MAD 6406 f20 lec1 ~~[illegible]~~ 08/31/20

- matrix notation
- inner products
- outer products

Notation: $A \in \mathbb{C}^{m \times n}$ or $A \in \mathbb{C}^{m,n}$: A is a matrix with

$\begin{cases} m \text{ rows} \\ n \text{ columns.} \end{cases}$

~~Also~~ Each entry a_{ij} is a complex number.

$$A = (a_{ij}) \qquad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix}$$

Complex number $z \in \mathbb{C}$: $z = a + ib$, $a, b \in \mathbb{R}$

Complex conjugate : $\bar{z} = a - ib$

Usually, vectors will be column vectors: $x \in \mathbb{C}^n$: $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, with

transpose: $x^T = (x_1 \ x_2 \ \cdots \ x_n)$ ← row vectors

conjugate - transpose: $x^* = x^H = (\bar{x}_1 \ \bar{x}_2 \ \cdots \ \bar{x}_n)$

matlab: $x' = x^*$

Column-view of matrix multiplication:

For $A \in \mathbb{C}^{m\times n}$, $x \in \mathbb{C}^n$, $Ax = b \in \mathbb{C}^m$

$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$, $a_j \in \mathbb{C}^m$, $j = 1, \ldots, n$, the columns of A.

Then: $Ax = \begin{pmatrix} a_1 \end{pmatrix} x_1 + \begin{pmatrix} a_2 \end{pmatrix} x_2 + \cdots + \begin{pmatrix} a_n \end{pmatrix} x_n = b$

$\Rightarrow$ b is a "linear combination" of the columns of A.

More general: $A \in \mathbb{C}^{m\times \ell}$, $B \in \mathbb{C}^{\ell\times n}$, then $AB = C \in \mathbb{C}^{m\times n}$

$$\underbrace{\begin{pmatrix} a_1 & \cdots & a_\ell \end{pmatrix}^{\ell}}_{m} \underbrace{\begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix}^{n}}_{\ell} = \underbrace{\begin{pmatrix} (Ab_1) & (Ab_2) & \cdots & (Ab_n) \end{pmatrix}^{n}}_{m},$$ where

$a_j \in \mathbb{C}^m$ are the cols of A, $j = 1, \ldots, \ell$

$b_j \in \mathbb{C}^\ell$ are the cols of B, $j = 1, \ldots, n$

$c_j = (Ab_j)$ are the cols of C, $j = 1, \ldots, n$.

Inner-products + orthogonality (super-important!)

Let $x, y \in X$, where X is a vector-space over $\mathbb{C}$. ~~Then $(\circ,\circ)$ is an inner product if it sats~~

An inner-product $(\circ,\circ)$ $X \times X \to \mathbb{C}$ satisfies the following properties:

(1) $(x, y) = \overline{(y, x)}$

(2) $(\alpha x, y) = \bar{\alpha}(x, y)$, $(x, \beta y) = \beta(x, y)$ for $\alpha, \beta \in \mathbb{C}$ ("sesquibilinear")

(3) $(x + y, z) = (x, z) + (y, z)$

(4) $(x, x) \geq 0$ and $(x, x) = 0 \Rightarrow x = 0$.

The inner-product $(,)$ induces a norm $\|\cdot\|$ by $\|x\|^2 = (x, x)$.

Cauchy-Schwartz ineq: $|(x,y)| \le \|x\| \cdot \|y\|$

Direction cosine: $\cos(x,y) = \dfrac{(x,y)}{\|x\| \cdot \|y\|}$.

Orthogonality: if $(x,y)=0$, we say x is orthogonal to y.

$\Rightarrow x \perp y$

If $x \perp y$ for all $y \in V$, we say $x \perp V$.

Back to $\mathbb{C}^n$: ~~that~~ An inner-product in $\mathbb{C}^n$ is given by

$$(x,y) = x^* y = \sum_{i=1}^{m} \bar{x}_i y_i = \underset{1}{\left(\bar{x}_1 \;\; \bar{x}_2 \cdots \bar{x}_m\right)^{n}} \underset{n}{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}^{1}} = \underset{1}{(\;\;)^{1}} \in \mathbb{C}.$$

For $x,y \in \mathbb{R}^n$ $x^T y$ is an inner-product.

Outer-product: xy^*

$$\underset{m}{\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_m \end{pmatrix}^{1}} \left(\bar{y}_1 \cdots \bar{y}_m\right)^{n} = \begin{pmatrix} x_1\bar{y}_1 & \cdots & x_1\bar{y}_m \\ x_2\bar{y}_1 & & \vdots \\ \vdots & & \vdots \\ x_m\bar{y}_1 & & x_m\bar{y}_m \end{pmatrix} = xy^* = \begin{pmatrix} x\bar{y}_1 & x\bar{y}_2 & \cdots & x\bar{y}_m \end{pmatrix}$$

The outer-product is a matrix!

It is a rank-1 matrix because every column is a multiple of x

(1 LI col).

Range, Nullspace + Rank. Let $A \in \mathbb{C}^{m \times n}$, then

$A: \mathbb{C}^n \to \mathbb{C}^m$

$$\underset{m}{\begin{pmatrix} & A & \end{pmatrix}^{n}} \underset{n}{\underbrace{(x)^{1}}_{x \in \mathbb{C}^n}} = \underset{m}{\underbrace{(b)^{1}}_{b \in \mathbb{C}^m}}$$