1 Question 1

1.1 part a

We assume that A is invertible such that $AA^{-1} = A^{-1}A = I$. Then

$$AXA(A^{-1}) = AA^{-1} \to AX = I \to X = A^{-1} = (U\Sigma V^*)^{-1} = V\Sigma^{-1}U^* \tag{1}$$

1.2 part b

Making the same assumptions as before, in addition to the fact that X is also invertible such that $XX^{-1} = X^{-1}X = I$ we see that

$$(X^{-1})XAX = (X^{-1})X \to AX = I \to X = V\Sigma^{-1}U^* \tag{2}$$

1.3 part c

Using ideas from the original equations in part a and b we note that

$$A^* = (AXA)^* = A^*(AX)^* = A^*(AX) \tag{3}$$

where we used the fact the operators commute as the equation in part c demands. At this point, we see that

$$X = (A^*A)^{-1}A^* \to X = (V\Sigma U^*U\Sigma V^*)^{-1}(V\Sigma U^*) \tag{4}$$

Consequently,

$$X = V\Sigma^{-2}V^*V\Sigma U^* = V\Sigma^{-1}U^* \tag{5}$$

1.4 part d

We note that we can also obtain the following from parts a and b as seen in part c:

$$X^* = (XAX)^* = X^*A^*X^* = X^*(XA)^* = X^*(XA) = U\Sigma^{-1}V^* \tag{6}$$

where the latter two terms arise by enforcing the condition seen in part d. Thus we are left with $X^* = X^*XA \to A = (X^*X)^{-1}X^* = U\Sigma V^*$. Insering the above expression in for X and X^* in terms of SVD, we see that

$$\left(U\Sigma^{-1}V^*V\Sigma^{-1}U^*\right)^{-1}(U\Sigma^{-1}V^* = U\Sigma V^* = A \tag{7}$$

Thus, we have shown that be enforcing condition d, we arrive at appropriate expressions for X and X^*.

2 Question 2 - Book 12.1

Using the definition of the Frobenius norm and the fact that $||A||_2 = \sigma_{max} = 100$, we note that

$$||A||_F = \left(\sum_{i=1}^{202}\sigma_i\right)^{\frac{1}{2}} = \left(\sigma_{max} + \sum_{i=2}^{202}\sigma_i\right)^{\frac{1}{2}} \tag{8}$$

We recall from the singular value decomposition that the singular values are ordered in a decreasing fashion. Furthermore, in a general situation, we note that the sum of remaining singular values must be equal to 201, since $||A||_F = 101$. And since we are summing over the remaining 201 singular values, we note that any general $\sigma_i \leq 1$ for $2 < i < 202$. A lower bound for the condition number can be achieved by maximizing σ_{202} such that

$$\kappa(A) = \frac{\sigma_{max}}{\sigma_{202}} \tag{9}$$

This maxima is obtain if $\sigma_2 = \cdots = \sigma_{202} = 1$. Thus the sharpest possible lower bound is $\frac{100}{1} = 100$.

3 Question 3

Noting a matrix $A \in \mathbb{C}^{m\times n}$ has full rank, we can decompose A via the SVD such that $A = U\Sigma V^*$, such that U and V^* are unitary matrices and

$$\Sigma = \begin{pmatrix} \sigma_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \sigma_m \end{pmatrix} \tag{10}$$

where the eigenvalues of placed along the diagonal of Σ and ordered such that $\sigma_1 > \cdots > \sigma_m$. Thus we note that the induced matrix 2 norm can be written such that

$$\frac{||Ax||_2}{||x||_2} = \frac{||(U\Sigma V^*)x||_2}{||x||_2} = \frac{||\Sigma x||_2}{||x||_2} \tag{11}$$

remembering the properties of said normal when unitary matrices are involved. Assuming we are dealing with vectors x that can assume any column of the $n \times n$ identiy matrix, we immediately note that $||Ax||_2$ assumes a maximum value whenever

$$x = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} \tag{12}$$

and a minimum value whenever

$$x = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \tag{13}$$

Thus, we have shown that

$$\sigma_1 \geq \frac{||Ax||_2}{||x||_2} \geq \sigma_m \tag{14}$$

where strict equality is achieved in the prior 2 examples, and inequalities are achieved when dealing with any of the remaining columns of the identity matrix.

4 Question 4

Let $\kappa(AB) = ||(AB)||_2||(AB)^{-1}||_2$, $\kappa(A) = ||A||_2||A^{-1}||_2$, and $\kappa(B) = ||B||_2||B^{-1}||_2$ be the appropriately defined conditions numbers for some nonsingular matrices A and B. Because of the identity

$$||XY||_p \leq ||X||_p||Y||_p \tag{15}$$

for $1 \leq p \leq \infty$, we note that separately, (and if we are working in the 2 norm)

$$||(AB)||_2 \leq ||A||_2||B||_2 \tag{16}$$

and

$$||(AB)^{-1}||_2 \leq ||A^{-1}||_2||B^{-1}||_2 \tag{17}$$

must be true. Therefore, we can combine these statements, such that

$$\kappa(AB) \leq \kappa(A)\kappa(B) \tag{18}$$

5 Question 5

A 2×2 matrix A and B that satisfy $||(AB)^{\dagger}|| \neq ||B^{\dagger}A^{\dagger}||$ is

$$A = \begin{pmatrix} 1 & -0.9888 \\ 1 & -1.00001 \end{pmatrix} \tag{19}$$

and

$$B = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix} \tag{20}$$

I attempting finding 3 by 3 matrices for part a, but to no avail. I recognize that I need to work with matrices that are singular.