SVD:

- left and right singular vectors
- constructive proof of existence + uniqueness.

~~(A) & (B) on next HW~~

Reduced SVD: for $A \in \mathbb{C}^{m\times n}$, $\operatorname{rank}(A) = r$

$$A = U\Sigma V^* = \left(u_1 \cdots u_r \,\middle|\, u_{r+1} \cdots u_m\right)\left(\begin{array}{c|c} \sigma_1 \ddots \sigma_r & 0 \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c} v_1^* \\ \vdots \\ v_r^* \\ \hline v_{r+1}^* \\ \vdots \\ v_n^* \end{array}\right)$$

$$= \underbrace{\left(u_1 \cdots u_r\right)}_{U_1} \underbrace{\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}}_{\hat{\Sigma}} \underbrace{\begin{pmatrix} v_1^* \\ \vdots \\ v_r^* \end{pmatrix}}_{V_1}$$

Reduced SVD: $A = U_1 \hat{\Sigma} V_1^*$, U_1 is $m \times r$, $\hat{\Sigma}$ is $r\times r$, invertible, V_1^* is $r \times n$.

Cols of U_1 form a basis for $\operatorname{col}(A)$

cols of V_1 " " " " $\operatorname{col}(A^*)$.

Back to the full SVD: Who are the unitary matrices U, V?

$A = U\Sigma V^*$

$$A^*A = V\Sigma^* U^* U \Sigma V^* = V\Sigma^*\Sigma V^* = V\left(\begin{array}{c|c} \sigma_1^2 \ddots \sigma_r^2 & 0 \\ \hline 0 & 0 \end{array}\right)V^*$$

$$A^*AV = V\left(\begin{array}{c|c} \sigma_1^2 \ddots \sigma_r^2 & 0 \\ \hline 0 & 0 \end{array}\right) \Rightarrow \text{columns of } V \text{ are the eigenvectors of } A^*A.$$

$\sigma_1^2, \dots, \sigma_r^2$ (singular values) are the nonzero eigenvalues of A^*A

Recall: A^*A is Hermitian $\Rightarrow$ Normal $\therefore$ has a full set of orthogonal eigenvectors

and A^*A " " $\Rightarrow$ eigenvalues of A^*A are real

AND $A^*Ax = \lambda x \Rightarrow x^*A^*Ax = \lambda x^* x \Rightarrow \lambda = \dfrac{x^*A^*Ax}{x^*x} = \dfrac{(Ax, Ax)}{(x,x)} = \dfrac{\|Ax\|^2}{\|x\|^2}$

eigenvalues of A^*A are non-negative.

$\sigma_1^2, \dots, \sigma_r^2 > 0$

We also have:

$$AA^* = U\Sigma \underbrace{V^*V}_{I}\Sigma^T U^* = U\Sigma\Sigma^T U^* = U \begin{pmatrix} \begin{matrix}\sigma_1^2 & & \\ & \ddots & \\ & & \sigma_r^2\end{matrix} & 0 \\ 0 & 0 \end{pmatrix}_{m\times m} U^*$$

$$AA^*U = U\begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_r^2 \end{pmatrix}$$

$\Rightarrow$ columns of U are the eigenvectors of AA^*

$\sigma_1^2, \dots, \sigma_r^2$ are also the nonzero eigenvalues of AA^*

$\boxed{\text{BUT}}$ ~~eigenvalues~~ eigenvectors, even normalized (length 1) are defined only up to arbitrary rotation:

If $AA^*x = \lambda x$, $AA^*(e^{i\theta}x) = \lambda(e^{i\theta}x)$.

How do the singular vectors relate?

$$Av_j = U\Sigma V^* v_j = U\Sigma e_j = \sigma_j u_j$$

$$\boxed{\Rightarrow u_j = \frac{1}{\sigma_j} Av_j}$$

$$u_j^* A = u_j^* U\Sigma V^* = e_j^T \Sigma V^* = \sigma_j v_j$$

$$\boxed{\Rightarrow v_j = \frac{1}{\sigma_j} u_j^* A}$$

Notice: if $A = U\Sigma V^*$ what is an SVD of $-A$?

$-A = (-U)\Sigma V^* = U\Sigma(-V^*)$

$= (e^{i\theta}U)\Sigma(e^{i(\pi-\theta)}V^*)$

SVD of zA?

$u_1, \dots, u_r$ are called the "left singular vectors"

$u_{r+1}, \dots, u_m$ " " " "left null vectors"

$v_1, \dots, v_r$ " " " "right singular vectors"

$v_{r+1}, \dots, v_n$ " " " "right null vectors"

Existence + uniqueness of the SVD

Th'm: Every matrix $A \in \mathbb{C}^{m \times n}$ has a singular value decomposition.

- The singular values $\{\sigma_j\}$ are uniquely determined.
- If A is square and the σ_j are distinct, the left and right singular vectors $\{u_j\}$ and $\{v_j\}$ are uniquely determined up to rotation (multiplication by $e^{i\theta}$).

Constructive proof:

First, consider $A^* A$. $A^* A$ is Hermitian: $(A^* A)^* = A^* A$.

Any Hermitian matrix is normal and has an o.n. set of eigenvectors: vectors $\{v_1, \dots, v_n\}$ satisfy $v_i^* v_j = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$

So: $A^* A$ has a spectral decomp:

$A^* A = V \Lambda V^*$, V unitary, each column v_j defined up to ~~[illegible]~~ mult. by $e^{i\theta_j}$

Λ diagonal matrix of eigenvalues $\lambda_1, \dots, \lambda_n$.

Set $r_i = A v_i$, $i = 1, \dots, n$. For each nonzero λ_i,

~~A^*~~ $AA^* r_i = A(A^* A v_i) = A \lambda_i v_i = \lambda_i r_i$.

$\therefore$ r_i is an eigenvector of AA^*.

$r_i^* r_j = (Av_i)^* A v_j = v_i^* \underbrace{A^* A v_j}_{\lambda_j v_j} = \lambda_j v_i^* v_j = \begin{cases} \lambda_i, & i = j \\ 0, & i \neq j \end{cases}$

$\therefore$ $\{r_1, \dots, r_n\}$ are orthogonal.

Let $p = \operatorname{rank}(A)$. Then indexing $\lambda_1, \dots, \lambda_p$ as nonzero eigenvalues of $A^* A$, define $u_i = \frac{1}{\sqrt{\lambda_i}} r_i$, $i = 1, \dots, p$. Then $u_i^* u_j = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$

~~AA^*~~ AA^* is Hermitian $\Rightarrow$ has an ⊥n set of eigenvectors.

Starting with $\{u_1, \dots, u_p\}$ ~~to~~ build an orthonormal eigendecomp: ~~for~~ $AA^* = UDU^*$, with D diagonal.

Since $AA^* r_i = \lambda_i r_i$

$AA^* u_i = \lambda_i u_i$, $i = 1, \dots, p$.

And $Av_j = r_j = \sigma_j u_j$, where $\sigma_j = \sqrt{\lambda_j}$, $j = 1, \dots, p$.

In matrix form:

$$\underbrace{AV}_{r_i\text{'s}} = U\Sigma = \left(u_1 \cdots u_p \,\middle|\, u_{p+1} \cdots u_m\right) \left(\begin{array}{c|c} \begin{matrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_p\end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right)$$

Since a permutation matrix P is unitary, we can reorder the diagonal elements of Σ to be in decreasing order.