- norm properties
- vector p-norms
- matrix norms

Vector norms

Let $x \in \mathbb{C}^n$, the p-norm of x is

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \quad p \geq 1.$$

For $p < 1$ this does not produce a norm.

Properties of norms:

(1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$

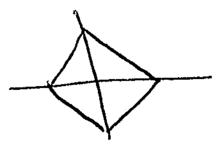
(2) $\|x + y\| \leq \|x\| + \|y\|$ "triangle ineq"

(3) $\|\alpha x\| = |\alpha| \|x\|$, $\alpha \in \mathbb{C}$.

Important p-norms:

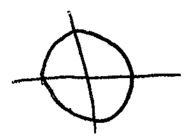
unit ball $(n=2)$ $\|x\|_p = 1$

$p=1$: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$



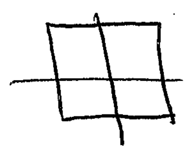
$|x_1| + |x_2| = 1$

$p=2$ $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} = \sqrt{x^* x}$



$(x_1^2 + x_2^2)^{1/2} = 1$

$p=\infty$ $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$



$\max \{ |x_1|, |x_2| \} = 1$

Making sense of the ∞-norm: we have the ineq:

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p} \|x\|_\infty$$

Suppose $\|x\|_\infty = |x_M|$ for some $1 \leq M \leq n$

Then $\|x\|_p \leq (n|x_M|^p)^{1/p} = n^{1/p}|x_M| = n^{1/p}\|x\|_\infty$

$\|x\|_p \geq (|x_M|^p)^{1/p} = \|x\|_\infty.$

Notice: as $p \to \infty$ $1/p \to 0$ so $n^{1/p} \to 1$

The ineq yields

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p}\|x\|_\infty \xrightarrow{p \to \infty} \|x\|_\infty \qquad \lim_{p \to \infty} \|x\|_p = \|x\|_\infty$$

$\|x\|_p$ increases as p decreases: $\|x\|_1 \geq \|x\|_p \geq \|x\|_\infty.$

Matrix norms

Matrix norms also satisfy the 3 norm-properties:

(1) $\|A\| \geq 0$, and $\|A\| = 0$ only if $A = 0$ (the zero matrix)

(2) $\|A + B\| \leq \|A\| + \|B\|$ (triangular)

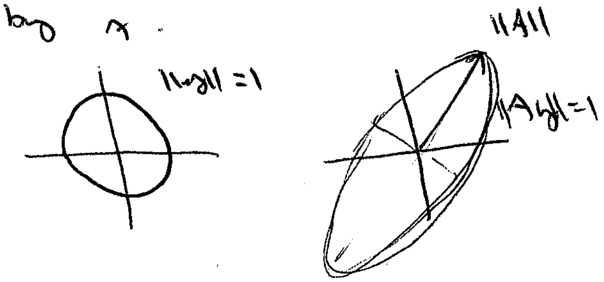
(3) $\|\alpha A\| = |\alpha| \|A\|$ for any $\alpha \in \mathbb{C}$.

An important class of matrix norms are those induced by vector norms:

$$\underbrace{\|A\|}_{\text{matrix}} = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\| \text{, where } \|x\| \text{ is some vector norm.}$$

This can be thought of the "maximum amplification" of A by x.

Ex: 2-norm



Check the 3 properties:

$\|A\| \geq 0$: clear ($\|Ax\| \geq 0$)

$\|A\| = 0 \Rightarrow Ay = 0$ for all $y \Rightarrow A = 0$

$\|\alpha A\| = \max_{\|x\|=1} \|\alpha A x\| = |\alpha| \max_{\|x\|=1} \|Ax\| = |\alpha| \|A\|$.

triangular ineq: ~~max ‖(A+x)‖~~

$$\|A+B\| = \max_{\|x\|=1} \|(A+B)x\| \leq \max_{\|x\|=1} \left(\|Ax\| + \|Bx\|\right)$$

$$\leq \max_{\|x\|=1} \|Ax\| + \max_{\|y\|=1} \|By\| = \|A\| + \|B\|.$$

A matrix norm is called <u>consistent</u> if it satisfies $\|AB\| \le \|A\|\cdot\|B\|$

The induced matrix norms are all consistent.

Notice first for any $v \in \mathbb{C}^n$, $A \in \mathbb{C}^{m\times n}$ satisfies

$\|Av\| \le \|A\|\|v\|$ because $\|A\| = \max\limits_{v\neq 0} \frac{\|Av\|}{\|v\|} \ge \frac{\|Av\|}{\|v\|} \Rightarrow \|A\|\|v\| \ge \|Av\|$

Thus: $\max\limits_{x\neq 0} \frac{\|(AB)x\|}{\|x\|} \le \max\limits_{x\neq 0} \frac{\|A\|\|Bx\|}{\|x\|} = \|A\|\|B\| \Rightarrow \|AB\| \le \|A\|\cdot\|B\|$

(Subtlety: if A is not square the vector ~~norms~~ norms above and below are for vectors of different dimensions.)

Diagonal matrices: For any p-norm : $\|D\|_p = \max\limits_{1\le j\le n} |d_{jj}|$, where D is diagonal $n\times n$.

Suppose $A = D$, D is diagonal, $d_m = \max\limits_{1\le j\le n} |d_{jj}|$ $\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$\|D\|_p = \max_{\|y\|_p=1} \|Dy\| = \max_{\|y\|_p=1} \left\| \begin{pmatrix} d_1y_1 \\ d_2y_2 \\ \vdots \\ d_ny_n \end{pmatrix} \right\|_p$$

$$= \max_{\|y\|_p=1} \left(\sum_{i=1}^n |d_iy_i|^p\right)^{1/p}$$

$$\le \max_{\|y\|_p=1} \left(\sum_{i=1}^n |d_m|^p |y_i|^p\right)^{1/p}$$

$$= |d_m| \max_{\|y\|_p=1} \left(\sum |y_i|^p\right)^{1/p} \Rightarrow \|D\|_p \le \max_{1\le j\le n} |d_{jj}|$$

However let $y = e_m \Rightarrow y_i = \begin{cases} 0, & i\neq m \\ 1, & i = m \end{cases}$ $\begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow m^{th}$ "m^{th} Euclidean basis vector" e_m

$\|D\|_p \ge \|Dy\|_p = |d_m|$

$\therefore$ $|d_m| \le \|D\|_p \le |d_m| \Rightarrow \|D\| = \max\limits_{1\le j\le n} |d_{jj}|$