The Pythagorean Theorem

❖ A 4,000-Year History

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Mesopotamia, 1800 BCE

We would more properly have to call "Babylonian" many things which the Greek tradition had brought down to us as "Pythagorean."

—Otto Neugebauer, quoted in Bartel van der Waerden, *Science Awakening*, p. 77

The vast region stretching from the Euphrates and Tigris Rivers in the east to the mountains of Lebanon in the west is known as the Fertile Crescent. It was here, in modern Iraq, that one of the great civilizations of antiquity rose to prominence four thousand years ago: Mesopotamia. Hundreds of thousands of clay tablets, found over the past two centuries, attest to a people who flourished in commerce and architecture, kept accurate records of astronomical events, excelled in the arts and literature, and, under the rule of Hammurabi, created the first legal code in history. Only a small fraction of this vast archeological treasure trove has been studied by scholars; the great majority of tablets lie in the basements of museums around the world, awaiting their turn to be deciphered and give us a glimpse into the daily life of ancient Babylon.

Among the tablets that have received special scrutiny is one with the unassuming designation "YBC 7289," meaning that it is tablet number 7289 in the Babylonian Collection of Yale University (fig. 1.1). The tablet dates from the Old Babylonian period of the Hammurabi dynasty, roughly 1800–1600 BCE. It shows a tilted square and its two diagonals, with some marks engraved along one side and under the horizontal diagonal. The marks are in cuneiform (wedge-shaped) characters, carved with a stylus into a piece of soft clay which was then dried in the sun or baked in an oven. They turn out to be numbers, written in the peculiar Babylonian numeration system that used the base 60. In this *sexagesimal system*, numbers up to 59 were written in essentially our modern base-ten numeration system, but without a zero. Units were written as vertical Y-shaped notches, while tens were marked with similar notches written horizontally. Let us denote these symbols by | and —, respectively. The number 23, for example, would be written as — — | | |. When a number exceeded 59,

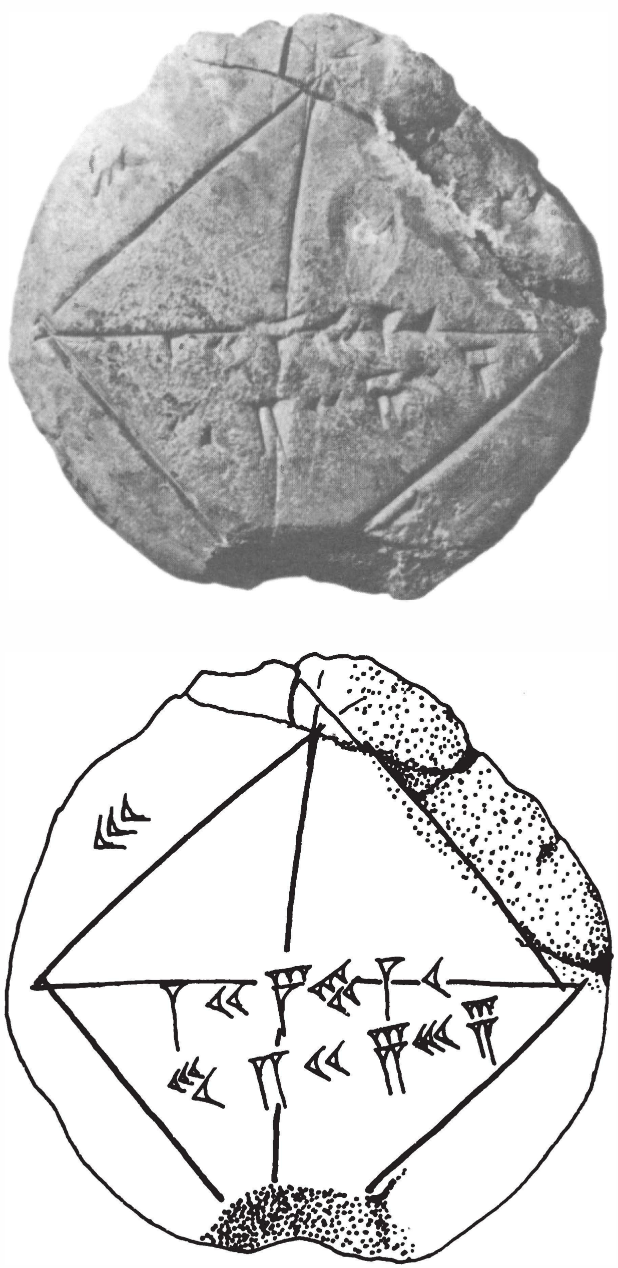

Figure 1.1. YBC 7289

it was arranged in groups of 60 in much the same way as we bunch numbers into groups of ten in our base-ten system. Thus, 2,413 in the sexagesimal system is $40 \times 60 + 13$, which was written as — — — —   — | | | (often a group of several identical symbols was stacked, evidently to save space).

Because the Babylonians did not have a symbol for the "empty slot"—our modern zero—there is often an ambiguity as to how the numbers should be grouped. In the example just given, the numerals — — — —   — | | | could also stand for $40 \times 60^2 + 13 \times 60 = 144{,}780$; or they could mean $40/60 + 13 = 13.166$, or any other combination of powers of 60 with the coefficients 40 and 13. Moreover, had the scribe made the space between — — — — and — | | | too small, the number might have erroneously been read as — — — — — | | |, that is, $50 \times 60 + 3 = 3{,}003$. In such cases the correct interpretation must be deduced from the context, presenting an additional challenge to scholars trying to decipher these ancient documents.

Luckily, in the case of YBC 7289 the task was relatively easy. The number along the upper-left side is easily recognized as 30. The one immediately under the horizontal diagonal is 1;24,51,10 (we are using here the modern notation for writing Babylonian numbers, in which commas separate the sexagesimal "digits," and a semicolon separates the integral part of a number from its fractional part). Writing this number in our base-10 system, we get $1 + 24/60 + 51/60^2 + 10/60^3 = 1.414213$, which is none other than the decimal value of $\sqrt{2}$, accurate to the nearest one hundred thousandth! And when this number is multiplied by 30, we get 42.426389, which is the sexagesimal number 42;25,35—the number on the second line below the diagonal. The conclusion is inescapable: the Babylonians knew the relation between the length of the diagonal of a square and its side, $d = a\sqrt{2}$. But this in turn means that they were familiar with the Pythagorean theorem—or at the very least, with its special case for the diagonal of a square ($d^2 = a^2 + a^2 = 2a^2$)—more than a thousand years before the great sage for whom it was named.

Two things about this tablet are especially noteworthy. First, it proves that the Babylonians knew how to compute the square root of a number to a remarkable accuracy—in fact, an accuracy equal to that of a modern eight-digit calculator.[1] But even more remarkable is the probable purpose of this particular document: by all likelihood, it was intended as an example of how to find the diagonal of *any* square: simply multiply the length of the side by 1;24,51,10. Most people, when given this task, would follow the "obvious" but more tedious route: start with 30, square it, double the result, and take the square root: $d = \sqrt{30^2 + 30^2} = \sqrt{1800} = 42.4264$, rounded to four places. But suppose you had to do this over and over for squares of different sizes; you would have to repeat the process each time with a new number, a rather tedious task. The anonymous scribe who carved these numbers into a clay tablet nearly four thousand years ago showed us a simpler way: just multiply the side of the square by $\sqrt{2}$ (fig. 1.2). Some simplification!

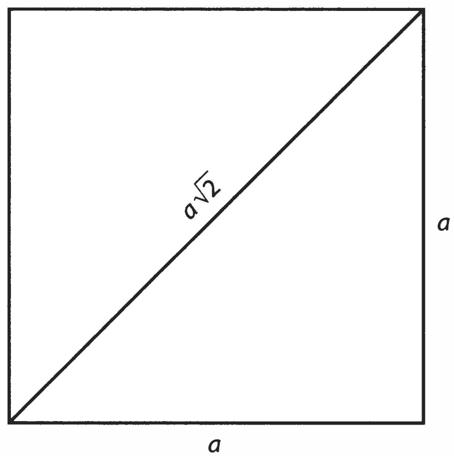

Figure 1.2. A square and its diagonal

But there remains one unanswered question: why did the scribe choose a side of 30 for his example? There are two possible explanations: either this tablet referred to some particular situation, perhaps a square field of side 30 for which it was required to find the length of the diagonal; or—and this is more plausible—he chose 30 because it is one-half of 60 and therefore lends itself to easy multiplication. In our base-ten system, multiplying a number by 5 can be quickly done by halving the number and moving the decimal point one place to the right. For example, $2.86 \times 5 = (2.86/2) \times 10 = 1.43 \times 10 = 14.3$ (more generally, $a \times 5 = \frac{a}{2} \times 10$). Similarly, in the sexagesimal system multiplying a number by 30 can be done by halving the number and moving the "sexagesimal point" one place to the right ($a \times 30 = \frac{a}{2} \times 60$).

Let us see how this works in the case of YBC 7289. We recall that 1;24,51,10 is short for $1 + 24/60 + 51/60^2 + 10/60^3$. Dividing this by 2, we get $\frac{1}{2} + \frac{12}{60} + \frac{25\frac{1}{2}}{60^2} + \frac{5}{60^3}$, which we must rewrite so that each coefficient of a power of 60 is an integer. To do so, we replace the 1/2 in the first and third terms by by 30/60, getting $\frac{30}{60} + \frac{12}{60} + \frac{25+\frac{30}{60}}{60^2} + \frac{5}{60^3} = \frac{42}{60} + \frac{25}{60^2} + \frac{35}{60^3} = 0;42,25,35$. Finally, moving the sexagesimal point one place to the right gives us 42;25,35, the length of the diagonal. It thus seems that our scribe chose 30 simply for pragmatic reasons: it made his calculations that much easier.

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If YBC 7289 is a remarkable example of the Babylonians' mastery of elementary geometry, another clay tablet from the same period goes even further: it shows that they were familiar with algebraic procedures as well.[2] Known as

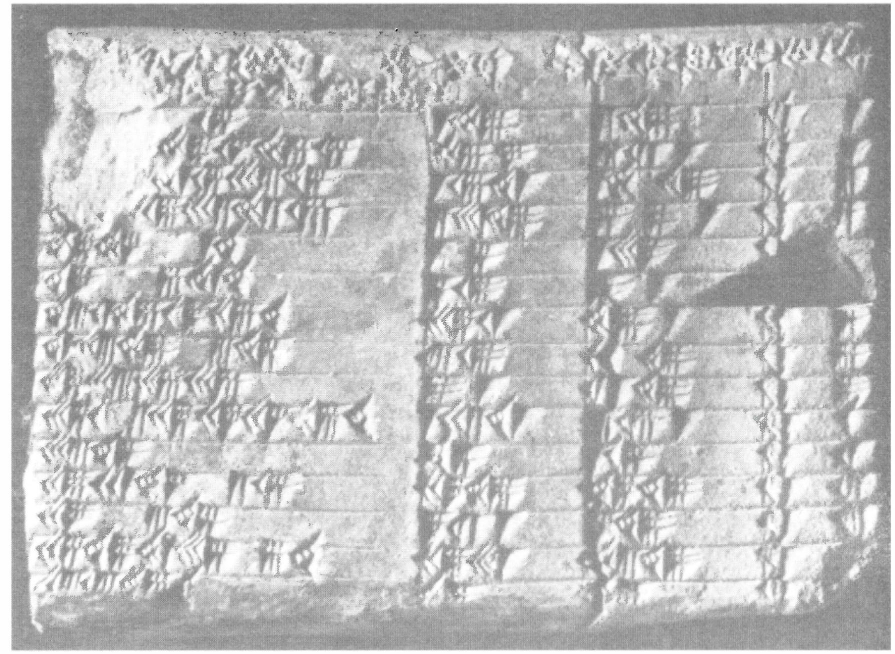

Figure 1.3. Plimpton 322

Plimpton 322 (so named because it is number 322 in the G. A. Plimpton Collection at Columbia University; see fig. 1.3), it is a table of four columns, which might at first glance appear to be a record of some commercial transaction. A close scrutiny, however, has disclosed something entirely different: the tablet is a list of *Pythagorean triples*, positive integers (a, b, c) such that $a^2 + b^2 = c^2$. Examples of such triples are (3, 4, 5), (5, 12, 13), and (8, 15, 17). Because of the Pythagorean theorem,[3] every such triple represents a right triangle with sides of integer length.

Unfortunately, the left edge of the tablet is partially missing, but traces of modern glue found on the edges prove that the missing part broke off after the tablet was discovered, raising the hope that one day it may show up on the antiquities market. Thanks to meticulous scholarly research, the missing part has been partially reconstructed, and we can now read the tablet with relative ease. Table 1.1 reproduces the text in modern notation. There are four columns, of which the rightmost, headed by the words "its name" in the original text, merely gives the sequential number of the lines from 1 to 15. The second and third columns (counting from right to left) are headed "solving number of the diagonal" and "solving number of the width," respectively; that is, they give the length of the diagonal and of the short side of a rectangle, or equivalently, the length of the hypotenuse and the short leg of a right triangle. We will label these columns with the letters c and b, respectively. As

Maor- The Pythag. Thm.

TABLE 1.1
Plimpton 322

$(c/a)^2$	b	c	
[1,59,0,]15	1,59	2,49	1
[1,56,56,]58,14,50,6,15	56,7	3,12,1	2
[1,55,7,]41,15,33,45	1,16,41	1,50,49	3
[1,]5[3,1]0,29,32,52,16	3,31,49	5,9,1	4
[1,]48,54,1,40	1,5	1,37	5
[1,]47,6,41,40	5,19	8,1	6
[1,]43,11,56,28,26,40	38,11	59,1	7
[1,]41,33,59,3,45	13,19	20,49	8
[1,]38,33,36,36	9,1	12,49	9
1,35,10,2,28,27,24,26,40	1,22,41	2,16,1	10
1,33,45	45	1,15	11
1,29,21,54,2,15	27,59	48,49	12
[1,]27,0,3,45	7,12,1	4,49	13
1,25,48,51,35,6,40	29,31	53,49	14
[1,]23,13,46,40	56	53	15

Note: The numbers in brackets are reconstructed.

an example, the first line shows the entries $b = 1{,}59$ and $c = 2{,}49$, which represent the numbers $1 \times 60 + 59 = 119$ and $2 \times 60 + 49 = 169$. A quick calculation gives us the other side as $a = \sqrt{169^2 - 119^2} = \sqrt{14400} = 120$; hence (119, 120, 169) is a Pythagorean triple. Again, in the third line we read $b = 1{,}16{,}41 = 1 \times 60^2 + 16 \times 60 + 41 = 4601$, and $c = 1{,}50{,}49 = 1 \times 60^2 + 50 \times 60 + 49 = 6649$; therefore, $a = \sqrt{6649^2 - 4601^2} = \sqrt{23\,040\,000} = 4800$, giving us the triple (4601, 4800, 6649).

The table contains some obvious errors. In line 9 we find $b = 9{,}1 = 9 \times 60 + 1 = 541$ and $c = 12, 49 = 12 \times 60 + 49 = 769$, and these do not form a Pythagorean triple (the third number a not being an integer). But if we replace the 9,1 by $8{,}1 = 481$, we do indeed get an integer value for a: $a = \sqrt{769^2 - 481^2} = \sqrt{360\,000} = 600$, resulting in the triple (481, 600, 769). It seems that this error was simply a "typo"; the scribe may have been momentarily distracted and carved nine marks into the soft clay instead of eight; and once the tablet dried in the sun, his oversight became part of recorded history.

Newsome Note: Table 1.1 doesn't make any corrections to the errors on Plimpton 322.

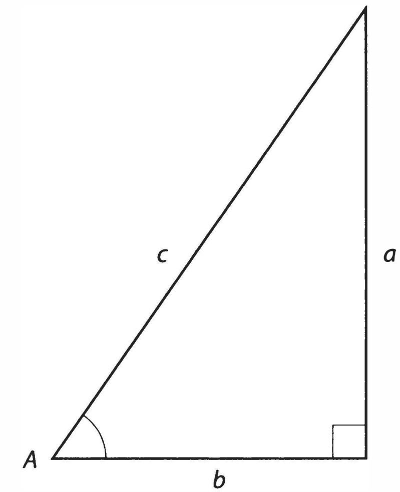

Figure 1.4. The cosecant of an angle: csc $A = c/a$

Again, in line 13 we have $b = 7{,}12{,}1 = 7 \times 60^2 + 12 \times 60 + 1 = 25\,921$ and $c = 4{,}49 = 4 \times 60 + 49 = 289$, and these do not form a Pythagorean triple; but we may notice that 25 921 is the square of 161, and the numbers 161 and 289 do form the triple (161, 240, 289). It seems the scribe simply forgot to take the square root of 25 921. And in row 15 we find $c = 53$, whereas the correct entry should be twice that number, that is, $106 = 1{,}46$, producing the triple (56, 90, 106).[4] These errors leave one with a sense that human nature has not changed over the past four thousand years; our anonymous scribe was no more guilty of negligence than a student begging his or her professor to ignore "just a little stupid mistake" on the exam.[5]

The leftmost column is the most intriguing of all. Its heading again mentions the word "diagonal," but the exact meaning of the remaining text is not entirely clear. However, when one examines its entries a startling fact comes to light: this column gives the square of the ratio c/a, that is, the value of $\csc^2 A$, where A is the angle opposite side a and csc is the cosecant function studied in trigonometry (fig. 1.4). Let us verify this for line 1. We have $b = 1{,}59 = 119$ and $c = 2{,}49 = 169$, from which we find $a = 120$. Hence $(c/a)^2 = (169/120)^2 = 1.983$, rounded to three places. And this indeed is the corresponding entry in column 4: $1;59{,}0{,}15 = 1 + 59/60 + 0/60^2 + 15/60^3 = 1.983$. (We should note again that the Babylonians did not use a symbol for the "empty slot" and therefore a number could be interpreted in many different ways; the correct interpretation must be deduced from the context. In the example just cited, we assume that the leading 1 stands for units rather than sixties.) The reader may check other entries in this column and confirm that they are equal to $(c/a)^2$.

Several questions immediately arise: Is the order of entries in the table random, or does it follow some hidden pattern? How did the Babylonians find those particular numbers that form Pythagorean triples? And why were they interested in these numbers—and in particular, in the ratio $(c/a)^2$—in the first place? The first question is relatively easy to answer: if we compare the values of $(c/a)^2$ line by line, we discover that they decrease steadily from 1.983 to 1.387, so it seems likely that the order of entries was determined by this sequence. Moreover, if we compute the square root of each entry in column 4—that is, the ratio $c/a = \csc A$—and then find the corresponding angle A, we discover that A *increases* steadily from just above 45° to 58°. It therefore seems that the author of this text was not only interested in finding Pythagorean triples, but also in determining the ratio c/a of the corresponding right triangles. This hypothesis may one day be confirmed if the missing part of the tablet shows up, as it may well contain the missing columns for a and c/a. If so, Plimpton 322 will go down as history's first trigonometric table.

As to how the Babylonian mathematicians found these triples—including such enormously large ones as (4601, 4800, 6649)—there is only one plausible explanation: they must have known an algorithm which, 1,500 years later, would be formalized in Euclid's *Elements*: Let u and v be any two positive integers, with $u > v$; then the three numbers

$$a = 2uv, \quad b = u^2 - v^2, \quad c = u^2 + v^2 \tag{1}$$

form a Pythagorean triple. (If in addition we require that u and v are of opposite parity—one even and the other odd—and that they do not have any common factor other than 1, then (a, b, c) is a *primitive* Pythagorean triple, that is, a, b, and c have no common factor other than 1.) It is easy to confirm that the numbers a, b, and c as given by equations (1) satisfy the equation $a^2 + b^2 = c^2$:

$$\begin{aligned} a^2 + b^2 &= (2uv)^2 + (u^2 - v^2)^2 \\ &= 4u^2v^2 + u^4 - 2u^2v^2 + v^4 \\ &= u^4 + 2u^2v^2 + v^4 \\ &= (u^2 + v^2)^2 = c^2. \end{aligned}$$

The converse of this statement—that *every* Pythagorean triple can be found in this way—is a bit harder to prove (see Appendix B).

Plimpton 322 thus shows that the Babylonians were not only familiar with the Pythagorean theorem, but that they knew the rudiments of number theory and had the computational skills to put the theory into practice—quite remarkable for a civilization that lived a thousand years before the Greeks produced their first great mathematician.

Notes and Sources

1. For a discussion of how the Babylonians approximated the value of $\sqrt{2}$, see Appendix A.

2. The text that follows is adapted from *Trigonometric Delights* and is based on

Otto Neugebauer, *The Exact Sciences in Antiquity* (1957; rpt. New York: Dover, 1969), chap. 2. See also Eves, pp. 44–47.

3. More precisely, its *converse*: if the sides of a triangle satisfy the equation $a^2 + b^2 = c^2$, the triangle is a right triangle.

4. This, however, is not a *primitive triple*, since its members have the common factor 2; it can be reduced to the simpler triple (28, 45, 53). The two triples represent similar triangles.

5. A fourth error occurs in line 2, where the entry 3,12,1 should be 1,20,25, producing the triple (3367, 3456, 4825). This error remains unexplained.