# COSC343: Artificial Intelligence

Lecture 11 : Recurrent neural networks

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## In today's lecture

- Examples of neural networks in action
- Softmax function – network output as a probability distribution
- Simple Recurrent Network (SRN)
- Teaching SRN to talk



## Recap: Notation

$w_{ij}^{[l]}$ weight; layer; neuron # in the layer; neuron # in previous layer

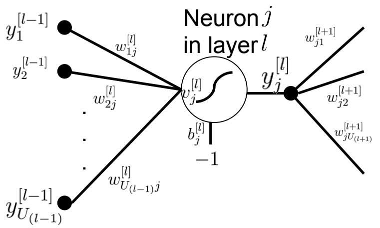
$L$ #of layers

$b_j^{[l]}$ bias; neuron # in the layer

$v_j^{[l]}$ activity; neuron # in the layer

$y_j^{[l]}$ output; neuron # in the layer

$U_l$ number of neurons in layer $l$




- Activity is the weighted sum of inputs from the previous layer minus the bias $$v_j^{[l]} = \sum_{i=1}^{U^{[l-1]}} w_{ij}^{[l]} y_i^{[l-1]} - b_j^{[l]}$$
- Output is a function of activity $$y_j^{[l]} = f_{\text{logsig}}(v_j^{[l]}) = \frac{1}{1+\exp^{-v_j^{[l]}}}$$

where $y_i^{[0]} = x_i$ and $U_0 = M$



## Recap: Backpropagation

Let's assume that we are minimising some cost function that depends on the network output and the desired output

$$J\left(y_j^{[L]}, \tilde{y}_j\right)$$

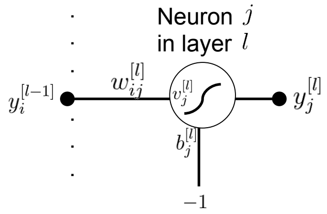
Steepest gradient descent update for weight connecting input i with neuron j in layer l is :

$$w_{ij}^{[l]} = w_{ij}^{[l]} - \alpha \Delta w_{ij}^{[l]}$$

Steepest gradient descent update for bias on neuron j in layer l is :

$$b_j^{[l]} = b_j^{[l]} - \alpha \Delta b_j^{[l]}$$

where...




The change in the weight connecting input i with neuron j in layer l is:

$$\Delta w_{ij}^{[l]} = y_i^{[l-1]} \frac{dy_j^{[l]}}{dv_j^{[l]}} \delta J_j^{[l]}$$

The change in the bias for neuron j in layer l is:

$$\Delta b_j^{[l]} = -\frac{dy_j^{[l]}}{dv_j^{[l]}} \delta J_j^{[l]}$$

where...

Sigmoid function has a derivative

$$y_j^{[l]} = f_{\text{logsig}}(v_j^{[l]}) = \frac{1}{1+\exp^{-v_j^{[l]}}}$$

$$\frac{\partial y_j^{[l]}}{\partial v_j^{[l]}} = y_j^{[l]}(1 - y_j^{[l]})$$

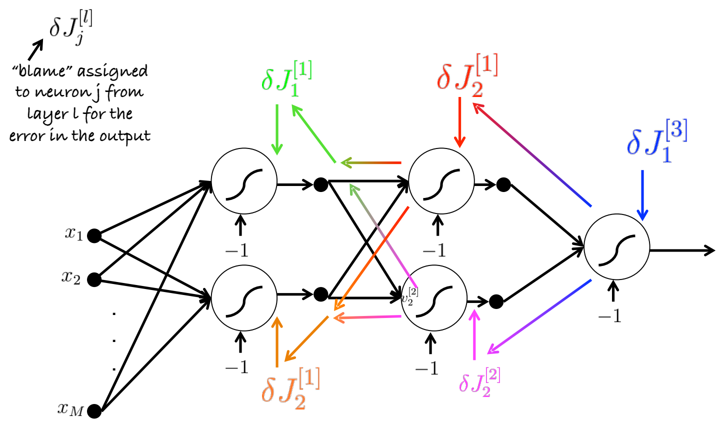
The cost blame for neuron i in layer l-1 is:

$$\delta J_i^{[l-1]} = \sum_{j=1}^{U_l} w_{ij}^{[l]} \frac{dy_j^{[l]}}{dv_j^{[l]}} \delta J_j^{[l]}$$

The cost blame for neuron j in the output layer is a derivative of the overall cost with respect to the neuron's output:

$$\delta J_j^{[L]} = \frac{dJ\left(y_j^{[L]}, \tilde{y}_j\right)}{dy_j^{[L]}}$$

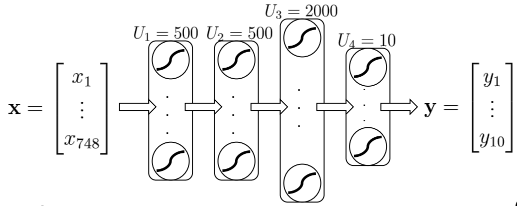
## Recap: Backpropagation






## An example: an artificial neural network for classification

748-500-500-2000-10 neural network:




$\text{class label} = \arg\max_i y_i$

Input:
- Each image is 28x28 pixels
- Digits are normalised for size, position and orientation.

Output:
- One-zero vector code for digit identified in the image

Mean squared error

Training with backpropagation of MSE:
- 60000 images
- 2.5% classification error

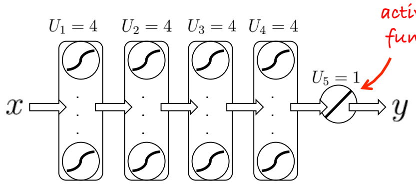
Testing:
- 10000 images
- 3% training error



## An example: an artificial neural network for regression

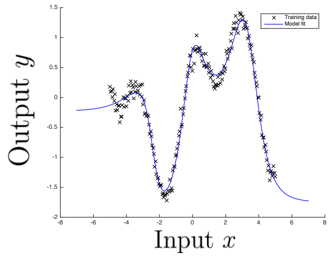
1-4-4-4-4-1 neural network:

Linear activation function




Input:
- Value between -5 and 5

Output:
- Inferred function of input x




Training with backpropagation of MSE:
- 100 samples
- RMSE=0.72

Root mean squared error

Testing:
- 100 samples
- RMSE=0.77



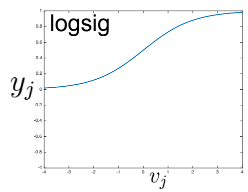
## Selection of activation functions

j indexes an individual neuron

- **Logistic sigmoid**

  $$y_j = \frac{1}{1+e^{-v_j}} \quad , \quad \frac{dy_j}{dv_j} = y_j(1-y_j)$$

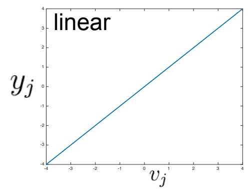
  - Output bounded between 0 and 1
  - Useful for classification, hidden units, and interpreting output as a probability




- **Linear**

  $$y_j = v_j \; , \; \frac{dy_j}{dv_j} = 1$$

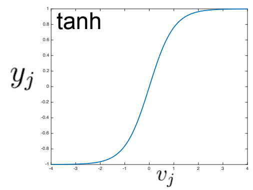
  - Output not bounded
  - Useful for output neurons in regression tasks




- **Hyperbolic tangent**

  $$y_j = \tanh(v_j) \; , \; \frac{dy_j}{dv_j} = (1+y_j)(1-y_j)$$

  - Output bounded between -1 and 1
  - Sometimes gives richer hidden layer representation than logistic sigmoid



# Network output as a probability distribution

A neural network with $K$ outputs can represent probability distribution of a discrete random variable with $K$ possible outcomes

- Most commonly used for classification, where each output represents probability of classifying the input a belonging to one of $K$ possible labels: $y_j^{[L]} = p(\text{label} = c_j)$ , where $\text{label} \in \{c_1, \ldots, c_K\}$.
- Probability distribution at the output is computed with the softmax function

$$y_j^{[L]} = \frac{e^{v_j}}{\sum_{k=1}^{K} e^{v_k}}$$ for which the derivative

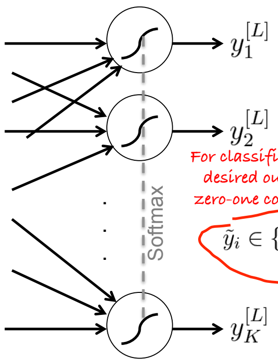
is $$\frac{dy_j^{[L]}}{dv_j^{[L]}} = y_j^{[L]}(1 - y_j^{[L]})$$

This is all you need to backpropagate the softmax function

- The training error at the output is $\delta J_j^{[L]} = \tilde{y}_j(y_j^{[L]} - 1) + (1 - \tilde{y}_j)y_j^{[L]}$, which

corresponds to minimisation of the following cost: $J = -\sum_j \left(\tilde{y}_j \ln y_j^{[L]} + (1 - \tilde{y}_j)\ln(1 - y_j^{[L]})\right)$

Output layer:




For classification, the desired output is a zero-one coded vector

$\tilde{y}_i \in \{0, 1\}$



# An example: Classification with softmax

A 784-500-500-2000-10 neural network trained with a the softmax function to recognise digit in an image.

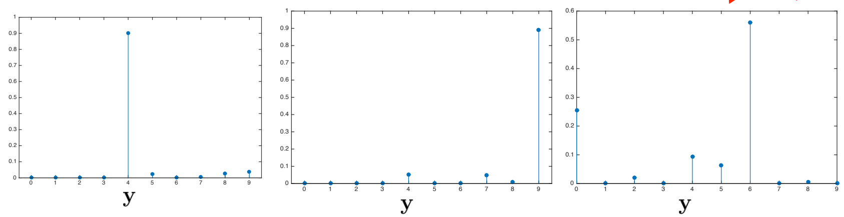
For the following test inputs,

this input is misclassified

target label is "4"

the network produces the following probability distributions over 10 possible labels:

$P(\mathbf{y}|\mathbf{x})$






# What about temporal patterns?

So far we have only considered learning tasks where the order of inputs (during training and testing) was of no consequence

- E.g. In digit recognition, input  should be recognised as a "2" regardless whether the previously processed input was a , , or .

What about a language related task?

- E.g. If we wanted to train a neural network to predict the next word in a sentence....the prediction depends on the context:

  "Dogs are ..."

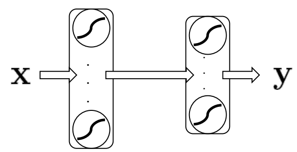
  "Cars are ..."



# Feed forward and recurrent neural networks

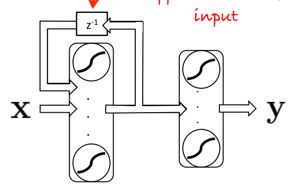
Networks we were looking at up to this point

A **feed forward neural network** is a directed acyclic graph:
- The internal state of the network depends only on its current input, which allows the model to perform static input-output mapping.



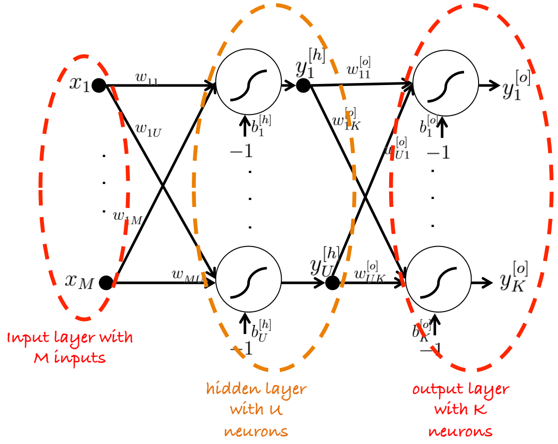

State of the output is saved and used to supplement the next input

A **recurrent neural network** is a directed graph with cycles (loops):
- The internal state of the network depends on its previous state, which allows the model to exhibit a dynamic temporal behaviour.



# Simple Recurrent Network (SRN)

- Also referred to as *Elman network* for its inventor Jeff Elman




$w_{ij}$ - weight connecting input i to neuron j in the hidden layer

$b_j^{[h]}$ - bias of neuron j in the hidden layer

$w_{ij}^{[o]}$ - weight connecting neuron i from the hidden layer to neuron j in the output layer

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$x_i$ - input i to the hidden layer

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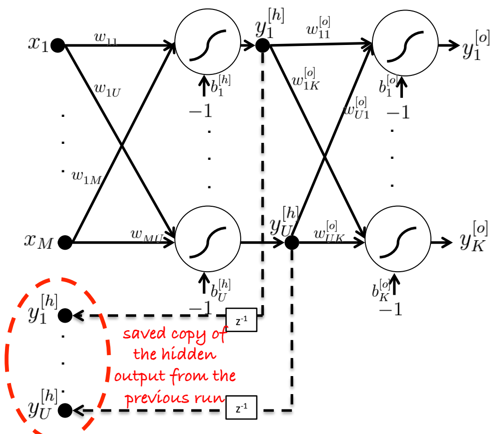
$M$ - # of inputs $U$ - # of hidden neurons $K$ - # of outputs



# Simple Recurrent Network (SRN)

- Output of the hidden layer feeds into the output layer as well as the hidden layer.

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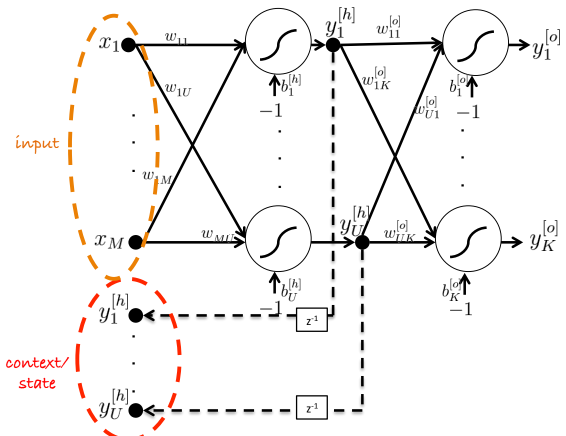


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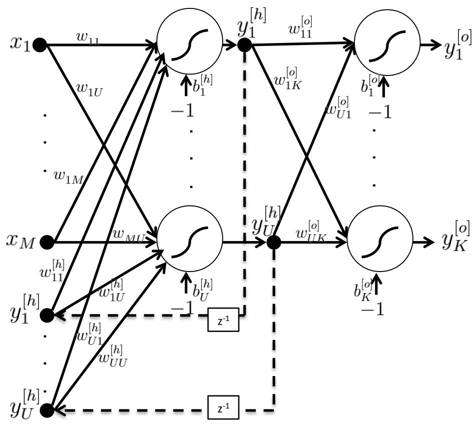

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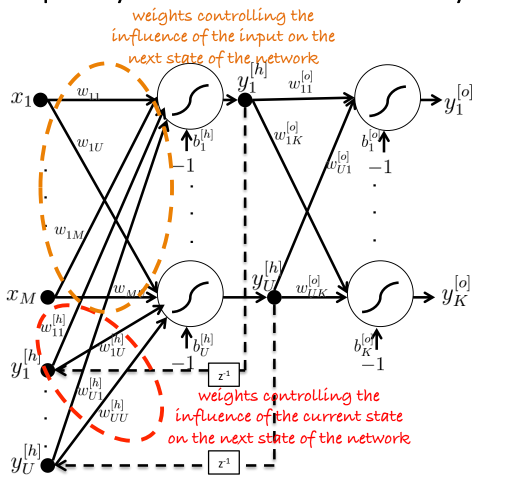


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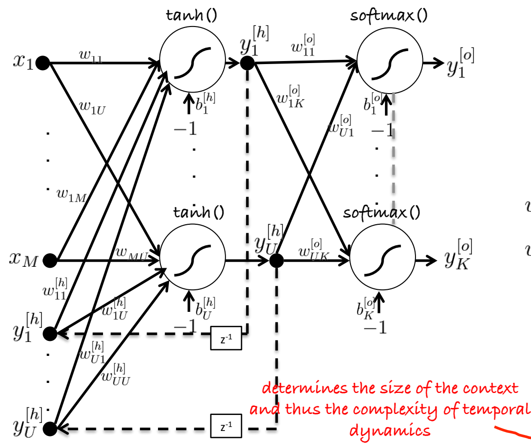


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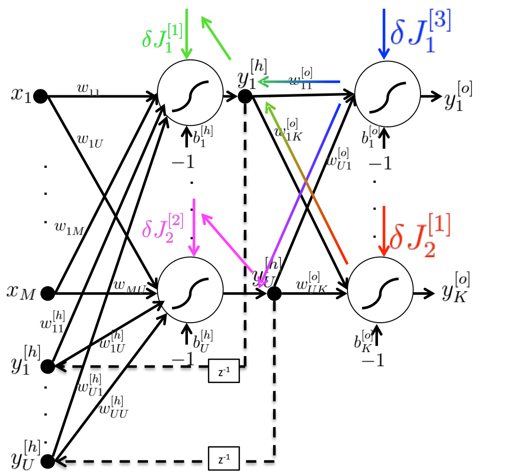
determines the size of the context and thus the complexity of temporal dynamics
$M$ - # of inputs $U$ - # of hidden neurons $K$ - # of outputs



# Simple Recurrent Network (SRN)

- Backpropagation works in SNR.



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$M$ - # of inputs $U$ - # of hidden neurons $K$ - # of outputs



# SNR training

Backpropagation works in SNR:

error "blame" on output neurons $$\delta J^{[o]} = \tilde{y}_j(y_{jt}^{[o]} - 1) + (1 - \tilde{y}_j)y_{jt}^{[o]}$$

error "blame" on hidden neurons $$\delta J_i^{[h]} = \sum_{j=1}^{U} w_{ij}^{o}\left(y_{jt}^{[o]}(1 - y_{jt}^{[o]})\right)\delta J_j^{[o]}$$

updates to hidden-output weights $$\Delta w_{ij}^{[o]} = y_{it}^{[h]}\left(y_{jt}^{[o]}(1 - y_{jt}^{[o]})\right)\delta J_j^{[o]}$$

updates to output layer biases $$\Delta b_j^{[o]} = -\left(y_{jt}^{[o]}(1 - y_{jt}^{[o]})\right)\delta J_j^{[o]}$$

updates to hidden-hidden weights $$\Delta w_{ij}^{[h]} = y_{i(t-1)}^{[h]}\left(1 + y_{jt}^{[h]}\right)(1 - y_{jt}^{[h]})\delta J_j^{[h]}$$

updates to hidden layer biases $$\Delta b_j^{[h]} = -\left(1 + y_{jt}^{[h]}\right)(1 - y_{jt}^{[h]})\delta J_j^{[h]}$$

updates to input-hidden weights $$\Delta w_{ij} = x_i\left(1 + y_{jt}^{[h]}\right)(1 - y_{jt}^{[h]})\delta J_j^{[h]}$$

## SNR training

Backpropagation works in SNR:

error "blame" on output neurons $$\delta J^{[o]} = \tilde{y}_j(y_{jt}^{[o]} - 1) + (1-\tilde{y}_j)\, y_{jt}^{[o]}$$

derivative of softmax with respect to network output

error "blame" on hidden neurons $$\delta J_i^{[h]} = \sum_{j=1}^{U} w_{ij}^{o} \left(y_{jt}^{[o]}(1-y_{jt}^{[o]})\right) \delta J_j^{[o]}$$

output of the hidden neuron i due to previous input (indexed t-1)

derivative of softmax with respect to output neuron's activity

updates to hidden-output weights $$\Delta w_{ij}^{[o]} = y_{it}^{[h]} \left(y_{jt}^{[o]}(1-y_{jt}^{[o]})\right) \delta J_j^{[o]}$$

output of the hidden neuron j due to current input (indexed t)

updates to output layer biases $$\Delta b_j^{[o]} = -\left(y_{jt}^{[o]}(1-y_{jt}^{[o]})\right) \delta J_j^{[o]}$$

updates to hidden-hidden weights $$\Delta w_{ij}^{[h]} = y_{i(t-1)}^{[h]} (1+y_{jt}^{[h]})(1-y_{jt}^{[h]})\, \delta J_j^{[h]}$$

updates to hidden layer biases $$\Delta b_j^{[h]} = -(1+y_{jt}^{[h]})(1-y_{jt}^{[h]})\, \delta J_j^{[h]}$$

derivative of tanh with respect to hidden neuron's activity

updates to input-hidden weights $$\Delta w_{ij} = x_i (1+y_{jt}^{[h]})(1-y_{jt}^{[h]})\, \delta J_j^{[h]}$$



## An example: sentence generation

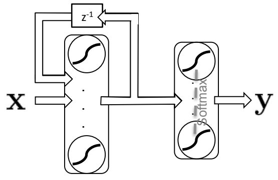
- A dictionary of 6 strings encoded as a 6-dim zero-one vectors (M=6)

$$\mathbf{x} \in \left\{ \begin{array}{cccccc} . & \text{Cars} & \text{Dogs} & \text{are} & \text{furry} & \text{red} \\ \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}, & \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}, & \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}, & \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}, & \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}, & \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} \end{array} \right\}$$

- Output is a 6-value probability distribution over all possible strings (K=6)

$$p(\text{string})$$
$$\mathbf{y} = \begin{bmatrix} y_1^{[o]} \\ y_2^{[o]} \\ y_3^{[o]} \\ y_4^{[o]} \\ y_5^{[o]} \\ y_6^{[o]} \end{bmatrix} \begin{array}{l} P(\text{string} = .) \\ P(\text{string} = \text{Cars}) \\ P(\text{string} = \text{Dogs}) \\ P(\text{string} = \text{are}) \\ P(\text{string} = \text{furry}) \\ P(\text{string} = \text{red}) \end{array}$$

SRN:
- 16 hidden neurons (U=16)
- Tanh activity function in the hidden layer
- Softmax function on the output



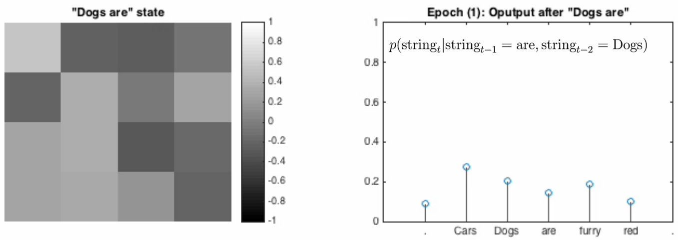

Training:
- Initial state $\mathbf{y}_h = \left[ y_1^{[h]} \ \ldots \ y_U^{[h]} \right]^T$ set to zeros.

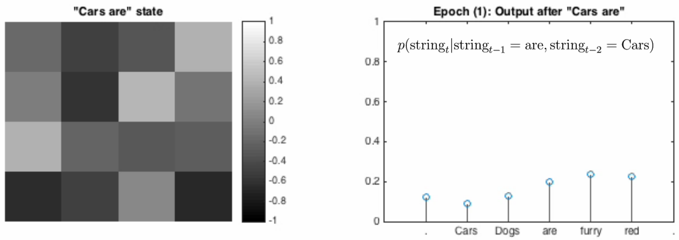
Input sequence 1: $\begin{array}{ccc} \text{Dogs} & \text{are} & \text{furry} \\ \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}, & \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}, & \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix} \end{array}$ Desired output 1: $\begin{array}{ccc} \text{are} & \text{furry} & . \\ \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}, & \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}, & \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix} \end{array}$

Input sequence 2: $\begin{array}{ccc} \text{Cars} & \text{are} & \text{red} \\ \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}, & \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}, & \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} \end{array}$ Desired output 2: $\begin{array}{ccc} \text{are} & \text{red} & . \\ \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}, & \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}, & \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix} \end{array}$



## An example: sentence generation






## An example: more complicated text generation

- A dictionary of 1038 words, zero-one coded into a 1038-dimensional input
- SRN with 64 hidden units and 1038-dimensional output
- Trained with softmax on the first 5,000 words of "Pride and Prejudice" to predict the next word in a sentence.
- Used to generate text by priming the network with a sentence fragment...
- ...then taking the predicted most likely word to follow and feeding it as the next input...repeating this over and over.

## An example: more complicated text generation

- Sample of the training data:

*"It is a truth universally acknowledged, that a single man in possession of a good fortune must be in want of a wife. However little known the feelings or views of such a man may be on his first entering a neighbourhood, this truth is so well fixed in the minds of the surrounding families, that he is considered as the rightful property of some one or other of their daughters.*

- Sentences generated after the starting sequence "**It is**":
  - After 1 epoch, J=4.331e+04

    *"**It is** rode seminaries no respectable enduring one little your does fortune bad everywhere please sensible information your views too"*
  - After 300 epochs, J=2.742e+04

    *"**It is** barefaced truth the I. Bingley. Bingley. Bingley. Bingley. Bingley. Bingley. Bingley. Bingley. Bingley. Bingley. Bingley. Bingley. Bingley. Bingley"*
  - After 3500 epochs, J=7.595e+03

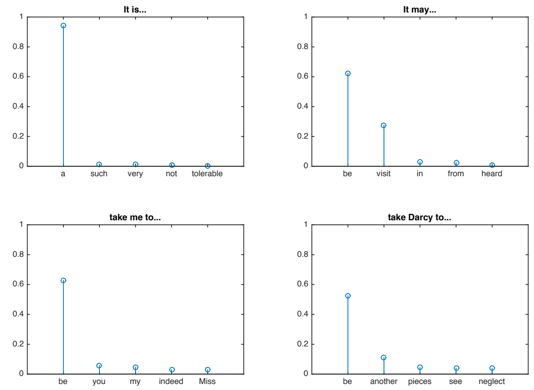
    *"**It is** a truth universally acknowledged, and no two for your husband day, and the two sixth of joy was a lively and it till the room"*
  - After 9900 epochs, J=2.310e+03

    *"**It is** a truth universally acknowledged, that a single man ought to be above his company, and above his father, my dear. I have a high respect for your nerves."*



## An example: more complicated text generation

- Probabilities of the top 5 most likely words after a certain word sequence, as given by the "Pride and Prejudice"-trained SRN after 10000 training epochs:






## Summary

- Feed forward neural networks can do classification and regression
- Softmax function gives network output as a probability distribution
- Recurrent neural networks find temporal patterns in the sequences of input data
  - Simple Recurrent Network can be trained using the backpropagation algorithm

Reading for the lecture: Andrej Karpathy, "**The Unreasonable Effectiveness of Recurrent Neural Networks**", http://karpathy.github.io/2015/05/21/rnn-effectiveness/

Reading for next lecture: AIMA Chapter 18.9