MA2101 - Linear Algebra II

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1 Vector spaces

1.1 Subspaces

Definition 1.1 (Subspace)
A subset U of V is a *subspace* of V if it is also a vector space using the same operations as V.

It is troublesome to prove all the axioms of a vector space for a subset. It turns out that we only need to prove that the following conditions hold.

Theorem 1.2 (Conditions for a subspace)
U is a subspace of V if and only if the following conditions hold:

1. $0 \in U$ (i.e. additive identity exists)
2. If $u, v \in U$, then $u + v \in U$. (i.e. closed under addition)
3. If $u \in U$, then $c \in \mathbb{F}$, $cu \in U$. (i.e. closed under scalar multiplication)

The notion of a direct sum (and hence, a sum) is important in the study of linear maps.

Definition 1.3 (Sum of subsets)
Let $U_1, \ldots, U_m$ be subsets of V. The sum of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is defined as

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_i \in U_i\}$$

That is, it is the set of all possible sums of elements from $U_1, \ldots, U_m$.

Theorem 1.4 (Sum of subspaces is smallest containing subspace)
Let $U_1, \ldots, U_m$ be subspaces of V. Then, $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.

Definition 1.5 (Direct sum)
Let $U_1, \ldots, U_m$ be subspaces of V. The sum $U_1 + \cdots + U_m$ is a *direct sum* if each element of $U_1 + \cdots + U_m$ can be written in exactly one way as $u_1 + \cdots + u_m$ with $u_i \in U_i$. The direct sum is denoted as $U_1 \oplus \cdots \oplus U_m$.

It is clear that sums of subspaces are analogous to unions of subsets, while direct sums of subspaces are analogous to disjoint unions of subsets.

Theorem 1.6 (Condition for a direct sum)
Let $U_1, \ldots, U_m$ be subspaces of V. Then, $U_1 + \cdots + U_m$ is a direct sum if and only if the only solution to $u_1 + \cdots + u_m = 0$ with $u_i \in U_i$ is $u_1 = \cdots = u_m = 0$.

With this, there is a simple condition for the sum of two subspaces to be a direct sum.

Theorem 1.7 (Direct sum of two subspaces)
Let U and W be subspaces of V. Then, $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

2 Linear maps

2.1 Isomorphism and invertibility

Theorem 2.1 (Inverse and invertible)
A linear map $S \in \mathcal{L}(W, V)$ is an *inverse* of a linear map $T \in \mathcal{L}(V, W)$ if $ST = I_V$ and $TS = I_W$. T is said to be *invertible* if it has an inverse.

Remark 2.2 (Inverses are unique)**.** *An invertible linear map has a unique inverse.*

Theorem 2.3 (Invertiblity is equivalent to bijectivity)
A linear map is invertible if and only if it is bijective.

Definition 2.4 (Isomorphism)
An *isomorphism* is an invertible linear map. Two vector spaces are *isomorphic* if there is an isomorphism from one to the other.

Theorem 2.5 (Isomorphic vector spaces have the same dimension)
Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.

Theorem 2.6 ($\mathcal{L}(V, W)$ is isomorphic to $M_{m\times n}(\mathbb{F})$)
Let V and W be finite-dimensional vector spaces with $\dim V = n$ and $\dim W = m$. Then, $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}_{m\times n}$.

Corollary 2.7 ($\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$)
Let V and W be finite-dimensional vector spaces. Then, $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$.

Definition 2.8 (Linear operator, $\mathcal{L}(V)$)
A linear map from a vector space V to itself is called a *linear operator* and is denoted by $\mathcal{L}(V)$.

Theorem 2.9 (Injectivity is equivalent to surjectivity)
Let $T \in \mathcal{L}(V)$ be a linear operator on a finite-dimensional vector space V. Then, T is injective if and only if it is surjective.

2.2 Duality

Definition 2.10 (Linear functional)
A *linear functional* $T \in \mathcal{L}(V, \mathbb{F})$ is a linear map from a vector space to its underlying field.

Example 2.11 (Examples of linear functionals)
We present some common examples of linear functionals.

1. Let $(c_1, \ldots, c_n) \in \mathbb{F}^n$. Then, the map $f(x_1, \ldots, x_n) = c_1x_1 + \cdots + c_nx_n$ is a linear functional on $\mathbb{F}^n$.
2. The *trace* of an $n \times n$ matrix A, defined as $\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$, is a linear functional on $\mathbb{F}_{n\times n}$.
3. Let $[a, b]$ be a closed interval in $\mathbb{R}$. Then, the map $L(f) = \int_a^b f(x)\,dx$ is a linear functional on the space of continuous real-valued functions on $[a, b]$, denoted $C([a, b])$.

Definition 2.12 (Dual space)
The *dual space* of a vector space V, denoted V^*, is the vector space of all linear functionals on V. That is, $V^* = \mathcal{L}(V, \mathbb{F})$.

Theorem 2.13 ($\dim V^* = \dim V$)
Let V be a finite-dimensional vector space. Then, $\dim V^* = \dim V$.

2.3 Matrix transpose

Definition 2.14 (Transpose of a matrix, A^t)
Let A be an $m \times n$ matrix. The *transpose* of A is the $n \times m$ matrix A^t defined by $A^t_{ij} = A_{ji}$. That is, the rows and columns of A are interchanged.

Theorem 2.15 (Transpose of matrix products)
Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then, $(AB)^t = B^tA^t$.

(Skipping some definitions here regarding the dual basis and going straight to more pertinent results – will revisit it later.)

Definition 2.16 (Row rank, column rank)
Let $A \in \mathbb{F}^{m\times n}$ be an $m \times n$ matrix.

- The *row rank* of A is the dimension of the span of the rows of A in $\mathbb{F}^{1,n}$.
- The *column rank* of A is the dimension of the span of the columns of A in $\mathbb{F}^{m,1}$.

Theorem 2.17 ($\dim R(T)$ equals to column rank of $\mathcal{M}(T)$)
Let V and W be finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. Then, $\dim R(T)$ equals the column rank of $\mathcal{M}(T)$.

Theorem 2.18 (Row rank equals column rank)
Let $A \in \mathbb{F}^{m \times n}$ be an $m \times n$ matrix. Then, the row rank of A is equal to the column rank of A.

Since the row rank and column rank are equal, we can simply refer to the rank of a matrix.

Definition 2.19 (Rank)
The *rank* of a matrix A, denoted $\operatorname{rank} A$, is the column rank of A.

3 Trace and Determinant

Trace and determinants are primarily used to establish the properties of eigenvalues in the next section.

Definition 3.1 (Characteristic polynomial)
Let T be an operator on a complex vector space. If T has distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ with multiplicities $d_1, \ldots, d_m$, then the *characteristic polynomial* of T is defined as

$$p_T(z) = (z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m} \tag{1}$$

3.1 Trace

Definition 3.2 (Trace of an operator)
The trace of an operator T, denoted $\operatorname{tr} T$, is defined as follows:

- If $\mathbb{F} = \mathbb{C}$, then $\operatorname{tr} T$ is the sum of the eigenvalues of T, counted with multiplicity.
- If $\mathbb{F} = \mathbb{R}$, then $\operatorname{tr} T$ is the sum of the eigenvalues of $T_{\mathbb{C}}$, counted with multiplicity.

Theorem 3.3 (Relation between trace to characteristic polynomial)
For any operator $T \in \mathcal{L}(V)$ with $n = \dim V$, $\operatorname{tr} T$ is equal to the negative of the coefficient of z^{n-1} in the characteristic polynomial of T.

Definition 3.4 (Trace of a matrix)
The *trace* of an $n \times n$ matrix A, denoted $\operatorname{tr} A$, is the sum of the diagonal entries of A. That is, $\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$.

We will show that the trace of an operator is equal to the trace of any of its matrix representation.

Lemma 3.5 (Trace is cyclic)
For any two square matrices A and B of the same size,

$$\operatorname{tr} AB = \operatorname{tr} BA \tag{2}$$

Lemma 3.6 (Trace of matrix is oblivious to basis)
Suppose T is a linear operator. For any two bases $\mathcal{B}$ and $\mathcal{C}$ of a finite-dimensional vector space,

$$\operatorname{tr} \mathcal{M}_{\mathcal{B}}(T) = \operatorname{tr} \mathcal{M}_{\mathcal{C}}(T) \tag{3}$$

Theorem 3.7 (Trace of linear operator and its matrix are equal)
For any operator T,

$$\operatorname{tr} T = \operatorname{tr} \mathcal{M}(T) \tag{4}$$

Corollary 3.8 (Trace is additive)
For any two operators S and T,

$$\operatorname{tr}(S+T) = \operatorname{tr} S + \operatorname{tr} T \tag{5}$$

Theorem 3.9 ($ST - TS \neq I$)
For any two operators S and T,

$$ST - TS \neq I \tag{6}$$

3.2 Determinant

Definition 3.10 (Determinant of an operator)
The determinant of an operator T, denoted $\det T$, is defined as follows:

- If $\mathbb{F} = \mathbb{C}$, then $\det T$ is the product of the eigenvalues of T, counted with multiplicity.
- If $\mathbb{F} = \mathbb{R}$, then $\det T$ is the product of the eigenvalues of $T_{\mathbb{C}}$, counted with multiplicity.

Similar to the trace, the determinant is closely related to the characteristic polynomial.

Theorem 3.11 (Relation between determinant and characteristic polynomial)
For any operator $T \in \mathcal{L}(V)$ with $n = \dim V$, $\det T$ is equal to $(-1)^n$ times the constant term of the characteristic polynomial of T.

Remark 3.12. *Combining the trace and determinant, we can rewrite the characteristic polynomial as*

$$p_T(t) = z^n - (\operatorname{tr} T)z^{n-1} + \cdots + (-1)^n \det T \tag{7}$$

Theorem 3.13 (Invertible iff nonzero determinant)
An operator T is invertible if and only if $\det T \neq 0$.

Theorem 3.14 (Invertible iff nonzero determinant)
A square matrix A is invertible if and only if $\det A \neq 0$.

4 Eigenvalues and Eigenvectors

Definition 4.1 (Invariant subspace)
A subspace $U \subseteq V$ is *invariant* under a linear operator T if $T(U) \subseteq U$.

Example 4.2 (Examples of trivially invariant subspaces)
For any linear operator $T \in \mathcal{L}(V)$, the following subspaces are invariant under T:

- $\{0\}$
- V
- $\ker T$
- $R(T)$

Of particular interest are the invariant subspaces of dimension 1.

Definition 4.3 (Eigenvalue and eigenvector)
A scalar $\lambda \in \mathbb{F}$ is an *eigenvalue* of a linear operator $T \in \mathcal{L}(V)$ if there exists a nonzero vector $v \in V$ such that $Tv = \lambda v$. The vector v is called an *eigenvector* of T corresponding to λ.

Remark 4.4 (Equivalent definition of eigenvector)**.** *A non-zero vector $v \in V$ is an eigenvector of T corresponding to λ if and only if $v \in \ker(T - \lambda I)$.*

With that remark, we arrive at the following equivalences for eigenvalues.

Theorem 4.5 (Equivalent conditions for eigenvalues)
Let V be finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then, the following are equivalent:

1. λ is an eigenvalue of T.
2. $T - \lambda I$ is not injective.
3. $T - \lambda I$ is not surjective.
4. $T - \lambda I$ is not invertible.

Theorem 4.6 (Eigenvectors corresponding to distinct eigenvalues are linearly independent)
Let $T \in \mathcal{L}(V)$. Every list of eigenvectors of T corresponding to distinct eigenvalues is linearly independent.

Theorem 4.7 (Number of eigenvalues is at most $\dim V$)
Any linear operator on a finite-dimensional vector space V has at most $\dim V$ distinct eigenvalues.

Definition 4.8 (T^m)
Let $T \in \mathcal{L}(V)$ and $m \in \mathbb{N}$.

- T^0 is defined as the identity operator I on V.
- $T^m \in \mathcal{L}(V)$ is defined as $T^m = T \circ T^{m-1}$.
- If T is invertible, then $T^{-m} \in \mathcal{L}(V)$ is defined as $T^{-m} = (T^{-1})^m$.

Remark 4.9. *$T^mT^n = T^{m+n}$ and $(T^m)^n = T^{mn}$.*

Definition 4.10 (Matrix similarity)
Two order-n square matrices A and B are *similar* if there is an invertible matrix P such that

$$B = P^{-1}AP \tag{8}$$

Similar matrices represent the same linear map under two different bases.

Remark 4.11 (Similarity equivalence)**.** *Similarity is an equivalence relation on the set of all order-n square matrices.*

Definition 4.12 (Characteristic polynomial equals $\det(zI - T)$)
The characteristic polynomial of an operator T is equal to $\det(zI - T)$.

Remark 4.13. *Similar matrices have the same characteristic polynomial.*

Theorem 4.14 (Multiplicies sum to $\dim V$)
For any operator T on a complex vector space V, the sum of the multiplicities of the eigenvalues of T is equal to $\dim V$.