1 Question 1 - Book 1.1

1.1 1.1 part a

The general idea is that all row operations acting on A are carried out by multiplying on the left of A and all column operations act on A from the right.

So, all column operations (operations 1,4,6,7 in order) can be given as

$$B\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\\0&0&1\end{pmatrix} \tag{1}$$

and all row operations (operations 2,3,5 in order) can be given as

$$\begin{pmatrix}1&-1&0&0\\0&1&0&0\\0&-1&1&0\\0&-1&0&1\end{pmatrix}\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{1}{2}&0\\0&0&0&1\end{pmatrix}B \tag{2}$$

Therefore, all operations on B can be given as

$$\begin{pmatrix}1&-1&0&0\\0&1&0&0\\0&-1&1&0\\0&-1&0&1\end{pmatrix}\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{1}{2}&0\\0&0&0&1\end{pmatrix}B\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\\0&0&1\end{pmatrix} \tag{3}$$

1.2 1.1 part b

We can use the above information and the rules of matrix multiplication to condense the left and right hand matrices on either side of B to single matrices. Thus, for the left hand side, we note that a matrix A is given as the product of the 3 matrices shown above, seen as

$$A=\begin{pmatrix}1&-1&\frac{1}{2}&0\\0&1&0&0\\0&-1&\frac{1}{2}&0\\0&-1&0&1\end{pmatrix} \tag{4}$$

Additionally, the matrix product of the four matrices on the right of B can be condensed using the rules of matrix multiplication such that

$$C=\begin{pmatrix}0&0&0\\2&0&0\\0&1&1\\0&0&0\end{pmatrix} \tag{5}$$

2 Question 2

Suppose $\mathbf{R}$ is given to be an upper triagular matrix such that its components are

$$R_{ij}=\begin{cases} r_{ij} & \text{if } j\geq i\\ 0 & \text{otherwise}\end{cases} \tag{6}$$

for a general $r_{ij} \in \mathbb{C}$. The full matrix can be written as

$$R_{m,m} = \begin{pmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,m} \\ 0 & r_{2,2} & \cdots & r_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{m,m} \end{pmatrix}$$

Suppose also that $\mathbf{R}$ has an inverse, $\mathbf{R}^{-1}$, given to be

$$R^{-1}_{m,m} = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix}$$

for general $x_{1\cdots m,1\cdots m} \in \mathbb{C}$. Furthermore, the matrix multiplication of these two matrices abides by the following relationship

$$\mathbf{R}\mathbf{R}^{-1} = \mathbf{R}^{-1}\mathbf{R} = \mathbf{I} \tag{7}$$

where $\mathbf{I}$ is the identity matrix. Using pg 6 of our text as reference, we further know that the column formula yields

$$\mathbf{I}_j = \mathbf{R}^{-1}\mathbf{R}_j = \sum_{k=1}^{m} r_{kj} x_k \tag{8}$$

with the interpretation that a column of the identity matrix is a linear combination of the columns of $\mathbf{R}^{-1}$ with coefficients r_{kj}. So we will prove that the matrix $\mathbf{R}^{-1}$ is upper triangular for cases $j = 1, 2, \cdots, m-n, m$ for an arbitrary n and therefore true for any m.

2.1 case $j = 1$

$$I_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} = r_{1,1} \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{m-n,1} \\ \vdots \\ x_{m,1} \end{bmatrix} \tag{9}$$

The only solution to this equation is one where $r_{1,1}x_{1,1} = 1 \rightarrow x_{1,1} = \frac{1}{r_{1,1}}$ and the other $x_{2\cdots m,1} = 0$

2.2 case $j = 2$

$$I_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} = r_{1,2} \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{m-n,1} \\ \vdots \\ x_{m,1} \end{bmatrix} + r_{2,2} \begin{bmatrix} x_{1,2} \\ x_{2,2} \\ \vdots \\ x_{m-n,2} \\ \vdots \\ x_{m,2} \end{bmatrix} = r_{1,2} \begin{bmatrix} \frac{1}{r_{1,1}} \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} + r_{2,2} \begin{bmatrix} x_{1,2} \\ x_{2,2} \\ \vdots \\ x_{m-n,2} \\ \vdots \\ x_{m,2} \end{bmatrix} \tag{10}$$

From this set of equations, we know that for any $x_{3\cdots m,2} = 0$, $x_{2,2} = \frac{1}{r_{2,2}}$, and $\frac{r_{1,2}}{r_{1,1}} + r_{2,2}x_{1,2} = 0$

2.3 case $j = m - n$

$$I_{m-n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = r_{1,m-n} \begin{bmatrix} x_{1,1} \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} + r_{2,m-n} \begin{bmatrix} x_{1,2} \\ x_{2,2} \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + r_{m-n,m-n} \begin{bmatrix} x_{1,m-n} \\ x_{2,m-n} \\ \vdots \\ x_{m-n,m-n} \\ \vdots \\ x_{m,m-n} \end{bmatrix} \tag{11}$$

We can follow the pattern and immediately note that for any $p > m-n$, $x_{p,m-n} = 0$ and $x_{m-n,m-n} r_{m-n,m-n} = 1$ must be true. Furthermore, back substitution can be used to determine the values of $x_{p,m-n}$ for $p < m-n$.

2.4 case $j = m$

$$I_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{bmatrix} = r_{1,m-n} \begin{bmatrix} x_{1,1} \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} + r_{2,m-n} \begin{bmatrix} x_{1,2} \\ x_{2,2} \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + r_{m-n,m-n} \begin{bmatrix} x_{1,m-n} \\ x_{2,m-n} \\ \vdots \\ x_{m-n,m-n} \\ \vdots \\ 0 \end{bmatrix} + \cdots + r_{m,m} \begin{bmatrix} x_{1,m} \\ x_{2,m} \\ \vdots \\ x_{m-n,m} \\ \vdots \\ x_{m,m} \end{bmatrix} \tag{12}$$

Again, we follow the pattern; namely that $x_{m,m} r_{m,m} = 1$, and follow the back substitution procedure to determine the other $x_{m-i,m}$ for any $1 < i < m$.

Thus substituting all of this information, we have shown that $\mathbf{R}^{-1}$ must have an upper triangular structure given to be

3 Question 3

We wish to prove the fact that if matrix A is both lower triangular and normal, then it is diagonal. We will skip the $m = 1$ case, as this matrix is by default triangular, normal, and diagonal. For the $m = 2$ case we see that A can be written

$$A = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \tag{13}$$

Then, we note that for A to be normal, $AA^* = A^*A$, so expanding this out we see that

$$AA^* = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \bar{a_{11}} & \bar{a_{21}} \\ 0 & \bar{a_{22}} \end{pmatrix} = \begin{pmatrix} |a_{11}|^2 & a_{11}\bar{a_{21}} \\ a_{21}\bar{a_{11}} & |a_{21}|^2 + |a_{22}|^2 \end{pmatrix} \tag{14}$$

and

$$A^*A = \begin{pmatrix} \bar{a_{11}} & \bar{a_{21}} \\ 0 & \bar{a_{22}} \end{pmatrix} \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} |a_{11}|^2 + |a_{21}|^2 & a_{22}\bar{a_{21}} \\ a_{21}\bar{a_{22}} & |a_{22}|^2 \end{pmatrix} \tag{15}$$

which further implies that $a_{21} = 0$ for the expression to hold. Note, if we plug this into our original matrix A, we see that it is diagonal.

Now suppose matrices of dimensions $l = 1, \cdots, m-1$ are diagonal, we will show that the relationship holds for dimension m. Given a lower triangular matrix of dimension $m-1 \times m-1$ called A_{m-1} and row vector b of dimension $m-1$, we can show that

$$AA^* = \begin{pmatrix} A_{m-1} & 0 \\ b & a_{mm} \end{pmatrix} \begin{pmatrix} A^*_{m-1} & b^* \\ 0 & a_{\bar{m}m} \end{pmatrix} = \begin{pmatrix} A^*_{m-1}A_{m-1} & A_{m-1}b^* \\ A^*_{m-1}b & bb^* + |a_{mm}|^2 \end{pmatrix} \tag{16}$$

and

$$A^*A = \begin{pmatrix} A^*_{m-1} & b^* \\ 0 & a_{\bar{m}m} \end{pmatrix} \begin{pmatrix} A_{m-1} & 0 \\ b & a_{mm} \end{pmatrix} = \begin{pmatrix} b^*b + A^*_{m-1}A_{m-1} & b^*a_{mm} \\ a_{\bar{m}m}b & |a_{mm}|^2 \end{pmatrix} \tag{17}$$

Note, 0 here denotes a column of 0 of dimension $m-1$ in the case of matrix A. Equating these two, we see that they are only equivalent if and only if $b = 0$. Again, plugging this into the general expression for matrix A of dimenion $m \times m$ we see that A is diagonal.

4 Question 4

Assuming $Ay = x$, we can write

$$x^*(Ay) = x^*(\sum_j y_j a_j) \tag{18}$$

where a_j represent the jth column of matrix A. We see that $x^* = (Ay)^* = y^*A^*$, and inserting into the above equation yields

$$(\sum_{j^*} a^*_{j^*} y^*_{j^*})(\sum_j y_j a_j) \tag{19}$$

Acknowledging that the inner product formula has the form $p^*q = \sum_i^m \bar{p}_i q_i$, we can equate $\bar{p}_i = a^*_{j^*} y^*_{j^*}$ and $q_i = y_j a_j$, meaning the above formula is in the form of an inner product, ie

$$x^*(Ay) = p^*q = \sum_i^m \bar{p}_i q_i \tag{20}$$

Therefore, $x^*(Ay)$ represents an inner product. NOTE:: The above holds also if $Ay = z$ for general column vector z $\in \mathbb{C}^m$; none of the derivations change fundamentally as A acting on y creates a column vector z which is acted upon by a row vector x^*. I just wanted to point this out.

5 Question 5 - Book 2.5

5.1 2.5.a

Assuming $Sx = \lambda x$, we can project this onto x^* such that

$$x^*Sx = \lambda x^*x \to \lambda = \frac{x^*Sx}{x^*x} \tag{21}$$

where the last part is justified since division by the scalar x^*x is allowed. This equation implies that

$$\bar{\lambda} = \frac{(x^*Sx)^*}{(x^*x)^*} = \frac{(x^*S^*x)}{x^*x} \tag{22}$$

and since we know $S^* = -S$, then

$$-\bar{\lambda} = \frac{x^*Sx}{x^*x} \tag{23}$$

which implies that $\lambda - \bar{\lambda} = 2\frac{x^*Sx}{x^*x}$. If both $\lambda, \bar{\lambda} \in \mathbb{C}$, then $\lambda = a + ic$ and $\bar{\lambda} = a - ic$. Therefore

$$\lambda - \bar{\lambda} = ic = \frac{x^*Sx}{x^*x} \tag{24}$$

Ergo the eigenvalues are purely imaginary.

5.2 2.5.b

Let $B \equiv I - S$. The idea is that we want to show B is invertible, which means the columns of B are linearly independent, which further implies that the column vectors of B, $x_i \perp x_j, i \neq j$

$$\begin{cases} Bx_i = \lambda_i x_i \\ Bx_j = \lambda_j x_j \end{cases} \tag{25}$$

, where we make the assumption that $\lambda_i \neq \lambda_j$ and $x_i \neq x_j$. We can project the above equations onto x_j^* and x_i^* such that

$$\begin{cases} x_j^* Bx_i = \lambda_i x_j^* x_i \\ x_i^* Bx_j = \lambda_j x_i^* x_j \end{cases} \tag{26}$$

We can then add the complex conjugate of the second equation from the top equation such that

$$x_j^* Bx_i + x_j^* B^* x_i = \lambda_i x_j^* x_i + \lambda_j x_j^* x_i \tag{27}$$

At this point, it is prudent to expand $B \equiv I - S$ and $B^* \equiv I^* - S^* = I + S$ such that

$$x_j^* x_i + x_j^* x_i - x_j^* Sx_i + x_j^* Sx_i = \lambda_i x_j^* x_i + \lambda_j^* x_j^* x_i \tag{28}$$

which can be simplified such that

$$0 = x_j^* x_i (\lambda_i - \lambda_j - 2) \tag{29}$$

Thus, two solutions exist:

$$\begin{cases} x_j^* x_i = 0 \\ (\lambda_i - \lambda_j - 2) = 0 \end{cases} \tag{30}$$

Remembering that $\lambda_i \neq \lambda_j$ and $x_i \neq x_j$, then the first relation implies that the column vectors must be perpendicular. In other words, B must be nonsingular and/or invertible.

5.3 2.5.c

To show $Q = (I - S)^{-1}(I + S)$ is unitary, we need to show that $QQ^* = Q^*Q = I$

In other words,

$$Q^*Q = \Big((I - S)^{-1}(I + S)\Big)^*(I - S)^{-1}(I + S) \tag{31}$$

Expanding this, we see that

$$(I + S)^*\Big((I - S)^*(I - S)\Big)^{-1}(I + S) = (I - S)\Big(I - S^2\Big)^{-1}(I + S) \tag{32}$$

We can multiply the above by 1, or $(I - S)(I - S)^{-1}$, yielding

$$Q^*Q = (I - S)\left(I - S^2\right)^{-1}(I - S^2)(I - S)^{-1} = I \tag{33}$$

Furthermore, by the properties of commuting operators, we note that $[Q, Q^*] = QQ^* - Q^*Q = 0$. Since we have already shown $Q^*Q = I$, we have $QQ^* - I = 0 \rightarrow QQ^* = I$. Ergo, Q is a unitary matrix.

6 Question 6 - Book 2.6

Assuming A is nonsingular, we wish to find A^{-1} such that $AA^{-1} = I$. So we choose to express $A^{-1} = |x_1, \cdots, x_m|$ in a column vector format for arbitrary and unknown x_i; inserting this into our expression, we see that

$$AA^{-1} = (I + uv^*)|x_1, \cdots, x_m| = |e_1, \cdots, e_m| \tag{34}$$

where e_i are the column vectors of the identity matrix. This system of equations can be simplified into a general framework:

$$x_i + uv^*x_i = e_i \rightarrow x_i = e_i - u(v^*x_i) \tag{35}$$

given for each column of A^{-1}. The final term in parenthesis is just a scalar $\beta = v^*x_i$, proving that

$$A^{-1} = I - u\beta \tag{36}$$

Expanding this out to solve for β, we see that

$$AA^{-1} = (I + uv^*)(I - u\beta) = (I + uv^* - u\beta - uv^*u\beta) = I \tag{37}$$

I on each side cancels, leaving $v^* = \beta(1 + v^*u) \rightarrow \beta = \frac{v^*}{1+v^*u}$. Plugging this final expression of β into our expression for A^{-1} we see that it is in the form

$$A^{-1} = I - \frac{uv^*}{1 + v^*u} \rightarrow I + \alpha uv^* \tag{38}$$

where the last expression holds assuming that $\alpha = \frac{-1}{1+v^*u}$

If A is singular, then

$$Ax = (I + uv^*)x = 0 \rightarrow x = u(-v^*x) \tag{39}$$

Since the term in parenthesis is a scalar, we acknowledge that the above can be rewritten such that $x = \alpha u$. Inserting this expression into Ax, we see that

$$A(\alpha u) = (I + uv^*)(\alpha u) = \alpha u + \alpha uv^*u = \alpha u(1 + v^*u) = 0 \tag{40}$$

This gives us two solutions: 1) $\alpha u = 0$ or 2) $1 + v^*u = 0 \rightarrow v^*u = -1$. The former is the trivial solution, so we assume the latter condition holds. As such, the nullspace of A would be any linear combination of αu.