Robust Monopoly Pricing

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1. Introduction
2. Model
3. Maximin Utility
4. Minmax Regret
5. Conclusion

Introduction

1. The optimal mechanisms under the theoretical literature have been widely used with the development of electronic marketplace and Internet trading.
2. How robust of such mechanism with respect to the model uncertainty?
3. In the robust version, the seller only knows that the true distribution is in the neighborhood of a given model distribution.
4. The optimal pricing policy of the seller in the presence of model uncertainty is an instance of decision-making with multiple priors.

Model

- The seller faces a single buyer with valuation $v \in [0, 1]$ for an unit of good.
- The net utility of the buyer is $v - p$.
- The profit for the seller given price p and buyer's valuation v is

$$\pi(p, v) \triangleq p\mathbb{I}_{\{v \geqslant p\}}$$

- If the seller chooses a randomized policy $\Phi \in \Delta\mathbb{R}_+$, the buyer's utility given v is

$$\pi(\Phi, v) \triangleq \int \pi(p, v) d\Phi(p) \, .$$

- The seller's expected profit given price p and demand distribution F is

$$\pi(p, F) \triangleq \int \pi(p, v) dF(v).$$

- With randomizing pricing, the seller's expected profit is

$$\pi(\Phi, F) \triangleq \iint \pi(p, v) dF(v) d\Phi(p).$$

Model

- The optimal random pricing policy given the demand distribution F solves

$$\Phi^*(F) \in \underset{\Phi \in \Delta\mathbb{R}_+}{\arg\max} \pi(\Phi, F).$$

- A well-known result states that for every distribution F, there exists a deterministic price $p^*(F)$ that maximizes profits.
- The uncertainty is represented by a set of possible distributions. Given the model distribution F_0, the ε neighborhood under the Prohorov metric, denoted by $\mathcal{P}_\varepsilon(F_0)$, is:

$$\mathcal{P}_\varepsilon(F_0) \triangleq \{F \mid F(A) \leqslant F_0(A^\varepsilon) + \varepsilon, \forall \text{ measurable } A \subseteq [0,1]\}.$$

where the set A^ε denotes the closed ε neighborhood of any measurable set A:

$$A^\varepsilon \triangleq \left\{x \in [0,1] \mid \underbrace{\min_{y \in A} d(x,y)}_{d(x,A)} \leqslant \varepsilon\right\}.$$

Model

- Example: if $A = [0, a] \subset [0, 1]$ and $d(x, y) = |x - y|$, so that

$$A^{\varepsilon} = \left\{ x \in [0,1] \mid d(x, A) \leqslant \varepsilon \right\} = [0, a + \varepsilon].$$

- The Prohorov metric is a metric for weak convergence of probability measures.

Definition (Topology Definition)

Let (M, d) be a metric space with its Borel sigma algebra $\mathcal{B}(M)$. Let $\mathcal{P}(M)$ denote the collection of all probability measures on the measurable space $(M, \mathcal{B}(M))$. For a subset $A \subseteq M$, define the ε-neighborhood of A by

$$A^{\varepsilon} := \{ p \in M \mid \exists q \in A, d(p, q) < \varepsilon \} = \bigcup_{p \in A} B_{\varepsilon}(p)$$

where $B_{\varepsilon}(p)$ is the open ball of radius ε centered at p. The Lévy-Prokhorov metric $\pi : \mathcal{P}(M)^2 \to [0, +\infty)$ is defined by setting the distance between two probability measures μ and ν to be $\pi(\mu, \nu) := \inf \{ \varepsilon > 0 \mid \mu(A) \leq \nu(A^{\varepsilon}) + \varepsilon$ and $\nu(A) \leq \mu(A^{\varepsilon}) + \varepsilon$ for all $A \in \mathcal{B}(M) \}$

Maximin Utility

- Under maximin utility, the seller maximizes the minimum utility

$$\Phi_m \in \arg\max_{\Phi \in \Delta\mathbb{R}_+} \min_{F \in \mathcal{P}_\varepsilon(F_0)} \pi(\Phi, F).$$

- Accordingly, we say that Φ_m attains maximin utility. We refer to F_m as a **least favorable demand** (for maximin utility) if

$$F_m \in \arg\min_{F \in \mathcal{P}_\varepsilon(F_0)} \max_{\Phi \in \Delta\mathbb{R}_+} \pi(\Phi, F).$$

- Occasionally, it is useful to explicitly state the dependence of the optimal policies on the size of the neighborhood ϵ, by $\Phi_{m,\varepsilon}$ and $F_{m,\varepsilon}$.

Minimax Regret

- The regret of the monopolist at a given price p and valuation v is:

$$r(p,v) \triangleq v - p\mathbb{I}_{\{v \geqslant p\}} = v - \pi(p,v)$$

  where v is the maximum profit the monopolist could make if she were to know the value v before setting her price and the profit she makes without this information.

- The regret is non-negative and can only vanish when $p = v$. Otherwise, if $v > p$, then $r(p,v) = v - p * 1 > 0$, or $v < p$, then $r(p,v) = v - p * 0 > 0$.
- The expected regret of a random pricing policy Φ given a demand distribution F is:

$$r(\Phi, F) \triangleq \iint r(p,v)d\Phi(p)dF(v) = \int vdF(v) - \int \pi(p,F)d\Phi(p).$$

Minimax Regret

- The pricing policy Φ_r attains minimax regret if it minimizes the maximum regret over all distributions F in the neighborhood of a model distribution F_0:

$$\Phi_r \in \arg\min_{\Phi \in \Delta\mathbb{R}_+} \max_{F \in \mathcal{P}_\varepsilon(F_0)} r(\Phi, F).$$

- F_r is called a least favorable demand if

$$F_r \in \arg\min_{F \in \mathcal{P}_\varepsilon(F_0)} \max_{\Phi \in \Delta\mathbb{R}_+} r(\Phi, F) = \arg\max_{F \in \mathcal{P}_\varepsilon(F_0)} \left(\int v dF(v) - \max_{\Phi} \pi(\Phi, F) \right).$$

- In particular, the decision maker does not need the information to become available ex post to evaluate his expected regret.

Robust Policy

Definition (Robust Pricing Policy)

A family of pricing policies $\{\Phi_\varepsilon\}_{\varepsilon>0}$ is called robust if, for each $\gamma > 0$, there is $\varepsilon > 0$ such that $F \in \mathcal{P}_\varepsilon(F_0) \Rightarrow \pi(\Phi^*(F), F) - \pi(\Phi_\varepsilon, F) < \gamma$.

- We can check $\Phi^*(F_0)$ is not a robust pricing policy.
- Consider a Dirac distribution which puts probability one on valuation v, i.e., $F = \delta_{\{v\}}$.
- The optimal monopoly price $\Phi^*(F) = v$. If the $F_0 = \delta_{\{v-\epsilon\}}$, then the revenue is 0. The difference is $v \nrightarrow 0$.

Maximin Utility

- The pricing rule that attains maximin utility is the equilibrium strategy in a game between the seller and adversarial nature.
- The seller chooses a probabilistic pricing policy, a distribution $\Phi \in \Delta\mathbb{R}_+$, and nature chooses a demand distribution $F \in \mathcal{P}_\varepsilon(F_0)$.
- In this game, the payoff of the seller is the expected profit while the payoff of nature is the negative of the expected profit.
- A Nash equilibrium of this zero-sum game is a solution (Φ_m, F_m) to the saddle point problem:

$$\underbrace{\pi(\Phi, F_m) \leqslant}_{\text{seller maximizes profit}} \quad \pi(\Phi_m, F_m) \quad \underbrace{\leqslant \pi(\Phi_m, F)}_{\text{nature minimizes profit}} \quad ,$$

for all $\Phi \in \Delta\mathbb{R}_+, \forall F \in \mathcal{P}_\varepsilon(F_0)$.

Maximin Utility

- For a given price p, the expected profit of the seller is

$$\pi(p, F) = \int \pi(p, v) dF(v) = p(1 - F(p)).$$

- The profit minimization demand should minimize $(1 - F(p))$ for all p within the neighborhood $\mathcal{P}_{\varepsilon}(F_0)$.
- This defines the order: first-order-stochastic dominance relationship. The profit minimizing demand is the **smallest** elements of $\mathcal{P}_{\varepsilon}(F_0)$ in FOSD sense.
- This results in the distribution $F_{m,\varepsilon}$ given by:

$$F_{m,\varepsilon}(v) \triangleq \min\left\{F_0(v + \varepsilon) + \varepsilon, 1\right\}$$

- Recall the definition of Prohorov metric, we have

$$F([0, a]) \leq F_0([0, a + \epsilon]) + \epsilon$$

Maximin Utility

- Given that the profit minimizing demand $F_{m,\varepsilon}$ does not depend on the offered prices, the monopolist acts as if the demand is given by $F_{m,\varepsilon}$.
- In consequence, the seller maximizes profits at $F_{m,\varepsilon}$ by choosing a **deterministic** price $p_{m,\varepsilon}$ where $p_{m,\varepsilon} \triangleq p^* (F_{m,\varepsilon})$.

Maximin Utility

Proposition (1. Maximin Utility)

For every $\varepsilon > 0$, there exists a pair $(p_{m,\varepsilon}, F_{m,\varepsilon})$, such that $p_{m,\varepsilon} \in [0, 1]$ attains maximin utility and $F_{m,\varepsilon}$ is a least favorable demand.

Maximin Utility

Proposition (2. Pricing under Maximin Utility)

The price $p_{m,\varepsilon}$ responds to an increase in uncertainty at $\varepsilon = 0$ by:

$$\left.\frac{dp_{m,\varepsilon}}{d\varepsilon}\right|_{\varepsilon=0} = -1 + \frac{1 - f_0(p_0)}{\partial \pi^2 (p_0, F_0) / \partial p^2} < -\frac{1}{2}.$$

Maximin Utility

Proposition (3. Robustness)

The family of pricing policies $\{p_{m,\varepsilon}\}_{\varepsilon>0}$ is a robust family of pricing policies.

Minimax Regret

- We consider the minimax regret problem of the seller, where a (probabilistic) pricing policy Φ_r and a least favorable demand F_r are the equilibrium policies of a zero-sum game.
- In this zero-sum game, the payoff of the seller is the negative of the regret while the payoff to nature is regret itself.
- A Nash equilibrium (Φ_r, F_r) is a solution to the saddle point problem:

$$\underbrace{r\left(\Phi_r, F\right) \leqslant}_{\text{nature maximizes regret}} \quad (\Phi_r, F_r) \quad \underbrace{\leqslant r\left(\Phi, F_r\right)}_{\text{seller minimizes regret}} \quad , \quad \forall \Phi \in \Delta\mathbb{R}_+, \forall F \in \mathcal{P}_\varepsilon\left(F_0\right).$$

Minimax Regret

- The deterministic policy can not resolve the follow confliction:
  - If setting a low price, nature can cause regret with a distribution which puts substantial probability on high valuation buyers.
  - If setting a high price, nature can cause regret with a distribution which puts substantial probability at valuations just below the offered price.
- To get more intuitions, we can consider the seller's expected regret given a single price p:

$$r(p,F) = \int v dF(v) - \int \pi(p,v) dF(v) = \underbrace{\int v dF(v)}_{\text{mean valuation}} - \underbrace{p(1-F(p))}_{\text{upper cumulative prob}}$$

- For nature can maximize the regret by increasing mean valuation or lower upper cumulative prob (choose a smaller F in FOSD). However, two actions are conflict!

Minimax Regret

Proposition (4. Existence of Minimax Regret)

A solution (Φ_r, F_r) to the saddlepoint condition exists.

Corollary

Suppose that $G = (X_i, u_i)_{i=1}^N$ is a compact, Hausdorff game. Then G possesses a mixed strategy Nash equilibrium if its mixed extension, $\bar{G}$, is better-reply secure. Moreover, $\bar{G}$ is better-reply secure if it is both reciprocally upper semicontinuous and payoff secure.

Definition

A game $G = (X_i, u_i)_{i=1}^N$ is payoff secure if for every $x \in X$ and every $\epsilon > 0$, each player i can secure a payoff of $u_i(x) - \epsilon$ at x.

Minimax Regret

- For the monopolist: payoff secure requires that for each (F_r, Φ_r) with $F_r \in \mathcal{P}_\epsilon(F_0)$ and for every $\delta > 0$, there exists $\gamma > 0$ and $\bar{\Phi}$ such that $F \in \mathcal{P}_\gamma(F_r)$ implies

$$r(\bar{\Phi}, F) \leq r(\Phi_r, F_r) + \delta.$$

- For the seller: we need to show that for each (Φ_r, F_r), with $F_r \in \mathcal{P}_\epsilon(F_r)$ and every $\delta > 0$, there exists $\gamma > 0$, and $\bar{F} \in \mathcal{P}_\epsilon(F_0)$, such that $\Phi \in \mathcal{P}_\gamma(\Phi_r)$ implies

$$r(\Phi, \bar{F}) \geq r(\Phi_r, F_r) - \delta.$$

Minimax Regret

Proposition (5. Minimax regret)

1. *Given* $\delta > 0$*, for every* ε *sufficiently small, there exist* a, b *and* c *with* $0 < a < b < c < 1$ *and* $p_0 - \delta < a < p_0 < c < p_0 + \delta$ *such that a minimax regret probabilistic price* Φ_r *is given by:*

$$\Phi_r(p) = \begin{cases} 0 & \text{if } 0 \leqslant p < a \\ \ln \frac{p}{a} & \text{if } a \leqslant p < b \\ 1 - \ln \frac{c}{p} & \text{if } b \leqslant p \leqslant c \\ 1 & \text{if } c < p \leqslant 1 \end{cases}.$$

2. *The boundary points* a, b *and* c *respond to an increase in uncertainty at* $\varepsilon = 0$ *:*
    1. $\lim_{\varepsilon \to 0} a'(0) = -\infty$;
    2. $\lim_{\varepsilon \to 0} b'(0)$ *is finite;*
    3. $\lim_{\varepsilon \to 0} c'(0) = \infty$;

Minimax Regret

Corollary (1. Comparative Statics with Minimax Regret)

1. *The expected price $\mathbb{E}\left[p_{r,\varepsilon}\right]$ responds to an increase in uncertainty at $\varepsilon = 0$ by:*

$$\frac{d}{d\varepsilon}\mathbb{E}\left[p_{r,\varepsilon}\right]\Big|_{\varepsilon=0} = \begin{cases} -1 - \frac{f_0(p_0)+1}{\partial\pi^2(p_0,F_0)/\partial p^2} > -1 & \text{if } p_0 \leqslant \frac{1}{2} \\ -1 - \frac{f_0(p_0)-1}{\partial\pi^2(p_0,F_0)/\partial p^2} < -\frac{1}{2} & \text{if } p_0 > \frac{1}{2} \end{cases}$$

2. *If ε is sufficiently small, then for any $v \in (a, c)\backslash b$,*

$$\frac{d}{d\varepsilon}\mathbb{E}\left[p_{r,\varepsilon} \mid p_{r,\varepsilon} \leqslant v\right] < 0.$$

Minimax Regret

Proposition (5. Robustness)

If $\{\Phi_{r,\varepsilon}\}_{\varepsilon>0}$ attains minimax regret at F_0 for all sufficiently small ε, then $\{\Phi_{r,\varepsilon}\}_{\varepsilon>0}$ is a robust family of pricing policies.

Discussion

- Axioms and behavioral implications:
  - Two criteria: the maximin utility and minimax regret criteria
  - The similarity and difference between maximin utility and minimax regret.
- The choice of metric (and neighborhoods):
  - different metrics: such as the Levy metric or the bounded Lipschitz metric which also metrize the weak topology.
  - the comparative static results near $\epsilon = 0$ are unaffected by the specific notion for the metric.
- Beyond small neighborhoods: how about large neighbor?
- Beyond monopoly pricing: general mechanism design problem.

Thanks!