MAD 6406 F20 lec 20 10/16/20

- Sensitivity of linear eqns ~~~~ $Ax=b$ to
  - perturbations in b
  - perturbations in A.

$\Rightarrow$ Sensitivity analysis provides a worst-case scenario.

~~$\Rightarrow$ the bound~~

$\Rightarrow$ We may want to ask if the bound can be attained.

Condition of a problem:

Given data d_1, d_2 and sol'ns $s(d_1), s(d_2)$, then the <u>condition</u> or <u>condition number</u> of a problem with exact sol'n $s(d)$ is

$$\max_{\|d_1-d_2\| \text{ "small"}} \frac{\|s(d_1)-s(d_2)\|}{\|d_1-d_2\|}$$

A problem is called "ill conditioned" if $\frac{\|s(d_1)-s(d_2)\|}{\|d_1-d_2\|}$ is "large"

for "small" $\|d_1-d_2\|$.

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Ex: Sensitivity of linear eq'ns: $Ax=b$; consider $A \in \mathbb{C}^{n\times n}$ invertible

Data: A, b

Perturbed data: $\tilde{A} = A+\Delta A$, $\tilde{b} = b+\Delta b$

Solution: x

Perturbed sol'n: $\tilde{x} = x+\Delta x$

Perturbed problem:

$(A+\Delta A)(x+\Delta x) = b+\Delta b$

(1) If only b (not A) is perturbed (e.g., measurement error)

$A(x+\Delta x) = b+\Delta b \Rightarrow Ax + A\Delta x = b+\Delta b \Rightarrow A\Delta x = \overbrace{b-Ax}^{0} + \Delta b$

$\Delta x = A^{-1}\Delta b$

$\frac{\|\tilde{x}-x\|}{\|x\|} = \frac{\|\Delta x\|}{\|x\|} = \frac{\|A^{-1}\Delta b\|}{\|x\|}$ and $b = Ax \Rightarrow \|b\| \le \|A\|\|x\| \Rightarrow \|x\| \ge \frac{\|b\|}{\|A\|}$

$\therefore \frac{\|\Delta x\|}{\|x\|} = \frac{\|A^{-1}\Delta b\|}{\|x\|} \le \|A\| \frac{\|A^{-1}\Delta b\|}{\|b\|} \le \|A^{-1}\|\|A\| \frac{\|\Delta b\|}{\|b\|}$

$$\underbrace{\frac{\|\Delta x\|}{\|x\|}}_{\text{soln pert}} \le \underbrace{\|A^{-1}\|\|A\|}_{\text{magnifies pert of } b} \underbrace{\frac{\|\Delta b\|}{\|b\|}}_{\text{data pert.}}$$

$\Rightarrow$ rel. pert. of b is magnified by $\|A^{-1}\|\|A\|$, called the condition number of A

$\cdot$ $\text{cond}(A) = \|A^{-1}\|\|A\|$ w.r.t. norm $\|\cdot\|$, for invertible matrix A.

$\Rightarrow$ For any subordinate matrix norm:

$1 = \|I\| = \|A^{-1}A\| \le \|A^{-1}\|\|A\| = \text{cond}(A) \Rightarrow \text{cond}(A) \ge 1.$

More generally: $\text{cond}(A) = \|A^{+}\|\|A\|$, where A^{+} is the pseudoinverse of A.

$\circ$ 2-norm: $\|A\|_2 = \sigma_1$

$A = U\Sigma V^* \Rightarrow A^{-1} = V\Sigma^{-1}U^*$, $\Sigma^{-1} = \begin{pmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n \end{pmatrix} \Rightarrow \sigma_{max}(A^{-1}) = 1/\sigma_n$

$\therefore$ $\text{cond}_2(A) = \dfrac{\sigma_1}{\sigma_n}$ | If $\text{rank}(A) = p$, $\text{cond}_2(A) = \dfrac{\sigma_1}{\sigma_p}$.

Is the bound ~~realistic~~ attainable? $\Rightarrow$ yes!

$A(x+\Delta x) = b + \Delta b \Rightarrow \Delta x = A^{-1}\Delta b$

Let $b = u_1$, $\Delta b = \varepsilon u_n$ (u_1 1st singular vector)

$x = A^{-1}b = V\Sigma^{-1}U^* u_1 = V\Sigma^{-1}e_1 = 1/\sigma_1 \, v_1$

$\Delta x = A^{-1}\Delta b = \varepsilon V\Sigma^{-1}U^* u_n = \varepsilon \Sigma^{-1} e_n = \varepsilon/\sigma_n \, v_n$

then $\dfrac{\|\Delta x\|_2}{\|x\|_2} = \dfrac{\|A^{-1}\Delta b\|_2}{\|x\|_2} = \dfrac{\varepsilon\sigma_1}{\sigma_n}\dfrac{\|v_n\|_2}{\|v_1\|_2} = \varepsilon\dfrac{\sigma_1}{\sigma_n} = \varepsilon\,\text{cond}_2(A) = \dfrac{\|\Delta b\|_2}{\|b\|_2}\text{cond}_2(A)$

Perturbations to A:

$$(A+\Delta A)(x+\Delta x)=b \Rightarrow (A+\Delta A)\Delta x = \overbrace{b-Ax}^{0}-\Delta A x \Rightarrow (A+\Delta A)\Delta x = -\Delta A x$$

bound Δx:

$$(A+\Delta A)\Delta x = -\Delta A x \Rightarrow A(I+A^{-1}\Delta A)\Delta x = -\Delta A x$$
$$(I+A^{-1}\Delta A)\Delta x = -A^{-1}\Delta A x$$

let $E = A^{-1}\Delta A \Rightarrow (I+E)\Delta x = -Ex$

Useful result:

Let $\|\cdot\|$ be a subordinate matrix norm.*

If E is $n\times n$ with $\|E\|<1$, then $I+E$ is nonsingular with

$$\|(I+E)^{-1}\| \le \frac{1}{1-\|E\|}$$

proof: let $x \in \mathbb{C}^n$

$$\|(I+E)x\| = \|x+Ex\| \ge \|x\|-\|Ex\|$$

Since $\|Ex\| \le \|E\|\|x\|$

$$\|x\|-\|Ex\| \ge \|x\|-\|E\|\cdot\|x\| = \|x\|(1-\|E\|)$$

so $\|x\|(1-\|E\|) \le \|(I+E)x\|$

since $\|E\|<1$, $(I+E)x=0$ only for $x=0$ so $I+E$ is nonsingular.

Next, let $C=(I+E)^{-1} \Rightarrow (I+E)C = I$ then

$$1 = \|I\| = \|(I+E)C\| = \|C+CE\| \ge \|C\|-\|CE\|$$

Again, $\|CE\| \le \|C\|\|E\|$, so

$$1 \ge \|C\|-\|CE\| \ge \|C\|-\|C\|\|E\| = \|C\|(1-\|E\|)$$

so $\|C\| = \|(I+E)^{-1}\| \le \dfrac{1}{1-\|E\|}$

*(result actually holds in any matrix norm)

Back to our problem:

$(I+E)\Delta x = -Ex$, with $E = A^{-1}\Delta A$

Assume the perturbation is small meaning $\|A^{-1}\|\|\Delta A\| < 1$

Thus: $\|E\| < 1$ and $I+E$ is invertible with $\|(I+E)^{-1}\| \le \frac{1}{1-\|E\|}$

So: $\Delta x = -(I+E)^{-1}Ex$

$$\frac{\|\Delta x\|}{\|x\|} \le \frac{1}{1-\|E\|}\|E\|\|x\| \le \frac{\|A^{-1}\|\|\Delta A\|}{1-\|A^{-1}\|\|\Delta A\|} = \frac{\frac{\|\Delta A\|}{\|A\|}\|A^{-1}\|\|A\|}{1-\frac{\|\Delta A\|}{\|A\|}\|A^{-1}\|\|A\|}$$

$$= \frac{1}{1-\frac{\|\Delta A\|}{\|A\|}\mathrm{cond}(A)} \cdot \frac{\|\Delta A\|}{\|A\|}\mathrm{cond}(A) \quad (*)$$

(1) let $\varepsilon = \frac{\|\Delta A\|}{\|A\|}$, rel. pert. in A, with $\varepsilon\,\mathrm{cond}(A) < 1/2$, thus:

$$\frac{\|\Delta x\|}{\|x\|} \le 2\,\mathrm{cond}(A)\cdot\varepsilon$$

(2) Can bound $(*)$ be attained? → yes.

let $\Delta A = -\varepsilon\|A\|_2 u_n v_n^*$, $b = \delta u_n$.

Thus $\frac{\|\Delta A\|_2}{\|A\|_2} = \varepsilon$

$$x = A^{-1}b = \delta(V\Sigma^{-1}U^*)u_n = \delta V\Sigma^{-1}e_n = \frac{\delta}{\sigma_n}v_n$$