MA2108 - Mathematical Analysis I

Liew Zhao Wei

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1 Preface

Here are just some quick scribbles of theorems that are used in the course. More complete notes will be written in the future.

2 Real Numbers

Theorem 2.1 (Triangle Inequality)
For any real numbers a and b, we have

$$||a| - |b|| \leq |a \pm b| \leq |a| + |b| \tag{1}$$

3 Sequences

Theorem 3.1 (Convergence Implie Boundedness)
Every convergent sequence is bounded.

Theorem 3.2 (Algebraic Properties of Limits)
Limits preserve addition, subtraction, multiplication, and division.

We will later see that limits also preserve square roots and absolute values.

Theorem 3.3 (Squeeze Theorem)
Suppose x_n and y_n converge to a. If there is a natural number K such that $x_n \leq y_n \leq z_n$ for all natural numbers $n \geq K$, then $y_n \to a$.

Theorem 3.4
If $|x_n| \to 0$, then $x_n \to 0$.

Theorem 3.5 (Bernoulli's Inequality)
For every real number $r \geq 1$ and $x \geq -1$, $(1 + x)^r \geq 1 + rx$.

Theorem 3.6
For any **fixed** $-1 < b < 1$, $b^n \to 0$.

Example 3.7
$\dfrac{1}{2^n} \to 0$ and $\left(\dfrac{2}{3}\right)^n \to 0$

Theorem 3.8
For a fixed $c > 0$, $c^{1/n} \to 1$.

Theorem 3.9
If $x_n \to x$, then $|x_n| \to |x|$.

Theorem 3.10
If $x_n \to x$, then $\sqrt{x_n} \to \sqrt{x}$.

Theorem 3.11
$n^{1/n} \to 1$.

Theorem 3.12 (Limits Preserve Inequalities)
Suppose $x_n \to x$ and $y_n \to y$.

If $x_n \geq y_n$ for all $n \in \mathbb{N}$, then $x \geq y$.

Corollary 3.13
By Theorem 3.12,

1. If $x_n \geq 0$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} x_n \geq 0$.
2. If $a, b \in \mathbb{R}$ and $a \leq x_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq \lim\limits_{n\to\infty} x_n \leq b$.

Theorem 3.14
$\left(1 + \frac{1}{n}\right)^n \to e$

Theorem 3.15 (Monotone Convergence Theorem)
If (x_n) is monotone and bounded, then x_n converges to:

1. $\sup\{x_n\}$ if (x_n) is increasing
2. $\inf\{x_n\}$ if (x_n) is decreasing

Remark 3.16. *If $(y_k) = (x_{n_k})$ is a subsequence of (x_n), then $n_k \geq k$.*

Theorem 3.17 (Sequence Converges Implies Subsequences Converge)
If (x_n) converges to x, then any subsequence (x_{n_k}) also converges to x.

Theorem 3.18 (Monotone Subsequence Theorem)
Every sequence has a monotone subsequence.

Theorem 3.19 (Bolzano-Weierstrass Theorem)
Every bounded sequence has a convergent subsequence.

3.1 Cauchy Convergence Criterion

Definition 3.20 (Cauchy Sequence)
A sequence $\{x_n\}$ is *Cauchy* if for every $\epsilon > 0$, there is an integer N such that $|s_n - s_m| < \epsilon$ whenever $n, k \geq N$.

Example 3.21
The sequence $x_n = \dfrac{1}{n}$ is Cauchy, but $x_n = (-1)^n$ is not.

Theorem 3.22 (Cauchy Implies Boundedness)
Every Cauchy sequence is bounded.

Theorem 3.23 (Cauchy and Convergent Subsequence Implies Convergence)
If $\{x_n\}$ is Cauchy and has a convergent subsequence, then $\{x_n\}$ converges.

Theorem 3.24 (Cauchy Convergence Criterion)
A sequence $\{x_n\}$ in $\mathbb{R}^k$ is Cauchy if and only if it converges.

Definition 3.25 (Contractive Sequence)
A sequence $\{x_n\}$ is *contractive* if there is a constant $0 \leq C < 1$ such that $|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$.

Theorem 3.26 (Contractive Sequences are Cauchy)
Every contractive sequence is Cauchy.

Theorem 3.27 (Increasing and Unbounded Sequences are Properly Divergent)
If a sequence $\{x_n\}$ is increasing and not bounded above, then $x_n \to \infty$.

Definition 3.28 (Properly Divergent Sequence)
A sequence $\{x_n\}$ is *properly divergent* if either $x_n \to \infty$ or $x_n \to -\infty$.

Theorem 3.29
If $k < \ell$ and $1 < a < b$, then $n^k << n^\ell << a^n << b^n << n!$.

4 Series

Analysis was mainly motivated by the study of infinite series, which are sums of infinitely many terms.

Definition 4.1 (Convergence of Series)
A series $\sum_{n=1}^{\infty} x_n$ *converges* if the sequence of partial sums $\{s_n\}$ converges.

Example 4.2
$\sum\limits_{i=1}^{\infty} 2^{-|i|}$ converges to 3. $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.

Theorem 4.3 (Convergence of Geometric Series)
If $|r| < 1$, then $\sum\limits_{n=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$.

Theorem 4.4
Let $\{x_n\}$ be a sequence and let $M \in \mathbb{N}$. $\sum\limits_{n=1}^{\infty} x_n$ converges if and only if $\sum\limits_{n=M}^{\infty} x_n$ converges.

Definition 4.5 (Cauchy Series)
A series $\sum x_n$ is *Cauchy* if the sequence of partial sums $\{s_n\}$ is Cauchy.

Theorem 4.6 (Cauchy Convergence Criterion for Series)
A series $\sum x_n$ converges if and only if it is Cauchy.

Theorem 4.7
A series $\sum x_n$ is Cauchy if and only if for every $\epsilon > 0$, there is an integer N such that $\left|\sum\limits_{k=n+1}^{m} x_k\right| < \epsilon$ whenever $n, m \geq N$.

Theorem 4.8 (nth Term Test)
If a series $\sum x_n$ converges, then $\lim\limits_{n\to\infty} x_n = 0$.

The converse of the nth term test is not true.

Theorem 4.9 (Divergence of Harmonic Series)
The *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Theorem 4.10
If $|r| \geq 1$, then $\sum_{n=0}^{\infty} r^n$ diverges.

Corollary 4.11
The series $\sum n = 0^{\infty} \alpha(r)^n$ converges if and only if $|r| < 1$.

Theorem 4.12 (Comparison Test)
Suppose that there is a natural number $K \in \mathbb{N}$ such that $0 \leq x_n \leq y_n$ for all $n \geq K$. Then,

1. If $\sum y_n$ converges, then so does $\sum x_n$.
2. If $\sum x_n$ diverges, then so does $\sum y_n$.

Theorem 4.13 (Convergence Criterion for p-series)
The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Definition 4.14 (Eventually Non-negative Series)
A series $\sum x_n$ is *eventually non-negative* if there is a natural number N such that $x_n \geq 0$ for all $n \geq N$.

Theorem 4.15
If $\sum x_n$ is eventually non-negative, then it converges if and only if $\{s_n\}$ is bounded above.

Definition 4.16 (Eventually Positive Series)
A series $\sum x_n$ is *eventually positive* if there is a natural number N such that $x_n > 0$ for all $n \geq N$.

Theorem 4.17 (Limit Comparison Test)
Consider two eventually positive series $\sum x_n$ and $\sum y_n$ where the limit

$$\rho = \lim_{n \to \infty} \frac{x_n}{y_n} \tag{2}$$

exists. Then,

1. If $\rho > 0$, then the two series either both converge or both diverge.
2. If $\rho = 0$ and $\sum y_n$ converges, then $\sum x_n$ converges.

Theorem 4.18 (Ratio Test)
Consider an eventually positive series $\sum a_n$ where the limit

$$\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \tag{3}$$

exists. Then,

1. If $\rho < 1$, then $\sum a_n$ converges.
2. If $\rho > 1$, then $\sum a_n$ diverges.
3. Otherwise, the test is inconclusive.