- partial + global methods
- partial methods:
  - power iteration
  - inverse iteration
  - ~~Rayleigh quotient iteration~~

Recall: 2 stages of eigenvalue computation

(1) Reduce A to upper-Hessenberg form: Householder reduction

(2) Apply an iterative method to converge to ~~eigenvector~~ eigenvalues (eigenvectors).

Two main classes of iterative methods for eval problems.

<u>partial methods</u>: compute select eigenvalue / eigenvector pairs

- power methods
- inverse iteration
- Rayleigh quotient iteration

<u>global methods</u>: compute entire spectrum

- QR iteration to Schur form $\Rightarrow$ eigenvalues, not eigenvectors
- variants of QR (~~for~~ using shifts)

Global methods generally compute Schur factorizations revealing eigenvalues.

The (approx) eigenvalues can then be used to start <u>partial methods</u> revealing eigenvectors.

Power methods

$A \in \mathbb{C}^{n\times n}$

x_0 = initial choice ("guess") for eigenvector

idea: $x_{k+1} = A x_k$

"Often" $x_k \to$ an eigenvector of A corr to an eigenvalue of A ~~ev~~ of largest magnitude.

Here's why: Suppose A is diagonalizable with $\{v_1, \dots, v_n\}$ LI eigenvectors where $\|v_i\| = 1$ and $Av_i = \lambda_i v_i$, $i = 1, \dots, n$; and $|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_n|$.

Expand $x_0 = \sum_{i=1}^{n} c_i v_i$, supposing $c_1 \neq 0$.

Obtain:

$$x_1 = A x_0 = A \sum c_i v_i = \sum c_i \lambda_i v_i$$

$$x_2 = A x_1 = A \sum c_i \lambda_i v_i = \sum c_i \lambda_i^2 v_i$$

$$\vdots$$

$$x_k = A x_{k-1} = A \sum c_i \lambda_i^{k-1} v_i = \sum c_i \lambda_i^k v_i$$

$$= \lambda_1^k \left[c_1 v_1 + \sum_{i=2}^{n} c_i \left(\frac{\lambda_i}{\lambda_1}\right)^k v_i \right]$$

$\left(\frac{\lambda_i}{\lambda_1}\right)^k \to 0$ as $k \to \infty$.

$\Rightarrow x_k$ approaches a multiple of v_1 as $k \to \infty$.

Power method:

Given x_0 with $\|x_0\|=1$

for $k=1,2,\ldots$

$w = Ax_{k-1}$ ← Apply A

$x_k = w/\|w\|_2$ ← normalize

$\lambda_k = x_k^* A x_k$ ← Rayleigh quotient

end

Convergence properties

By induction, one can show $x_k = \dfrac{A^k x_0}{\|A^k x_0\|_2}$ (HW).

Assuming A is diagonalizable: $x_0 = \sum_{i=1}^{n} c_i v_i$

Then $A^k x_0 = c_1 \lambda_1^k \left(v_1 + \sum_{i=2}^{n} \frac{c_i}{c_1} \left(\frac{\lambda_i}{\lambda_1}\right)^k v_i \right)$, $k=1,2,\ldots$

and $x_k = \dfrac{c_1 \lambda_1^k (v_1 + e_k)}{\|c_1 \lambda_1^k (v_1 + e_k)\|}$, $e_k = \underbrace{\sum_{i=2}^{n} \frac{c_i}{c_1} \left(\frac{\lambda_i}{\lambda_1}\right)^k v_i}_{\to 0 \text{ as } k \text{ incr.}}$

Thm: Let $A \in \mathbb{C}^{n\times n}$ diagonalizable with $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$.

Then assuming $c_1 \neq 0$ $\exists$ a constant $C>0$ s.t.

$\|\tilde{v}_k - v_1\|_2 \leq C \left|\frac{\lambda_2}{\lambda_1}\right|^k$, $k \geq 1$, $\tilde{v}_k = \dfrac{x_k \|A^k x_0\|}{c_1 \lambda_1^k} = v_1 + \sum_{i=2}^{n} \frac{c_i}{c_1}\left(\frac{\lambda_i}{\lambda_1}\right)^k v_i$

pf: Assuming $\{v_i\}_{i=1}^{n}$ are a basis of eigenvectors, $\|v_i\|_2 = 1$, $i=1,\ldots,n$,

$\|\tilde{v}_k - v_1\| = \left\| \sum_{i=2}^{n} \frac{c_i}{c_1}\left(\frac{\lambda_i}{\lambda_1}\right)^k v_i \right\|$

$$\overset{HW}{\leq} \left(\sum_{i=2}^{n} \left(\frac{c_i}{c_1}\right)^2 \left(\frac{\lambda_i}{\lambda_1}\right)^{2k} \right)^{1/2} \leq \left|\frac{\lambda_2}{\lambda_1}\right|^k \underbrace{\left(\sum_{i=2}^{n} \left(\frac{c_i}{c_1}\right)^2 \right)^{1/2}}_{C} = C \left|\frac{\lambda_2}{\lambda_1}\right|^k$$

→ C depends on the initial vector x_0.

Inverse iteration

If the eigenvalues of A are $\lambda_1, \dots, \lambda_n$ are all nonzero, the eigenvalues of A^{-1} are $1/\lambda_1, \dots, 1/\lambda_n$, ~~the~~ with the same corr. eigenvectors.

$$Ax = \lambda x \Rightarrow \lambda^{-1} x = A^{-1} x.$$

If $\lambda_i - \rho_i \neq 0$, $i = 1, \dots, n$

$$Ax = \lambda x \Leftrightarrow (A - \rho I)x = (\lambda - \rho)x \qquad \frac{1}{\lambda - \rho} x = (A - \rho I)^{-1} x$$

If ρ_i, a good approx to λ_i is known, the Power iteration applied to $(A - \rho I)^{-1}$ can rapidly converge.

Algorithm: inverse iteration (with shift σ)

Choose σ, x_0 with $\|x_0\| = 1$
for $k = 1, 2, \dots$
  solve $(A - \sigma I)w = x_{k-1}$ (e.g., by LU)
  $x_k = w / \|w\|$
  $\lambda_k = x_k^* A x_k$ (Rayleigh quot.)
end.

Note: If $Ax = \lambda x$, x_k is an approx to x.

The least-squares sol'n $\hat{\lambda}$ to $Ax_k = \hat{\lambda} x_k$ is

$$x_k^* x_k \hat{\lambda} = x_k^* A x_k$$

$$\left.\hat{\lambda} = \frac{x_k^* A x_k}{x_k^* x_k}\right\} \text{Rayleigh quotient.}$$