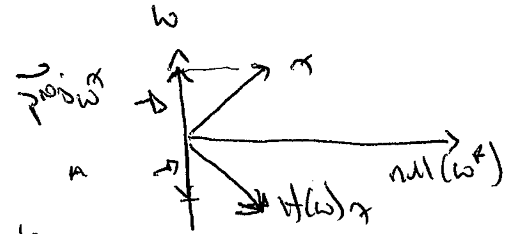
- triangularization by Householder reflection.

In order to use Householder reflectors to determine a QR decomposition, let's see how to choose $w \in \mathbb{C}^n$ s.t. for $x, y \in \mathbb{C}^n$, $H(w)x = y$.

Recall: $H(w) = I - \frac{2}{w^*w} ww^*$

First notice: $H(w)x = x - \frac{2}{w^*w} w(w^*x) = x - \underbrace{2\frac{w^*x}{w^*w}w}_{2 \times \text{ projection of } x \text{ along } w}$



H(w)x ~~projects~~ reflects x about null(w^*)

Now, consider solving $H(w)x = y$: Let $H = H(w)$

$Hx = x - 2\frac{w^*x}{w^*w} w = y \Rightarrow x - y = 2\underbrace{\left(\frac{w^*x}{w^*w}\right)}_{\text{a number}} w$ $\Rightarrow$ w must be a multiple of $x-y$ to satisfy $Hx = y$.

$\Rightarrow$ H is indep. of scaling of w, so take $w = (x-y)$

To be useful ~~to~~ for forming a QR factorization, design H (choose w) s.t. $(H(w)x)_i = \begin{cases} x_i & \text{for } i<k \\ 0, & \text{for } i>k \\ \underset{\uparrow \text{something}}{*} & \text{for } i=k \end{cases}$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow Hx = \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Ex: $A = (a_1 | a_2 | \cdots | a_n)$

H_1 should satisfy $H a_1 = \alpha \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Who is α ?

$$\underbrace{\|a_1\|_2 = \|Ha_1\|_2}_{\text{unitary}} = \|\alpha e_1\| = |\alpha| \Rightarrow \alpha = e^{i\theta_1}\|a_1\|_2, \text{ for some } \theta_1.$$

$$\Rightarrow w_1 = a_1 - e^{i\theta} e_1 \|a_1\|_2 = \begin{pmatrix} a_{11} - e^{i\theta_1}\|a_1\|_2 \\ a_{12} \\ \vdots \\ a_{1n} \end{pmatrix}$$

$\Rightarrow \theta_1$ should be chosen so $|a_{11} - e^{i\theta_1}\|a_1\|_2|$ is as large as possible: $-e^{i\theta_1} = \text{sign}(a_{11})$

This avoids a "critical cancellation" and preserves numerical stability.

$\Rightarrow w_1 = a_1 + \text{sign}(a_{11})\|a_1\|_2 e_1$.

Next: Householder transformation to triangular form.

Householder transformation to triangular forms

Given $A \in \mathbb{C}^{m\times n}$, define $A^{(1)} = A$

Choose $H_1 = I - \frac{2}{w_1^* w_1} w_1 w_1^*$ with $w_1 = a_1 + \text{sign}(a_{11}^{(1)}) \|a_1^{(1)}\|_2 e_1$

$$A^{(2)} = H_1 A^{(1)} = \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} & & \\ 0 & a_{22}^{(2)} & & \\ | & | & \cdots & a_n^{(2)} \\ 0 & a_{m2}^{(2)} & & \end{pmatrix}, \quad a_2^{(2)} = \begin{pmatrix} a_{12}^{(2)} \\ a_{22}^{(2)} \\ \vdots \\ a_{m2}^{(2)} \end{pmatrix}, \text{ let } \tilde{a}_2^{(2)} = \begin{pmatrix} a_{22}^{(2)} \\ \vdots \\ a_{m2}^{(2)} \end{pmatrix}$$

$a_2^{(2)}$

$$H_2 = I - \frac{2}{w_2^* w_2} w_2 w_2^*, \quad w_2 = \begin{pmatrix} 0 \\ \hline \tilde{w}_2 \end{pmatrix}, \quad \tilde{w}_2 = \tilde{a}_2^{(2)} + \text{sign}(a_{22}^{(2)}) \|\tilde{a}_2^{(2)}\| e_1$$

there $e_1 \in \mathbb{C}^{m-1}$

$$A^{(3)} = H_2 A^{(2)} = \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} & | & | \\ & a_{22}^{(3)} & a_3^{(3)} \cdots & a_n^{(3)} \\ & 0 & | & | \\ & \vdots & & \\ & 0 & & \end{pmatrix}$$

After $n-1$ iterations:

$$A^{(n)} = R = \underbrace{H_{n-1} H_{n-2} \cdots H_1 A}_{Q^* A}, \quad R \text{ upper-trapezoidal.}$$

Algorithm:

$e_1 \in \mathbb{C}^{m-k+1}$

for $k = 1:n$
  $x = A_{k:m,k}$
  $v_k = x + \text{sign}(x_1)\|x\|_2 e_1$
  $v_k = v_k / \|v_k\|_2$
  $A_{k:m,k:n} = A_{k:m,k:n} - 2v_k(v_k^* A_{k:m,k:n})$
end

operation count (see book): $2mn^2 - \frac{2}{3}n^3$ flops

block formulation only operates on entries that change.

"Q-less QR": Q is not explicitly computed, but Q^*b is computed by applying each H to vector b.