1 Problem 1

We note that since A is hermitian positive definite, it admits a unique Cholesky decomposition given to be

$$A = \begin{pmatrix} \alpha & 0 \\ \frac{\omega}{\alpha} & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & K - \frac{\omega\omega^*}{a_{11}} \end{pmatrix} \begin{pmatrix} \alpha & \frac{\omega^*}{\alpha} \\ 0 & I \end{pmatrix} = R_1^* A_1 R_1 \tag{1}$$

We wish to show that all principal submatrices of A (ie all A_j for $j = 1, \cdots, m$) are Hermitian positive definite via induction. We start for the case $j = 1$. We let X be of full rank $\in \mathbb{C}^{m \times m-1}$ with Euclidean basis vectors in each column. We know the principal submatrix of A_1 is Hermitian, because $(X^*AX)^* = X^*AX$. We also know since $Xx \neq 0$, that it is positive definite because $x^*(X^*AX)x = (Xx)^*A(Xx) > 0$. Thus, we have shown that the first principal submatrix of A, A_1, is hermitian positive definite. Now, assume that the principle submatrices of $A_2, \cdots, A_{m-1}$ are also Hermitian positive definite. We need to show that A_m is Hermitian positive definite. In this case, let $X \in \mathbb{C}^{m \times 1}$ and containing the relevant Euclidean basis vector. Again, we see that A_m is Hermitian, since $(X^*AX)^* = X^*AX$. Indeed since this is effectively a 1 by 1 matrix, this makes sense. Similarly, it is positive definite, since $x^*(X^*AX)x = (Xx)^*A(Xx) > 0$. Thus, this shows that every principal submatrix of A is Hermitian positive definite.

2 Problem 2

Let $A = LDL^*$, for some lower triangular matrix L and diagonal matrix D. If we assume that A is positive definite, this implies that $x^*Ax > 0, \forall x \neq 0$. By letting $y = L^*x$, and using the fact that since L is invertible implies that $x = 0$ if and only if $y = 0$, we see that

$$x^*Ax = x^*(LDL^*)x = y^*Dy = \sum_i |y_i|^2 d_{ii} > 0 \tag{2}$$

Now, the squared magnitude of y is always positive, but this does not necessarily put any constraint on any arbitrary element of the set of diagonal elements of D. But suppose that a particular $d_{ii} \leq 0$ for some index j, $1 \leq j \leq n$ and assume we create a y such that only the $d_{ii} \leq 0$ term survives. This would implies that $y^*Dy \leq 0$, which contradicts our original assertion that A is Hermitian positive definite. Ergo, every diagonal element of must be larger that 0.

3 Problem 3

We want to show that if A is a Hermitian positive definite matrix, then an element of A with largest magnitude lies on the diagonal. Assuming A is HPD, recall that we can factor A using a Cholesky factorization such that

$$A = R^*R = \begin{pmatrix} r_1^* \\ r_2^* \\ \vdots \\ r_m^* \end{pmatrix} \begin{pmatrix} r_1 & r_2 & \cdots & r_m \end{pmatrix} \tag{3}$$

for rows and columns r_j^* and r_j respectively. We have the following knowledge regarding the relationship between the inner product of these rows and columns and individual elements of matrix A: $(r_j, r_j) = a_{jj}$, $(r_k, r_k) = a_{kk}$, and $(r_j, r_k) = a_{jk}$. Using the Cauchy-Schwartz inequality ($|x^*y| \leq ||x||_2||y||_2$), we can set $x = r_j$ and $y = r_k$. Then we are left with

$$|r_j^* r_k| \leq \sqrt{(r_j^* r_j)}\sqrt{(r_k^* r_k)} \to |a_{jk}| \leq \sqrt{a_{jj}}\sqrt{a_{kk}} \to |a_{jk}|^2 \leq a_{jj}a_{kk} \tag{4}$$

We note that the strict equality is achieved whenever $j = k$, otherwise the inequality is always satisfied.

Now, let's assume that $|a_{jk}|$ is the largest element out of row/column j, as well as row/column k, where $j \neq k$. This would mean that $|a_{jk}|^2 > a_{jj}a_{kk}$, which is a contradiction since we have already shown that $|a_{jk}|^2 < a_{jj}a_{kk}$. Thus we see that elements on the diagonal of a HPD matrix are largest in magnitude.

4 Problem 4

We wish to prove the the product of two lower triangular matrices is a lower triangular matrix. We will use a proof by induction. Acknowledging that this is true in the case the matrix 1×1, we choose to start by showing this is true for the case where we are dealing with a 2×2 matrix.

$$\begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & 0 \\ a_{21}b_{11} + a_{22}b_{21} & a_{22}b_{22} \end{pmatrix} \tag{5}$$

The inductive step assumes that this is true for the case where the matrices are $\in \mathbb{C}^{m \times m}$. We then must show that the assumption is true for the case where matrices A and $B \in \mathbb{C}^{m+1\times m+1}$. So our problem looks like

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & & \ddots & 0 \\ a_{m+1,1} & a_{m+1,2} & \cdots & a_{m+1,m+1} \end{pmatrix} \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \vdots & & \ddots & 0 \\ b_{m+1,1} & b_{m+1,2} & \cdots & b_{m+1,m+1} \end{pmatrix} \tag{6}$$

So an arbitrary column c_j will be a linear combination of the columns of A, such that

$$c_j = \sum_{i}^{m+1} a_i b_{ij} \tag{7}$$

So column 1 of the resulting matrix C will look like

$$c_1 = b_{11} \begin{pmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m+1,1} \end{pmatrix} + b_{21} \begin{pmatrix} 0 \\ a_{22} \\ \cdots \\ a_{m+1,2} \end{pmatrix} + \cdots + b_{m+1,1} \begin{pmatrix} 0 \\ 0 \\ \cdots \\ a_{m+1,m+1} \end{pmatrix} \equiv \begin{pmatrix} x \\ x \\ \cdots \\ x \end{pmatrix} \tag{8}$$

and columns 2 of the results matrix C will look like

$$c_2 = 0 \begin{pmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m+1,1} \end{pmatrix} + b_{22} \begin{pmatrix} 0 \\ a_{22} \\ \cdots \\ a_{m+1,2} \end{pmatrix} + \cdots + b_{m+1,2} \begin{pmatrix} 0 \\ 0 \\ \cdots \\ a_{m+1,m+1} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ x \\ \vdots \\ x \end{pmatrix} \tag{9}$$

and column $m+1$ of the resulting matrix will look like

$$c_{m+1} = b_{m+1,m+1} \begin{pmatrix} 0 \\ 0 \\ \cdots \\ a_{m+1,m+1} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x \end{pmatrix} \tag{10}$$

Combining columns $1, \cdots, m+1$ into a complete matrix C shows us that the matrix multiplication of 2 lower triagular matrices must yield a triangular matrix.

5 Problem 5

Consider a Hermitian positive definite matrix $A \in \mathbb{C}^{m\times m}$ defined as

$$A = \begin{pmatrix} a_{11} & w^* \\ w & K \end{pmatrix} \tag{11}$$

for $w \in \mathbb{C}^{m-1\times 1}$, $w^* \in \mathbb{C}^{1\times m-1}$, and $K \in \mathbb{C}^{m-1\times m-1}$. We want to show that such a matrix has a Cholesky factorization using induction. We will use the following two criteria established in earlier proofs: diagonal elements of a Hermitian positive definite matrix are greater than zero, and each principal submatrix of a Hermitian positive definite matrix is itself Hermitian positive definite. Since this matrix has $a_{11} > 0$ and is defined as a Hermitian positive definite matrix, a symmetric triangular reduction technique can be used to factorize the matrix. Consequently our first step of induction shows A can be factored such that

$$A = R_1 A_1 R_1^* = \begin{pmatrix} \alpha & 0 \\ \frac{\omega}{\alpha} & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & K - \frac{ww^*}{a_{11}} \end{pmatrix} \begin{pmatrix} \alpha & \frac{w^*}{\alpha} \\ 0 & I \end{pmatrix} \tag{12}$$

for $\alpha = \sqrt{a}_{11}$. Relying on previously established proofs that the principal submatrices of Hermitian positive definite matrices are themselves Hermitian positive definite, we can continue this factorization for $A_1 = R_2 A_2 R_2^*$, and again for $A_2 = R_3 A_3 R_3^*$ and so on. Thus our inductive hypothesis is that each principal submatrices in $A_1, A_2, \cdots, A_{m-1}$ of the form $K - \frac{ww^*}{a_{jj}} \in \mathbb{C}^{j\times j}$ for $j = m-1, m-2, \cdots, 2$ can be factorized using an equivalent triangular reduction technique. We need to show that the submatrix $\in \mathbb{C}^{1\times 1}$ also can be factored using the same algorithm. As we have already established that all diagonal elements > 0, and this submatrix contains only the diagonal element $a_{m,m}$, it meets the factorization criteria. This show that there exists a Cholesky factorization for any Hermitian positive definite matrix A.

6 Problem 6 - Book 23.1

First we note that since A is nonsignular, $Ax \neq 0 \forall x \neq 0$. Consequently, we see that

$$x^* A^* A x = ||Ax||_2^2 > 0 \tag{13}$$

meaning that A^*A is positive definite and therefore A^*A has a unique cholesky factorization. Given the fact that $A = QR$ and $A^*A = U^*U$, it is true that $R = U$. This can be seen since

$$A^*A = R^*Q^*QR = R^*R = U^*U \tag{14}$$

since Q is unitary.