# Patch Mobility Method

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## 1 Patch Mobility Definition

Patch-Mobility method is defined by:

$$Y_{R-I} = \frac{\langle V \rangle_R}{\langle F \rangle_I} \tag{1}$$

where:

$$\langle V \rangle_R = \frac{\iint_{S_R} V(x, y) \, \mathrm{d}S}{S_R} \tag{2}$$

and:

$$\langle F \rangle_I = \frac{\iint_{S_I} F(x, y) \, \mathrm{d}S}{S_I} = \iint_{S_I} P(x, y) \, \mathrm{d}S = \langle P \rangle_I S_I \tag{3}$$

The subscript $I$ indicates Incident, and $R$ indicates Response.

## 2 Panel Patch Mobility

### 2.1 General Function Solution

Panel equation:

$$D\nabla^4 w(x, y) - \omega^2 \rho h w(x, y) = P(x, y) \tag{4}$$

panel displacement modal expansion:

$$w(x, y) = \sum_{n=1}^{\infty} W_n \psi_n(x, y) \tag{5}$$

substitute Eq. (5) into Eq. (4):

$$D\nabla^4 \sum_{n=1}^{\infty} W_n \psi_n(x, y) - \omega^2 \rho h \sum_{n=1}^{\infty} W_n \psi_n(x, y) = P(x, y) \tag{6}$$

Eq. (6)'s characteristic equation:

$$D\nabla^4 W_n \psi_n(x, y) - \omega_n^2 \rho h W_n \psi_n(x, y) = 0 \tag{7}$$

We can obtain:

$$\rho h \sum_{n=1}^{\infty} \omega_n^2 W_n \psi_n(x, y) - \omega^2 \rho h \sum_{n=1}^{\infty} W_n \psi_n(x, y) = P(x, y) \tag{8}$$

Use modal orthogonality decouple Eq. (8), times panel modal $\psi_m(x,y), m=1,2,\ldots$ on both side and then integral, obtain:

$$\rho h\omega_m^2 W_m \iint_S \psi_m^2(x,y)\,\mathrm{d}S - \omega^2\rho h W_m \iint_S \psi_m(x,y)\,\mathrm{d}S = \iint_S P(x,y)\,\psi_m(x,y)\,\mathrm{d}S \tag{9}$$

Solve Eq. (9):

$$W_m = \frac{\iint_S P(x,y)\,\psi_m(x,y)\,\mathrm{d}S}{\rho h\left(\omega_m^2-\omega^2\right)\iint_S \psi_m^2(x,y)\,\mathrm{d}S} \tag{10}$$

Where $P(x,y)$ the mean pressure on the incident Patch:

$$P(x,y) = \begin{cases} \langle P\rangle_i\,, & (x,y)\in S_i \\ 0\,, & \text{other} \end{cases} \tag{11}$$

Put it into Eq. (10):

$$W_m = \frac{\langle P\rangle_i \iint_{S_i} \psi_m(x,y)\,\mathrm{d}S}{\rho h\left(\omega_m^2-\omega^2\right)\iint_S \psi_m^2(x,y)\,\mathrm{d}S} \tag{12}$$

Substitute Eq. (12) into Eq. (5):

$$\begin{aligned} w(x,y) &= \langle P\rangle_i \sum_{n=1}^{\infty} \frac{\iint_{S_i}\psi_n(x,y)\,\mathrm{d}S}{\rho h\left(\omega_n^2-\omega^2\right)\iint_S\psi_n^2(x,y)\,\mathrm{d}S}\psi_n(x,y) \\ &= \frac{\langle F\rangle_i}{S_i}\sum_{n=1}^{\infty} \frac{\iint_{S_i}\psi_n(x,y)\,\mathrm{d}S}{\rho h\left(\omega_n^2-\omega^2\right)\iint_S\psi_n^2(x,y)\,\mathrm{d}S}\psi_n(x,y) \end{aligned} \tag{13}$$

Velocity:

$$v(x,y) = \mathrm{i}\omega w(x,y) = \mathrm{i}\omega\frac{\langle F\rangle_i}{S_i}\sum_{n=1}^{\infty} \frac{\iint_{S_i}\psi_n(x,y)\,\mathrm{d}S}{\rho h\left(\omega_n^2-\omega^2\right)\iint_S\psi_n^2(x,y)\,\mathrm{d}S}\psi_n(x,y) \tag{14}$$

Patch mean velocity:

$$\langle V\rangle_j = \frac{\iint_{S_j} v(x,y)\,\mathrm{d}S}{S_j} = i\omega\frac{\langle F\rangle_i}{S_iS_j}\sum_{n=1}^{\infty} \frac{\iint_{S_i}\psi_n(x,y)\,\mathrm{d}S}{\rho h\left(\omega_n^2-\omega^2\right)\iint_S\psi_n^2(x,y)\,\mathrm{d}S}\iint_{S_j}\psi_n(x,y)\,\mathrm{d}S \tag{15}$$

Panel patch-mobility:

$$YP_{ij} = \frac{\langle V\rangle_j}{\langle F\rangle_i} = \frac{\mathrm{i}\omega}{S_iS_j}\sum_{n=1}^{\infty} \frac{\iint_{S_i}\psi_n(x,y)\,\mathrm{d}S}{\rho h\left(\omega_n^2-\omega^2\right)\iint_S\psi_n^2(x,y)\,\mathrm{d}S}\iint_{S_j}\psi_n(x,y)\,\mathrm{d}S \tag{16}$$

## 2.2 Simply Support Boundary Solution

For simply support boundary:

$$\psi_n(x,y) = \sin\left(\frac{p\pi}{L_x}x\right)\sin\left(\frac{q\pi}{L_y}y\right) \tag{17}$$

then

$$\begin{aligned} &\iint_{S_i}\psi_n(x,y)\,\mathrm{d}S \\ &= \int_{S_{ix1}}^{S_{ix2}} \sin\left(\frac{p\pi}{L_x}x\right)\mathrm{d}x \int_{S_{iy1}}^{S_{iy2}} \sin\left(\frac{q\pi}{L_y}y\right)\mathrm{d}y \\ &= \left[-\frac{L_x}{p\pi}\cos\left(\frac{p\pi}{L_x}x\right)\Bigg|_{S_{ix1}}^{S_{ix2}}\right]\left[-\frac{L_y}{q\pi}\cos\left(\frac{q\pi}{L_y}y\right)\Bigg|_{S_{iy1}}^{S_{iy2}}\right] \\ &= \frac{L_xL_y}{pq\pi^2}\left[\cos\left(\frac{p\pi}{L_x}S_{ix2}\right)-\cos\left(\frac{p\pi}{L_x}S_{ix1}\right)\right]\left[\cos\left(\frac{q\pi}{L_y}S_{iy2}\right)-\cos\left(\frac{q\pi}{L_y}S_{iy1}\right)\right] \end{aligned} \tag{18}$$

apply the same process:

$$\begin{aligned}&\iint_{S_j} \psi_n(x,y)\,\mathrm{d}S \\ &= \frac{L_x L_y}{pq\pi^2}\left[\cos\left(\frac{p\pi}{L_x}S_{jx2}\right)-\cos\left(\frac{p\pi}{L_x}S_{jx1}\right)\right]\left[\cos\left(\frac{q\pi}{L_y}S_{jy2}\right)-\cos\left(\frac{q\pi}{L_y}S_{jy1}\right)\right]\end{aligned} \tag{19}$$

and:

$$\iint_S \psi_n^2(x,y)\,\mathrm{d}s = \int_0^{L_x}\sin^2\left(\frac{p\pi}{L_x}x\right)\mathrm{d}x\int_0^{L_y}\sin^2\left(\frac{q\pi}{L_y}y\right)\mathrm{d}y = \frac{1}{4}L_xL_y \tag{20}$$

Substitute Eq. (17), (18), (19) and (20) into Eq. (16):

$$\begin{aligned}YP_{ij} &= \frac{\mathrm{i}\omega}{S_iS_j}\sum_{n=1}^{\infty}\frac{\frac{L_xL_y}{pq\pi^2}\left[\cos\left(\frac{p\pi}{L_x}S_{ix2}\right)-\cos\left(\frac{p\pi}{L_x}S_{ix1}\right)\right]\left[\cos\left(\frac{q\pi}{L_y}S_{iy2}\right)-\cos\left(\frac{q\pi}{L_y}S_{iy1}\right)\right]}{\frac{1}{4}\rho h\left(\omega_n^2-\omega^2\right)L_xL_y} \\ &\quad * \frac{L_xL_y}{pq\pi^2}\left[\cos\left(\frac{p\pi}{L_x}S_{jx2}\right)-\cos\left(\frac{p\pi}{L_x}S_{jx1}\right)\right]\left[\cos\left(\frac{q\pi}{L_y}S_{jy2}\right)-\cos\left(\frac{q\pi}{L_y}S_{jy1}\right)\right] \\ &= \frac{4i\omega L_xL_y}{\rho h\pi^4S_iS_j}\sum_{n=1}^{\infty}\frac{\left[\cos\left(\frac{p\pi}{L_x}S_{ix2}\right)-\cos\left(\frac{p\pi}{L_x}S_{ix1}\right)\right]\left[\cos\left(\frac{q\pi}{L_y}S_{iy2}\right)-\cos\left(\frac{q\pi}{L_y}S_{iy1}\right)\right]}{pq\left(\omega_n^2-\omega^2\right)} \\ &\quad * \frac{1}{pq}\left[\cos\left(\frac{p\pi}{L_x}S_{jx2}\right)-\cos\left(\frac{p\pi}{L_x}S_{jx1}\right)\right]\left[\cos\left(\frac{q\pi}{L_y}S_{jy2}\right)-\cos\left(\frac{q\pi}{L_y}S_{jy1}\right)\right]\end{aligned} \tag{21}$$

where:

$$\omega_n^2 = \frac{D^*}{\rho h}\left[\left(\frac{p\pi}{L_x}\right)^2+\left(\frac{q\pi}{L_y}\right)^2\right]^2 \tag{22}$$

### 2.3 Numerical Calculation

Use accumulation instead integral in Eq. (16):

$$\begin{aligned}YP_{ij} &= \frac{\mathrm{i}\omega}{S_iS_j}\sum_{n=1}^{\infty}\frac{\iint_{S_i}\psi_n(x,y)\,\mathrm{d}S}{\rho h\left(\omega_n^2-\omega^2\right)\iint_S\psi_n^2(x,y)\,\mathrm{d}S}\iint_{S_j}\psi_n(x,y)\,\mathrm{d}S \\ &= \frac{\mathrm{i}\omega}{S_iS_j}\sum_{n=1}^{\infty}\frac{S_i\psi_n(x_i,y_i)}{\rho h\left(\omega_n^2-\omega^2\right)\sum_{k=1}^{patchNum}S_k\psi_n^2(x_k,y_k)}S_j\psi_n(x_j,y_j) \\ &= \mathrm{i}\omega\sum_{n=1}^{\infty}\frac{\psi_n(x_i,y_i)}{\rho h\left(\omega_n^2-\omega^2\right)\sum_{k=1}^{patchNum}S_k\psi_n^2(x_k,y_k)}\psi_n(x_j,y_j)\end{aligned} \tag{23}$$

## 3 Cavity Patch Mobility

### 3.1 General Function Solution

Wave equation:

$$\nabla^2 p(x,y,z)+k^2p(x,y,z) = -2\mathrm{i}\omega\rho_0V(x,y)\,\delta(z-z_0) \tag{24}$$

Modal expansion:

$$p(x,y,z) = \sum_{s=0}^{\infty}P_s\Phi(x,y,z) \tag{25}$$

Incorporating Eq. (25) into Eq. (24):

$$\nabla^2 \sum_{s=0}^{\infty} P_s \Phi(x,y,z) + k^2 \sum_{s=0}^{\infty} P_s \Phi(x,y,z) = -2\mathrm{i}\omega\rho_0 V(x,y)\,\delta(z-z_0) \tag{26}$$

Eq. (26)'s characteristic equation:

$$\nabla^2 P_l \Phi_l(x,y,z) + k_l^2 P_l \Phi_l(x,y,z) = 0 \tag{27}$$

Substitute Eq. (27) into Eq. (26):

$$-\sum_{s=0}^{\infty} k_s^2 P_s \Phi(x,y,z) + k^2 \sum_{s=0}^{\infty} P_s \Phi(x,y,z) = -2\mathrm{i}\omega\rho_0 V(x,y)\,\delta(z-z_0) \tag{28}$$

Using modal orthogonality:

$$\left(k^2 - k_l^2\right) P_l \iiint_\Omega \Phi_l^2(x,y,z)\,\mathrm{d}\Omega = -2\mathrm{i}\omega\rho_0 \iiint_\Omega \Phi_l(x,y,z)\,V(x,y)\,\delta(z-z_0)\,\mathrm{d}\Omega \tag{29}$$

where:

$$V(x,y) = \begin{cases} \langle V\rangle_j\,, & (x,y)\in S_j \\ 0\,, & \text{other} \end{cases} \tag{30}$$

then:

$$\begin{aligned}\iiint_\Omega \Phi_l(x,y,z)\,V(x,y)\,\delta(z-z_0)\,\mathrm{d}\Omega &= \frac{1}{2}\iint_S \Phi_l(x,y,z_0)\,V(x,y)\,\mathrm{d}S \\ &= \frac{1}{2}\langle V\rangle_j \iint_{S_j} \Phi_l(x,y,z_0)\,\mathrm{d}S\end{aligned} \tag{31}$$

Put it into Eq. (29), become:

$$\left(k^2 - k_l^2\right) P_l \iiint_\Omega \Phi_l^2(x,y,z)\,\mathrm{d}\Omega = -\mathrm{i}\omega\rho_0\langle V\rangle_j \iint_{S_j} \Phi_l(x,y,z_0)\,\mathrm{d}S \tag{32}$$

We can obtain the modal answer:

$$P_l = -\mathrm{i}\omega\rho_0\langle V\rangle_j \frac{\iint_{S_j}\Phi_l(x,y,z_0)\,\mathrm{d}S}{\left(k^2-k_l^2\right)\iiint_\Omega \Phi_l^2(x,y,z)\,\mathrm{d}\Omega} \tag{33}$$

Substitute Eq. (33) into Eq. (25):

$$p(x,y,z) = -\mathrm{i}\omega\rho_0\langle V\rangle_j \sum_{s=0}^{\infty} \frac{\iint_{S_j}\Phi_s(x,y,z_0)\,\mathrm{d}S}{\left(k^2-k_l^2\right)\iiint_\Omega \Phi_s^2(x,y,z)\,\mathrm{d}\Omega}\Phi_s(x,y,z) \tag{34}$$

Combine Eq. (34) and Eq. (3), we can obtain cavity impedance, which is the reciprocal of mobility:

$$ZC_{ij} = \frac{\langle F\rangle_i}{\langle V\rangle_j} = -\mathrm{i}\omega\rho_0 \sum_{s=0}^{\infty} \frac{\iint_{S_j}\Phi_s(x,y,z_0)\,\mathrm{d}S}{\left(k^2-k_l^2\right)\iiint_\Omega \Phi_s^2(x,y,z)\,\mathrm{d}\Omega} \iint_{S_i} \Phi_s(x,y,z_1)\,\mathrm{d}S \tag{35}$$

### 3.2 Hexahedral Cavity Solution

Cavity Modal:

$$\Psi_s(x,y,z) = \cos\left(\frac{p\pi}{L_x}x\right)\cos\left(\frac{q\pi}{L_y}y\right)\cos\left(\frac{r\pi}{L_z}z\right) \tag{36}$$

where $s = 0, 1, 2, \cdots$. Then:

$$\begin{aligned}\iint_{S_j} \Phi_s\left(x, y, z_0\right) \mathrm{d}S &= \iint_{S_j} \cos\left(\frac{p\pi}{L_x}x\right) \cos\left(\frac{q\pi}{L_y}y\right) \cos\left(\frac{r\pi}{L_z}z_0\right) \mathrm{d}S \\ &= \cos\left(\frac{r\pi}{L_z}z_0\right) \int_{S_{jx1}}^{S_{jx2}} \cos\left(\frac{p\pi}{L_x}x\right) \mathrm{d}x \int_{S_{jy1}}^{S_{jy2}} \cos\left(\frac{q\pi}{L_y}y\right) \mathrm{d}y\end{aligned} \tag{37}$$

where

$$\int_{S_{jx1}}^{S_{jx2}} \cos\left(\frac{p\pi}{L_x}x\right) \mathrm{d}x = \begin{cases} S_{jx2} - S_{jx1} \ , & p = 0 \\ \frac{L_x}{p\pi}\left[\sin\left(\frac{p\pi}{L_x}S_{jx2}\right) - \sin\left(\frac{p\pi}{L_x}S_{jx1}\right)\right] \ , & p \neq 0 \end{cases} ,$$

and

$$\int_{S_{jy1}}^{S_{jy2}} \cos\left(\frac{q\pi}{L_y}y\right) \mathrm{d}y = \begin{cases} S_{jy2} - S_{jy1} \ , & q = 0 \\ \frac{L_y}{q\pi}\left[\sin\left(\frac{q\pi}{L_y}S_{jy2}\right) - \sin\left(\frac{q\pi}{L_y}S_{jy1}\right)\right] \ , & q \neq 0 \end{cases} .$$

$$\begin{aligned}\iiint_{\Omega} \Phi_s^2\left(x, y, z\right) \mathrm{d}\Omega &= \iiint_{\Omega} \cos^2\left(\frac{p\pi}{L_x}x\right) \cos^2\left(\frac{q\pi}{L_y}y\right) \cos^2\left(\frac{r\pi}{L_z}z\right) \mathrm{d}\Omega \\ &= \int_0^{L_x} \cos^2\left(\frac{p\pi}{L_x}x\right) \mathrm{d}x \int_0^{L_y} \cos^2\left(\frac{q\pi}{L_y}y\right) \mathrm{d}y \int_0^{L_z} \cos^2\left(\frac{r\pi}{L_z}z\right) \mathrm{d}z \\ &= L_x L_y L_z \varepsilon_x \varepsilon_y \varepsilon_z\end{aligned} \tag{38}$$

where

$$\varepsilon_x = \begin{cases} 1 \ , & x = 0 \\ \frac{1}{2} \ , & x \neq 0 \end{cases} , \qquad \varepsilon_y = \begin{cases} 1 \ , & y = 0 \\ \frac{1}{2} \ , & y \neq 0 \end{cases} \quad \text{and} \quad \varepsilon_z = \begin{cases} 1 \ , & z = 0 \\ \frac{1}{2} \ , & z \neq 0 \end{cases} .$$

Cavity modal circular frequency:

$$\omega_l^2 = c^2 \left[\left(\frac{p\pi}{L_x}\right)^2 + \left(\frac{q\pi}{L_y}\right)^2 + \left(\frac{r\pi}{L_z}\right)^2\right] \tag{39}$$

Cavity modal wave number:

$$k_l^2 = \left(\frac{p\pi}{L_x}\right)^2 + \left(\frac{q\pi}{L_y}\right)^2 + \left(\frac{r\pi}{L_z}\right)^2 \tag{40}$$

# 4 Source Room

## 4.1 General Function Solution

Wave equation with point source:

$$\nabla^2 p^{\mathrm{S}}\left(x, y, z\right) + k^2 p^{\mathrm{S}}\left(x, y, z\right) = -\mathrm{i}\omega\rho_0 Q \delta\left(x - x_s\right) \delta\left(y - y_s\right) \delta\left(z - z_s\right) \tag{41}$$

Where $(x_s, y_s, z_s)$ is the point source location and the upper script "S" indicate "Source". Modal expansion:

$$p^{\mathrm{S}}\left(x, y, z\right) = \sum_{t=0}^{\infty} P_t^{\mathrm{S}} \phi_t\left(x, y, z\right) \tag{42}$$

Refer to Eq. (33), we can write pressure modal answer directly:

$$\begin{aligned} P_t^{\mathrm{S}} &= \frac{-\mathrm{i}\omega\rho_0 Q \iiint_\Omega \phi_t\left(x,y,z\right)\delta\left(x-x_s\right)\delta\left(y-y_s\right)\delta\left(z-z_s\right)\,\mathrm{d}\Omega}{\left(k^2-k_t^2\right)\iiint_\Omega \phi_t^2\left(x,y,z\right)\,\mathrm{d}\Omega} \\ &= \frac{-\mathrm{i}\omega\rho_0 Q\phi_t\left(x_s,y_s,z_s\right)}{\left(k^2-k_t^2\right)\iiint_\Omega \phi_t^2\left(x,y,z\right)\,\mathrm{d}\Omega} \end{aligned} \tag{43}$$

Incorporating Eq. (43) into Eq. (42):

$$p^{\mathrm{S}}\left(x,y,z\right) = -\mathrm{i}\omega\rho_0 Q\sum_{t=0}^{\infty}\frac{\phi_t\left(x_s,y_s,z_s\right)}{\left(k^2-k_t^2\right)\iiint_\Omega \phi_t^2\left(x,y,z\right)\,\mathrm{d}\Omega}\phi_t\left(x,y,z\right) \tag{44}$$

For hexahedral room:

$$\phi_t\left(x,y,z\right) = \cos\left(\frac{p\pi}{L_x}x\right)\cos\left(\frac{q\pi}{L_y}y\right)\cos\left(\frac{r\pi}{L_z}z\right) \tag{45}$$

then

$$\begin{aligned} \iiint_\Omega \phi_t^2\left(x,y,z\right)\,\mathrm{d}\Omega &= \int_0^{L_x}\cos^2\left(\frac{p\pi}{L_x}x\right)\,\mathrm{d}x\int_0^{L_y}\cos^2\left(\frac{q\pi}{L_y}y\right)\,\mathrm{d}y\int_0^{L_z}\cos^2\left(\frac{r\pi}{L_z}z\right)\,\mathrm{d}z \\ &= \varepsilon_p\varepsilon_q\varepsilon_r L_x L_y L_z \end{aligned} \tag{46}$$

where

$$\varepsilon_p = \begin{cases} 1\,, & p=0 \\ \frac{1}{2}\,, & p\neq 0 \end{cases}\,,\qquad \varepsilon_q = \begin{cases} 1\,, & q=0 \\ \frac{1}{2}\,, & q\neq 0 \end{cases}\qquad\text{and}\qquad \varepsilon_r = \begin{cases} 1\,, & r=0 \\ \frac{1}{2}\,, & r\neq 0 \end{cases}\,.$$

Incorporating Eq. (46) into Eq. (43) and Eq. (44):

$$P_t^{\mathrm{S}} = \frac{-\mathrm{i}\omega\rho_0 Q\phi_t\left(x_s,y_s,z_s\right)}{\left(k^2-k_t^2\right)\varepsilon_p\varepsilon_q\varepsilon_r L_x L_y L_z} \tag{47}$$

$$p^{\mathrm{S}}\left(x,y,z\right) = -\mathrm{i}\omega\rho_0 Q\sum_{t=0}^{\infty}\frac{\phi_t\left(x_s,y_s,z_s\right)}{\left(k^2-k_t^2\right)\varepsilon_p\varepsilon_q\varepsilon_r L_x L_y L_z}\phi_t\left(x,y,z\right) \tag{48}$$

## 4.2 Quadratic Room Pressure

Quadratic room pressure:

$$\begin{aligned} P_r^2 &= \frac{\iiint_\Omega \left[p^{\mathrm{S}}\left(x,y,z\right)\right]^2\,\mathrm{d}\Omega}{L_x L_y L_z} \\ &= \frac{\iiint_\Omega \left[\sum_{t=0}^{\infty} P_t^{\mathrm{S}}\phi_t\left(x,y,z\right)\right]^2\,\mathrm{d}\Omega}{L_x L_y L_z} \\ &= \frac{\sum_{t=0}^{\infty}\left|P_t^{\mathrm{S}}\right|^2\iiint_\Omega \phi_t^2\left(x,y,z\right)\,\mathrm{d}\Omega}{L_x L_y L_z} \\ &= \sum_{t=0}^{\infty}\left|P_t^{\mathrm{S}}\right|^2\varepsilon_p\varepsilon_q\varepsilon_r \\ &= \sum_{t=0}^{\infty}\frac{\omega^2\rho_0^2 Q^2\phi_t^2\left(x_s,y_s,z_s\right)}{\left(k^2-k_t^2\right)^2\varepsilon_p\varepsilon_q\varepsilon_r L_x^2 L_y^2 L_z^2} \end{aligned} \tag{49}$$

### 4.3 Blocked Patch Pressure

Blocked patch pressure:

$$
\begin{aligned}
\langle P\rangle_i &= \frac{1}{S_i}\iint_{S_i} p^{\mathrm{S}}\left(x, y_w, z\right)\,\mathrm{d}S_i \\
&= \frac{1}{S_i}\iint_{S_i}\left[\sum_{t=0}^{\infty} P_t^{\mathrm{S}}\phi_t\left(x, y_w, z\right)\right]\mathrm{d}S_i \\
&= \frac{1}{S_i}\sum_{t=0}^{\infty}\left[P_t^{\mathrm{S}}\iint_{S_i}\phi_t\left(x, y_w, z\right)\,\mathrm{d}S_i\right] \\
&= \frac{1}{S_i}\sum_{t=0}^{\infty}\left[P_t^{\mathrm{S}}\cos\left(\frac{q\pi}{L_y}y_w\right)\int_{\mathrm{patch}x1}^{\mathrm{patch}x2}\cos\left(\frac{p\pi}{L_x}x\right)\mathrm{d}x\int_{\mathrm{patch}z1}^{\mathrm{patch}z2}\cos\left(\frac{r\pi}{L_z}z\right)\mathrm{d}z\right]
\end{aligned}
\tag{50}
$$

where

$$
\int_{\mathrm{patch}.x1}^{\mathrm{patch}.x2}\cos\left(\frac{p\pi}{L_x}x\right)\mathrm{d}x = \begin{cases} \dfrac{L_x}{p\pi}\sin\left(\dfrac{p\pi}{L_x}x\right)\Big|_{\mathrm{patch}.x1}^{\mathrm{patch}.x2} , & p \neq 0 \\ \mathrm{patch}.x2 - \mathrm{patch}.x1 , & p = 0 \end{cases}
$$

and

$$
\int_{\mathrm{patch}.z1}^{\mathrm{patch}.z2}\cos\left(\frac{r\pi}{L_z}z\right)\mathrm{d}z = \begin{cases} \dfrac{L_z}{r\pi}\sin\left(\dfrac{r\pi}{L_z}z\right)\Big|_{\mathrm{patch}.z1}^{\mathrm{patch}.z2} , & r \neq 0 \\ \mathrm{patch}.z2 - \mathrm{patch}.z1 , & r = 0 \end{cases}
$$

# 5 Semi-Infinite Medium

### 5.1 Radiated Pressure and Impedance

Radiation impedance is defined as the ratio of averaged patch $i$ radiated pressure to averaged patch $j$ velocity:

$$
\langle Z\rangle_{ij} = \frac{\langle P_{\mathrm{rad}}\rangle_i}{\langle V\rangle_j} \tag{51}
$$

Rayleigh's formula [David Feit. Sound, Structures, and Their Interaction: p89, Eq. (4.29)]:

$$
p\left(\boldsymbol{R}\right) = \frac{\rho_0}{2\pi R}\int_{S_0}\exp\left(\mathrm{j}k\left|\boldsymbol{R}-\boldsymbol{R}_0\right|\right)\ddot{w}\left(\boldsymbol{R}_0\right)\mathrm{d}S\left(\boldsymbol{R}_0\right) \quad , \qquad \boldsymbol{R} \gg \boldsymbol{R}_0 \tag{52}
$$

Where the vibrating surface is planar and infinitely extend, the Rayleigh integral applies:

$$
\tilde{p}\left(\mathbf{r}\right)e^{\mathrm{j}\omega t} = \frac{\mathrm{j}\omega\rho_0}{4\pi}e^{\mathrm{j}\omega t}\int_S\left[2\tilde{v}_n\left(\mathbf{r}_s\right)\frac{e^{-\mathrm{j}kR}}{R}\right]\mathrm{d}S \tag{53}
$$

then, consider the patch as a rigid circular disc of radius $a$ vibrating in a coplanar rigid baffle. The pressure at the centre due to the motion of an annulus of radius $R$ is given by Eq. (53) as (Fahy P247):

$$
\delta\tilde{p}\left(0\right) = \frac{\mathrm{j}\omega\rho_0}{4\pi R}2\tilde{v}_n e^{-\mathrm{j}kR}2\pi R\delta R = \mathrm{j}\omega\rho_0\tilde{v}_n e^{-\mathrm{j}kR}\delta R \tag{54}
$$

The total pressure at the centre is given by the integral over the limits 0 to $a$:

$$
\tilde{p}\left(0\right) = \mathrm{j}\omega\rho\tilde{v}_n\int_0^a e^{-\mathrm{j}kR}\,\mathrm{d}R = \rho_0 c\tilde{v}_n\left(1 - e^{-\mathrm{j}ka}\right) \tag{55}
$$

the patch impedance:

$$\langle\langle Z\rangle_i\rangle_i = \frac{\langle P_{\mathrm{rad}}\rangle_i}{\langle V\rangle_i} = \frac{\tilde{p}\,(0)}{\tilde{v}_n} = \rho_0 c\left(1 - e^{-\mathrm{j}ka}\right) \tag{56}$$

Consider the patch $j$, if the patch dimensions are small, the integral can be approximated by the value at the central point times the patch area. The pressure at patch $j$ can be write from Eq. (53):

$$\tilde{p}\,(d_{ij}) = \frac{\mathrm{j}\omega\rho_0}{4\pi} 2\tilde{v}_n \frac{e^{\mathrm{j}kd_{ij}}}{d_{ij}} S_j \tag{57}$$

the patch impedance:

$$\langle\langle Z\rangle_i\rangle_j = \frac{\langle P_{\mathrm{rad}}\rangle_j}{\langle V\rangle_i} = \frac{\tilde{p}\,(d_{ij})}{\tilde{v}_n} = \mathrm{j}\omega\rho_0 \frac{e^{-\mathrm{j}kd_{ij}}}{2\pi d_{ij}} S_j \tag{58}$$

Radiation patch mobilities are obtained by inversion of impedance matrix calculated from previous Eq. (56) and Eq. (58).

## 5.2 Radiated Power

Radiated power is calculated from patch velocities and radiated patch pressures, and can be written using radiation patch mobility method (Fahy P124):

$$I_{\mathrm{rad}} = \frac{1}{2}\sum_i \mathrm{Re}\{\langle V\rangle_i^* \langle P_{\mathrm{rad}}\rangle_i\} = \frac{1}{2}\left(\left[Y_{\mathrm{rad}}\right]^{-1}\{V\}\right)'\{V\}^* = \frac{1}{2}\{V\}'\left[Z_{\mathrm{rad}}\right]\{V\}^* \tag{59}$$