MAD 6406 f20 — lec 6 — 09/14/20

- dual norms
- matrix norm computation and the ∞-norm.

Consider the outer-product uv^*:

$$\|uv^*\|_p = \max_{\|w\|_p=1} \|uv^*w\|_p = \max_{\|w\|_p=1} |v^*w|\cdot\|u\|_p = \|u\|_p \max_{\|w\|_p=1} |v^*w|.$$

The 2nd term: $\max_{\|w\|_p=1} |v^*w| = \max_{w\neq 0} \frac{|v^*w|}{\|w\|}$ is called the (vector)

dual-norm of v: The norm dual to $\|v\|_p$:

$$\|v\|_p' = \max_{w\neq 0} \frac{|v^*w|}{\|w\|}.$$

<u>Some dual-norms</u>:

dual-norm of $\|\cdot\|_2$: $\max_{w\neq 0} \frac{|v^*w|}{\|w\|_2} \geq \frac{v^*v}{\|v\|_2} = \|v\|_2$

The Cauchy-Schwarz ineq gives

$|v^*w| \leq \|v\|_2\|w\|_2 \Rightarrow \frac{|v^*w|}{\|w\|_2} \leq \|v_2\|$

so $\|v\|_2' = \max_{w\neq 0} \frac{|v^*w|}{\|w\|_2} = \|v\|_2$: the 2-norm is self-dual.

The dual of the ∞-norm: $\max_{w\neq 0} \frac{|v^*w|}{\|w\|_\infty}$:

$$|v^*w| = \left|\sum_{i=1}^n \bar{v}_i w_i\right| \leq \max_{1\leq i\leq n} |w_i| \sum_{i=1}^n |\bar{v}_i| = \|w\|_\infty \sum_{i=1}^n |v_i| = \|w\|_\infty \|v\|_1$$

$$\Rightarrow \frac{\|v^*w\|}{\|w\|_\infty} \leq \|v\|_1$$

Now, construct w where the upper bound is realized

Let $w = \begin{cases} v_i/|v_i|, & v_i \neq 0 \\ 1, & v_i = 0 \end{cases}$. Then $\frac{|v^*w|}{\underbrace{\|w\|_\infty}_{=1}} = \|v^*w\| = \left|\sum_{i=1}^n \bar{v}_i w_i\right| = \left|\sum_{i=1}^n |v_i|\right| = \|v\|_1$

$\therefore \max_{w\neq 0} \frac{|v^*w|}{\|w\|_\infty} \geq \|v\|_1$

conclude: $\|v\|_\infty' = \|v\|_1$.

In general (see Horn + Johnson, Matrix Analysis)

~~*~~ The dual-norm of the vector p-norm is the vector q-norm where $1/p + 1/q = 1$ $\quad \|\cdot\|_p' = \|\cdot\|_q, \ \frac{1}{p}+\frac{1}{q}=1$

Back to our outer product:

$\|uv^*\|_p = \|u\|_p \max_{\|y\|=1} |v^* y| = \|u\|_p \|v\|_p' = \|u\|_p \|v\|_q, \quad \frac{1}{p}+\frac{1}{q}=1.$

We also ~~have~~ have the Hölder ineq. for the inner product:

$\|x\|' = \max_{y\neq 0} \frac{|x^* y|}{\|y\|}$, so for any vector w, $\frac{|x^* w|}{\|w\|} \le \|x\|'$

so $|x^* w| \le \|x\|' \|w\|$, where $\|x\|_q' = \|x\|_p$

For vector p-norms: $|x^* w| \le \|x\|_p \|w\|_q, \quad 1/p + 1/q = 1.$ (Hölder).

Computing matrix norms: in general it is hard.

For matrix norms induced by vector p-norms with $p = 1, 2, \infty$, it is relatively easy.

$\|A\|_1 = \max_{\|x\|_1=1} \|Ax\|_1 =$ max column-sum of A: $\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|$ (HW)

$\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty =$ max row-sum of A: $\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$

$\|A\|_2 = \max_{1\le j\le n} \sqrt{\lambda_j (A^* A)} = \max_{1\le j\le n} \sigma(A)$

recall: A^*A has all ~~positive~~ non-negative eigenvalues

$\cancel{\|A\|_\infty = \sum \max}$ $\|A\|_\infty = \max\limits_{1 \le i \le m} \sum\limits_{j=1}^{n} |a_{ij}|$. "$\|A\|_\infty$ is the max row sum" 3-6

If $A = 0$, this is clear. Assume $A \neq 0$.

Let $a^1, a^2, \dots, a^m$ denote the rows of A. $Ax = \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$$\|Ax\|_\infty = \max_{1 \le i \le m} \left\| \begin{pmatrix} a^1 x \\ a^2 x \\ \vdots \\ a^m x \end{pmatrix} \right\|_\infty = \max_{1 \le i \le m} \left| \sum_{j=1}^{n} a_{ij} x_j \right|$$

$$\le \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij} x_j| \le \underbrace{\max_{1 \le j \le n} |x_j|}_{\|x\|_\infty} \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|$$

$$\therefore \|A\|_\infty = \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \le \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|$$

~~Thu~~ For any fixed vector $u \neq 0$, $\|A\|_\infty \ge \frac{\|Au\|_\infty}{\|u\|_\infty}$, so to finish proof construct u with $\|Au\|_\infty = \max\limits_{1 \le i \le m} \sum\limits_{j=1}^{n} |a_{ij}|$.

Let k be an index with $\sum\limits_{j=1}^{n} |a_{kj}| = \max\limits_{1 \le i \le m} \sum\limits_{j=1}^{n} |a_{ij}|$ (k is ~~row~~ index of row where max sum is attained)

$$\text{Let } u_j = \begin{cases} \dfrac{\bar{a}_{kj}}{|a_{kj}|}, & a_{kj} \neq 0 \\ 1, & a_{kj} = 0. \end{cases}$$

Then $\|u\|_\infty = 1$ and $\sum\limits_{j=1}^{n} a_{kj} u_j = \sum\limits_{j=1}^{n} |a_{kj}|$

$$\therefore \|A\|_\infty \le \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}| \le \|A\|_\infty$$