- Arnoldi & Lanczos iterations.

Recall: phase 1 of eigenvalue iterations:
reduction to upper-Hessenberg form (general matrices)
for symmetric problems, upper-Hessenberg is tridiagonal

We discussed obtaining the factorization $A = QHQ^*$ using Householder transforms.

You may recall: Earlier, when we discussed computing the QR factorization, we discussed:

- Gram-Schmidt (unstable)
- Modified Gram-Schmidt (stable)
- Householder reflections (stable)
- Givens rotations (stable).

Between Householder + Modified G.S., Householder is more work, but stable.

The Arnoldi iteration is the "modified G.S." version of ~~the~~ computing $A = QHQ^*$.

- disadvantage: less stable
- advantage: Householder is <u>dense</u>, it does not preserve sparse structure of A (if A has such structure).
  Arnoldi requires repeated multiplications of A against vectors, so if A is large + sparse (common situation), it may be much faster!
- advantage: sometimes it is sufficient to run it for "a while" but not until convergence (restarted Arnoldi is a standard PageRank algorithm)

We want: Q (unitary), H upper-Hess s.t. $A = QHQ^*$, or

$AQ = QH$ for $m \times m$ matrix A.

For $n \leq m-1$, let Q_n be the 1st n columns of Q

$\tilde{H}_{n+1}$ be the $(n+1)\times n$ upper-left section of H.

$$\begin{pmatrix} A \end{pmatrix}\left(\underbrace{q_1\ q_2 \cdots q_n}_{Q_n} \;\middle|\; q_{n+1} \cdots q_m\right) = \left(q_1 \cdots q_n \mid q_{n+1} \mid \cdots q_m\right)\begin{pmatrix} \circ & \circ & \cdots \\ \circ & \circ & \\ & \circ & \ddots \end{pmatrix}$$

Thus

$$AQ_n = (Q_n \mid q_{n+1})\,\tilde{H}_{n+1} \Rightarrow AQ_n = Q_{n+1}\tilde{H}_{n+1}$$

The n^{th} column of this iteration can be written:

$$Aq_n = h_{1n}q_1 + \cdots + h_{nn}q_n + h_{n+1,n}q_{n+1}$$

$$Aq_n = \sum_{j=1}^{n+1} h_{jn} q_j$$

Notice, this says:

$$h_{n+1,n}q_{n+1} = Aq_n - \sum_{j=1}^{n} q_j h_{jn} \Rightarrow q_{n+1} = \left(Aq_n - \sum_{j=1}^{n} q_j h_{jn}\right)\Big/ h_{n+1,n}$$

But the q's are orthogonal! so $h_{jn} = q_j^*(Aq_n)$.

This process ~~oss~~ produces an orthonormal basis for the Krylov space $K_n := \{b, Ab, \ldots, A^{n-1}b\}$, where $q_1 = b/\|b\|$

$\Rightarrow$ the process should be terminated when $|h_{n+1,n}| < tol$, if not before.

Arnoldi iteration:

Given some b, $q_1 = b/\|b\|$

$\text{for } n = 1, 2, \ldots$

$\quad v = A q_n$

$\quad \text{for } j = 1:n$

$\quad\quad h_{jn} = q_j^t v$

$\quad\quad v \;\; = v - h_{jn} q_j$

$\quad \text{end}$

$\quad h_{n+1,n} = \|v\|$

$\quad q_{n+1} = v / h_{n+1,n}$

end

Notice q_{n+1} is $A^n q_1 = A^n b$, orthogonalized against $b, Ab, A^2 b \ldots$

so $q_n \to$ dominant eigenvector, like in the power iteration.

~~The matrices H_n can~~

The upper-Hessenberg matrix H_n generated are used to approx eigenvalues of A

If this is run to some fixed n, the dominant eigenvector of H_n is found, and used as the initial vector b to start a new Arnoldi process, the method is called "restarted Arnoldi."

For instance, if $m = 10^6$, $n = 10$, this can be very efficient!

<u>Lanczos</u>: suppose A is symmetric. (and real)

Then entrywise, $A = QHQ^T$ yields

$$h_{ij} = q_i^T A q_j = h_{ji}$$

Then $H_n = T_n = \begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & \alpha_3 & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{pmatrix}$

The Arnoldi iteration reduces to the Lanczos iteration.

Set $\beta_0 = 0$, $q_0 = 0$, b some vector, $q_1 = b/\|b\|$

for $n = 1, 2, 3, \ldots$

$v = Aq_n$
$\alpha_n = q_n^T v$
$v = v - \beta_{n-1} q_{n-1} - \alpha_n q_n$
$\beta_n = \|v\|$
$q_{n+1} = v/\beta_n$

end