From DRS to ADMM

In this note, we consider the step-by-step derivation of ADMM from Douglas-Rachford Splitting (DRS). This translation is adapted from the book and completes the skipped steps.

https://link.springer.com/content/pdf/10.1007/978-981-16-9840-8_2

1 Problem Setup

Consider using ADMM to solve the following problem,

$$\begin{aligned} &\min_{x,y} f(x) + g(y) \\ &\text{s.t. } Ax + By = b \end{aligned}$$

The ADMM update is,

$$\begin{aligned} x^{k+1} &= \arg\min_x \left\{ f(x) + \langle v^k, Ax \rangle + \frac{\beta}{2}\|Ax + By^k - b\|^2 \right\} \\ y^{k+1} &= \arg\min_y \left\{ g(y) + \langle v^k, By \rangle + \frac{\beta}{2}\|Ax^{k+1} + By - b\|^2 \right\} \\ v^{k+1} &= v^k + \beta(Ax^{k+1} + By^{k+1} - b) \end{aligned}$$

In this note, we derive the ADMM update from DRS. To apply DRS, consider the Fenchel dual problem,

$$\min_\lambda \underbrace{f^*(-A^\top\lambda) + b^\top\lambda}_{\varphi_1(\lambda)} + \underbrace{g^*(-B^\top\lambda)}_{\varphi_2(\lambda)}$$

Applying DRS yields,

$$\begin{aligned} v^k &= \mathrm{prox}_{\beta\varphi_2}(y^k) \\ u^{k+1} &= \mathrm{prox}_{\beta\varphi_1}(2v^k - y^k) \\ y^{k+1} &= y^k + u^{k+1} - v^k \end{aligned}$$

2 Switched DRS

Consider the start of next DRS iteration,

$$v^{k+1} = \mathrm{prox}_{\beta\varphi_2}(y^{k+1}) = \mathrm{prox}_{\beta\varphi_2}(y^k + u^{k+1} - v^k)$$

Then consider the DRS iteration $(u^{k+1}, y^{k+1}, v^{k+1})$ and switch the update of y^{k+1}, v^{k+1}, we derive an equivalent algorithm,

$$\begin{aligned} u^{k+1} &= \mathrm{prox}_{\beta\varphi_1}(2v^k - y^k) \\ v^{k+1} &= \mathrm{prox}_{\beta\varphi_2}(y^k + u^{k+1} - v^k) \\ y^{k+1} &= y^k + u^{k+1} - v^k. \end{aligned}$$

Apply a change of variable, $w^k := v^k - y^k$,

$$\begin{aligned} u^{k+1} &= \mathrm{prox}_{\beta\varphi_1}(v^k + w^k) \\ v^{k+1} &= \mathrm{prox}_{\beta\varphi_2}(u^{k+1} - w^k) \\ w^{k+1} &= w^k + v^{k+1} - u^{k+1} \end{aligned}$$

3 Recovering ADMM

Step 1: optimality condition of u-update.

Consider the optimality condition of $u^{k+1} = \mathrm{prox}_{\beta\varphi_1}(v^k + w^k)$ and by $\varphi_1(\lambda) := f^*(-A^\top\lambda) + b^\top\lambda$,

$$\begin{aligned} 0 &\in \partial\varphi_1(u^{k+1}) + \tfrac{1}{\beta}(u^{k+1} - (v^k + w^k)) \\ &= -A\partial f^*(-A^\top u^{k+1}) + b + \tfrac{1}{\beta}(u^{k+1} - (v^k + w^k)) \end{aligned}$$

Then there exists $x^{k+1} \in \partial f^*(-A^\top u^{k+1})$, (which implies that $-A^\top u^{k+1} \in \partial f(x^{k+1})$), such that

$$\begin{aligned} 0 &= -Ax^{k+1} + b + \tfrac{1}{\beta}(u^{k+1} - (v^k + w^k)) \\ (\Rightarrow)\, u^{k+1} &= v^k + w^k + \beta(Ax^{k+1} - b) \end{aligned} \tag{1}$$

By $0 \in \partial f(x^{k+1}) + A^\top u^{k+1}$ and (1),

$$0 \in \partial f(x^{k+1}) + A^\top(v^k + w^k) + \beta A^\top(Ax^{k+1} - b) \tag{2}$$

Step 2: optimality condition of v-update.

Consider the optimality condition of $v^{k+1} = \mathrm{prox}_{\beta\varphi_2}(u^{k+1} - w^k)$ and by $\varphi_2(\lambda) := g^*(-B^\top\lambda)$,

$$\begin{aligned} 0 &\in \partial\varphi_2(v^{k+1}) + \tfrac{1}{\beta}(v^{k+1} - (u^{k+1} - w^k)) \\ &= -B\partial g^*(-B^\top v^{k+1}) + \tfrac{1}{\beta}(v^{k+1} - u^{k+1} + w^k) \end{aligned}$$

Then there exists $y^{k+1} \in \partial g^*(-B^\top v^{k+1})$, (which implies that $-B^\top v^{k+1} \in \partial g(y^{k+1})$), such that

$$\begin{aligned} 0 &= -By^{k+1} + \tfrac{1}{\beta}(v^{k+1} - u^{k+1} + w^k) \\ (\Rightarrow)\, v^{k+1} &= u^{k+1} - w^k + \beta By^{k+1} \end{aligned} \tag{3}$$

From the update of w that $w^{k+1} = w^k + v^{k+1} - u^{k+1}$, we also have,

$$0 = -By^{k+1} + \tfrac{1}{\beta}w^{k+1} \tag{4}$$

By $0 \in \partial g(y^{k+1}) + B^\top v^{k+1}$ and (3),

$$0 \in \partial g(y^{k+1}) + B^\top(u^{k+1} - w^k) + \beta B^\top By^{k+1} \tag{5}$$

Step 3: Recover ADMM updates.

From (2) and (4),

$$0 \in \partial f(x^{k+1}) + A^\top v^k + \beta A^\top(Ax^{k+1} - b + By^k),$$

which is equivalent to the x-update of ADMM,

$$x^{k+1} = \arg\min_x \left\{ f(x) + \langle v^k, Ax\rangle + \tfrac{\beta}{2}\|Ax + By^k - b\|^2 \right\}.$$

From (5) and (4),

$$\begin{aligned} 0 &\in \partial g(y^{k+1}) + B^\top(u^{k+1} - w^k + \beta By^{k+1}) \\ &= \partial g(y^{k+1}) + B^\top(v^k + \beta(Ax^{k+1}By^{k+1} - b)\,) \end{aligned} \tag{6}$$

where the equality is by (1). This is the y-update of ADMM,

$$y^{k+1} = \arg\min_y \left\{ g(y) + \langle v^k, By\rangle + \tfrac{\beta}{2}\|Ax^{k+1} + By - b\|^2 \right\}.$$

From (3) and (6), we recovered the dual update of ADMM,

$$\begin{aligned} v^{k+1} &= u^{k+1} - w^k + \beta By^{k+1} \\ &= v^k + \beta(Ax^{k+1}By^{k+1} - b). \end{aligned}$$