CS231a HW2

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*Adapted from last year's slides

Overview

Focus on

1. Problem 2 - Image Rectification
2. Problem 4 - Structure from Motion

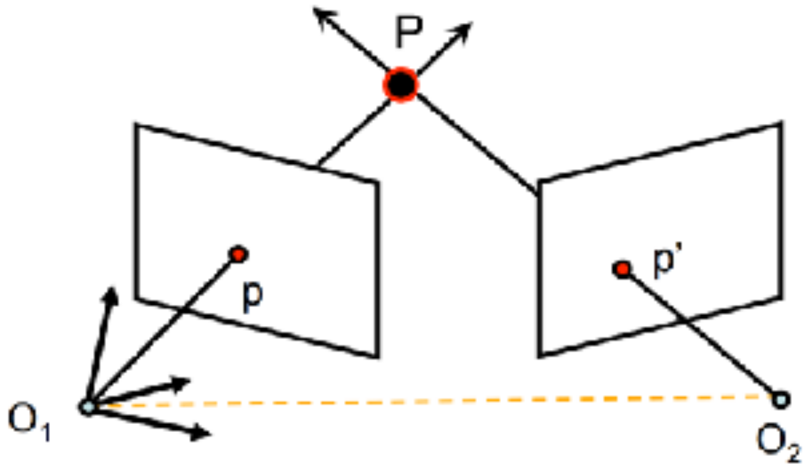
Will briefly cover Problem 1 - Fundamental Matrix Estimation

Will not cover Problem 3 - Factorization Method
(Enough details covered in class. Please visit our office hour if you need help)

Problem 1- Fundamental Matrix Estimation

Fundamental Matrix

- A matrix which maps the relationship of correspondences between stereo images



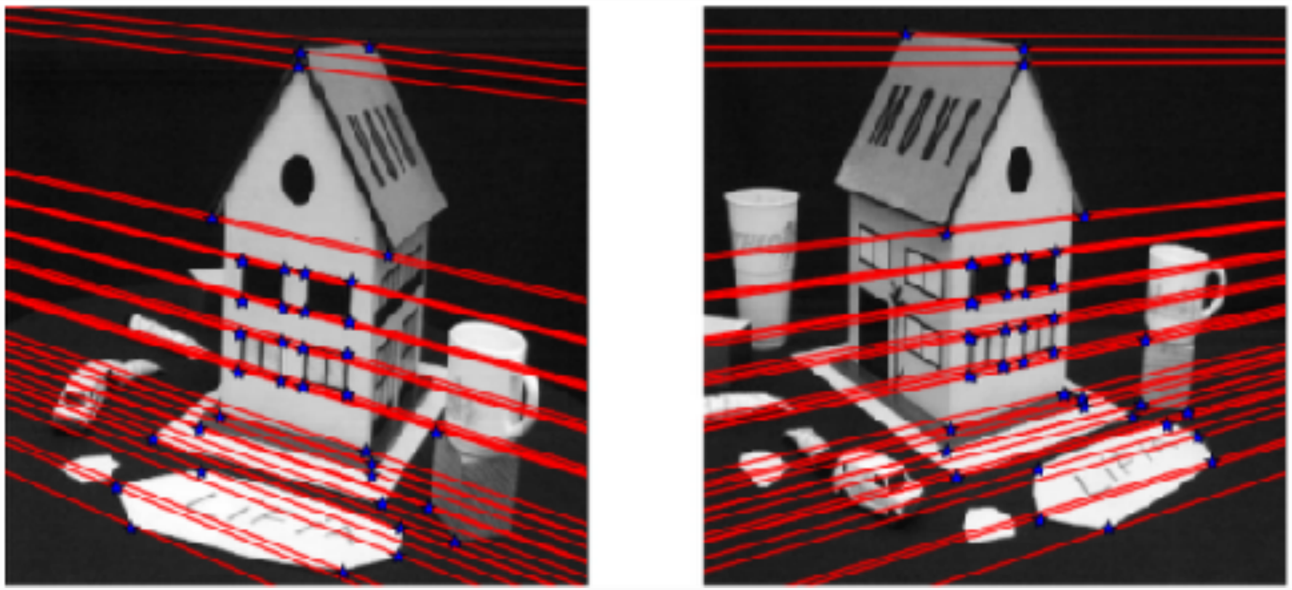
[Eq. 13] $\boxed{p^T F p' = 0}$

$$F = K^{-T} \cdot [T_\times] \cdot R K'^{-1}$$

Problem 1- Fundamental Matrix Estimation

Ex)

Image and correspondences given in the homework



Problem 1- Fundamental Matrix Estimation

How to compute F?

- Eight point algorithm

[Eq. 13] $\mathrm{p}^{\mathrm{T}}\,\mathrm{F}\,\mathrm{p}' = 0 \quad \Rightarrow$

$$p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

$$(u, v, 1)\begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}\begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \quad (uu', uv', u, vu', vv', v, u', v', 1)\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

[Eq. 14]

Let's take 8 corresponding points

Problem 1- Fundamental Matrix Estimation

How to compute F?

- Eight point algorithm

Problem?

- W is highly unbalanced (not well conditioned)

$$\mathbf{W}\mathbf{f} = 0$$

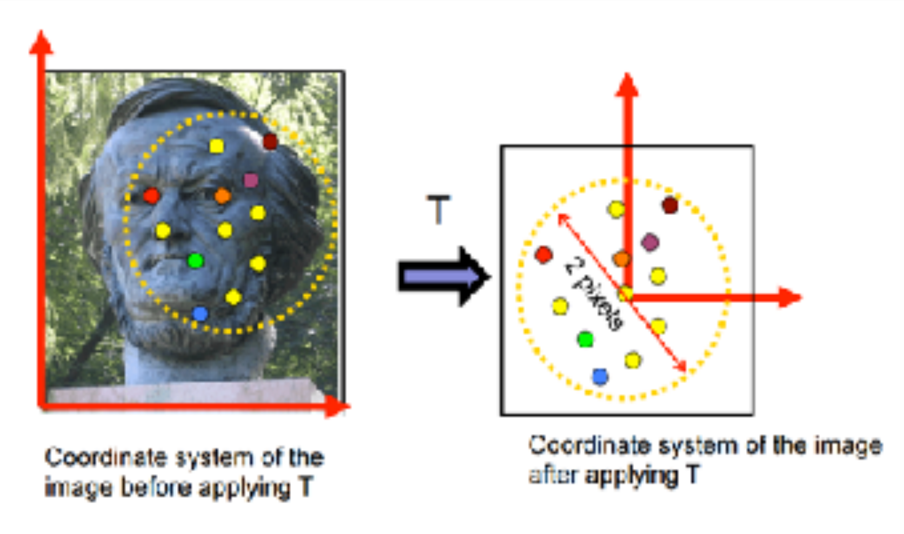
$$\begin{pmatrix}
u_1u'_1 & u_1v'_1 & u_1 & v_1u'_1 & v_1v'_1 & v_1 & u'_1 & v'_1 & 1 \\
u_2u'_2 & u_2v'_2 & u_2 & v_2u'_2 & v_2v'_2 & v_2 & u'_2 & v'_2 & 1 \\
u_3u'_3 & u_3v'_3 & u_3 & v_3u'_3 & v_3v'_3 & v_3 & u'_3 & v'_3 & 1 \\
u_4u'_4 & u_4v'_4 & u_4 & v_4u'_4 & v_4v'_4 & v_4 & u'_4 & v'_4 & 1 \\
u_5u'_5 & u_5v'_5 & u_5 & v_5u'_5 & v_5v'_5 & v_5 & u'_5 & v'_5 & 1 \\
u_6u'_6 & u_5v'_6 & u_6 & v_6u'_6 & v_6v'_6 & v_6 & u'_6 & v'_5 & 1 \\
u_7u'_7 & u_7v'_7 & u_7 & v_7u'_7 & v_7v'_7 & v_7 & u'_7 & v'_7 & 1 \\
u_8u'_8 & u_8v'_8 & u_8 & v_8u'_8 & v_8v'_8 & v_8 & u'_8 & v'_8 & 1
\end{pmatrix}
\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

Problem 1- Fundamental Matrix Estimation

Possible improvement?

Pre-condition our linear system to get more stable result

- origin = centroid of the image
- mean square distance of the image points from origin is ~2px



Problem 1- Fundamental Matrix Estimation

Final step

- Reduce rank(F) to 2

Find F that minimizes $\left\| \mathrm{F} - \hat{\mathrm{F}} \right\| = 0$

Frobenius norm (*)

Subject to det(F)=0

$$F = U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

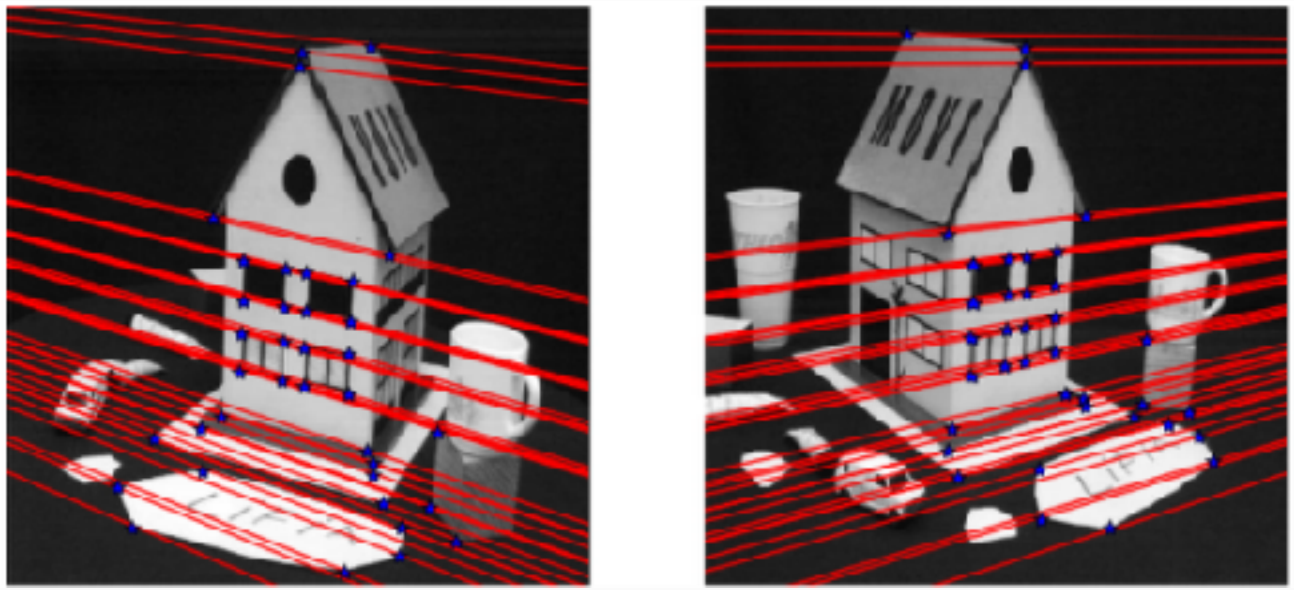
Where:

$$U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} V^T = SVD(\hat{F})$$

[HZ] pag 281, chapter 11, "Computation of F"

Problem 1- Fundamental Matrix Estimation

Epipolar lines

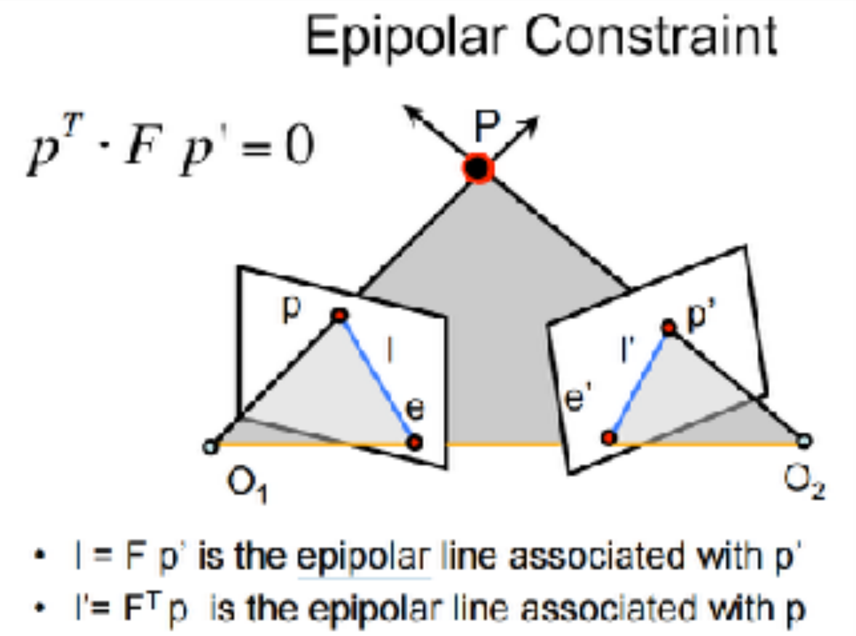


Problem 1- Fundamental Matrix Estimation

Computing epipolar lines from F

l = Fp’

l’ = F^Tp

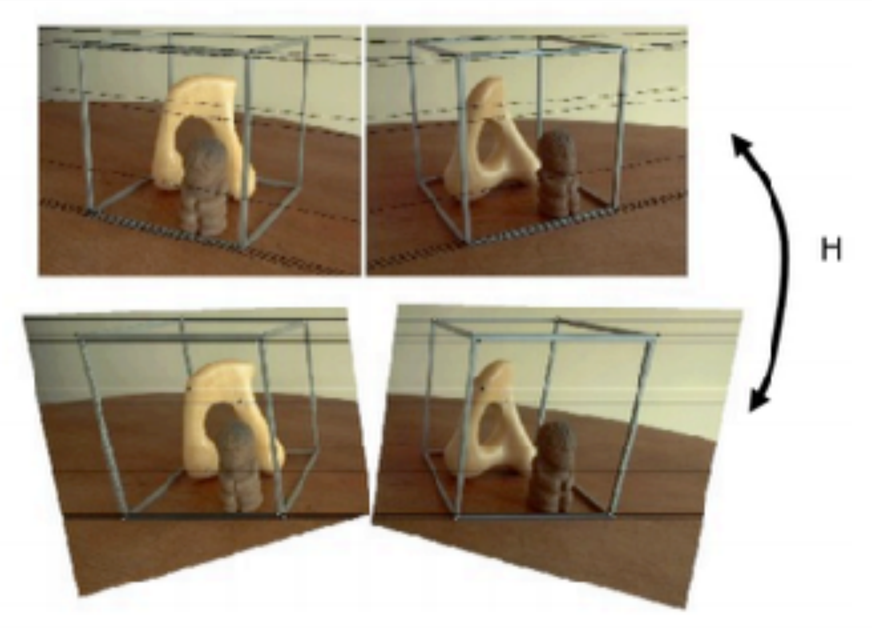


Problem 2 - Image Rectification

Image Rectification

Making two images “parallel”

All correspondences lie on the same y axis



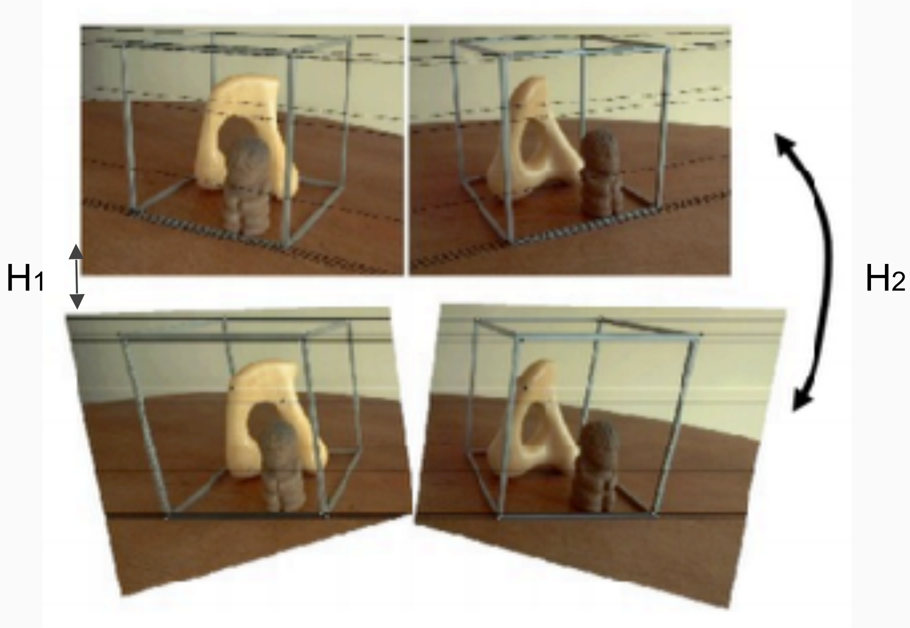
Problem 2 - Image Rectification

Application

Novel view synthesis (view morphing) based on rectified images

Problem 2 - Image Rectification

Find a homography H_1 and H_2 such that corresponding points of image 1 and image 2 lie on the same y axis



Problem 2 - Image Rectification

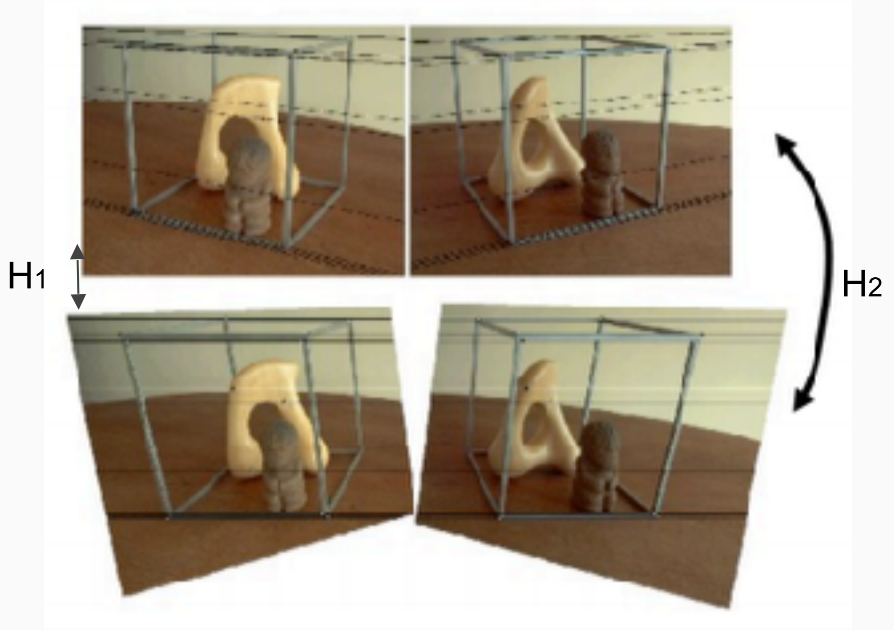
Overview

- Find H_2: morph image 2 to make all epipolar lines parallel
  - Find epipole of image 2 (e_2)
  - Find homography H_2 which brings e_2 to a point a infinity
- Find H_1: minimize square distance from rectified image 2 to image 1

Problem 2 - Image Rectification

Find a homography H_1 and H_2 such that corresponding points of image 1 and image 2 lie on the same y axis

1. Find H_2 such that epipolar lines are all horizontally aligned (parallel)
2. Find H_1 which minimizes square distance between corresponding points of the image



Problem 2 - Image Rectification

Find a homography H_1 and H_2 such that corresponding points of image 1 and image 2 lie on the same y axis

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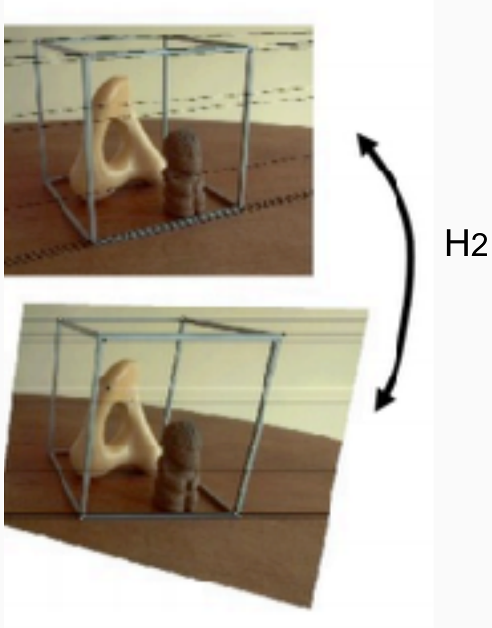
2. Find H_1 which minimizes square distance between corresponding points of the image



Problem 2 - Image Rectification

Find a homography H_1 and H_2 such that corresponding points of image 1 and image 2 lie on the same y axis

1. Find H_2 such that epipolar lines are all horizontally aligned (parallel)
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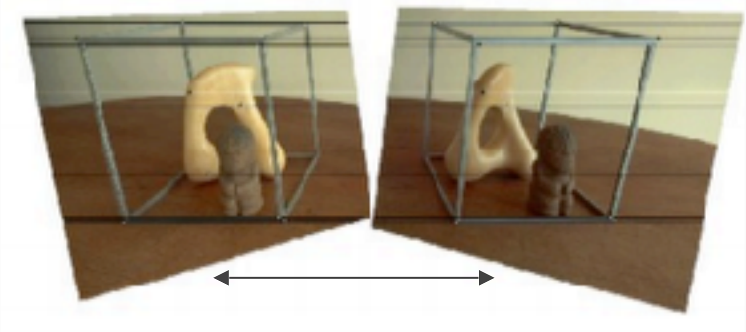


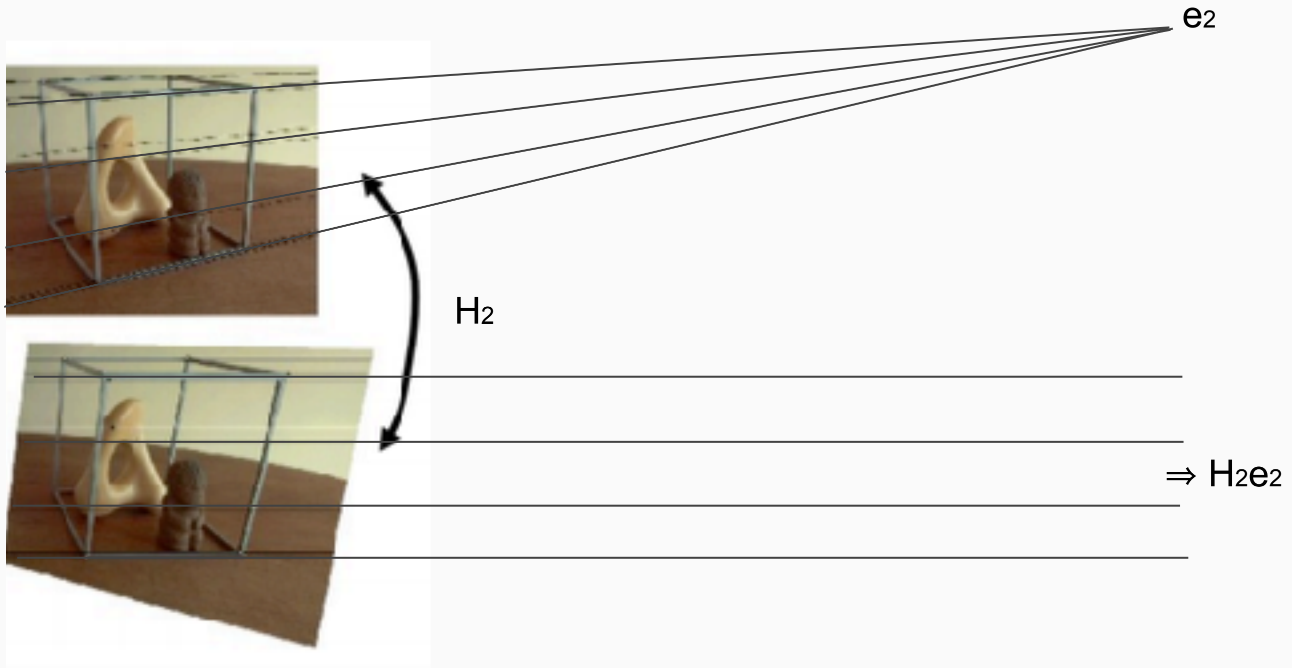
Problem 2 - Image Rectification

Overview

- **Find H_2: morph image 2 to make all epipolar lines parallel**
  - Find epipole of image 2 (e_2)
  - Find homography H_2 which brings e_2 to a point a infinity
- Find H_1: minimize square distance from rectified image 2 to image 1

Problem 2 - Image Rectification

1. Find epipole of image 2, e_2
2. Find homography H_2 which brings e_2 to a point at infinity



Problem 2 - Image Rectification

Overview

- Find H_2: morph image 2 to make all epipolar lines parallel
  - **Find epipole of image 2 (e_2)**
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Problem 2 - Image Rectification

Find epipole of image 2 (e_2)

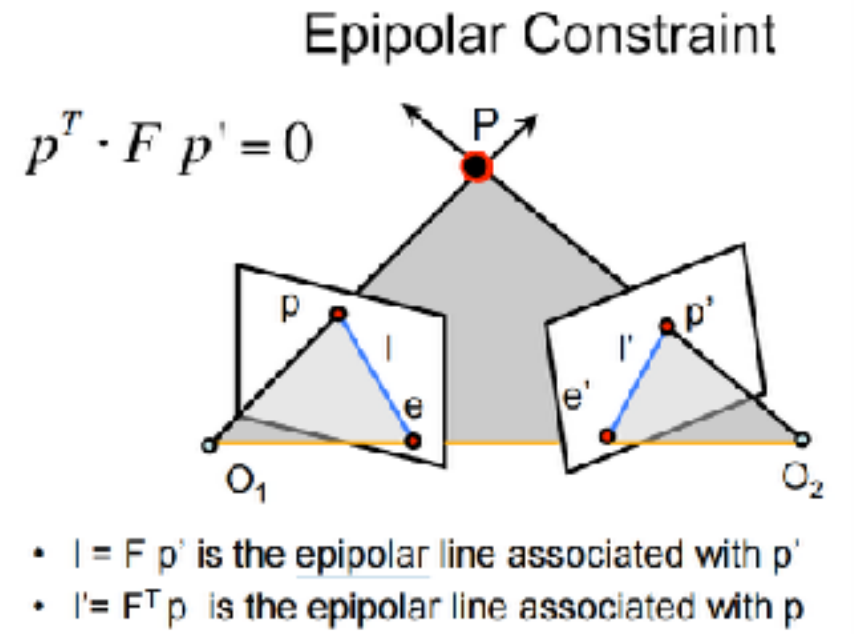
- Epipole is an intersection of epipolar lines
  - ⇔ epipole lies on every epipolar lines
  - defining epipolar line ℓ such that all points on the line are in set $\{x | \ell^T x = 0\}$, formulate a linear system of equations on the right
  - Solve using SVD

$$\begin{bmatrix} \ell_1^T \\ \vdots \\ \ell_n^T \end{bmatrix} e = 0$$

Problem 2 - Image Rectification

Computing epipolar lines from F

$$l = Fp'$$
$$l' = F^T p$$



Problem 2 - Image Rectification

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- Find H_2: morph image 2 to make all epipolar lines parallel
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Problem 2 - Image Rectification

Find homography H_2 which brings e_2 to a point at infinity

Intuition (from Hartley, Zisserman):

To keep image as realistic as possible after transformation, we want to keep H_2 to act as a rigid transformation in the neighborhood of a given selected point x_0 of the image

Center of the image is often a good choice for x_0

Problem 2 - Image Rectification

Find homography H_2 which brings e_2 to a point at infinity

1. **Translate image coordinate so that the origin will be at the image center**
2. Rotate the image so that e_2 will lie horizontal axis at some point $(f, 0, 1)$
3. Bring e_2 to infinity on $(f, 0, 0)$
4. Translate image coordinate back to the original origin

$$T = \begin{bmatrix} 1 & 0 & -\frac{width}{2} \\ 0 & 1 & -\frac{height}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 2 - Image Rectification

Find homography H_2 which brings e_2 to a point at infinity

1. Translate image coordinate so that the origin will be at the image center
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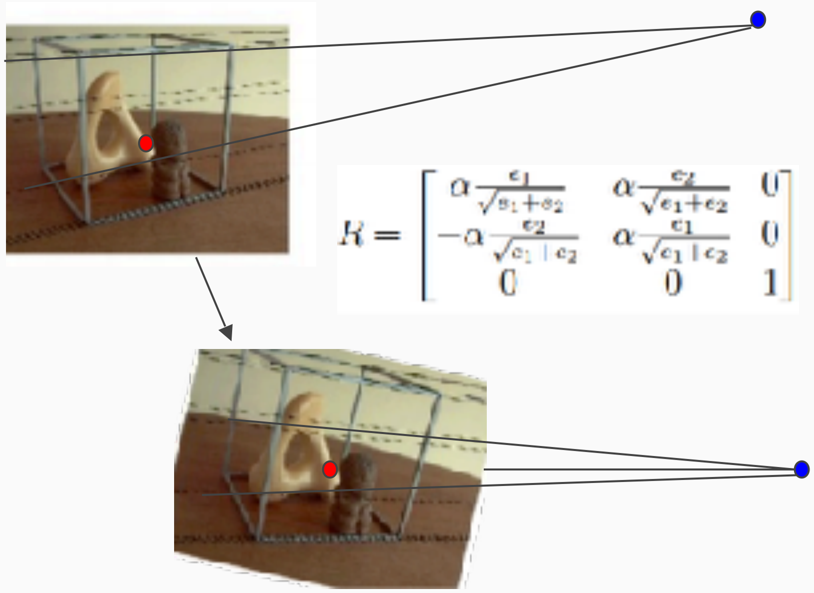

$$R = \begin{bmatrix} \alpha\frac{e_1}{\sqrt{e_1+e_2}} & \alpha\frac{e_2}{\sqrt{e_1+e_2}} & 0 \\ -\alpha\frac{e_2}{\sqrt{e_1+e_2}} & \alpha\frac{e_1}{\sqrt{e_1+e_2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 2 - Image Rectification

Find homography H_2 which brings e_2 to a point at infinity

1. Translate image coordinate so that the origin will be at the image center
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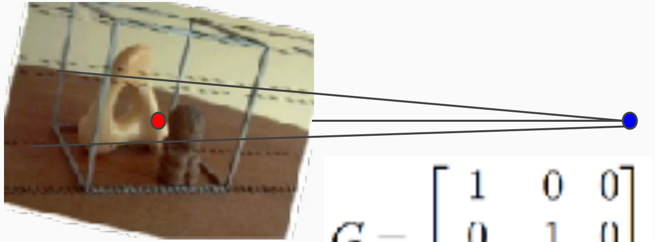

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{f} & 0 & 1 \end{bmatrix}$$

Problem 2 - Image Rectification

Find homography H_2 which brings e_2 to a point at infinity

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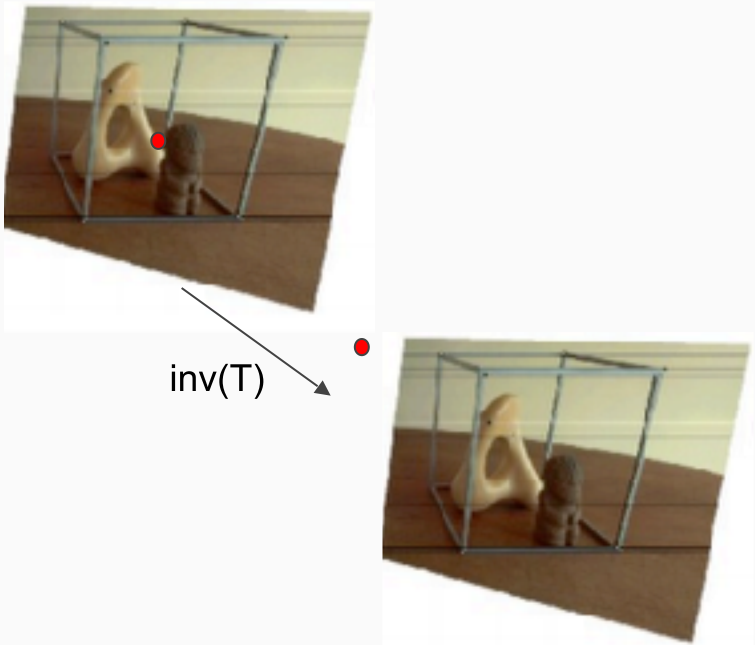

Problem 2 - Image Rectification

Find homography H_2 which brings e_2 to a point at infinity

$$H_2 = T^{-1}GRT$$

Problem 2 - Image Rectification

Overview

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- **Find H_1: minimize square distance from rectified image 2 to image 1**

Problem 2 - Image Rectification

Find H_1: minimize square distance from rectified image 2 to image 1

$$\arg\min_{H_1} \sum_i \|H_1 p_i - H_2 p_i'\|^2$$

Why this? $H_1^{-T} \ell_i = H_2^{-T} \ell_i'$ Reduce matching lines to matching points

We will take a simpler approach for this problem

Problem 2 - Image Rectification

Find H_1: minimize square distance from rectified image 2 to image 1

We can show that $H_1 = H_A H_2 M$

where $$H_A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This form of H_A allows a transformation of H_1 where its epipole is at (1, 0, 0) - a page long proof of this in Chapter 11 of Hartley & Zisserman's textbook

Problem 2 - Image Rectification

Find H_1: minimize square distance from rectified image 2 to image 1

We can also show that $M = [e]_\times F + ev^T$

because 1) $M = [e]_\times F$

- we know F up to scale
- any skew-symmetric matrix X (including $[e]_x$) is $A = A^3$ up to scale

$$F = [e]_\times M = [e]_\times [e]_\times [e]_\times M = [e]_\times [e]_\times F$$

Problem 2 - Image Rectification

Find H_1: minimize square distance from rectified image 2 to image 1

We can also show that $M = [e]_\times F + ev^T$

because 2) if columns of M are added by any scalar multiple of e,

up to scale, $F = [e]_\times M$

Therefore, $M = [e]_\times F + ev^T$ is more general case of defining M

Where in practice, we use $v = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

Problem 2 - Image Rectification

Find H_1: minimize square distance from rectified image 2 to image 1

Which reduces to

$$\arg\min_{H_A} \sum_i \|H_A \hat{p}_i - \hat{p}'_i\|^2$$

where $\hat{p}_i = H_2 M p_i$ $\hat{p}'_i = H_2 p_i$

Problem 2 - Image Rectification

Find H_1: minimize square distance from rectified image 2 to image 1

Which reduces down to solving least squares $W[a_1, a_2, a_3]^T = b$

Where $$W = \begin{bmatrix} \hat{x}_1 & \hat{y}_1 & 1 \\ & \vdots & \\ \hat{x}_n & \hat{y}_n & 1 \end{bmatrix} \qquad b = \begin{bmatrix} \hat{x}'_1 \\ \vdots \\ \hat{x}'_n \end{bmatrix}$$

and $\hat{p}_i = H_2 M p_i$ $\hat{p}'_i = H_2 p_i$

Problem 2 - Image Rectification

Find a homography H_1 and H_2 such that corresponding points of image 1 and image 2 lie on the same y axis

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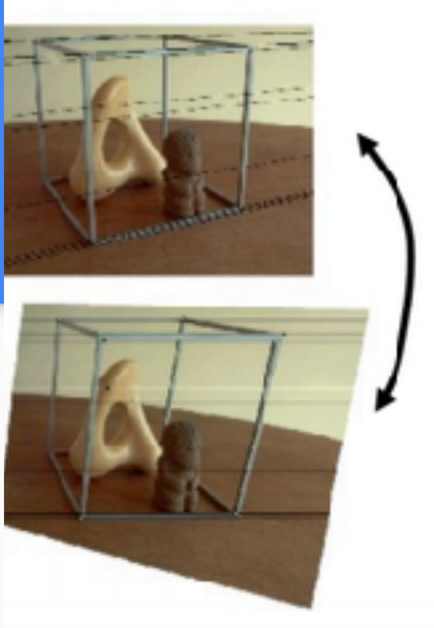
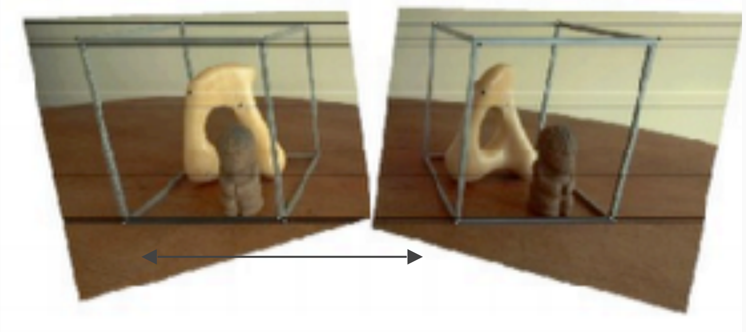




Problem 4: Structure from Motion

Structure from Motion (SfM)

Estimating 3D structure from 2D images that may be coupled with local motions

Input: 2D images

Output: 3D structure
(+ camera extrinsic)

Problem 4: Structure from Motion

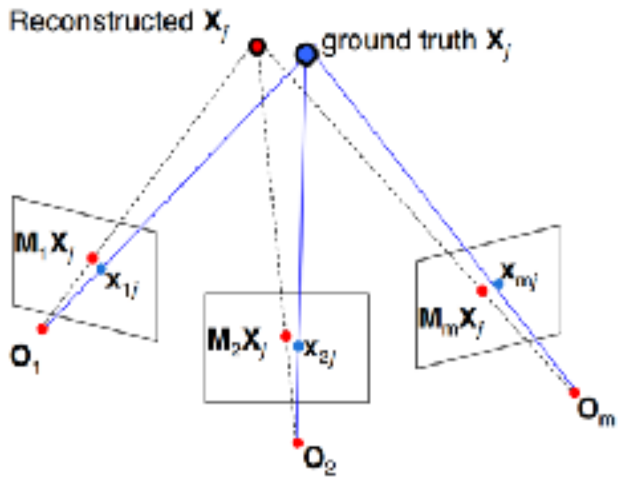
In this homework, we explore two different approaches for SfM

- Factorization Method (problem 3) - Tomasi & Kanade algorithm
- **Bundle Adjustment (problem 4)**

Problem 4: Structure from Motion

1. Compute essential matrix E from two views
2. Use E to make initial estimate of relative rotation R and translation T
3. Estimate 3D location of the reconstruction given RT
4. Optimize (bundle adjustment)
    - Jointly optimize all relative camera motions (R's and T's)
    - Minimize total reprojection error with respect to all 3D point and camera parameters
5. Repeat 3 and 4 for pairs of frames

$$E(M, X) = \sum_{i=1}^{m} \sum_{j=1}^{n} D\left(x_{ij}, M_i X_j\right)^2$$



Problem 4: Structure from Motion

1. Compute essential matrix E from two views
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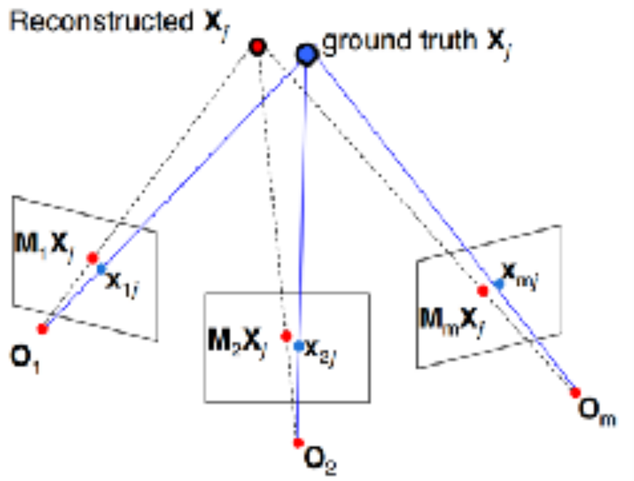

Problem 4: Structure from Motion

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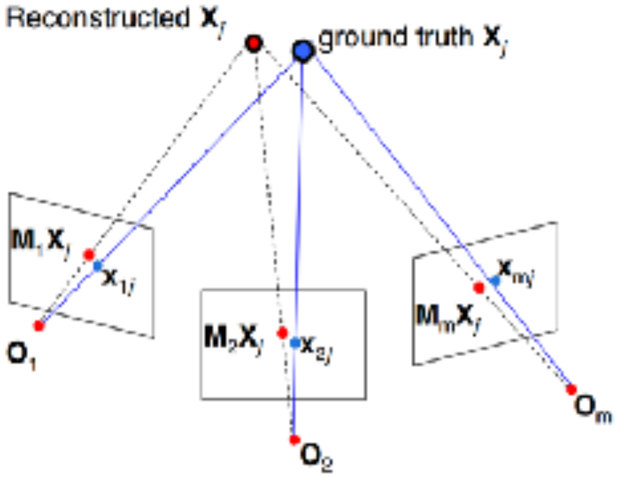
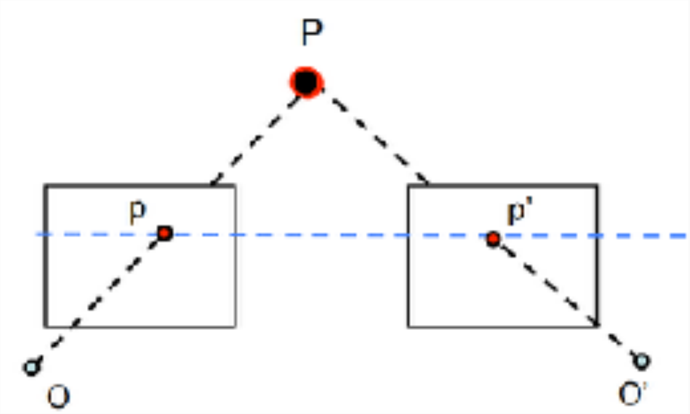


Problem 4: Structure from Motion

Estimate 3D location of the reconstruction given 1. projective camera matrix 2. their image coordinates



Two different possible approaches:

1. Formulating a linear equation to solve
2. Nonlinear optimization to minimize reprojection error

Problem 4: Structure from Motion

Estimate 3D location of the reconstruction
given 1. projective camera matrix 2. their image coordinates

Formulating a linear equation to solve, using [p=MP]
Solve for AP=0 (using SVD) where:

$$A = \begin{bmatrix} p_{1,1} m^{3\top} - m^{1\top} \\ p_{1,2} m^{3\top} - m^{2\top} \\ \vdots \\ p_{n,1} m^{3\top} - m^{1\top} \\ p_{n,2} m^{3\top} - m^{2\top} \end{bmatrix}$$

$p_{i,j}$: (x, y)[j] coordinate of *i*th image
m^{kT}: *k*-th row of M

Problem 4: Structure from Motion

Estimate 3D location of the reconstruction
given 1. projective camera matrix 2. their image coordinates

Nonlinear optimization to minimize reprojection error

Gauss-Newton algorithm

$$\hat{P} = \hat{P} - (J^T J)^{-1} J^T e$$

Begin from linear estimation for better initialization
How to define error and Jacobian?

Problem 4: Structure from Motion

(reprojection) error: difference between the projected point (M_iP) and ground-truth image coordinate p_i

Jacobian:

$$J = \begin{bmatrix} \frac{\partial e_1}{\partial P_1} & \frac{\partial e_1}{\partial P_2} & \frac{\partial e_1}{\partial P_3} \\ \vdots & \vdots & \vdots \\ \frac{\partial e_m}{\partial P_1} & \frac{\partial e_m}{\partial P_2} & \frac{\partial e_m}{\partial P_3} \end{bmatrix}$$

Problem 4: Structure from Motion

1. Compute essential matrix E from two views
2. **Use E to make initial estimate of relative rotation R and translation T**
3. Estimate 3D location of the reconstruction given RT
4. Optimize (bundle adjustment)
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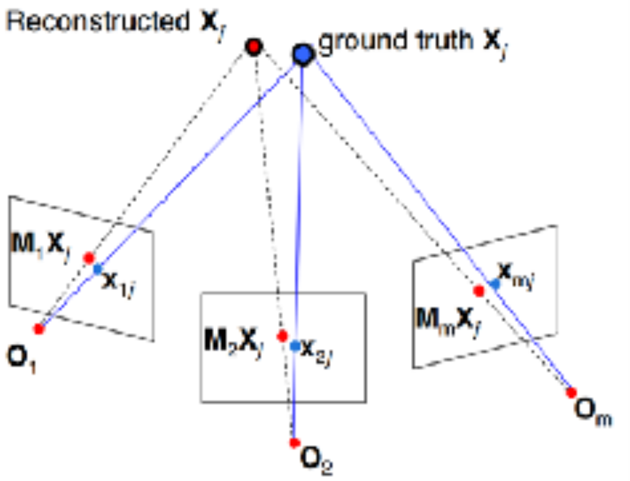
$$E(M, X) = \sum_{i=1}^{m} \sum_{j=1}^{n} D\left(x_{ij}, M_i X_j\right)^2$$



Problem 4: Structure from Motion

Use E to make initial estimate of relative rotation R and translation T

$$E = [T_\times] \cdot R$$

1. To compute R: Given the singular value decomposition $E = UDV^T$, we can rewrite $E = MQ$ where $M = UZU^T$ and $Q = UWV^T$ or UW^TV^T, where

$$Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Note that this factorization of E only guarantees that Q is orthogonal. To find a rotation, we simply compute $R = (\det Q)Q$.

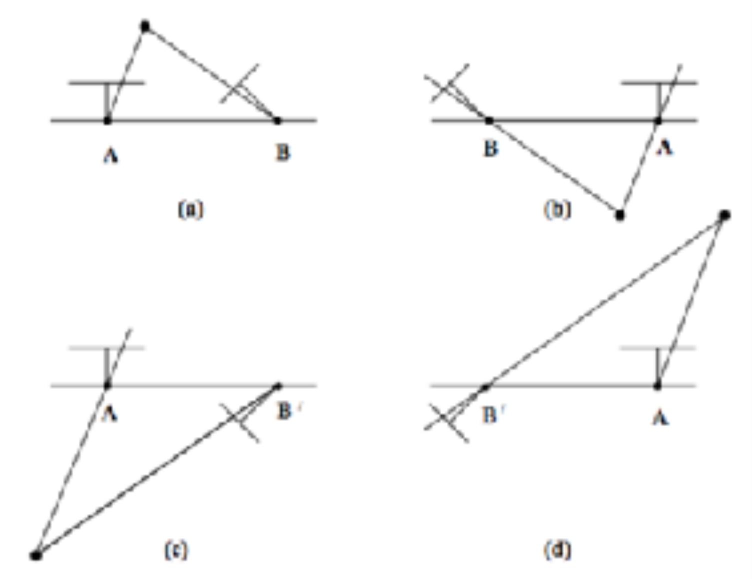
2. To compute T: Given that $E = U\Sigma V^T$, T is simply either u_3 or $-u_3$, where u_3 is the third column vector of U.

Problem 4: Structure from Motion

Use E to make initial estimate of relative rotation R and translation T

However, this gives four pairs of rotation and translation, $(R_1, R_2) \times (T, -T)$

How do we find out which R and T is the correct one?

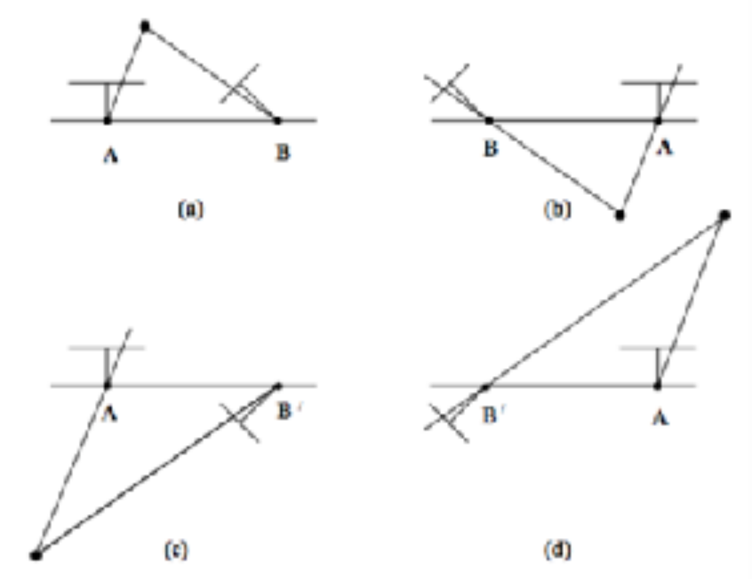


Problem 4: Structure from Motion

There exists only **one** solution that will consistently produce 3D points which are both in front of camera

Compute 3D point's location in the RT frame!

- Find 3D location of the image points given RT frame
- Chose the one which has the most 3D points with positive depth (z-coordinate) with respect to both camera frame



Problem 4: Structure from Motion

1. Compute fundamental matrix F from two views
2. **Use F to make initial estimate of relative rotation R and translation T**
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