1 Question 1 - Book 20.1

We will use a proof by induction to show that A has a LU factorization if for each k with $1 \le k \le m$, the upper-left $k \times k$ block is nonsingular. The $k = 1$ portion of the proof is the trivial case as $A_{1:1,1:1} = L_{1:1,1:1}U_{1:1,1:1}$. We assume the following is true: $A_{1:k,1:k} = L_{1:k,1:k}U_{1:k,1:k}$ for $k \le m$.

We want to prove the case $k = m + 1$. For this, we see that

$$A_{1:m+1,1:m+1} = \begin{pmatrix} L_{1:m,1:m} & 0 \\ x_m & 1 \end{pmatrix} \begin{pmatrix} U_{1:m,1:m} & y_m \\ 0 & u_{m+1} \end{pmatrix} \tag{1}$$

We allude to ideas illustrated in part b of question 2 to see that the x_m,y_m, and u_{m+1} is shorthand notation for $x_m = (a_{m+1,1} \cdots a_{m+1,m})U^{-1}_{1:m,1:m}$,

$$y_m = L^{-1}_{1:m,1:m} \begin{pmatrix} a_{1,m+1} \\ \vdots \\ a_{m,m+1} \end{pmatrix} \tag{2}$$

and $u_{m+1} = -x_m y_m$. Since the $det(A_{1:m+1,1:m+1}) = det(U_{1:m,1:m})u_{m+1} \neq 0$, $u_{m+1} \neq 0$ and the LU decomposition is unique.

2 Question 2 - Book 20.3

2.1 part a

So if i understand this question correct, we can simply multiply the LHS out, block by block. So the 1,1 block on the right hand side should be equivalent to IA_{11}. The 1,2 block on the RHS should be equal to IA_{12}. The 2,1 block will be equivalent to $-A_{21}A_{11}^{-1}A_{11} + IA_{21} = 0$, since $A_{11}^{-1}A_{11} = I$. And finally the 2,2 block with be equal to $-A_{21}A_{11}^{-1}A_{12} + IA_{22}$.

2.2 part b

After n steps of Gaussian elimination, A has been factorized such that

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & I \end{pmatrix} \begin{pmatrix} U_{11} & U_{21} \\ 0 & U_{22} \end{pmatrix} \tag{3}$$

If we expand this out, we get a linear system of equations: $L_{11}U_{11} = A_{11}$, $L_{11}U_{12} = A_{12}$, $L_{21}U_{11} = A_{21}$, and $L_{21}U_{12} + U_{22} = A_{22}$. Solving the third equation for L_{21}, we see that $L_{21} = A_{21}U_{11}^{-1}$, and solving the second equation we see that $U_{12} = L_{11}^{-1}A_{12}$. Using this information to solve the fourth equation we see that:

$$U_{22} = A_{22} - L_{21}U_{12} = A_{22} - A_{21}U_{11}^{-1}L_{11}^{-1}A_{12} = A_{22} - A_{21}A_{11}^{-1}A_{21} \tag{4}$$

where we have used the fact that $A_{11}^{-1} = U_{11}^{-1}L_{11}^{-1}$

3 Question 3 - Book 21.6

We know that Gaussian elimination has the following effect on matrix A:

$$A = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \to \begin{pmatrix} a_{11} & A_{12} \\ 0 & A_{22} - \frac{A_{21}}{a_{11}}A_{21} \end{pmatrix} \tag{5}$$

In order to avoid row swapping, we need to show that elimination creates submatrices that are diagonally dominant. I will try using a proof by induction for this. So when the dimension is $k = 1$ of these submatrices, this is trivial as each block is already inherently diagonal. We assume that the result of elimination yields diagonally dominant submatrices for $k < n$. We then need to show that this is also true for any matrix A of dimension n. We see that

$$\sum_{j\neq k} |\big(A_{22} - \frac{A_{21}}{a_{11}} A_{21}\big)_{jk}| \leq \sum_{j\neq k} |\big(A_{22}\big)_{jk} + \sum_{j\neq k} |\frac{1}{a_{11}}(A_{21})_j (A_{12})_k| \tag{6}$$

Furthermore, we note that we have assumed already the diagonal elements of A are dominant. We use this idea to write

$$\sum_{j\neq k} |\big(A_{22}\big)_{jk}| < |(A_{22})_{kk}| - |(A_{12})_k| \tag{7}$$

and

$$\sum_{j\neq k} |(A_{21})_j| < |a_{11}| - |(A_{21})_k| \tag{8}$$

Inserting these relations into our original equation we see that

$$\sum_{j\neq k} |\big(A_{22} - \frac{A_{21}}{a_{11}} A_{21}\big)_{jk}| < |(A_{22})_{kk}| - |(A_{12})_k| + \frac{|(A_{12})_k|}{|a_{11}|}\Bigg(|a_{11} - |(A_{21})_k|\Bigg) \leq |(A_{22})_{kk} - \frac{(A_{21})_k (A_{12})_k}{a_{11}}| \tag{9}$$

where we finally arrive at the desired result that $\sum_{j\neq k} |\big(A_{22} - \frac{A_{21}}{a_{11}} A_{21}\big)_{jk}| \leq |\Bigg((A_{22}) - \frac{(A_{21})(A_{12})}{a_{11}}\Bigg)_{kk}|$

Goal: Given $A \in \mathbb{R}^{n\times n}$, guess vector $v_1 \in \mathbb{R}^n$ and number of iterations m, find a tridiagonal $T \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times m}$ such that $T = V^* AV$. Diagonalize T (hopefully of smaller dimension than A) to approximate an extremal eigenvalue of A.

$$\omega_1' = Av_1 \tag{10}$$

$$\alpha_1 = \omega_1'^* v_1 \tag{11}$$

$$\omega_1 = \omega_1' - \alpha_1 v_1 \tag{12}$$

$$\beta_j = ||\omega_{j-1}|| \tag{13}$$

$$\beta_j \neq 0 \rightarrow v_1 = \frac{\omega_{j-1}}{\beta_j} \tag{14}$$

$$\omega_j' = Av_j \tag{15}$$

$$\alpha_j = \omega_j'^* v_j \tag{16}$$

$$\omega_j = \omega_j' - \alpha_j v_j - \beta_j v_{j-1} \tag{17}$$

$$V = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \end{pmatrix} \tag{18}$$

$$T = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ \beta_2 & \alpha_2 & \beta_3 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \\ \\ \end{pmatrix} \tag{19}$$

Proof: Suppose A is Hermitian, λ is an extremal eigenvalue of A we are searching for, a factorization $V^*AV = T$ has been found, and $Tx = \lambda x$. Given the corresponding eigenvector of A, $y = Vx$,

$$Ay = (VTV^*)y = (VTV^*)(Vx) = V(Tx) = \lambda(Vx) = \lambda y \tag{20}$$

So by diagonalizing T, we effectively find the eigenvalues of A.

Application: Use Rayleigh quotient and $y \neq 0$ to generate v_k such that

$$r(y) = \frac{y^T A y}{y^T y} \to m_k = \min_{y \neq 0} \frac{x^T (V_k^T A V_k) x}{x^T x} = \min_{||x||_2 = 1} r(V_k x) \geq \lambda_n(A) \tag{21}$$

where m_k is an increasingly better approximation to λ upon successive iteration. Equating columns k of $AQ = QT$ and using a little algebra , we see that for iteration $k = 1, \cdots, n-1$

$$A v_k = \beta_{k-1} v_{k-1} + \alpha_k v_k + \beta_k v_{k+1} \tag{22}$$

Projecting this on to v_k and using the fact that the columns of V are orthonormal shows that $\alpha_k = v_k^T A v_k$, $\beta_k = v_{k+1}^T A v_k$ for scalars α, β.

Implementation: If choose a random starting $r_0 = v_0$, we can define a vector $r_k = (A - \alpha_k I)v_k - \beta_{k-1}v_{k-1}$ where $v_{k+1} = \beta_k^{-1} r_k$ and $\beta_k = ||r_k||_2$. We iterate this up to n times, or until $r_k = 0$. This implies that our signal for convergence is when $AV_k - V_k T_k = r_k e_k^T$ or $\beta_k = 0$. In the limit $k \to \infty$, $\alpha_k = v_k^T A^k v_k \approx \lambda$.