- Range + nullspace
- Properties of matrix mult.
- Properties of Hermitian matrices.

Range of A = $\mathrm{col}(A)$ = $\{ b \in \mathbb{C}^m \mid Ax = b \text{ for some } x \in \mathbb{C}^n \}$

Nullspace of A = $\mathrm{Null}(A)$ = $\{ x \in \mathbb{C}^n \mid Ax = 0 \}$

the zero vector

Rank(A) = rank of A = no. linearly indep. (LI) cols of A

= no. LI rows of A.

Recall: The set $\{v_1, v_2, \ldots, v_m\}$ is LI if $\sum_{i=1}^{m} \alpha_i v_i = 0$ has only the sol'n $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$.

In matrix form: if $v_1, v_2, \ldots, v_m \in \mathbb{C}^m$

$$\underbrace{\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}}_{\alpha} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0$$

$V\alpha = 0$ has only the sol'n $\alpha = 0$

$\Leftrightarrow$ the columns of V are LI.

$A \in \mathbb{C}^{m \times n}$ has full-rank if $\mathrm{rank}(A) = \min\{m, n\}$.

Invertibility: Suppose $A \in \mathbb{C}^{n \times n}$. A is invertible if $\exists$ matrix B s.t. $AB = BA = I$. Then $B = A^{-1}$.

Moreover, there is a unique sol'n to $Ax = b$ for any $b \in \mathbb{R}^n$, which satisfies $x = A^{-1}b$

BUT: We do not invert A to solve $Ax = b$!!! (usually...)

Two important properties of matrix mult's

1) If $A \in \mathbb{C}^{m \times l}$, $B \in \mathbb{C}^{l \times n}$ then $(AB)^* = B^* A^*$

2) If $A, B \in \mathbb{C}^{n \times n}$ are both invertible, then $(AB)^{-1} = B^{-1} A^{-1}$.

Two important properties of matrix mult:

(1) $(AB)^* = B^* A^*$

(2) $(AB)^{-1} = B^{-1} A^{-1}$.

To show (1), first lets show for $z_1, z_2 \in \mathbb{C}$: $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

Then, show $(AB)^T = B^T A^T$

$\Rightarrow$ let $z_1 = a + ib$, $z_2 = c + id$

$$\overline{z_1 z_2} = \overline{(a+ib)(c+id)} = \overline{(ac-bd)+i(ad+bc)} = (ac-bd) - i(ad+bc)$$

$$\overline{z_1}\,\overline{z_2} = (a-ib)(c-id) = (ac-bd) - i(ac+bc).$$

[Notice: this also shows $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ by setting $z_3 = \frac{1}{z_2}$]

Next: $(AB)^T = B^T A^T$. Suppose A is $m \times l$, B is $l \times n$

$$(AB)_{ij} = \underbrace{\left(\; i\text{th row} \;\right)}_{A} \underbrace{\left(\; j\text{th col} \;\right)}_{B} = \sum_{k=1}^{l} a_{ik} b_{kj}$$

$$(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{l} a_{jk} b_{ki}$$

$$(B^T A^T)_{ij} = \sum_{k=1}^{l} (B^T)_{ik} (A^T)_{kj} = \sum_{k=1}^{l} b_{ki} a_{jk} = (AB)_{ji}$$

Finally: $(AB)^* = (\overline{AB})^T = (\bar{A}\bar{B})^T = \bar{B}^T \bar{A}^T = B^* A^*$.

For (2): $(AB)^{-1}$ satisfies

$$(AB)(AB)^{-1} = I$$
$$B(AB)^{-1} = A^{-1}$$
$$(AB)^{-1} = B^{-1} A^{-1}.$$

Ex: If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$, $A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & \overline{a_{31}} \\ \overline{a_{12}} & \overline{a_{22}} & \overline{a_{32}} \end{pmatrix}$

If $A^* = A$ then A is Hermitian
(For $A \in \mathbb{R}^{m \times m}$, if $A^* = A$, then A is symmetric).

Properties of Hermitian matrices:

(1) All Eigenvalues are real

(2) Eigenvectors corresponding to distinct eigenvalues are orthogonal.

Recall: An eigenvalue of A is a scalar λ for which

$Ax = \lambda x$ for $x \neq 0$, called the corresponding eigenvector.

(1) $Ax = \lambda x$

$x^* A x = \lambda x^* x \Rightarrow \lambda = \dfrac{x^* A x}{x^* x}$

Then $\bar{\lambda} = \dfrac{(x^* A x)^*}{(x^* x)^*}$, $(x^* x)^* = x^* x^{**} = x^* x$ ($x^* x$ is always real)

$(x^* A x)^* = x^* A^* x^{**} = x^* A x$

$\therefore \bar{\lambda} = \lambda$, eigenvalues must be real.

(2) orthogonality:

$$\begin{cases} Ax_1 = \lambda_1 x_1 \\ Ax_2 = \lambda_2 x_2 \end{cases} \quad \lambda_1 \neq \lambda_2$$

$x_2^* A x_1 = \lambda_1 x_2^* x_1 \rightarrow (x_2^* A x_1)^* = (\lambda_1 x_2^* x_1)^* \Rightarrow x_1^* A x_2 = \lambda_1 x_1^* x_2$

$x_1^* A x_2 = \lambda_2 x_1^* x_2$

$x_1^* A x_2 = \lambda_1 x_1^* x_2$

$0 = (\lambda_2 - \lambda_1)(x_1^* x_2)$, but $\lambda_2 \neq \lambda_1$ so $x_1^* x_2 = 0$. $\therefore x_1 \perp x_2$.