MAD 6406 F20 lec 37 12/02/20

- QR iteration with shifts
  - shifting + deflation

QR with shift μ

$A_1 = Q_0^* A Q_0 \leftarrow A_1$ upper Hessenberg.

for $k = 1, 2, \ldots$

factor $A_k - \mu I = Q_k R_k$

set $A_{k+1} = R_k Q_k + \mu I$

end

$$\Rightarrow R_k = Q_k^* A_k - \mu Q_k^*$$

Thm: ~~[illegible]~~

$$A_{k+1} = R_k Q_k + \mu I$$

$$= Q_k^* A_k Q_k - \cancel{\mu Q_k^* Q_k} + \cancel{\mu I}$$

$$= (Q_k^* \cdots Q_1^*) Q_0^* A Q_0 Q_1 \cdots Q_k$$

and A_{k+1} is still unitarily similar to A.

Convergence: If $|\lambda_1 - \mu| \geq |\lambda_2 - \mu| \geq \cdots \geq |\lambda_n - \mu|$

then the p^{th} subdiagonal entry in A_k converges to zero at rate

$$\left| \frac{\lambda_{p+1} - \mu}{\lambda_p - \mu} \right|^k.$$

Usual shift: $\mu = A_k(n,n)$ (approx of n^{th} eigenvalue from A_k)

To understand <u>shifts</u> we need to understand <u>deflation</u>.
(Golub + van Loan, Matrix Computations).

We start w/ $A_1 = Q_0^T A Q_0$ where A_1 is in upper-Hess form.

We assume A_1 is "unreduced." Otherwise, at some stage we have

$$A_k = \begin{array}{cc} \left(\begin{array}{c|c} H_{11} & H_{12} \\ \hline 0 & H_{22} \end{array}\right) & \begin{array}{l} p \\ n-p \end{array} \\ \begin{array}{cc} p \quad & \quad n-p \end{array} & \end{array}$$

, $1 \le p < n$, and the problem <u>decouples</u> into two smaller problems, for H_{11} and H_{22}.

Ex: $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$, $|A - \lambda I| = (3-\lambda)\det\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, so it suffices to seek eigenvalues of $H_{11} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

$a_{32} = 0$

In practice, in the std eigenvalue solver Eispack, decoupling occurs when a subdiagonal entry is below a given tolerance, e.g., for $H = A_k$,

$|h_{p+1,p}| \le c\,\epsilon_m (|h_{pp}| + |h_{p+1,p+1}|)$ for a given constant c.

Plan: consider unreduced Hessenberg matrix ~~A~~ $H = A_1$:
perform shifted QR until $a^{(k)}_{n,n-1} \to 0$, then "reduce" or deflate $H = A^k(1:n-1, 1:n-1)$ and continue...

Thm (7.5.1, G+VL)

Let p be an eigenvalue of an $n \times n$ unreduced upper-Hess matrix H.

If $\bar{H} = RQ + pI$ where $H - pI = QR$ is the QR-factorization of $H - pI$, then $\bar{h}_{n,n-1} = 0$ and $\bar{h}_{nn} = p$.

pf: Since H is unreduced upper-Hess, the 1st $n-1$ cols of $H - pI$ are LI.

$\therefore$ If $QR = H - pI$ is the QR factorization, $r_{ii} \neq 0$, $i = 1, \dots, n-1$.

BUT: if $H - pI$ is singular (which it is, if p is an eval), then

$r_{11} \cdots r_{nn} = 0$ so $r_{nn} = 0$.

Then $\bar{H} = RQ + pI = \begin{pmatrix} \diagdown & | \\ 0 & 0 \end{pmatrix} (q_1 \cdots q_n) + pI \Rightarrow$ n^{th} row of $\bar{H}$ is $(0 \cdots 0\, p)$

$\Rightarrow$ This says if we shift by an exact eigenvalue, deflation occurs in one step

Single-shift strategy: choose $p = h_{nn}$ at each iteration:

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for k=1,2,...
    p=H(n,n)
    H-pI = QR        (QR factorization)
    H = RQ+pI        update.
end
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"If the $n,n-1$ entry converges to zero, it is likely to do so at a quadratic rate"

There are also double-shift strategies to account for conjugate eval pairs.