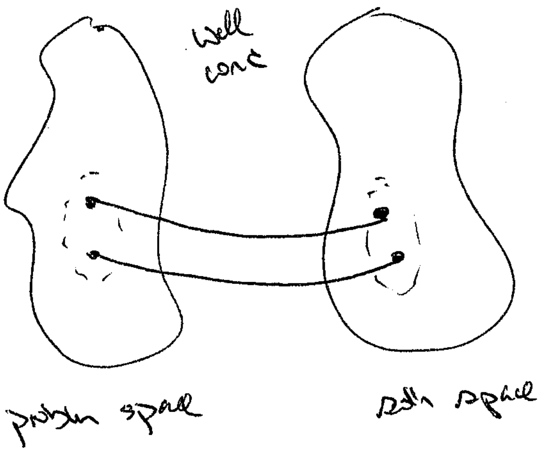
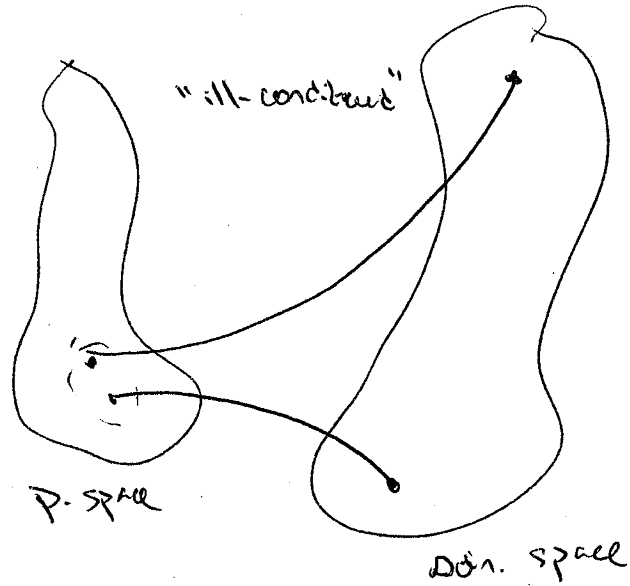
MAD 6406 F20 Lec 19 10/14/20

- Conditioning + stability
- sensitivity to perturbed data
- Ex: scalar addition.

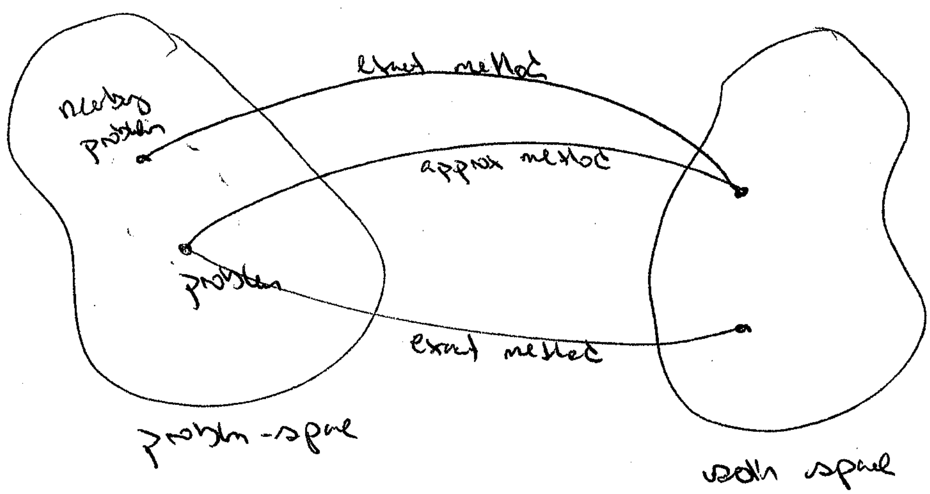
# Part III: Conditioning + Stability:

A **problem** is well-conditioned if small changes in the data lead to small changes in the soln




An **algorithm** or method is called stable if it always produces a soln to a nearby problem.




If a problem is ill-conditioned, stability of the algorithm is ~~not~~ ~~very~~ less useful.

Problem conditioning is essential to numerical methods b/c we work with rounded data.

If an algorithm is applied to a problem with data $d$ to get a sol'n $s$, then $s = s(d)$.

We are interested in the relative error $\frac{\|s(d) - s(\hat{d})\|}{\|s\|}$ where $\hat{d}$ is rounded (or otherwise polluted) data.

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Ex 1: scalar addition

- data: $a, b \in \mathbb{R}$
- solution: $s = a + b$
- perturbed data: $\tilde{a} = a + \Delta a$, $\tilde{b} = b + \Delta b$
- perturbed soln: $\tilde{s} = \tilde{a} + \tilde{b}$

relative error: $\frac{|\tilde{s} - s|}{|s|} = \frac{|\tilde{a} + \tilde{b} - (a+b)|}{|s|} = \frac{|\Delta a + \Delta b|}{|a+b|} \le \frac{|\Delta a| + |\Delta b|}{|a+b|}$

$$= \frac{1}{|a+b|}\left(|a|\frac{|\Delta a|}{|a|} + |b|\frac{|\Delta b|}{|b|}\right)$$

$$\Rightarrow \underbrace{\frac{|\tilde{s} - s|}{|s|}}_{\text{rel. pert. in soln}} \le \underbrace{\frac{|a| + |b|}{|a+b|}}_{\text{amplification}} \times \underbrace{\max\left\{\frac{|\Delta a|}{|a|}, \frac{|\Delta b|}{|b|}\right\}}_{\text{relative pert. in data.}}$$

$c = \frac{|a| + |b|}{|a+b|}$ magnifies the perturbation to the data.

if $\text{sign}(a) = \text{sign}(b)$, $c = 1$

if $\text{sign}(a) = -\text{sign}(b)$, $c > 1$

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if $|a+b| \ll |a| + |b|$ (cancellation error), $c \gg 1$: $c$ "blows up" perturbation of data.

$\Rightarrow$ subtraction with cancellation is ill-conditioned!

More precise: <u>(back notation)</u>

let $\delta(x)$ be a small perturbation ~~..~~ of $x$

$$\delta(f) = f(x+\delta x) - f(x) \qquad (\text{"}f\text{" is soln, "}x\text{" is data})$$

The <u>absolute</u> condition number $\hat{\kappa} = \lim_{\delta \to 0} \sup_{\|\delta x\| \le \delta} \frac{\|\delta f\|}{\|\delta x\|}$

It <u>looks</u> like a ~~condition number~~ derivative.

If ~~..~~ $J(x)$ is the Jacobian of $f$ at $x$: $J_{ij} = \frac{\partial f_i}{\partial x_j}$

then $\delta f \approx J(x)\delta x$ ~~..~~ and $\hat{\kappa} = \|J(x)\|$.

Usually we are concerned with the <u>relative</u> condition number

$$\kappa = \kappa(x) = \lim_{\delta \to 0} \sup_{\|\delta x\| \le \delta} \left( \frac{\|\delta f\|}{\|f(x)\|} \Big/ \frac{\|\delta x\|}{\|x\|} \right)$$

$$\Rightarrow \frac{\|\delta f\|}{\|f\|} \le \kappa \frac{\|\delta x\|}{\|x\|}$$

change in soln

magnified at most by

change in data