1 Question 1

The Frobenius norm is defined as

$$||A||_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij}|^2 \right)^{\frac{1}{2}} \tag{1}$$

Furthermore, from the definition of outer product, we know some matrix $C = uv^*$ for $u \in \mathbb{C}^m, v \in \mathbb{C}^n$, and $C \in \mathbb{C}^{m \times n}$ whose elements $c_{ij} = u_i v_j$. Inserting this into the above definition yields

$$||C||_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |c_{ij}|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |u_i|^2 |v_j|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{m} |u_i|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} |v_j|^2 \right)^{\frac{1}{2}} = ||u||_F ||v||_F \tag{2}$$

Ergo, we have shown that $||C||_F = ||u||_F ||v||_F$

2 Question 2

2.1 part a

To show that the Col(A) $\perp$ Nul(A^*) for some $A \in \mathbb{C}^{m \times n}$, we first define $y \in col(A)$. This implies that $Ax = y$ for some $x \in \mathbb{C}^n$. Furthermore, we note that there exists some $v \in nul(A^*)$, implying that $A^*v = 0$. To be perpendicular, we show in the following expression that the inner product of (y, v) is zero.

$$y^*v = (Ax)^*v = x^*A^*v = 0 \tag{3}$$

where we note from the above assertions that $A^*v = 0$. Therefore, we have proven that the $Nul(A^*) \perp col(A)$.

2.2 part b

We wish to show that for any $v \in \mathbb{C}^m$, $v \perp Col(A) \implies v^*(Ax) = 0$ for some $x \in \mathbb{C}^n$. Taking the adjoint of this expression yields

$$0^* = 0 = \Big(v^*(Ax) \Big)^* \rightarrow x^*(A^*v) = 0 \tag{4}$$

Thus we note that $\forall x \in \mathbb{C}^n$, the above implies that $A^*v = 0 \rightarrow v \in Nul(A^*)$. In other words, we have shown that v being orthogonal to Col(A) means that v must be an element of the Nul(A^*).

2.3 part c

Consider a matrix $X \in \mathbb{C}^{m \times m}$. Suppose the columns of this matrix serve as a basis for $Col(A)$. By the rank-nullity theorem, we know the $dim(Nul(A^*)) = m - r$. So we let the columns $x_{r+1}, \cdots, x_m$ be a basis for $Nul(A^*)$. Then

$$X = \begin{pmatrix} x_1 & \cdots & x_r & x_{r+1} & \cdots & x_m \end{pmatrix} \tag{5}$$

where we note that X is invertible and the columns of X are linearly independent vectors in $\mathbb{C}^m$. Therefore, the problem has a unique solution for any $v \in \mathbb{C}^m$

$$Xv = v_R + v_N = \sum_{x \in Col(A)}^{r} x_i v_i + \sum_{x \in Nul(A^*)}^{m} x_i v_i \tag{6}$$

3 Question 3

3.1 part a

We will first show that $\text{Col}(A^*) = \text{Span}\{\ v_1, \cdots, v_r\}$. Suppose $A^* \in \mathbb{C}^{n \times m}$ and $(A)^* = (U\Sigma V^*)^* = V\Sigma^* U^* = A^*$. This looks like

$$A^* = \left(\begin{array}{c|c} V_1 & V_2 \end{array}\right) \left(\begin{array}{ccc|c} \sigma_1^* & & & \\ & \ddots & & \\ & & \sigma_r^* & \\ \hline & & & \end{array}\right) \left(\begin{array}{c} U_1^* \\ \hline U_2^* \end{array}\right) \tag{7}$$

where $V \in \mathbb{C}^{n \times n}, \Sigma^* \in \mathbb{C}^{n \times m}$ and $U^* \in \mathbb{C}^{m \times m}$ and the vertical lines denote the boundary for the index corresponding to the rank of A. We note that the columns of U are a basis for $\mathbb{C}^m$, so for some $y \in \mathbb{C}^m$, $x = Uy$. So

$$b = A^*x = V\Sigma^*U^*x = V\Sigma^*U^*Uy = V\Sigma^*y = V \begin{pmatrix} \sigma_1^* y_1 \\ \vdots \\ \sigma_r^* y_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \sum_{j=1}^{r} v_j(\sigma_j^* y_j) \tag{8}$$

Therefore, $\text{Col}(A^*) \subseteq \text{span}\{v_1, \cdots, v_m\}$. Furthermore, for each $j = 1, \cdots, r$, $A^*u_j = V\Sigma^*U^*u_j = v_j\sigma_j^*$, thereby showing that $\text{span}\{v_1, \cdots, v_r\} \in \text{Col}(A^*)$. In conclusion this proves that $\text{Col}(A^*) = \text{Span}\{\ v_1, \cdots, v_r\}$.

3.2 part b

For the next part, we will show that $\text{Nul}(A^*) = \text{span}\{u_{r+1}, \cdots, u_m\ \}$. We build off the foundations laid in the prior subsection and we note that $\text{Nul}(A^*) = \{\ A^*x = 0\}$. For $x = Uy$, we see that

$$A^*x = V\Sigma^*U^*(Uy) = V\Sigma^*y = 0 = \sum_{j=1}^{r} v_j(\sigma_j^* y_j) + \sum_{j=r+1}^{m} 0 \tag{9}$$

where we note that because the columns of V_1 are linearly independent, the only way the equality is held is if $y_1, \cdots, y_r = 0$. Consequently $A^*x = 0 \to x \in \text{span}\{u_{r+1}, \cdots, u_m\}$ alluding to $\text{Nul}(A^*) \subseteq \text{span}\{u_{r+1}, \cdots, u_m\}$. Furthermore, since $A^*u_j = V\Sigma^*U^*u_j = 0$ for $j = r+1, \cdots, m$, we know $\text{Nul}(A^*)$=$\text{span}\{u_{r+1}, \cdots, u_m\}$

4 Question 4

4.1 part a

See attached jupyter-notebook script.

4.2 part b

My custom build function that computes the induced matrix 3 norm completes in 0.0006237030029296875 seconds. Perhaps a more interesting comparison is Numpy's in house matrix infinity norm and my custom built infinity norm: the former runs in 0.00011086463928222656 seconds while the ladder runs in 0.00011014938354492188 seconds. These are obviously very comparable. I suspect the induced matrix 3 norm runs comparatively slower because it is expensive to compute powers.

4.3 part c

Suppose a matrix $A \in \mathbb{C}^{m \times n}$ and some vector $x \in \mathbb{C}^n$. We want to define α in the relation $||A||_\infty = \alpha||A||_3$. To begin, we will start with the definitions of the relevant vector p-norms:

$$||x||_3 = \left(\sum_{i=1}^{m} |x_i|^3\right)^{\frac{1}{3}} \tag{10}$$

and

$$||x||_\infty = \max_{1 \leq i \leq m} |x_i| = x_{max} \tag{11}$$

Expanding the vector 3-norm and comparing with the vector infinity-norm, we immediately see that

$$x_{max} \leq \left(|x_1|^3 + \cdots + |x_{max}|^3 + \cdots + |x_m|^3\right)^{\frac{1}{3}} \tag{12}$$

which implies that $||x||_\infty \leq ||x||_3$, where we note pure equality is obtain if dealing with the Euclidean unit vectors.

Additionally, we note that

$$\left(\sum_{i=1}^{m} |x_i|^3\right)^{\frac{1}{3}} \leq \left(\sum_{i=1}^{m} |x_{max}|^3\right)^{\frac{1}{3}} \rightarrow ||x||_3 \leq m^{\frac{1}{3}}||x||_\infty \tag{13}$$

We then use the above two relations in addition to the definition of an induced matrix norm to compare $||A||_\infty$ and $||A||_3$:

$$||A||_\infty = \max_{||x||=1} \frac{||Ax||_\infty}{||x||_\infty} \leq \frac{||Ax||_3}{||x||_\infty} \leq (m)^{\frac{1}{3}} \frac{||Ax||_3}{||x||_3} \tag{14}$$

Thus, we see that $||A||_\infty \leq (m)^{\frac{1}{3}}||A||_3$ where $\alpha = (m)^{\frac{1}{3}}$ as stated in the problem.

4.4 part d

As my approximate revolves around using Euclidean unit vectors for x, the resulting induced matrix 3 norm will be a poor approximation to the true matrix 3 norm. In fact, this is such a poor approximation that none of the 10 runs verify the inequality proven in part c.