MAD 6406 f20 lec 16 10/07/20

- Givens rotations
- Reduction to triangular form by Givens (plane) rotations

Recall: QR decomp of full rank m×n matrix A ($m \geq n$)

$$\underset{m}{\begin{pmatrix} A \\ \\ \\ \end{pmatrix}}^{n} = \underset{Q}{\begin{pmatrix} \underbrace{Q_1}_{n} & \underbrace{Q_2}_{m-n} \end{pmatrix}}^{m} \underset{m \quad R}{\begin{pmatrix} \diagdown \\ \\ \\ \end{pmatrix}}^{n} \quad \text{or} \quad \underset{m}{\begin{pmatrix} A \\ \\ \\ \end{pmatrix}}^{n} = \underset{m}{\begin{pmatrix} Q_1 \\ \\ \\ \end{pmatrix}}^{n} \underset{n \quad R_1}{\begin{pmatrix} \diagdown \\ 0 \end{pmatrix}}^{n}$$

Q unitary (orthogonal)

R upper trapezoidal

Q_1 spans col(A)

Q_2 spans null(A^*)

Plane Rotations (Givens rot.) ⇒ consider real case: Q orthogonal $A \in \mathbb{R}^{m \times n}$,

Consider n×n matrix P_{ij} s.t.

$P_{ij} = I_n$ then $P_{ij}(\{i,j\},\{i,j\}) = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$. $P_{ij} = $

$$\begin{array}{cccccc} & & i & & j & \\ \ddots_1 & & & & & \\ & & c_1 & & s & \leftarrow i \\ & & & \ddots_1 & & \\ & & -s & & c_1 & \leftarrow j \\ & & & & \ddots_1 & \end{array}$$

$i < j$

$c = \cos(\theta)$, $s = \sin(\theta)$ for some θ.

If $i > j$: $P_{ij} = $

$$\begin{array}{cccccc} & & j & & i & \\ \ddots_1 & & & & & \\ & & c_1 & & -s & \leftarrow j \\ & & & \searrow & & \\ & & s & & c_1 & \leftarrow i \\ & & & & \ddots_1 & \end{array}$$

P_{ij} is orthogonal: $P_{ij}(\{i,j\},\{i,j\})^T P_{ij}(\{i,j\},\{i,j\}) =$

$$= \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} c^2+s^2 & cs-cs \\ cs-cs & c^2+s^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$\Rightarrow P_{ij}^T P_{ij} = I$

{from 517 L6}

QR-3c

Let $y = P_{ij}x$.

Then $\|y\|_2^2 = (P_{ij}x)^T(P_{ij}x) = x^T P_{ij}^T P_{ij} x = \|x\|^2$

Given nonzero $x \in \mathbb{R}^n$, P_{ij} rotates x through θ radians in the ij-plane.

Ex: for $i < j$

$$y = P_{ij}x = \begin{pmatrix} x_1 \\ \vdots \\ x_{i-1} \\ cx_i + sx_j \\ x_{i+1} \\ \vdots \\ x_{j-1} \\ -sx_i + cx_j \\ x_{j+1} \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{row } i \\ \\ \\ \\ \leftarrow \text{row } j \\ \\ \\ \\ \end{matrix}$$

~~$y_i \neq x_i$~~

$y_i = cx_i + sx_j$

$y_j = -sx_i + cx_j$

$y_k = x_k$, $k \neq i$, $k \neq j$

Use: make $y_j = 0$ by choosing c and s:

$$-sx_i + cx_j = 0 \Rightarrow \frac{s}{c} = \tan\theta = \frac{x_j}{x_i} \Rightarrow s = \frac{x_j}{\sqrt{x_i^2 + x_j^2}}, \quad c = \frac{x_i}{\sqrt{x_i^2 + x_j^2}}$$

Then
$$y = P_{ij}x = \begin{pmatrix} \vdots \\ \sqrt{x_i^2 + x_j^2} \\ \vdots \\ 0 \\ \vdots \end{pmatrix} \begin{matrix} \\ \leftarrow \text{row } i \\ \\ \leftarrow \text{row } j \\ \\ \end{matrix}$$

Notes: another soln is ~~counterclockwise rotation: by $\theta \to -\theta$~~

$$s = \frac{-x_j}{\sqrt{x_i^2 + x_j^2}}, \quad c = \frac{-x_i}{\sqrt{x_i^2 + x_j^2}}$$

$y = P_{ij}x \Rightarrow y_i = -\sqrt{x_i^2 + x_j^2}$

$y_j = 0$

Choice of sign: choose c positive.

Orthogonal reduction to triangular form:

$$A = \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix} \longrightarrow P_{12}A \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix} \longrightarrow P_{13}P_{12}A \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ x & x & x & x \end{pmatrix}$$

$$P_{13}P_{12}A = \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{pmatrix} \longrightarrow P_{23}P_{14}P_{13}P_{12}A \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & x & x & x \end{pmatrix}$$

$$\longrightarrow P_{24}P_{23}P_{14}P_{13}P_{12}A \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{pmatrix}$$

$$\longrightarrow P_{34}P_{24}P_{23}P_{14}P_{13}P_{12}A = \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{pmatrix} = R$$

$$R = (\nabla) = P_{34}P_{24}P_{23}P_{14}P_{13}P_{12}A$$

By orthogonality of each P_{ij}

$$A = \underbrace{\left(P_{12}^T P_{13}^T P_{14}^T P_{23}^T P_{24}^T P_{34}^T\right)}_{Q} R = QR$$

Work: Standard ordering: $2n^3 + O(n^2)$ flops: $n(n-1)/2$ rotations

Householder: $\frac{4}{3}n^3 + O(n^2)$

BUT: Rotations can be applied in parallel

AND: greater efficiency for matrices of certain sparse structures.

See: Quarteroni 5.6: Householder to upper-Hessenberg form:
zeros below 1st subdiagonal

Then Givens rotations.