Monotone Additive Statistics

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Introduction

- How should a random quantity be summarized by a single number?
- We study the **statistics** that are monotone with respect to first-order stochastic dominance, and additive for sums of independent random variables.
- Our main result establishes that every monotone additive statistic Φ is of the form

$$\Phi(X) = \int K_a(X) \mathrm{d}\mu(a)$$

for some probability measure μ and $K_a(X) = \frac{1}{a} \log \mathbb{E}\left[\mathrm{e}^{aX}\right]$.

Monotone additive statistics

- Let L^∞ denotes the collection of bounded real random variables, defined over a nonatomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ denotes the extended real numbers.

Definition (Statistic)

A map $\Phi : L^\infty \to \mathbb{R}$ is a statistic if it satisfies (i) $\Phi(X) = \Phi(Y)$ whenever $X, Y \in L^\infty$ have the same distribution, and (ii) $\Phi(c) = c$ for every $c \in \mathbb{R}$; that is, Φ assigns c to the constant random variable c.

Definition (Monotone and Additive)

- Φ is additive if $\Phi(X + Y) = \Phi(X) + \Phi(Y)$ whenever X and Y are independent, and
- Φ is monotone if $X \geq_1 Y$ implies $\Phi(X) \geq \Phi(Y)$, where $\geq_1$ denotes first-order stochastic dominance.

Monotone additive statistics

- Given $X \in L^\infty$ and $a \in \overline{\mathbb{R}} \backslash \{0, \pm\infty\}$, we consider the statistic

$$K_a(X) = \frac{1}{a} \log \mathbb{E}\left[e^{aX}\right].$$

- $K_a(X)$ satisfies additive and monotone. If X, Y are independent, we have

$$\mathbb{E}\left[e^{a(X+Y)}\right] = \mathbb{E}\left[e^{aX}\right] \cdot \mathbb{E}\left[e^{aY}\right].$$

- Let $K_0(X), K_\infty(X), K_{-\infty}(X)$ to be the expectation, the essential maximum, and the essential minimum of X.
- Let L^∞_+ denotes the set of bounded non-negative random variables.
- Let $L^\infty_{\mathbb{N}}$ represents the set of bounded non-negative integer-valued random variables.
- Let L_M denotes the set of random variables X for which $K_a(X)$, as defined in the same way, is finite for all $a \in \mathbb{R}$.
- The domain L_M contains those unbounded random variables whose distributions have sub-exponential tails, as in the case of the normal distribution.

Representation Theorem

Theorem (1)

$\Phi : L^\infty \to \mathbb{R}$ is a monotone additive statistic if and only if there exists a unique Borel probability measure μ on $\overline{\mathbb{R}}$ such that for every $X \in L^\infty$,

$$\Phi(X) = \int_{\overline{\mathbb{R}}} K_a(X) \mathrm{d}\mu(a).$$

Theorem (2)

Let L be either $L^\infty_+, L^\infty_{\mathbb{N}}$ or L_M. Then $\Phi : L \to \mathbb{R}$ is a monotone additive statistic if and only if it admits a (unique) representation of the form in Theorem 1, where, in the case of L_M, the measure μ has compact support in $\mathbb{R}$.

Proof of Theorem 1

Definition (Catalytic Stochastic Order)

Given $X, Y \in L^\infty$, we say that X dominates Y in the catalytic stochastic order on L^∞ if there exists a $Z \in L^\infty$, independent of X and Y, such that $X + Z$ dominates $Y + Z$ in first-order stochastic dominance.

- Any monotone additive Φ is monotone with respect to catalytic order, because

$$\Phi(X+Z) \geq \Phi(Y+Z) \Leftrightarrow \Phi(X) + \Phi(Z) \geq \Phi(Y) + \Phi(Z) \Leftrightarrow \Phi(X) \geq \Phi(Y).$$

Theorem (7)

Let $X, Y \in L^\infty$ satisfy $K_a(X) > K_a(Y)$ for all $a \in \overline{\mathbb{R}}$. Then there exists an independent $Z \in L^\infty$ such that $X + Z \geq_1 Y + Z$.

Proof of Theorem 1

- First, we can add the same constant b to both X and Y so that $\min[Y + b] = -N$ and $\max[X + b] = N$ for some $N > 0$.
- Since translating both X and Y leaves the existence of an appropriate Z unchanged (and also does not affect $K_X > K_Y$), we henceforth assume without loss of generality that $\min[Y] = -N$, and $\max[X] = N$.
- Since $K_X > K_Y$, we know that $\min[X] > -N$ and $\max[Y] < N$.
- Denote the c.d.f.s of X and Y by F and G, respectively.
- Let $\sigma(x) = G(x) - F(x)$. Note that σ is supported on $[-N, N]$ and bounded in absolute value by 1.
- Moreover, by choosing $\varepsilon > 0$ sufficiently small, we have that

$$\min[X] > -N + \varepsilon \text{ and } \max[Y] < N - \varepsilon.$$

- So $\sigma(x)$ is positive on $[-N, -N + \varepsilon]$ and on $[N - \varepsilon, N]$. In fact, there exists $\delta > 0$ such that $\sigma(x) \geq \delta$ whenever $x \in \left[-N + \frac{\varepsilon}{4}, -N + \frac{\varepsilon}{2}\right]$ and $x \in \left[N - \frac{\varepsilon}{2}, N - \frac{\varepsilon}{4}\right]$. We also fix a large constant A such that

$$\mathrm{e}^{\frac{\varepsilon A}{4}} \geq \frac{8N}{\varepsilon\delta}.$$

Proof of Theorem 1

- Define
$$M_\sigma(a) = \int_{-N}^{N} \sigma(x)\mathrm{e}^{ax}\ \mathrm{d}x$$
- Note that for $a \neq 0$, integration by parts shows
$$M_\sigma(a) = \frac{1}{a}\left(\mathbb{E}\left[\mathrm{e}^{aX}\right] - \mathbb{E}\left[\mathrm{e}^{aY}\right]\right),$$
and that $M_\sigma(0) = \mathbb{E}[X] - \mathbb{E}[Y]$.
- Therefore, since $K_X > K_Y$, we have that M_σ is strictly positive everywhere.
- Since $M_\sigma(a)$ is clearly continuous in a, it is in fact bounded away from zero on any compact interval.
- We will use these properties of σ to construct a truncated Gaussian density h such that
$$[\sigma * h](y) = \int_{-N}^{N} \sigma(x)h(y-x)\mathrm{d}x \geq 0$$
for each $y \in \mathbb{R}$. If we let Z be a random variable independent from X and Y, whose distribution has density function h, then $\sigma * h = (G - F) * h$ is the difference between the c.d.f.s of $Y + Z$ and $X + Z$. Thus $[\sigma * h](y) \geq 0$ for all y would imply $X + Z \geq_1 Y + Z$.

Proof of Theorem 1

- To do this, we write $h(x) = \mathrm{e}^{-\frac{x^2}{2V}}$ for all $|x| \leq T$, where V is the variance and T is the truncation point to be chosen. We will show that given the above constants N and A, $[\sigma * h](y) \geq 0$ holds for each y when V is sufficiently large and $T \geq AV + N$.
- First consider the case where $y \in [-AV, AV]$. In this region, $|y - x| \leq T$ is automatically satisfied when $x \in [-N, N]$. So we can compute the convolution $\sigma * h$ as follows:

$$\int \sigma(x) h(y-x) \mathrm{d}x = \mathrm{e}^{-\frac{y^2}{2V}} \cdot \int_{-N}^{N} \sigma(x) \cdot \mathrm{e}^{\frac{y}{V} \cdot x} \cdot \mathrm{e}^{-\frac{x^2}{2V}} \, \mathrm{d}x.$$

Note that $\frac{y}{V}$ in the exponent belongs to the compact interval $[-A, A]$. So for our fixed choice of A, the integral $M_\sigma\left(\frac{y}{V}\right) = \int_{-N}^{N} \sigma(x) \cdot \mathrm{e}^{\frac{y}{V} \cdot x} \, \mathrm{d}x$ is uniformly bounded away from zero when y varies in the current region. Thus,

$$\begin{aligned} \int_{-N}^{N} \sigma(x) \cdot \mathrm{e}^{\frac{y}{V} \cdot x} \cdot \mathrm{e}^{-\frac{x^2}{2V}} \, \mathrm{d}x &= M_\sigma\left(\frac{y}{V}\right) - \int_{-N}^{N} \sigma(x) \cdot \mathrm{e}^{\frac{y}{V} \cdot x} \cdot \left(1 - \mathrm{e}^{-\frac{x^2}{2V}}\right) \mathrm{d}x \\ &\geq M_\sigma\left(\frac{y}{V}\right) - 2N \cdot \mathrm{e}^{AN} \cdot \left(1 - \mathrm{e}^{\frac{-N^2}{2V}}\right) \end{aligned}$$

which is positive when V is sufficiently large. So the right-hand side of (6) is positive.

Proof of Theorem 1

- Next consider the case where $y \in (AV, T+N-\varepsilon]$; the case where $-y$ is in this range can be treated symmetrically. Here the convolution can be written as

$$[\sigma * h](y) = \int_{\max\{-N, y-T\}}^{N} \sigma(x) \cdot \mathrm{e}^{\frac{-(y-x)^2}{2V}} \, \mathrm{d}x.$$

- We break the range of integration into two sub-intervals: $I_1 = [\max\{-N, y-T\}, N-\varepsilon]$ and $I_2 = [N-\varepsilon, N]$.
- On I_1 we have $\sigma(x) = G(x) - F(x) \geq -1$, so

$$\int_{x \in I_1} \sigma(x) \cdot \mathrm{e}^{\frac{-(y-x)^2}{2V}} \, \mathrm{d}x \geq -2N \cdot \mathrm{e}^{\frac{-(y-N+\varepsilon)^2}{2V}}.$$

- On I_2 we have $\sigma(x) \geq 0$ by our choice of ε, and furthermore $\sigma(x) \geq \delta$ when $x \in \left[N - \frac{\varepsilon}{2}, N - \frac{\varepsilon}{4}\right]$. Thus

$$\int_{x \in I_2} \sigma(x) \cdot \mathrm{e}^{\frac{-(y-x)^2}{2V}} \, \mathrm{d}x \geq \frac{\varepsilon}{4} \cdot \delta \cdot \mathrm{e}^{\frac{-\left(y-N+\frac{\varepsilon}{2}\right)^2}{2V}} \geq 2N \cdot \mathrm{e}^{\frac{-\left(y-N+\frac{\varepsilon}{2}\right)^2}{2V} - \frac{\varepsilon A}{4}},$$

where the second inequality holds by the choice of A. Observe that when $y > AV$ and V is large, the exponent $\frac{-\left(y-N+\frac{\varepsilon}{2}\right)^2}{2V} - \frac{\varepsilon A}{4}$ is larger than $\frac{-(y-N+\varepsilon)^2}{2V}$.

Proof of Theorem 1

- Finally, if $y \in (T + N - \varepsilon, T + N]$, then the range of integration in computing $[\sigma * h](y)$ is from $x = y - T$ to $x = N$, where $\sigma(x)$ is always positive. So the convolution is positive. And if $y > T + N$, then clearly the convolution is zero. These arguments symmetrically apply to $-y \in (T + N - \varepsilon, T + N]$ and $-y > T + N$. We therefore conclude that $[\sigma * h](y) \geq 0$ for all y, completing the proof.

Integral Representation

- Let $\mathcal{L}$ denote the set of functions $\{K_X : X \in L^\infty\}$.
- If Φ is a monotone additive statistic and $K_X = K_Y$, then X and Y have the same distribution and $\Phi(X) = \Phi(Y)$.
- Thus there exists some functional $F : \mathcal{L} \to \mathbb{R}$ such that $\Phi(X) = F(K_X)$.
- It follows from the additivity of Φ and the additivity of K_a that F is additive: $F(K_X + K_Y) = F(K_X) + F(K_Y)$.
- Moreover, F is monotone in the sense that $F(K_X) \geq F(K_Y)$ whenever $K_X \geq K_Y$ (i.e., $K_X(a) \geq K_Y(a)$ for all $a \in \overline{\mathbb{R}}$); this follows from Lemma 1 which in turn is proved by Theorem 7.
- The main goal is to show that the monotone additive functional F on $\mathcal{L}$ can be extended to a positive linear functional on the entire space of continuous functions $\mathcal{C}(\overline{\mathbb{R}})$.

Integral Representation

Lemma (3)

$F : \mathcal{L} \to \mathbb{R}$ is 1-Lipschitz:

$$|F(K_X) - F(K_Y)| \leq \|K_X - K_Y\|$$

- Let $\|K_X - K_Y\| = \varepsilon$. Then $K_{X+\varepsilon} = K_X + \varepsilon \geq K_Y$. Hence, by Lemma 1, $F(K_Y) \leq F(K_{X+\varepsilon})$, and so

$$F(K_Y) - F(K_X) \leq F(K_{X+\varepsilon}) - F(K_X) = F(K_\varepsilon) = \Phi(\varepsilon) = \varepsilon$$

- Symmetrically we have $F(K_X) - F(K_Y) \leq \varepsilon$, as desired.

Integral Representation

Lemma (4)

Any monotone additive functional F on $\mathcal{L}$ can be extended to a positive linear functional on $\mathcal{C}(\overline{\mathbb{R}})$.

- First consider the rational cone spanned by $\mathcal{L}$:

$$\text{Cone}_{\mathbb{Q}}(\mathcal{L}) = \{qL : q \in \mathbb{Q}_+, L \in \mathcal{L}\}.$$

- G is additive, Lipschitz, positively homogeneous over $\mathbb{Q}_+$ and monotone.
- Thus G can be extended to a Lipschitz functional H defined on the closure of $\text{Cone}_{\mathbb{Q}}(\mathcal{L})$ with respect to the sup norm. In particular, H is defined on the convex cone spanned by $\mathcal{L}$:

$$\text{Cone}(\mathcal{L}) = \{\lambda_1 K_1 + \cdots + \lambda_k K_k : k \in \mathbb{N} \text{ for each } 1 \leq i \leq k, \lambda_i \in \mathbb{R}_+, K_i \in \mathcal{L}\}.$$

- It is immediate to verify that the properties of additivity, positive homogeneity (now over $\mathbb{R}_+$), and monotonicity extend, by continuity, from G to H.

Integral Representation

Lemma (4)

Any monotone additive functional F on $\mathcal{L}$ can be extended to a positive linear functional on $\mathcal{C}(\overline{\mathbb{R}})$.

- Consider the vector subspace $\mathcal{V} = \text{Cone}(\mathcal{L}) - \text{Cone}(\mathcal{L}) \subset \mathcal{C}(\overline{\mathbb{R}})$ and define $I : \mathcal{V} \to \mathbb{R}$ as
$$I(g_1 - g_2) = H(g_1) - H(g_2)$$
for all $g_1, g_2 \in \text{Cone}(\mathcal{L})$. The functional I is well defined and linear (because H is additive and positively homogeneous). Moreover, by monotonicity of $H, I(f) \geq 0$ for any nonnegative function $f \in \mathcal{V}$.
- The result then follows from the next theorem of Kantorovich (1937), a generalization of the Hahn-Banach Theorem. It applies not only to $\mathcal{C}(\overline{\mathbb{R}})$ but to any Riesz space.

Theorem

If $\mathcal{V}$ is a vector subspace of $\mathcal{C}(\overline{\mathbb{R}})$ with the property that for every $f \in \mathcal{C}(\overline{\mathbb{R}})$ there exists a function $g \in \mathcal{V}$ such that $g \geq f$. Then every positive linear functional on $\mathcal{V}$ extends to a positive linear functional on $\mathcal{C}(\overline{\mathbb{R}})$.

Monotone Stationary Time Preferences

- Next, we apply monotone additive statistics to the study of time preferences.
- A time lottery is a monetary reward received by a decision maker at a future, random time.
- Formally, it consists of a pair (x, T), where $x \in \mathbb{R}_{++}$ is a positive payoff and $T \in L^\infty_+$ is the random time at which it realizes.

Definition (MSTP)

We say that a preference relation $\succeq$ on $\mathbb{R}_{++} \times L^\infty_+$ is a monotone stationary time preference (MSTP) if it satisfies the following axioms:

1. Axiom 3.1 (More is Better). If $x > y$ then $(x, T) \succ (y, T)$.
2. Axiom 3.2 (Earlier is Better). If $s > t$ then $(x, t) \succ (x, s)$, and if $S \geq_1 T$ then $(x, T) \succeq (x, S)$.
3. Axiom 3.3 (Stationarity). If $(x, T) \succeq (y, S)$ then $(x, T + D) \succeq (y, S + D)$ for any D that is independent from T and S.
4. Axiom 3.4 (Continuity). For any (y, S), the sets $\{(x, t) : (x, t) \succeq (y, S)\}$ and $\{(x, t) : (x, t) \preceq (y, S)\}$ are closed in $\mathbb{R}_{++} \times \mathbb{R}_+$.

Representation Theorem

Theorem (3)

A preference relation $\succeq$ over time lotteries is an MSTP if and only if there exist a monotone additive statistic Φ, a constant $r > 0$, and a continuous and increasing function $u : \mathbb{R}_{++} \to \mathbb{R}_{++}$such that $\succeq$ is represented by

$$V(x, T) = u(x) \cdot \mathrm{e}^{-r\Phi(T)}.$$

- Over the domain of deterministic time lotteries, V coincides with an exponentially discounted utility representation with discount rate r.
- For a general measure μ, the statistic $\Phi(T) = \int K_a(T)\mathrm{d}\mu(a)$ aggregates different discount rates by mixing over their corresponding certainty equivalents.
- The resulting representation is behaviorally distinct from expected discounted utility whenever μ is not a point mass.

Implications for Risk Attitudes toward Time

- Standard expected discounted utility preferences are **risk-seeking** over time, in the sense that a decision maker prefers receiving a reward at a random time T rather than at the deterministic expected time $t = \mathbb{E}[T]$.
- But other monotone additive statistics lead to stationary time may be not risk-seeking.
- For example, for every $a > 0$ the statistic

  $$\Phi(T) = K_a(T) = \frac{1}{a} \log \mathbb{E}\left[\mathrm{e}^{aT}\right]$$

  leads, with the normalization $r = a$, to the representation

  $$V(x, T) = \frac{u(x)}{\mathbb{E}\left[\mathrm{e}^{aT}\right]},$$

  which is in fact **risk-averse** over time.
- Another key distinctive property of monotone stationary time preferences is their flexibility in allowing for risk attitudes that are not uniform across time lotteries.

Aggregation of Preferences over Time Lotteries

- We formulate a collective decision problem as a problem of aggregating preferences over time lotteries that display different degrees of impatience.
- We take as primitive a group represented by n preference relations $\succeq_1, \ldots, \succeq_n$ over time lotteries, each admitting a standard expected discounted utility representation

$$u(x)\mathbb{E}\left[\mathrm{e}^{-r_i T}\right],$$

where $u : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is a common utility function that is increasing and continuous, and $r_i > 0$ is agent i 's discount rate.
- These preferences must be aggregated into a social preference relation $\succeq$. We require $\succeq$ to be an MSTP, and to agree with individual preferences whenever there is a consensus among the agents:
- Axiom 3.5 (Pareto). If $(x, T) \succeq_i (y, S)$ for every i, then $(x, T) \succeq (y, S)$.

Aggregation of Preferences over Time Lotteries

Theorem (4)

Let $(\succeq_1, \ldots, \succeq_n, \succeq)$ be preference relations over time lotteries, where each $\succeq_i$ is represented by $u(x)\mathbb{E}\left[\mathrm{e}^{-r_i T}\right]$ and $\succeq$ is an MSTP. The Pareto axiom is satisfied if and only if there exists a probability vector $(\lambda_1, \ldots, \lambda_n)$ such that $\succeq$ can be represented by $u(x)\mathrm{e}^{-r\Phi(T)}$ with

$$\Phi = \sum_{i=1}^{n} \lambda_i K_{-r_i} \quad \text{and} \quad \frac{1}{r} = \sum_{i=1}^{n} \frac{\lambda_i}{r_i}.$$

- Any social preference that satisfies the Pareto axiom and admits an expected discounted utility representation must coincide with one of the individual preferences.
- **Dictatorship** becomes the only admissible aggregation procedure if one insists that the social preference must conform to expected discounted utility.
- We demonstrate that Paretian aggregation and stationarity are **compatible**, and do not necessarily result in a dictatorship, if we allow the social preference to belong to the larger class of **MSTPs**.

Wealth and Risk Invariance

- We consider a preference relation $\succeq$ over the set L^∞ of bounded gambles, that is complete, transitive, and non-trivial (i.e., $X \succ Y$ for some pair of gambles).
- We maintain two basic conditions on the preference:
- Axiom 4.1 (Monotonicity). If $X \geq_1 Y$ then $X \succeq Y$.
- Axiom 4.2 (Continuity). If $X \succ Y$ then there exists $\varepsilon > 0$ such that $X \succ Y + \varepsilon$ and $X - \varepsilon \succ Y$.
- Axiom 4.3 (Wealth Invariance). $X \succeq Y$ if and only if $X + z \succeq Y + z$ for all $z \in \mathbb{R}$.
- Axiom 4.4 (Risk Invariance). Suppose Z is mean-zero and independent of X and Y. Then $X \succeq Y$ if and only if $X + Z \succeq Y + Z$.

Disentangling the Two Invariance Properties

- Under a monotone expected utility preference, wealth invariance and risk invariance each imply CARA utility functions.
- For general preferences over gambles, the two properties are logically independent.
- We characterize all preferences that are invariant to mean-zero background risks, and additionally exhibit risk aversion.
- Axiom 4.5 (Second-order Monotonicity). $X \succeq Y$ whenever X dominates Y in secondorder stochastic dominance.
- To state the next condition, for each $c > 0$ we denote by W_c a random variable that is equal to $\pm c$ with equal probabilities.
- As c becomes large, W_c is a mean-zero risk of increasing magnitude. We require the decision maker to regard W_c as arbitrarily undesirable as $c \to \infty$
- Axiom 4.6 (Archimedeanity). For every $x \in \mathbb{R}$ there exists $c > 0$ such that $W_c \prec x$ and $x + W_c \prec 0$.

Disentangling the Two Invariance Properties

Theorem (5)

A preference $\succeq$ on L^∞ satisfies Axioms 4.2, 4.4, 4.5, 4.6 (i.e., continuity, risk invariance, second-order monotonicity and Archimedeanity) if and only if there exist a continuous and non-decreasing function $v : \mathbb{R} \to \mathbb{R}$ and a probability measure μ supported on $[-\infty, 0)$ such that $\succeq$ is represented by

$$V(X) = v(\mathbb{E}[X]) + \int_{[-\infty,0)} K_a(X)\mathrm{d}\mu(a).$$

- The first part is a monotone function of the expectation.
- The second part is a monotone additive statistic.
- Two parts are unaffected by mean-zero risks.

Combined Choices

- In large organizations, risky prospects are not always chosen through a deliberate, centralized process. Rather, they are combinations of independent choices, often carried out with limited coordination among the different actors.
- Under what conditions the agents' combined choices respect first-order stochastic dominance?
- Our main result shows this is true if and only if individual preferences are identical and represented by a monotone additive statistic.
- We study the following model. We are given two preference relations $\succeq_1$ and $\succeq_2$ over L^∞, the set of bounded gambles, that are complete and transitive (our result immediately generalizes to three or more agents).

Combined Choices

- Our main axiom requires that whenever the two agents face independent decision problems, their choices, when combined, do not violate stochastic dominance:
- Axiom 4.7 (Consistency of Combined Choices). Suppose X, X' are independent of Y, Y'. If $X \succ_1 X'$ and $Y \succ_2 Y'$, then $X' + Y'$ does not strictly dominate $X + Y$ in first-order stochastic dominance.
- In addition to this axiom, we assume individual preference relations $\succeq_i$ satisfy a basic continuity condition, Axiom 4.2, as well as the next monotonicity assumption:
- Axiom 4.8 (Responsiveness). $X + \varepsilon \succ_i X$ for every $\varepsilon > 0$.

Theorem (6)

Two preference $\succeq_1, \succeq_2$ on L^∞ satisfy Axioms 4.2, 4.7 and 4.8 (i.e., continuity, consistency of combined choices and responsiveness) if and only if there exists a monotone additive statistic that represents both $\succeq_1$ and $\succeq_2$.

Comparative Risk Attitudes

- We characterize risk-averse and risk-seeking behavior for preferences that are represented by monotone additive statistics.
- A preference relation $\succeq$ over gambles is risk-averse if its certainty equivalent Φ satisfies $\Phi(X) \leq \mathbb{E}[X]$ for every gamble X, and risk-seeking if the opposite inequality holds.
- Risk aversion translates into a property of the support of the corresponding mixing measure μ:

Proposition (1)

A monotone additive statistic satisfies $\Phi(X) \leq \mathbb{E}[X]$ for every $X \in L^\infty$ if and only if

$$\Phi(X) = \int_{\overline{\mathbb{R}}} K_a(X) \mathrm{d}\mu(a)$$

for a Borel probability measure μ supported on $[-\infty, 0]$. Symmetrically, $\Phi(X) \geq \mathbb{E}[X]$ for every X if and only if the measure μ is supported on $[0, \infty]$.

Comparative Risk Attitudes

- A corollary of Proposition 1 is that an additive statistic Φ is monotone with respect to second-order (or any higher-order) stochastic dominance if and only if

$$\Phi(X) = \int K_a(X) \mathrm{d}\mu(a),$$

for a probability measure μ supported on $[-\infty, 0]$.

Compare the Risk Attitudes

- For two preference relations $\succeq_1$ and $\succeq_2$ over gambles, with corresponding certainty equivalents Φ_1 and Φ_2, the preference $\succeq_1$ is more risk-averse than $\succeq_2$ if $\Phi_1(X) \leq \Phi_2(X)$ for every gamble X.

Proposition (2)

Let Φ_1, Φ_2 be monotone additive statistics, characterized by measures μ_1 and μ_2 respectively. Then $\Phi_1(X) \leq \Phi_2(X)$ for all $X \in L^\infty$ if and only if

1. For every $b > 0$, $\int_{[b,\infty]} \frac{a-b}{a} \, \mathrm{d}\mu_1(a) \leq \int_{[b,\infty]} \frac{a-b}{a} \, \mathrm{d}\mu_2(a)$.
2. For every $b < 0$, $\int_{[-\infty,b]} \frac{a-b}{a} \, \mathrm{d}\mu_1(a) \geq \int_{[-\infty,b]} \frac{a-b}{a} \, \mathrm{d}\mu_2(a)$.

Conclusion

- We provide a complete characterization of such statistics, and explore a number of applications to models of individual and group decision-making.
- We characterize of risk-averse preferences over monetary gambles that are invariant to mean-zero background risks.

Thanks!