

CS5234 - Algorithms at Scale

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1 Probability and Hashing

1.1 Probability

We start by stating some crucial probability results that will be used throughout the course.

Lemma 1.1 (Union Bound)

For a countable set of events A_1, A_2, \dots , we have

$$\Pr \left[\bigcup_{i=1}^{\infty} A_i \right] \leq \sum_{i=1}^{\infty} \Pr[A_i] \quad (1)$$

Lemma 1.2 (Linearity of Expectation)

For any random variables X_1, X_2, \dots, X_n , we have

$$\mathbb{E} \left[\sum_{i=1}^n X_i \right] = \sum_{i=1}^n \mathbb{E}[X_i] \quad (2)$$

Lemma 1.3 (Markov's Inequality)

For any *non-negative* random variable X and $t > 0$, we have

$$\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t} \quad (3)$$

Lemma 1.4 (Chebyshev's Inequality)

For any random variable X with mean μ and variance σ^2 , we have

$$\Pr[|X - \mu| \geq t] \leq \frac{\sigma^2}{t^2} \quad (4)$$

In fact, this holds for any moment p instead of 2.

Lemma 1.5 (Chernoff-Hoeffding Bounds)

Suppose X_1, \dots, X_n are *independent* random variables with $X_i \in [0, 1]$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$ such that $\mu_L \leq \mu \leq \mu_H$.

(Chernoff) For any $0 \leq \delta \leq 1$,

$$\Pr[X \geq (1 + \delta)\mu] \leq \exp\left\{-\frac{\delta^2\mu}{3}\right\} \quad (5)$$

$$\Pr[X \leq (1 - \delta)\mu] \leq \exp\left\{-\frac{\delta^2\mu}{2}\right\} \quad (6)$$

$$\Pr[|X - \mu| \geq \delta\mu] \leq 2 \exp\left\{-\frac{\delta^2\mu}{3}\right\} \quad (7)$$

$$(8)$$

(Hoeffding) For any $\delta \geq 0$,

$$\Pr[X \geq \mu + \delta] \leq \exp\left\{-\frac{2\delta^2}{n}\right\} \quad (9)$$

$$\Pr[X \leq \mu - \delta] \leq \exp\left\{-\frac{2\delta^2}{n}\right\} \quad (10)$$

$$\Pr[|X - \mu| \geq \delta] \leq 2 \exp\left\{-\frac{2\delta^2}{n}\right\} \quad (11)$$

More generally, if $a_i \leq X_i \leq b_i$, then

$$\Pr[X \geq \mu + \delta] \leq \exp\left\{-\frac{2\delta^2}{\sum_{i=1}^n (b_i - a_i)^2}\right\} \quad (12)$$

$$\Pr[X \leq \mu - \delta] \leq \exp\left\{-\frac{2\delta^2}{\sum_{i=1}^n (b_i - a_i)^2}\right\} \quad (13)$$

1.2 Hashing

Definition 1.6 (k -Universal Hash)

A hash family $\mathcal{H} = \{h: \mathcal{U} \rightarrow S\}$ is *k -universal* if for any distinct k elements $x_1, \dots, x_k \in \mathcal{U}$ and any k elements $y_1, \dots, y_k \in S$, we have

$$\Pr_{h \in \mathcal{H}}[h(x_1) = y_1 \wedge \dots \wedge h(x_k) = y_k] = \frac{1}{|S|^k} \quad (14)$$

Such a hash family is also called a *k -wise independent hash family*.

Lemma 1.7 (Construction of k -Universal Hash Family)

There is a construction of a k -universal hash family from $[n]$ to $[n]$ that takes $O(k \log n)$ space.

2 Simple Techniques

2.1 Reservoir Sampling

We can use *reservoir sampling* to uniformly sample an element from a stream without having to store the stream in advance.

Algorithm 1 Reservoir Sampling algorithm

```
1: procedure RESERVOIRSAMPLING(stream  $s$ )
2:   for  $i = 1, 2, \dots$  do
3:     Flip a coin with probability  $1/i$ 
4:     if coin was heads then
5:        $x \leftarrow s_i$ 
6:     end if
7:   end for
8:   return  $x$ 
9: end procedure
```

By a simple proof by induction, we can show that the probability of any element x being sampled is $1/n$.

2.2 Mean Trick to Reduce Variance

We can take the average of multiple independent estimators to reduce the variance of the final estimator. This is useful when the variance of the single basic estimator is too large to apply concentration bounds.

Lemma 2.1 (Reduce Variance by Taking Mean of k Samples)

Let $X = \frac{1}{k} \sum_{i=1}^k X_i$ where each X_i has mean μ and variance σ^2 . Then, $E[X] = \mu$ and $\text{Var}[X] = \sigma^2/k$.

2.3 Median Trick to Amplify Success Probability

We can use the median of multiple independent estimators to reduce the probability of the final estimator being far from the true value (typically by Chernoff-Hoeffding bounds).

Algorithm 2 Median Trick for Amplifying Success Probability

```
1: procedure AMPLIFYBYMEDIAN(algorithm  $\mathcal{A}$ , integer  $\ell$ )
2:   for  $i = 1, \dots, \ell$  do
3:     Let  $Z_i$  be the output of  $\mathcal{A}$  with new independent random values
4:   end for
5:   return Median( $Z_1, \dots, Z_\ell$ )
6: end procedure
```

This trick can only be used when the success probability of the original estimator is more than 50%. We commonly combine both the Mean and the Median tricks to form the Median of Means trick. The typical space blow-up of such a technique is $O(\epsilon^{-2} \log(1/\delta))$.

A slightly different version of the Mean Trick is the Min Trick, where we take the minimum instead of the median. This is useful when there is only one-sided error – think of the median as a two-sided error version of the min trick. An example of this is the *Count-Min-Sketch* data structure.

3 Sketches

Definition 3.1 (Sketches and Linear Sketches)

A data structure S is a *sketch* if there is a space-efficient combining algorithm COMB such that for any two streams s_1 and s_2 , we have $S(s_1 \cdot s_2) = \text{COMB}(S(s_1), S(s_2))$. A sketch is *linear* if $S(s_1 \cdot s_2) = S(s_1) + S(s_2)$.

3.1 Misra-Gries

The *Misra-Gries* algorithm solves the heavy hitters problem:

- If x is a heavy hitter, then x must be in the output.
- If x is not a heavy hitter, then x may or may not be in the output.

Algorithm 3 Misra-Gries heavy hitters algorithm

```

1: procedure MISRAGRIES(stream  $s$ )
2:   Let  $C$  be an empty hash table
3:   for  $i = 1, \dots, m$  do
4:     if  $a_i \in C$  then
5:        $C[a_i] \leftarrow C[a_i] + 1$ 
6:     else if  $|C| < k - 1$  then
7:        $C[a_i] \leftarrow 1$ 
8:     else
9:       for  $j \in C$  do
10:         $C[j] \leftarrow C[j] - 1$ 
11:        if  $C[j] = 0$  then
12:          Delete  $C[j]$ 
13:        end if
14:      end for
15:    end if
16:  end for
17:  return Keys( $C$ )
18: end procedure

```

Lemma 3.2 (Misra-Gries Analysis)

The Misra-Gries algorithm with parameter k uses one pass and $O(k(\log m + \log n))$ bits of space and provides, for any token j , an estimate \hat{f}_j satisfying

$$f_j - \frac{m - \hat{m}}{k} \leq \hat{f}_j \leq f_j \quad (15)$$

3.2 Count-Min-Sketch

The *Count-Min-Sketch* data structure solves the frequency estimation problem.

This algorithm takes $O(\epsilon^{-1} \log(1/\delta) \cdot (\log m + \log n))$ space and provides, for any token j , an estimate \hat{f}_j satisfying $f_a \leq \hat{f}_a \leq \epsilon \|f_{-a}\|$ with probability at least $1 - \delta$.

Algorithm 4 Count-Min-Sketch frequency estimation algorithm

Require: $C[1 \dots t][1 \dots k] \leftarrow \vec{0}$, where $k = 2/\epsilon$ and $t = \lceil \log_2(1/\delta) \rceil$

Require: Choose t independent hash functions $h_1, \dots, h_t: [n] \rightarrow [k]$, each from a 2-universal family

```
1: procedure PROCESS(token  $(j, c)$ )
2:   for each token  $(j, c)$  in  $s$  do
3:     for  $i = 1, \dots, t$  do
4:        $C[i][h_i(j)] \leftarrow C[i][h_i(j)] + c$ 
5:     end for
6:   end for
7: end procedure
8: procedure OUTPUT(query  $a$ )
9:   return  $\hat{f}_a = \min_{i=1, \dots, t} C[i][h_i(a)]$ 
10: end procedure
```

4 Dimensions and Distances

Lemma 4.1 (Johnson-Lindenstrauss Lemma)

For any set $S \subseteq \mathbb{R}^d$ of n -points, there is an embedding $f: \mathbb{R}^d \rightarrow \mathbb{R}^m$ for $m = O(\epsilon^{-2} \log n)$ such that

$$\forall u, v \in S \quad (1 - \epsilon)\|u - v\|_2^2 \leq \|f(u) - f(v)\|_2^2 \leq (1 + \epsilon)\|u - v\|_2^2 \quad (16)$$

In other words, we can embed S into a lower-dimensional space while approximately preserving ℓ_2 norms. Some observations:

- The embedding has only a logarithmic dependence on n and *no* dependence on d .
- The embedding is can be generated using a Gaussian distribution.
- The embedding can be represented as a linear transformation, or in other words, a matrix.

Definition 4.2 (Locality Sensitive Hash)

A hash family $\mathcal{H} = \{h: \mathcal{U} \rightarrow S\}$ is a (r_1, r_2, p_1, p_2) -locally sensitive if for all points $p, p' \in \mathcal{U}$,

1. if $d(p, p') \leq r_1$, then $\Pr_{h \in \mathcal{H}}[h(p) = h(p')] \geq p_1$,
2. if $d(p, p') > r_2$, then $\Pr_{h \in \mathcal{H}}[h(p) = h(p')] \leq p_2$.

In other words, a *locality sensitive hash* (LSH) is a hash family where similar items are more likely to collide. Note that the definition makes sense only if $r_1 < r_2$ and $p_1 > p_2$.

WIP. 1. ANN, PLEB, how to solve them