Lecture 2

SEARCH-BASED PATH FINDING

基于搜索的方法

Outline

1. Graph Search Basis
2. Dijkstra and A*
3. Jump Point Search
4. Homework

Graph Search Basis
图搜索基础知识

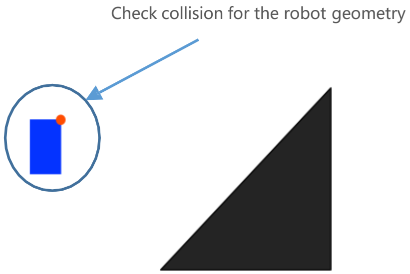
Configuration Space
配置空间

Configuration Space
配置空间

- **Robot configuration:** a specification of the positions of all points of the robot
- **机器人配置:** 机器人位置的所有点空间
- **Robot degree of freedom (DOF):** The minimum number n of real-valued coordinates needed to represent the robot configuration
- **机器人自由度:**表示机器人配置所需的实际坐标的最小数量n。
- **Robot configuration space:** a n-dim space containing all possible robot configurations, denoted as C-space
- **机器人配置空间:** 一个包含机器人所有可能配置的n维空间， C-space
- **Each robot pose is a point in the C-space**
- **每个机器人姿势是C空间的一个点**

Configuration Space Obstacle
配置空间障碍物

- Planning in workspace 工作空间规划
  - Robot has different shape and size 不同尺寸和形状
  - Collision detection requires knowing the robot geometry - time consuming and hard 耗时、难

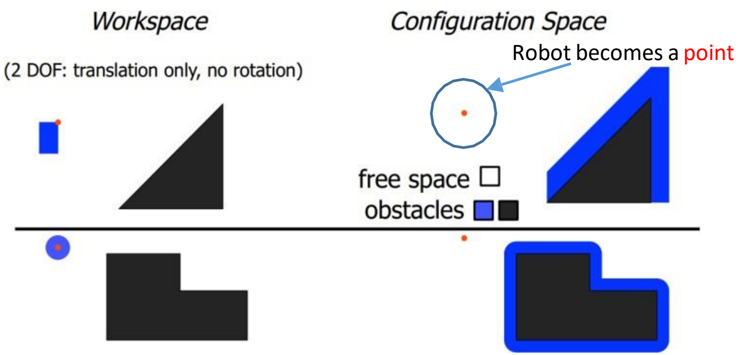


(1) Rectangular mobile robot

(2) Circular mobile robot

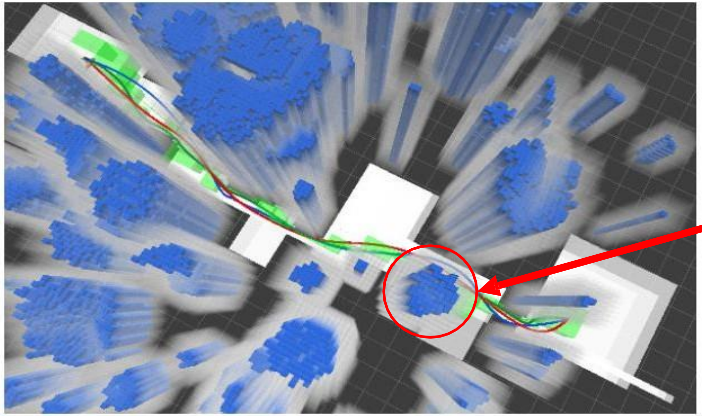
Configuration Space Obstacle
配置空间障碍物

- Planning in configuration space 配置空间规划
  - Robot is represented by a point in C-space, e.g. position (a point in R^3), pose (a point in $SO(3)$),etc.
  - Obstacles need to be represented in configuration space (one-time work prior to motion planning), called configuration space obstacle, or C-obstacle
  - C-space = (C-obstacle) ∪ (C-free)
  - The path planning is finding a path between start point q_{start} and goal point q_{goal} within C-free



Workspace and Configuration Space Obstacle

- In workspace 工作空间
  - Robot has shape and size (i.e. hard for motion planning) 机器人有形状和尺寸
- In configuration space: C-space 配置空间
  - Robot is a point (i.e. easy for motion planning) 机器人是个点
  - Obstacle are represented in C-space prior to motion planning 运动规划之前障碍物在C空间表示
- Representing an obstacle in C-space can be extremely complicated. So approximated (but more conservative) representations are used in practice. 实际工作中近似表示



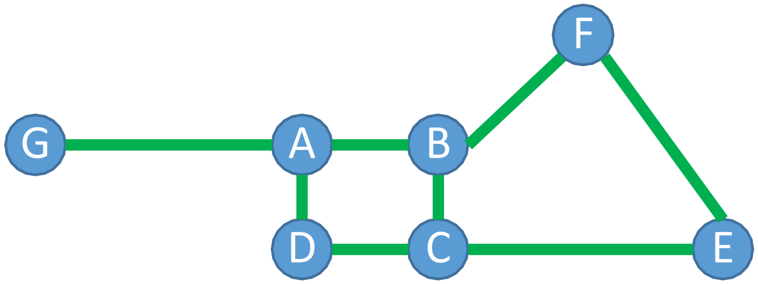
If we model the robot conservatively as a ball with radius δ_r,
将机器人表示为一个半径为δ_r的球
then the C-space can be constructed by inflating obstacle at all directions by δ_r.
C-空间的所有障碍物在所有方向拓宽δ_r。

Graph and Search Method
基于图搜索的方法

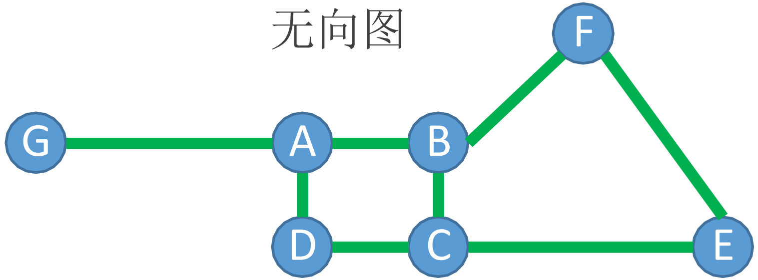
Search-based Method 搜索法

Graphs

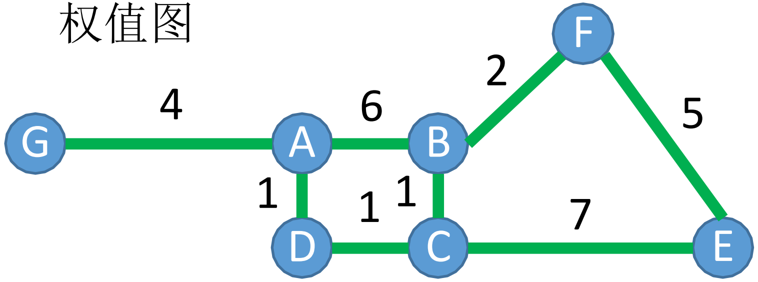
Graphs have nodes and edges



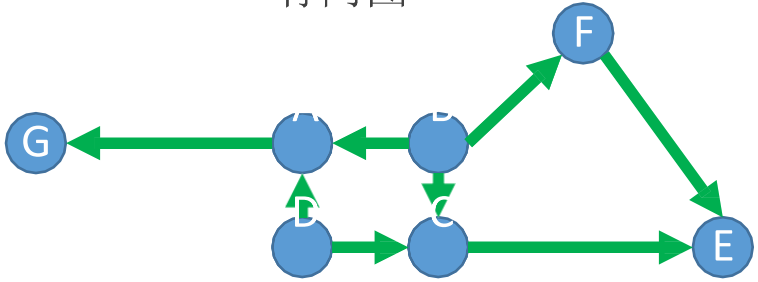
Undirected
无向图



Weighted
权值图

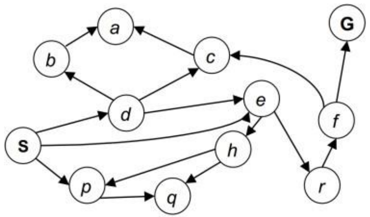


Directed
有向图



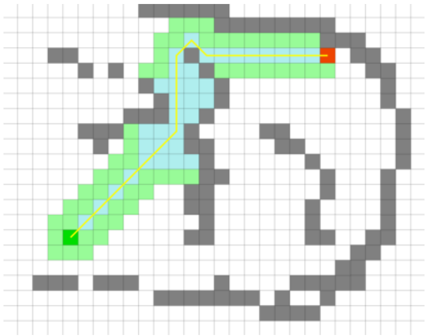
Search-based Method 搜索法

- State space graph （状态空间图）: a mathematical representation of a search algorithm
  搜索算法的一种数学表达
  - For every search problem, there's a corresponding state space graph
    对于每一个搜索问题，都有对应的状态空间图
  - Connectivity between nodes in the graph is represented by (directed or undirected) edges
    图中的点连接可以表示为（有向或无向）边

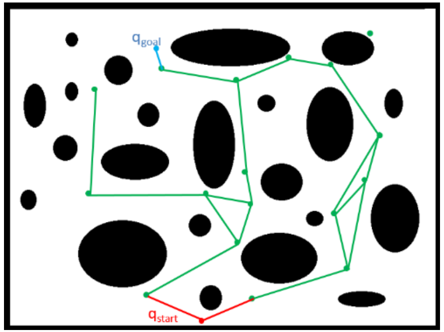


Ridiculously tiny search graph for a tiny search problem

网格图



Grid-based graph: use grid as vertices and grid connections as edges



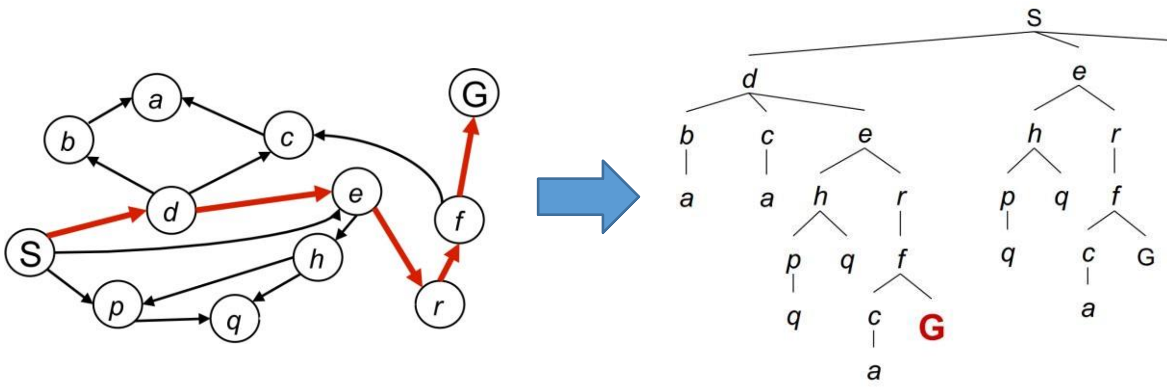
概率路线图

The graph generated by probabilistic roadmap (PRM)

Graph Search Overview 图搜索简介

- The search always start from start state X_S 从起始点开始的路径搜索
  - Searching the graph produces a search tree 搜索图生成搜索树
  - Back-tracing a node in the search tree gives us a path from the start state to that node

    回溯搜索树中的节点提供了从起始状态到该节点的路径
  - For many problems we can never actually build the whole tree, too large or inefficient – we only want to reach the goal node asap.



Graph Search Overview 图搜索简介

- Maintain a container to store all the nodes to be visited

  维护一个容器
- The container is initialized with the start state X_S

  容器以初始状态开始
- Loop 循环回路
  - Remove a node from the container according to some pre-defined score function 访问
    - Visit a node
  - Expansion: Obtain all neighbors of the node 扩展
    - Discover all its neighbors
  - Push them (neighbors) into the container 加入
- End Loop 回路结束

Graph Search Overview 图搜索简介

- Question 1: When to end the loop?

  问题1： 为什么循环？

  - Possible option: End the loop when the container is empty

- Question 2: What if the graph is cyclic?

  问题2：如果图是循环的怎么办

  - When a node is removed from the container (expanded / visited), it should never be added back to the container again

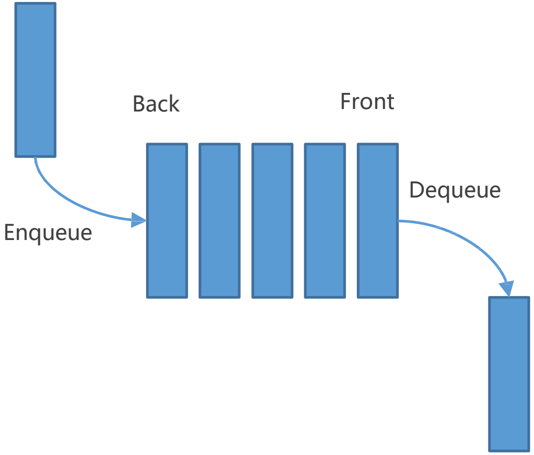
- Question 3: In what way to remove the right node such that the goal state can be reached as soon as possible, which results in less expansion of the graph node.

  问题3: 以哪种方式移除正确的节点，使得能够尽快达到目标状态，从而导致图节点的膨胀较少?

Graph Traversal 图遍历

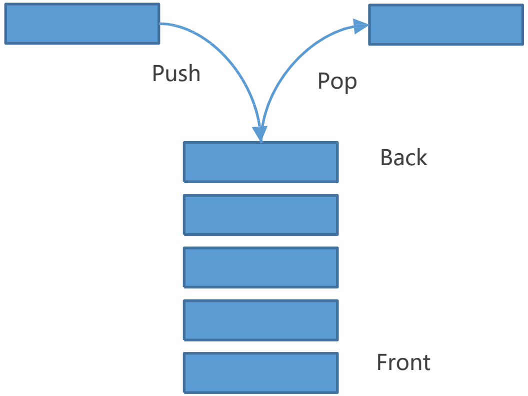
- Breadth First Search (BFS) vs. Depth First Search (DFS)
  广度优先搜索和深度优先搜索

BFS uses “first in first out”



This is a queue
队列

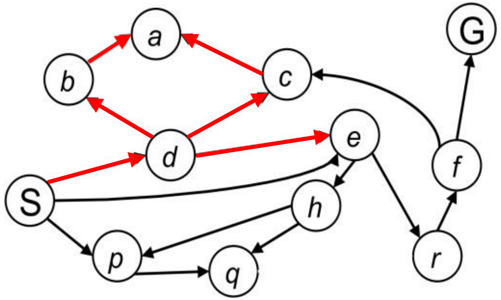
DFS uses “last in first out”

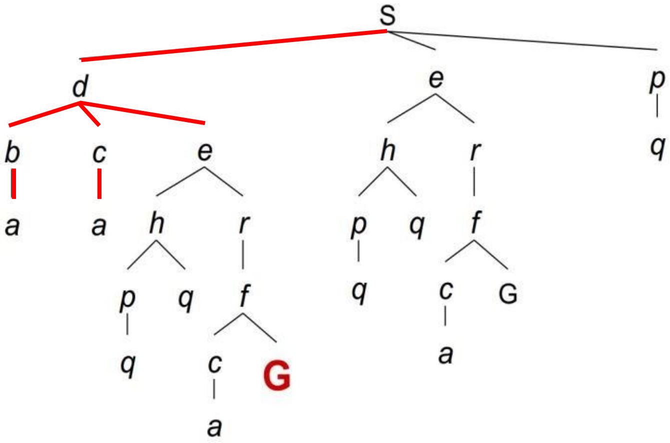


This is a stack
堆栈

Depth First Search (DFS) 深度优先搜索

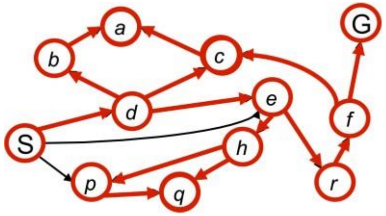
- Strategy: remove / expand the deepest node in the container
- 策略：移除/扩展最深的节点

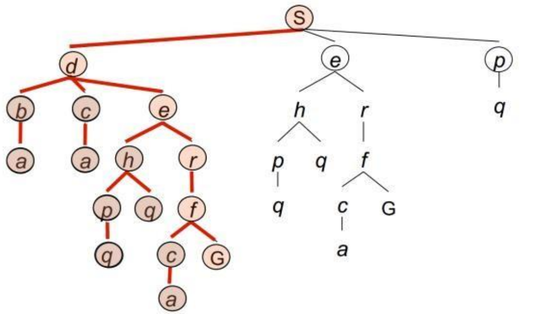


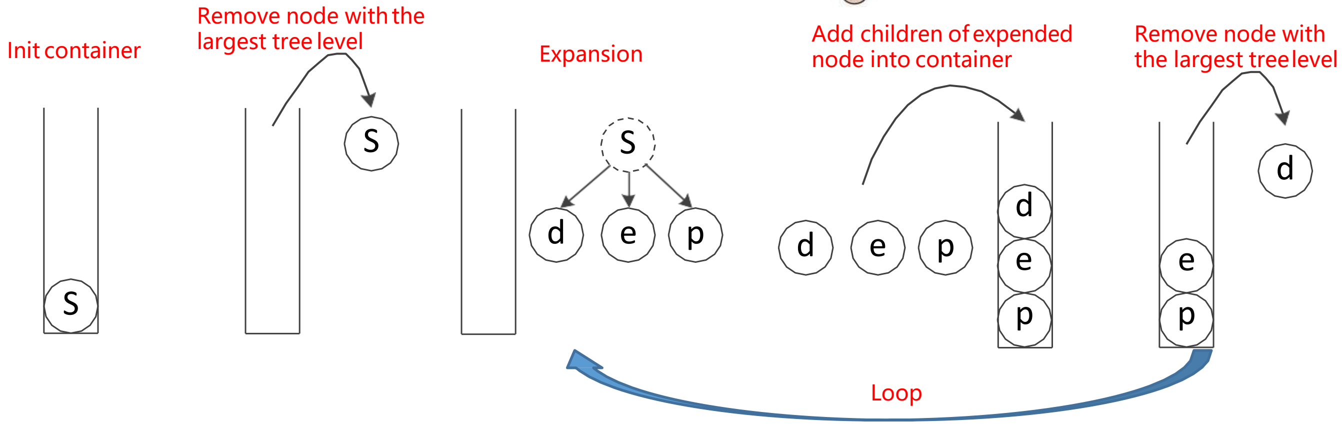


Depth First Search (DFS) 深度优先搜索

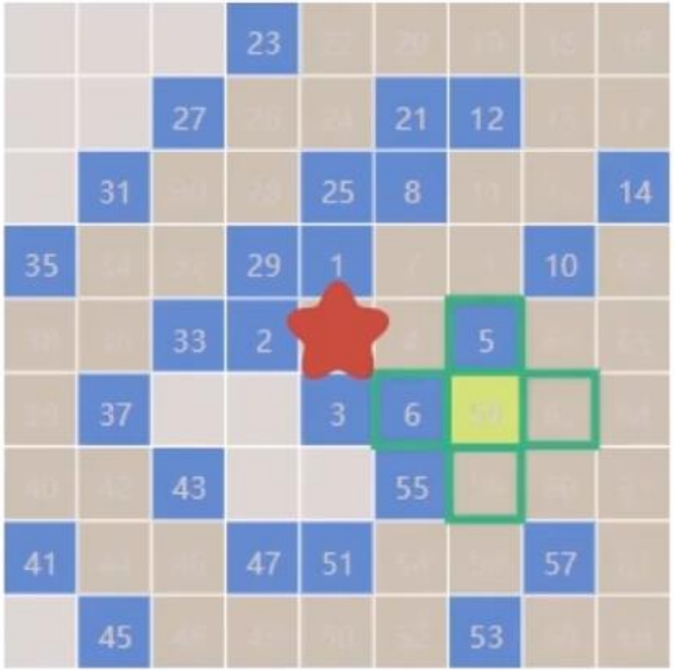
- Implementation: maintain a last in first out (LIFO) container (i.e. stack)
- 实行：后入先出







Depth First Search (DFS) 深度优先搜索

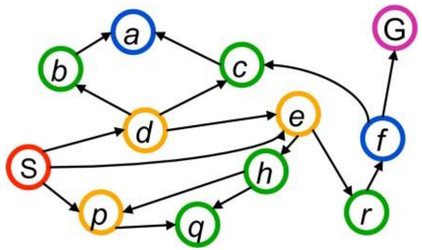


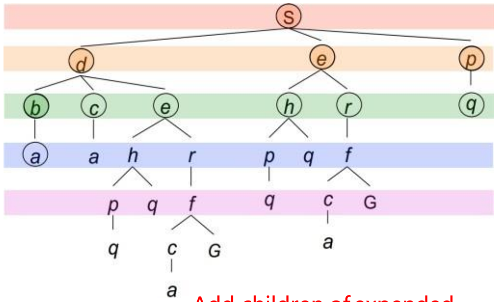
Courtesy: Amit Patel's Introduction to A*，Stanford

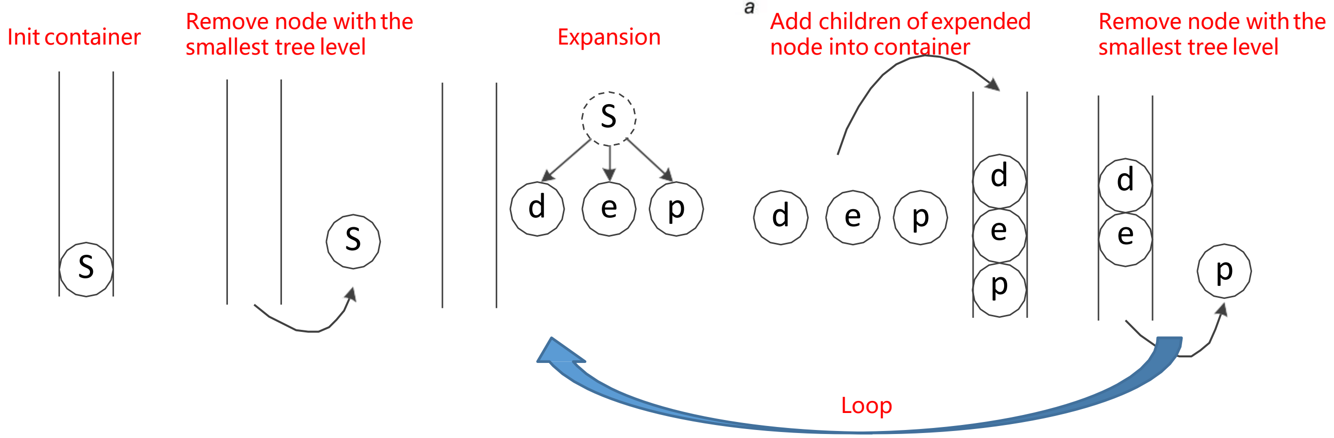
https://www.redblobgames.com/pathfinding/a-star/introduction.html

Breadth First Search (BFS) 广度优先搜索

- Implementation: maintain a first in first out (FIFO) container (i.e. queue)
- 实行：先入先出



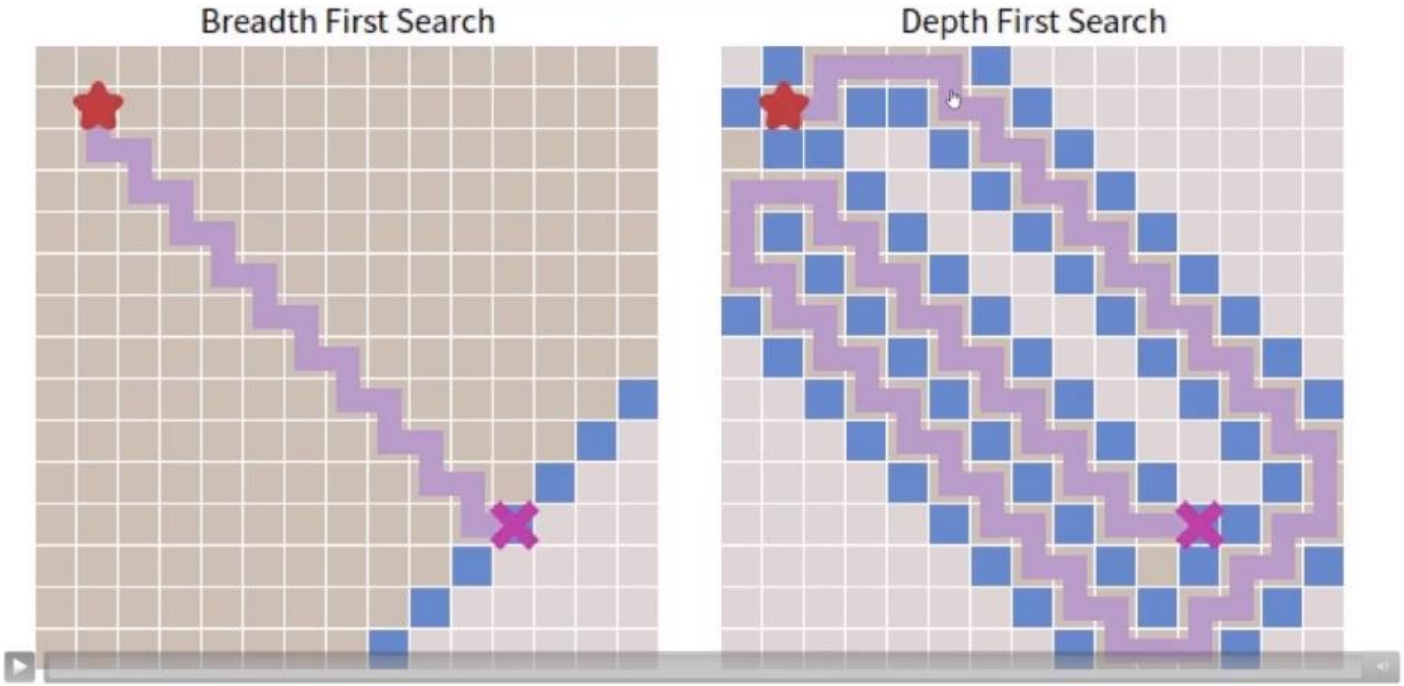




Breadth First Search (BFS)广度优先搜索

Courtesy: Amit Patel's Introduction to A*, Stanford

BFS vs. DFS: which one is useful?



广度优先搜索 Remember BFS. 深度优先搜索

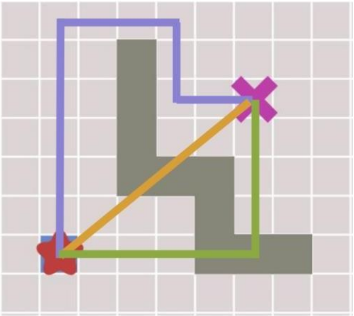
Heuristic search

Greedy Best First Search

- BFS and DFS pick the next node off the frontiers based on which was “first in” or “last in”.
- Greedy Best First picks the “best” node according to some rule, called a heuristic.

- Definition: A heuristic is a guess of how close you are to the target.

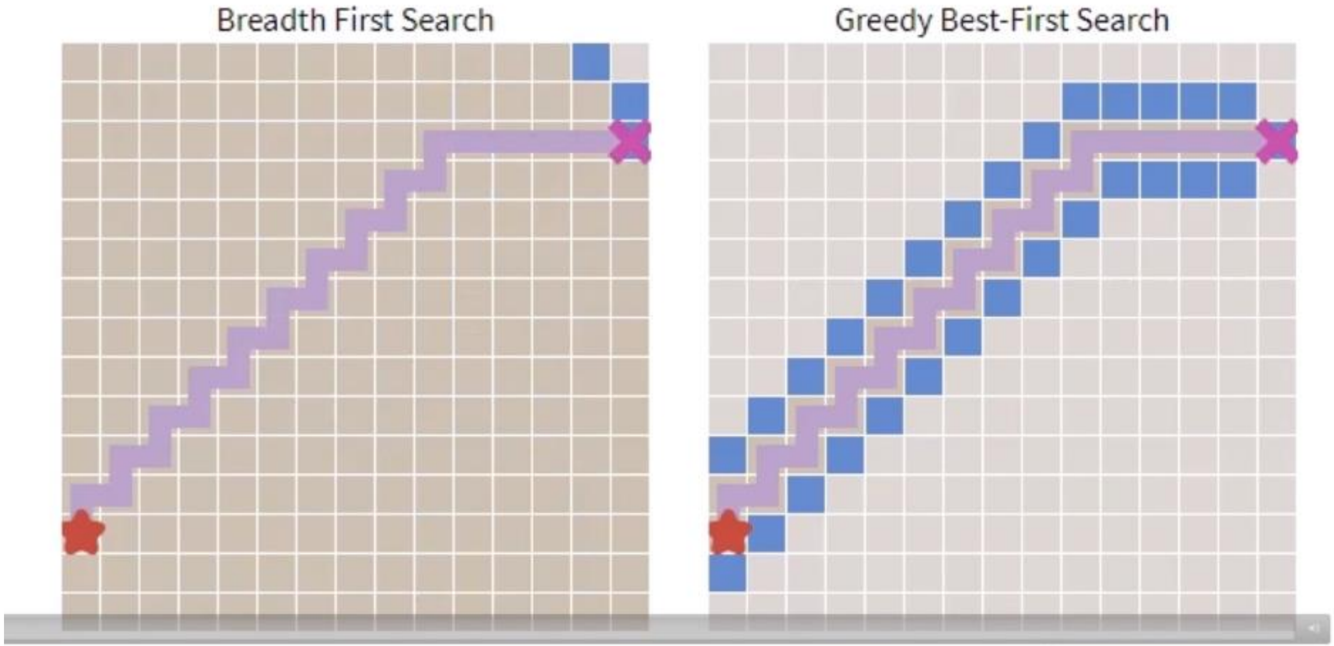
- A heuristic guides you in the right direction.
- A heuristic should be easy to compute.



- Euclidean Distance
- Manhattan Distance

Both are approximations for the actual shortest path.

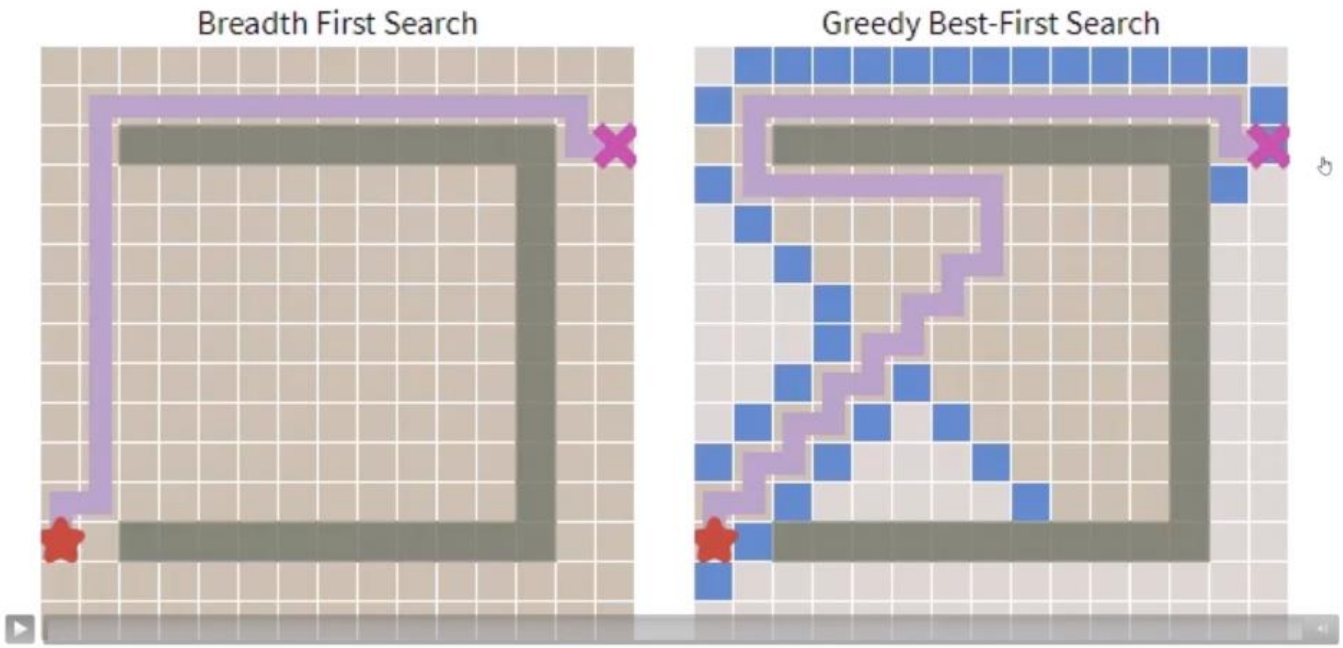
Greedy Best First Search



Looks pretty good.

Courtesy: Amit Patel's Introduction to A*, Stanford

Greedy Best First Search



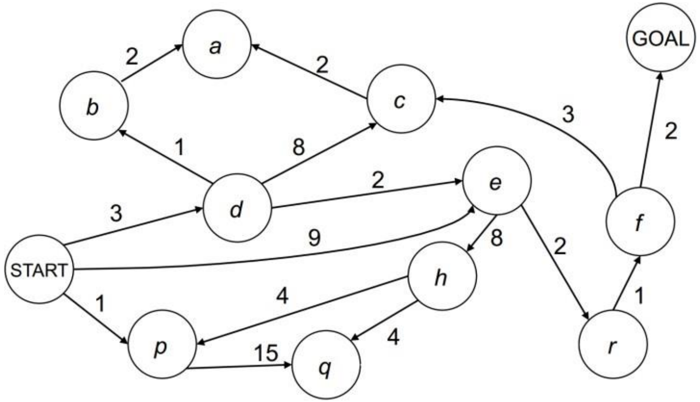
But with obstacles ...

Courtesy: Amit Patel's Introduction to A*, Stanford

https://www.redblobgames.com/pathfinding/a-star/introduction.html

Costs on Actions

- A practical search problem has a cost “C” from a node to its neighbor
  - Length, time, energy, etc.
- When all weight are 1, BFS finds the optimal solution
- For general cases, how to find the least-cost path as soon as possible?

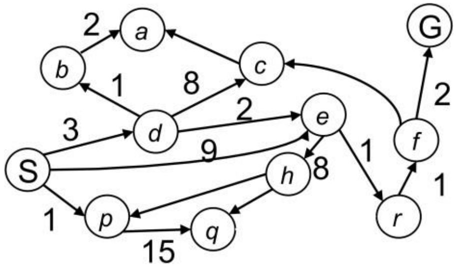


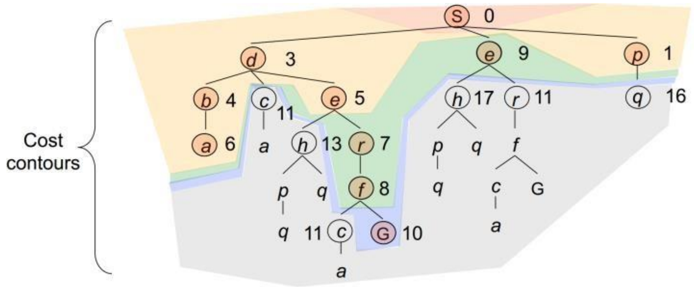
Dijkstra and A*

Algorithm Workflow

Dijkstra’ s Algorithm

- Strategy: expand/visit the node with cheapest accumulated cost g(n)
  - g(n): The current best estimates of the accumulated cost from the start state to node “n”
  - Update the accumulated costs g(m) for all unexpanded neighbors “m” of node “n”
  - A node that has been expanded/visited is guaranteed to have the smallest cost from the start state

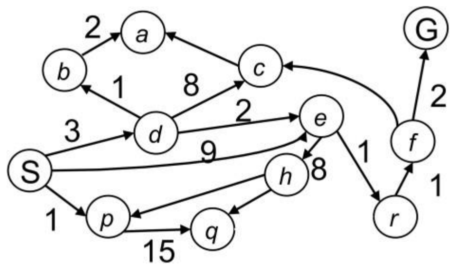


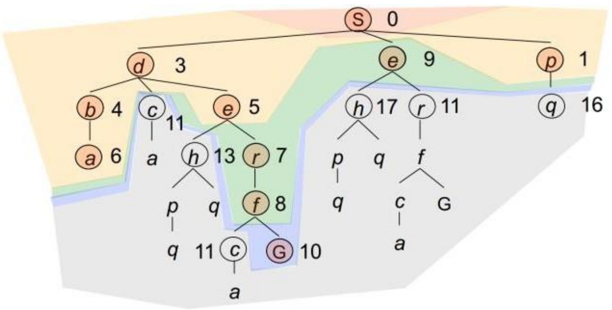


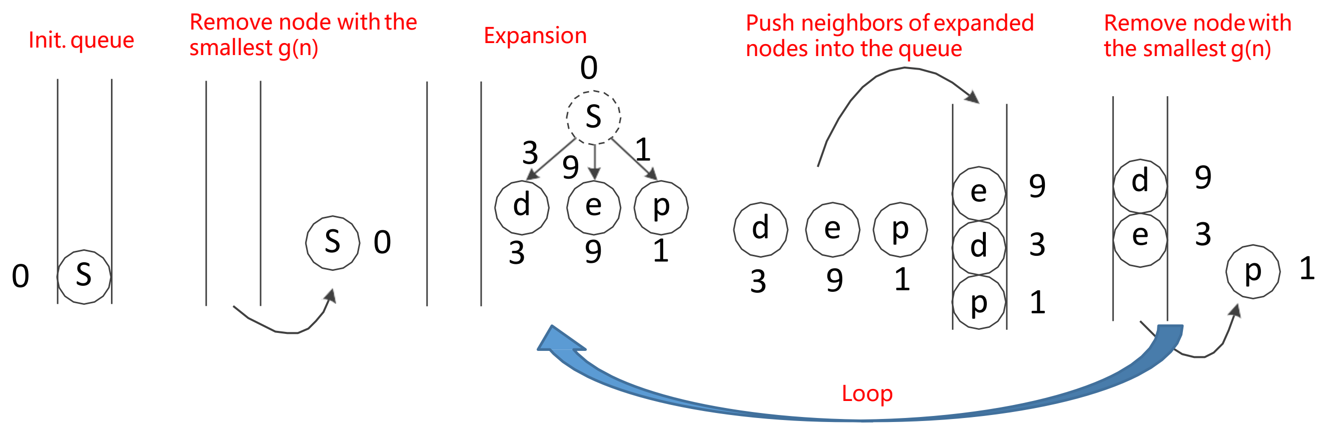
Dijkstra’s Algorithm

- Maintain a priority queue to store all the nodes to be expanded
- The priority queue is initialized with the start state X_S
- Assign $g(X_S)=0$, and $g(n)$=infinite for all other nodes in the graph
- Loop
    - If the queue is empty, return FALSE; break;
    - Remove the node “n” with the lowest g(n) from the priority queue
    - Mark node “n” as expanded
    - If the node “n” is the goal state, return TRUE; break;
    - For all unexpanded neighbors “m” of node “n”
        - If g(m) = infinite
            - g(m)= g(n) + Cnm
            - Push node “m” into the queue
        - If $g(m) > g(n) + C_{nm}$
            - $g(m)= g(n) + C_{nm}$
    - end
- End Loop

Dijkstra’ s Algorithm

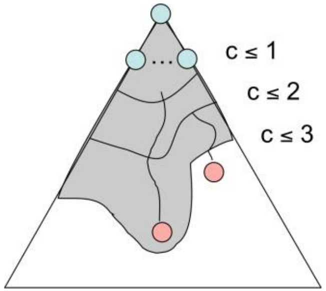


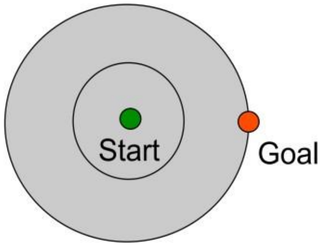




Pros and Cons of Dijkstra’s Algorithm

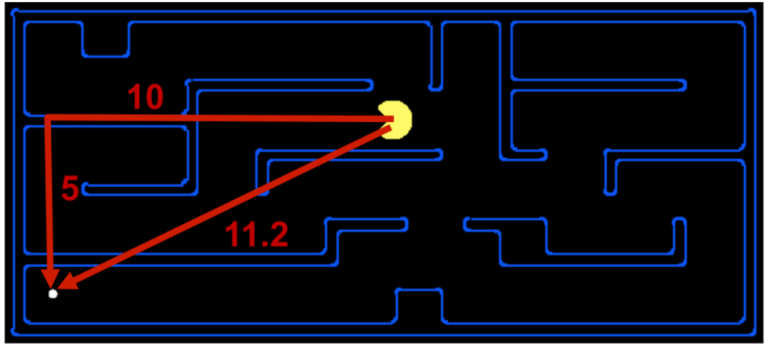
- The good:
  - Complete and optimal
- The bad:
  - Can only see the cost accumulated so far (i.e. the uniform cost), thus exploring next state in every “direction”
  - No information about goal location





Search Heuristics

- Recall the heuristic introduced in **Greedy Best First Search**
- Overcome the shortcomings of uniform cost search by inferring the least cost to goal (i.e. goal cost)
- Designed for particular search problem
- Examples: Manhattan distance VS. Euclidean distance



A*: Dijkstra with a Heuristic

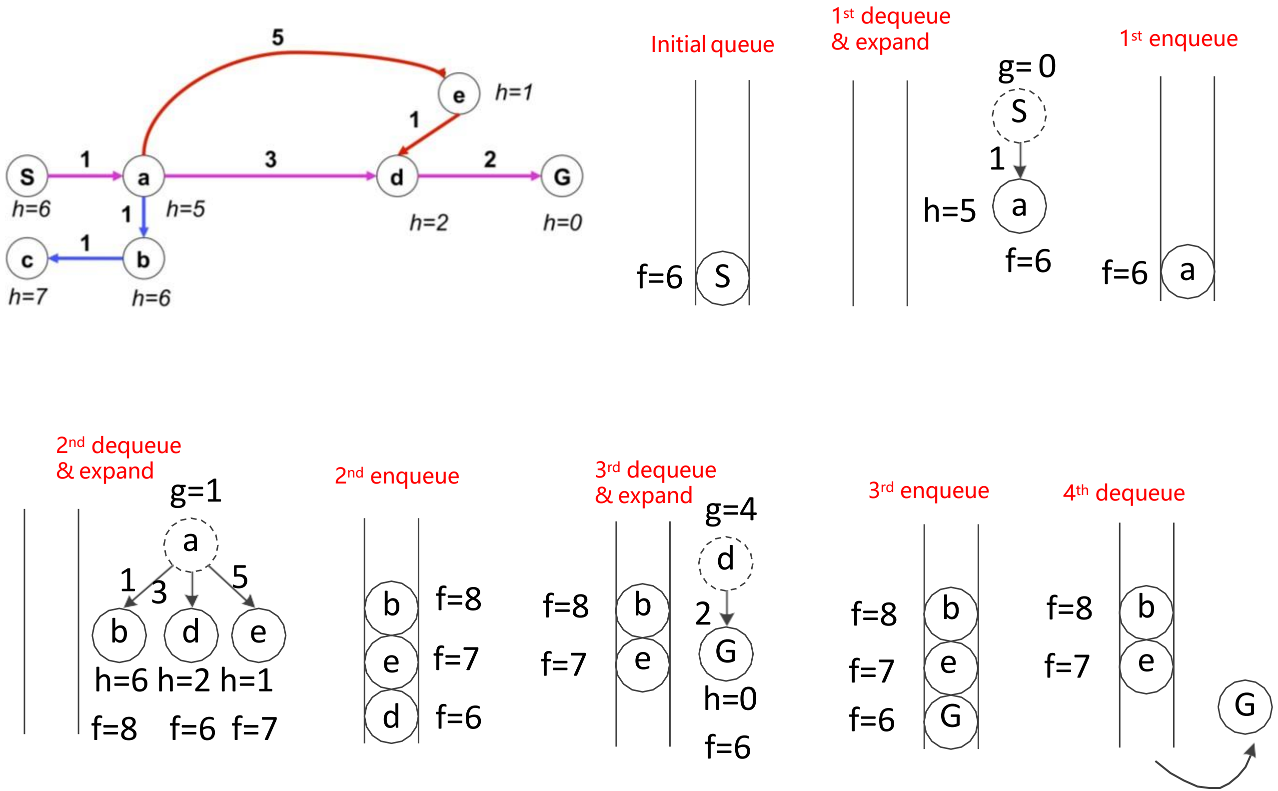
- Accumulated cost
  - g(n): The current best estimates of the accumulated cost from the start state to node "n"
- Heuristic
  - h(n): The **estimated least cost** from node n to goal state (i.e. goal cost)
- The least estimated cost from start state to goal state passing through node "n" is f(n) = g(n) + h(n)
- Strategy: expand the node with cheapest f(n) = g(n) + h(n)
  - Update the accumulated costs g(m) for all unexpanded neighbors "m" of node "n"
  - A node that has been expanded is guaranteed to have the smallest cost from the start state

A* Algorithm

- Maintain a priority queue to store all the nodes to be expanded
- The heuristic function h(n) for all nodes are pre-defined
- The priority queue is initialized with the start state X_S
- Assign $g(X_S)=0$, and g(n)=infinite for all other nodes in the graph
- Loop
  - If the queue is empty, return FALSE; break;
  - Remove the node "n" with the lowest $f(n)=g(n)+h(n)$ from the priority queue
  - Mark node "n" as expanded
  - If the node "n" is the goal state, return TRUE; break;
  - For all unexpanded neighbors "m" of node "n"
    - If g(m) = infinite
      - g(m)= g(n) + Cnm
      - Push node "m" into the queue
    - If $g(m) > g(n) + C_{nm}$
      - g(m)= g(n) + Cnm
  - end
- End Loop

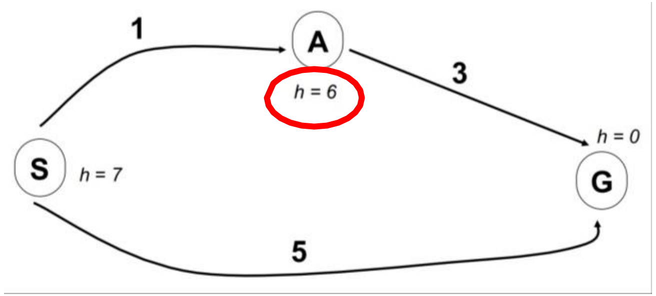
Only difference comparing to Dijkstra's algorithm

A* Example



5
e
h=1
1
1
3
2
S
a
d
G
h=6
1
h=5
h=2
h=0
1
c
b
h=7
h=6
Initial queue
f=6
S
1st dequeue
& expand
g= 0
S
1
h=5
a
f=6
1st enqueue
f=6
a
2nd dequeue
& expand
g=1
a
1
3
5
b
d
e
h=6 h=2 h=1
f=8 f=6 f=7
2nd enqueue
b f=8
e f=7
d f=6
3rd dequeue
& expand
g=4
d
f=8 b
2
f=7 e
G
h=0
f=6
3rd enqueue
f=8 b
f=7 e
f=6 G
4th dequeue
f=8 b
f=7 e
G

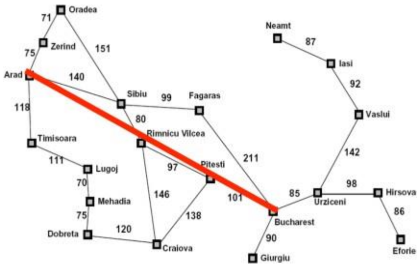
A* Optimality

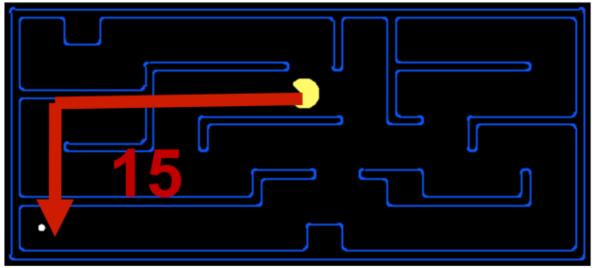


- What went wrong?
- For node A: actual least cost to goal (i.e. goal cost) < estimated least cost to goal (i.e. heuristic)
- We need the estimate to be less than actual least cost to goal (i.e. goal cost) for all nodes!

Admissible Heuristics

- A Heuristic h is admissible (optimistic) if:
  - h(n) <= h*(n) for all node "n", where h*(n) is the true least cost to goal from node "n"
- If the heuristic is admissible, the A* search is optimal
- Coming up with admissible heuristics is most of what's involved in using A* in practice.
- Example:

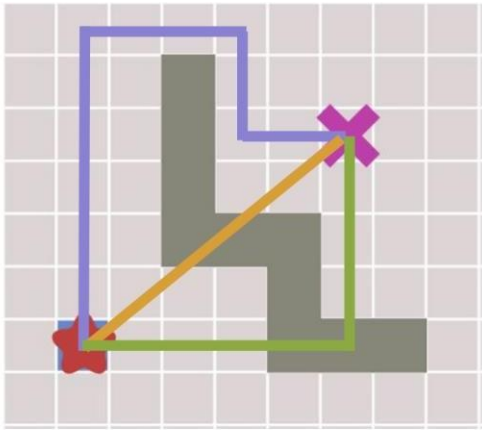




Heuristic Design

An admissible heuristic function has to be designed case by case.

- Euclidean Distance
- Manhattan Distance



Is Euclidean distance (L2 norm) admissible? Always

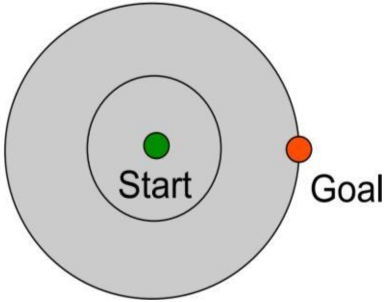
Is Manhattan distance (L1 norm) admissible? Depends

Is L∞ norm distance admissible? Always

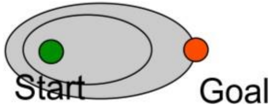
Is 0 distance admissible? Always

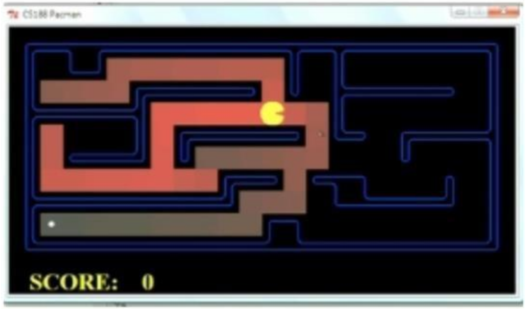
Dijkstra’ s VS A*

- Dijkstra’ s algorithm expanded in all directions



- A* expands mainly towards the goal, but does not hedge its bets to ensure optimality





Sub-optimal Solution

What if we intend to use an over-estimate heuristic?

- Suboptimal path
- Faster

Weighted A*: Expands states based on $f = g + \varepsilon h, \varepsilon > 1$ =bias towards states that are closer to goal.

Shallow local minima

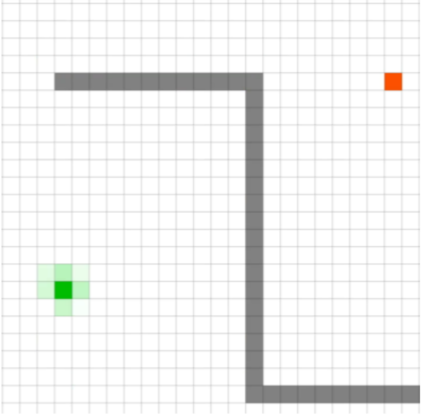
- Weighted A* Search:

➢ Optimality vs. speed
➢ ε-suboptimal:
cost(solution) <= εcost (optimal solution)
➢ It can be orders of magnitude faster than A*

Weighted A*-> $\underbrace{\text{Anytime A*-> ARA*->D*}}_{\text{Beyond the scope of this course}}$

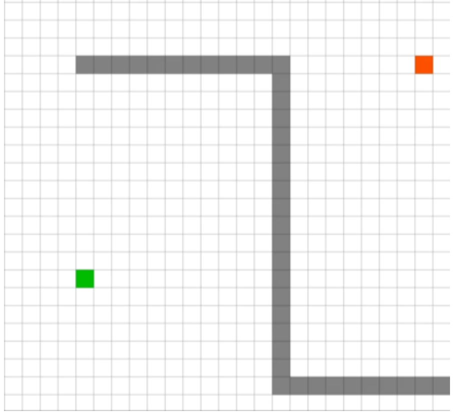
Greedy Best First Search vs. Weighted A* vs. A*

$$f = \color{red}{a} \cdot g + \color{red}{b} \cdot h$$



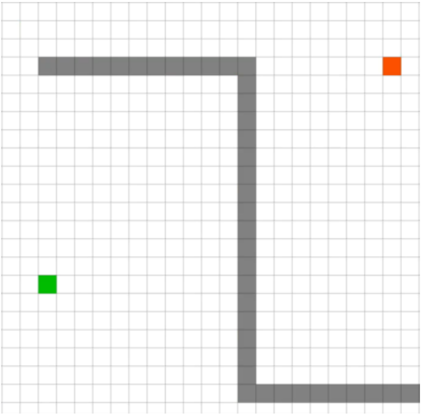
Most Greedy

$a = 0, b = 1$



Tunable Greediness

$a = 1, b = \varepsilon > 1$



Optimal

$a = 1, b = 1$

Animation from: http://qiao.github.io/PathFinding.js/visual/

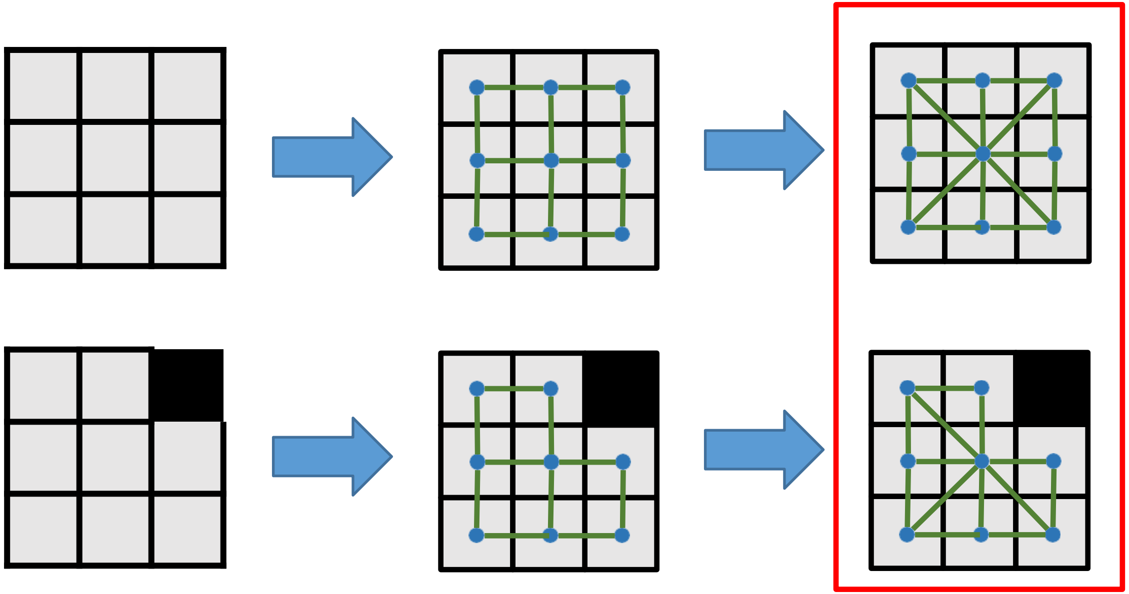
Dijkstra: $a = 1, b = 0$

Engineering Considerations

Example: Grid-based Path Search

How to represent grids as graphs?

Each cell is a node. Edges connect adjacent cells.

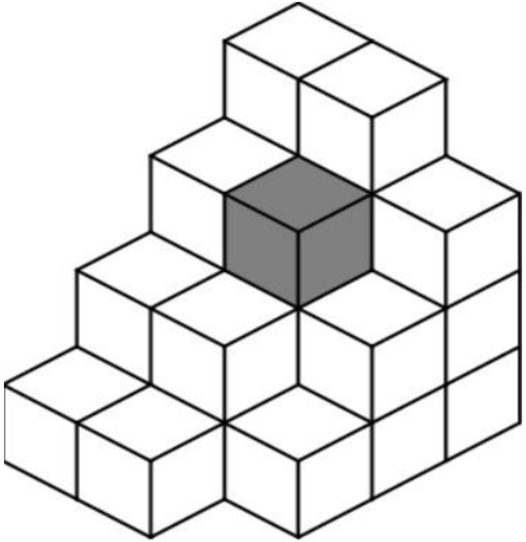


Common Choice!

4 connection

8 connection

Grid-based Path Search: Implementation



- Create a dense graph.
- Link the occupancy status stored in the grid map.
- Neighbors discovered by grid index.
- Perform A* search.

- Priority queue in C++
  - std::priority_queue
  - std::make_heap
  - std::multimap

The Best Heuristic

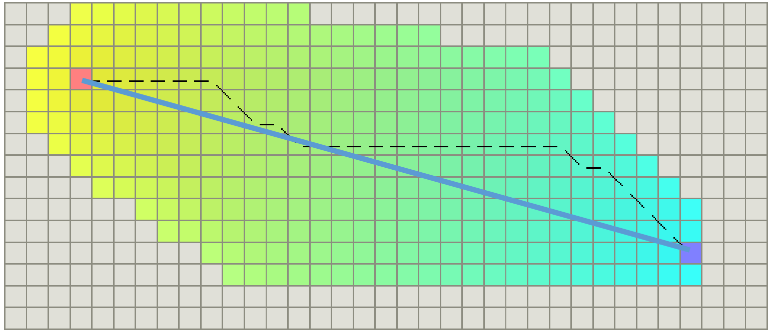
- Recall:
  - Is Euclidean distance (L2 norm) admissible?
  - Is Manhattan distance (L1 norm) admissible?
  - Is L∞ norm distance admissible?
  - Is 0 distance admissible?

They are useful, but none of them is the best choice, why?

Because none of them is tight.

Tight means who close they measure the true shortest distance.

Euclidean Heuristic



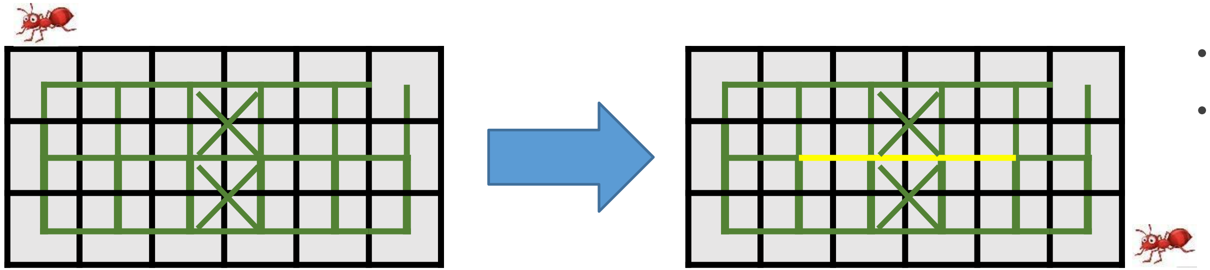
Why so many nodes expanded?

Because Euclidean distance is far from the truly theoretical optimal solution.

The Best Heuristic

How to get the truly theoretical optimal solution?

Fortunately, the grid map is highly structural.



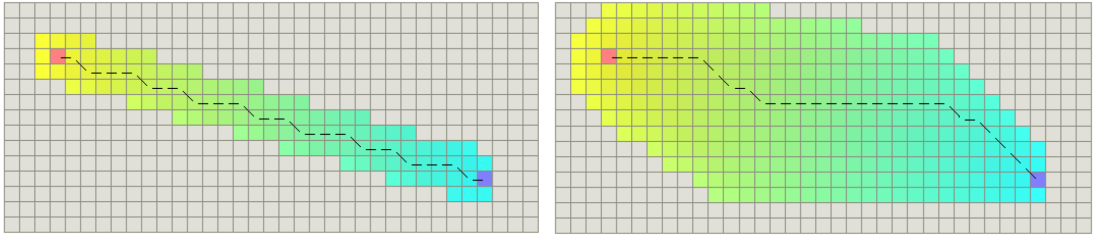
- You don’t need to search the path.
- It has the closed-form solution!

dx=abs(node.x − goal.x)
dy=abs(node.y − goal.y)
h=(dx+dy)+(√2−2)∗min(dx,dy)

For 3D case, we also have a similar version of this.

Compare

Diagonal Heuristic



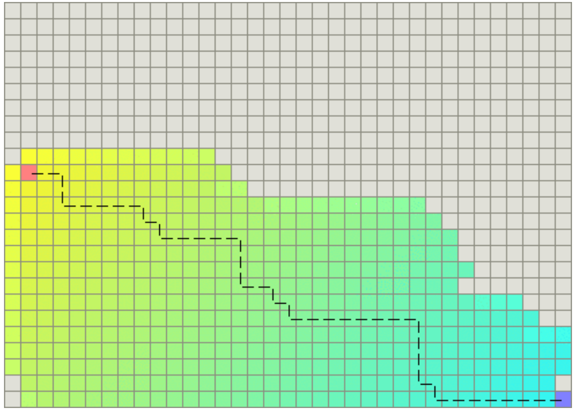
Tie Breaker

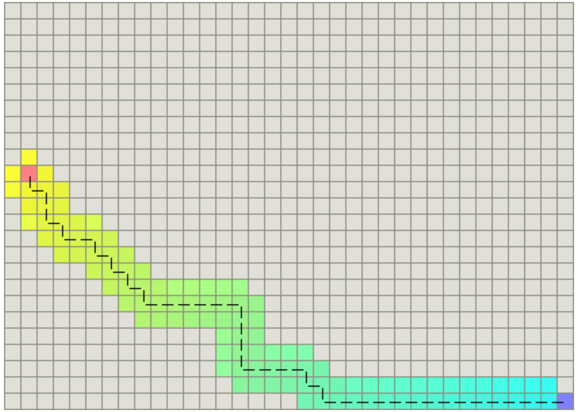
- Many paths have the same f value.
- No differences among them making them explored by A* equally.

- Manipulate the f value breaks the tie.
- Make same f values differ.
- Interfere h slightly.

$$h = h \times (1.0 + p)$$

$$p < \frac{minimum\ cost\ of\ one\ step}{expected\ maximum\ path\ cost}$$





Slightly breaks the admissibility of h, does it matter?

Tie Breaker

Core idea of tie breaker:

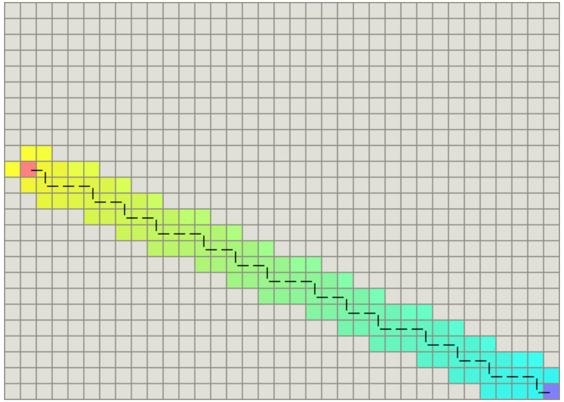
Find a preference among same cost paths

- When nodes having same f, compare their h.
- Add deterministic random numbers to the heuristic or edge costs (A hash of the coordinates).
- Prefer paths that are along the straight line from the starting point to the goal.

$$dx1 = abs(node.x - goal.x)$$
$$dy1 = abs(node.y - goal.y)$$
$$dx2 = abs(start.x - goal.x)$$
$$dy2 = abs(start.y - goal.y)$$
$$cross = abs(dx1 \times dy2 - dx2 \times dy1)$$
$$\mathrm{h} = h + cross \times 0.001$$

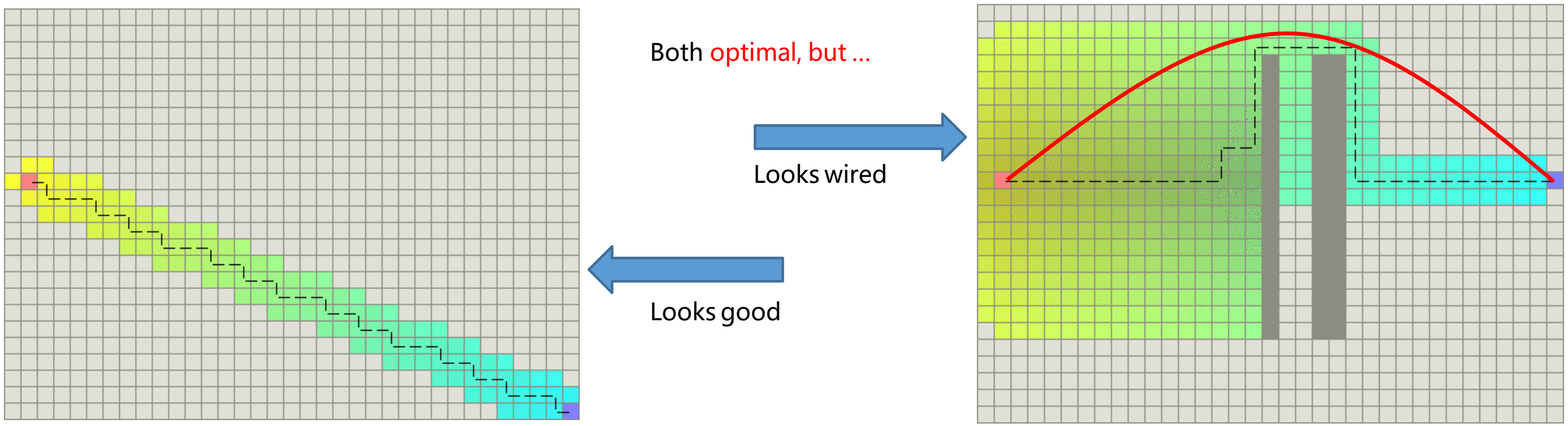
- … Many customized ways

Better/other **tie breaker**?



Tie Breaker

- Prefer paths that are along the straight line from the starting point to the goal.



It's the shortest path, but harm for trajectory generation (smoothing).

Or a systematic approach: Jump Point Search (JPS)

Jump Point Search

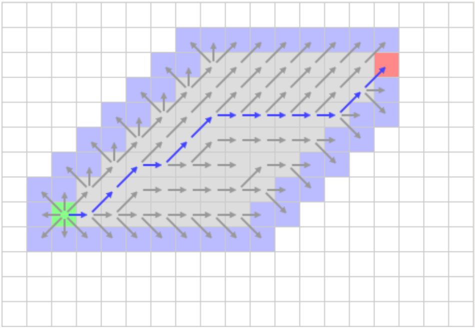
Algorithm Workflow

Jump Point Search

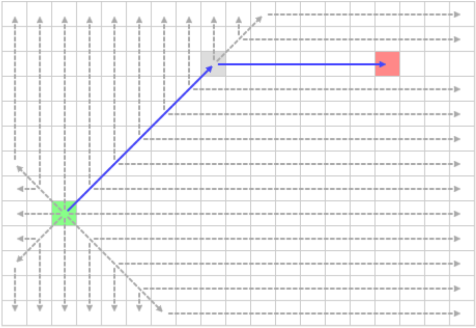
Core idea of JPS:

Find symmetry and break them.

A* explore all symmetric path.



JPS choose one path.

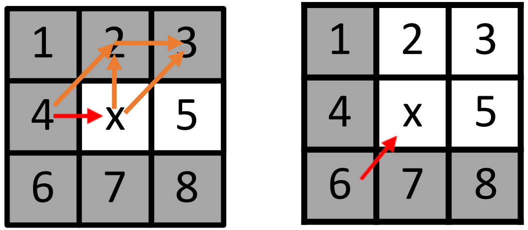


Grey node: Added in the Open List.

JPS explores intelligently, because it always looks ahead based on a rule.

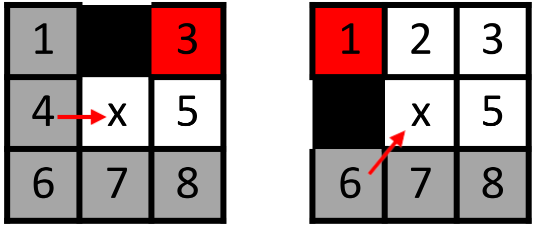
Look Ahead Rule

straight diagonal



Consider:

- current node x
- x's expanded direction



Neighbor Pruning

- Gray nodes: inferior neighbors, when going to them, the path without x is cheaper. Discard.
- White nodes: natural neighbors.
- We only need to consider natural neighbors when expand the search.

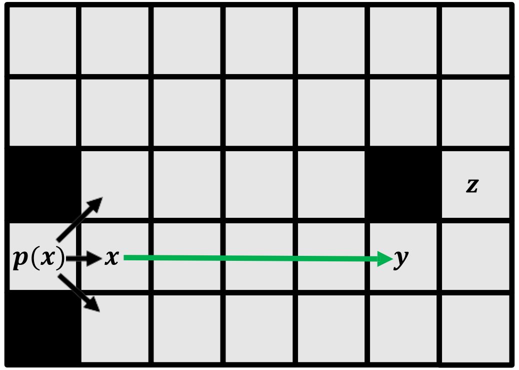
Forced Neighbors

- There is obstacle adjacent to x
- Red nodes are forced neighbors.
- A cheaper path from x's parent to them is blocked by obstacle.

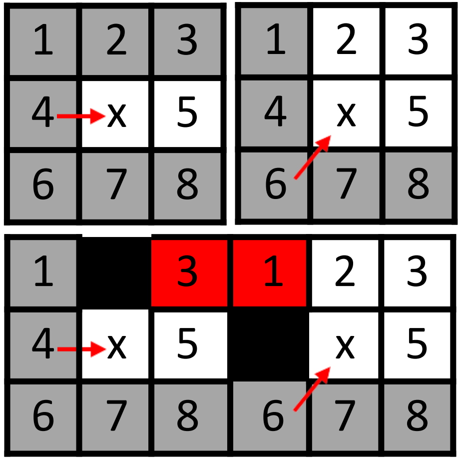
See: http://users.cecs.anu.edu.au/~dharabor/data/papers/harabor-grastien-aaai11.pdf Equation 1/2

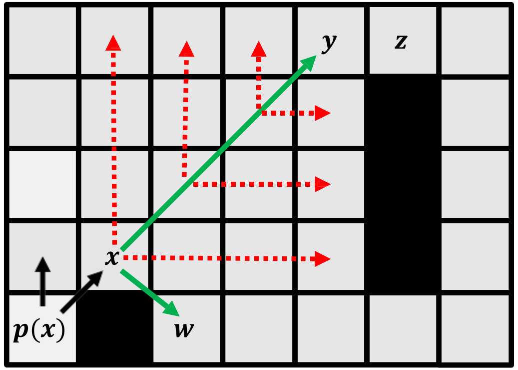
Jumping Rules

Look Ahead Rule



Jumping Straight

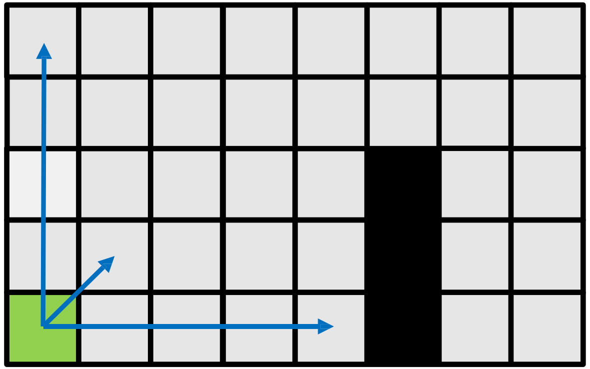




Jumping Diagonally

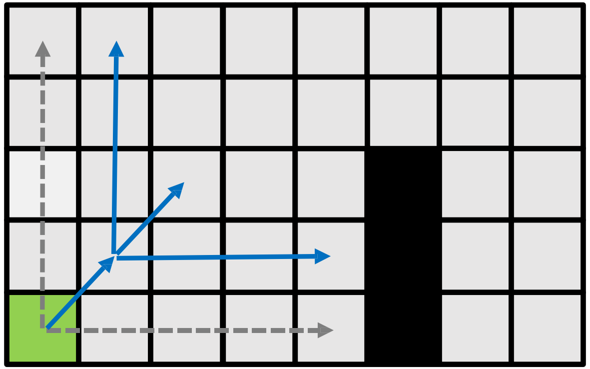
- Recursively apply straight pruning rule and identify y as a jump point successor of x. This node is interesting because it has a neighbor z that cannot be reached optimally except by a path that visits x then y.
- Recursively apply the diagonal pruning rule and identify y as a jump point successor of x.
- Before each diagonal step we first recurse straight. Only if both straight recursions fail to identify a jump point do we step diagonally again.
- Node w, a forced neighbor of x, is expanded as normal. (also push into the open list, the priority queue)

Jump Point Search



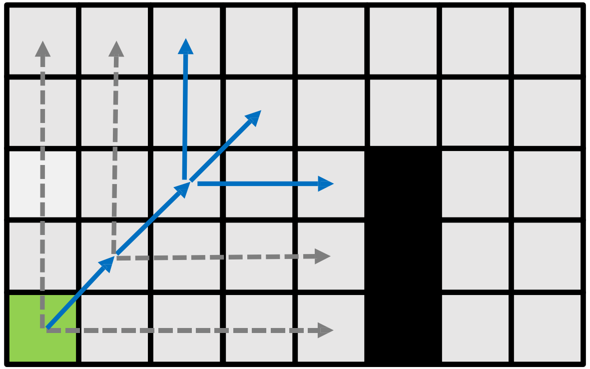
- Expand horizontally and vertically.
- Both jumps end in obstacles.
- Move diagonally.

Jump Point Search



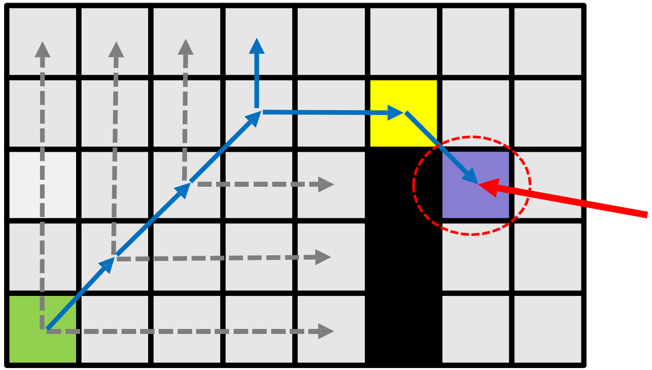
- Expand horizontally and vertically.
- Both jumps end in obstacles.
- Move diagonally.

Jump Point Search



- Expand horizontally and vertically.
- Both expansions end in obstacles.
- Move diagonally.

Jump Point Search

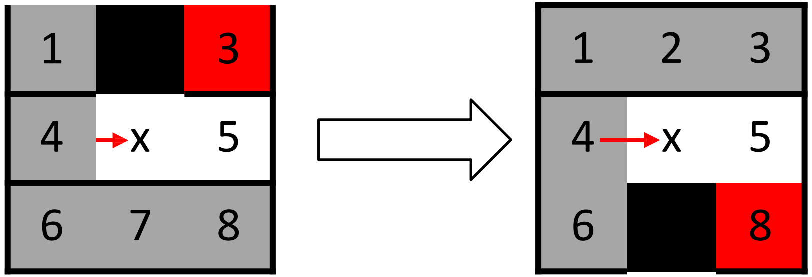


- Remember: you can only jump straight or diagonally; never piecewise jump

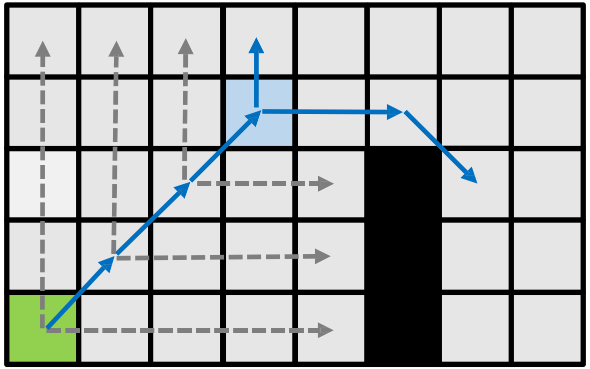
- Vertically expansion end in obstacle.
- Right-ward expansion finds a node with a forced neighbor.

Recall the rule

So we have



Jump Point Search



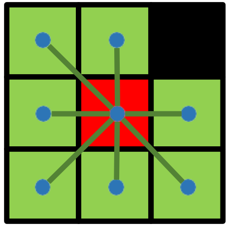
- Now this node is of interested.
- Put it to open list.

Jump Point Search

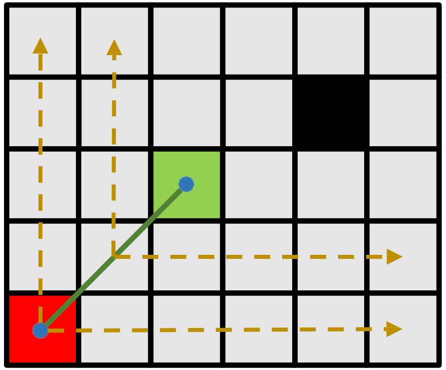
Recall A*' s pseudo-code, JPS' s is all the same!

- Maintain a priority queue to store all the nodes to be expanded
- The heuristic function h(n) for all nodes are pre-defined
- The priority queue is initialized with the start state X_S
- Assign $g(X_S)=0$, and g(n)=infinite for all other nodes in the graph
- Loop
  - If the queue is empty, return FALSE; break;
  - Remove the node "n" with the lowest f(n)=g(n)+h(n) from the priority queue
  - Mark node "n" as expanded
  - If the node "n" is the goal state, return TRUE; break;
  - For all unexpanded neighbors "m" of node "n"
    - If g(m) = infinite
      - $g(m)= g(n) + C_{nm}$
      - Push node "m" into the queue
    - If $g(m) > g(n) + C_{nm}$
      - $g(m)= g(n) + C_{nm}$
  - end
- End Loop

A*: "Geometric" neighbors



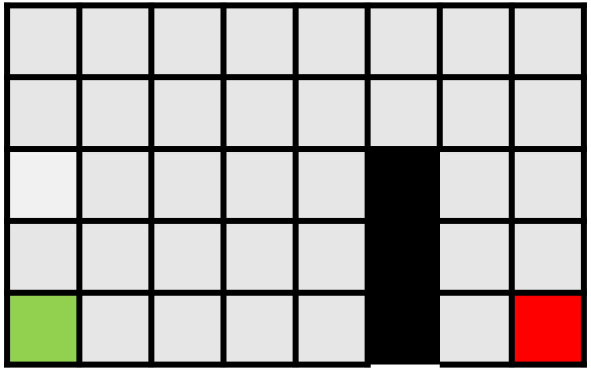
JPS: "Jumping" neighbors



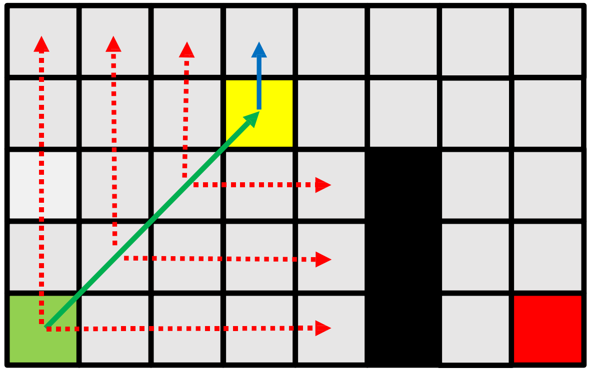
Example

Jumping Example

Planning Case

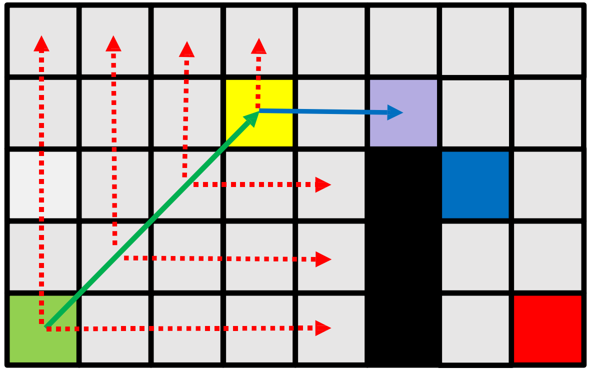


Jumping Example



- Expand—>move diagonally
- Find a critical node finally, add it into open list.
- Pop it（the only one） from the open list.
- Expand vertically, end at obstacles.

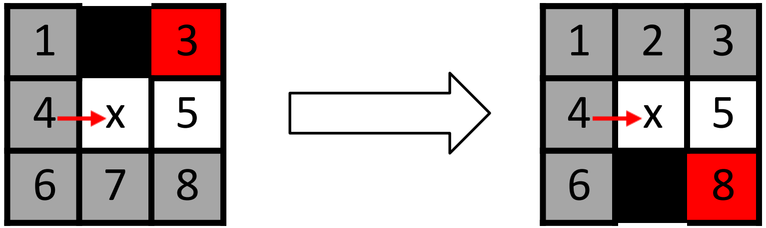
Jumping Example



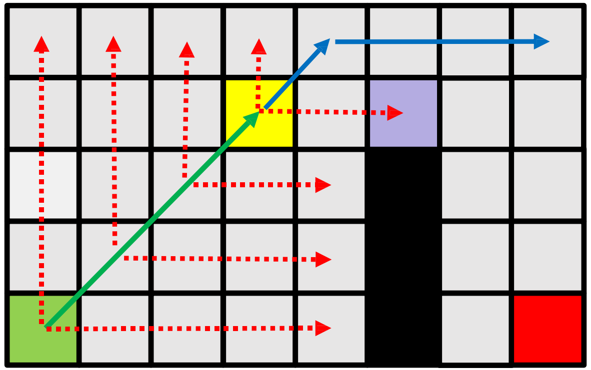
- Expand horizontally, meets a node with a forced neighbor.
- Add it to open list

Recall the rule

So we have

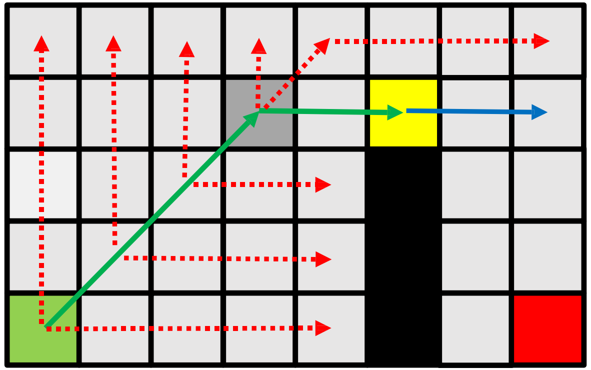


Jumping Example



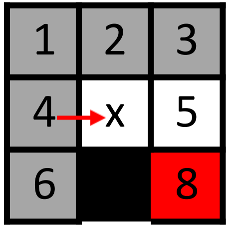
- Expand diagonally, expand, find nothing.
- Finish the expansion of the current node.

Jumping Example

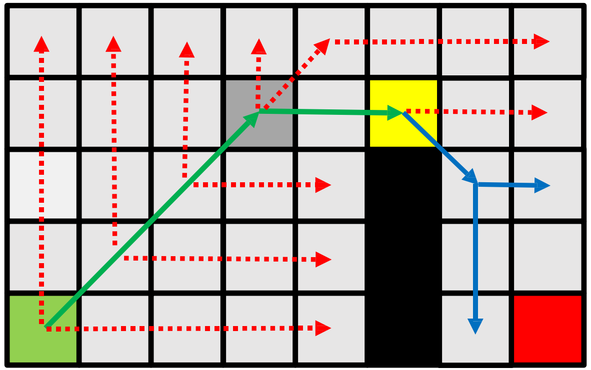


- Examine the “new best” node in the open list.
- Expand horizontally.
- Finds nothing.

Remember the rule

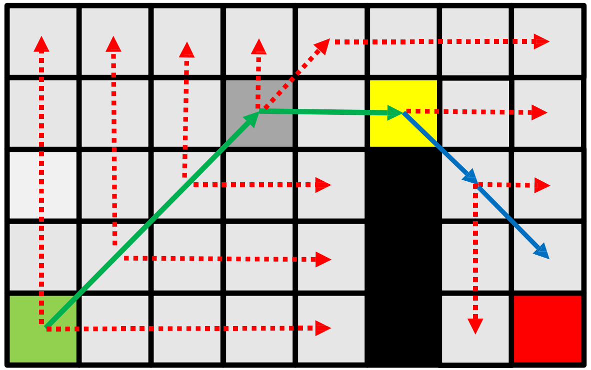


Jumping Example



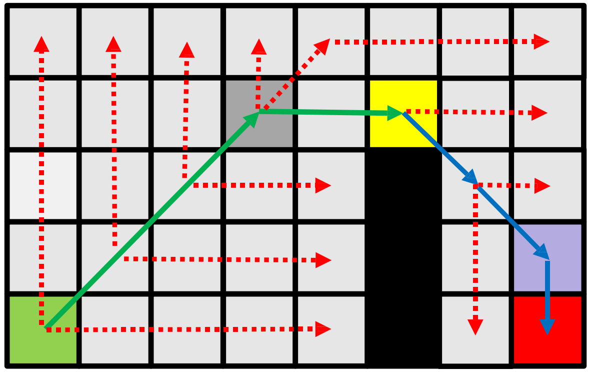
- Move diagonally.
- Expand along vertical and horizontal first.

Jumping Example



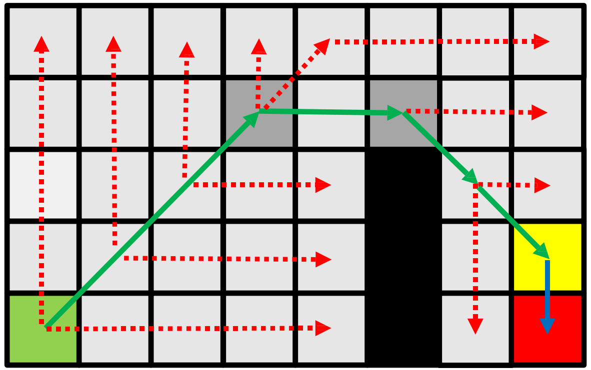
- Finds nothing.
- Move diagonally.

Jumping Example



- Expand horizontally and vertically.
- Finds the goal. Equally interested as finding a node with a forced neighbor.
- Add this node to open list.
- Finish the expand of the current node (No naturally neighbors left).
- Pop it out of the open list.

Jumping Example

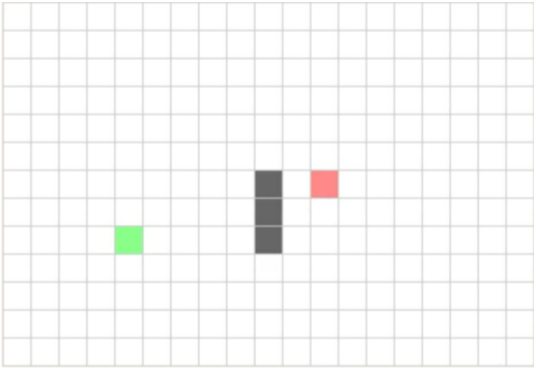


- Examine the “new best” node in the open list.
- Expand horizontally (nowhere), and vertically (finds the goal).
- The end.

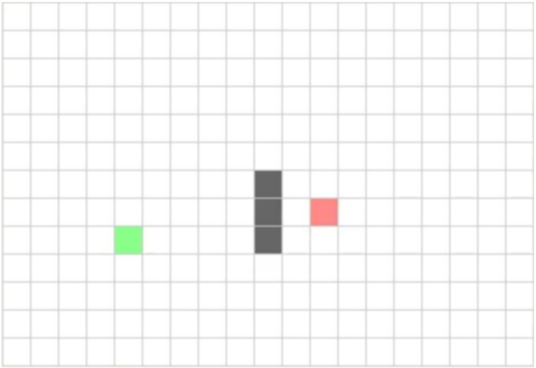
Jumping Example

Final Path

Example



(1)



(2)

Thanks:

https://zerowidth.com/2013/a-visual-explanation-of-jump-point-search.html

Extension

3D JPS

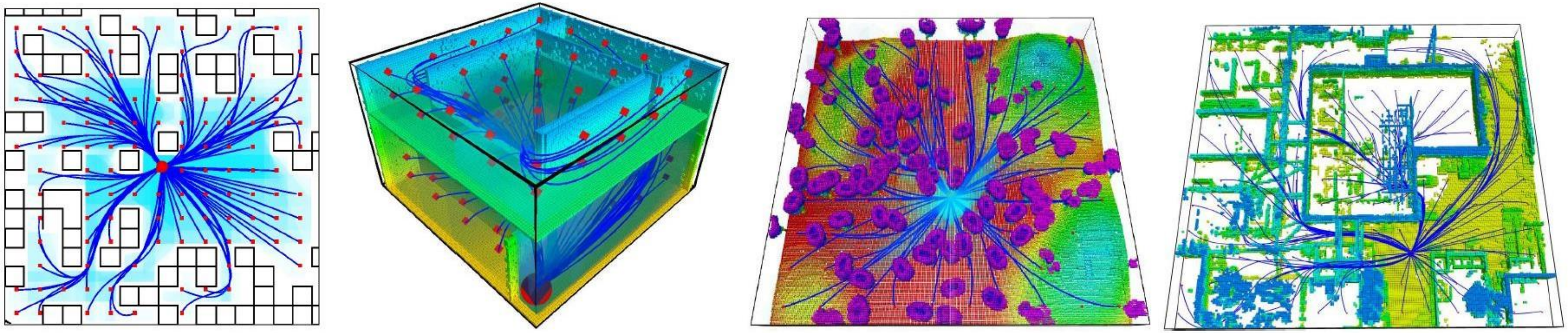


Table 1. Trajectory Generation Run Time (sec)

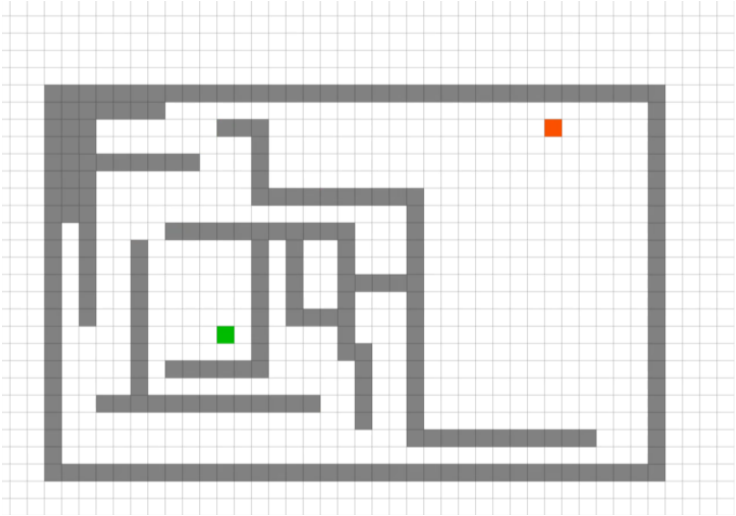
Map	Size	# of Cells	# of Trajs	Time (s)	Path Planning A*	Path Planning JPS	Convex Decomp	Traj Opt	Replan (JPS)
Random Blocks	$40 \times 40 \times 1$	1.4×10^6	130	Avg	0.57	0.034	0.0021	0.028	0.065
				Std	1.26	0.034	0.0028	0.022	0.051
				Max	9.98	0.19	0.020	0.099	0.27
Multiple Floors	$10 \times 10 \times 6$	5.9×10^5	147	Avg	6.12	0.039	0.0064	0.082	0.13
				Std	15.77	0.046	0.0038	0.041	0.081
				Max	84.56	0.22	0.021	0.23	0.45
The Forest	$50 \times 50 \times 6$	1.8×10^6	89	Avg	0.65	0.033	0.0039	0.055	0.094
				Std	1.57	0.044	0.0024	0.031	0.068
				Max	7.78	0.20	0.010	0.12	0.30
Outdoor Buildings	$100 \times 110 \times 7$	6.2×10^5	127	Avg	0.54	0.028	0.0066	0.099	0.14
				Std	1.46	0.045	0.0053	0.064	0.10
				Max	10.96	0.27	0.027	0.24	0.47

Planning Dynamically Feasible Trajectories for Quadrotors using Safe Flight Corridors in 3-D Complex Environments, Sikang Liu, RAL 2017

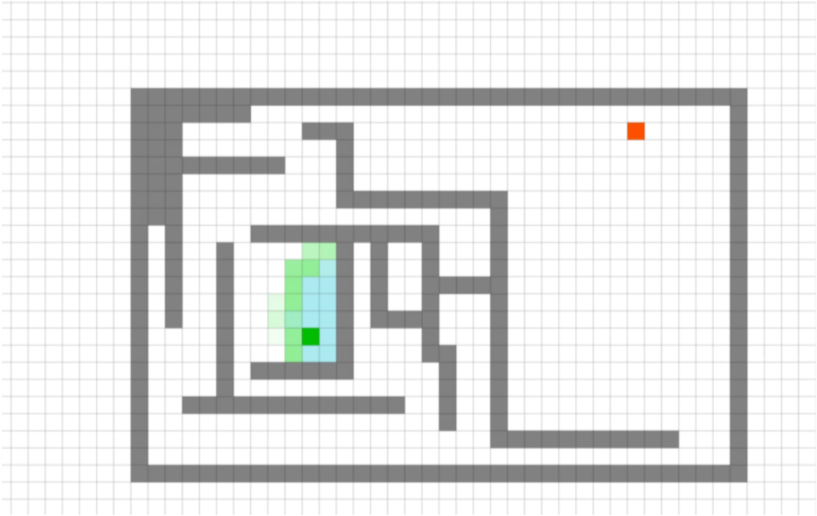
https://github.com/KumarRobotics/jps3d

Is JPS always better?

Maze-like environments



JPS

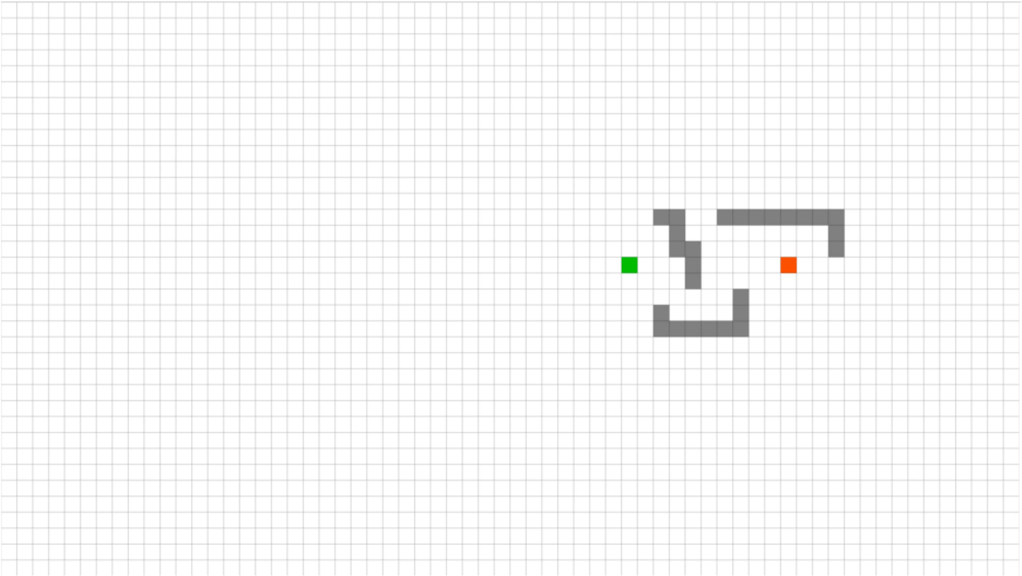


A*

Do more tests by yourself!

Thanks: http://qiao.github.io/PathFinding.js/visual/

Is JPS always better?



- This is a simple example saying "No."
- This case may commonly occur in robot navigation.
- Robot with limited FOV, but a global map/large local map.

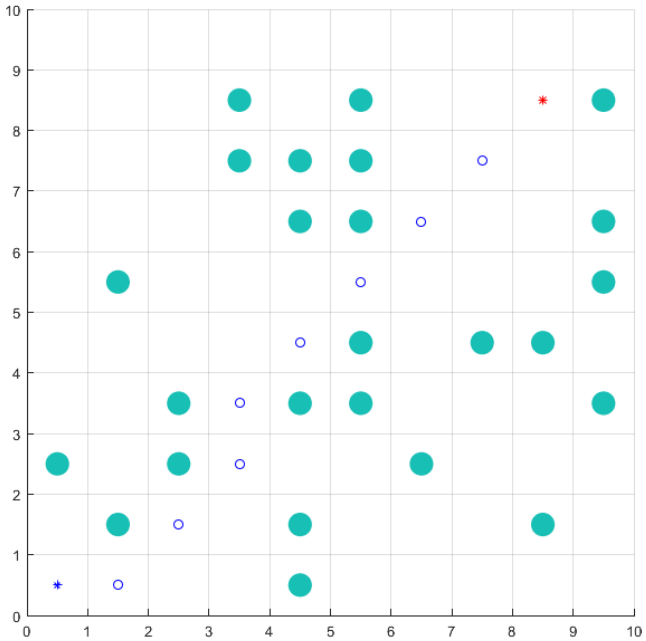
Conclusion:

- Most time, especially in complex environments, JPS is better, but far away from "always". Why?
- JPS reduces the number of nodes in Open List, but increases the number of status query.
- You can try JPS in Homework 2.
- JPS's limitation: only applicable to uniform grid map.

Homework

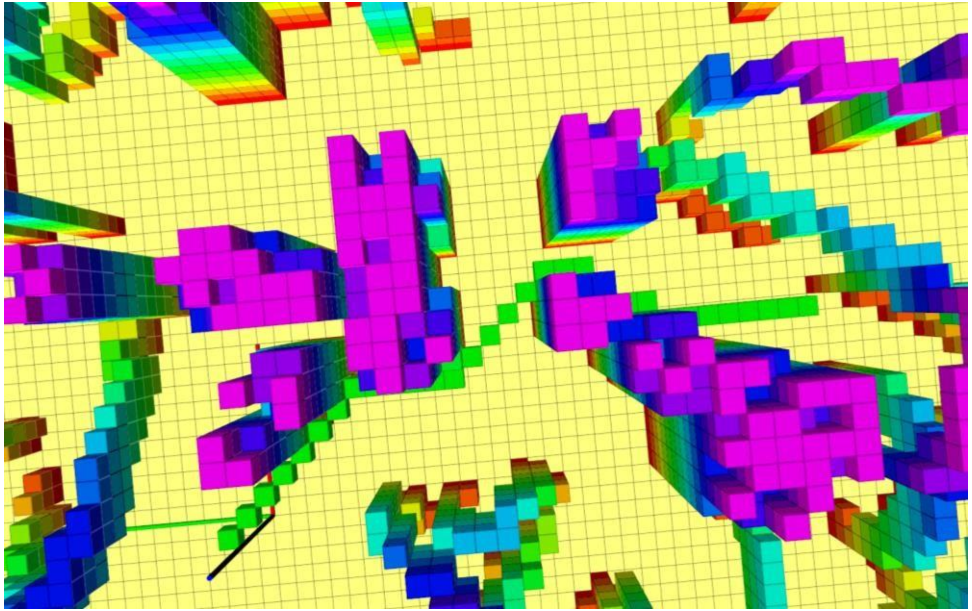
Testing Environment

Matlab



Basic

C++



Advance

Assignment：Basic

- This project work will focus on path Þnding and obstacle avoidance in a 2D grid map.
- A 2D grid map is generated randomly every time the Project is run, which contains the obstacles, start point and target point locations will be provided. You can also change the probability of obstacles in the map in obstacle_map.m
- You need to implement a 2D A* path search method to plan an optimal path with safety guarantee.

Assignment: Advance

- I highly suggest you implement Dijkstra/A* with C++/ROS.
- Complex 3d map can be generated randomly. The sparsity of obstacles in this map is tunable.
- An implementation of JPS is also provided. Comparisons can be made between A* and JPS in different map set-up.

Thanks for Listening!