

# Recitation 3

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## 1 Contour Integration

### 1.1 The Residue Theorem

It's commonly said in math that the shortest path between 2 points in the real domain passes through the complex plane. One such path is contour integration, a powerful technique using the tools of complex analysis that can be applied to evaluate difficult real integrals. It has incredible importance in QFT for reasons I'll discuss, so I'd like to start by reviewing it.

#### Simple Poles

First, consider a function  $f$ , holomorphic in some region  $D \subset C$  with  $C = \partial D$ . **Draw figure.** Then Cauchy's integral theorem tells us that

$$\oint_C dz f(z) = 0$$

This is a corollary of Green's theorem in the plane, combined with the Cauchy-Riemann equations.

We take something more interesting, take  $g(z) = \frac{f(z)}{z - z_0}$ , where  $f(z)$  is holomorphic. There are 2 cases.

1.  $z_0 \notin D$ . Then  $g$  is holomorphic on  $D$ , so by Cauchy's theorem above the integral vanishes
2.  $z_0 \in D$ . **Draw figure.** By Cauchy the integral only picks up a contribution at the pole. Here the numerator is regular, and we can compute the integral of the denominator using the curve  $z(t) = z_0 + \epsilon e^{it}$ . We then have

$$\oint_C \frac{f(z)}{z - z_0} = 2\pi i f(z_0) =: 2\pi i \text{Res}_g(z_0)$$

The story is very similar if there are many poles in  $D$ : we can split up the domain of integration and evaluate each pole using the technique above. **Draw figure.** For instance,

$$h(z) := \frac{f(z)}{(z - z_0) \cdots (z - z_n)} = \frac{1}{z - z_0} \frac{f(z)}{(z - z_1) \cdots (z - z_n)} = \frac{1}{z - z_0} \text{Res}_h(z_0) = \frac{1}{z - z_k} \text{Res}_h(z_k)$$

$$\oint_C dz h(z) = 2\pi i \sum_k \text{Res}_h(z_k)$$

The way to intuitively understand the residue of a function at some point  $z_i$  is what you get when you block out the pole at  $z_i$ , and evaluate the rest of the function there.

#### Higher Order Poles

Technically this is only true for simple poles; for higher order poles (e.g.  $1/(z - z_0)^2$ ), the story is a bit more complicated. For an  $n$ th-order pole, one has the formula

$$\text{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}}((z - z_0)^n f(z))$$

For simple poles  $n = 1$  this gives what we expect,

$$\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0)f(z)$$

What does this mean? One can expand a function  $f$  in terms of a Laurent series about a pole of order  $n$ . This is just a Taylor series with negative exponents, to account for the singular structure.

$$f(z) = \sum_{k \geq -n} \frac{a_k}{(z - z_0)^k}$$

Then, the residue of  $f$  is the coefficient of the  $(z - z_0)^{-1}$  term—again, it's what happens when you block out the pole at  $z_i$ . One can now make sense of this formula. Say  $f$  has a pole of order  $n$ —how do we isolate the  $-1$  term of the Laurent series when there are terms more singular? There is a trick: multiply the series by  $(z - z_0)^n$ , so we have no more singularities. Then, differentiate the regular series  $n-1$  times so that terms corresponding to higher singular behavior vanish. The constant term is now just the coefficient of the  $-1$  coefficient (with some combinatorial factor). Finally, one evaluates at  $z = z_0$  to kill off every other regular term, isolating  $a_{-1}$ .

Usually you will only have to work with simple poles, but this formula is good to know.

Now, we can state the residue theorem in its full generality. Let  $h(z)$  be some function, analytic except at a discrete collection of poles. Pick some closed curve  $C$  bounding a region  $D$ . One has

$$\oint_C dz h(z) = 2\pi i \sum_{z_j \in D} \text{Res}_h(z_j)$$

## 1.2 Applications

### Example 1

Let's do an example. Consider the following real integral. The trick is to write this as a complex integral.

$$I = \int_{\mathbb{R}} \frac{e^{ix}}{x^2 + 1} = \int_{\mathbb{R}} dx \frac{e^{ix}}{(x+i)(x-i)}$$

This has poles at  $x = \pm i$ , with residues  $\pm e^{\mp 1}/2i$ .

We want to use the residue theorem. However, right now we can't because this requires a closed path of integration. There is a trick to get it of this form, known as closing the contour.

- We consider the semicircle of radius  $R \gg 1$  in the upper half plane, going counterclockwise,  $C = L_R \cup C_R$ . **Draw figure.** By the residue theorem,

$$\oint_{L_R \cup C_R} dz \frac{e^{iz}}{z^2 + 1} = 2\pi i \text{Res}(i) = \frac{\pi}{e}$$

- We now take  $R \rightarrow \infty$ :

$$\lim_{R \rightarrow \infty} \oint_{L_R \cup C_R} dz \frac{e^{iz}}{z^2 + 1} = \oint_{\mathbb{R}} dz f(z) + \lim_{R \rightarrow \infty} \int_{C_R} dz f(z) = \frac{\pi}{e}$$

This is almost what we want, but the presence of this second term is annoying. Let's study it. for  $f(z) = e^{ix}/(x^2 + 1)$ , the integrand is exponentially suppressed along this contour, and vanishes:

$$\lim_{R \rightarrow \infty} \int_{C_R} dz f(z) \sim \lim_{R \rightarrow \infty} (\pi R) \frac{e^{-R}}{R^2} = 0$$

This is why we closed the contour upwards: if we closed in the other direction, the integrand would have been exponentially amplified, making our method useless.

We have recovered  $I = \pi/e$ .

We can summarize this 3-step procedure for integrals  $\int_{\mathbb{R}}$ . First, identify the poles and residues. Next, identify if closing upwards or downwards will give a vanishing contribution. Lastly, close in this direction and use the residue theorem.

### Example 2

Now let's do an integral in QFT which is extremely important:

$$\Pi(x - y) = i \int d^4 p \frac{e^{ip \cdot (x-y)}}{p^2 + m^2 - i\epsilon}$$

Here  $\epsilon > 0$  is a generic, arbitrarily small number. In particular, we'll see why this  $i\epsilon$  is so important in QFT. Let's look for the poles of the integrand. We have

$$p^2 + m^2 - i\epsilon = -(p^0)^2 + \mathbf{p}^2 - i\epsilon = -(p^0)^2 + \omega_{\mathbf{p}}^2 - i\epsilon = (p^0 + \omega_{\mathbf{p}} - i\epsilon)(-p^0 + \omega_{\mathbf{p}} - i\epsilon) = 0$$

Where in the last step we don't care about quadratic terms, and renamed  $2\omega_{\mathbf{p}}\epsilon \rightarrow \epsilon$ . It is important when doing this that  $\omega_{\mathbf{p}} > 0$ : the positive sign of  $\epsilon$  is sacred. Therefore, the poles are located at  $p^0 = \pm\omega \mp i\epsilon$ .

**Draw figure.**

$$\Pi(x - y) = -i \int d^3 \mathbf{p} e^{i\mathbf{p} \cdot (\mathbf{x}-\mathbf{y})} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 + \omega_{\mathbf{p}} - i\epsilon)(-p^0 + \omega_{\mathbf{p}} - i\epsilon)}$$

The  $dp^0$  integrand  $f$  has  $\text{Res}_f(-\omega_{\mathbf{p}} + i\epsilon) = \frac{1}{2\pi} \frac{1}{2\omega_{\mathbf{p}}} e^{i\omega_{\mathbf{p}}(x^0-y^0)}$ , and  $\text{Res}_f(\omega_{\mathbf{p}} - i\epsilon) = \frac{1}{2\pi} \frac{1}{2\omega_{\mathbf{p}}} e^{-i\omega_{\mathbf{p}}(x^0-y^0)}$

Now we close the contour. We want exponential decay in  $e^{-ip^0(x^0-y^0)}$ , so the half-plane we pick actually depends on  $\text{sign}(x^0 - y^0)$ .

1.  $x^0 > y^0$ . We close down, picking up the pole at  $\omega_{\mathbf{p}} - i\epsilon$ :

$$I(x^0 - y^0) = 2\pi i \frac{1}{2\pi} \frac{1}{2\omega_{\mathbf{p}}} e^{-i\omega_{\mathbf{p}}(x^0-y^0)} = \frac{i}{2\omega_{\mathbf{p}}} e^{-i\omega_{\mathbf{p}}(x^0-y^0)}$$

2.  $x^0 < y^0$ . We close up, picking up the pole at  $-\omega_{\mathbf{p}} + i\epsilon$ . A similar calculation shows

$$I(x^0 - y^0) = \frac{i}{2\omega_{\mathbf{p}}} e^{-i\omega_{\mathbf{p}}(y^0-x^0)}$$

Finally, we can substitute this into  $\Pi(x - y)$  to get our result:

$$\Pi(x - y) = \begin{cases} \int \frac{d^3 \mathbf{p}}{2\omega_{\mathbf{p}}} e^{ip \cdot (x-y)}|_{p^0=\omega_{\mathbf{p}}}, & x^0 - y^0 > 0 \\ \int \frac{d^3 \mathbf{p}}{2\omega_{\mathbf{p}}} e^{ip \cdot (y-x)}|_{p^0=\omega_{\mathbf{p}}}, & x^0 - y^0 < 0 \end{cases}$$

We will shortly see how this is equivalent to the time-ordered 2-point function,  $\langle 0 | T\phi(x)\phi(y) | 0 \rangle$ .

## 2 Green's Functions

When learning QFT, it can be difficult to grasp the true value of Green's functions. I'd like to make this very clear in the following statement, which we'll spend some time trying to understand. Vacuum expectation values of products of free field operators can be written in terms of Green's functions of the wave equation (equation of motion). Today I'd like to give you some intuition playing around with these things, and relate them to the contour integrals we just talked about.

There are a lot of Green's-adjacent functions we can write down, and here are some of them. Note that these expressions depend on 4-vectors  $x$  and  $x'$ , which can have different time components.

- Wightman functions  $G^+(x, x') = \langle 0|\phi(x)\phi(x')|0\rangle$ ,  $G^-(x, x') = \langle 0|\phi(x')\phi(x)|0\rangle$
- Schwinger/Hadamard functions  $iG(x, x') = \langle 0|[\phi(x), \phi(x')]|0\rangle$ ,  $G^{(1)}(x, x') = \langle 0|\{\phi(x), \phi(x')\}|0\rangle$
- Feynman ( $T$ -ordered) function  $iG_F(x, x') = \langle 0|T\phi(x)\phi(x')|0\rangle = \theta(t - t')G^+ + \theta(t' - t)G^-$
- Retarded/advanced function  $G_R(x, x') = -\theta(t - t')G(x, x')$ ,  $G_A(x, x') = \theta(t - t')G(x, x')$

The fact that there are so many of basically the same thing, and that all of them have the names of some of the most important figures in QFT, should tell you how important these are.

Using the Klein-Gordon equation  $(\partial^2 + m^2)\phi(x) = 0$ , we can see that  $G^\pm$ , and consequentially  $G$ ,  $G^{(1)}$  satisfy the homogenous equation

$$(\partial_x^2 + m^2)\mathcal{G}(x, x') = 0$$

We can also evaluate the Klein-Gordon operator on the Feynman Green's function:

$$(\partial_x^2 + m^2)G_F(x, x') = (\partial_x^2 + m^2)(\theta(t - t')G^+(x, x') + \theta(t' - t)G^-(x, x'))$$

It is very instructive to do this calculation. On each term, the derivative can act in 3 ways by the product rule. If both  $\partial_x$  act on  $G^+$  or  $G^-$ , the term added to the  $m^2$  term vanishes by the equation of motion. The other terms can be evaluated by  $\partial_t\theta(t - t') = \delta(t - t')$  and the equal time commutation relations to find the Feynman Green's function solves the inhomogenous Klein-Gordon equation with  $\delta$ -function source.

$$\begin{aligned} (\partial_x^2 + m^2)G_F(x, x') &= -\delta^{(4)}(x - x') \\ (\partial_x^2 + m^2)G_{R,A}(x, x') &= \delta^{(4)}(x - x') \quad (\text{similar computation}) \end{aligned}$$

In this way the Feynman, retarded, and advanced functions are ‘true’ Green’s functions, while the Wightman, Schwinger, and Hadamard functions are not, being solutions of the homogenous equation of motion. In the general way we can build general solutions to the inhomogenous equations of motion out of these Green’s functions.

### 2.1 Boundary Conditions

This begs the question: what is the difference between all of these Green's functions? The answer lies in the boundary condition.

Observe that due to the presence of the  $\theta(t - t')$ , the retarded function  $G_R(x, x') = -\theta(t - t')(G^+ - G^-)$  vanishes unless  $t > t'$ . Looking at the equation of motion, the Green's function measures the response to a  $\delta$ -function which stimulates the field at  $t = t'$ . We could have instead chosen time to run backwards: the

Klein-Gordon equation is agnostic, since it is invariant under  $t \rightarrow -t$ . Doing so gives you the advanced Green's function: instead of everything travelling forwards in time, the field disturbances travel backwards.

The Feynman function is an average of the retarded and advanced solution. In QFT this is the right thing to do because the positive and negative frequency solutions to the equations of motion are on equal footing. That is, we have particles AND antiparticles, which can be mathematically treated as particles travelling backwards in time. Causality must be imposed on both. If I disturb my particle field at time  $t$ , the response at time  $t'$  will be zero until  $t' > t$ . For the antiparticle, the reverse is true.

To summarize: the Feynman, retarded, and advanced functions describe the propagation of field disturbances subject to certain boundary conditions. In QFT, causality, along with the existence of antiparticles, make the Feynman Green's function the right one to use.

## 2.2 Integral Representations

Integral representations for Green's functions can be obtained by substituting the mode decomposition of  $\phi(x)$  and  $\phi(x')$  into the definitions, and using properties of creation and annihilation operators. Let's do this for the Feynman Green's function, first for  $t > t'$ .

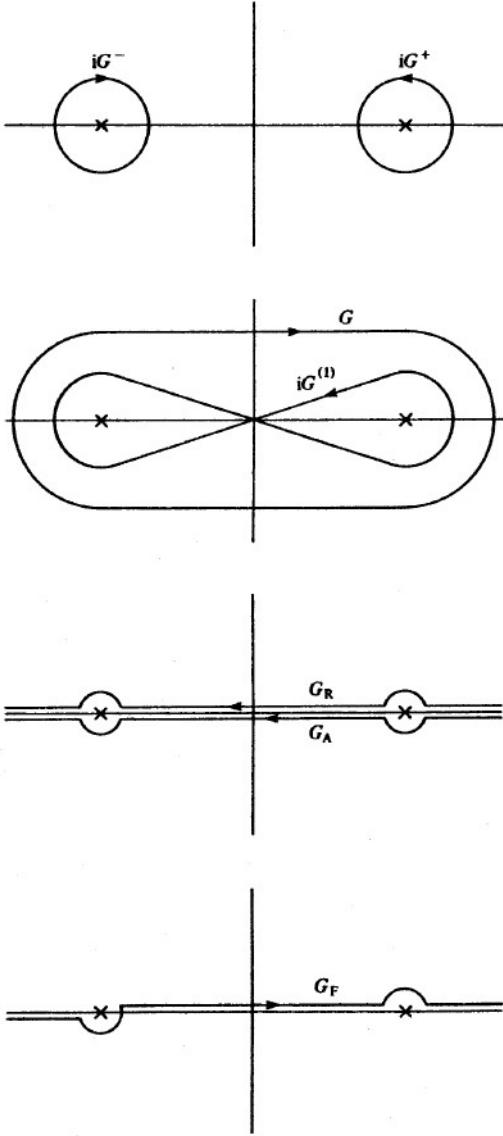
$$\begin{aligned} G^+(x, x')|_{t>t'} &= \langle 0 | \phi(x) \phi(x') | 0 \rangle = \int d^3\mathbf{k} d^3\mathbf{k}' \frac{1}{(2\omega_{\mathbf{k}} 2\omega_{\mathbf{k}'})^{1/2}} e^{-i\omega_{\mathbf{k}} t + i\mathbf{k} \cdot \mathbf{x}} e^{i\omega_{\mathbf{k}'} t' i\mathbf{k}' \cdot \mathbf{x}'} \langle 0 | a_{\mathbf{k}} a_{\mathbf{k}'}^\dagger | 0 \rangle \\ &= \int \frac{d^3\mathbf{k}}{2\omega_{\mathbf{k}}} e^{-ik \cdot (x-x')}, \quad k^0 = \omega_{\mathbf{k}} \\ G^+(x, x')|_{t<t'} &= \int \frac{d^3\mathbf{k}}{2\omega_{\mathbf{k}}} e^{-ik \cdot (x'-x)}, \quad k^0 = \omega_{\mathbf{k}} \end{aligned}$$

But this should be familiar! In fact, this was the answer of something we computed earlier! We thus have

$$G_F(x, x') = \Pi(x - y) = i \int d^4 p \frac{e^{ip \cdot (x-y)}}{p^2 + m^2 - i\epsilon}$$

To the Feynman Greens function we have associated a very specific integration contour, specified by the  $i\epsilon$  prescription. Note that because  $i\epsilon$  is so small, we can either deform the contour away from the real line and keep the poles real, or deform the poles away from the real line and integrate over  $\mathbb{R}$ .

We can do the exact same thing for the advanced and retarded Greens functions, and compare them to the value of  $i \int d^4 p \frac{e^{ip \cdot (x-y)}}{p^2 + m^2}$  with different choices of integration contour. To summarize: we can obtain each Greens function by performing some contour integral, with a specific choice of integration contour specifying the boundary condition of our theory. They are given below. **Draw figure.**



We may have done something quite peculiar to you today. We started with certain objects we like to study, these 2-point correlators, that are initially real-valued in position coordinates. But the choice of boundary conditions has imposed a specific complex (or pole) structure in our theory, and so we add complex analysis to our QFT toolbox. By considering analytic continuations, these objects have a very profound analytic structure universal to all QFTs. This is an incredibly deep rabbit hole that has birthed entire fields, none of which we'll have the time for.

I will mention one mindblowing (to me) application. We've discussed that in QFT, the Feynman Green's function is the right one to use. By the residue theorem, no integrals change if we deform our contour, so long as we don't cross any poles. We can therefore deform the contour from the real plane to the imaginary plane, and therefore compute integrals like the ones we've seen by integrating over purely imaginary time. If time is imaginary, then our metric becomes  $(-i^2, +, +, +) = (+, +, +, +)$  which is now Euclidean. This is known as a Wick rotation. In this way, we can compute quantities in a Minkowski theory by converting to a theory in Euclidean space, which is essentially just a statistical field theory at zero-temperature. That last part requires more explanation, which I'll provide next week when you start talking about path-integrals. I emphasize this is only possible with the Feynman  $i\epsilon$  prescription: if we used a retarded or advanced contour, then the poles obstruct us from deforming our contour to the imaginary axis.

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## 8.323 Relativistic Quantum Field Theory I

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