# 1 Question 1

To be considered a norm, a norm has to satisfy the three conditions given on pg. 17. So, by proving - for example - the expression in question does not satisfy the triangle inequality we will have shown that it does not satisfy all 3 properties and therefore can not be considered a norm. Given the expression

$$f(x) = \big(\sum_{j=1}^{n} |x_j|^p\big)^{\frac{1}{p}} \tag{1}$$

for p on the interval $0 < p < 1$, we will show that the converse of the triangle inequality, namely $||x + y|| > ||x|| + ||y||$, is true, and therefore the expression does not constitute a norm.

We start by assuming the there exists some $x, y \in \mathbb{C}^n$ and that $p = \frac{a}{b}$ such that the interval we are concerned with is still $0 < \frac{a}{b} < 1$.We further assume that $x = e_i$ and $y = e_j$ - a particular column of the identity matrix - such that $i \neq j$. So the left-hand side of the triangle inequality will yield

$$||x + y|| = \left(\sum_{j=1}^{n} |x_j + y_j|^{\frac{a}{b}}\right)^{\frac{b}{a}} \tag{2}$$

Because $x$ and $y$ are orthogonal based on the above assumption this can be simplified such that

$$||x + y|| = \left(1^{\frac{a}{b}} + 1^{\frac{a}{b}}\right)^{\frac{b}{a}} = 2^{\frac{b}{a}} \tag{3}$$

where we note that the ratio $\frac{b}{a} > 1$, and consequently the lefthand side of the triangle inequality for this example is $> 2$.

The right hand side reduces to the following expression

$$||x|| + ||y|| = (1) + (1) = 2 \tag{4}$$

Therefore, we have proven that on the interval $0 < p < 1$ the triangle inequality does not hold because $||x+y|| > ||x||+||y||$ and the original expression does not satisfy all the necessary requirements to be considered a norm.

# 2 Question 2 - Book 3.1

We seek to prove that $||x||_W = ||Wx||$ is a vector norm for any general $W \in \mathbb{C}^{m \times m}$. To be considered a norm, all 3 conditions must be satisfied.

We will show that $||Wx|| \geq 0$ and $||Wx|| = 0$ only if $x = 0$ as the first condition. By assuming that $W$ is nonsingular, we note that this means the columns of $W$ are linearly independent. By definition, this means that no linear combination of the columns of $W$ can equal 0 except $x = 0$. In other words, nullspace of $W$ only has the zero vector in it. Furthemore, we know $||Wx|| \geq 0$ must be true and is a trivial statement that arises from the absolute value in the definition of a norm.

We next show that $||\alpha Wx|| = |\alpha|||Wx||$ as the second condition. We note

$$||\alpha Wx|| = \left(\sum_{i=1}^{m} |\alpha w_i x_i|^p\right)^{\frac{1}{p}} = \left(|\alpha|^p \sum_{i=1}^{m} |w_i x_i|^p\right)^{\frac{1}{p}} = |\alpha| \left(\sum_{i=1}^{m} |w_i x_i|^p\right)^{\frac{1}{p}} \tag{5}$$

where we note the the final term $\left(\sum_{i=1}^{m} |w_i x_i|^p\right)^{\frac{1}{p}} = ||Wx||$. Therefore we have shown that $||\alpha Wx|| = |\alpha|||Wx||$.

The final condition we will show is the triangle inequality is satisfied, namely $||W(x+y)|| \leq ||Wx|| + ||Wy||$. We let $x = e_i$ and $y = e_j$ where $i = j$ ( ie the same column of the identity matrix ). We note $x, y \in \mathbb{C}^m$. So, using the definition of a norm, the expression of the left hand side of the inequality looks like

$$||Wx + Wy|| = \left((w_i)^p + (w_i)^p\right)^{\frac{1}{p}} = (2w_i^p)^{\frac{1}{p}} = w_i 2^{\frac{1}{p}} \tag{6}$$

where $w_i, w_j$ are columns $i, j$ of $W$, respectively. The right hand side of the expression looks like

$$||Wx|| + ||Wy|| = w_i + w_i = 2w_i \tag{7}$$

So putting this information together, we see that

$$2^{\frac{1}{p}} w_i \leq 2w_i \tag{8}$$

For the interval $1 \leq p < \infty$, we note that the equality holds for $p = 1$ but that $2^{\frac{1}{p}} < 2$ for $p > 1$. Thus the triangle identity holds, and the above expression represents a vector norm for a general $W$.

# 3 Question 3 - Book 3.2

We wish to show that the maximum eigenvalue of matrix $A$ is less than or equal to the norm of $A$, ie $max(\lambda) \leq ||A||$. We know the right hand side of this expression can be defined as

$$||A|| = \max_{y \neq 0} \frac{||Ay||}{||y||} = \max_{||y||=1} ||Ay|| \geq ||Ax|| \tag{9}$$

We know the eigenvalue equation $Ax = \lambda x$. Thus, plugging this into the expression on right hand side of the above expression describing a vector norm on $x \in \mathbb{C}^m$ yields

$$||Ax|| = ||\lambda x|| = |\lambda| ||x|| \rightarrow |\lambda| = \frac{||Ax||}{||x||} \tag{10}$$

Supposing $||x|| = 1$, taking the $max$ on both sides, and inserting this back into our starting expression yields

$$|\lambda_{max}| \leq ||A|| \tag{11}$$

where $\lambda_{max}$ is the largest eigenvalue of $A$.

To see why the strict inequality is true, we can use a simple matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{12}$$

which has a maximal eigenvalue of 0, but the operator norm is 1.

# 4 Question 4 - Book 3.3

## 4.1 part a

We note that from the definitions

$$||x||_\infty = max_{1 \leq i \leq m} |x_i| = x_{max} \tag{13}$$

where $x_{max}$ is the maximum element of the vector $x$ and

$$||x||_2 = \Big(\sum_i^m |x_i|^2\Big)^{\frac{1}{2}} = \big(|x_1|^2 + |x_2|^2 + \cdots + |x_{max}|^2 + \cdots + |x_m|^2\big)^{\frac{1}{2}} \tag{14}$$

so clearly we can rearrange the inequality such that

$$x_{max}^2 \leq \big(|x_1|^2 + |x_2|^2 + \cdots + |x_{max}^2 + \cdots |x_m|^2| + |x_m|^2\big) \to 0 \leq \big(|x_1|^2 + |x_2|^2 + \cdots + |x_m|^2 \tag{15}$$

This is expression is always - and by extension $||x||_\infty \leq ||x||_2$ - true as a result of the absolute value and subsequent squaring. An example of a vector where a strict equality is held would be if $x = \alpha e_i$ where $e_i$ is a column of the identiy matrix and $\alpha \in \mathbb{C}$.

## 4.2 part b

We know the following expression must be true

$$\Big(\sum_i^m |x_i|^2\Big)^{\frac{1}{2}} \leq \Big(\sum_i^m |x_{max}|^2\Big)^{\frac{1}{2}} = \big(m|x_{max}|^2\big)^{\frac{1}{2}} = \sqrt{m}|x_{max}| \tag{16}$$

Ergo from the definitions of the respective vector norms in the part a, we see that $||x||_2 \leq \sqrt{m}||x||_\infty$. An example of a vector that achieves strict equality is

$$x = \alpha \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{17}$$

where every entry of the vector is a 1 and $\alpha \in \mathbb{C}$.

## 4.3 part c

We know that the general form of the matrix p norm is

$$||A||_p = max_{x\neq 0}\frac{||Ax||_p}{||x||_p} \tag{18}$$

From parts a and b, we have shown that $||x||_\infty \leq ||x||_2 \leq \sqrt{m}||x||_\infty$. So starting with the definition of the matrix infinity norm we see

$$||A||_\infty = \max_{x\neq 0}\frac{||Ax||_\infty}{||x||_\infty} \leq \max_{x\neq 0}\frac{||Ax||_2}{\frac{1}{\sqrt{n}}||x||_2} \tag{19}$$

Taking the scalar out, and inserting the definition for the matrix 2 norm yields $||A||_\infty \leq \sqrt{n}||A||_2$

## 4.4 part d

We know that the general form of the matrix p norm is

$$||A||_p = max_{x\neq 0}\frac{||Ax||_p}{||x||_p} \tag{20}$$

From parts a and b, we have shown that $||x||_\infty \leq ||x||_2 \leq \sqrt{m}||x||_\infty$. So starting with the definition of the matrix 2 norm we see

$$||A||_2 = max_{x\neq 0}\frac{||Ax||_2}{||x||_2} \leq max_{x\neq 0}\frac{\sqrt{m}||Ax||_\infty}{||x||_2} \leq max_{x\neq 0}\frac{\sqrt{m}||Ax||_\infty}{||x||_\infty} \tag{21}$$

We can pull the scalar $\sqrt{m}$ out of the expression and insert the definition for the infinity norm on the right hand side, showing that

$$||A||_2 \leq \sqrt{m}||A||_\infty \tag{22}$$

# 5 Question 5

From the expansion of $Ax = \sum_{j=1}^{n} a_{ij}x_j = \sum_{j=1}^{n} x_j a_j$ (for $A \in \mathbb{C}^{m\times n}$) and the induced matrix norm, $||A||_1 = ||Ax||_1$, we define the expression

$$||Ax||_1 = ||\sum_{j=1}^{n} x_j a_j||_1 \leq \sum_{j=1}^{n} |x_j|||a_j||_1 \leq \max_{1\leq j\leq n} ||a_j||_1 \rightarrow ||A||_1 \leq \max_{1\leq j\leq n} ||a_j||_1 \tag{23}$$

where we have brought $\sum_{j=1}^{n} |x_j| = ||x||_1$ to the left handside. Therefore, we have shown in the above that $||A||_1 \leq \max_{1\leq j\leq n} ||a_j||_1$

The next step is to define any vector $u \neq 0$ such that $||A||_1 \geq \frac{||Au||_1}{||u||_1}$. So we wish to construct $u$ such that $||Au||_1 = \max_{1\leq j\leq n} ||a_j||_1$. The $u$ that does this is the particular column of the identity matrix $e_j$ where the index $j$ corresponds to the maximum column of $A$ that has a maximal sum: $||a_j||_1$. This shows a tight bound and proves that

$$||A||_1 = \max_{1\leq j\leq n} \sum_{i}^{m} |a_{ij}| \tag{24}$$

aka that the 1-norm of a matrix is equivalent to the maximal column sum.