Stochastic Differential Equations and Simulation of Brownian Motion

- Brownian motion (Wiener process)
- Martingales
- Itô and Stratonovich forms for SDEs
- Numerical solution of Itô SDE
- Examples: Black–Scholes and Ornstein–Uhlenbeck models
- Weak and strong accuracy measures for numerical solutions
- Stability of SDEs and numerical implementations

Introduction

- Stochastic differential equation (SDE):
  - Differential equation with at least one term in equation being stochastic process $\Rightarrow$ Solution of SDE is itself stochastic process.
- SDEs used to model diverse phenomena such as fluctuating stock prices, interest rates, physical systems subject to thermal fluctuations, etc.
- Early work by Maxwell and Boltzmann into random nature of gaseous motion; later work by luminaries such as Rayleigh, Einstein, Planck, Fokker, Kolmogorov, etc.
- SDEs are good example of “flaw of averages” principle from week 1: “Danger of Replacing Random Variables by Their Means”
  - Methods and results in SDEs very different from that in deterministic differential equations

2

Brownian Motion and Wiener Process

- "Brownian motion" named after botanist Robert Brown who, in 1827, observed random motion of pollen grains in water and hypothesized Gaussian-type model
- Brownian motion is continuous-time stochastic process
  - Time derivative does not exist (as arises in SDE)
  - Approximation to actual random physical processes that have time derivatives necessarily finite
- Mathematical form of Brownian motion: **Wiener process**
  - Terms "Brownian motion" and "Wiener process" used interchangeably, but Brownian motion often used for physical processes and Wiener process used for general stochastic process
- Standard Wiener process $W(t)$ is non-differentiable, continuous-time stochastic process over $[0, T]$ such that

  1. $W(0) = 0$,
  2. For $0 \leq s \leq t$, $W(t) - W(s) \sim N(0, t - s)$
  3. For $0 \leq s \leq t \leq u \leq v$, $W(t) - W(s)$ and $W(v) - W(u)$ are independent random variables (note: strict inequalities $s < t < u < v$ in Higham, 2001, are not needed)

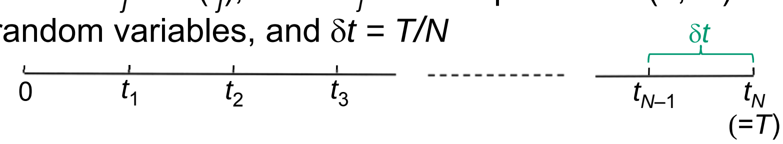
3

Discretization and Simulation of Wiener process

- Practical computation for $W(t)$ is available from discretized process:

$$W_j = W_{j-1} + dW_j\,,\ j = 1, 2, \ldots, N, \qquad (*)$$

  where $W_j = W(t_j)$, the dW_j are independent $N(0, \delta t)$ random variables, and $\delta t = T/N$

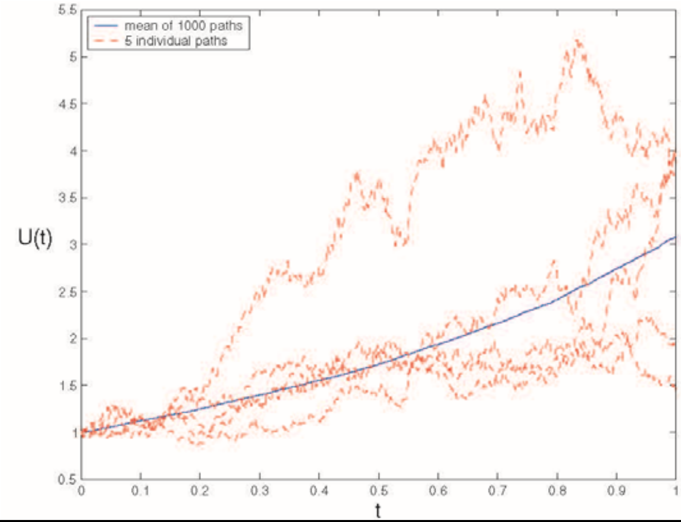


- Note that var(W_j) from (*) above is same as var($W(t_j)$) from definition of Wiener process
- Recursion (*) provides convenient mechanism for simulating Brownian motion
  - Generate dW_j by Monte Carlo and produce values of W_j according to (*)
- Example next slide

4

Example Simulation of Brownian Motion

- Discretized solution for $U(t) \equiv e^{[t+W(t)/2]}$; $\delta t = T/N = 1/500$
- $E[U(t)] = e^{9t/8}$; blue (solid) curve (average of 1000 runs) has maximum error of 0.0504 relative to exact mean $e^{9t/8} \geq 1$



Source: Fig. 2 in Higham (2001)

5

Martingales

- Martingales somewhat analogous to Markov processes; but conditions (and results) weaker than Markov
- Martingales provide powerful theory for convergence and bounds in stochastic processes
  - Brownian motion (above) is martingale; Itô integrals (defined below) ***may*** be martingale
- Consider scalar-valued stochastic process $Z(t)$. Then $Z(t)$ is a martingale if for any set of times $\{t_0, t_1, \ldots, t_n\}$, following hold for all $0 \leq k \leq n$:

  $E[|Z(t_k)|] < \infty,$

  $E[Z(t_{k+1})|Z(t_0), Z(t_1), \ldots, Z(t_k)] = Z(t_k)$
- Note: above definition is a discretized version of formal definition of martingales in continuous time, which relies on measure-theoretic concepts
- Example: Discretized Brownian motion (Weiner process)

6

Itô and Stratonovich Forms for Use in SDEs

- Path of Wiener process not differentiable w.r.t. time (i.e., $dW(t)/dt$ does not exist)
- Above fact has strong implications for interpretation and solution of SDEs
- Two methods for coping with above non-differentiability:
  - Itô calculus (K. Itô, 1944, *Proc. Imperial Acad. Tokyo*)
  - Stratonovich calculus (R. Stratonovich, 1966, *SIAM J. Control*)
- Itô calculus usual choice in applied mathematics; Stratonovich integral frequently used in physics
- Neither “better” than other:
  - Itô allows use of powerful martingale theory
  - Stratonovich allows standard calculus arguments like chain rule (Itô does not)
- Itô form more natural in “real-time” applications (i.e., no information about future)
  - Financial mathematics usually relies on Itô form
- Itô discussed below

Standard Itô Form of SDE

- Standard Itô form of SDE is

$$\boxed{dX(t) = f(X(t))dt + g(X(t))dW(t)\ ,}$$

  where f and g are specified scalar functions
- Above is largely a notational device to represent the integral equation:

$$X(t) = X(0) + \int_0^t f(X(s))ds + \int_0^t g(X(s))dW(s)$$

$$= \text{i.c.} + \text{"regular" integral} + \text{Itô integral}$$

- Not a priori obvious that the term labeled “Itô integral” above is meaningful
- However, under conditions defining Wiener process (earlier slide), know that $\int g(X(s))dW(s)$ can be properly interpreted as limiting value of sum of increments
- Note that $X(t)$ is stochastic process and $X(t)$ is a random variable at any t

Example 1 of Itô SDE: Linear SDE

- Special case of above SDE is model that is linear in $X(t)$, sometimes called geometric Brownian motion:

$$dX(t) = \lambda X(t)dt + \mu X(t)dW(t), \qquad (\#)$$

  where λ and μ are real-valued constants
- Eqn (#) linear in X (general Itô SDE allows nonlinearity)
- Eqn (#) used to model stock prices in **Black–Scholes model** in mathematical finance for option pricing (X represents asset price)
  - Published by F. Black and M. Scholes in 1973, "The Pricing of Options and Corporate Liabilities," *J. of Political Economy*
  - Basis of 1997 Nobel Prize in Economics to Merton and Scholes
- Closed-form solution to (#):

$$X(t) = X(0)e^{\left[(\lambda-\mu^2/2)t+\mu W(t)\right]}$$

- Note "geometric" (exponential) function of Brownian motion

9

Example 2 of Itô SDE: Ornstein–Uhlenbeck Process

- Ornstein–Uhlenbeck process describes velocity of Brownian particle under influence of friction:

$$dX(t) = \theta(\mu - X(t))dt + \sigma dW(t), \qquad (\wedge)$$

  where $\theta > 0$, μ is mean value, and $\sigma > 0$ are constants
- O-U process applied in physics and finance
  - In finance, O-U used to model interest rates, etc.: μ represents equilibrium value, σ is degree of volatility around mean due to shocks, θ is rate that shocks dissipate and process reverts towards mean
- Solution to (^) is (Jazwinski, 1970, p. 123):

$$X(t) = X(0)e^{-\theta t} + \mu(1-e^{-\theta t}) + \int_0^t \sigma e^{\theta(s-t)}dW(s) \qquad (\wedge\wedge)$$

- Process mean converges to fixed value μ ("mean-reverting")
  - In particular, from (^^) above, we find $E[X(t)] \to \mu$ and $\text{var}[X(t)] \to \sigma^2/(2\theta)$ as $t \to \infty$

10

Numerical Solution of Itô SDE

- Consider time interval $[0, T]$
- Euler–Maruyama (E-M) method provides "natural" means for numerical solution
- Consider equidistant time points $0 = \tau_0 < \tau_1 < \ldots\ldots < \tau_L = T$; let $\Delta t = \tau_j - \tau_{j-1}$ for all j
- Let X_j be numerical approximation to exact value $X(\tau_j)$
- Then, Monte Carlo simulation-based approximation to solution of SDE for $j = 1, 2, \ldots, L$ is:

$$\boxed{X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{j-1})\Delta W_j\,, \qquad (*)}$$

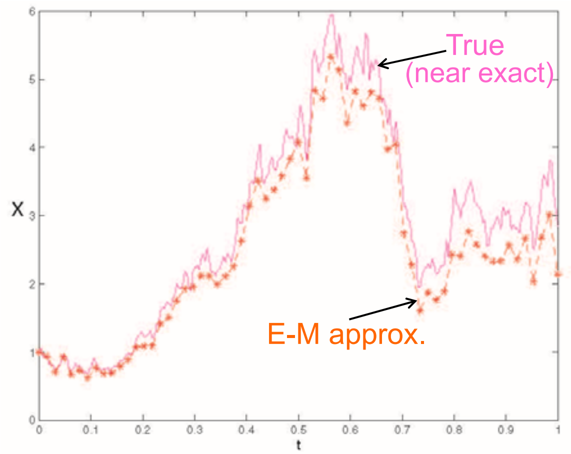
where $\Delta W_j = W(\tau_j) - W(\tau_{j-1}) \sim N(0, \Delta t)$

- Approximation (*) is Markov process driven by i.i.d. ΔW_j
- Note: if $g \equiv 0$, then (*) is standard Euler's method for solving (deterministic) ODE

11

Example of Euler–Maruyama Method: Linear SDE (see Example 1 above)

- Compare E-M method with near-exact ("true") solution to above geometric Brownian motion (#) over time [0, 1]
- Generated discretized Brownian motion with intervals = 1/256; used $\lambda = 2$, μ =1 in (#) and $\Delta t = 4/256$ in E-M



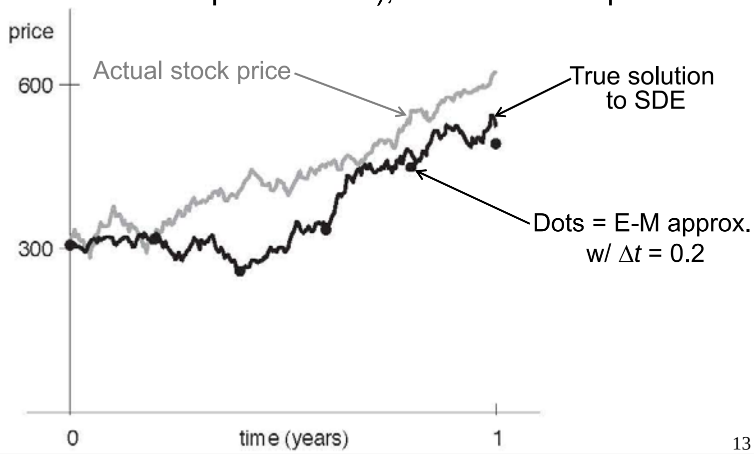
Source: Fig. 3 in Higham (2001)

12

Example of Euler–Maruyama Method: Linear SDE with Real Data (Google Stock Price)

- Consider model with λ = 0.75 and μ = 0.30, inferred from market close share prices of Google (NYSE symbol GOOG) during 250 trading days in 2009
- Plot below compares E-M solution, exact solution (using Brownian motion with step size << Δt), and real stock price

Conclusion: One realization of SDE (a stochastic process) may not match exact stock price and E-M is accurate in this case



Source: Fig. 1 in Sauer (2012)

13

Weak and Strong Measures of Accuracy to Numerical Solutions of Itô SDE

- In above, discussed popular numerical method (E-M) for solving SDEs, leading to $X_j \approx X(\tau_j)$
- Two main types of accuracy to consider:
  1. **Strong:** Approximations to individual ***sample paths*** for $X(t)$
  2. **Weak:** Approximations to ***distributional properties*** for $X(t)$ (such as mean value across time)
- Example where need strong convergence: Tasks involving need for a particular solution path for $X(t)$, such as generation of path for particular stock price, generation of data for state estimate (e.g., Kalman filter), etc.
- Example where only need weak convergence: Tasks involving need for general characteristics of $X(t)$, such as moments (means and variances), probabilities associated with $X(t)$, etc.

14

Weak and Strong Measures of Accuracy to Numerical Solutions of Itô SDE, cont'd

- Recall step size $\Delta t = \tau_j - \tau_{j-1}$ used in E-M method
- Strong convergence states that for some $C > 0$ and all $j = 0, 1, \ldots, L$,

$$E(|X_j - X(\tau_j)|) \leq C(\Delta t)^{1/2}$$

- Weak convergence states

$$|E(X_j) - E(X(\tau_j))| \leq C\Delta t$$

  (C here not necessarily same as C for strong convergence)
- Can use above strong convergence to make probabilistic statements on accuracy via Markov inequality (p. 537 of Higham, 2001)
- Methods other than E-M may yield higher rates of convergence
  - Higher-order method (Milstein) can yield strong convergence with exponent = 1 (vs. ½)

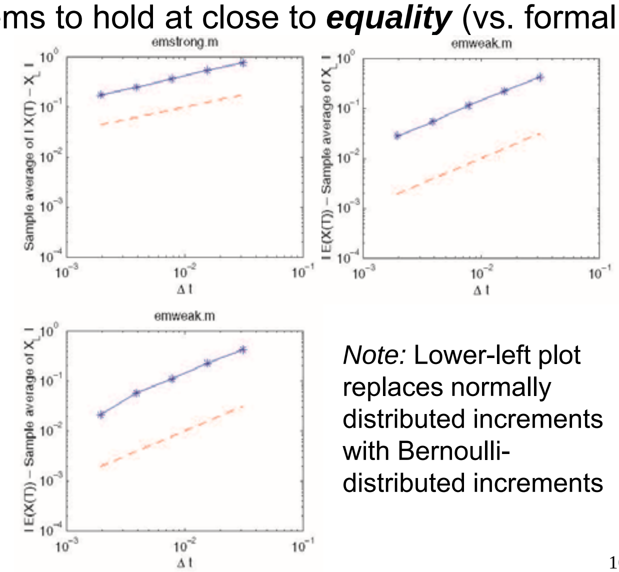
15

Illustration of Weak and Strong Measures of Accuracy for Linear SDE in Example 1

- Log-log plots below show that exponent ½ in strong law and 1 in weak law seems to hold at close to ***equality*** (vs. formal inequality)

Blue lines are empirical values; dashed red lines show exact slope ½ (for strong) or 1 (for weak)

Source: Fig. 4 in Higham (2001)



Note: Lower-left plot replaces normally distributed increments with Bernoulli-distributed increments

16

Stability in Case of Linear SDE in Example 1

- Recall linear model $dX(t) = \lambda X(t)dt + \mu X(t)dW(t)$
- Interested in conditions under which $X(t) \to 0$ in some stochastic sense as $t \to \infty$
- Special deterministic case of model ($\mu = 0$) implies $X(t) \to 0$ if $\mathrm{Re}(\lambda) < 0$, when λ is complex
- Stochastic analogue:
  - Mean-squared stability: $E[X(t)^2] \to 0$
  - Asymptotic stability: $X(t) \to 0$ with probability one (sometimes called "almost surely")
- Numerical convergence with EM scheme must account for both fundamental SDE stability and numerical algorithm
  - Stability of SDE is necessary condition
  - Problems that are stable with EM are subset of those that have stable SDEs

17

Stability in Case of Linear SDE (cont'd)

- Stability of SDE uses solution from previous slide:

$$X(t) = X(0)e^{\left[(\lambda - \mu^2/2)t + \mu W(t)\right]}$$

- If $X(0) \neq 0$, then $X(t)$ is mean-squared or asymptotically stable according to:

$$\lim_{t\to\infty} \mathbb{E}\, X(t)^2 = 0 \Leftrightarrow \Re\{\lambda\} + \tfrac{1}{2}|\mu|^2 < 0$$

$$\lim_{t\to\infty} |X(t)| = 0, \text{ with probability } 1 \Leftrightarrow \Re\{\lambda - \tfrac{1}{2}\mu^2\} < 0$$

- Stability for EM method follows from special case of implementation for linear model:

$$X_j = X_{j-1} + \lambda X_{j-1}\Delta t + \mu X_{j-1}\Delta W_{j-1}$$

- Conditions for mean-squared and asymptotic stability of EM implementation:

$$\lim_{j\to\infty} \mathbb{E}\, X_j^2 = 0 \Leftrightarrow |1 + \Delta t\lambda|^2 + \Delta t|\mu|^2 < 1$$

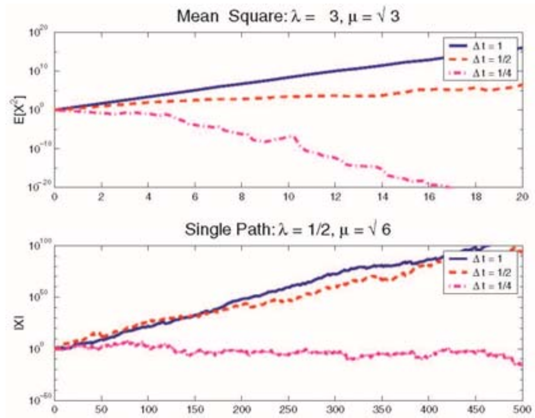
$$\lim_{j\to\infty} |X_j| = 0, \text{ with probability } 1 \Leftrightarrow \mathbb{E}\, \log\left|1 + \Delta t\lambda + \sqrt{\Delta t}\mu N(0,1)\right| < 0$$

18

Example of Stability and Instability

- Consider special cases of linear model $dX(t) = \lambda X(t)dt + \mu X(t)dW(t)$ with $\lambda = 3$, $\mu = 3^{1/2}$ for mean-squared stability and $\lambda = 1/2$, $\mu = 6^{1/2}$ for asymptotic stability
- Using conditions for EM stability (previous slide), can show that from $\Delta t \in \{1/4, 1/2, 1\}$, only $\Delta t = 1/4$ satisfies conditions for stability

Plots show numerical experiments on setting above. Only $\Delta t = 1/4$ shows stable behavior



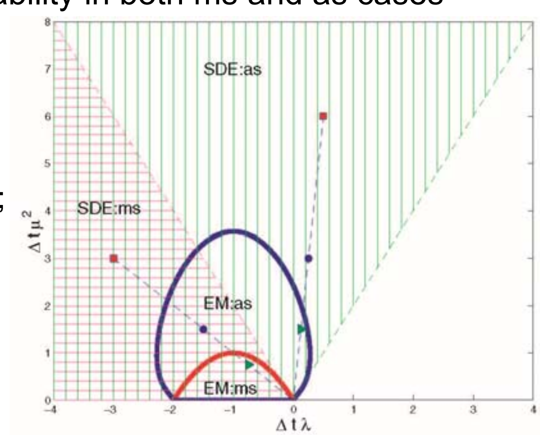
Source: Fig. 5 in Higham (2001)

19

Example of Stability and Instability (cont'd)

- Plot below shows region for mean-squared (ms) and asymptotic (as) stability in linear model
- Note that EM regions for stability are subsets of SDE regions for stability in both ms and as cases

Rays and points in plot pertain to specific example on previous slide; green triangles correspond to stable cases with $\Delta t = 1/4$



Source: Fig. 6 in Higham (2001)

20

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21