

MA2101 - Linear Algebra II

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1 Linear maps

1.1 Isomorphism and invertibility

Theorem 1.1 (Inverse and invertible)

A linear map $S \in \mathcal{L}(W, V)$ is an *inverse* of a linear map $T \in \mathcal{L}(V, W)$ if $ST = I_V$ and $TS = I_W$. T is said to be *invertible* if it has an inverse.

Remark 1.2 (Inverses are unique). *An invertible linear map has a unique inverse.*

Theorem 1.3 (Invertibility is equivalent to bijectivity)

A linear map is invertible if and only if it is bijective.

Definition 1.4 (Isomorphism)

An *isomorphism* is an invertible linear map. Two vector spaces are *isomorphic* if there is an isomorphism from one to the other.

Theorem 1.5 (Isomorphic vector spaces have the same dimension)

Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.

Theorem 1.6 ($\mathcal{L}(V, W)$ is isomorphic to $M_{m \times n}(\mathbb{F})$)

Let V and W be finite-dimensional vector spaces with $\dim V = n$ and $\dim W = m$. Then, \mathcal{M} is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}_{m \times n}$.

Corollary 1.7 ($\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$)

Let V and W be finite-dimensional vector spaces. Then, $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$.

Definition 1.8 (Linear operator, $\mathcal{L}(V)$)

A linear map from a vector space V to itself is called a *linear operator* and is denoted by $\mathcal{L}(V)$.

Theorem 1.9 (Injectivity is equivalent to surjectivity)

Let $T \in \mathcal{L}(V)$ be a linear operator on a finite-dimensional vector space V . Then, T is injective if and only if it is surjective.

1.2 Duality

Definition 1.10 (Linear functional)

A *linear functional* $T \in \mathcal{L}(V, \mathbb{F})$ is a linear map from a vector space to its underlying field.

Example 1.11 (Examples of linear functionals)

We present some common examples of linear functionals.

1. Let $(c_1, \dots, c_n) \in \mathbb{F}^n$. Then, the map $f(x_1, \dots, x_n) = c_1x_1 + \dots + c_nx_n$ is a linear functional on \mathbb{F}^n .
2. The *trace* of an $n \times n$ matrix A , defined as $\text{tr } A = \sum_{i=1}^n a_{ii}$, is a linear functional on $\mathbb{F}_{n \times n}$.
3. Let $[a, b]$ be a closed interval in \mathbb{R} . Then, the map $L(f) = \int_a^b f(x) dx$ is a linear functional on the space of continuous real-valued functions on $[a, b]$, denoted $C([a, b])$.

Definition 1.12 (Dual space)

The *dual space* of a vector space V , denoted V^* , is the vector space of all linear functionals on V . That is, $V^* = \mathcal{L}(V, \mathbb{F})$.

Theorem 1.13 ($\dim V^* = \dim V$)

Let V be a finite-dimensional vector space. Then, $\dim V^* = \dim V$.

1.3 Matrix transpose

Definition 1.14 (Transpose of a matrix, A^t)

Let A be an $m \times n$ matrix. The *transpose* of A is the $n \times m$ matrix A^t defined by $A_{ij}^t = A_{ji}$. That is, the rows and columns of A are interchanged.

Theorem 1.15 (Transpose of matrix products)

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then, $(AB)^t = B^t A^t$.

(Skipping some definitions here regarding the dual basis and going straight to more pertinent results – will revisit it later.)

Definition 1.16 (Row rank, column rank)

Let $A \in \mathbb{F}^{m \times n}$ be an $m \times n$ matrix.

- The *row rank* of A is the dimension of the span of the rows of A in $\mathbb{F}^{1,n}$.
- The *column rank* of A is the dimension of the span of the columns of A in $\mathbb{F}^{m,1}$.

Theorem 1.17 ($\dim R(T)$ equals to column rank of $\mathcal{M}(T)$)

Let V and W be finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. Then, $\dim R(T)$ equals the column rank of $\mathcal{M}(T)$.

Theorem 1.18 (Row rank equals column rank)

Let $A \in \mathbb{F}^{m \times n}$ be an $m \times n$ matrix. Then, the row rank of A is equal to the column rank of A .

Since the row rank and column rank are equal, we can simply refer to the rank of a matrix.

Definition 1.19 (Rank)

The *rank* of a matrix A , denoted $\text{rank } A$, is the column rank of A .

2 Trace and Determinant

Trace and determinants are primarily used to establish the properties of eigenvalues in the next section.

Definition 2.1 (Characteristic polynomial)

Let T be an operator on a complex vector space. If T has distinct eigenvalues $\lambda_1, \dots, \lambda_m$ with multiplicities d_1, \dots, d_m , then the *characteristic polynomial* of T is defined as

$$p_T(z) = (z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m} \quad (1)$$

2.1 Trace

Definition 2.2 (Trace of an operator)

The trace of an operator T , denoted $\text{tr } T$, is defined as follows:

- If $\mathbb{F} = \mathbb{C}$, then $\text{tr } T$ is the sum of the eigenvalues of T , counted with multiplicity.
- If $\mathbb{F} = \mathbb{R}$, then $\text{tr } T$ is the sum of the eigenvalues of $T_{\mathbb{C}}$, counted with multiplicity.

Theorem 2.3 (Relation between trace to characteristic polynomial)

For any operator $T \in \mathcal{L}(V)$ with $n = \dim V$, $\text{tr } T$ is equal to the negative of the coefficient of z^{n-1} in the characteristic polynomial of T .

Definition 2.4 (Trace of a matrix)

The *trace* of an $n \times n$ matrix A , denoted $\text{tr } A$, is the sum of the diagonal entries of A . That is, $\text{tr } A = \sum_{i=1}^n a_{ii}$.

We will show that the trace of an operator is equal to the trace of any of its matrix representation.

Lemma 2.5 (Trace is cyclic)

For any two square matrices A and B of the same size,

$$\text{tr } AB = \text{tr } BA \quad (2)$$

Lemma 2.6 (Trace of matrix is oblivious to basis)

Suppose T is a linear operator. For any two bases \mathcal{B} and \mathcal{C} of a finite-dimensional vector space,

$$\text{tr } \mathcal{M}_{\mathcal{B}}(T) = \text{tr } \mathcal{M}_{\mathcal{C}}(T) \quad (3)$$

Theorem 2.7 (Trace of linear operator and its matrix are equal)

For any operator T ,

$$\operatorname{tr} T = \operatorname{tr} \mathcal{M}(T) \quad (4)$$

Corollary 2.8 (Trace is additive)

For any two operators S and T ,

$$\operatorname{tr}(S + T) = \operatorname{tr} S + \operatorname{tr} T \quad (5)$$

Theorem 2.9 ($ST - TS \neq I$)

For any two operators S and T ,

$$ST - TS \neq I \quad (6)$$

2.2 Determinant

Definition 2.10 (Determinant of an operator)

The determinant of an operator T , denoted $\det T$, is defined as follows:

- If $\mathbb{F} = \mathbb{C}$, then $\det T$ is the product of the eigenvalues of T , counted with multiplicity.
- If $\mathbb{F} = \mathbb{R}$, then $\det T$ is the product of the eigenvalues of $T_{\mathbb{C}}$, counted with multiplicity.

Similar to the trace, the determinant is closely related to the characteristic polynomial.

Theorem 2.11 (Relation between determinant and characteristic polynomial)

For any operator $T \in \mathcal{L}(V)$ with $n = \dim V$, $\det T$ is equal to $(-1)^n$ times the constant term of the characteristic polynomial of T .

Remark 2.12. Combining the trace and determinant, we can rewrite the characteristic polynomial as

$$p_T(t) = z^n - (\operatorname{tr} T)z^{n-1} + \cdots + (-1)^n \det T \quad (7)$$

Theorem 2.13 (Invertible iff nonzero determinant)

An operator T is invertible if and only if $\det T \neq 0$.

Theorem 2.14 (Invertible iff nonzero determinant)

A square matrix A is invertible if and only if $\det A \neq 0$.

3 Eigenvalues and Eigenvectors

Definition 3.1 (Invariant subspace)

A subspace $U \subseteq V$ is *invariant* under a linear operator T if $T(U) \subseteq U$.

Example 3.2 (Examples of trivially invariant subspaces)

For any linear operator $T \in \mathcal{L}(V)$, the following subspaces are invariant under T :

- $\{0\}$
- V
- $\ker T$
- $R(T)$

Of particular interest are the invariant subspaces of dimension 1.

Definition 3.3 (Eigenvalue and eigenvector)

A scalar $\lambda \in \mathbb{F}$ is an *eigenvalue* of a linear operator $T \in \mathcal{L}(V)$ if there exists a nonzero vector $v \in V$ such that $Tv = \lambda v$. The vector v is called an *eigenvector* of T corresponding to λ .

Remark 3.4 (Equivalent definition of eigenvector). *A non-zero vector $v \in V$ is an eigenvector of T corresponding to λ if and only if $v \in \ker(T - \lambda I)$.*

With that remark, we arrive at the following equivalences for eigenvalues.

Theorem 3.5 (Equivalent conditions for eigenvalues)

Let V be finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then, the following are equivalent:

1. λ is an eigenvalue of T .
2. $T - \lambda I$ is not injective.
3. $T - \lambda I$ is not surjective.
4. $T - \lambda I$ is not invertible.

Theorem 3.6 (Eigenvectors corresponding to distinct eigenvalues are linearly independent)

Let $T \in \mathcal{L}(V)$. Every list of eigenvectors of T corresponding to distinct eigenvalues is linearly independent.

Theorem 3.7 (Number of eigenvalues is at most $\dim V$)

Any linear operator on a finite-dimensional vector space V has at most $\dim V$ distinct eigenvalues.

Definition 3.8 (T^m)

Let $T \in \mathcal{L}(V)$ and $m \in \mathbb{N}$.

- T^0 is defined as the identity operator I on V .
- $T^m \in \mathcal{L}(V)$ is defined as $T^m = T \circ T^{m-1}$.
- If T is invertible, then $T^{-m} \in \mathcal{L}(V)$ is defined as $T^{-m} = (T^{-1})^m$.

Remark 3.9. $T^m T^n = T^{m+n}$ and $(T^m)^n = T^{mn}$.

Definition 3.10 (Matrix similarity)

Two order- n square matrices A and B are *similar* if there is an invertible matrix P such that

$$B = P^{-1}AP \tag{8}$$

Similar matrices represent the same linear map under two different bases.

Remark 3.11 (Similarity equivalence). *Similarity is an equivalence relation on the set of all order- n square matrices.*

Definition 3.12 (Characteristic polynomial equals $\det(zI - T)$)

The characteristic polynomial of an operator T is equal to $\det(zI - T)$.

Remark 3.13. *Similar matrices have the same characteristic polynomial.*

Theorem 3.14 (Multiplicities sum to $\dim V$)

For any operator T on a complex vector space V , the sum of the multiplicities of the eigenvalues of T is equal to $\dim V$.