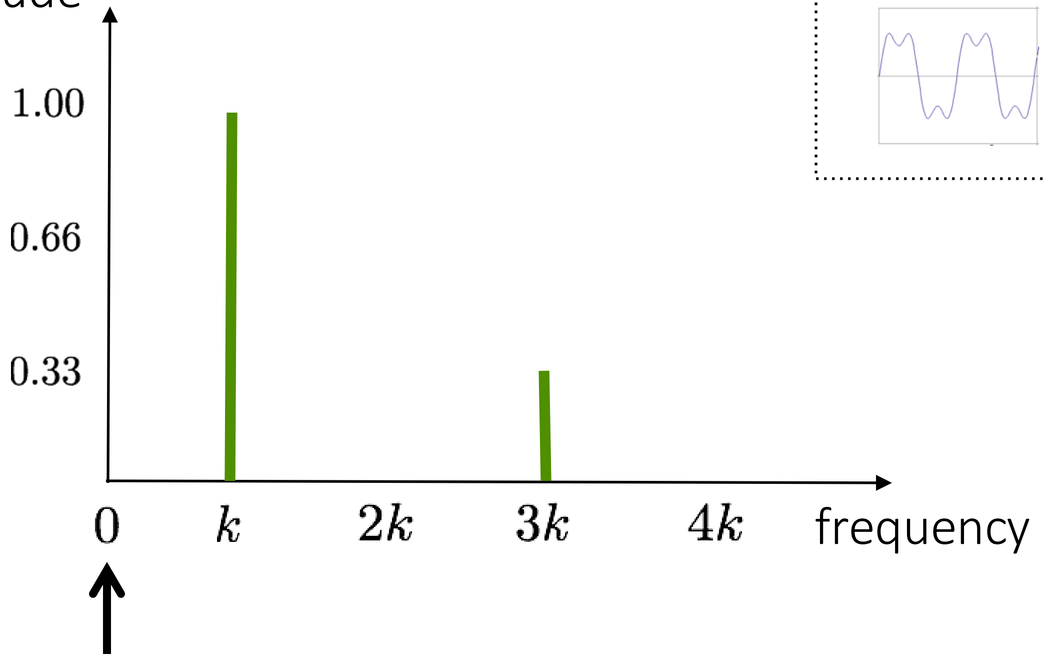


not visualizing the symmetric negative part



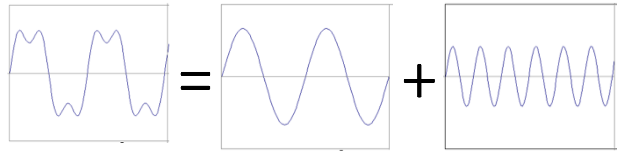
What is at zero frequency?

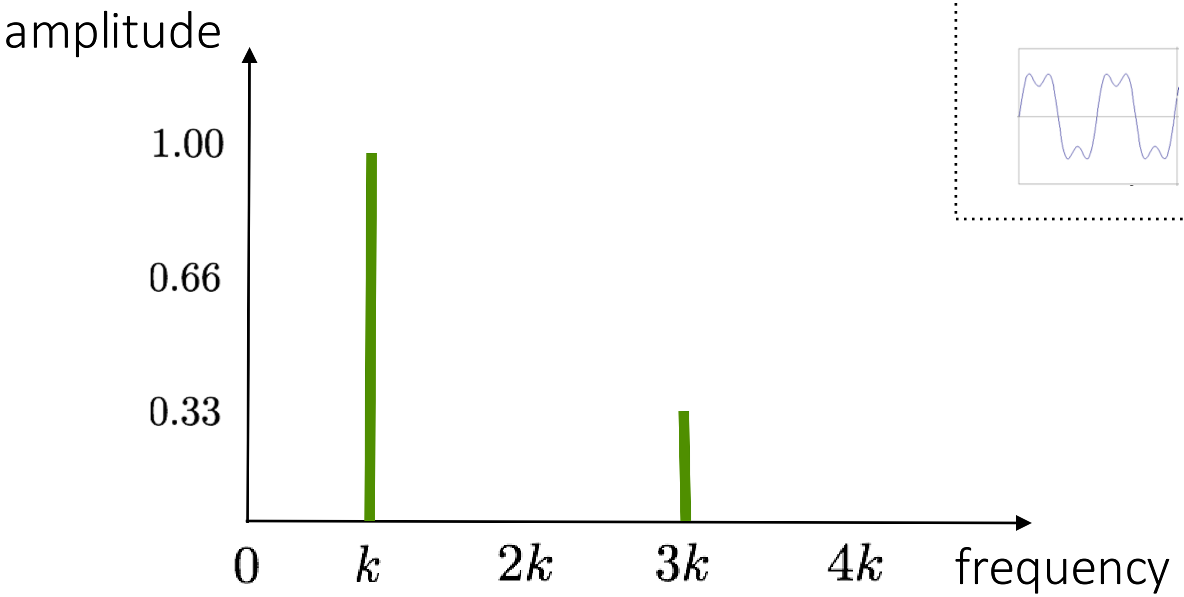
Need to understand this to understand the 2D version!

Visualizing the frequency spectrum

Recall the temporal domain visualization

$$f(x) = \sin(2\pi k x) + \frac{1}{3}\sin(2\pi 3 k x)$$





not visualizing the symmetric negative part

signal average (zero for a sine wave with no offset)

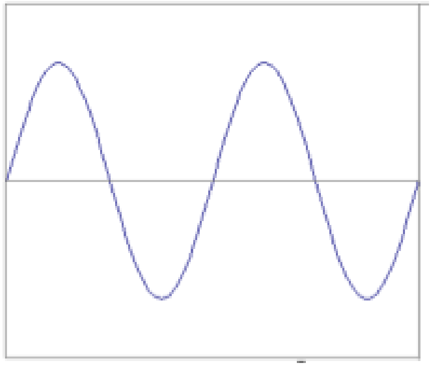
Need to understand this to understand the 2D version!

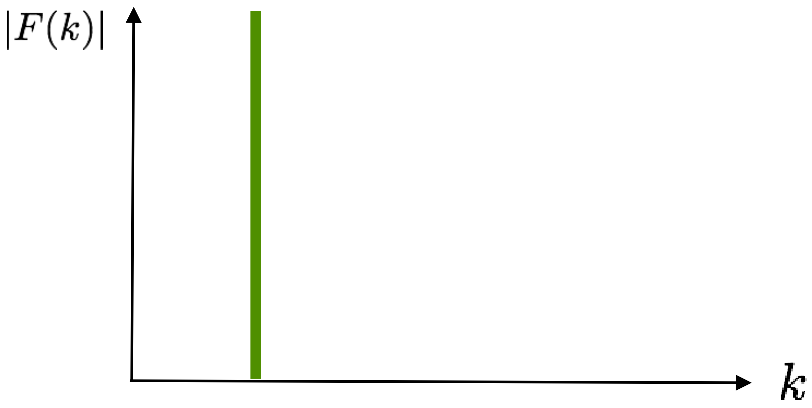
Examples

Spatial domain visualization

Frequency domain visualization

1D





2D

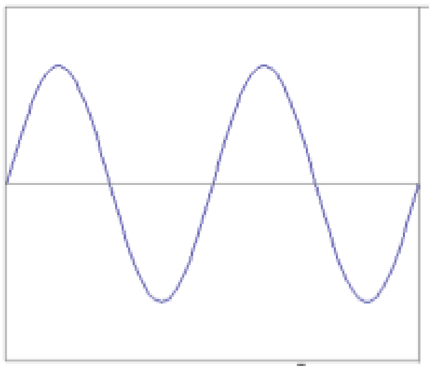
?

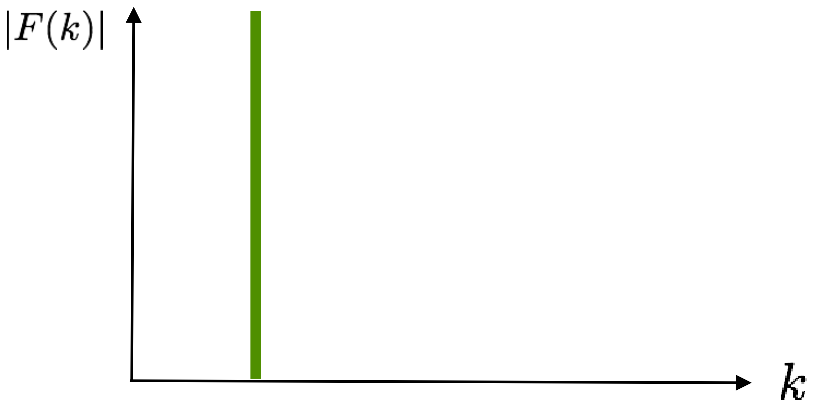
Examples

Spatial domain visualization

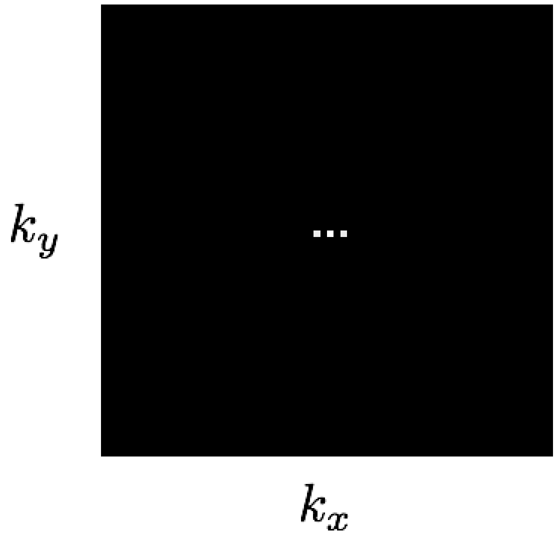
Frequency domain visualization

1D





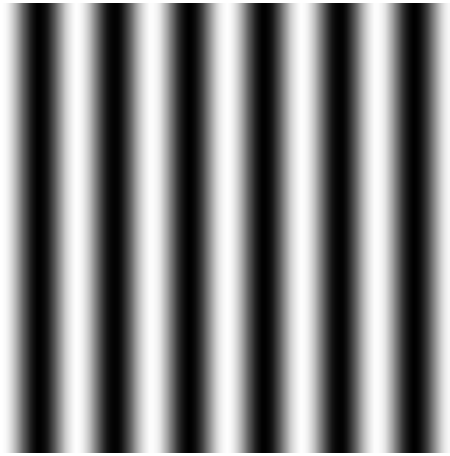
2D

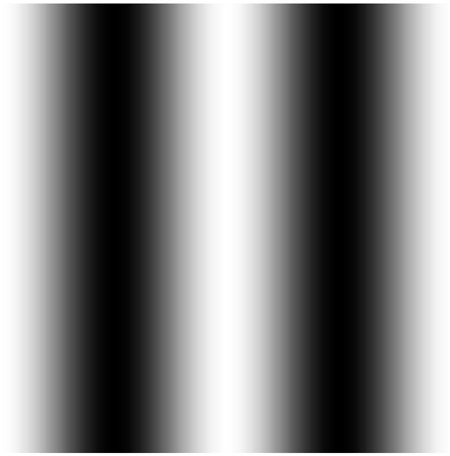


What do the three dots correspond to?

Examples

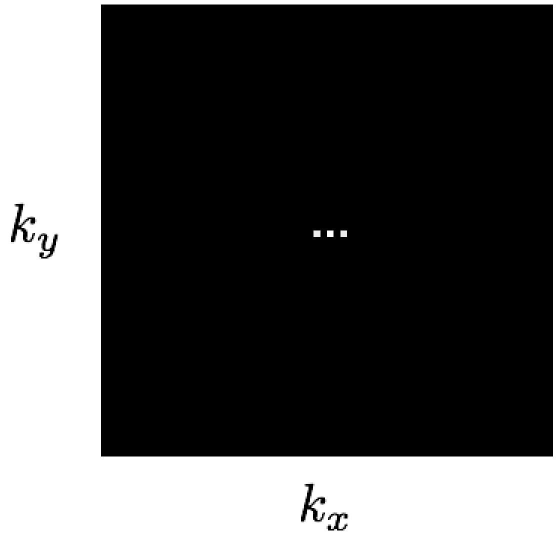
Spatial domain visualization





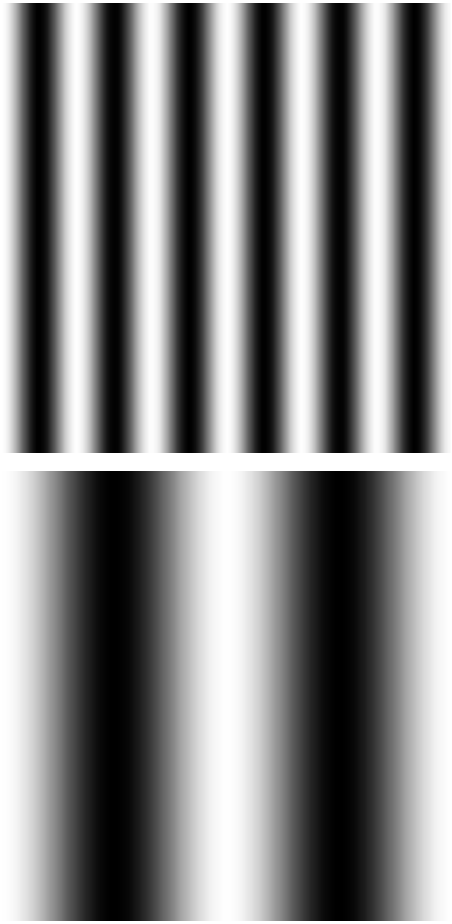
Frequency domain visualization

?

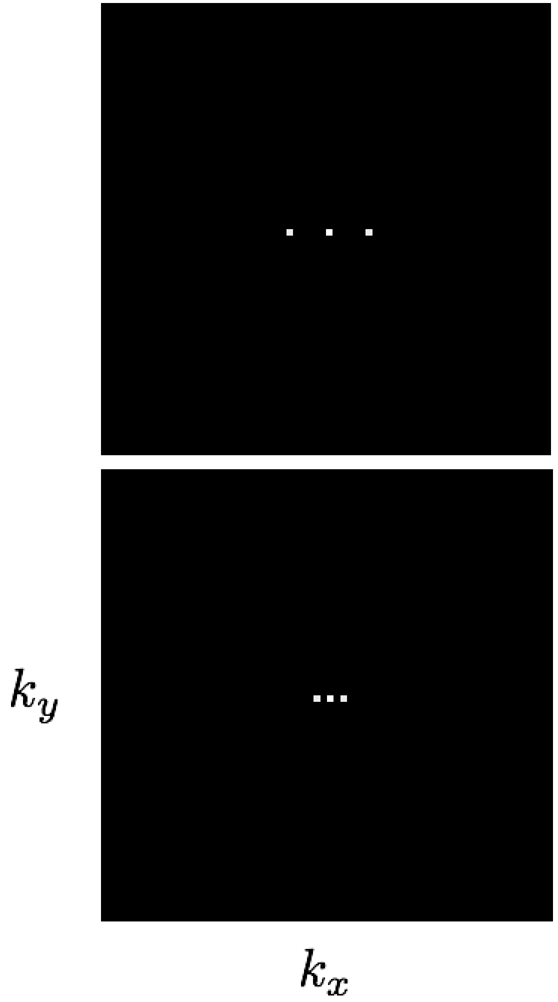


Examples

Spatial domain visualization

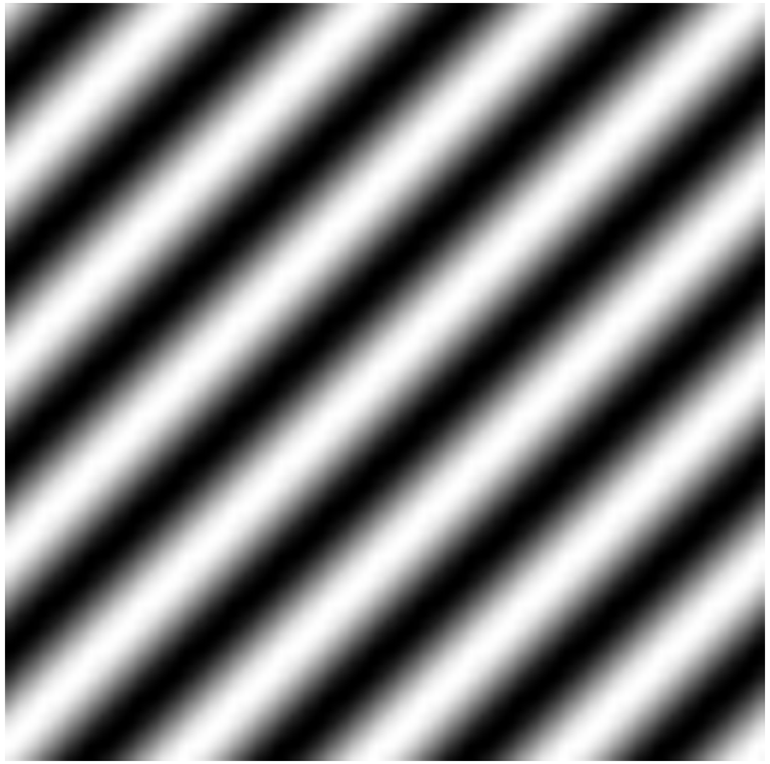


Frequency domain visualization



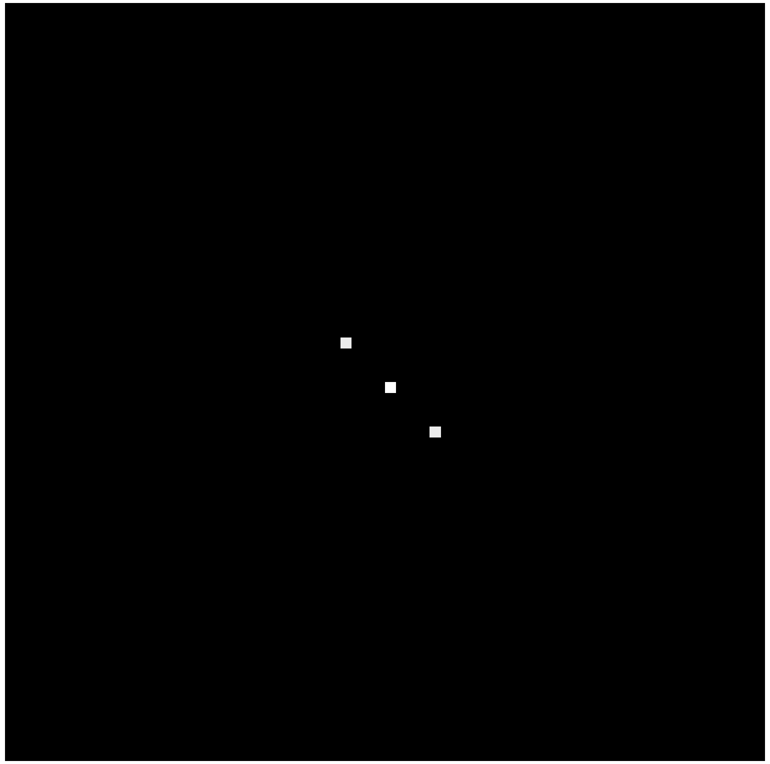
Examples

How would you generate this image with sine waves?



Examples

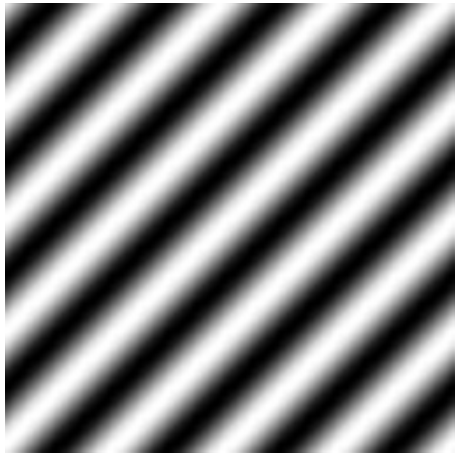
How would you generate this image with sine waves?



Has both an x and y components

Examples

+



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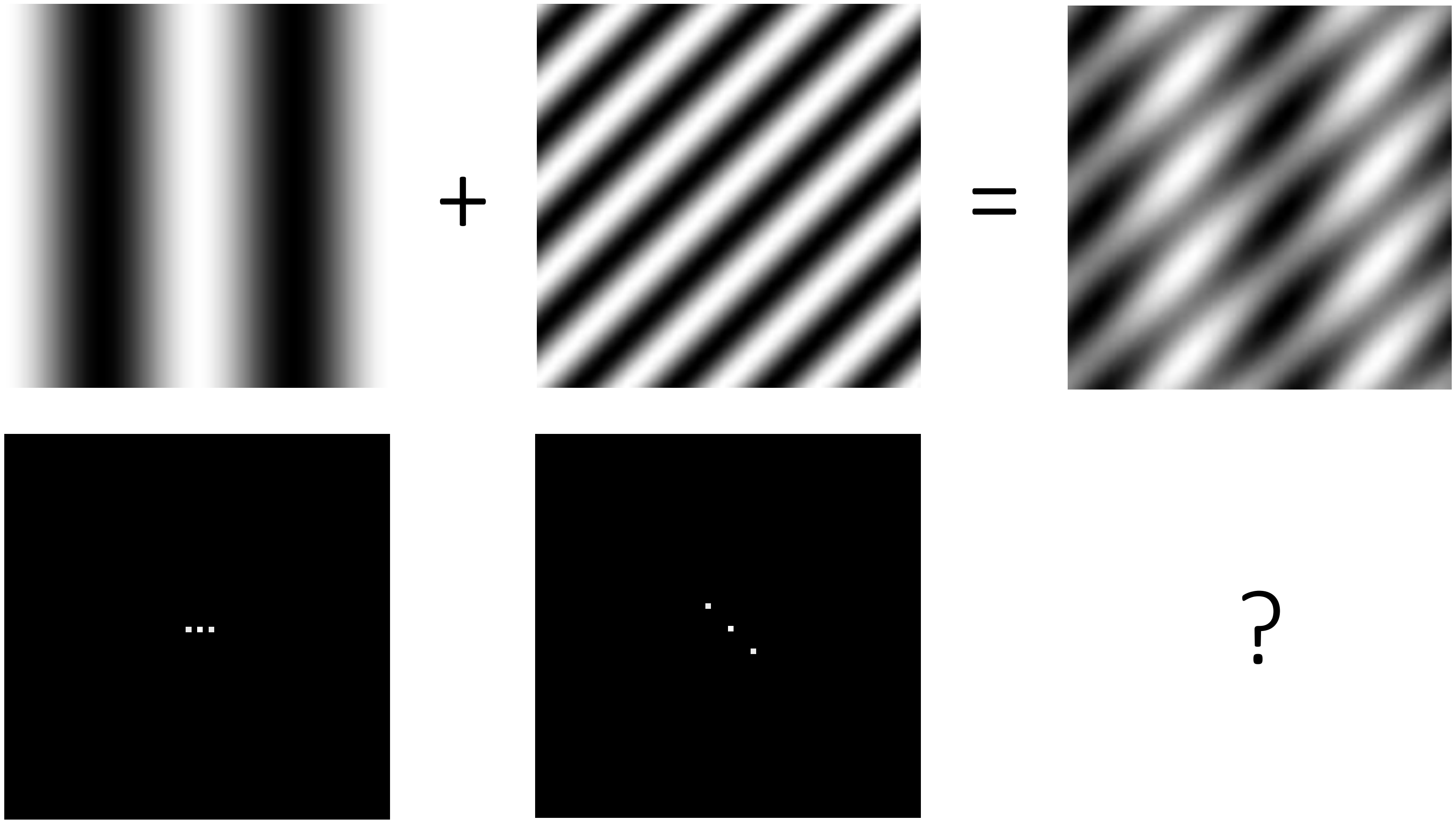
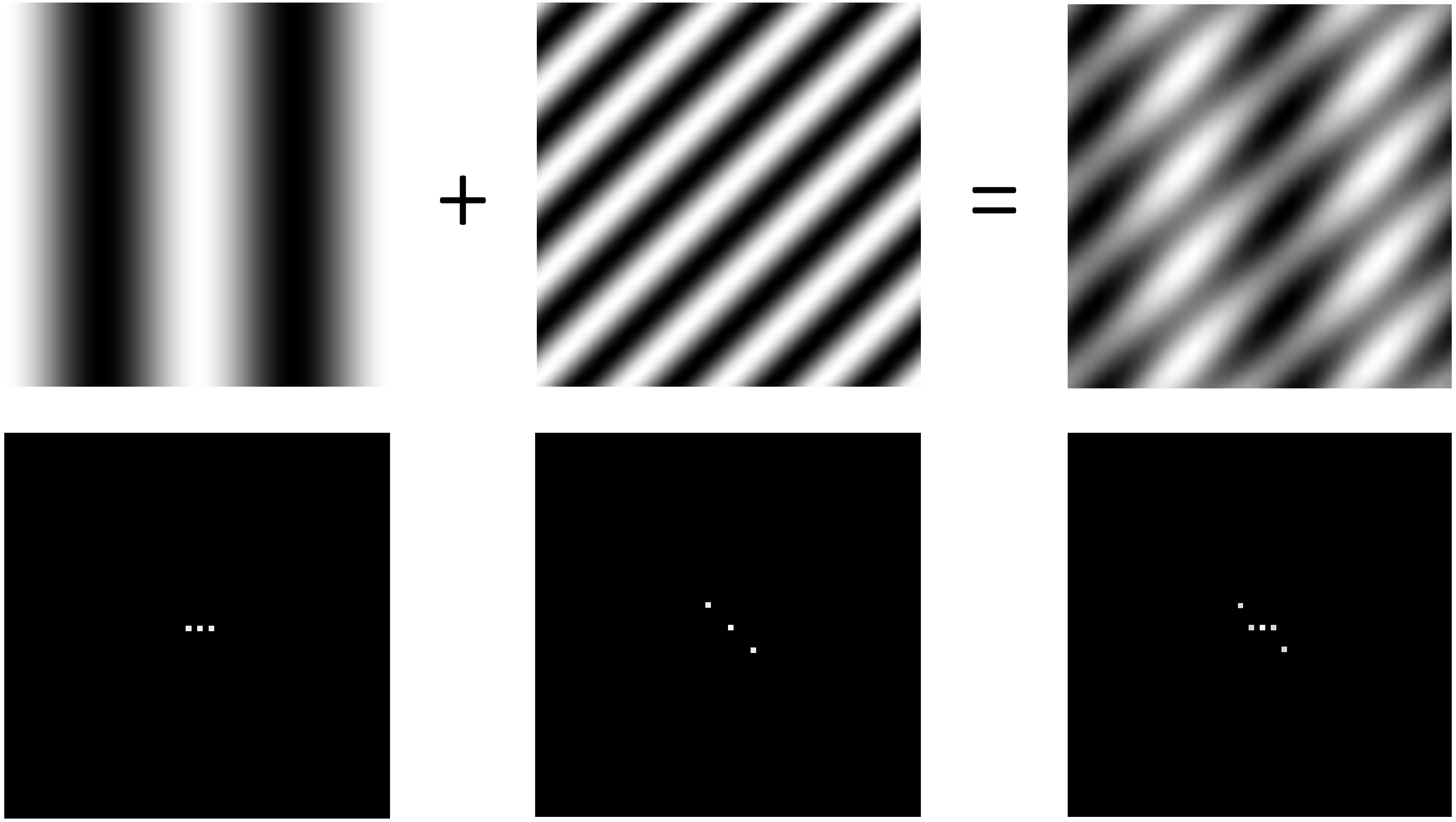
Examples



Examples



Basic building block

$$A \sin(\omega x + \phi)$$

amplitude

sinusoid

angular frequency

variable

phase

What about non-periodic signals?

Fourier's claim: Add enough of these to get <u>any</u> *periodic* signal you want!

Fourier transform

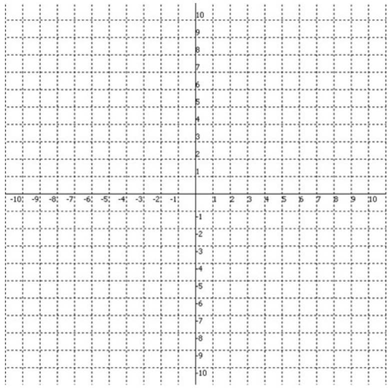
Recalling some basics

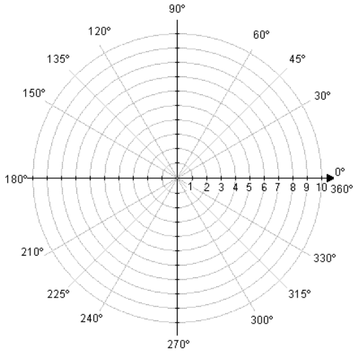
Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

what's this? what's this?



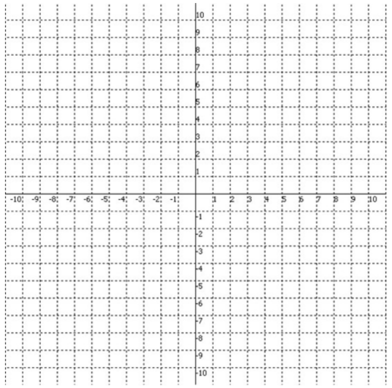


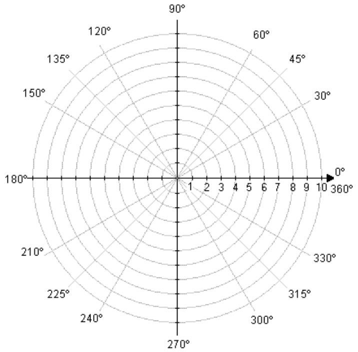
Recalling some basics

Complex numbers have two parts:

rectangular coordinates

$$\underbrace{R}_{\text{real}} + \underbrace{jI}_{\text{imaginary}}$$





Recalling some basics

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

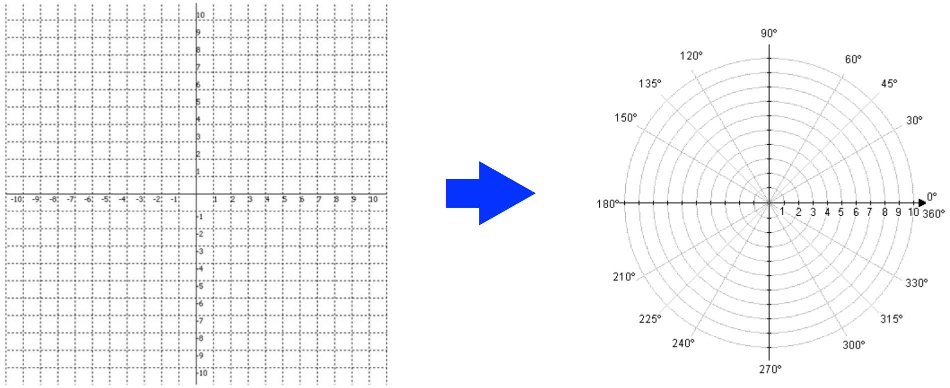
real imaginary

Alternative reparameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

how do we compute these?



polar transform

Recalling some basics

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

real imaginary

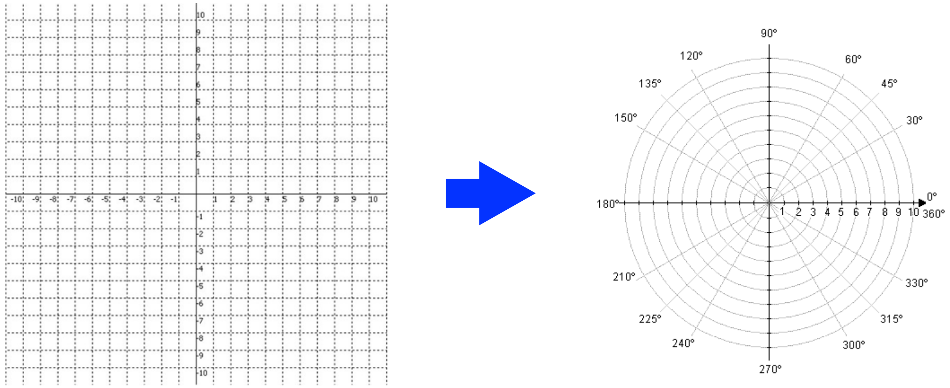
Alternative reparameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

polar transform

$$\theta = \tan^{-1}\left(\frac{I}{R}\right) \quad r = \sqrt{R^2 + I^2}$$



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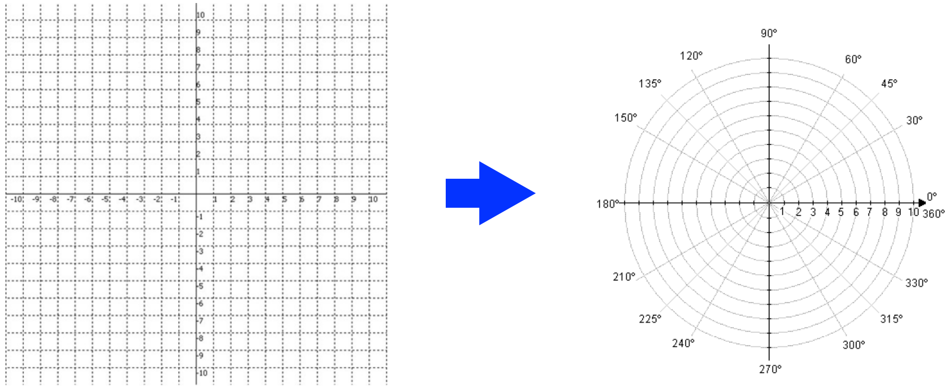


polar transform

How do you write these in exponential form?

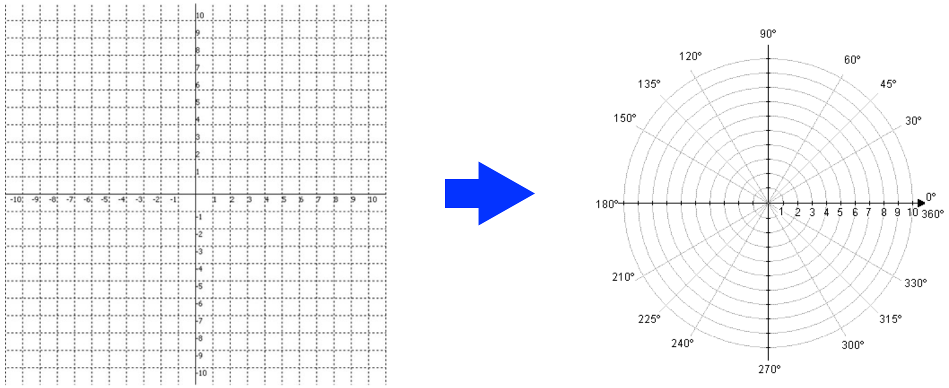
Recalling some basics

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

real imaginary



Alternative reparameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

polar transform

$$\theta = \tan^{-1}(\frac{I}{R}) \quad r = \sqrt{R^2 + I^2}$$

or equivalently

$$re^{j\theta}$$

how did we get this?

exponential form

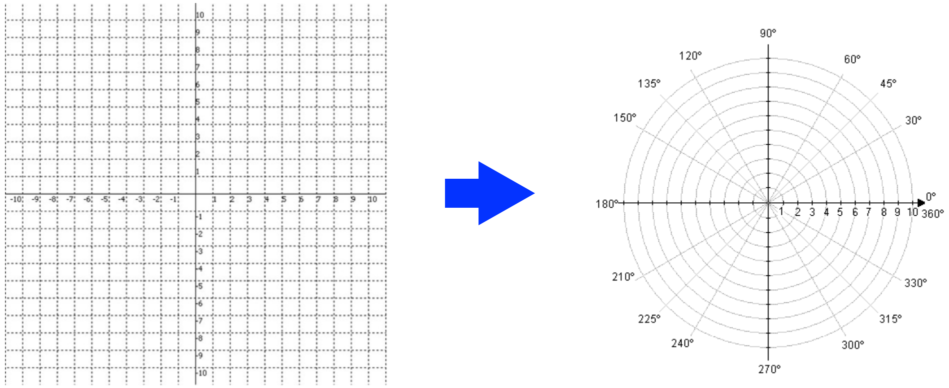
Recalling some basics

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

real imaginary



Alternative reparameterization:

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polar transform

$$\theta = \tan^{-1}\left(\frac{I}{R}\right) \quad r = \sqrt{R^2 + I^2}$$

or equivalently

$$re^{j\theta}$$

Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

exponential form

This will help us understand the Fourier transform equations

Fourier transform

Fourier transform | inverse Fourier transform

continuous

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi kx} \, dx \qquad f(x) = \int_{-\infty}^{\infty} F(k) e^{j2\pi kx} \, dk$$

discrete

$$F(k) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi kx/N} \qquad f(x) = \frac{1}{N} \sum_{k=0}^{N-1} F(k) e^{j2\pi kx/N}$$

$k = 0, 1, 2, \ldots, N-1$ $\qquad$ $x = 0, 1, 2, \ldots, N-1$

Where is the connection to the *"summation of sine waves"* idea?

Fourier transform

Where is the connection to the *"summation of sine waves"* idea?

$$f(x) = \frac{1}{N} \sum_{k=0}^{N-1} F(k) e^{j2\pi kx/N}$$

Euler's formula

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

sum over frequencies

$$f(x) = \frac{1}{N} \sum_{k=0}^{N-1} F(k) \left\{ \cos\left(\frac{2\pi kx}{N}\right) + j\sin\left(\frac{2\pi kx}{N}\right) \right\}$$

scaling parameter

wave components

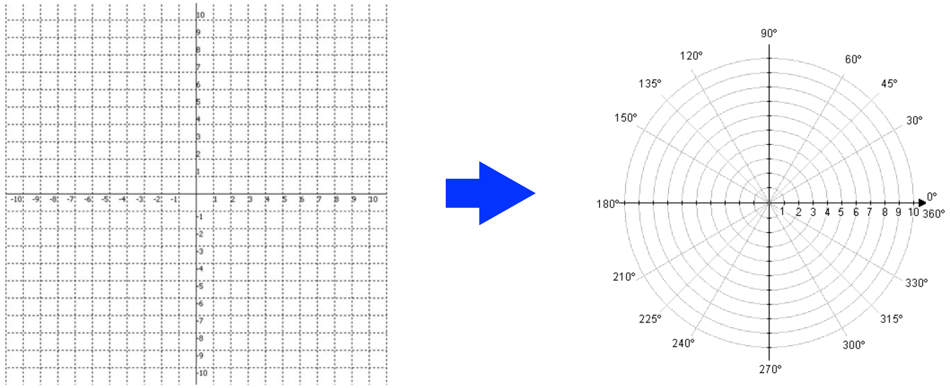
Recalling some basics

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

real imaginary



Alternative reparameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

polar transform

$$\theta = \tan^{-1}\left(\frac{I}{R}\right) \quad r = \sqrt{R^2 + I^2}$$

or equivalently

$$re^{j\theta}$$

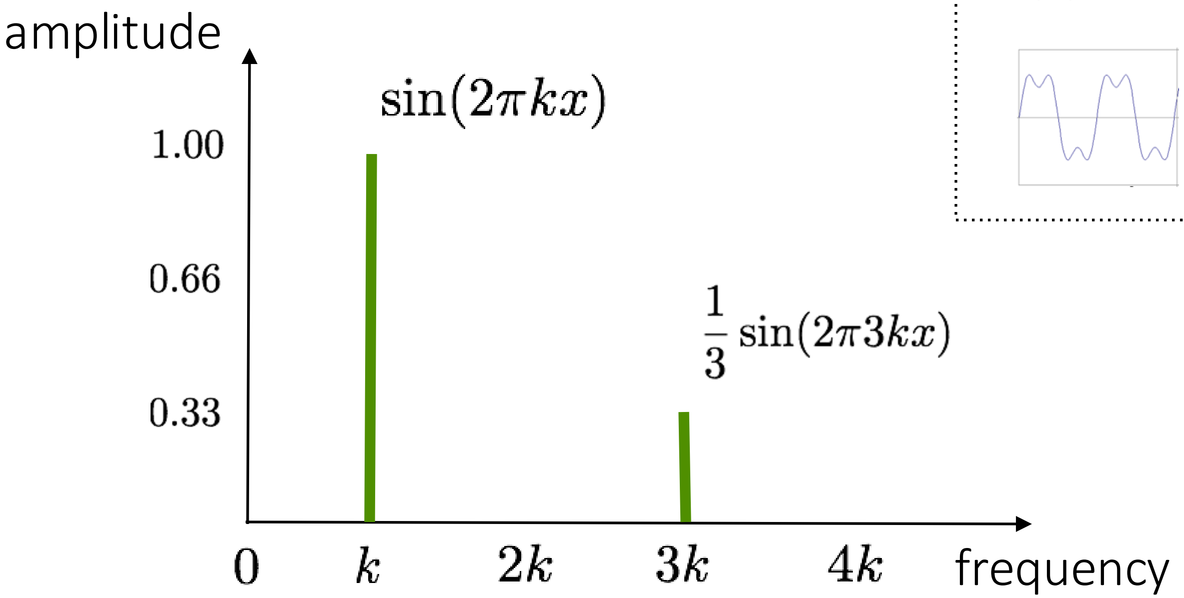
Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

exponential form

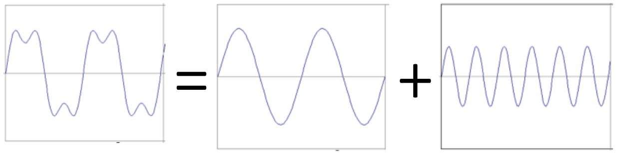
This will help us understand the Fourier transform equations

Visualizing the frequency spectrum



Recall the temporal domain visualization

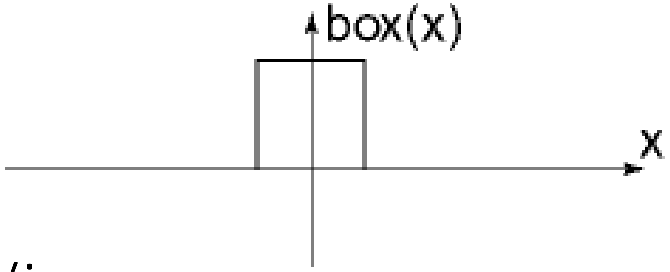
$$f(x) = \sin(2\pi kx) + \frac{1}{3}\sin(2\pi 3kx)$$

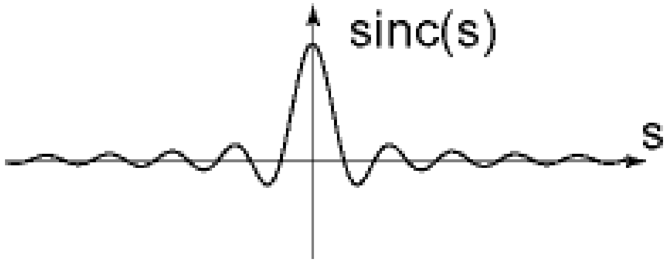


Fourier transform pairs

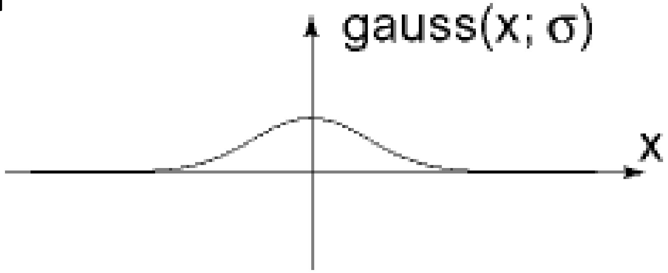
spatial domain

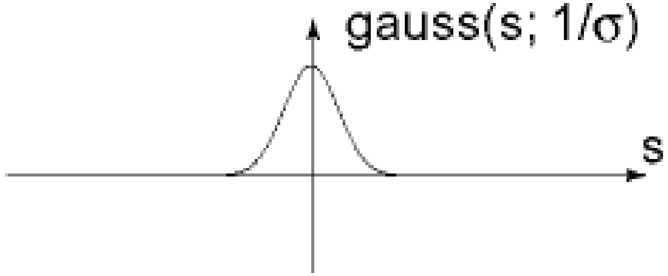
frequency domain



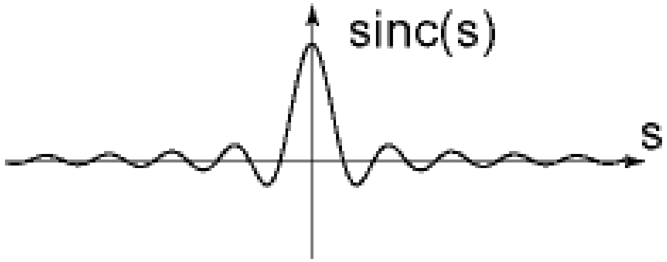


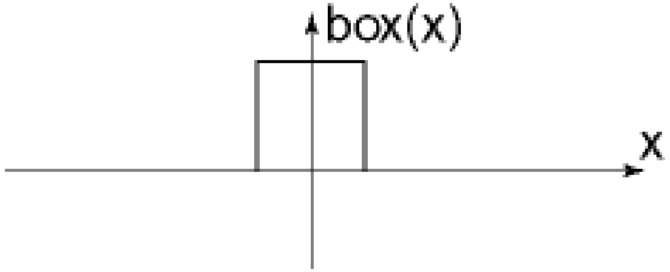
Note the symmetry: duality property of Fourier transform





Fun Exercise: Find other FT pairs!





Computing the discrete Fourier transform (DFT)

$$F(k) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)e^{-j2\pi kx/N}$$ is just a matrix multiplication:

$$\boldsymbol{F} = \boldsymbol{W}\boldsymbol{f}$$

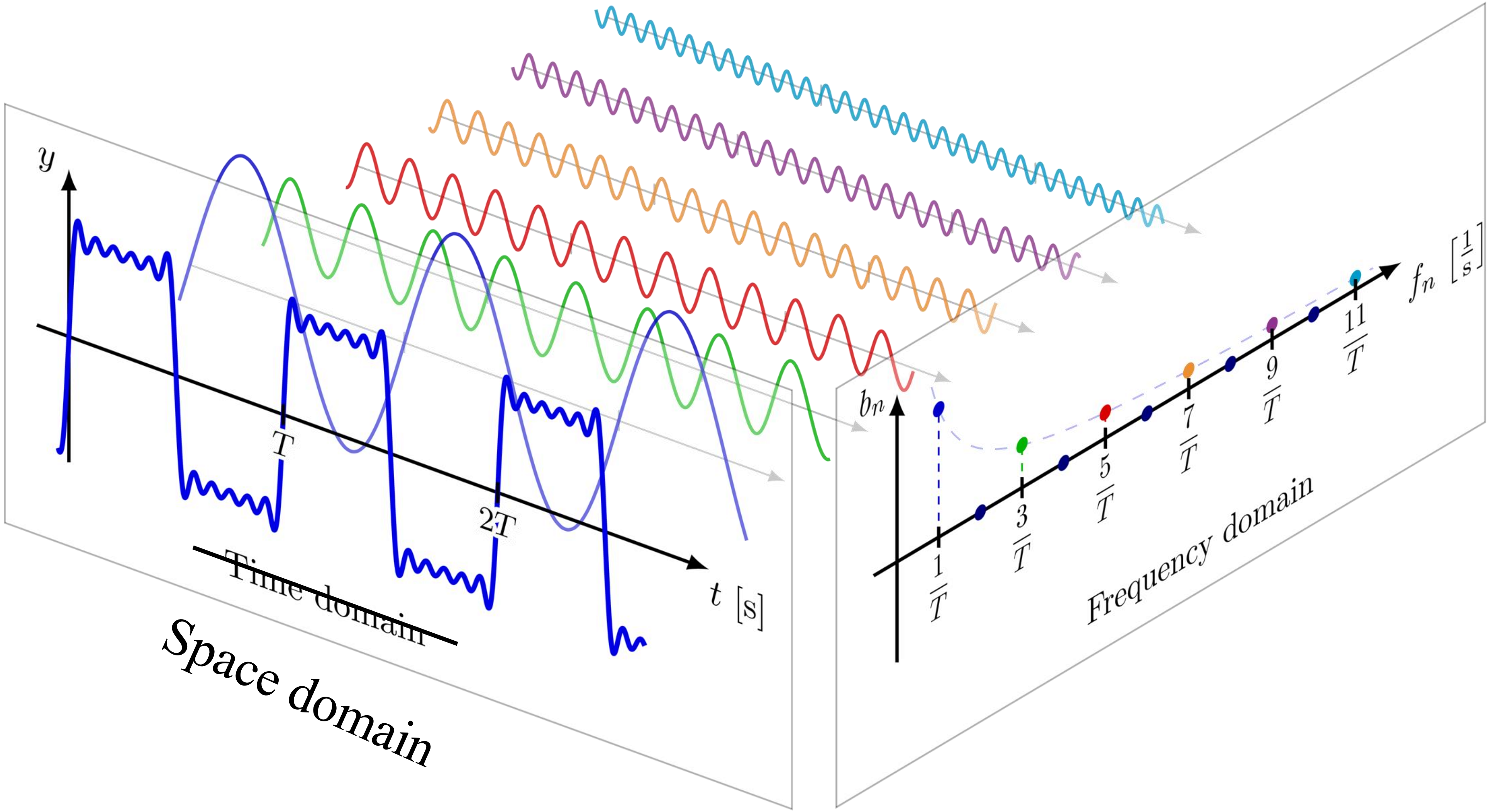
$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix} \qquad W = e^{-j2\pi/N}$$

In practice this is implemented using the *fast Fourier transform* (FFT) algorithm.

Recap: FT

Wikipedia: Fourier Transform

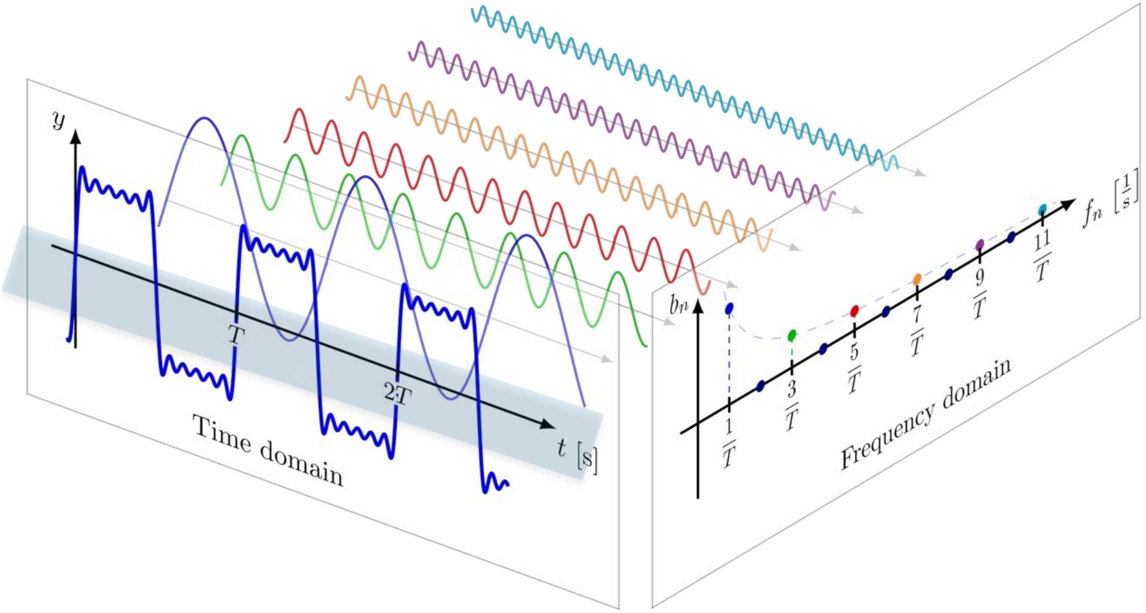
Recap: FT



y
T
$2T$
t [s]
~~Time domain~~
Space domain
b_n
$\frac{1}{T}$
$\frac{3}{T}$
$\frac{5}{T}$
$\frac{7}{T}$
$\frac{9}{T}$
$\frac{11}{T}$
f_n $\left[\frac{1}{s}\right]$
Frequency domain

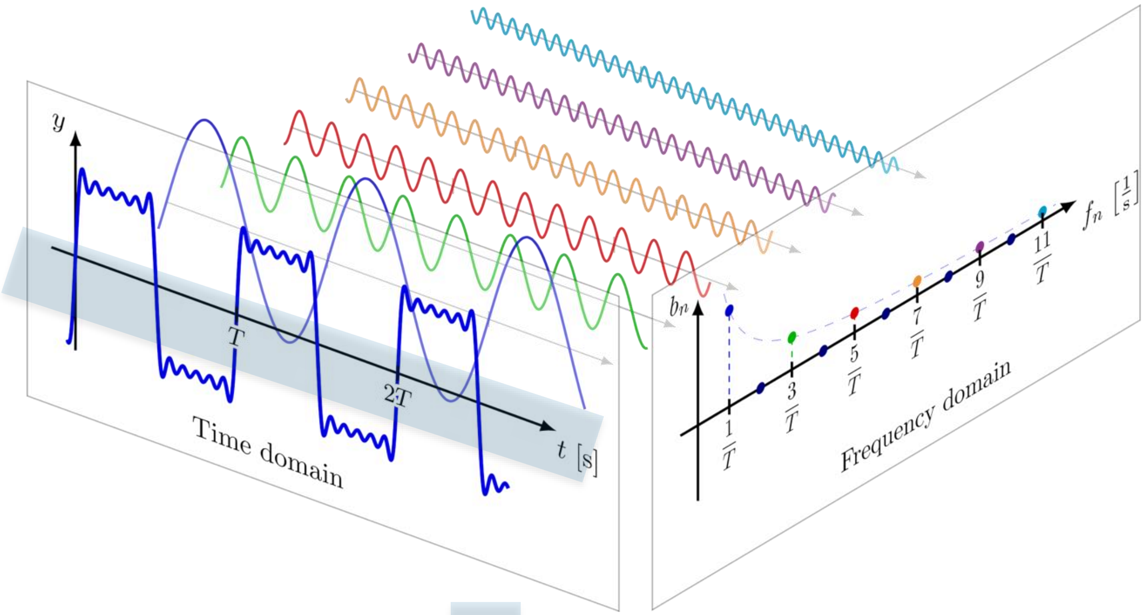
Recap: FT

What's this in the matrix?



$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix}$$

Recap: FT



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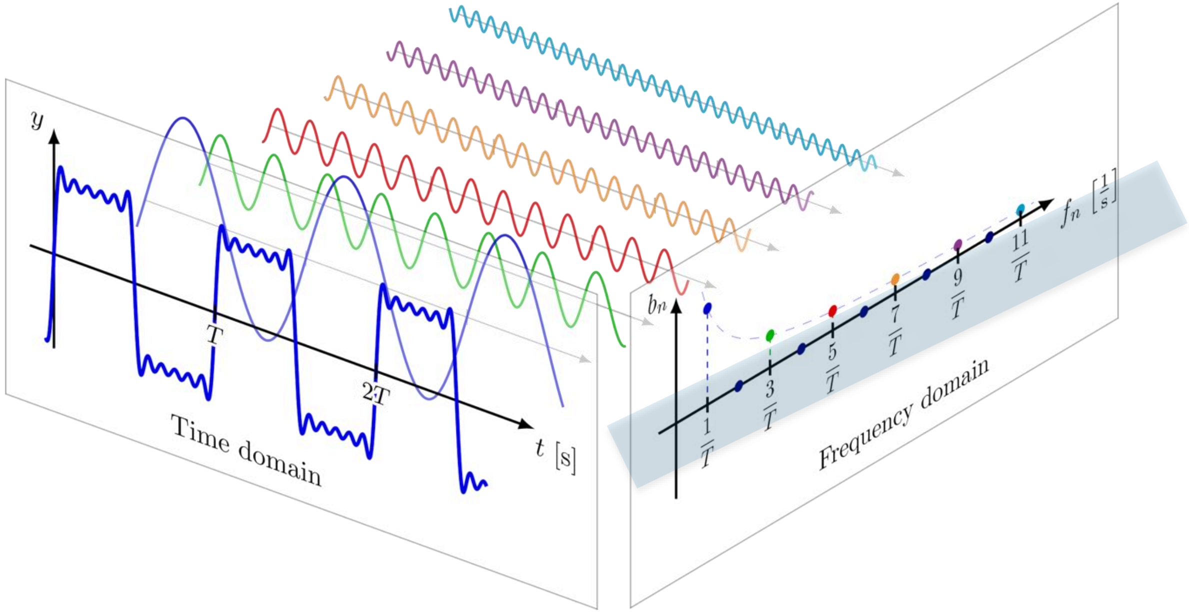
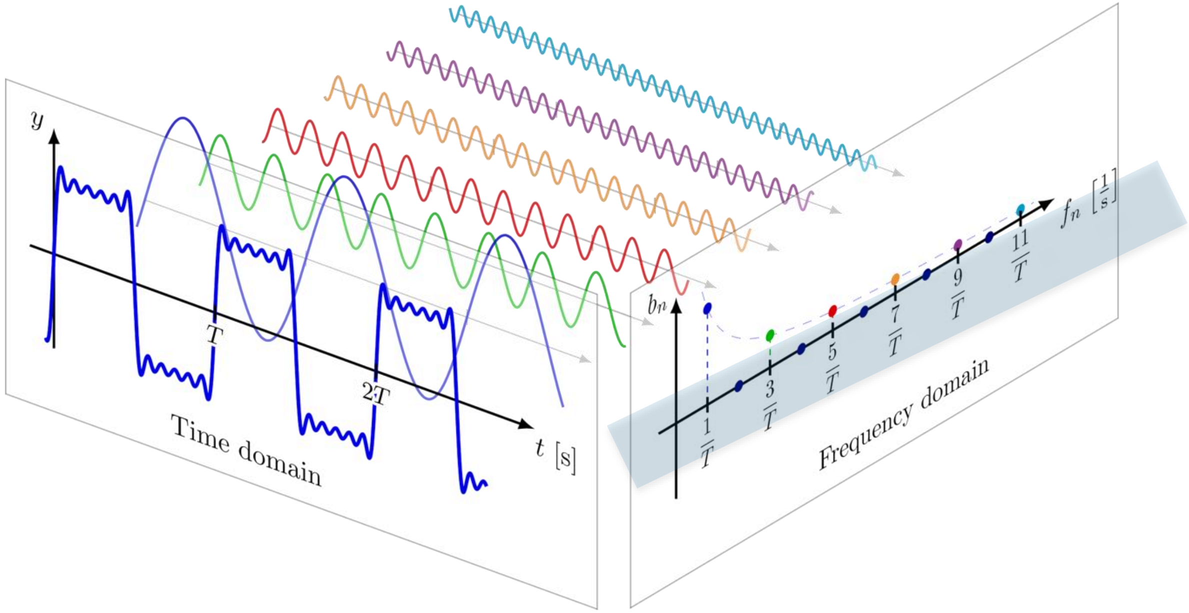
Recap: FT



$$
\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} =
\begin{bmatrix}
W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\
W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\
W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\
W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\
\vdots & & & & \ddots & \vdots \\
W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1
\end{bmatrix}
\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix}
$$

What's this in the matrix?

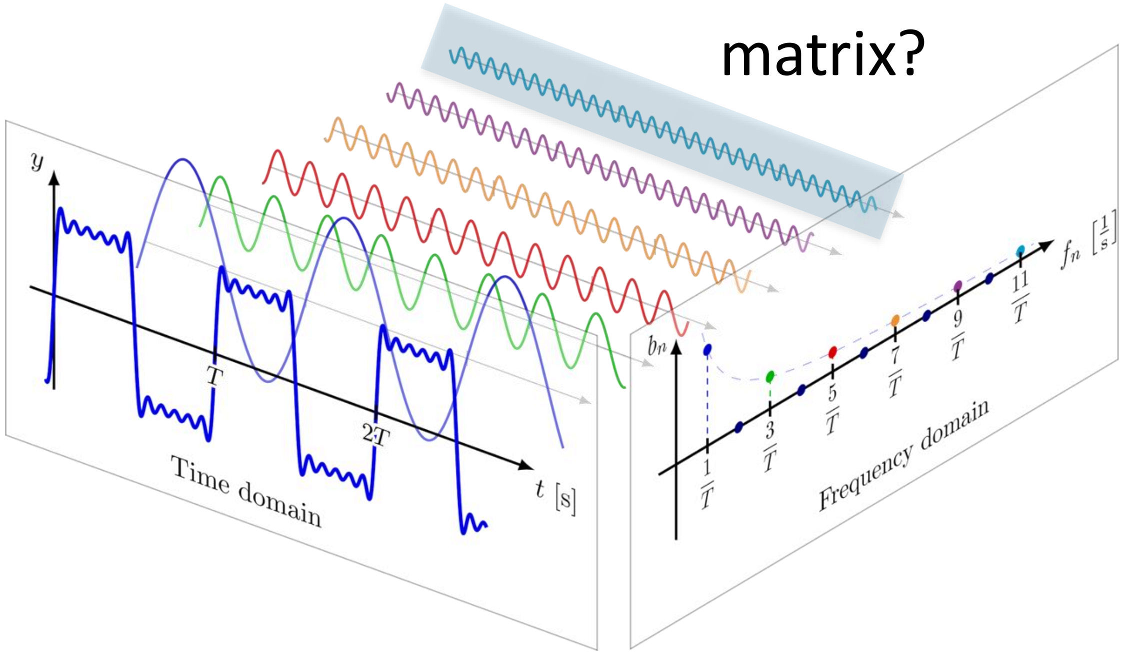
Recap: FT



$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix}$$

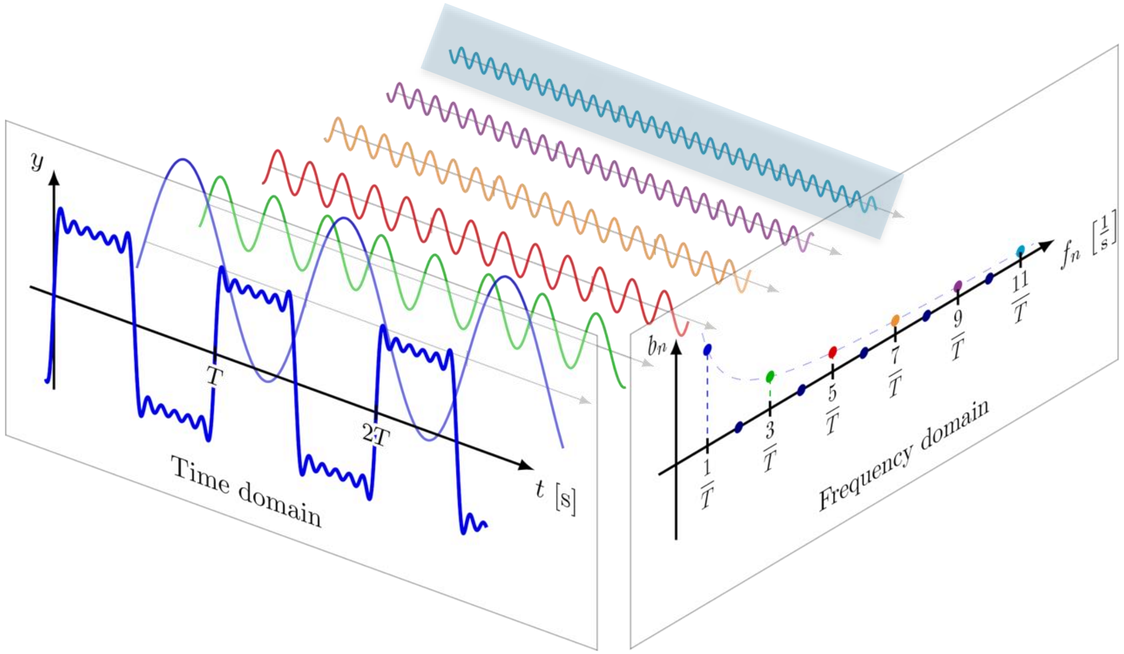
Recap: FT

What's this in the matrix?



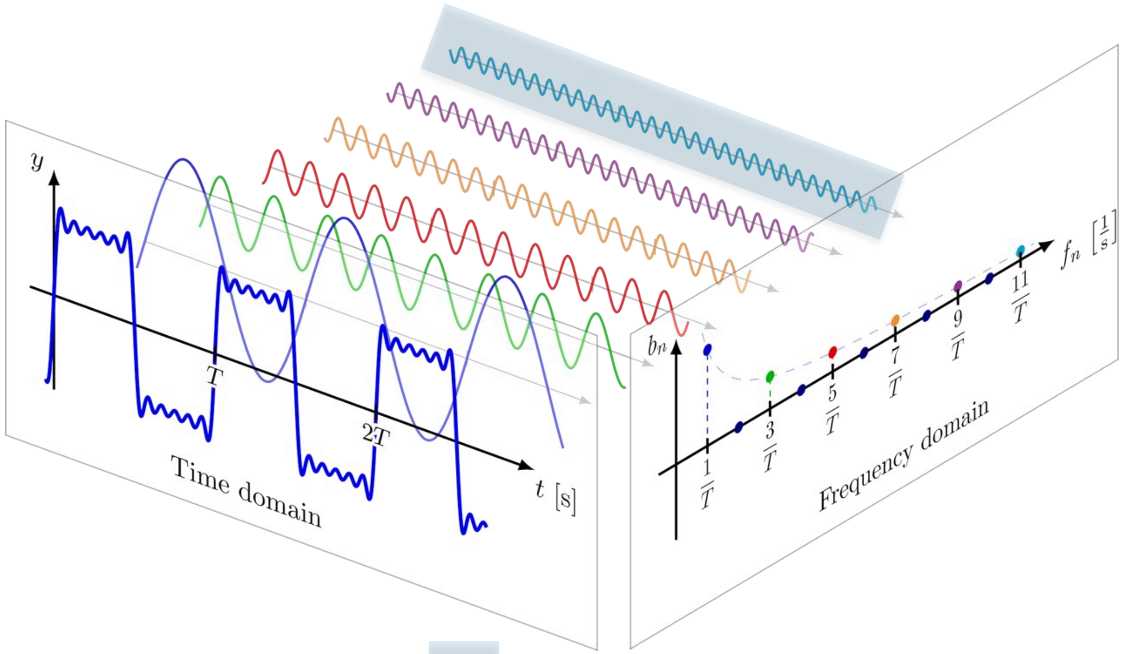
$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix}$$

Recap: FT



$$
\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix}
$$

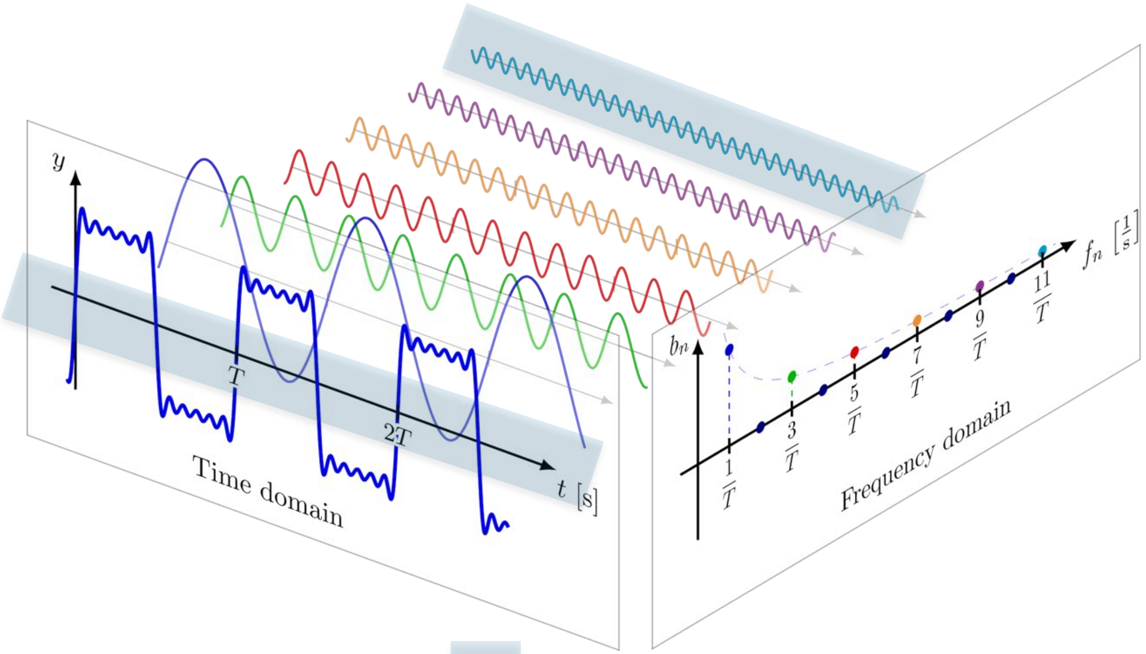
Recap: FT



$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix}$$

What's this in the diagram?

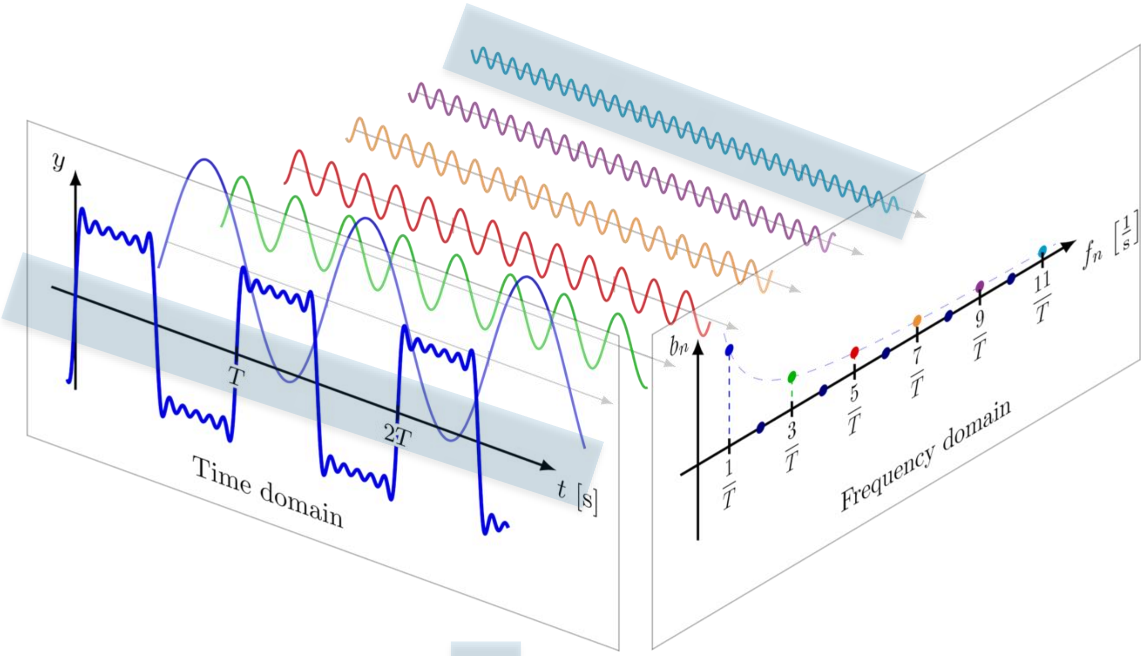
Recap: FT



$$
\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix}
$$

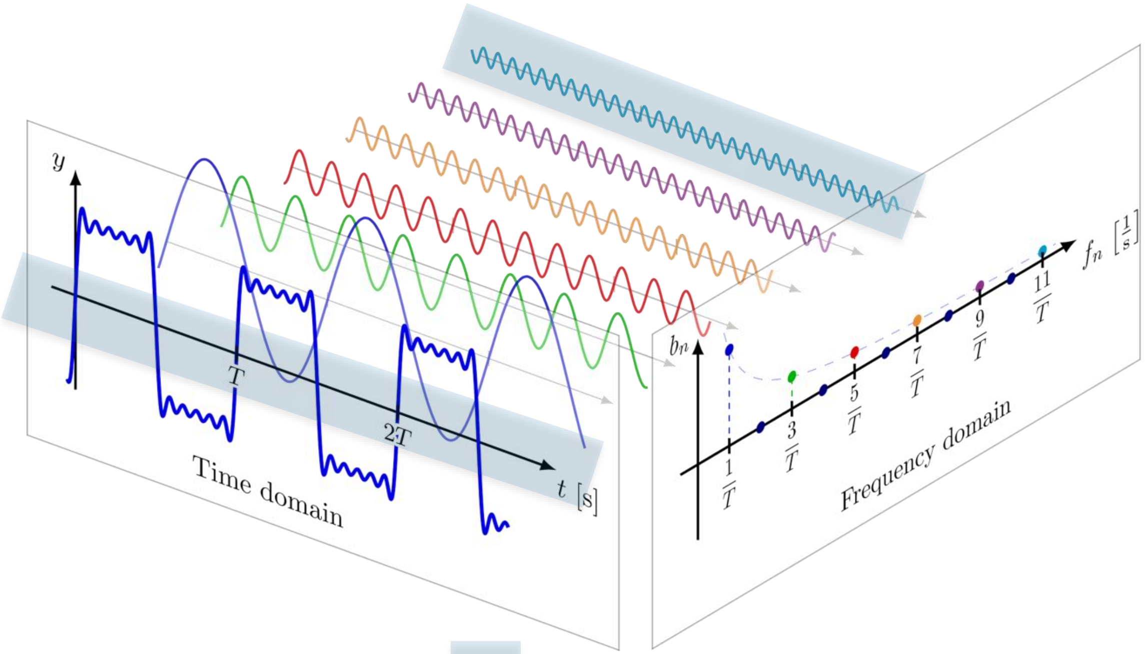
Multiplying it with this row, what do you get?

Recap: FT



$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix}$$

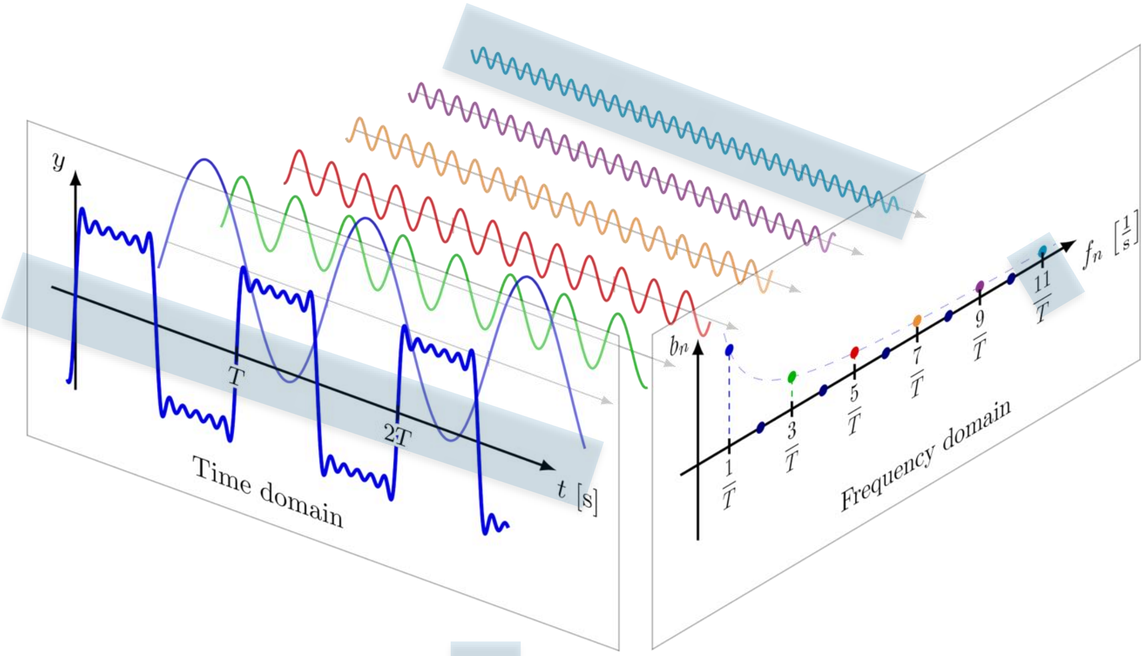
Recap: FT



What's this item in the diagram?

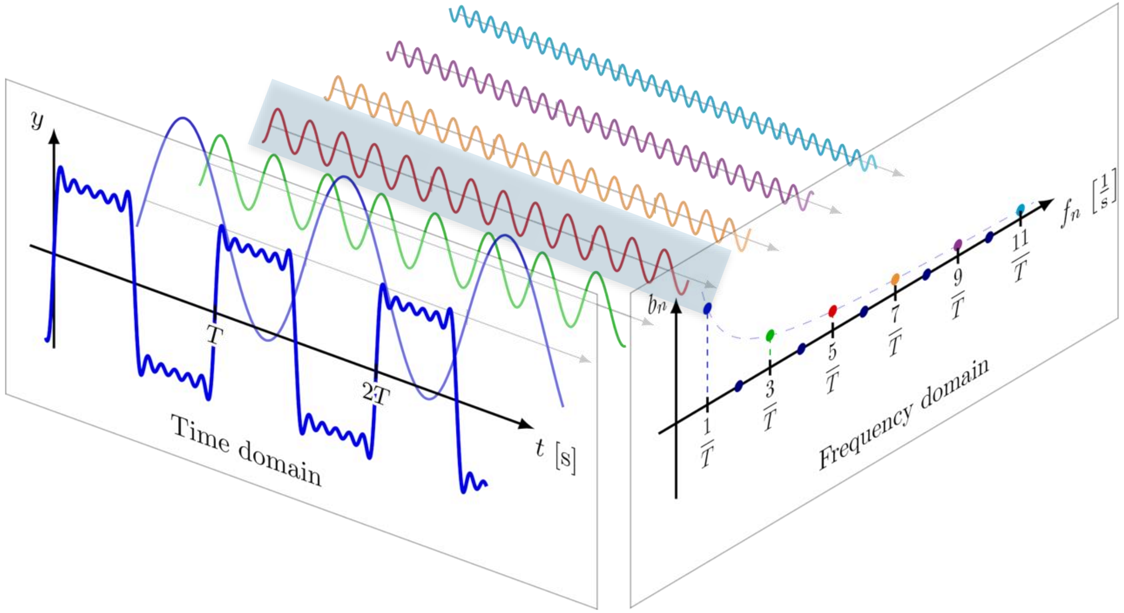
$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix}$$

Recap: FT



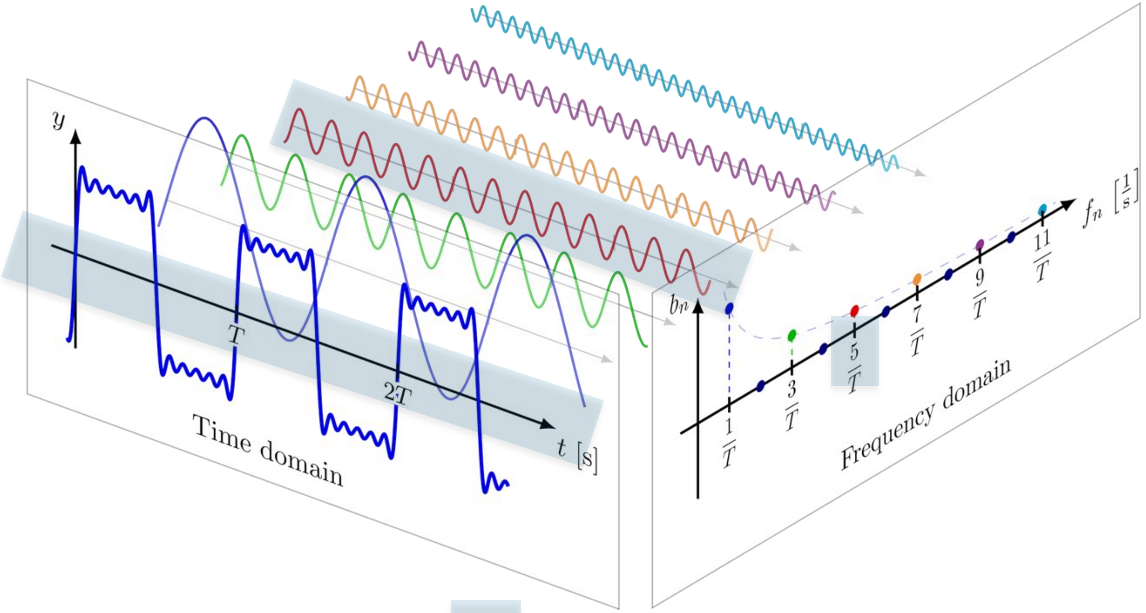
$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix}$$

Recap: FT



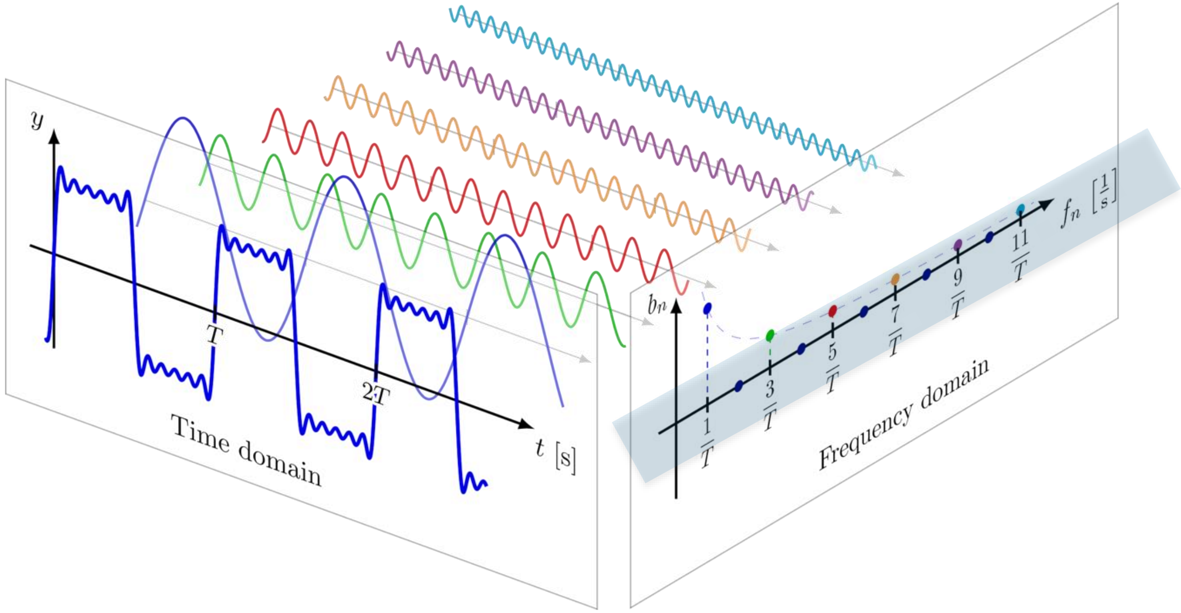
$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix}$$

Recap: FT



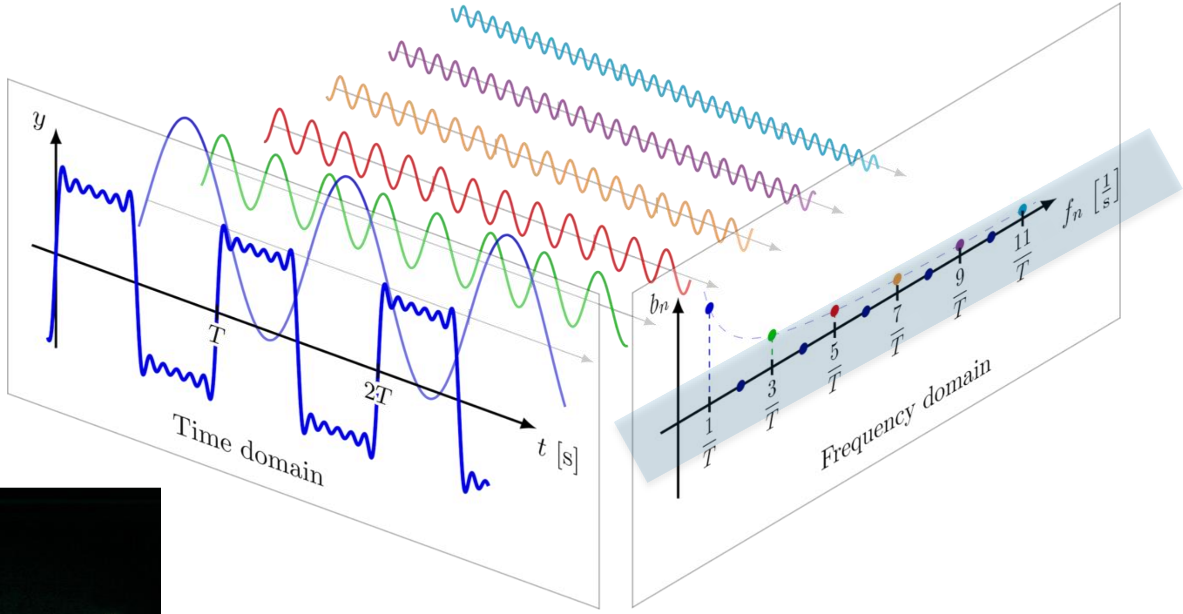
$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix}$$

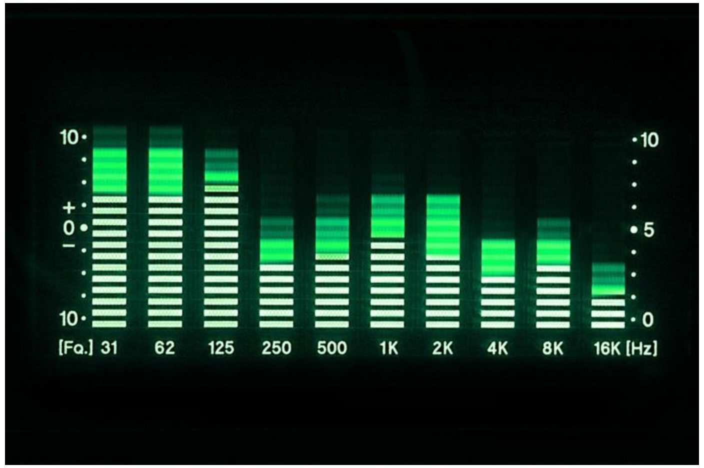
Recap: FT



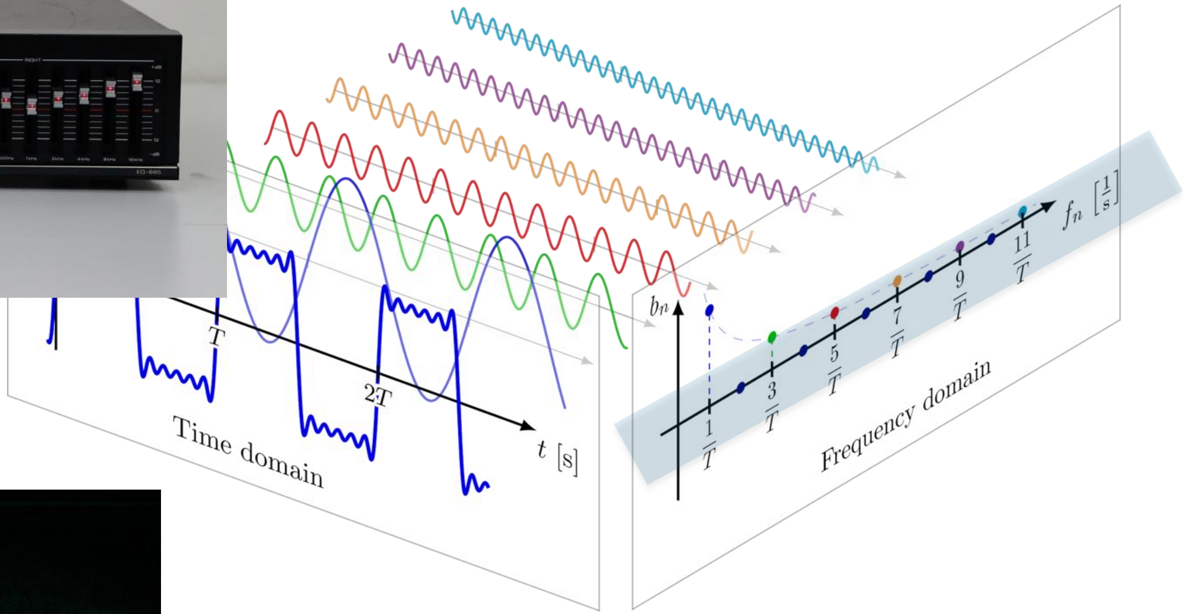
y
T
$2T$
t [s]
Time domain
b_n
$\frac{1}{T}$
$\frac{3}{T}$
$\frac{5}{T}$
$\frac{7}{T}$
$\frac{9}{T}$
$\frac{11}{T}$
f_n $[\frac{1}{s}]$
Frequency domain

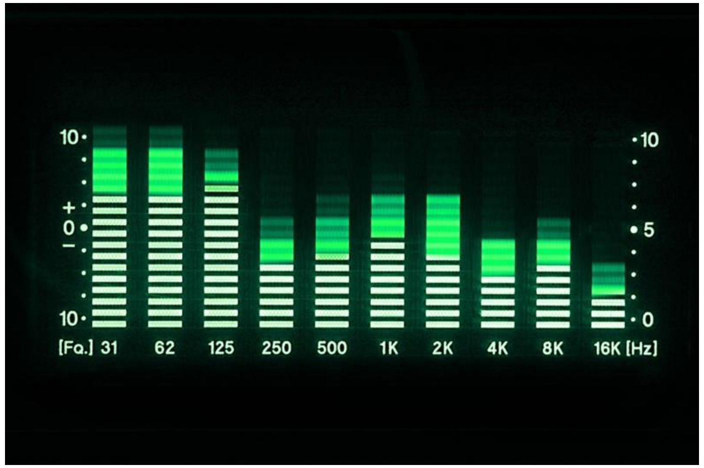
Recap: FT


Time domain
Frequency domain
y
t [s]
T
$2T$
b_n
$f_n\ [\frac{1}{s}]$
$\frac{1}{T}$ $\frac{3}{T}$ $\frac{5}{T}$ $\frac{7}{T}$ $\frac{9}{T}$ $\frac{11}{T}$


10
+
0
–
10
10
5
0
[Fq.] 31 62 125 250 500 1K 2K 4K 8K 16K [Hz]

Recap: FT





What happens when you tune the nob?

Treble
High-freq

Bass

Low-freq

Treble
High-freq

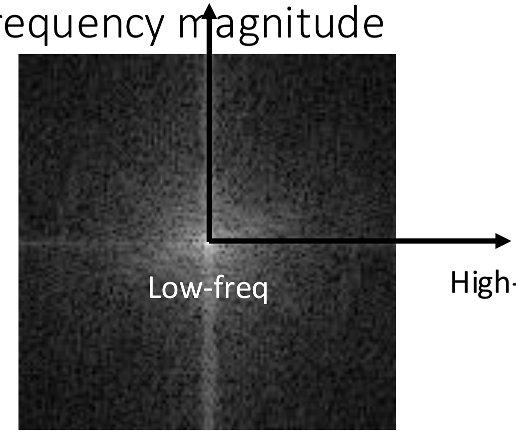
Bass
Low-freq

What are “bass” and ”treble” in images?

More filtering examples

original image

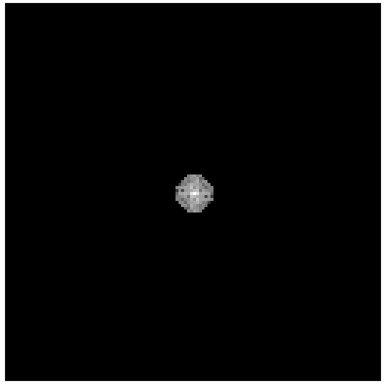
frequency magnitude



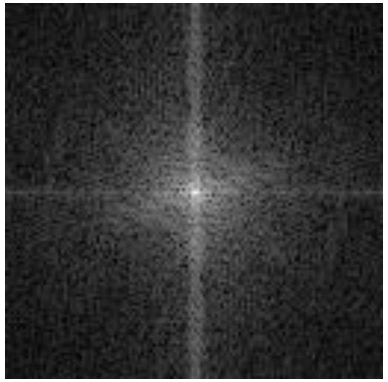
More filtering examples

original image

low-pass filter



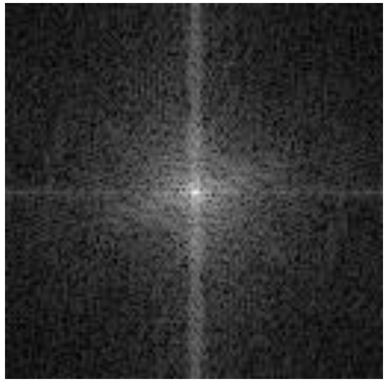
frequency magnitude



More filtering examples

original image

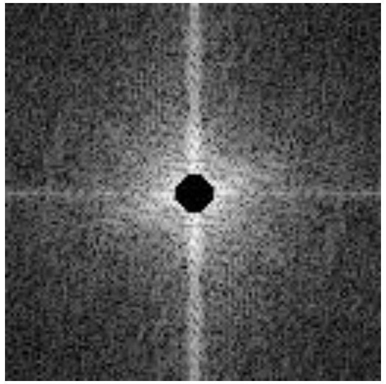
frequency magnitude



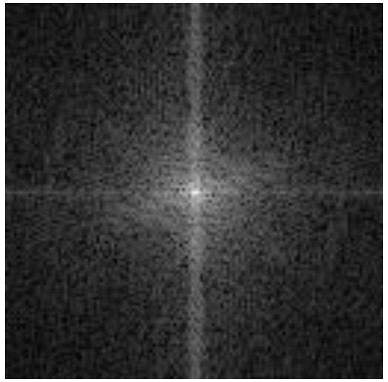
More filtering examples

original image

high-pass filter



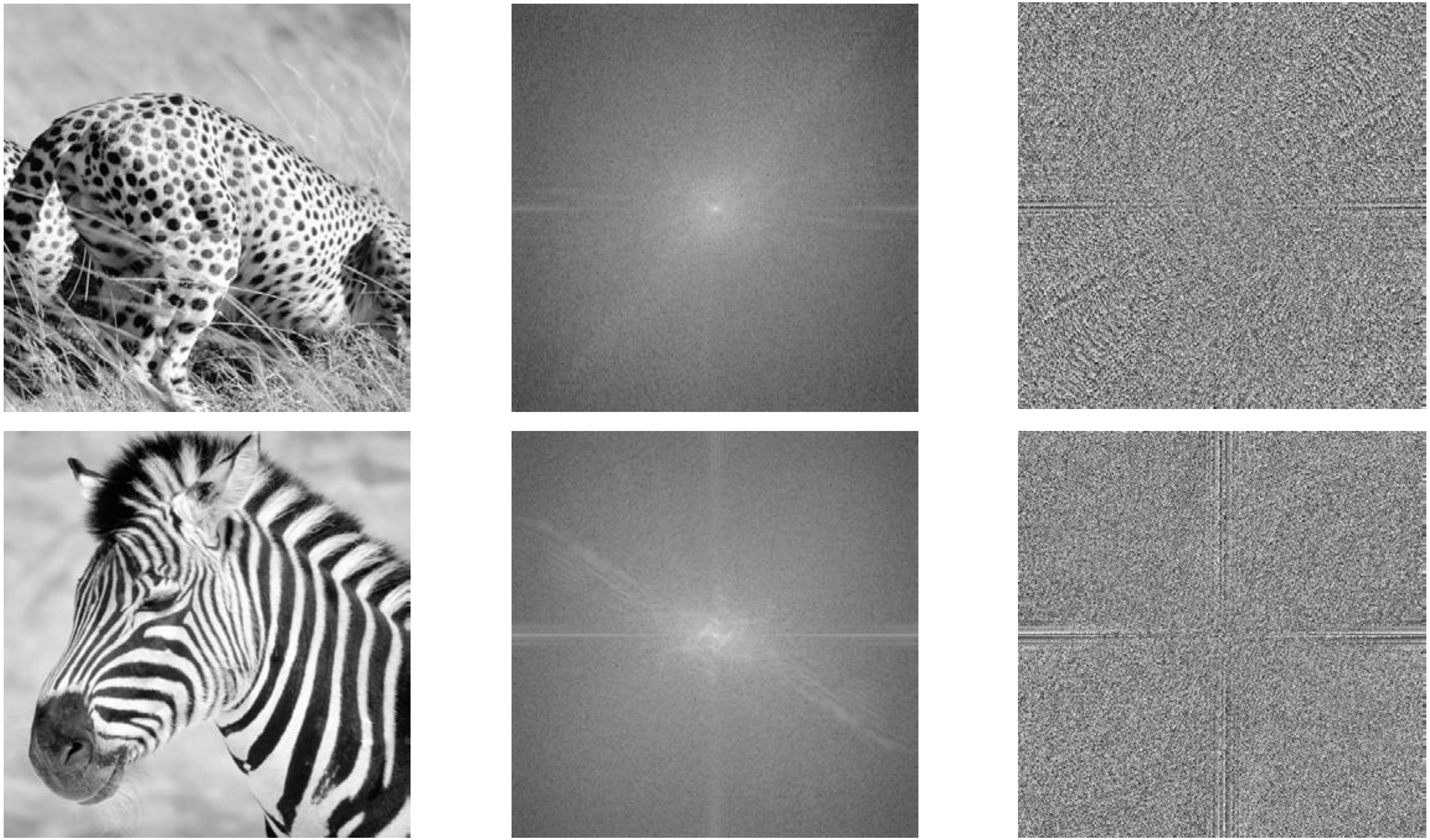
frequency magnitude



More resourses

- https://gru.stanford.edu/doku.php/tutorials/fouriertransform
- https://homepages.inf.ed.ac.uk/rbf/CVonline/LOCAL_COPIES/OWENS/LECT4/node2.html

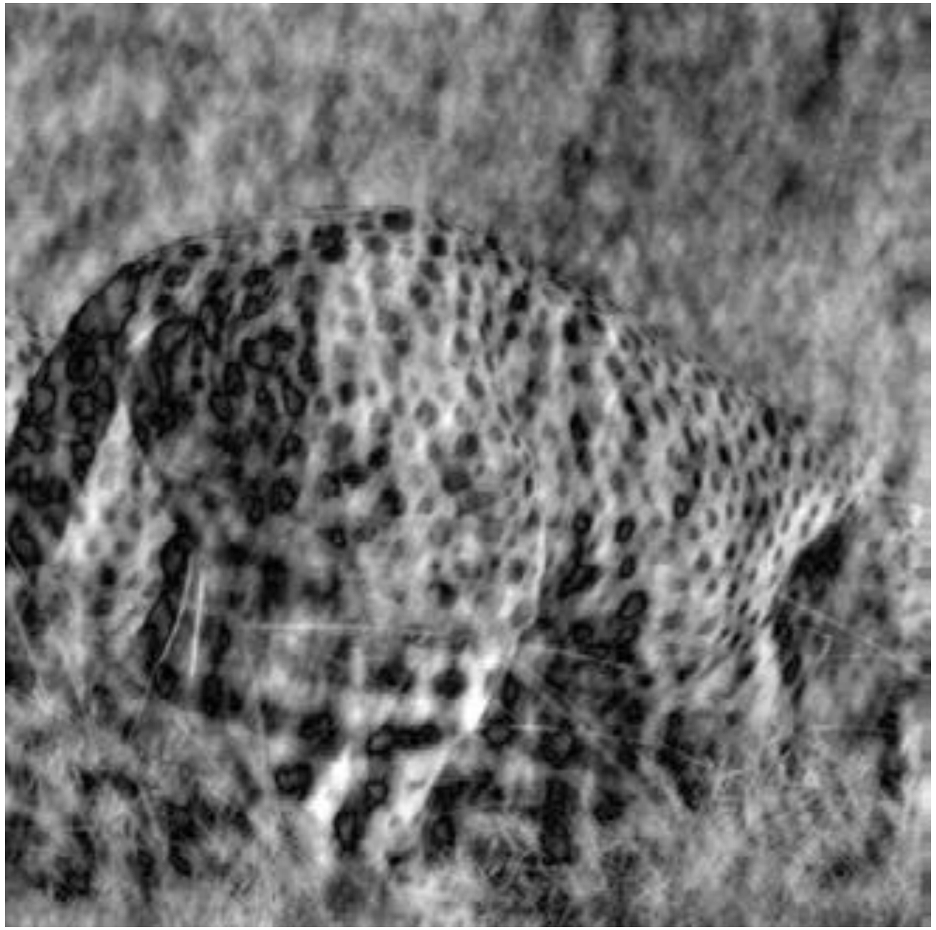
Fourier transforms of natural images



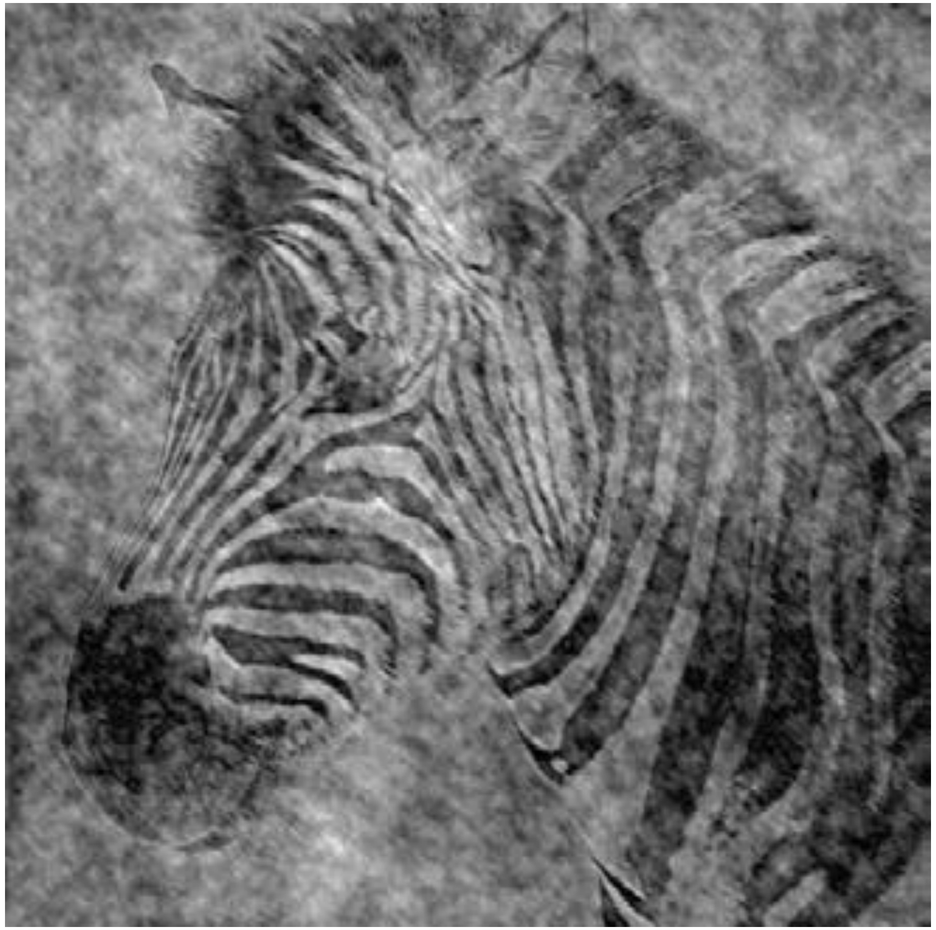
original amplitude phase

Fourier transforms of natural images

Image phase matters!



cheetah phase with zebra amplitude



zebra phase with cheetah amplitude

Frequency-domain filtering

The convolution theorem

The Fourier transform of the convolution of two functions is the product of their Fourier transforms:

$$\mathcal{F}\{g * h\} = \mathcal{F}\{g\}\mathcal{F}\{h\}$$

The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms:

$$\mathcal{F}^{-1}\{gh\} = \mathcal{F}^{-1}\{g\} * \mathcal{F}^{-1}\{h\}$$

Convolution in spatial domain is equivalent to multiplication in frequency domain!

What do we use convolution for?

Convolution for 1D continuous signals

Definition of linear shift-invariant filtering as convolution:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x - y) dy$$

filtered signal

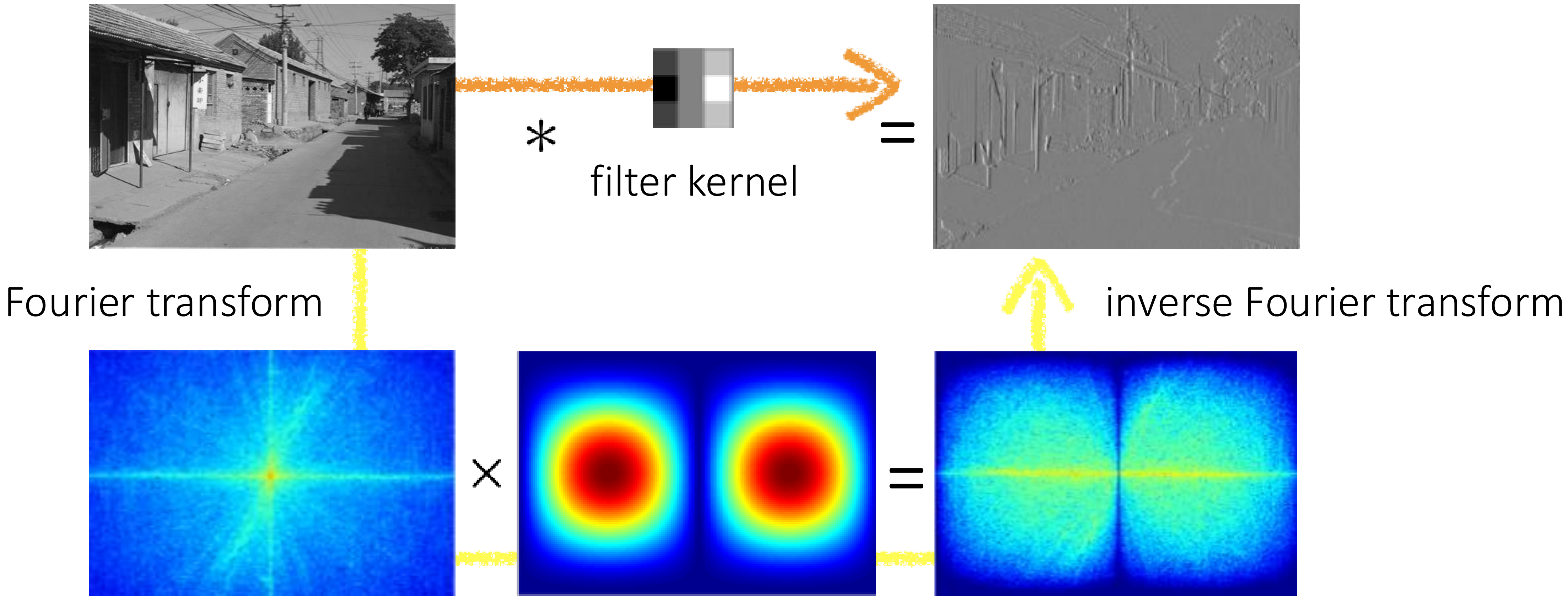
filter

input signal

Using the convolution theorem, we can interpret and implement all types of linear shift-invariant filtering as multiplication in frequency domain.

Why implement convolution in frequency domain?

Spatial domain filtering



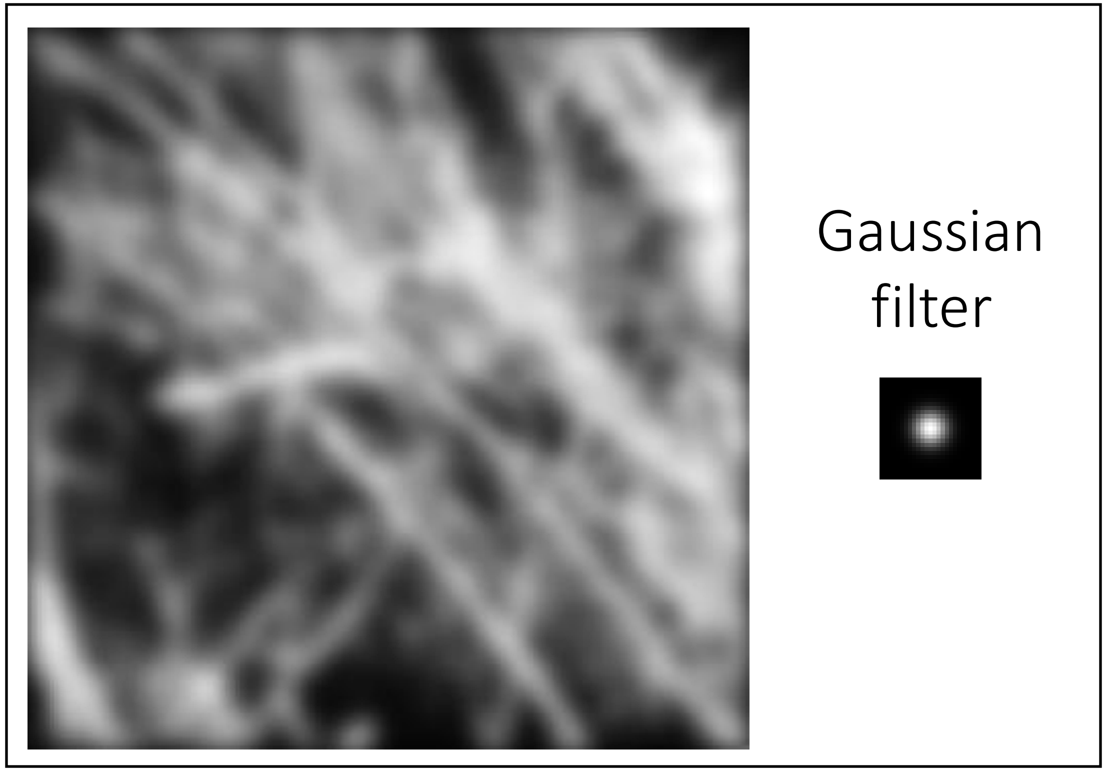
Frequency domain filtering

Frequency-domain filtering in Python

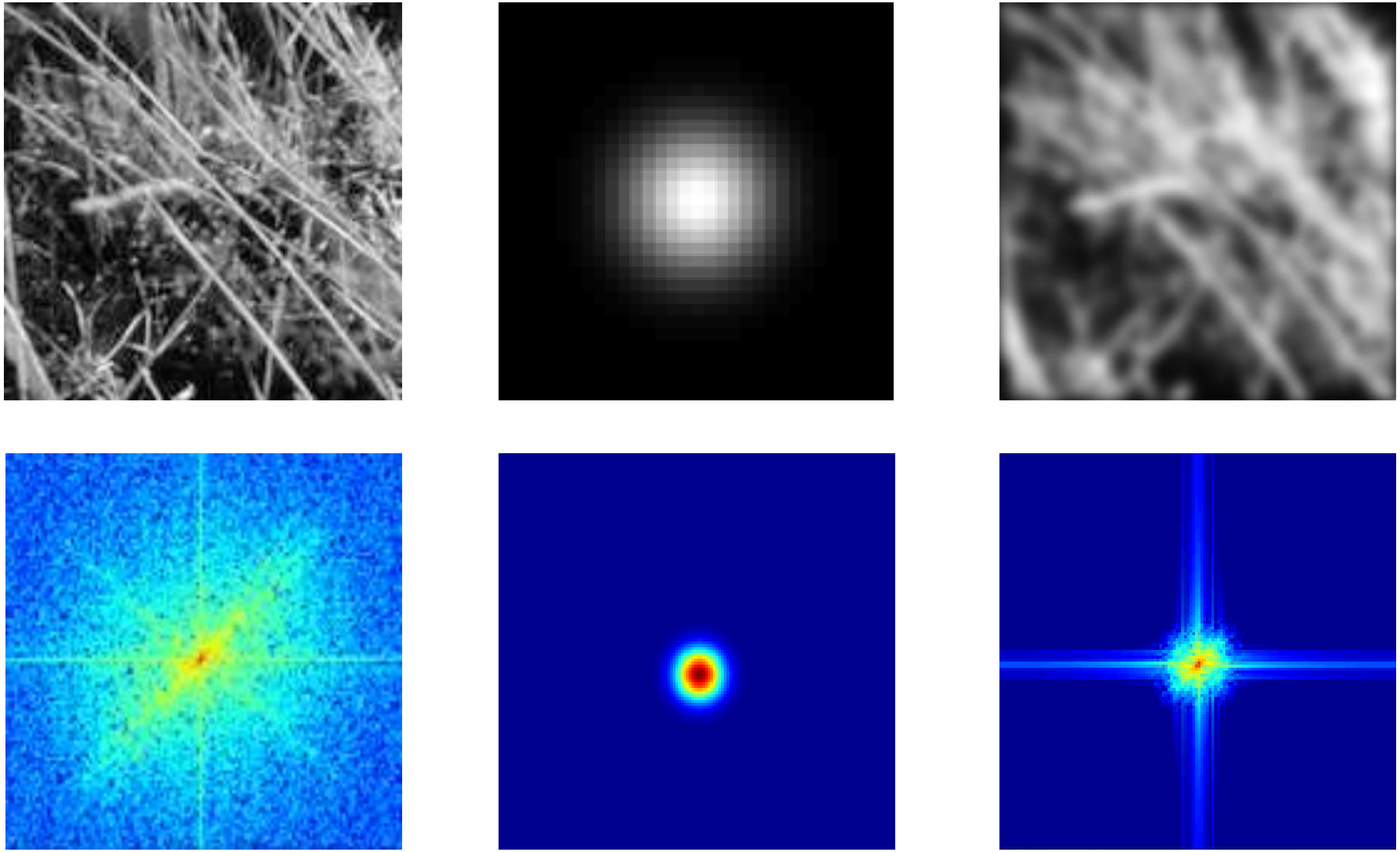
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.fft import fft2, ifft2, fftshift

# Create a sample image
image = np.zeros((100, 100))
image[40:60, 40:60] = 1

# Compute the Fourier Transform
fft_image = fft2(image)
fft_shifted = fftshift(fft_image)

# Create a low-pass filter
rows, cols = image.shape
crow, ccol = rows // 2, cols // 2
mask = np.zeros((rows, cols), np.uint8)
mask[crow - 10: crow + 10, ccol - 10: ccol + 10] = 1

# Apply the filter
fft_filtered = fft_shifted * mask

# Inverse Fourier Transform
filtered_image = np.real(ifft2(fftshift(fft_filtered)))

# Display the results
plt.figure(figsize=(10, 5))

plt.subplot(1, 3, 1)
plt.imshow(image, cmap='gray')
plt.title('Original Image')

plt.subplot(1, 3, 2)
plt.imshow(np.log(np.abs(fft_shifted)), cmap='gray')
plt.title('Fourier Spectrum')

plt.subplot(1, 3, 3)
plt.imshow(filtered_image, cmap='gray')
plt.title('Filtered Image')

plt.tight_layout()
plt.show()
```

Revisiting blurring

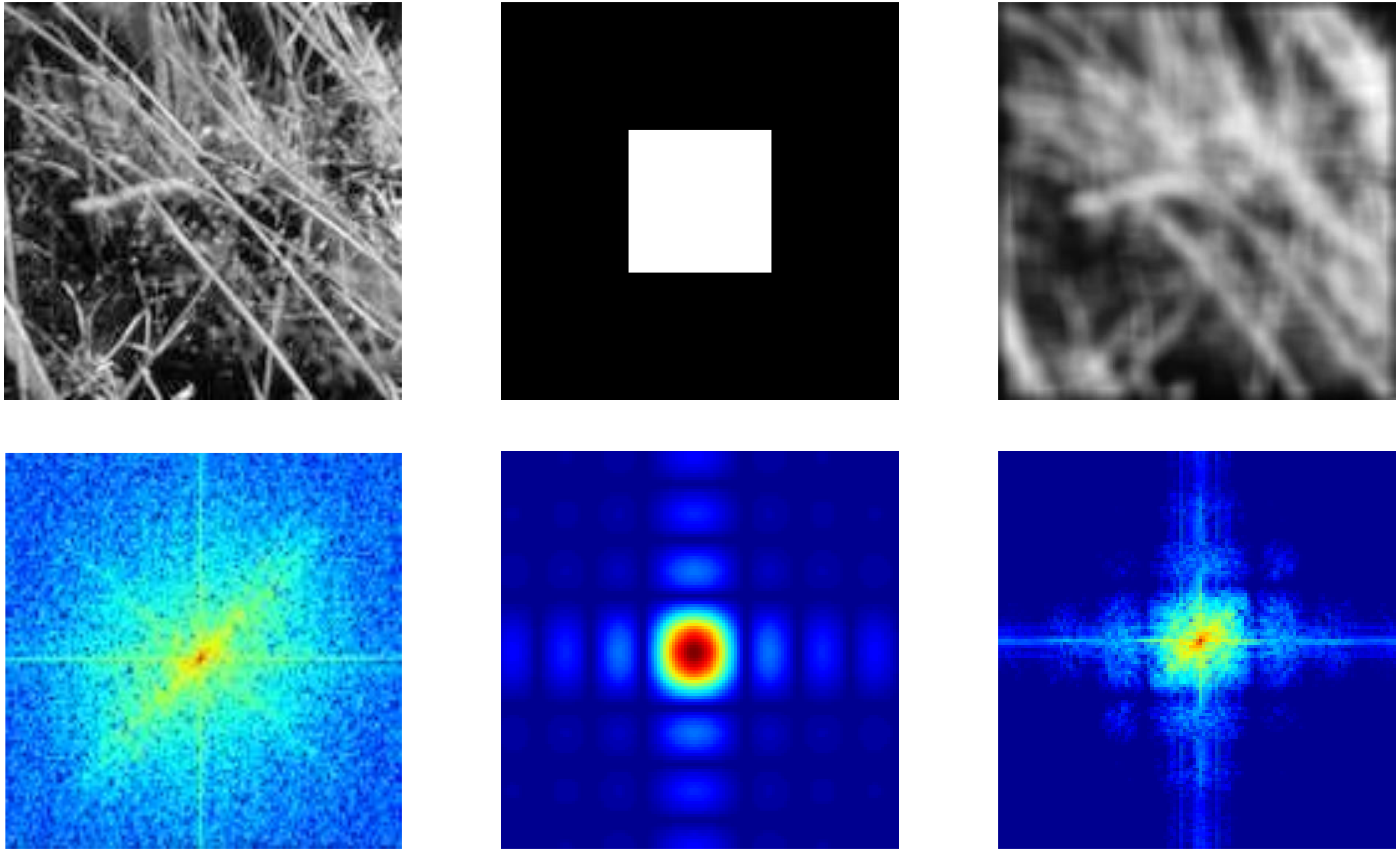
Why does the Gaussian give a nice smooth image, but the square filter give edgy artifacts?



Gaussian blur

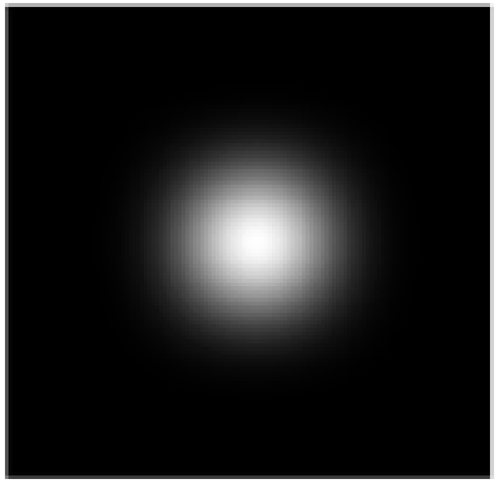


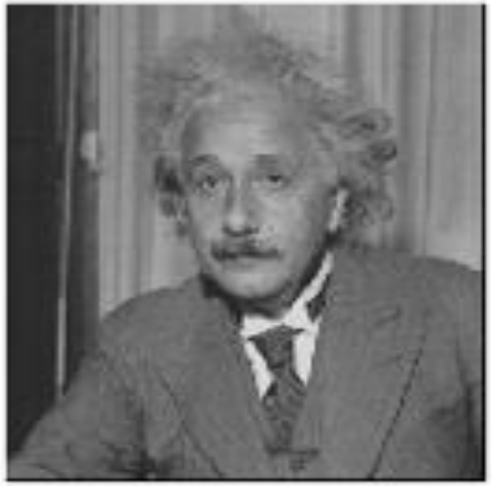
Box blur



More filtering examples

?

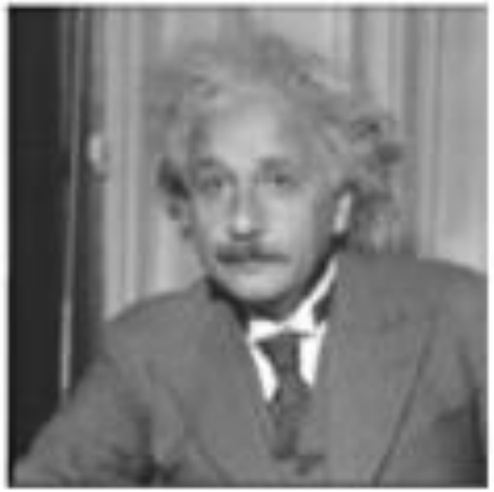




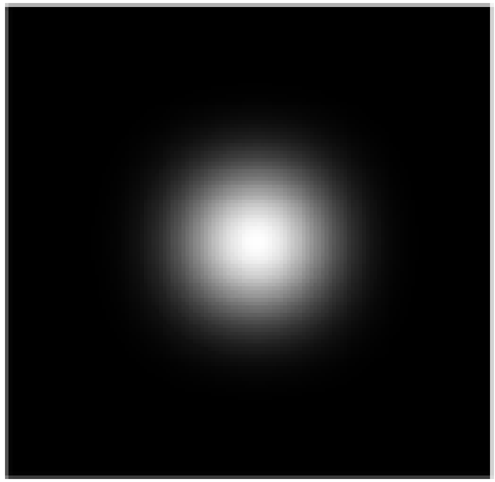
?

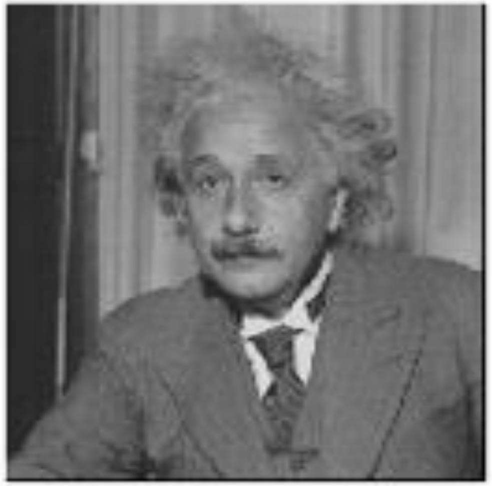
filters shown in frequency-domain

More filtering examples



low-pass





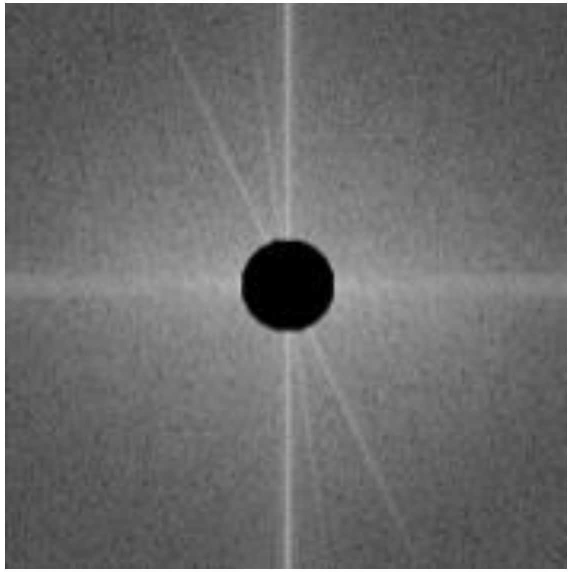
band-pass

filters shown in frequency-domain

More filtering examples

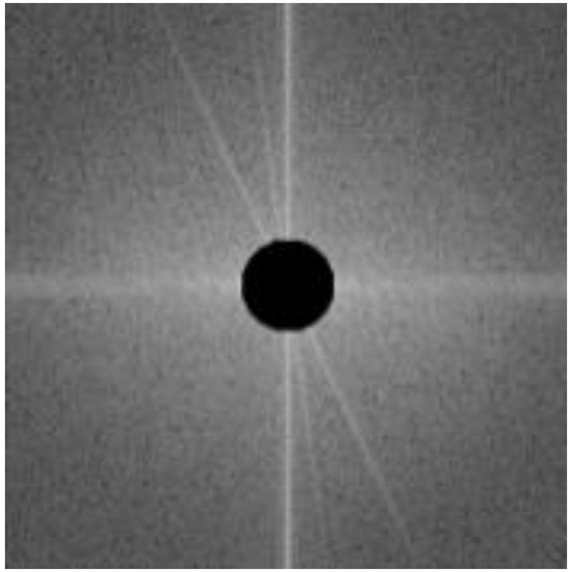
?

high-pass



More filtering examples

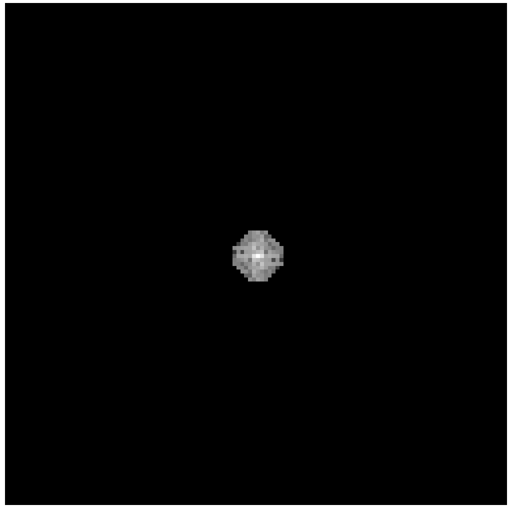
high-pass



More filtering examples

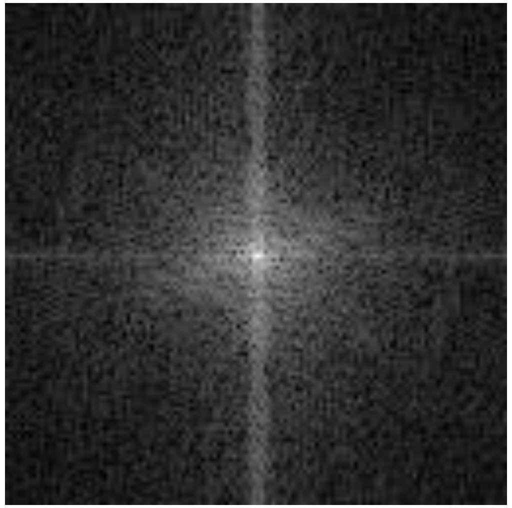
original image

low-pass filter



?

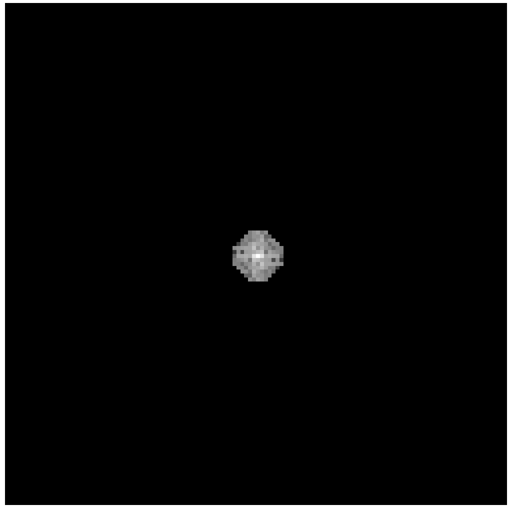
frequency magnitude



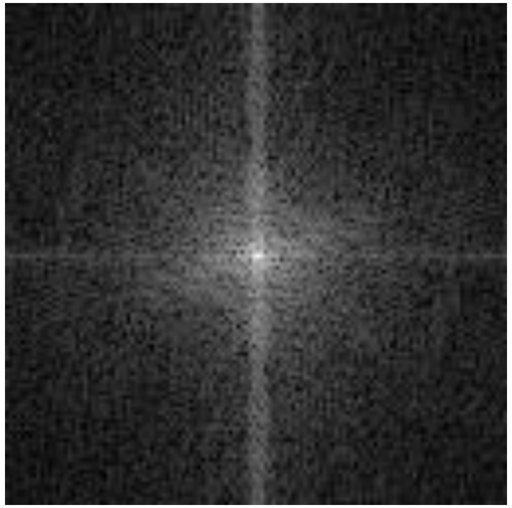
More filtering examples

original image

low-pass filter



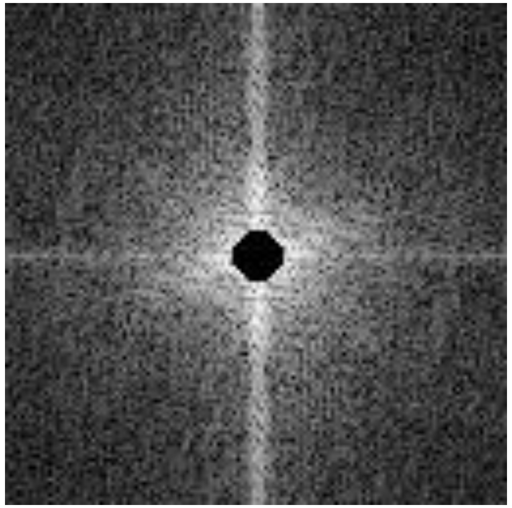
frequency magnitude



More filtering examples

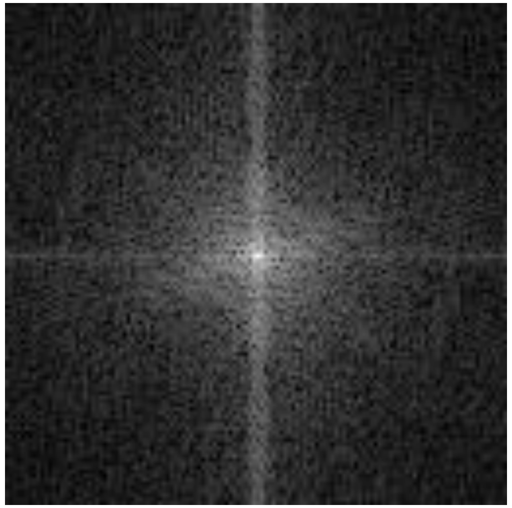
original image

high-pass filter



?

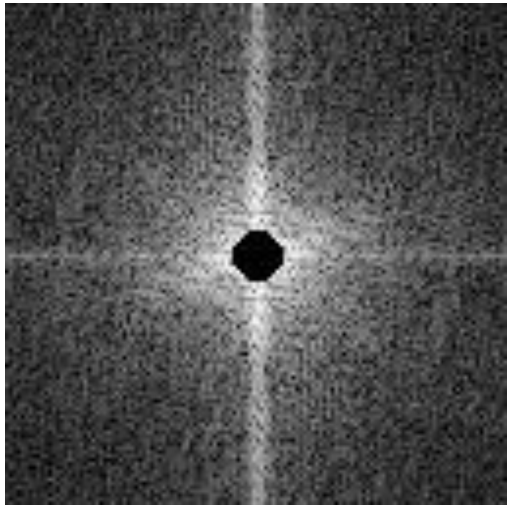
frequency magnitude



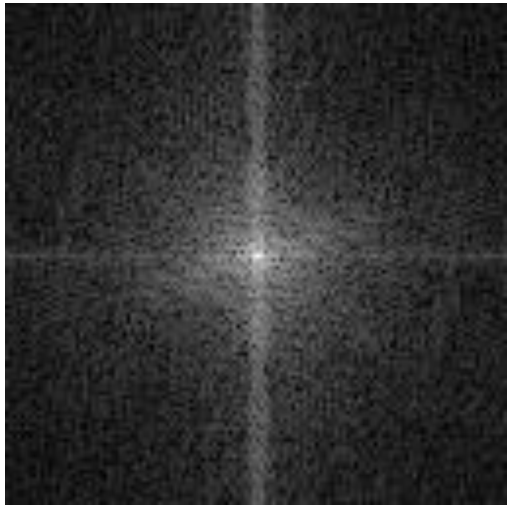
More filtering examples

original image

high-pass filter



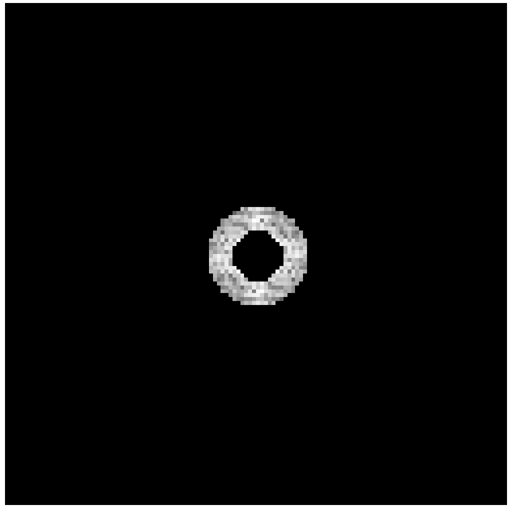
frequency magnitude

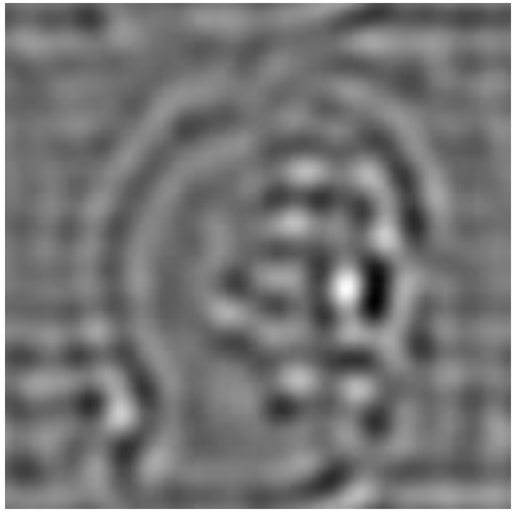


More filtering examples

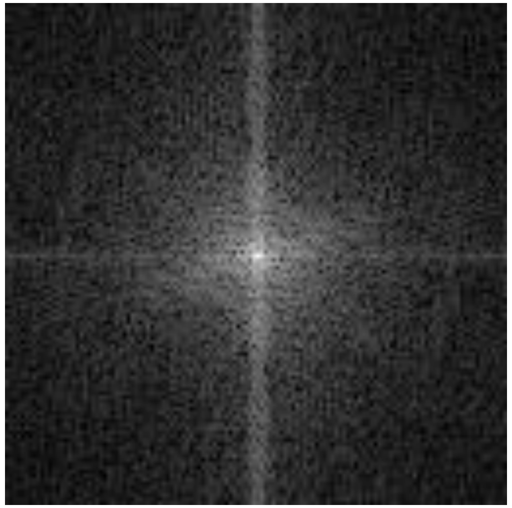
original image

band-pass filter





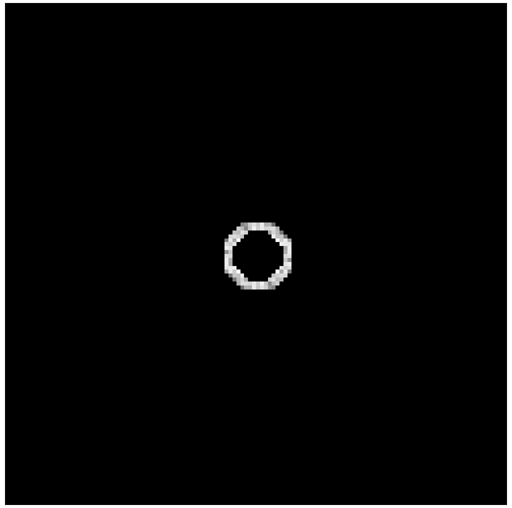
frequency magnitude

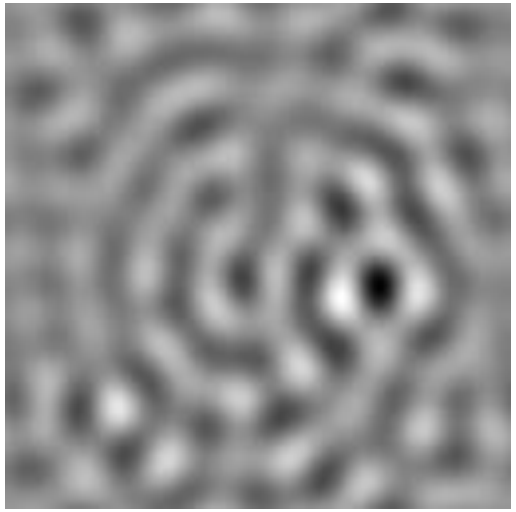


More filtering examples

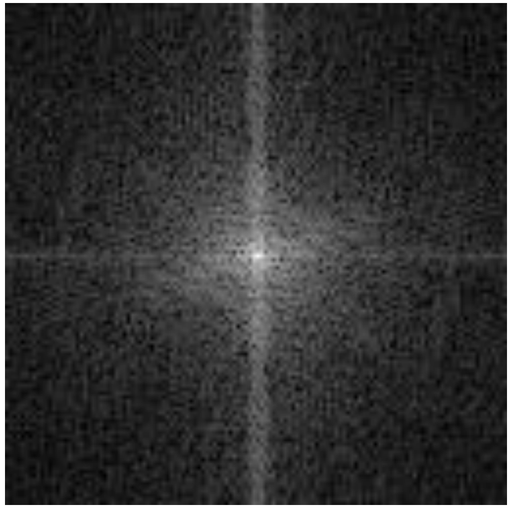
original image

band-pass filter





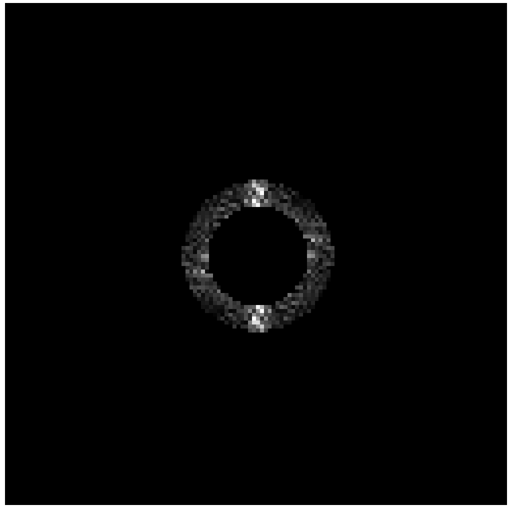
frequency magnitude



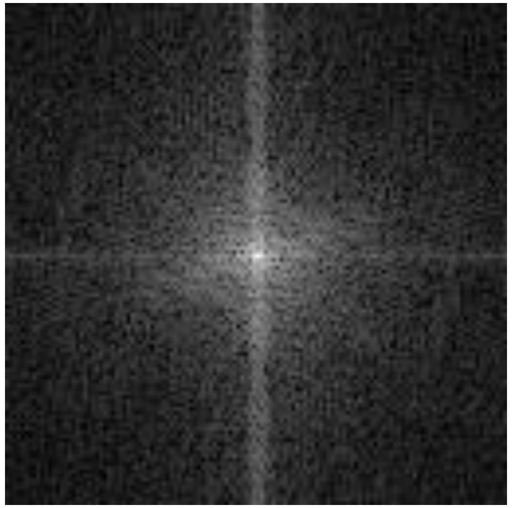
More filtering examples

original image

band-pass filter



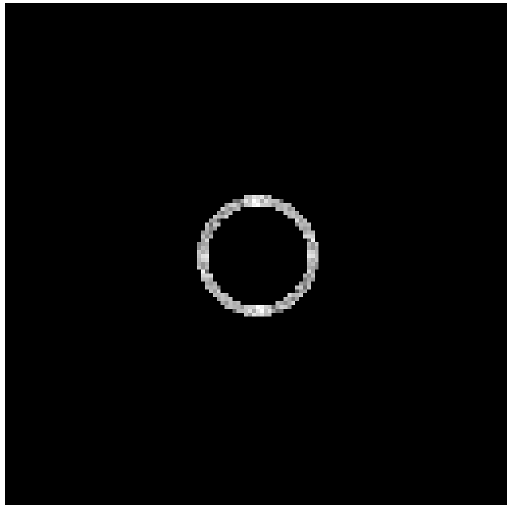
frequency magnitude

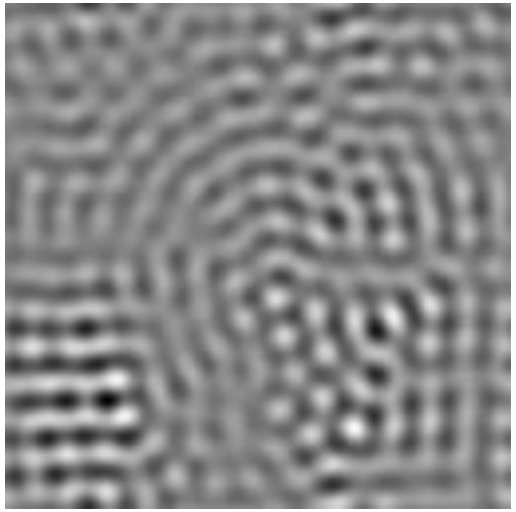


More filtering examples

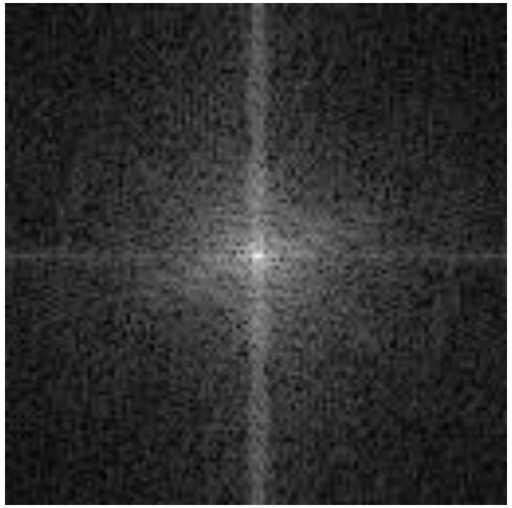
original image

band-pass filter





frequency magnitude



Revisiting sampling

The Nyquist-Shannon sampling theorem

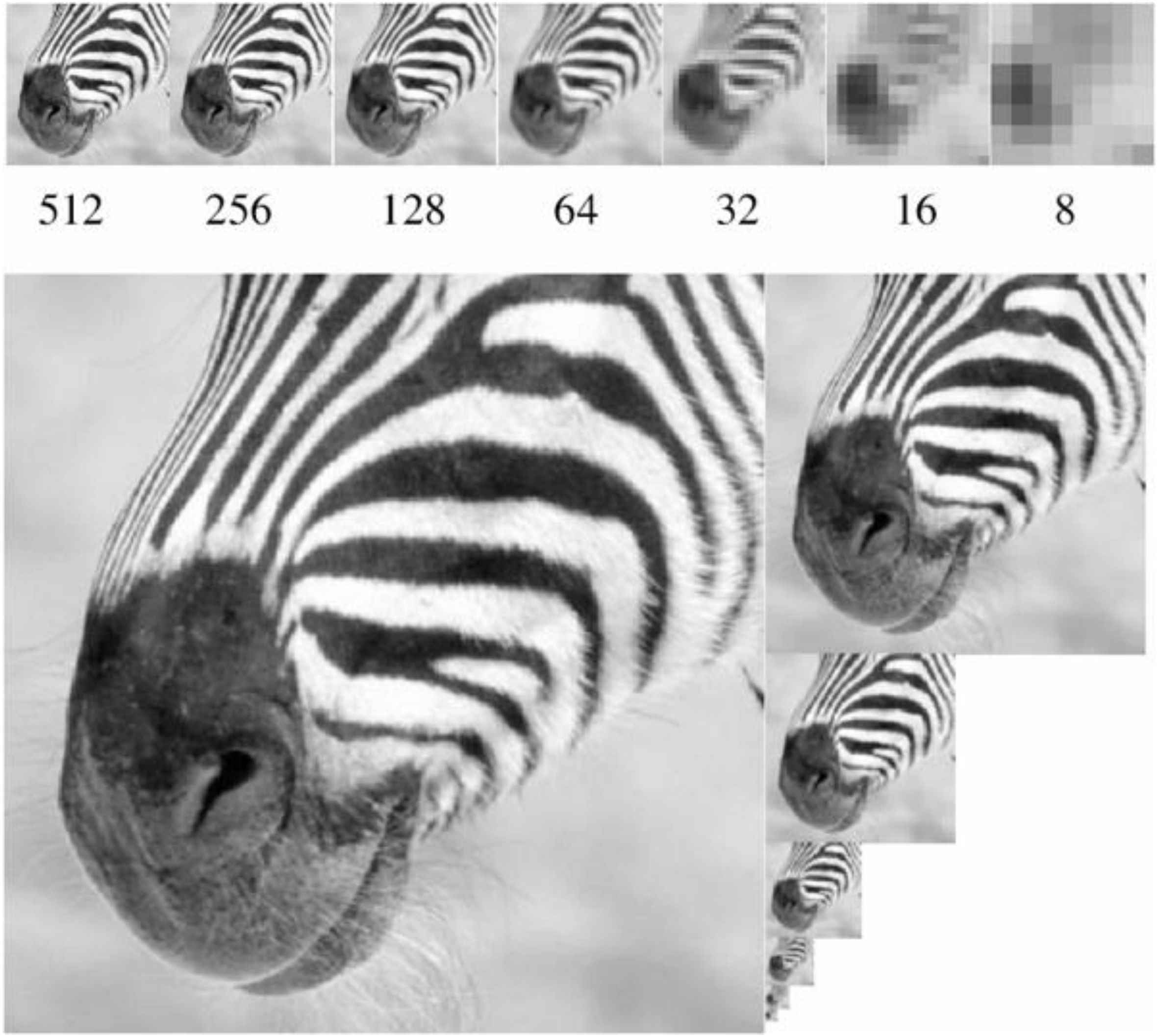
A continuous signal can be perfectly reconstructed from its discrete version using linear interpolation, if sampling occurred with frequency:

$$f_s \geq 2 f_{\max} \longleftarrow$$

This is called the Nyquist frequency

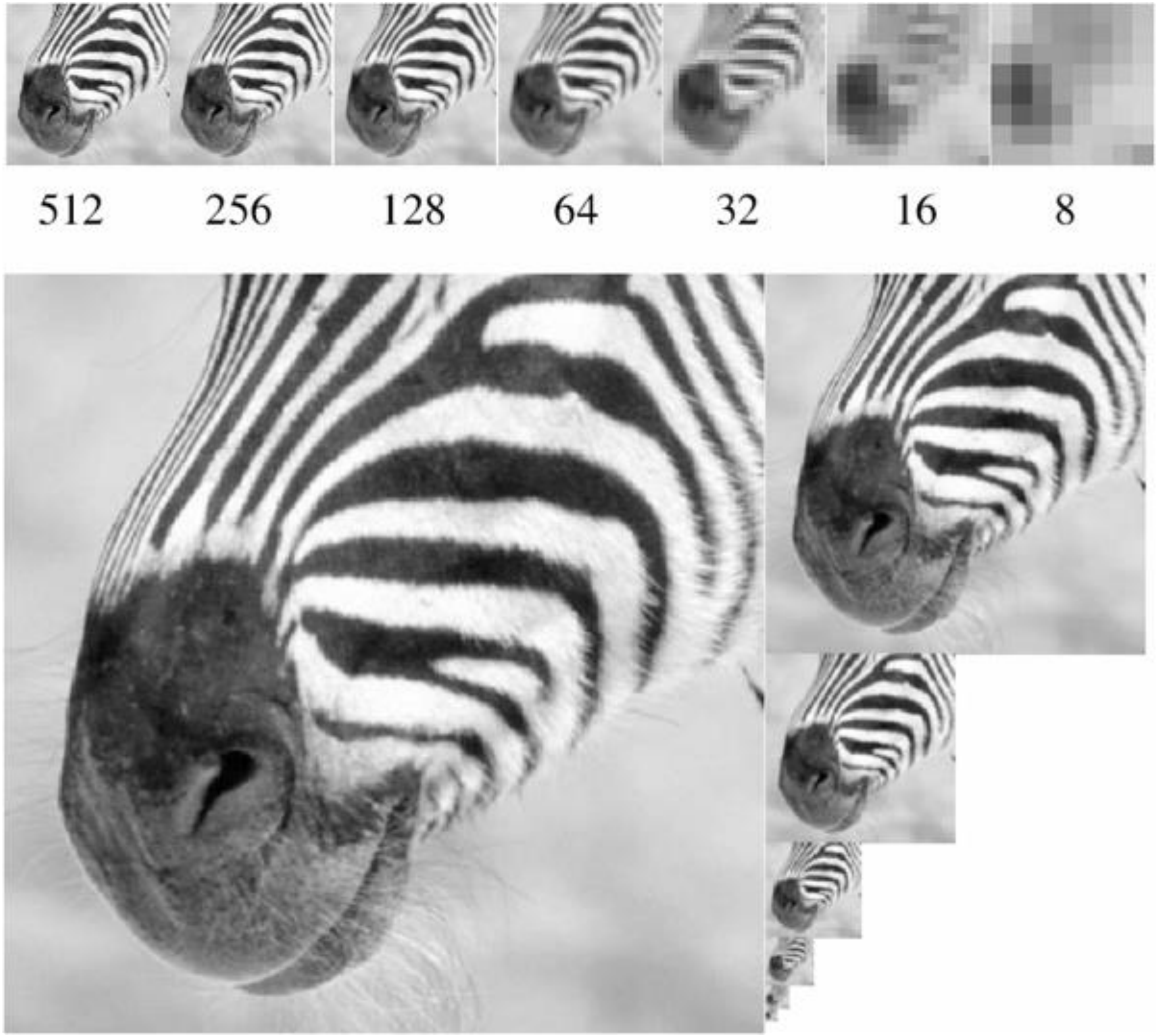
Equivalent reformulation: When downsampling, aliasing does not occur if samples are taken at the Nyquist frequency or higher.

Gaussian pyramid



How does the Nyquist-Shannon theorem relate to the Gaussian pyramid?

Gaussian pyramid



How does the Nyquist-Shannon theorem relate to the Gaussian pyramid?

- Gaussian blurring is low-pass filtering.
- By blurring we try to sufficiently decrease the Nyquist frequency to avoid aliasing.

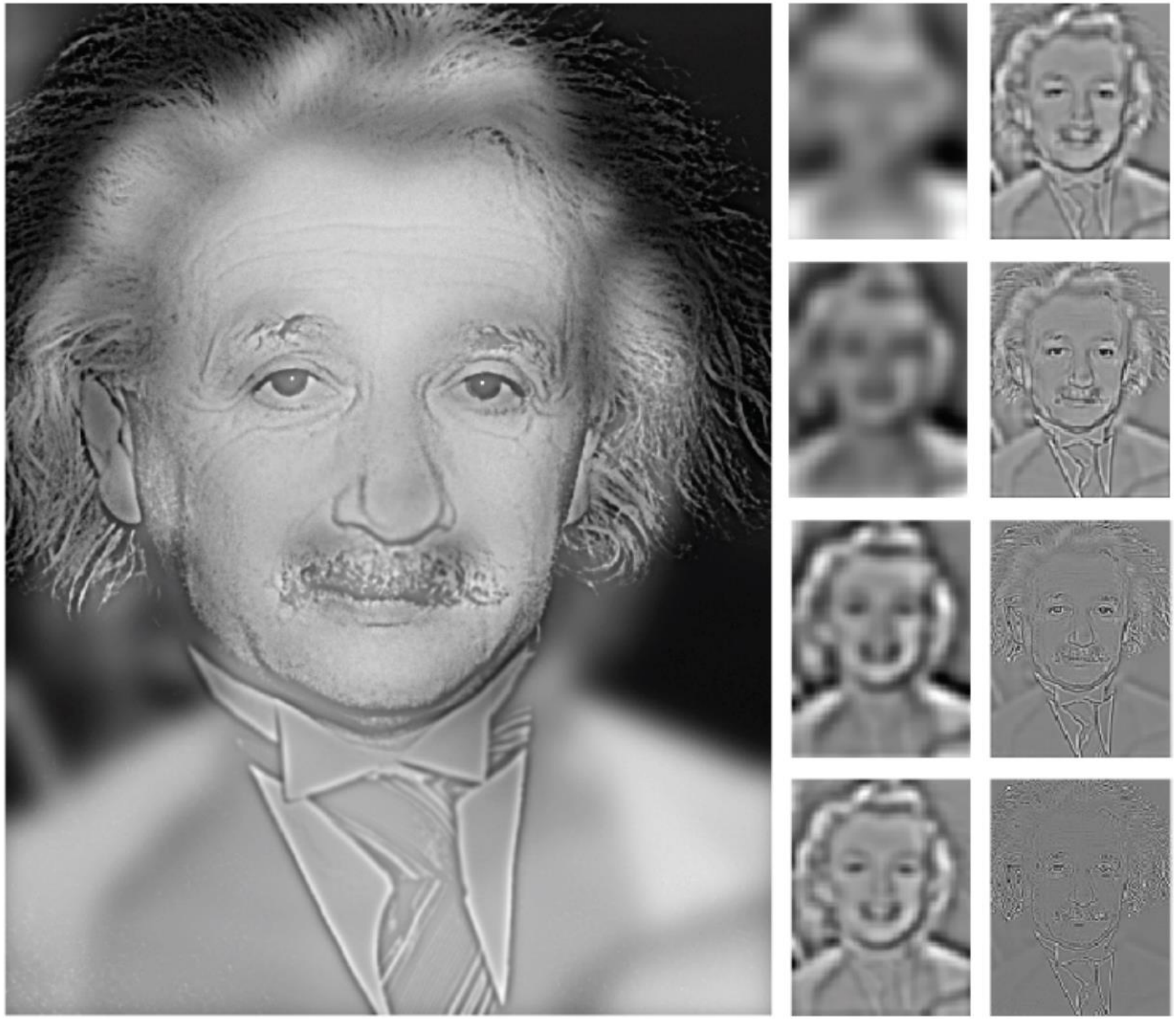
How large should the Gauss blur we use be?

Frequency-domain filtering in human vision

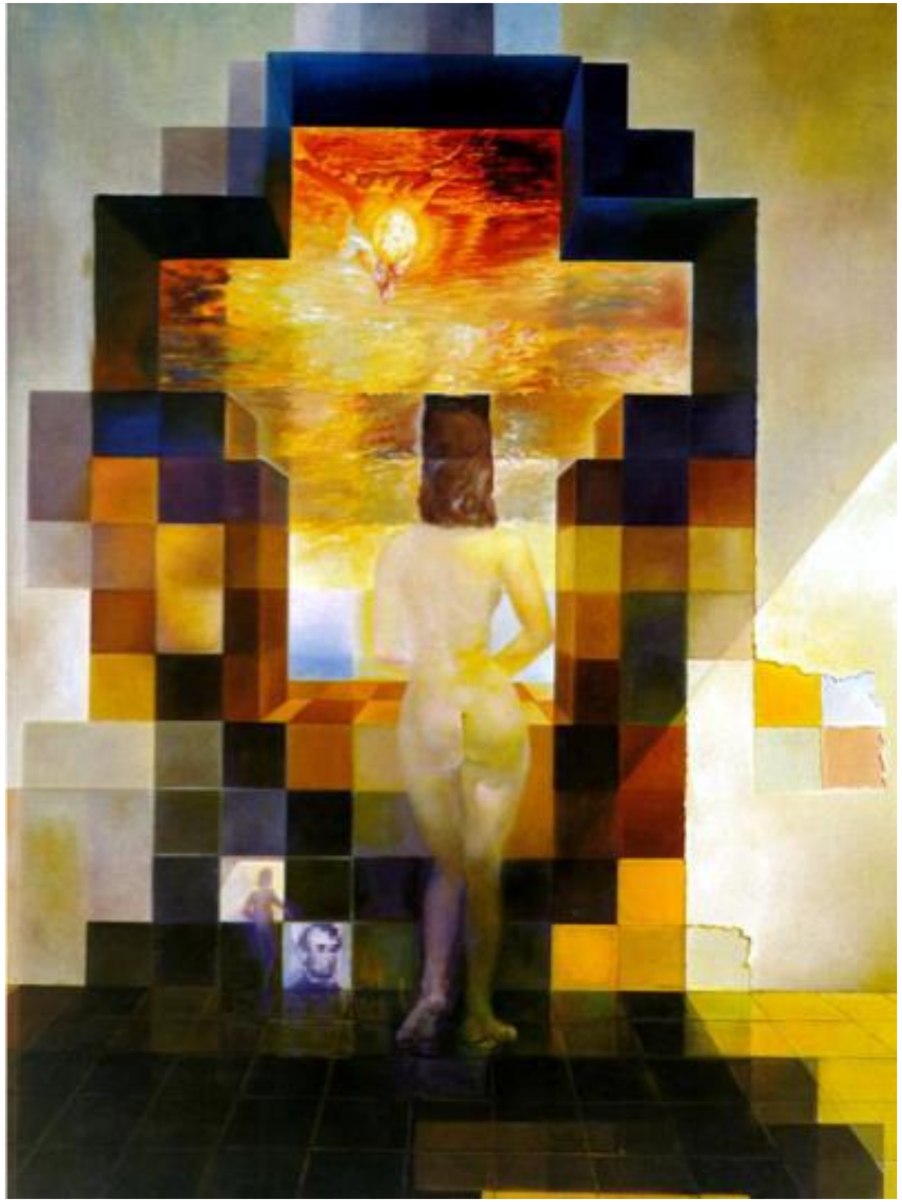
"Hybrid image"

Aude Oliva and Philippe Schyns

Frequency-domain filtering in human vision



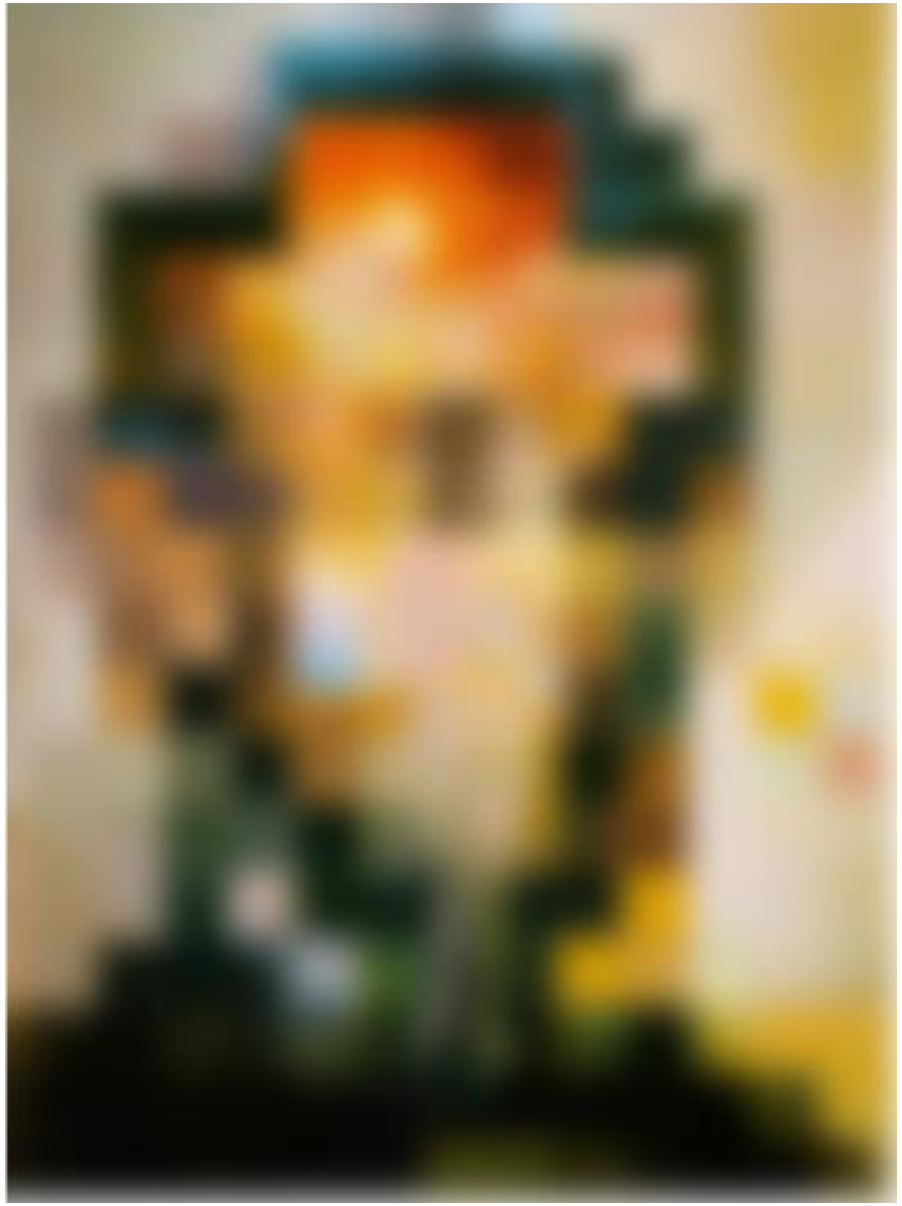
Frequency-domain filtering in human vision



Gala Contemplating the Mediterranean Sea Which at Twenty Meters Becomes the Portrait of Abraham Lincoln (Homage to Rothko)

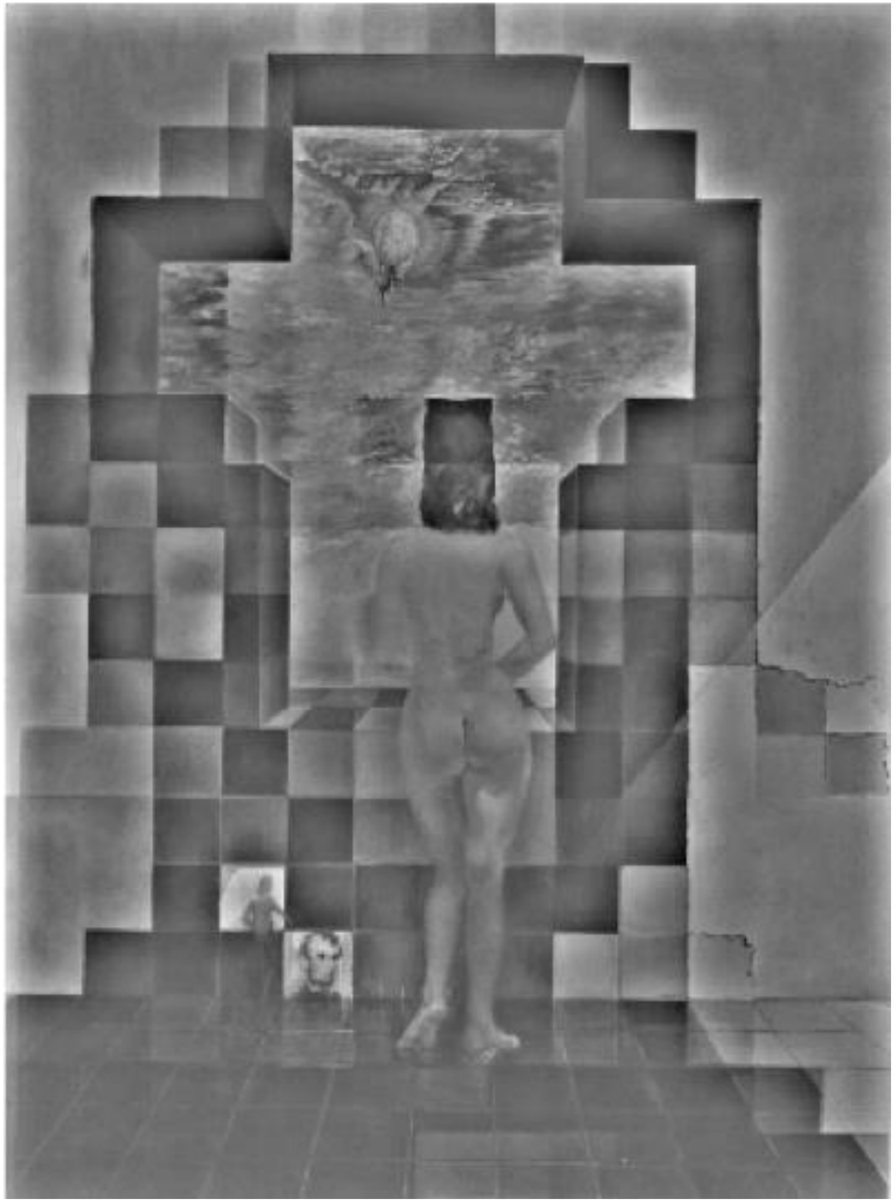
Salvador Dali, 1976

Frequency-domain filtering in human vision



Low-pass filtered version

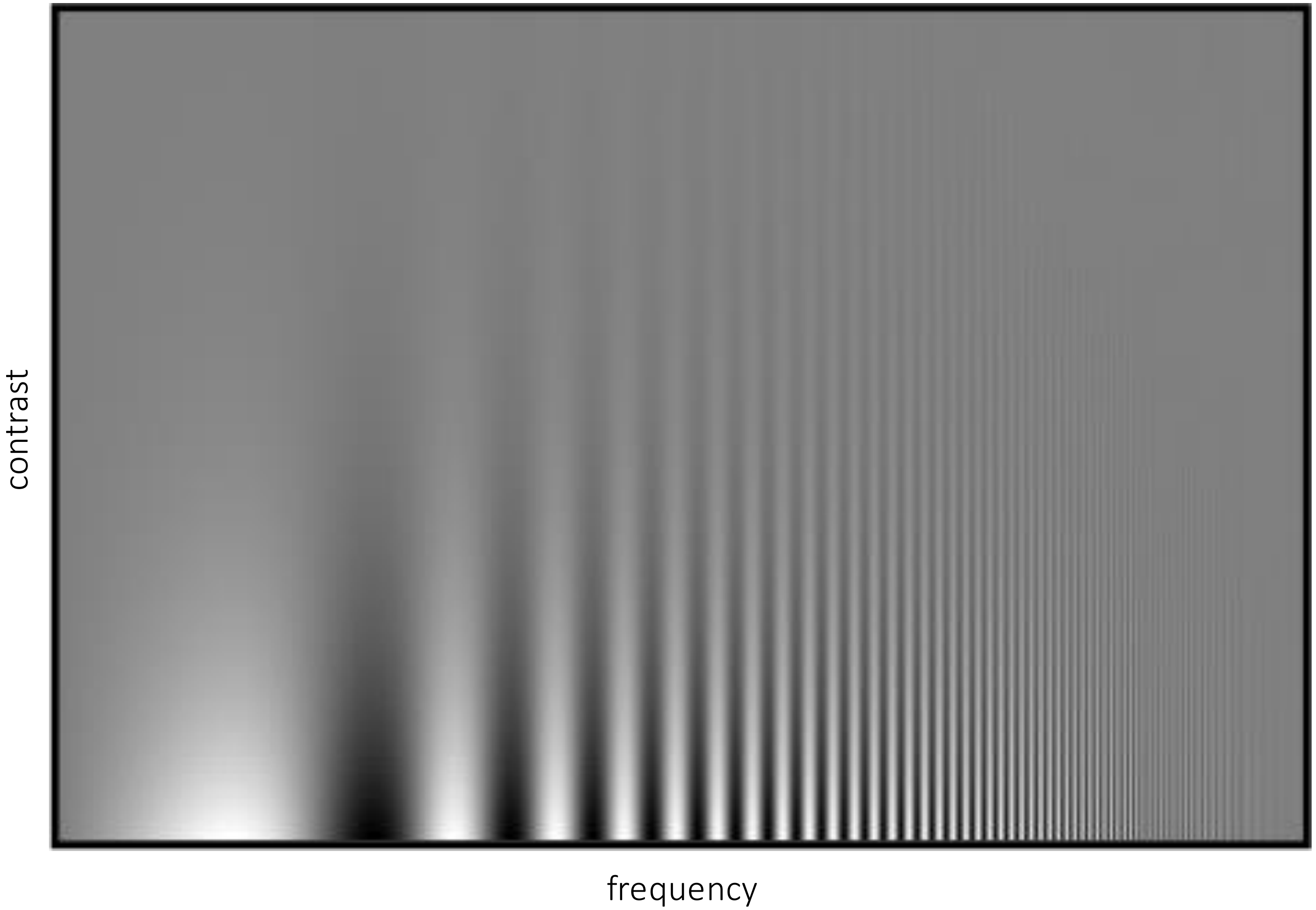
Frequency-domain filtering in human vision



High-pass filtered version

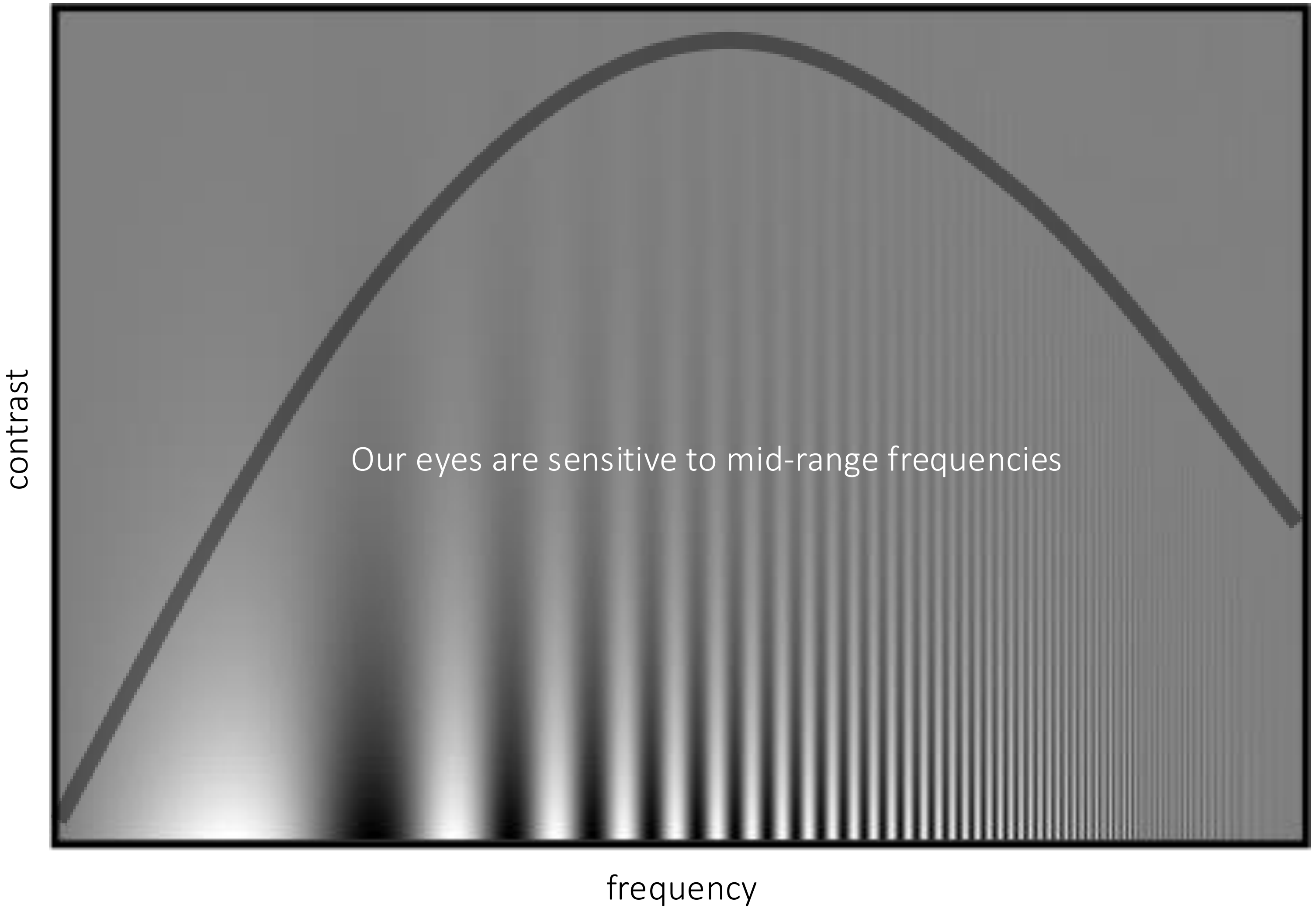
Variable frequency sensitivity

Experiment: Where do you see the stripes?



Variable frequency sensitivity

Campbell-Robson contrast sensitivity curve



- Early processing in humans filters for various orientations and scales of frequency
- Perceptual cues in the mid frequencies dominate perception

Other Properties of FT

- https://dspillustrations.com/pages/posts/misc/properties-of-the-fourier-transform.html

Properties of FT: Convolution theorem

The Fourier transform of the convolution of two functions is the product of their Fourier transforms:

$$\mathcal{F}\{g * h\} = \mathcal{F}\{g\}\mathcal{F}\{h\}$$

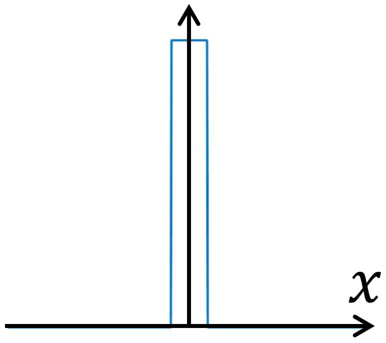
The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms:

$$\mathcal{F}^{-1}\{gh\} = \mathcal{F}^{-1}\{g\} * \mathcal{F}^{-1}\{h\}$$

Convolution in spatial domain is equivalent to multiplication in frequency domain!

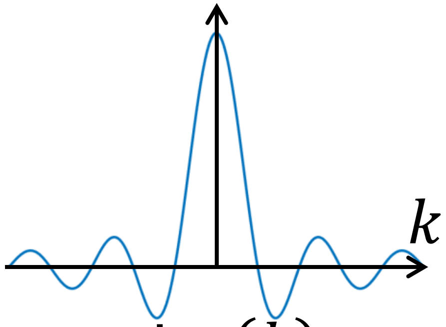
Properties of FT

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi k x} dx \qquad f(x) = \int_{-\infty}^{\infty} F(k) e^{j2\pi k x}\ dk$$



rect(x)

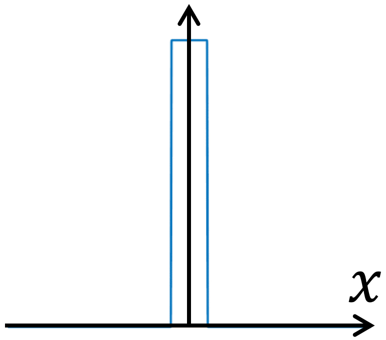
$$f(x) \xleftrightarrow{\text{Fourier}} F(k)$$



sinc(k)

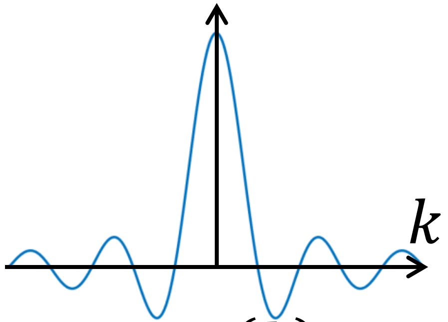
Properties of FT: scaling

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi k x} dx \qquad f(x) = \int_{-\infty}^{\infty} F(k) e^{j2\pi k x} \; dk$$



rect(x)

$$f(x) \xleftrightarrow{\text{Fourier}} F(k)$$

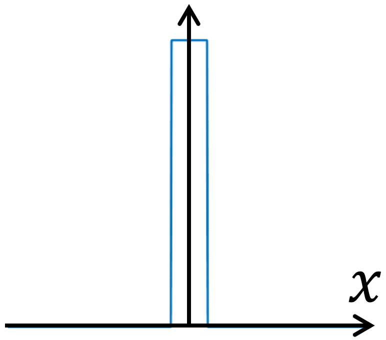


sinc(k)

$$f(sx) \xleftrightarrow{\text{Fourier}} ??$$

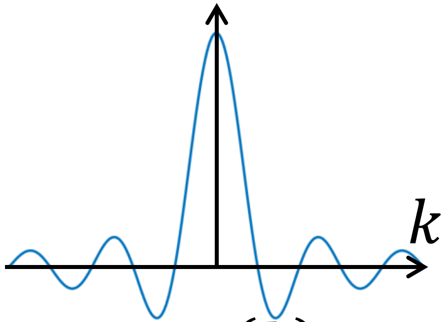
Properties of FT: scaling

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi kx} dx \qquad f(x) = \int_{-\infty}^{\infty} F(k) e^{j2\pi kx}\ dk$$



$\mathrm{rect}(x)$

$$f(x) \xleftrightarrow{\text{Fourier}} F(k)$$



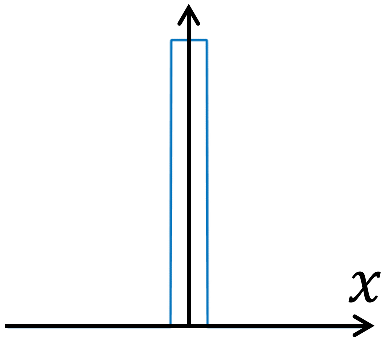
$\mathrm{sinc}(k)$

??

$$f(sx) \xleftrightarrow{\text{Fourier}} F(k/s)$$

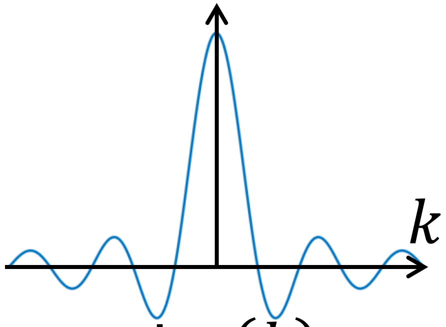
Properties of FT: scaling

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi k x} dx \qquad f(x) = \int_{-\infty}^{\infty} F(k) e^{j2\pi k x}\ dk$$

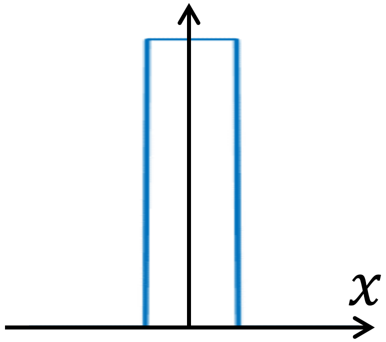


$\mathrm{rect}(x)$

$$f(x) \xleftrightarrow{\text{Fourier}} F(k)$$



$\mathrm{sinc}(k)$



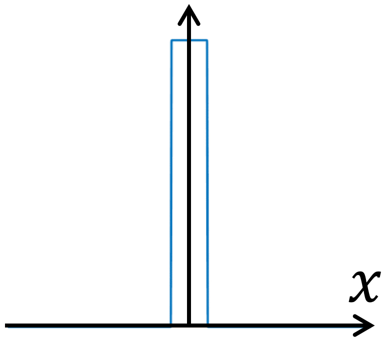
$\mathrm{rect}(sx)$

$$f(sx) \xleftrightarrow{\text{Fourier}} F(k/s)$$

??

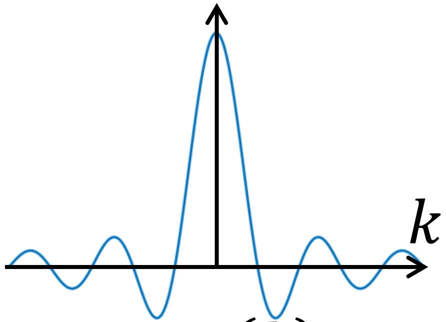
Properties of FT: scaling

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi k x} dx \qquad f(x) = \int_{-\infty}^{\infty} F(k) e^{j2\pi k x} \, dk$$

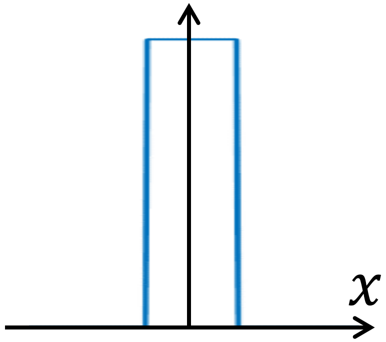


rect(x)

$$f(x) \xleftrightarrow{\text{Fourier}} F(k)$$

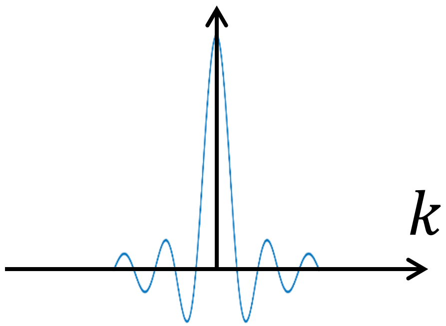


sinc(k)



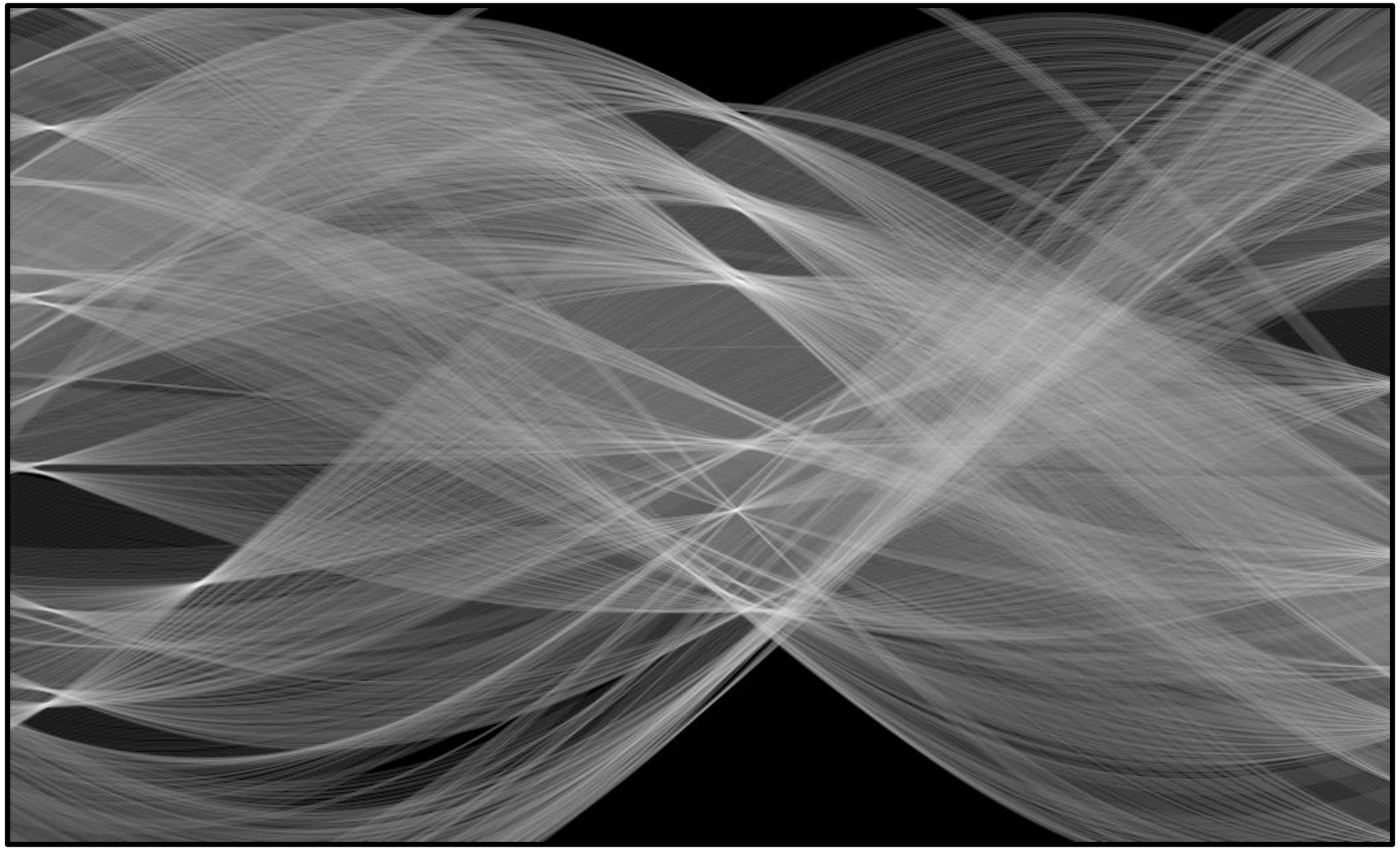
rect(sx)

$$f(sx) \xleftrightarrow{\text{Fourier}} F(k/s)$$



sinc(k/s)

Hough transform



Overview of today's lecture

- Finding boundaries.
- Line fitting.
- Line parameterizations.
- Hough transform.
- Hough circles.
- Some applications.

Slide credits

Most of these slides were adapted directly from:

- Matt O’Toole (CMU, 15-463, Fall 2022).

- Ioannis Gkioulekas (CMU, 15-463, Fall 2020).

- Kris Kitani (CMU, 15-463, Fall 2016).

Some slides were inspired or taken from:

- Fredo Durand (MIT).
- Bernd Girod (Stanford University).
- James Hays (Georgia Tech).
- Steve Marschner (Cornell University).
- Steve Seitz (University of Washington).

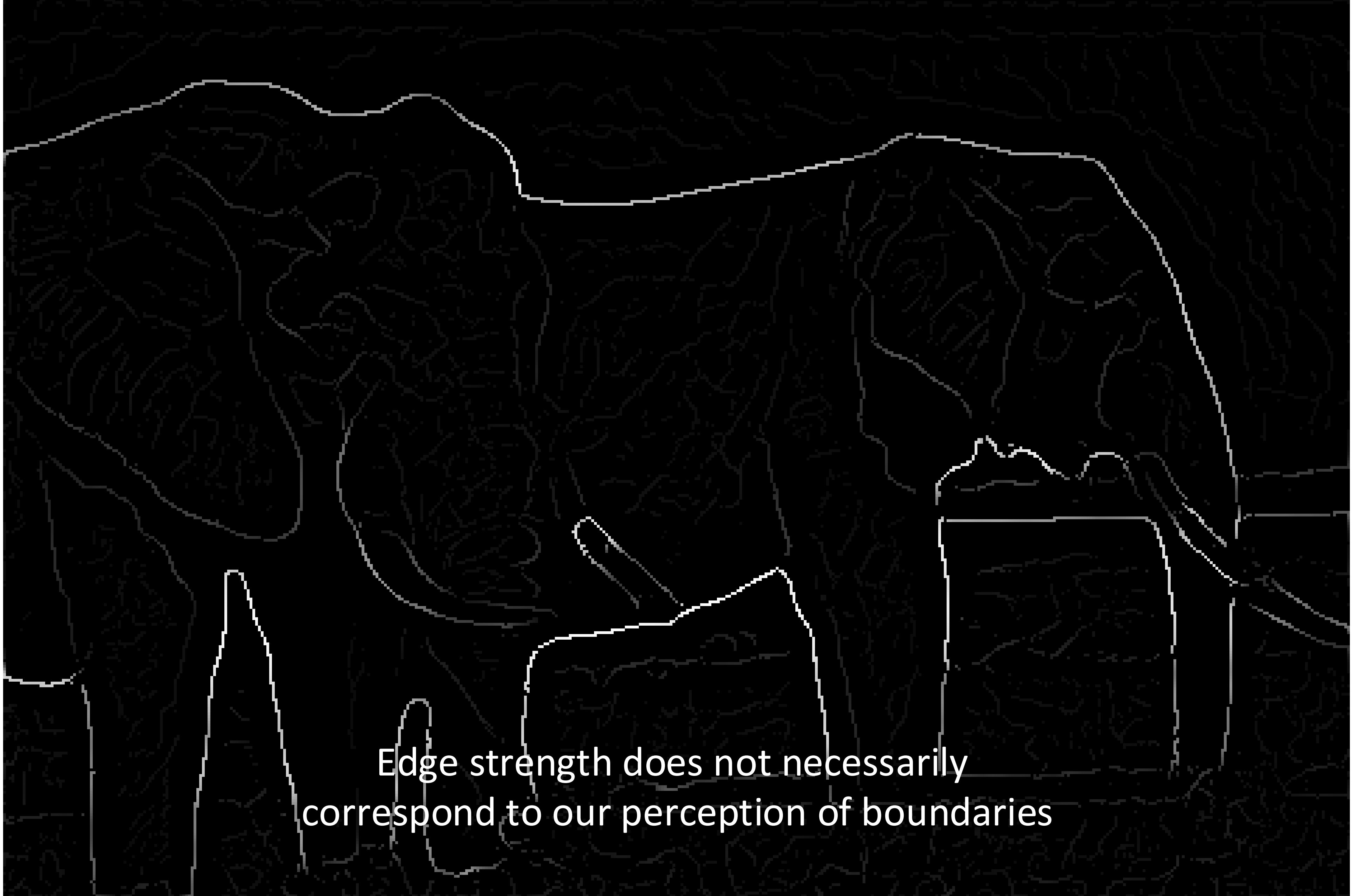
Finding boundaries

Where are the object boundaries?

Human annotated boundaries

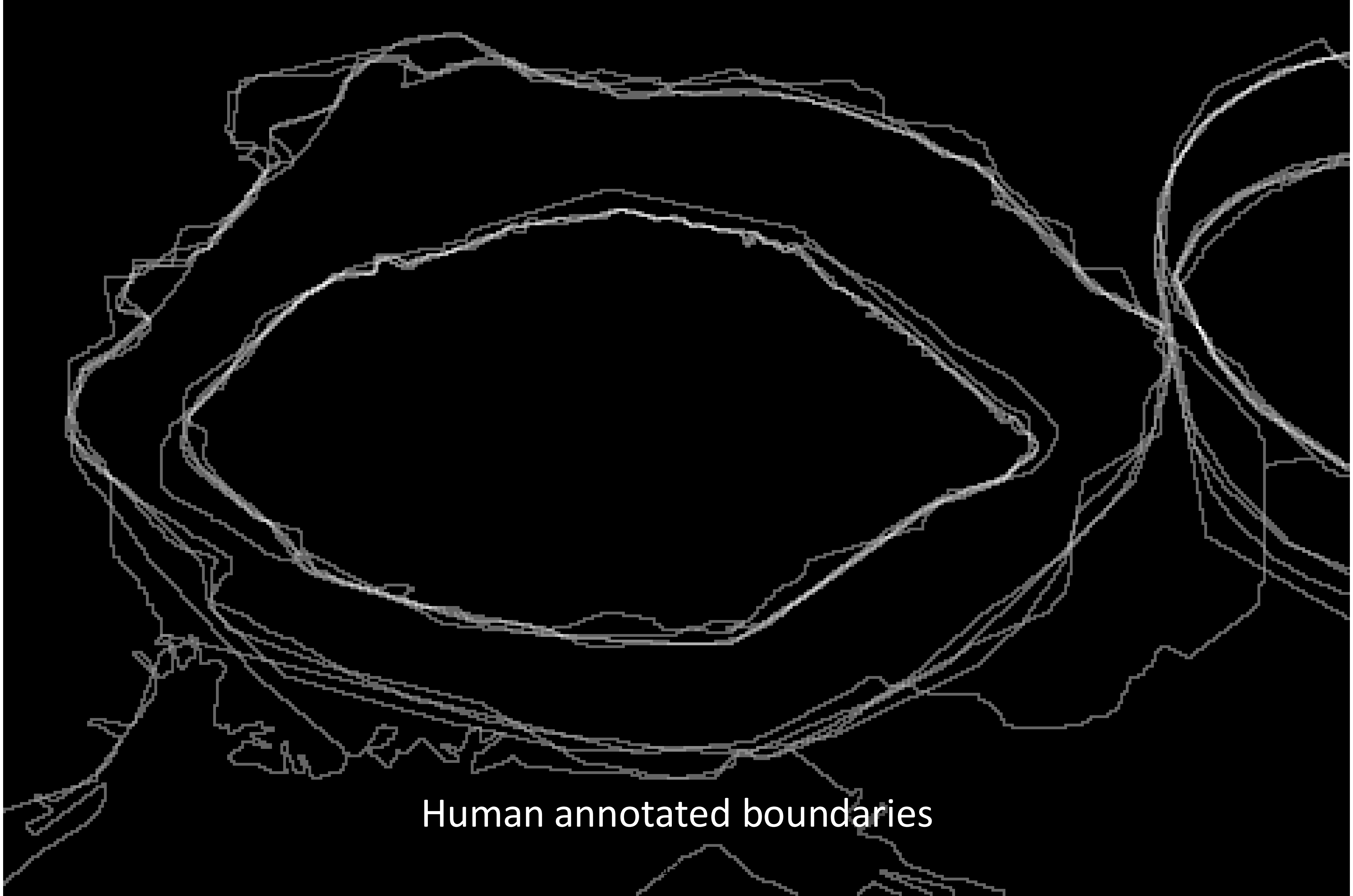

Edge detection


Multi-scale edge detection



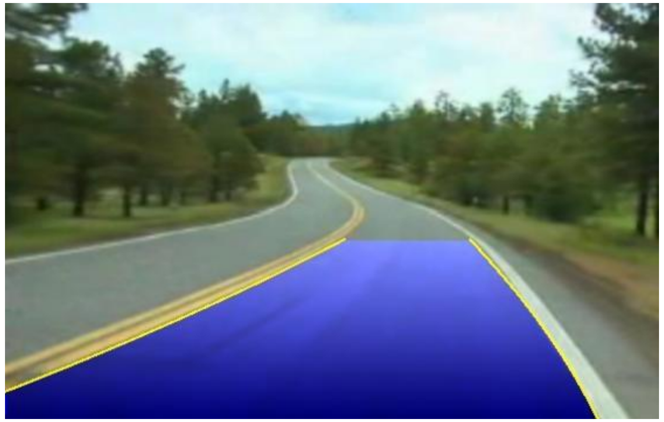
Edge strength does not necessarily correspond to our perception of boundaries

Where are the object boundaries?

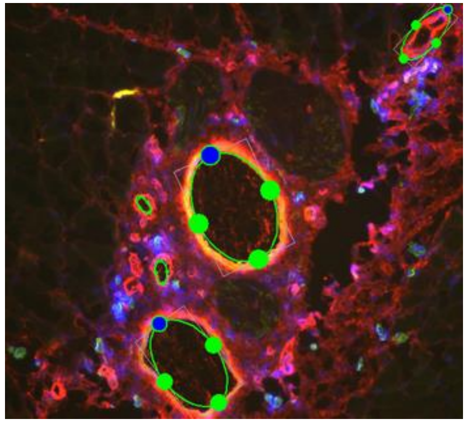

Human annotated boundaries


Edge detection

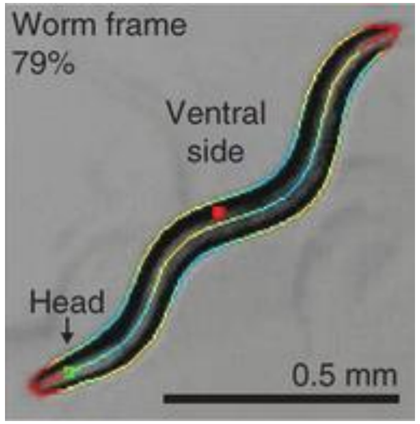
Applications



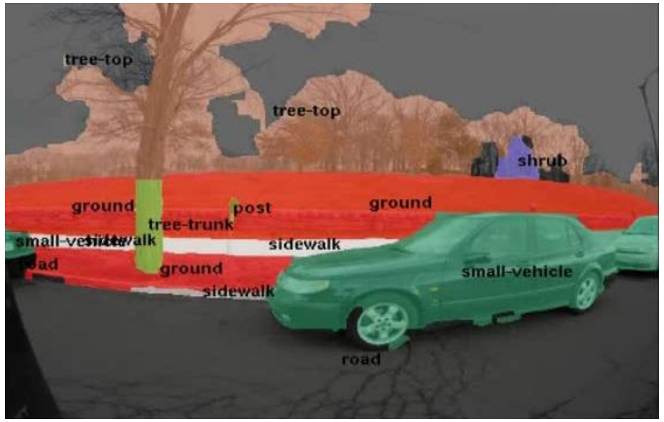
Autonomous Vehicles
(lane line detection)



tissue engineering
(blood vessel counting)



behavioral genetics
(earthworm contours)



Autonomous Vehicles
(semantic scene segmentation)

Computational Photography
(image inpainting)

Line fitting

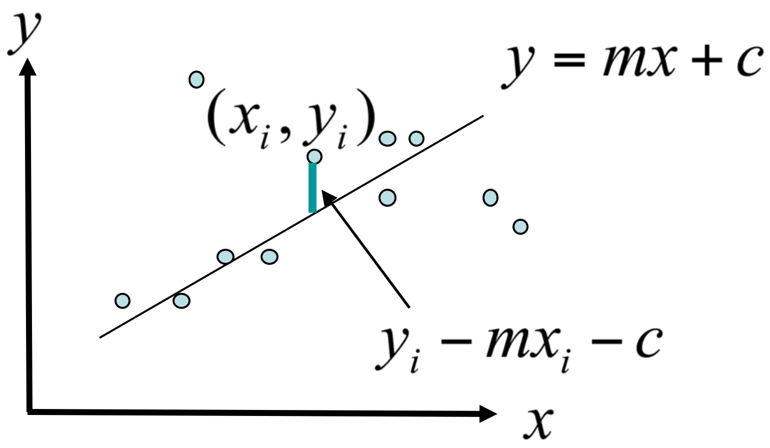
Line fitting

Given: Many (x_i, y_i) pairs

Find: Parameters (m, c)

Minimize: Average square distance:

$$E = \sum_i \frac{(y_i - mx_i - c)^2}{N}$$



How can we solve this minimization?

Line fitting

Given: Many (x_i, y_i) pairs

Find: Parameters (m, c)

Minimize: Average square distance:

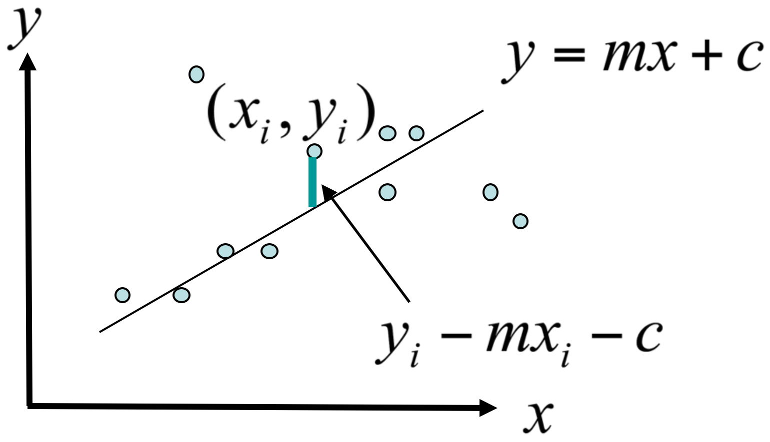
$$E = \sum_i \frac{(y_i - mx_i - c)^2}{N}$$

Using:

$$\frac{\partial E}{\partial m} = 0 \quad \& \quad \frac{\partial E}{\partial c} = 0$$

Note:

$$\bar{y} = \frac{\sum_i y_i}{N} \qquad \bar{x} = \frac{\sum_i x_i}{N}$$



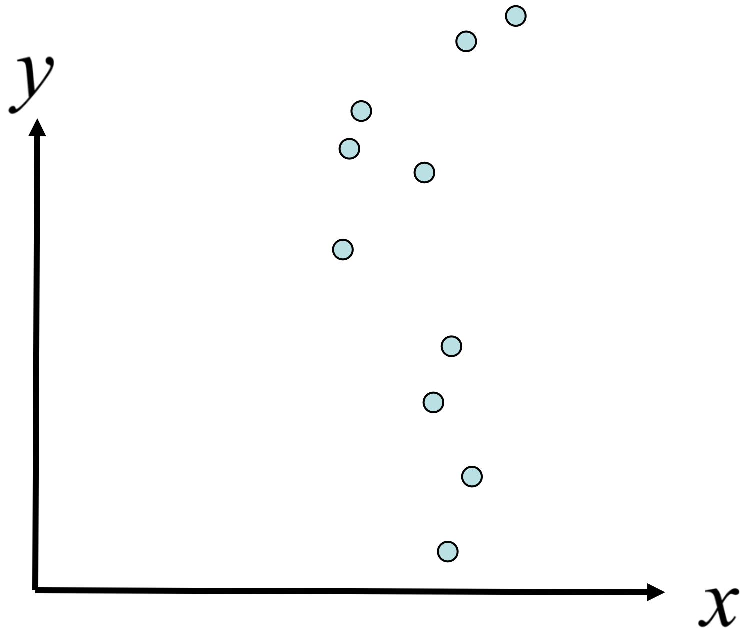
$$c = \bar{y} - m\,\bar{x}$$

$$m = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2}$$

What are some problems with the approach?

Problems with parameterizations

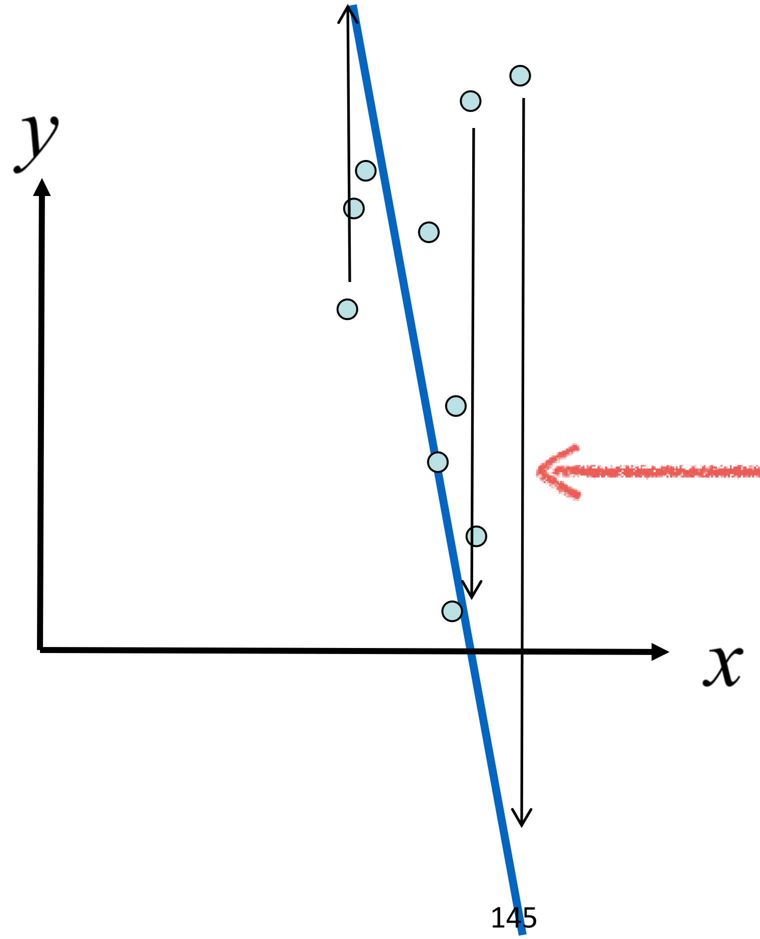
Where is the line that minimizes E? $E = \sum_{i=1}^{n} (y_i - mx_i - b)^2$



Problems with parameterizations

Where is the line that minimizes E?

$$\boxed{E = \sum_{i=1}^{n} (y_i - mx_i - b)^2}$$

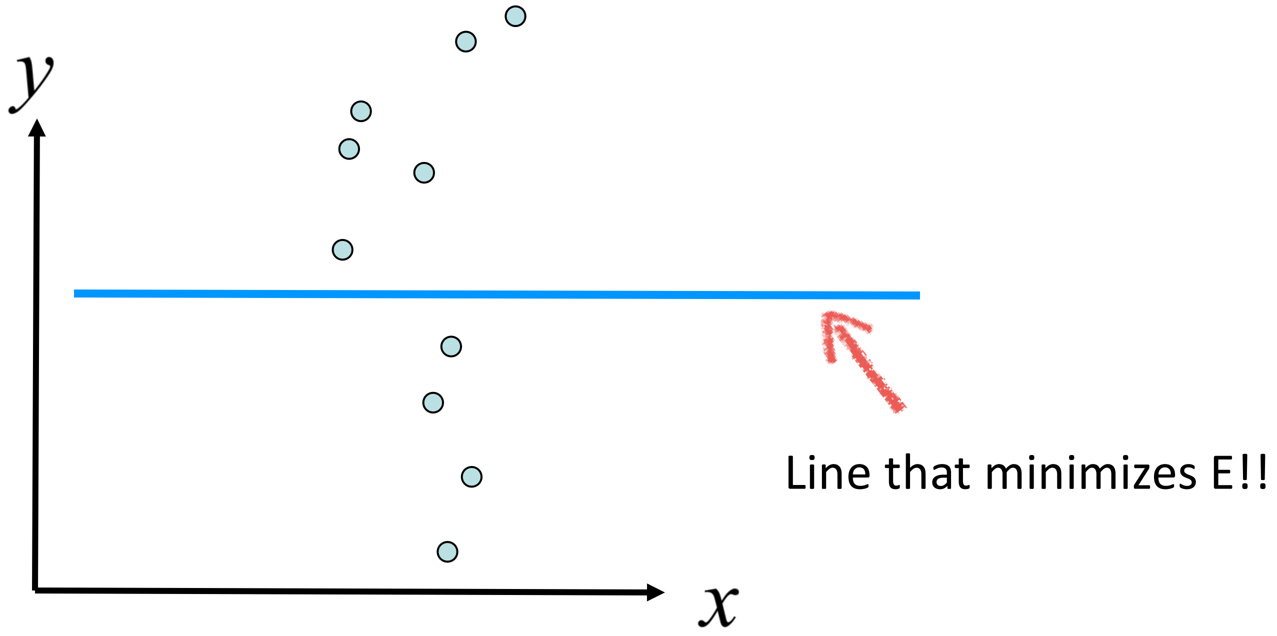


Huge E!

Problems with parameterizations

Where is the line that minimizes E?

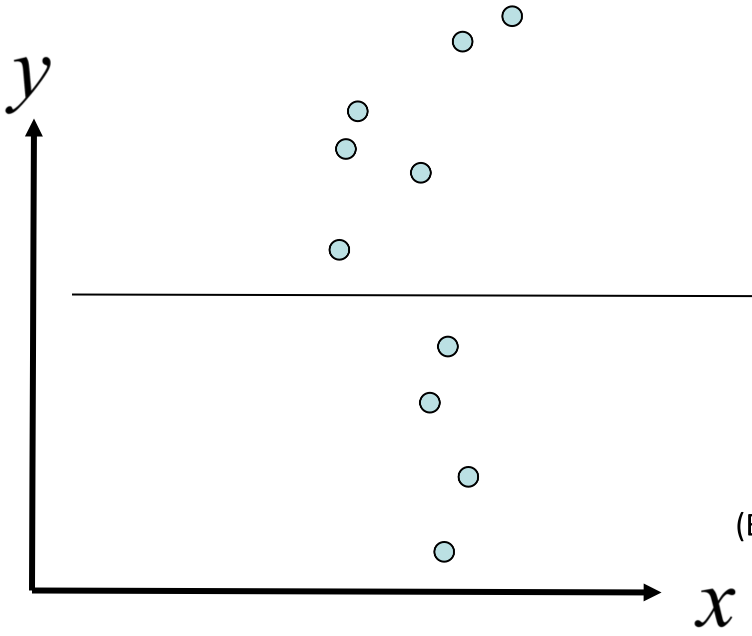
$$E = \sum_{i=1}^{n} (y_i - m x_i - b)^2$$



Problems with parameterizations

Where is the line that minimizes E?

$$\boxed{E = \sum_{i=1}^{n} (y_i - m x_i - b)^2}$$

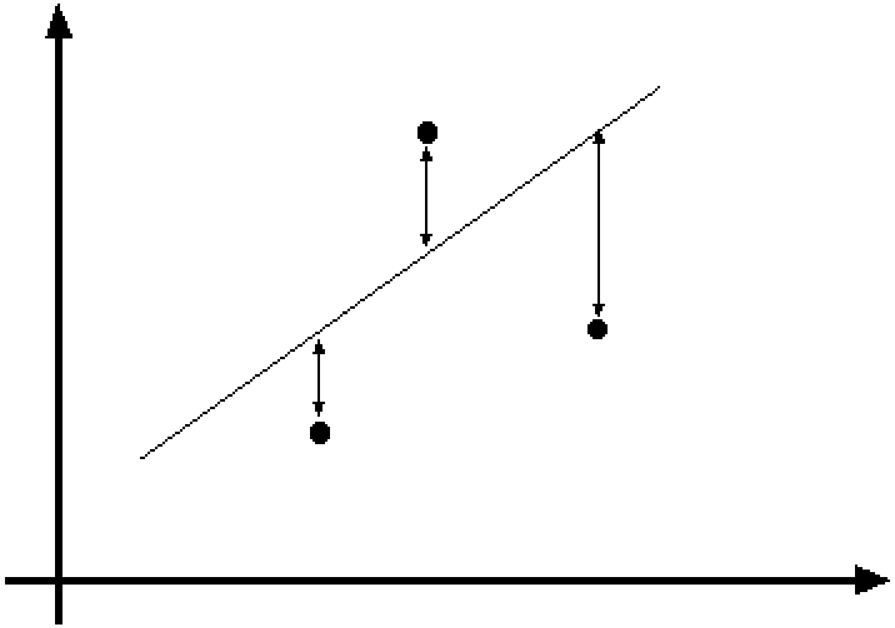


Line that minimizes E!!

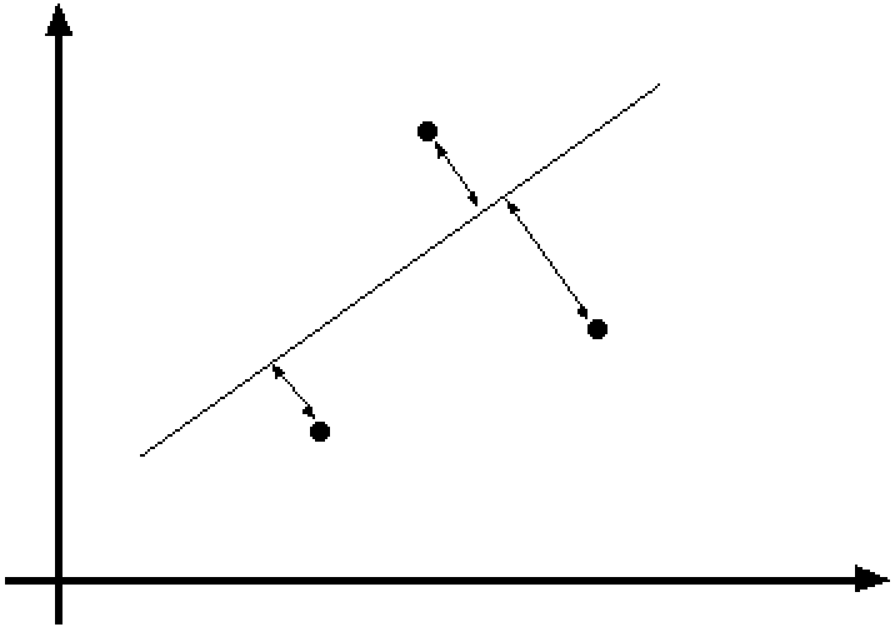
(Error E must be formulated carefully!)

How can we deal with this?

Line fitting is easily setup as an optimization problem
… but choice of model is important

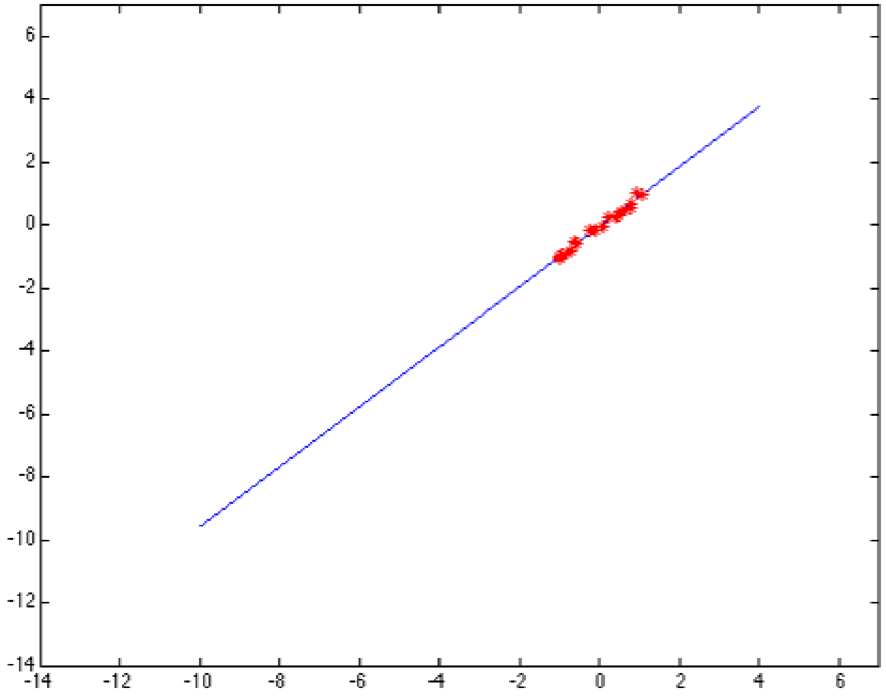


$$E = \sum_{i=1}^{n} (y_i - mx_i - b)^2$$

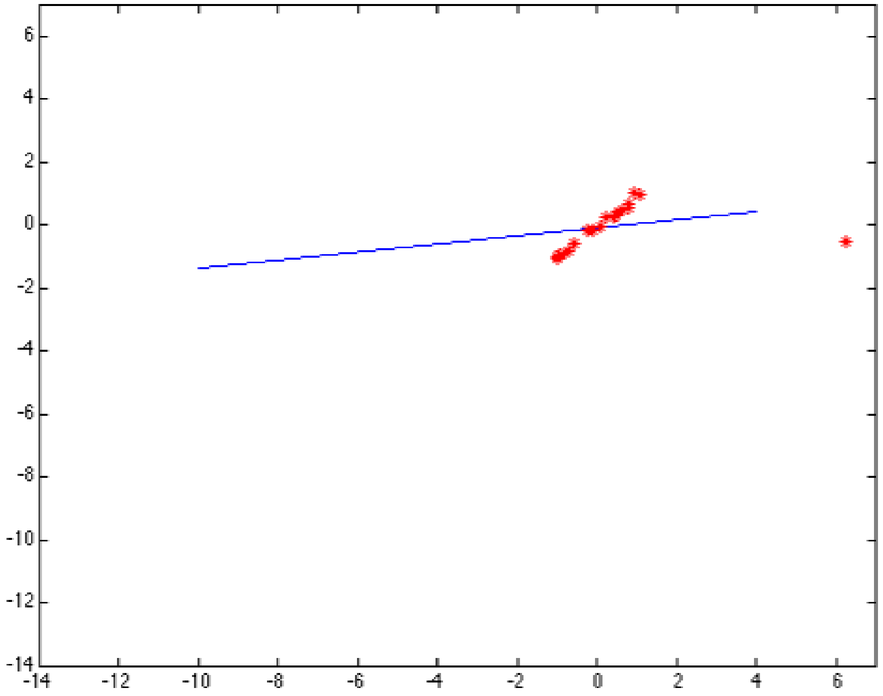


What optimization are we solving here?

Problems with noise



Least-squares error fit



Squared error heavily penalizes outliers

Line parameterizations

Slope intercept form

$$y = mx + b$$

What are m and b?

Slope intercept form

$$\underbrace{y = mx + b}$$

$$y = mx + b$$

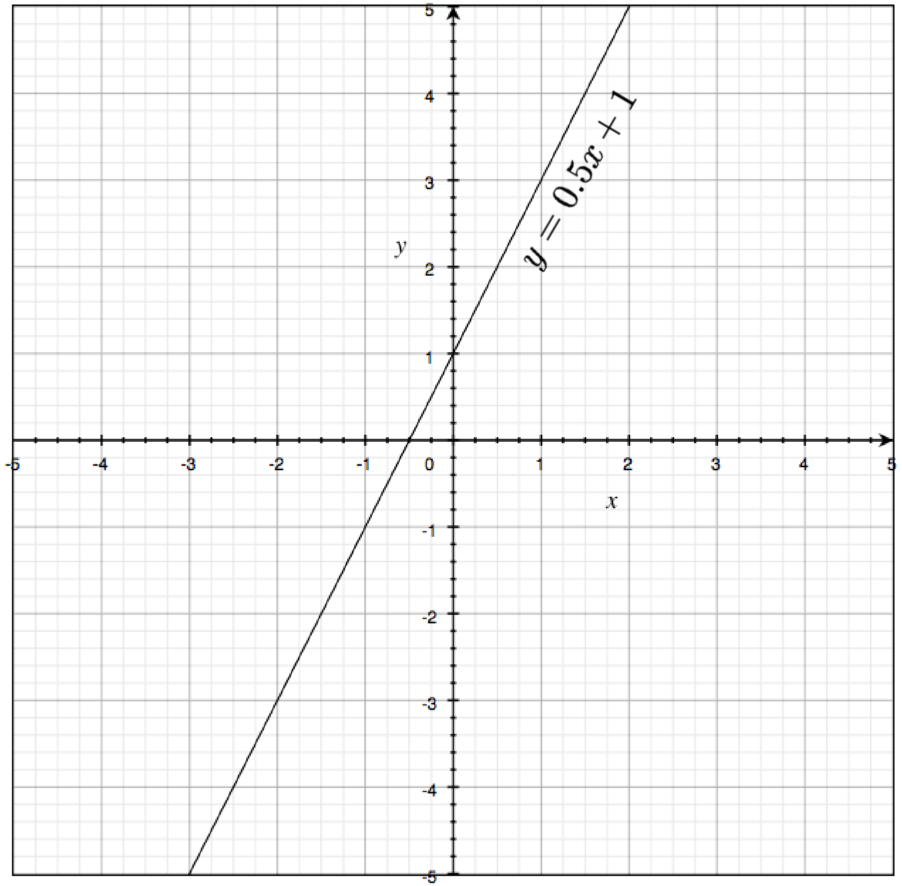
slope y-intercept

What are m and b?

Slope intercept form

$$y = mx + b$$

slope y-intercept



Double intercept form

$$\frac{x}{a} + \frac{y}{b} = 1$$

What are a and b?

Double intercept form

$$\frac{x}{a} + \frac{y}{b} = 1$$

x-intercept y-intercept

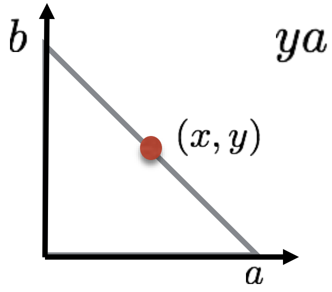
What are a and b?

Double intercept form

$$\frac{x}{a} + \frac{y}{b} = 1$$

x-intercept y-intercept

Derivation:

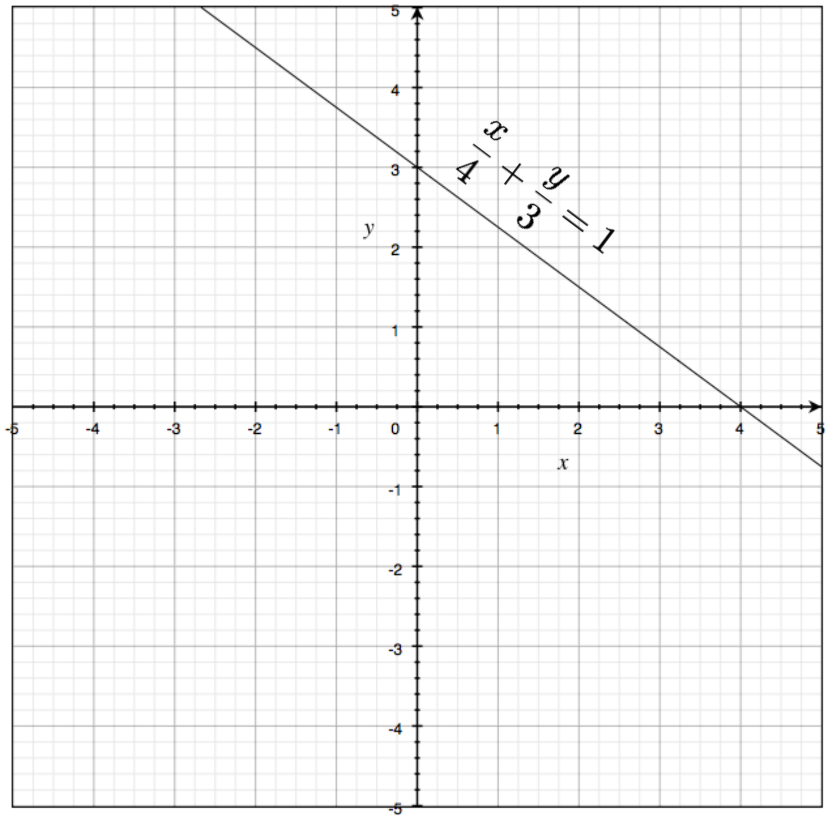


(Similar slope) $$\frac{y-b}{x-0} = \frac{0-y}{a-x}$$

$$ya + yx - ba + bx = -yx$$

$$ya + bx = ba$$

$$\frac{y}{b} + \frac{x}{a} = 1$$



Normal Form

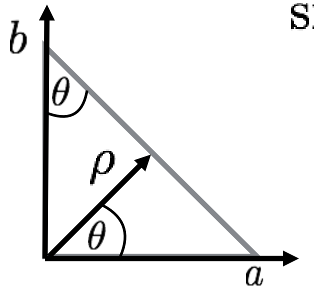
$$x \cos \theta + y \sin \theta = \rho$$

What are rho and theta?

Normal Form

$$x \cos\theta + y \sin\theta = \rho$$

Derivation:



$$\cos\theta = \frac{\rho}{a} \rightarrow a = \frac{\rho}{\cos\theta}$$

$$\sin\theta = \frac{\rho}{b} \rightarrow b = \frac{\rho}{\sin\theta}$$

plug into: $$\frac{x}{a} + \frac{y}{b} = 1$$

$$x \cos\theta + y \sin\theta = \rho$$