MAD 6406 f20 lec 35 11/23/20

- Rayleigh quotient iteration
- stopping criteria

Rayleigh quotient iteration

x = approx to eigenvector

$R(x) = x^t A x / x^t x$ = approx to eigenvalue.

Suppose $A^t = A$, ~~[illegible]~~ $x^t x = 1$,

and (λ, v) is an eigenpair that $(R(x), x)$ approximates, with $\|v\| = 1$.

x has a decomposition into col v and null v^t, so we can write $x = cv + w$, $w^t v = 0$.

Then:

$$v^t A w = (A^t v)^t w = (\lambda v)^t w = \lambda v^t w = 0$$

$$w^t A v = \lambda w^t v = 0$$

$$\begin{aligned} R(x) &= (cv + w)^t A (cv + w) \\ &= c^2 v^t A v + w^t A w + c v^t A w + w^t A c v, \\ &= c^2 \lambda v^t v + w^t A w \\ &= c^2 \lambda + w^t A w \end{aligned}$$

~~but $Av = \lambda v$, $v^t A w = (A^t w)^t = w^t A v$, $w^t A v = \lambda w^t v = 0$~~

Notice: $1 = \|x\|^2 = (cv + w)^t (cv + w) = c^2 v^t v + w^t w = c^2 + w^t w \Rightarrow c^2 = 1 - \|w\|^2$

$$\begin{aligned} \Rightarrow R(x) &= (1 - \|w\|^2)\lambda + w^t A w \\ &= \lambda - \lambda \|w\|^2 + w^t A w \\ &= \lambda + w^t (A - \lambda I) w \\ &= \lambda + O(\|w\|^2) \end{aligned}$$

If A is not Hermitian $y \sim$ left eigenvector, $x \sim$ right eigenvector

$$R(x, y) = y^t A x / y^t x = \lambda + O(\|\text{error in } y\|^2 + \|\text{error in } x\|^2)$$

Recall: inverse power method:

$$(A - \sigma I) x_{k+1} = x_k \quad , \sigma \text{ fixed}$$

Accelerate method by updating $\sigma = R(x_k)$ at each iteration.

$$(A - R(x_k) I) x_{k+1} = x_k$$

Rayleigh quotient iteration

Choose x_0 with $\|x_0\| = 1$

set $\lambda_0 = x_0^t A x_0$

for $k = 1, 2, \ldots$

~~solve $(A - \lambda_{k-1} I) x$~~

solve $(A - \lambda_{k-1} I) w = x_{k-1}$ Apply $(A - \lambda_{k-1} I)^{-1}$

$x_k = w / \|w\|$ normalize

$\lambda_k = x_k^t A x_k$ Rayleigh quotient

end

$\Rightarrow$ If $A^* = A$ and λ is "simple" (alg. mult. 1) then the algorithm is cubically convergent if x_0 is sufficiently close to the eigenvalue assoc. with λ.

If $A^* \neq A$, convergence is quadratic.

Cubically convergent: let $\mathcal{E}_\lambda$ = the eigenspace of λ.

$$\operatorname{dist}\{x_{k+1}, \mathcal{E}_\lambda\} \leq c \operatorname{dist}\{x_k, \mathcal{E}_\lambda\}^3$$

Assume $A^* = A$: convergence of eigenvalue:

$$|\lambda_k - \lambda_{k+1}| \leq c \cdot \operatorname{dist}\{x_k, \mathcal{E}_\lambda\}^3$$

For $x_k = \sum_{i=1} c_i v_i = c_1 v_1 + \sum_{i=2}^{n} c_i v_i$, $\operatorname{dist}\{x_k, \mathcal{E}_\lambda\} = \left\| \sum_{i=2}^{n} c_i v_i \right\|$,

where v_1 is eigenvector assoc w/ λ.

Stopping criteria

See: Numerical methods for large eigenvalue problems,
Yousef Saad

Two common stopping criteria are

(1) The difference between successive iterates:

$\|x_{k+1} - x_k\|_2$

~~(ii) $\|Ax_k - R(x_k)\|$ the residual, where $R(x) = x_k^* A x_k$.~~

(2) The residual
$\|Ax_k - \lambda_k x_k\|_2$ where $x_k^* x_k = 1$ and $\lambda_k = x_k^* A x_k$.

Usually the vector 2-norm is used

How good are these stopping criteria? It depends...

Ex: Assume A is Hermitian, let x_k be an approx eigenvector with $\|x_k\| = 1$, and let $\lambda_k = x_k^* A x$.
Assume we know an interval (a, b) that contains λ_k, and precisely one eigenvalue λ of A. Let $r = \|Ax_k - \lambda_k x_k\|_2$

Thm:
$$\frac{-\|r\|_2^2}{\lambda_k - a} \leq \lambda_k - \lambda \leq \frac{\|r\|_2^2}{b - \lambda_k}$$

$\therefore$ as $\|r\| \to 0$, $\lambda_k \to \lambda$.

Convergence criteria where eigenvalues are clustered or of higher mult. are hard problems.