A Reputational Theory of Firm Dynamics

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Introduction

- This work studies the firm's investment and exit decisions over life cycles.
- Applications: restaurant, surgeon, and investor.

Model

- A long-lived firm faces a mass of short-lived consumers.
- Time is continuous: $t \in [0, \infty)$.
- The firm chooses the investment levels: $A_t \in [0, \bar{a}]$ ($\bar{a} < 1$).
- The firm also chooses an exiting time $T \in [0, \infty]$.

Technology

- The firm's product quality is $\theta_t \in \{L, H\}$, where $L = 0$ and $H = 1$.
- Initial quality θ_0 is exogenous; subsequent quality depends on investment and technology shocks.
- Specifically, shocks are generated according to a Poisson process with arrival rate $\lambda > 0$.
- Quality θ_t is constant between shocks and determined by the firm's investment at the most recent technology shock $s \leq t$; i.e., $\theta_t = \theta_s$ and $\Pr(\theta_s = H) = A_s$.
- This captures the idea that quality is a lagged function of past investments.

Information

- Consumers observe neither quality nor investment but learn about quality through public breakthroughs.
- Given quality θ, breakthroughs are generated according to a Poisson process with arrival rate $\mu\theta$; that is, breakthroughs only occur when $\theta = H$.
- We write h^t for histories of breakthrough arrival times before time t, and h for infinite histories.
- The firm does not observe product quality, either, but does recall its past actions.
- Its investment plan $\{A_t\}_{t\geq 0}$ and exit time T are thus progressively measurable with respect to the filtration induced by public histories h^t.
- In turn, the investments $A := \{A_t\}_{t\geq 0}$ control the distribution of quality $\{\theta_t\}_{t\geq 0}$ and thereby the histories of breakthroughs h; we write E^A for expectations under this measure and call $Z_t = E^A[\theta_t | h^t]$ the firm's **self-esteem** at time $t < T$. This reflects the firm's belief in its own quality given its past investment and the history of breakthroughs.
- We write (pure) market beliefs over investment and exit as $\tilde{A} = \{\tilde{A}_t\}_{t\geq 0}$ and $\tilde{T}$. The firm's **reputation** is given by $X_t := E^{\tilde{A}}[\theta_t | h^t]$.

Payoffs

- The firm has flow profits $\pi(X_t) - c(A_t)$ and discount rate $r > 0$. The firm's income $\pi(\cdot)$ is smooth and strictly increasing with boundaries $\pi(0) < 0$ and $\pi(1) > 0$;
- The firm's investment cost $c(\cdot)$ is smooth, strictly increasing, and convex, with $c(0) = 0$
- Given the firm's strategy (A, T) and market beliefs $(\tilde{A}, \tilde{T})$, its expected present value equals

$$E^A \left[\int_{t=0}^{T} e^{-rt} \left(\pi(X_t) - c(A_t) \right) dt \right].$$

- To highlight the distinct roles of market beliefs $\tilde{A}$ and actual investment A, note that $\tilde{A}$ determines the firm's reputation $X_t = E^{\tilde{A}} [\theta_t | h^t]$ for a given history h^t, while A determines the distribution over histories h^t.

Dynamic Programming Formulation

- Truncating the integral at the first breakthrough (which arrives at rate μz_t), the firm's continuation value at time t is

$$V(t, z_t) = \sup_{a,\tau} \int_{s=t}^{\tau} e^{-\int_t^s (r+\mu z_u)du} \left[\pi(x_s) - c(a_s) + \mu z_s V(0,1)\right] ds$$

- We write optimal strategies as (a^*, τ^*) and the associated self-esteem as $z^* = \{z_t^*\}$.

Reputational Dynamics

- Self-esteem is governed by the firm's investment and the history of breakthroughs. At a breakthrough, self-esteem jumps to one. Absent a breakthrough, self-esteem is governed by $\dot{z}_t = g\left(a_t, z_t\right)$, where the drift is given by

$$g(a, z) = \lambda(a - z) - \mu z(1 - z).$$

- The first term derives from the technology process: with probability λdt a technology shock hits in $[t, t + dt)$, previous quality becomes obsolete, and the current quality is determined by the firm's investment.
- This term is positive if investment exceeds the firm's self-esteem, $a > z$, and negative otherwise.
- The second term derives from the absence of breakthroughs and is always negative.
- Analogously, reputation is governed by believed investment $\tilde{a}$ and the history of breakthroughs, jumping to 1 at a breakthrough and in its absence following $\dot{x}_t = g\left(\tilde{a}_t, x_t\right)$.

Condition for Exits

- We want to guarantee that in the absence of a breakthrough, the firm eventually exits.
- Assume
$$\pi(z^\dagger) + \mu z^\dagger \pi(1)/r < 0,$$
where $z^\dagger \in (0,1)$ is the unique level of self-esteem where reputational drift vanishes under maximal investment, $g\left(\bar{a}, z^\dagger\right) = 0$.
- So defined, drift $g(a,z)$ is strictly **negative** on $\left[z^\dagger, 1\right]$ for any beliefs $(\tilde{a}, \tilde{\tau})$, and (absent a breakthrough) reputation and self-esteem eventually drop below $z^\dagger$. At that point, the condition ensures that the integrand is negative and the firm exits, where $\pi(1)/r$ serves as an upper bound for $V(0,1)$.

The Firm's Problem

Lemma (1)

For any pure beliefs $(\tilde{a}, \tilde{\tau})$,

1. reputation $\{x_t\}$ is continuous and strictly decreasing,
2. an optimal strategy (a^*, τ^*) exists and $\tau^* < \infty$, and
3. firm value $V(t, z)$ strictly decreases in t and strictly increases in z.

- The optimal strategy is characterized by the firm's Hamilton Jacobi Bellman (HJB) equation

$$rV(t,z) = \max_a [\pi(x_t) - c(a) + g(a,z)V_z(t,z) \\ + V_t(t,z) + \mu z(V(0,1) - V(t,z))]^+,$$

where $y^+ := \max\{y, 0\}$, capturing the firm's ability to exit for a continuation value of 0.

- The optimal investment balances

$$\lambda V_z(t, z_t^*) = c'(a_t^*).$$

with $a_t^* = 0$ if $\lambda V_z(t, z_t^*) < c'(0)$ and $a_t^* = \bar{a}$ if $\lambda V_z(t, z_t^*) > c'(\bar{a})$.

Marginal Value of Self-Esteem

Lemma (2)

If $V_z\left(t, z_t^*\right)$ exists, it equals

$$\Gamma(t) := \int_t^{\tau *} e^{-\int_t^s \left(r+\lambda+\mu\left(1-z_u^*\right)\right) du} \mu \left[V(0,1) - V\left(s, z_s^*\right)\right] ds.$$

More generally, $V_{z-}\left(t, z_t^*\right) \leq \Gamma(t) \leq V_{z+}\left(t, z_t^*\right)$.

- Fix time t, self-esteem z_t, firm strategy (a, τ) (not necessarily optimal), write $z = \{z_s\}_{s \geq t}$ for future self-esteem, and let

$$\Pi\left(t, z_t\right) = \int_{s=t}^{\tau} e^{-\int_t^s (r+\mu z_u) du} \left[\pi\left(x_s\right) - c\left(a_s\right) + \mu z_s \Pi(0,1)\right] ds$$

  be the firm's continuation value, where the integral of the cash flows is truncated at the first breakthrough.

- We will show that $\Pi(t, z)$ is differentiable in z with derivative

$$\Pi_z\left(t, z_t\right) = \int_{s=t}^{\tau} e^{-\int_t^s (r+\lambda+\mu(1-z_u)) du} \mu \left[\Pi(0,1) - \Pi\left(s, z_s\right)\right] ds.$$

Marginal Value of Self-Esteem

Claim (2)

For any times $s > t$ and fixed investment a, time-s self-esteem z_s is differentiable in time-t self-esteem z_t. The derivative is

$$\frac{dz_s}{dz_t} = \exp\left(-\int_{u=t}^{s} (\lambda + \mu(1 - 2z_u))\, du\right)$$

- Taking the derivative with respect to z at $z = z_t$ and applying Claim 2 , we get

$$\Pi_z(t, z_t) = \int_{s=t}^{\tau} e^{-r(s-t)} \frac{dz_s}{dz_t} [\mu(\Pi(0,1) - \Pi(s, z_s)) - \mu z_s \Pi_z(s, z_s)]\, ds$$
$$= \int_{s=t}^{\tau} e^{-\int_t^s (r+\lambda+\mu(1-2z_u))du} [\mu(\Pi(0,1) - \Pi(s, z_s)) - \mu z_s \Pi_z(s, z_s)]\, ds.$$

- We can solve the integral equation with solution $\Pi_z(t, z)$.

Marginal Value of Self-Esteem

- The optimal strategy is characterized by the firm's Hamilton Jacobi Bellman (HJB) equation

$$rV(t,z) = \max_a [\pi(x_t) - c(a) + g(a,z)V_z(t,z) + V_t(t,z) + \mu z(V(0,1) - V(t,z))]^+ ,$$

where $y^+ := \max\{y, 0\}$, capturing the firm's ability to exit for a continuation value of 0.

- The optimal investment balances

$$\lambda V_z(t, z_t^*) = c'(a_t^*) .$$

with $a_t^* = 0$ if $\lambda V_z(t, z_t^*) < c'(0)$ and $a_t^* = \bar{a}$ if $\lambda V_z(t, z_t^*) > c'(\bar{a})$.

Optimal Investment and Exit Time

Lemma (3)

Any optimal strategy (a^*, τ^*) satisfies

$$\lambda\Gamma(t) = c'\left(a_t^*\right)$$

with $a_t^* = 0$ if $\lambda\Gamma(t) < c'(0)$ and $a_t^* = \bar{a}$ if $\lambda\Gamma(t) > c'(\bar{a})$ for almost all t.

Theorem (1)

For any pure beliefs $(\tilde{a}, \tilde{\tau})$, optimal investment $\{a_t^*\}$ is single-peaked in the time since a breakthrough t at the exit threshold, $a_{\tau^*}^* = 0$. The optimal exit time τ^* satisfies

$$\pi\left(x_{\tau^*}\right) + \mu z_{\tau^*} V(0,1) = 0.$$

Optimal Investment and Exit Time

- We wish to show that $\Gamma(t)$ is single-peaked in t with boundary conditions

$$\Gamma(0) > 0, \quad \dot{\Gamma}(0) > 0, \quad \Gamma\left(\tau^*\right) = 0.$$

- Taking the derivative of investment incentives and setting $\rho(t) := r + \lambda + \mu\left(1 - z_t^*\right)$ yields the adjoint equation

$$\dot{\Gamma}(t) = \rho(t)\Gamma(t) - \mu\left(V(0,1) - V\left(t, z_t^*\right)\right).$$

- Now assume that $\rho(t)$ and $V\left(t, z_t^*\right)$ are differentiable. Then $\dot{\rho}(t) = -\mu \dot{z}_t^*$ and $(d/dt)V\left(t, z_t^*\right) = \dot{z}_t^* \Gamma(t) + V_t\left(t, z_t^*\right)$.
- Differentiating $\dot{\Gamma}(t)$

$$\ddot{\Gamma}(t) = \rho(t)\dot{\Gamma}(t) + \dot{\rho}(t)\Gamma(t) + \mu \frac{d}{dt} V\left(t, z_t^*\right) = \rho(t)\dot{\Gamma}(t) + \mu V_t\left(t, z_t^*\right).$$

- Since $V_t\left(t, z_t^*\right) < 0, \dot{\Gamma}(t) = 0$ implies $\ddot{\Gamma}(t) < 0$; hence $\Gamma(t)$ is single-peaked.

Equilibrium

- To illustrate the equilibrium dynamic, suppose cost is linear: $c(a) = ca$ for $a \in [0, \bar{a}]$.
- By Lemma 3, optimal investment is **bang-bang** with $a_t^* = \bar{a}$ when $\lambda\Gamma(t) > c$ and $a_t^* = 0$ when $\lambda\Gamma(t) < c$.
- In any pure strategy equilibrium, Theorem 1 tells us that investment incentives $\Gamma(t)$ are single-peaked, so there are two cases.
- When costs are low, $\Gamma(0) > c = \Gamma(t_1)$ for some t_1, the firm chooses $a = \bar{a}$ on $[0, t_1]$ and $a = 0$ on $[t_1, \tau]$.
- We call this a "probationary equilibrium" since the market assumes a firm invests for a fixed period of time after each breakthrough but then grows suspicious if no breakthrough is forthcoming.
- The firm's reputation initially drifts down slowly, as the favorable beliefs about investment offset the bad news conveyed by the lack of breakthroughs. After enough time without a breakthrough, market beliefs turn against the firm and the perceived disinvestment hastens the firm's decline.

Equilibrium

- When costs are high, $\Gamma(0) < c = \Gamma(t_0) = \Gamma(t_1)$ for some $t_0 < t_1$, the firm chooses $a = 0$ on $[0, t_0]$, $a = \bar{a}$ on $[t_0, t_1]$, and $a = 0$ on $[t_1, \tau]$.
- Here, the firm's initial incentives $\Gamma(0)$ are insufficient to motivate effort. After a breakthrough, the firm rests on its laurels because it has little to gain from an additional breakthrough.
- As its reputation and self-esteem drop, it starts investing and works hard for its survival, but eventually gives up and shirks before exiting the market.

Theorem (2)

An equilibrium exists.

Steady-State Distribution

- Consider a continuum of price-taking firms, such as that studied above, and assume that new firms enter the market continuously at rate ϕ with reputation drawn with density h on $[x^e, 1]$, where $x^e = x_{\tau^*}$ is the reputation at the exit time.
- Writing $g(x) = g\left(a^*(x), x\right)$ for equilibrium drift, the density of firms $f(x,t)$ with reputation $x \in [x^e, 1]$ at time t is governed by the Kolmogorov forward equation

$$\frac{\partial}{\partial t} f(x,t) = \underbrace{-\frac{\partial}{\partial x}[f(x,t)g(x)]}_{\text{drift}} - \underbrace{\mu x f(x,t)}_{\text{break through}} + \underbrace{\phi h(x)}_{\text{entering}}.$$

- In steady state, the density of firms is constant, $f(x,t) \equiv f(x)$, and the measure of entering firms exactly compensates for the measure of exiting firms whose reputation drifts into the exit threshold, $\phi = -g\left(x^e\right) f\left(x^e\right)$. The total measure of firms depends on the choice of ϕ, which we normalize so that $\int_{x_e}^1 f(x)dx = 1$. Setting the LHS to 0 and rearranging, we get

$$f'(x) = \underbrace{\frac{g'(x)}{-g(x)} f(x)}_{\text{drift } (-1+)} + \underbrace{\frac{\mu x}{-g(x)} f(x)}_{\text{jumps } (+)} - \underbrace{\frac{\phi h(x)}{-g(x)}}_{\text{entry } (-)}.$$

Consumers Observe Firm's Investment

- Since the market has the same information as the firm, reputation and self-esteem coincide, $x_t = z_t$; we can thus write firm value as a function of self-esteem alone.
- We truncate the firm's flow payoffs at a breakthrough, yielding

$$\hat{V}(z_t) = \sup_{a,\tau} \int_t^{\tau} e^{-\int_t^s (r+\mu z_u)du} \left[\pi(z_s) - c(a_s) + \mu z_s \hat{V}(1)\right] ds.$$

- The value function is strictly convex with derivative

$$\hat{V}'(z_t) = \hat{\Gamma}(t) := \int_t^{\hat{\tau}} e^{-\int_t^s (r+\lambda+\mu(1-\hat{z}_u))du} \left[\pi'(z_s) + \mu\left(\hat{V}(1) - \hat{V}(\hat{z}_s)\right)\right] ds.$$

- In any optimal strategy, investment satisfies

$$\lambda\hat{\Gamma}(t) = c'(\hat{a}_t)$$

with $\hat{a}_t = 0$ if $\lambda\hat{\Gamma}(t) < c'(0)$ and $\hat{a}_t = \bar{a}$ if $\lambda\hat{\Gamma}(t) > c'(\bar{a})$.

Consumers Observe Firm's Investment

Theorem (3)

Investment $\hat{a}_t$ decreases in the time since a breakthrough; at the exit threshold, $\hat{a}_{\hat{\tau}} = 0$. Moreover, the optimal exit time satisfies

$$\pi\left(\hat{z}_{\hat{\tau}}\right) + \mu \hat{z}_{\hat{\tau}} \hat{V}(1) = 0.$$

- Given assumption on exiting, drift $g\left(a_t, z_t\right)$ is boundedly negative on $\left[z^{\dagger}, 1\right]$ and the firm exits before its reputation hits $z^{\dagger}$. Since z_t decreases and the value function is strictly convex, $\hat{\Gamma}(t) = \hat{V}'\left(z_t\right)$ strictly decreases in t.

Firm Knows Its Own Quality

- We focus on strategies that depend on the time since the last breakthrough, t, and on current quality.
- Formally, such a strategy consists of an investment plan $a^\theta = \{a_t^\theta\}$ and an exit time τ^θ for $\theta = L, H$.
- To analyze firm value for an arbitrary trajectory of reputation $\{x_t\}$, we truncate its cash flow expansion at the first technology shock, obtaining

$$\check{V}(t,\theta) = \sup_{a^\theta,\tau^\theta} \int_t^{\tau^\theta} e^{-(r+\lambda)(s-t)} \left[\pi(x_s) - c\left(a_s^\theta\right) + \lambda\left(a_s^\theta \check{V}(s,H) + \left(1-a_s^\theta\right)\check{V}(s,L)\right)\right. \\ \left. +\mu\theta(\check{V}(0,H) - \check{V}(s,H))\right]ds,$$

  where the last term captures the value of breakthroughs with present value $\check{V}(0,1) - \check{V}(s,1)$ and arrival rate $\mu\theta$.
- Writing $\check{\Gamma}(t) = \check{V}(t,H) - \check{V}(t,L)$ for the value of quality, optimal investment $\check{a}_t^\theta$ is thus characterized by the first-order condition

$$\lambda\check{\Gamma}(t) = c'\left(\check{a}_t^\theta\right),$$

  with $\check{a}_t^\theta = 0$ if $\lambda\check{\Gamma}(t) < c'(0)$ and $\check{a}_t^\theta = \bar{a}$ if $\lambda\check{\Gamma}(t) > c'(\bar{a})$. Importantly, optimal investment $\check{a}_t^\theta$ is independent of the firm's quality, allowing us to drop the superscript θ.

Firm Knows Its Own Quality

Theorem (4)

There exists an equilibrium with continuous and weakly decreasing reputation $\{x_t\}$ and pure investment $\{\check{a}_t\}$. In any such equilibrium, the exit time of the low-quality firm $\check{\tau}^L$ has support $[\underline{\tau}, \infty)$ for some $\underline{\tau} > 0$. Reputation and firm value are constant for $t \in [\underline{\tau}, \infty)$ and satisfy

$$\pi(x_t) + \max_a [a\lambda \check{V}(t, H) - c(a)] = 0.$$

The high-quality firm never exits; i.e., $\check{\tau}^H = \infty$. Investment $\check{a}_t$ increases over $[0, \underline{\tau}]$ and remains constant thereafter.

- Subtracting the value of high-value and low-value firms, we obtain the following expression for the equilibrium value of quality:

$$\check{\Gamma}(t) = \check{V}(t, H) - \check{V}(t, L) = \int_t^\infty e^{-(r+\lambda)(s-t)} \mu [\check{V}(0, H) - \check{V}(s, H)] ds.$$

- As $s \in [0, \underline{\tau}]$ rises, the firm's value $\check{V}(s, 1)$ falls and reputational dividends $\check{V}(0, 1) - \check{V}(s, 1)$ grow. Hence, an increase in t leads to an increase in the value of quality and in investment via the first-order condition.

Conclusion

- We have proposed a model in which firms make optimal investment and exit decisions while the market learns about the quality of the firm's product.
- We characterize investment incentives and show they are single-peaked in the firm's reputation. This yields predictions about the distribution of firms' reputations and the turnover rate.
- The model can be calibrated using real-data and can be extended to deal with competition, Brownian signals of quality.

Thanks!