- SVD outer product form
- $\|A\|_2 = \sigma_1$
- Best rank-p approximation

Outer-product form

$$A = (u_1 \cdots u_r) \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix} \begin{pmatrix} v_1^t \\ \vdots \\ v_r^t \end{pmatrix}$$

$$= (u_1 \cdots u_r) \left(\begin{pmatrix} \sigma_1 & 0 & \\ & \ddots & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & & \\ & \sigma_2 & & \\ & & 0 & \\ & & & \ddots \\ & & & & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & & \\ & \ddots & \\ & 0 & \\ & & \sigma_r \end{pmatrix} \right) \begin{pmatrix} v_1^t \\ \vdots \\ v_r^t \end{pmatrix}$$

$$= u_1 \sigma_1 v_1^t + u_2 \sigma_2 v_2^t + \cdots + u_r \sigma_r v_r^t$$

$$= \sum_{j=1}^{r} u_j \sigma_j v_j^t .$$

Property: $\|A\|_2 = \sigma_1 \quad \left(\|A\|_2^2 = \lambda_{max}(A^t A) \right)$

$\|A\|_2$ is invariant under unitary transformation:

$$\|A\|_2 = \|U \Sigma V^t\|_2 = \|U \Sigma\|_2 = \|\Sigma\|_2$$

Since $\Sigma = \left(\begin{array}{ccc|c} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array} \right)$, $\hat{\Sigma} = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}$, $\max_{\|x\|=1} \|\Sigma x\| = \max_{\|x\|=1} \|\hat{\Sigma} x\|$

$$\therefore \|A\|_2 = \|\hat{\Sigma}\|_2 = \sigma_1 .$$

Best rank-p approximation: $A = \sum_{j=1}^{r} \sigma_j u_j v_j^t$. Let $A_p = \sum_{j=1}^{p} \sigma_j u_j v_j^t$.

From above: $\|A - A_p\|_2 = \left\| \sum_{j=p+1}^{r} \sigma_j u_j v_j^t \right\|_2 = \sigma_{p+1}$

In fact, $\|A-A_p\|_2 = \inf_{\substack{B\in\mathbb{C}^{m\times n}\\ \mathrm{rank}(B)\le p}} \|A-B\|_2$.

By contradiction:

Proof: Suppose there is some B with $\mathrm{rank}(B)\le p$ and $\|A-B\|_2 < \|A-A_p\|_2 = \sigma_{p+1}$.

Since $\dim(\mathrm{Col}(B)) = \dim(\mathrm{Col}(B^*))\le p$, $\dim(\mathrm{null}(B)) \ge n-p$.

Let W be an $n-p$ dim'l subspace of $\mathrm{null}(B)$.

For $w\in W$:

$\|Aw\|_2 = \|(A-B)w\|_2 \le \|A-B\|_2\|w\|_2 < \sigma_{p+1}\|w\|_2$.
by assumption

So: $\|Aw\|_2 < \sigma_{p+1}\|w\|_2$ for $w\in W$.

But: $\tilde{V} = \mathrm{span}\{v_1, v_2, \dots, v_{p+1}\}$ is a $p+1$ dim'l space with ~~$\|Aw\|_2 \ge \sigma_{p+1}\|w\|_2$~~ $\|Aw\|_2 \ge \sigma_{p+1}\|w\|_2$,

for $w\in\tilde{V}$:

Notice: for $w\in\tilde{V}$, $w=\sum_{i=1}^{p+1}\alpha_i v_i$, $Aw = \sum_{i=1}^{p+1}\alpha_i U\Sigma V^* v_i = \sum_{i=1}^{p+1}\alpha_i\sigma_i u_i$.

So: $\|Aw\|_2^2 = \left(\sum_{i=1}^{p+1}\sigma_i\bar{\alpha}_i u_i^*\right)\left(\sum_{j=1}^{p+1}\sigma_j\alpha_j u_j\right) = \sum_{j=1}^{p+1}\sigma_j^2|\alpha_j|^2 \ge \sigma_{p+1}^2\sum_{j=1}^{p+1}|\alpha_j|^2 = \sigma_{p+1}^2\|w\|_2^2$,

as $\|w\|_2^2 = \left(\sum_{i=1}^{p+1}\bar{\alpha}_i v_i^*\right)\left(\sum_{j=1}^{p+1}\alpha_i v_i\right) = \sum_{j=1}^{p+1}|\alpha_j|^2$.

Since $\dim(\tilde{V}) = p+1$, $\dim(W) = n-p$, $\tilde{V}\subset\mathbb{C}^n$, $W\subset\mathbb{C}^n$

The intersection between $\tilde{V}$ and W has at least dimension 1:

There is some nonzero vector ~~with $\|Aw\|_2 \ge \sigma_{p+1}\|w\|_2$ and~~
$w\in W\cap\tilde{V}$ with $\|Aw\|_2\ge\sigma_{p+1}\|w\|_2$ and $\|Aw\|_2 < \sigma_{p+1}\|w\|_2$, a contradiction $\Rightarrow\!\Leftarrow$.

$\therefore \|A-A_p\|_2 = \inf_{\substack{B\in\mathbb{C}^{m\times n}\\ \mathrm{rank}(B)\le p}}\|A-B\|_2 = \sigma_{p+1}$.