MAD 6406 f20 lec 8 09/18/20

<u>SVD</u> + 4 fund. subspaces

- orthogonality of the subspaces
- subspaces spanned by singular vectors.

Orthogonality of the subspaces: for $A \in \mathbb{C}^{m\times n}$

① $\text{null}(A) \perp \text{col}(A^*)$: for $v \in \text{null}(A)$, $w \in \text{col}(A^*)$, $v^* w = 0$.

pf: $v \in \text{null}(A) \Rightarrow Av = 0$
for $y \in \text{col}(A^*)$, ~~$\exists$~~ $y = A^* w$ for some $w \in \mathbb{C}^m$
Thus: $y^* v = (A^* w)^* v = w^* A v = 0$
$\therefore$ $\text{null}(A) \perp \text{col}(A^*)$

② Any $v \in \mathbb{C}^n$ orthogonal to $\text{col}(A^*)$ is in $\text{null}(A)$:
$v \perp \text{col}(A^*) \Rightarrow v^*(A^* y) = 0 \quad \forall\, y \in \mathbb{C}^m$
Thus: $0 = (v^* A^* y)^* = y^* A v \quad \forall\, y \in \mathbb{C}^m \Rightarrow Av = 0 \Rightarrow v \in \text{null}(A)$.

① + ② shows $\mathbb{C}^n = \text{col}(A^*) \oplus \text{null}(A)$

Prop: Each $w \in \mathbb{C}^n$ has the unique decomp: $w = w_R + w_N$, $w_R \in \text{col}(A^*)$, $w_N \in \text{null}(A)$.

Consider $n\times n$ matrix X whose columns $x_1, \dots, x_r$ are a basis for $\text{col}(A^*)$.
By the rank-nullity thm, ~~$\dim(\text{null}(A^*))$~~ $\dim(\text{null}(A)) = n - r$, so let
$x_{r+1}, \dots, x_n$ be a basis for $\text{null}(A)$

Thus $X = \left(x_1 \cdots x_r \,\middle|\, x_{r+1} \cdots x_n \right)$ X is invertible b/c its n columns are LI vectors in $\mathbb{C}^n$.

$Xv = w$

$\therefore$ The problem ~~[illegible]~~ has a unique soln for any $w \in \mathbb{C}^n$:

$$Xw = \underbrace{\sum_{i=1}^{r} (x_i) v_i}_{\in \text{col}(A^*)} + \underbrace{\sum_{i=r+1}^{n} (x_i) v_i}_{\in \text{null}(A)} = w_R + w_N$$

On next HW:

① $\mathrm{col}(A) \perp \mathrm{null}(A^*)$

② Any $v \in \mathbb{C}^m$ orthogonal to $\mathrm{col}(A)$ is in $\mathrm{null}(A^*)$

③ Each $v \in \mathbb{C}^m$ has a unique decomposition $v = v_R + v_N$ with $v_R \in \mathrm{col}(A)$, $v_N \in \mathrm{null}(A^*)$

Back to the SVD

$$ {}_mA^n = \begin{pmatrix} U_1 & \Big| & U_2 \end{pmatrix}_m^m \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ \hline & & & \end{pmatrix}_m^n \begin{pmatrix} V_1^* \\ \hline V_2^* \end{pmatrix}_n^n $$

r $\quad U \qquad \Sigma$

We'll show

(1) $\mathrm{Col}(A) = \mathrm{span}\{u_1, \dots, u_r\}$	Cols of U_1 form $\perp$n basis of $\mathrm{col}(A)$
(2) $\mathrm{Null}(A^*) = \mathrm{span}\{u_{r+1}, \dots, u_m\}$	Cols of U_2 form $\perp$n basis of $\mathrm{null}(A^*)$
(3) $\mathrm{Col}(A^*) = \mathrm{span}\{v_1, \dots, v_r\}$	Cols of V_1 form $\perp$n basis of $\mathrm{col}(A^*)$
(4) $\mathrm{Null}(A) = \mathrm{span}\{v_{r+1}, \dots, v_n\}$	Cols of V_2 form $\perp$n basis of $\mathrm{null}(A)$

Recall: $\{w_1, \dots, w_r\}$ form a basis for W if:
(1) $W = \mathrm{span}\{w_1, \dots, w_r\}$, and (2) $\{w_1, \dots, w_r\}$ is an LI set.

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(1) proof: For $b \in \mathrm{col}(A)$, $b = Ax = U\Sigma V^* x$ for some $x \in \mathbb{C}^n$.

The cols of V are a basis for $\mathbb{C}^n$, so $x = Vy$ for some $y \in \mathbb{C}^n$.

So: $b = Ax = U\Sigma V^* V y = U\Sigma y = U \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & 0 \\ \hline & 0 & & 0 \end{pmatrix} y$

$= U \begin{pmatrix} \sigma_1 y_1 \\ \vdots \\ \sigma_r y_r \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \end{pmatrix} \sigma_1 y_1 + \dots + \begin{pmatrix} u_r \end{pmatrix} \sigma_r y_r = \sum_{j=1}^{r} \begin{pmatrix} u_j \end{pmatrix} \sigma_j y_j$

$\therefore$ $\mathrm{col}(A) \subseteq \mathrm{span}\{u_1, \dots, u_r\}$. Also: for each $j = 1, \dots, r$ $Av_j = U\Sigma V^* v_j = \sigma_j u_j$,
which shows $\mathrm{span}\{u_1, \dots, u_r\} \subseteq \mathrm{col}(A)$.
$\therefore$ $\mathrm{col}(A) = \mathrm{span}\{u_1, \dots, u_r\}$.

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(4) pf: $\mathrm{null}(A) = \{x \in \mathbb{C}^n \mid Ax = 0\}$. For $x = Vy$, $Ax = U\Sigma V^* V y = U\Sigma y$.

Setting $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = U\Sigma y = \begin{pmatrix} u_1 \dots u_r & \Big| & u_{r+1} \dots u_m \end{pmatrix}_m^m \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & 0 \\ \hline & 0 & & 0 \end{pmatrix}^n \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} =$

$= \sum_{j=1}^{r} u_j \sigma_j y_j + \sum_{j=r+1}^{m} 0$. Cols of U_1 are LI, so $y_1, \dots, y_r = 0$, so

$Ax = 0 \Rightarrow x \in \mathrm{span}\{v_{r+1}, \dots, v_n\}$ $\therefore$ $\mathrm{null}(A) \subseteq \mathrm{span}\{v_{r+1}, \dots, v_n\}$. Since $Av_j = U\Sigma V^* v_j = 0$, for $j = r+1, \dots, n$, $\mathrm{null}(A) = \mathrm{span}\{v_{r+1}, \dots, v_n\}$.