1 Question 1

Functionally, the end result is the same, however, the

A Householder reflection matrix H -defined as $H(\omega) = I - \frac{2}{\omega^*\omega}\omega\omega^*$ -can be understood as a reflection of the vector x across the nullspace of ω^*. When it acts on a vector $x \in \mathbb{C}^n$ yields a map

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{pmatrix} \to Hx = \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ XX_k \\ \vdots \\ 0 \end{pmatrix} \tag{1}$$

that essentially leaves the components of the vector x unchanged if below some index k, changes the index at k, and all indices beyond k are 0. So in this case, $\omega = x + sign(x)||x||_2 e_1$, which can be used to define the Householder matrix H that transforms x into αe_1 for some constant α. The net result is some column of $R = Q^* x$. Furthermore we see that since the Householder matrix is defined to be unitary such that

$$H = \begin{pmatrix} I & 0 \\ 0 & F \end{pmatrix} \tag{2}$$

where I is the (k-1)x(k-1) identiy matrix, F is the unitary matrix seen to be $F = I - 2\frac{vv^*}{v^*v}$ for some $v = sign(x_1)||x||_2 e_1 + x$. Thus, we note that it would be possible for the determinant for such a matrix to be -1, based on the form of F.

The Givens rotation matrix essentially incorporates rotation elements via trig functions into the Identity matrix at locations relevant to particular rows/columns needing to be zeroed out. It can be shown that P_{ij} rotates x through some angle θ in the ij plane. So the set of $P_{1:}$ matrices define the orthogonal matrix $Q^* = P_{1n} \cdots P_{13}P_{12}$, which should yield the same column of R as the relevant Householder reflection matrix did. However from its definition, we know the Householder matrix will have the all 1's on the diagonal with two pairs of cos insertion at two locations, in addition to a symmetrically distributed pair of sin and $-sin$ somewhere off the main diagonal. Thus if we were to take the determinate of such a matrix, we note that we would essentially get $sin^2(x) + cos^2(x) = 1$.

In summary, although the two matrices perform the same function, because their determinants are different, they can not be equivalent. On a side note, this problem has traumatized me. I naiively assumed these transformations - as long as they perform the same function - are equivalent. In other words, there should only be one unique matrix Q^* that transforms A into R. But I suppose since these are fundamentally rotation matrices, we can pick any rotation increment through some theta as long as we maintain orthogonality. Nice.

2 Question 2

Upon distributing and rearranging terms - as well as the fact that $b - Ax = 0$ - we see that $A(x + \Delta x) = b + \Delta b \to Ax + A\Delta x = b + \Delta b \to A\Delta x = (b - Ax) + \Delta b \to A\Delta x = \Delta b \to x = A^{-1}\Delta b$. We can use the fact that $\frac{||\hat{x} - x||}{||x||}$ to see that the above can be rewritten such that

$$\frac{||A^{-1}\Delta b||}{||x||} = \frac{||\Delta x||}{||x||} \tag{3}$$

Thus we are tasked with finding $||x||$. Recalling $b = Ax$, we can use the Cauchy Swartz inequality to see that $||b|| = ||Ax|| \leq ||A||||x|| \rightarrow \frac{||b||}{||A||} \leq ||x||$. Inserting this into our relation shows us that

$$\frac{||\Delta x||}{||x||} \leq \frac{||A||||A^{-1}\Delta b||}{||b||} \leq \frac{||A||||A^{-1}||||\Delta b||}{||b||} = Cond(A)\frac{||\Delta b||}{||b||} \tag{4}$$

3 Question 3

3.1 part a

Knowing A is invertible, we begin by supposing $A + E$ is singular and $||A^{-1}||||E|| < 1$. Then $(A + E)x = 0$ for some $x \neq 0$. This implies that

$$Ax + Ex = 0 \rightarrow (A^{-1}A)x + A^{-1}Ex = 0 \rightarrow x = -A^{-1}Ex \tag{5}$$

using the fact that $A^{-1}A = I$. Taking the p-norm on both sides and realizing $||AB|| \leq ||A||||B||$ for matrices A, B yields

$$||x||_p = ||A^{-1}Ex||_p \rightarrow ||x||_p \leq ||A^{-1}||_p||E||_p||x||_p \rightarrow 1 \leq ||A^{-1}||_p||E||_p \tag{6}$$

The final step was achieved by dividing through by the p-norm of x. So, we see that this contradicts our original assertion that $||A^{-1}||||E|| < 1$. Therefore, we know $A + E$ must be nonsingular.

3.2 part b

To begin, we let $(A+E)^{-1} = C$, then we know $(A+E)C = I$. Using the identities $||X+Y|| \geq ||X|| + ||Y||$ and $||XY|| \leq ||X||||Y||$, we see that $1 = ||I|| = ||(A+E)C|| \geq ||AC|| + ||EC|| \geq ||AC|| - ||EC|| \geq ||A||||C|| - ||E||||C|| \rightarrow 1 \geq ||C||(||A|| - ||E||)$. We rearrange, multiply by a factor of 1, then again use the identity $||A^{-1}||||A|| \geq ||AA^{-1}|| = 1$ such that

$$||C|| \leq \frac{1}{||A|| + ||E||} \cdot \frac{||A^{-1}||}{||A^{-1}||} \leq \frac{||A^{-1}||}{1 - ||A^{-1}||||E||} \tag{7}$$

Remembering $C = (A+E)^{-1} =$, we see that $||(A+E)^{-1}|| \leq \frac{||A^{-1}||}{1-||A^{-1}||||E||}$

4 Question 4

The problem was solved by reducing the multiplications in the numerator and denominator intro a multiplication of ratios such that

$$\frac{n(n-2)(n-4)\cdots 2}{(n-1)(n-3)\cdots 1} = \frac{n}{(n-1)} \cdot \frac{n-2}{n-3} \cdot \frac{n-4}{n-5} \cdots \tag{8}$$

For the value of R(4,000,000) I get 2506.628.

5 Question 5

The solution to this problem requires the following equivalence

$$\frac{cosh(x)}{1 + sinh(x)} = \frac{e^x + e^{-x}}{1 + e^x - e^{-x}} = \frac{e^x(1 + e^{-2x})}{e^x(\frac{1}{e^x} + 1 - e^{-2x})} = \frac{(1 + e^{-2x})}{(\frac{1}{e^x} + 1 - e^{-2x})} \tag{9}$$

So I performed the sum on this quantity. This was rearranged with the intent so that as $x \rightarrow \infty$ e^x doesn't blow up. I get the value of the sum to be 1.000730.