- Unitary matrices
- Normal matrices
- Triangular normal matrices

Last time: if $A^* = A$, then eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

We will show a ~~...~~ stronger result: if $A \in \mathbb{C}^{m \times m}$ with $A^* = A$, then A has a set of m orthogonal eigenvectors, which can be normalized so that $\{q_1, \ldots, q_m\}$, eigenvectors of A satisfy $q_i^* q_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$.

We will prove this for all <u>normal</u> matrices, using the <u>Schur</u> factorization.

First: another type of normal matrix ($A^*A = AA^*$) is a <u>unitary</u> matrix $U^* = U^{-1} \Rightarrow U^*U = UU^* = I$.

Unitary is the complex analog of <u>orthogonal</u>: $Q^T = Q^{-1}$ (e.g., rotation).

Let $q_1, \ldots, q_m$ be the columns of U:

$$U^*U = \begin{pmatrix} - & q_1^* & - \\ & \vdots & \\ - & q_m^* & - \end{pmatrix} \begin{pmatrix} | & & | \\ q_1 & \cdots & q_m \\ | & & | \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{I_m} \quad \text{or } q_i^* q_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Properties:

(1) every eigenvalue λ of a unitary matrix has $|\lambda| = 1$

(2) eigenvalues corresponding to distinct eigenvalues are orthogonal.

<u>proof of (1)</u>: For λ an eigenvalue of unitary U with eigenvector x:

$Ux = \lambda x$, so $\|Ux\| = \|\lambda x\|$

$$\left.\begin{aligned} \|Ux\|^2 &= (Ux)^*(Ux) = x^*U^*Ux = x^*x = \|x\|^2 \\ \|\lambda x\|^2 &= (\lambda x)^*(\lambda x) = \bar{\lambda}\lambda \|x\|^2 = |\lambda|^2 \|x\|^2 \end{aligned}\right\} \therefore |\lambda| = 1, \text{ or } \lambda = e^{i\theta} \text{ for some } \theta \in \mathbb{R}.$$

(c) ~~Eigenvalues~~ Eigenvectors corresponding to distinct eigenvalues are orthogonal:

pf:
$$Ux_1 = \lambda_1 x_1, \qquad Ux_2 = \lambda_2 x_2, \quad \text{with } \lambda_1 \neq \lambda_2$$

$$\bar{\lambda}_1 x_1^* U = x_1^*$$

Thus
$$\left.\begin{aligned} (Ux_1)^* &= \bar{\lambda} x_1^* \\ (Ux_1)^* &= x_1^* U^* = x_1^* U^{-1} \end{aligned}\right\} \bar{\lambda}_1 x_1^* = x_1^* U^{-1} \Rightarrow$$

$$x_1^* x_2 = \bar{\lambda}_1 x_1^* U x_2 = \bar{\lambda}_1 \lambda_2 x_1^* x_2$$

$$\Rightarrow x_1^* x_2 (1 - \bar{\lambda}_1 \lambda_2) = 0$$

Notice: $\bar{\lambda}_1 \lambda_2 = e^{-i\theta_1} e^{i\theta_2} \neq 1$ for $\lambda_1 \neq \lambda_2$, so $x_1^* x_2 = 0$

Normal matrices: A satisfies $A^* A = A A^*$

Normal matrices include:

real symmetric: $A^T = A$, skew symmetric: $A^T = -A$

Hermitian: $A^* = A$, skew Hermitian: $A^* = -A$

Unitary: $A^* A = A A^* = I$

real orthogonal: $A^T A = A A^T = I$

circulant: $\left(\quad \right)$ (look it up on Wikipedia!)

We will show that any normal matrix has a complete set of orthonormal eigenvectors (it is "unitarily diagonalizable")

First: a lemma.

Lemma: If matrix A is both <u>triangular</u> and <u>normal</u>, then it is diagonal.

<u>proof</u>: by induction on m (size of matrix). We'll suppose A is upper-Δ.

$m=1$: trivial, b/c every 1×1 matrix is triangular, normal + diagonal.

It is <u>instructive</u> to consider $m=2$ before inductive step.

<u>$m=2$</u>: $A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$.

$$A^*A = \begin{pmatrix} \overline{a_{11}} & 0 \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} |a_{11}|^2 & \overline{a_{11}}a_{12} \\ \overline{a_{12}}a_{11} & |a_{12}|^2+|a_{22}|^2 \end{pmatrix}$$

$$AA^* = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}\begin{pmatrix} \overline{a_{11}} & 0 \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix} = \begin{pmatrix} |a_{11}|^2+|a_{12}|^2 & a_{12}\overline{a_{22}} \\ a_{22}\overline{a_{12}} & |a_{22}|^2 \end{pmatrix}$$

If A is normal, $A^*A = AA^*$ so $a_{12}=0$ $\therefore$ A is diagonal.

Inductive hypothesis (IH): suppose triangular + normal $\Rightarrow$ diagonal for matrices of dim $l=1,\dots,m-1$. Show it holds for dim. m.

$$AA^* = \begin{pmatrix} A_{m-1} & b \\ 0 & a_{mm} \end{pmatrix}\begin{pmatrix} A_{m-1}^* & 0 \\ b^* & \overline{a_{mm}} \end{pmatrix} = \begin{pmatrix} A_{m-1}A_{m-1}^*+bb^* & b\overline{a_{mm}} \\ a_{mm}b^* & |a_{mm}|^2 \end{pmatrix},$$

where A_{m-1} is upper-Δ

$$A^*A = \begin{pmatrix} A_{m-1}^* & 0 \\ b^* & \overline{a_{mm}} \end{pmatrix}\begin{pmatrix} A_{m-1} & b \\ 0 & a_{mm} \end{pmatrix} = \begin{pmatrix} A_{m-1}^*A_{m-1} & A_{m-1}^*b \\ b^*A_{m-1} & \|b\|^2+|a_{mm}|^2 \end{pmatrix}.$$

If A is normal then $b=0$.

By IH, $A_{m-1}^*A_{m-1} = A_{m-1}A_{m-1}^* \Rightarrow A_{m-1}$ is normal $\therefore$ diagonal.

$\therefore$ A is diagonal.

Same idea works if A is lower-Δ (on HW).