- QR iteration
- Relation to power method + simultaneous iteration

<u>Global methods</u> : for approximating Schur factorization + entire spectrum.

QR algorithm:

$A_1 = Q_0 A Q_0 \to Q_0 A Q_0$ is Householder reduction to upper-Hessenberg form.

~~Repeat~~

For $j = 1, 2, \ldots$

~~&~~ factor $A_j = Q_j R_j$

set $A_{j+1} = R_j Q_j$

end

$A_j \to$ upper Δ matrix with eigenvalues on diagonal.

Notice: $R_k = Q_k^* A_k$

$$A_{k+1} = \cancel{Q_k} R_k Q_k$$

$$= Q_k^* A_k Q_k$$

$$= \underbrace{(Q_k^* \cdots Q_1^*)}_{(\tilde{Q}_k)^*} \cancel{Q} A_1 \underbrace{Q_1 \cdots Q_k}_{\tilde{Q}_k}$$

$$= (Q_k^* \cdots Q_1^*) Q_0 A Q_0 Q_1 \cdots Q_k$$

$\therefore A_{k+1}$ is unitarily similar to A.

Convergence: If $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$

then the p^{th} subdiagonal entry in A_k converges ~~at a rate~~ to zero at rate

$$\left| \frac{\lambda_{p+1}}{\lambda_p} \right|^k$$

We can understand convergence in some sense by relating the QR iteration to the power method.

~~Do for A~~ Suppose (for simplicity) A is ~~~~ upper Hess. ~~~~
QR iteration gives

$$A_{k+1} = (Q_k^* \cdots Q_1^*)\, A\, (Q_1 \cdots Q_k).$$

Next, note

$$
\begin{aligned}
&(Q_1 \cdots Q_k)(R_k \cdots R_1) \\
&= (Q_1 \cdots Q_{k-1})\, Q_k R_k\, (R_{k-1} \cdots R_1) && \leftarrow Q_k R_k = A_k \\
&= (Q_1 \cdots Q_{k-1})\, A_k\, (R_{k-1} \cdots R_1) && \uparrow A_k = (Q_{k-1}^* \cdots Q_1) A (Q_1 \cdots Q_{k-1}) \\
&= A (Q_1 \cdots Q_{k-1})(R_{k-1} \cdots R_1) \\
&= A (Q_1 \cdots Q_{k-2})\, Q_{k-1} R_{k-1}\, (R_{k-2} \cdots R_1) && \leftarrow Q_{k-1} R_{k-1} = A_{k-1} \\
&= A (Q_1 \cdots Q_{k-2})\, A_{k-1}\, (R_{k-2} \cdots R_1) && \uparrow A_{k-1} = (Q_{k-2}^* \cdots Q_1) A (Q_1 \cdots Q_{k-2}) \\
&= A^2 (Q_1 \cdots Q_{k-2})(R_{k-2} \cdots R_1) \\
&\;\;\vdots \\
&= A^k
\end{aligned}
$$

$\therefore$ $(Q_1 \cdots Q_k)(R_k \cdots R_1) = A^k$, k^{th} power of A.

$$\tilde{Q}_k \tilde{R}_k = A^k$$

Relation to simultaneous / orthogonal iteration: For simplicity, suppose A is ~~normal~~ Hermitian

If we start power method with $x_0 = e_1$

for $k = 1, 2, \ldots$
  $w = A x_{k-1}$
  $x_k = w / \|w\|$
end

If we start the power method with $\hat{Q}_0 = \{e_1\, e_2\}$

for $k = 1, 2, \ldots$
  $W_k = A\hat{Q}_{k-1}$
  $(\hat{Q}_k \hat{R}_k) = W_k$ & orthonormalize
end

We are running a power iteration on multiple vectors, orthogonalizing at each step.

If $\hat{Q}_0 = I_n$:

<u>Simultaneous iteration</u>:

for $k = 1, 2, \ldots$
  $W_k = A\hat{Q}_{k-1}$
  $\hat{Q}_k R_k = W_k$
  $A_k = \hat{Q}^*_k A \hat{Q}_k$
end

Thus: let $\hat{q}_i^k$: the ith column of $\hat{Q}_k$: $\hat{q}_i^k \to$ evector assoc with λ_i.

Relation to QR iteration:

Simultaneous and QR iterations generate identical sequences, defined by the QR factorization of the k^{th} power of A.

~~[illegible]~~ Showed already: for QR: $\underbrace{(Q_1 \cdots Q_k)}_{\tilde{Q}_k}\underbrace{(R_k \cdots R_1)}_{\tilde{R}_k} = A^k$

and $A_{k+1} = \tilde{Q}_k^* A Q_k$

for simultaneous iteration, show $\hat{Q}_k \hat{R}_k = A^k$, $\hat{R}_k = R_k R_{k-1} \cdots R_1 \overset{I}{\hat{R}_0}$

Base step ($k=0$)

~~$Q_0 \tilde{Q}_0 = I$~~ $\quad I = A^0 = \tilde{Q}_0 \tilde{R}_0$ ($\tilde{Q}_0 = I$, $R_0 = I$ for QR)

assume $\hat{Q}_0 = I$ for simult.

~~Inductive step~~
~~Need to show for Simultaneous~~

IH: for simultaneous, $A^{k-1} = \hat{Q}_{k-1} \hat{R}_{k-1}$

I.S.

$$A^k = A A^{k-1} = \underbrace{A \hat{Q}_{k-1}}_{W_k} \hat{R}_{k-1} = W_k \hat{R}_{k-1} = \hat{Q}_k R_k \hat{R}_{k-1} = \hat{Q}_k \hat{R}_k$$

$$\therefore \; A^k = \underbrace{\hat{Q}_k \hat{R}_k}_{\text{Simult.}} = \underbrace{\tilde{Q}_k \tilde{R}_k}_{QR}$$

And: $A_k = \hat{Q}_k^* A \hat{Q}_k = \tilde{Q}_k^* A \tilde{Q}_k$

$\Rightarrow$ diagonals of A_k are the Rayleigh quotients of A corr. to the $\tilde{q}_k$ vectors.

Comparing the QR iteration to the simultaneous iteration:

QR (no shift):

$A_0 = A$

for $k = 1, 2, \ldots$

$A_{k-1} = Q_k R_k$ (factor)

$A_k = R_k Q_k$ (multiply)

$\tilde{Q}_k = Q_1 \cdots Q_k$

Simultaneous:

$\tilde{Q}_0 = I$

for $k = 1, 2, \ldots$

$W = A\tilde{Q}_{k-1}$ (multiply)

$W = \tilde{Q}_k R_k$ (QR factor)

$A_k = \tilde{Q}_k^* A \tilde{Q}_k$

Both algorithms compute QR factorizations of A^k by

$A^k = \tilde{Q}_k \tilde{R}_k, \qquad \tilde{R}_k = R_k R_{k-1} \cdots R_1$

and $A_k = \tilde{Q}_k^* A \tilde{Q}_k$

"Simple case" A is real + symmetric, n simple eigenvalues.

As $k \to \infty$, $\tilde{Q}_k(:,j) \to j^{th}$ eigenvector of A (orthog. of power method)

$A_k(j,j) = (\tilde{Q}_k(:,j))^T A (\tilde{Q}_k(:,j))$ is the Rayleigh quot that gives e.v. approx to λ_j

$i \neq j$: $A_k(i,j) = \tilde{Q}_k(:,i)^T A (\tilde{Q}_k(:,j)) \to 0$ by orthogonality of eigenvectors

$\Rightarrow$ If A is nonsym (non-Hermitian), and has nonsimple evals, the situation is more complicated.