Review of DFT's Usage

- We have seen that DFT can be used to compute the spectrum of DTFT.
  - Note that CFT (or DTFT) integrates from $-\infty$ to ∞, they are not computationally feasible in practice.
- However, what is computed by DFT is only an approximation of the spectrum in most cases.

Approximations by DFT

- How it approximations?
  - DFT only evaluates in a finite-length rectangle window. (windowing effect)
  - DFT only evaluates on discrete-time samples. (aliasing effect).
- So,
  - **Approximation of CFT by DFT**: both windowing effect and aliasing effect
  - **Approximation of DTFT by DFT**: only windowing effect.
  - **(Approximation of CFT by DTFT**: aliasing effect)

Approximate DTFT by DFT

- DFT is an approximation for DTFT for infinite-long sequences or sequences with the durations longer than the rectangular window.
- Exact recovery case: However, for a finite-duration signal, its DTFT can be exactly recovered by its DFT.
  - In this case, DFT computes the samples of DTFT in the locations $2\pi k/N$ $(k = 0, \dots, (N-1))$ in the range [0,2π] along the unit circle in the polar-coordinat plane, where N is the length of the finite-duration signal.
  - To increase the sampling points, zero-padding can be used.

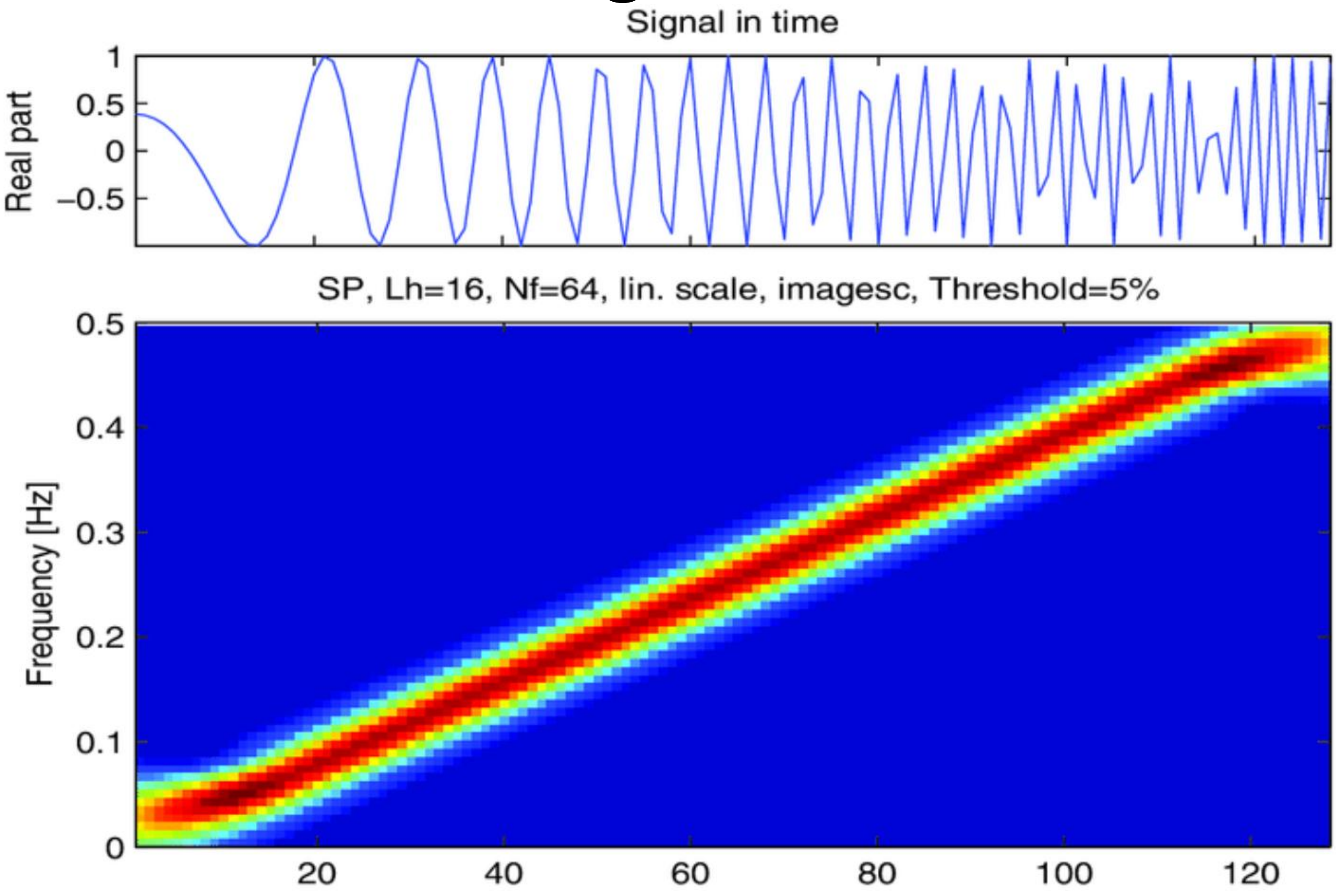
Approximation by better windowing function

- Instead of the rectangle window, we have mentioned that other windowing functions could be used, so that a better approximation or tradeoff of the spectrum can be achieved.
- One of the major shortcomings of the rectangular window is that its sidelobes are relatively high and could be confused with low amplitude peaks caused by other frequencies.
- See below

Review of Spectrogram

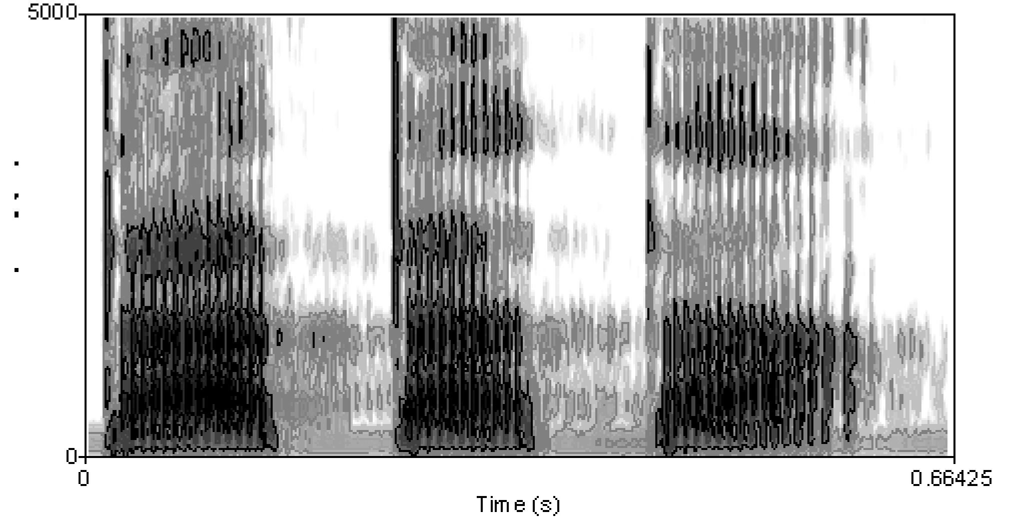
- A discrete-time version of the short-time Fourier Transform (STFT).
- Separating a discrete-time signal into **multiple segments** in time, and compute the DFT/FFT for each segment along the time axis.
- The segments can be either **non-overlapping** or **overlapping**.
- Spectrogram is a two-dimensional map
- We use rectangular window before, and other window functions can be used as well.

Example: Spectrogram of a Chirp Signal

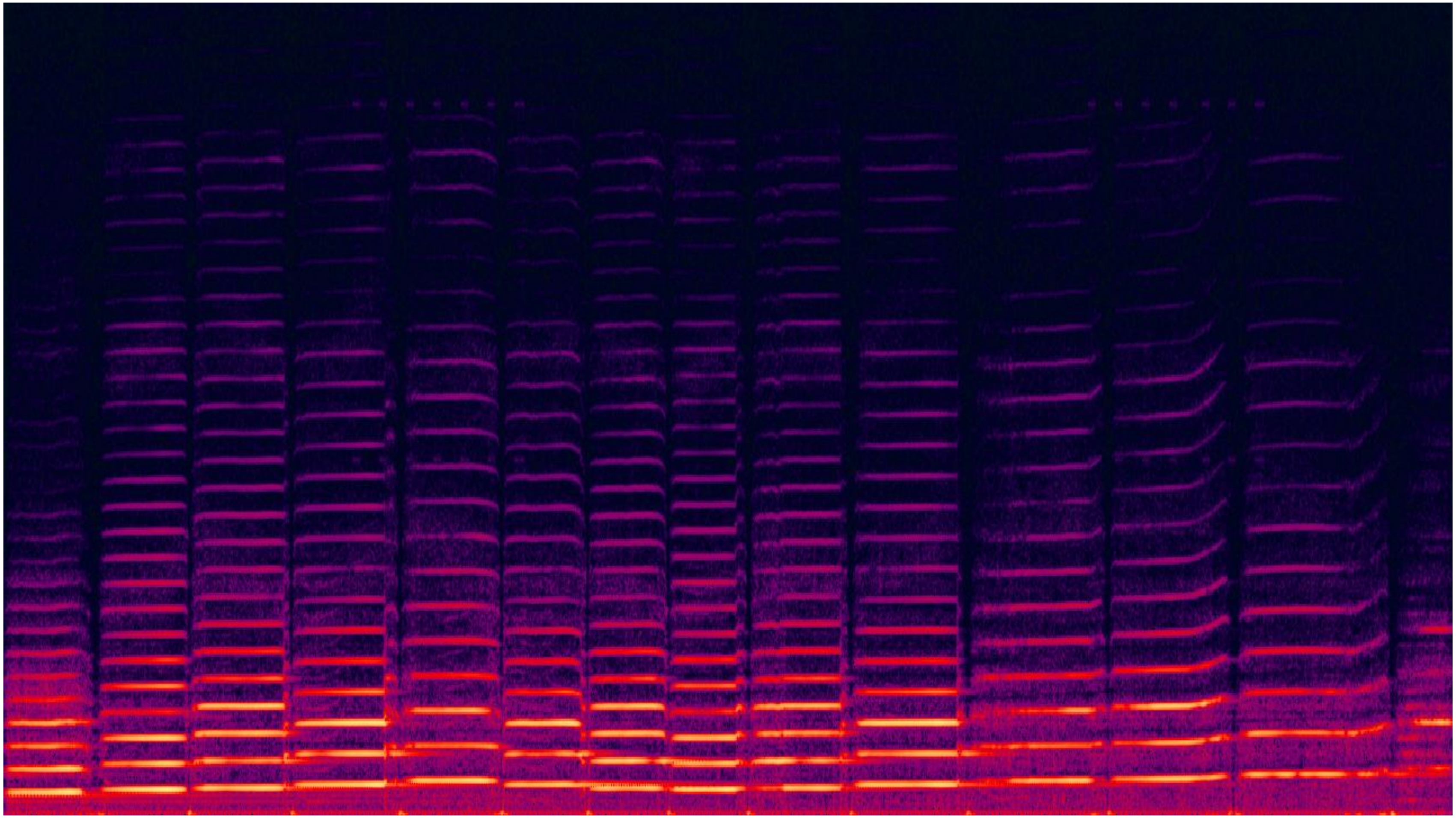


From [EEG Signal Processing for Epilepsy]

Example: Spectrogram of a male voice saying 'ta ta ta'.



Example: Spectrogram of a violin playing [this recording of a violin playing]



Mainlobe and sidelobes of rectangular window

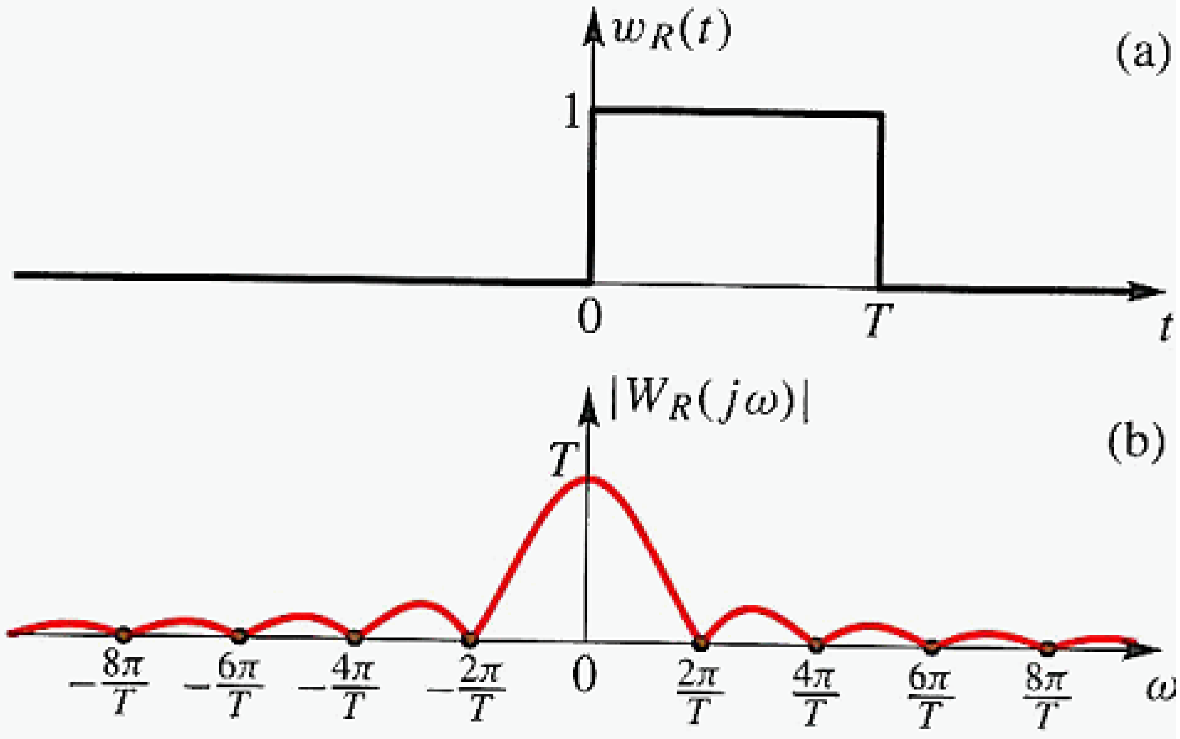
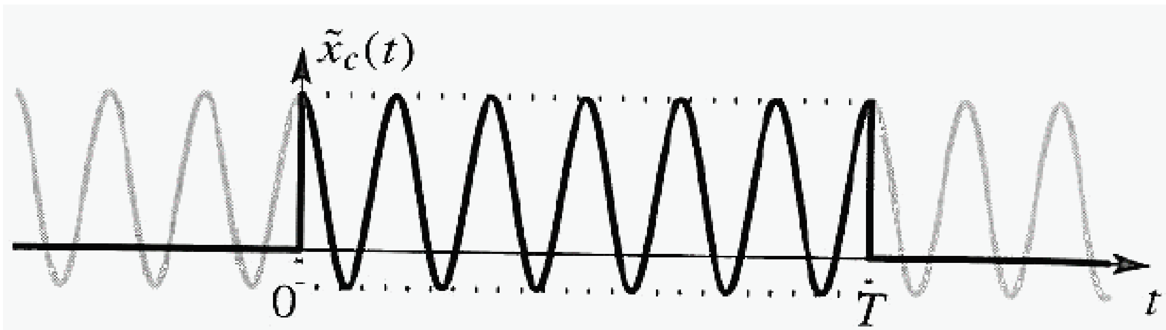


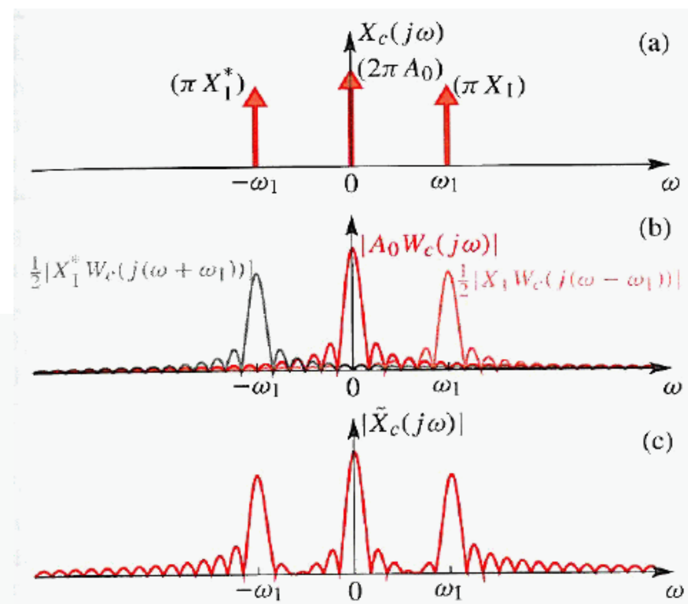
Figure 13-4: (a) Rectangular time window and (b) its Fourier transform. The part of the plot from $-2\pi/T$ to $2\pi/T$ is the *mainlobe*, while the *sidelobes* are evident for $|\omega| > 2\pi/T$.

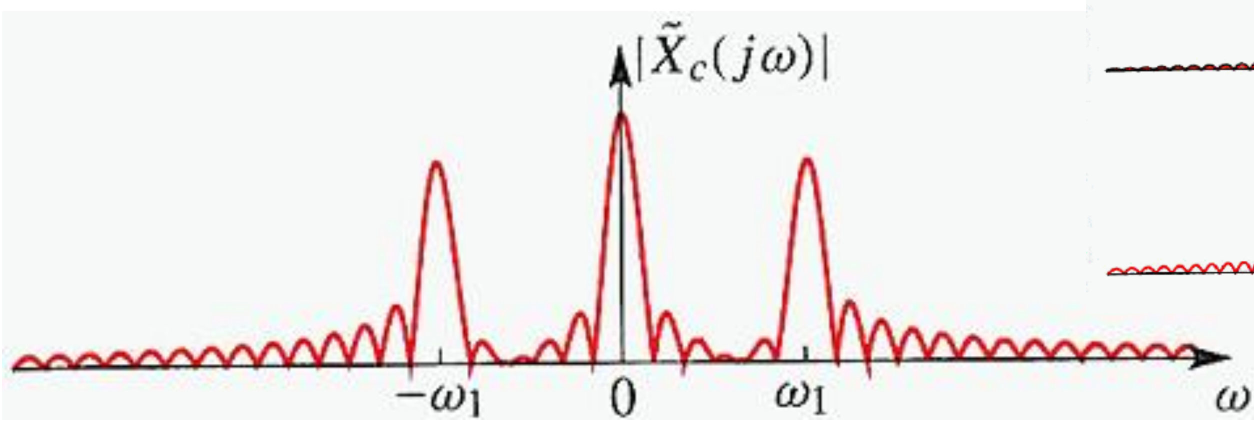
Effect of rectangular window

- The DTFT amplitude of a finite-length sinusoidal signal



through a rectangular window,
the frequency domain looks like





Other Windows

- Instead of rectangular window, other windows could be used as well.
- For example, Hamming window can ease this sidelobe effect at the expense of a wider mainlobe.
- Hamming widow (that utilizes a cosine function):

$$w_H(t) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi}{T}t\right) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

- The window does not drop abruptly at the edges like the rectangular window.

Hamming Windows – Time and Frequency Domain

- In the frequency domain, the mainlobe is twice as that of the rectangular window.
- On the other hand, the sidelobe is pretty much lower.

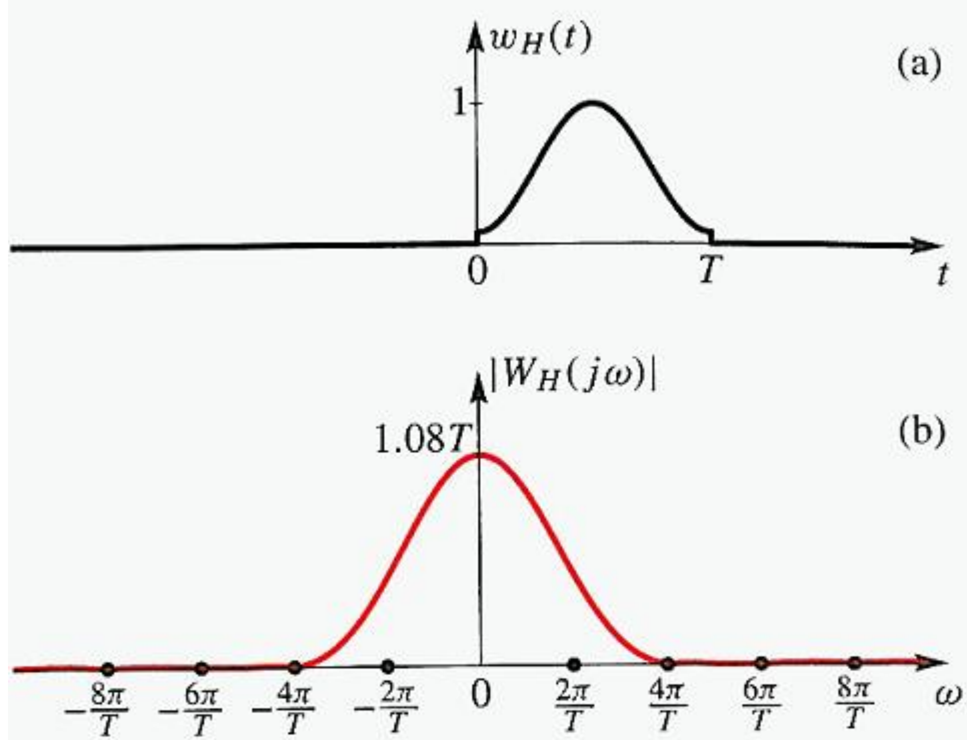
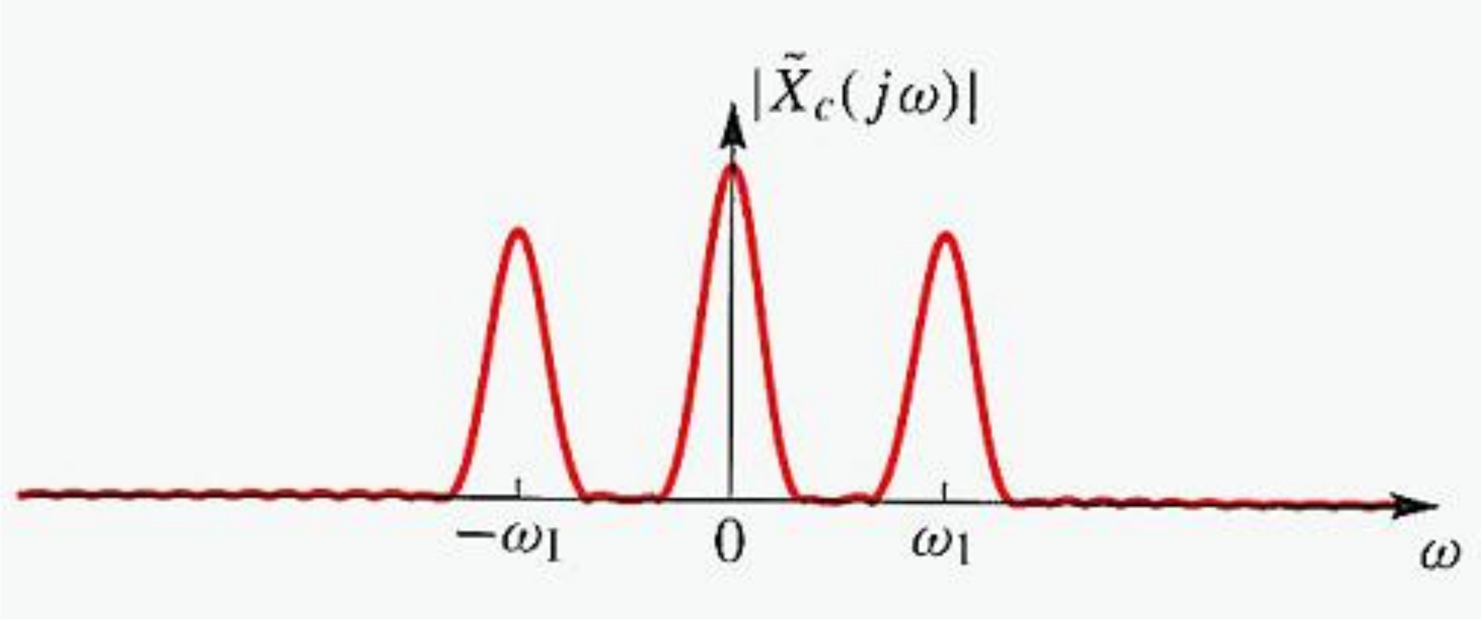


Figure 13-7: (a) Hamming time window and (b) its Fourier transform magnitude. The part of the plot from $-4\pi/T$ to $4\pi/T$ is the *mainlobe*, while the *sidelobes* are hardly visible over the range $|\omega| > 4\pi/T$.

Effect of using Hamming Window

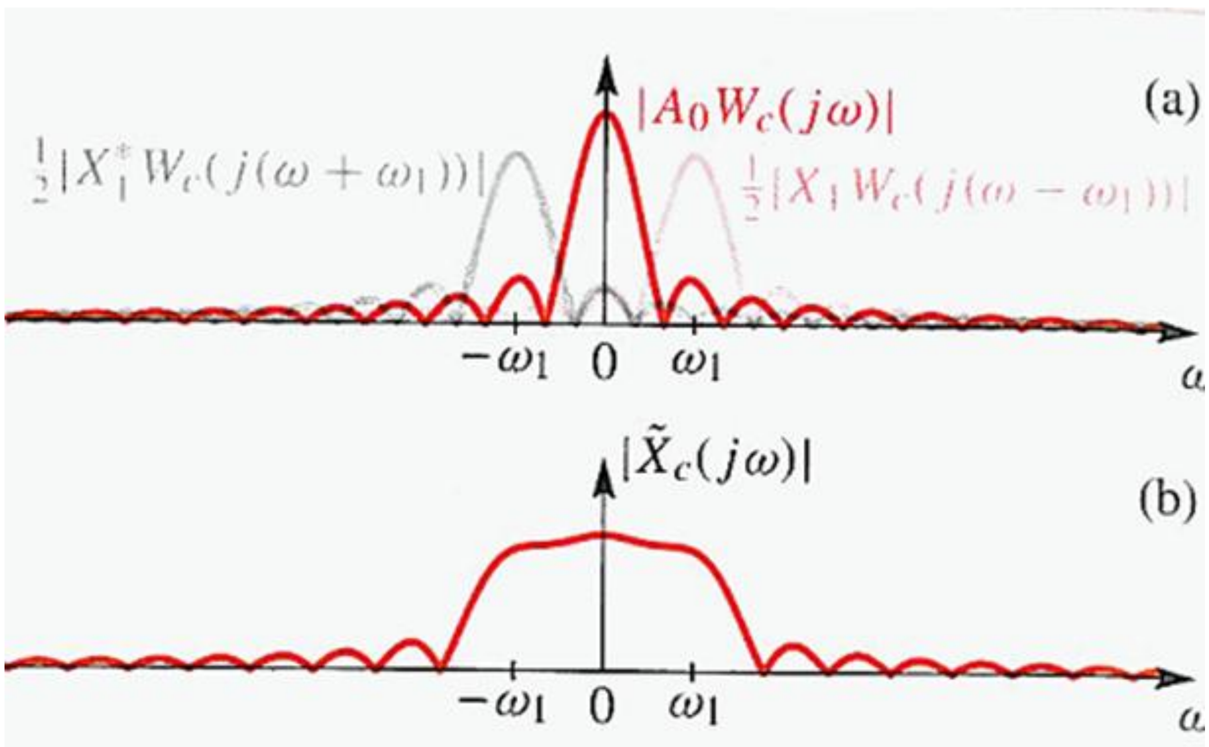
- The output of Hamming window then looks like



- Hence, the spectrum is different depending upon the window used.

Effect of Windowing - Discussion

- No matter using rectangular or Hamming windows, if too much overlap occurs, we will no longer see distinct peaks around each of the original frequencies.
- Eg., an example of using rectangular window:



sum

Effect of Windowing

- Note that the order of sampling and windowing can be exchanged.
- Equivalent systems can be obtained:

C/D (Sampling) first vs. Windowing first

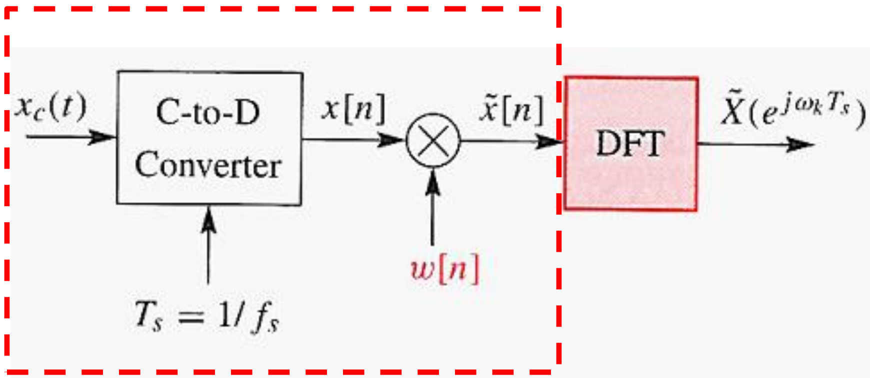


Figure 13-2: Discrete-time spectrum analysis using time-domain windowing and the DFT. The window $w[n]$ should be a finite-length sequence.

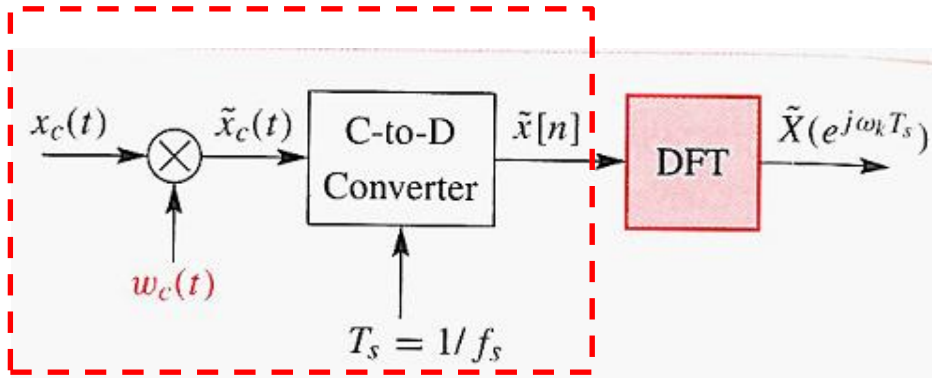


Figure 13-3: Equivalent system for discrete-time spectrum analysis using continuous-time windowing.

Spectral Analysis

- Previously, we use the right-hand system for discussing the effect of windowing.
- In the following, we use the left-hand system for the study of practical spectral analysis.

- Materials are from Boaz Porat's Book.

Recall: Definition of DFT in Boaz Porat's Book

Remark

- There could be different notations from different books.
- The Fourier transform is referred to as DTFT below.

Let the discrete-time signal $x[n]$ have finite duration, say in the range $0 \leq n \leq N-1$. The Fourier transform of this signal is

$$\boxed{X^{\mathrm{f}}(\theta) = \sum_{n=0}^{N-1} x[n] e^{-j\theta n}.} \quad \text{DTFT} \qquad (4.1)$$

Let us sample the frequency axis using a total of N equally spaced samples in the range $[0, 2\pi)$, so the sampling interval is $2\pi/N$; in other words, we use the frequencies

$$\theta[k] = \frac{2\pi k}{N}, \quad 0 \leq k \leq N-1. \qquad (4.2)$$

The result is, by definition, the discrete Fourier transform. Mathematically,

$$\boxed{X^{\mathrm{d}}[k] = \{\mathcal{D}x\}[k] = \sum_{n=0}^{N-1} x[n] \exp\left(-\frac{j2\pi kn}{N}\right), \quad 0 \leq k \leq N-1.} \quad \text{DFT} \qquad (4.3)$$

Windowing

The effect of rectangular window in discrete-time domain:

- Assume we are given a long, possibly infinite-duration signal $y[n]$. We pick a relatively short segment of $y[n]$, say

$$x[n] = \begin{cases} y[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

- We can describe the operation of getting $x[n]$ from $y[n]$ as a multiplication of $y[n]$ by a rectangular window, $w_r[n]$, that is,

$$x[n] = y[n]w_r[n], \quad \text{where } w_r[n] = \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

- Recall that multiplication in the time domain equals to convolution in the frequency domain, so

$$X^{\mathrm{f}}(\theta) = \frac{1}{2\pi}\{Y^{\mathrm{f}} * W_{\mathrm{r}}^{\mathrm{f}}\}(\theta),$$

where $W_{\mathrm{r}}^{\mathrm{f}}(\theta)$ is the Fourier transform of the rectangular window, given by

$$W_{\mathrm{r}}^{\mathrm{f}}(\theta) = \sum_{n=0}^{N-1} e^{-j\theta n} = \frac{1-e^{-j\theta N}}{1-e^{-j\theta}} = \frac{\sin(0.5\theta N)}{\sin(0.5\theta)} e^{-j0.5\theta(N-1)}.$$

- The function

$$D(\theta, N) = \frac{\sin(0.5\theta N)}{\sin(0.5\theta)}$$

is called the Dirichlet kernel.

Note that this kernel also appeared in the constructing DTFT from DFT for a final-length signal.

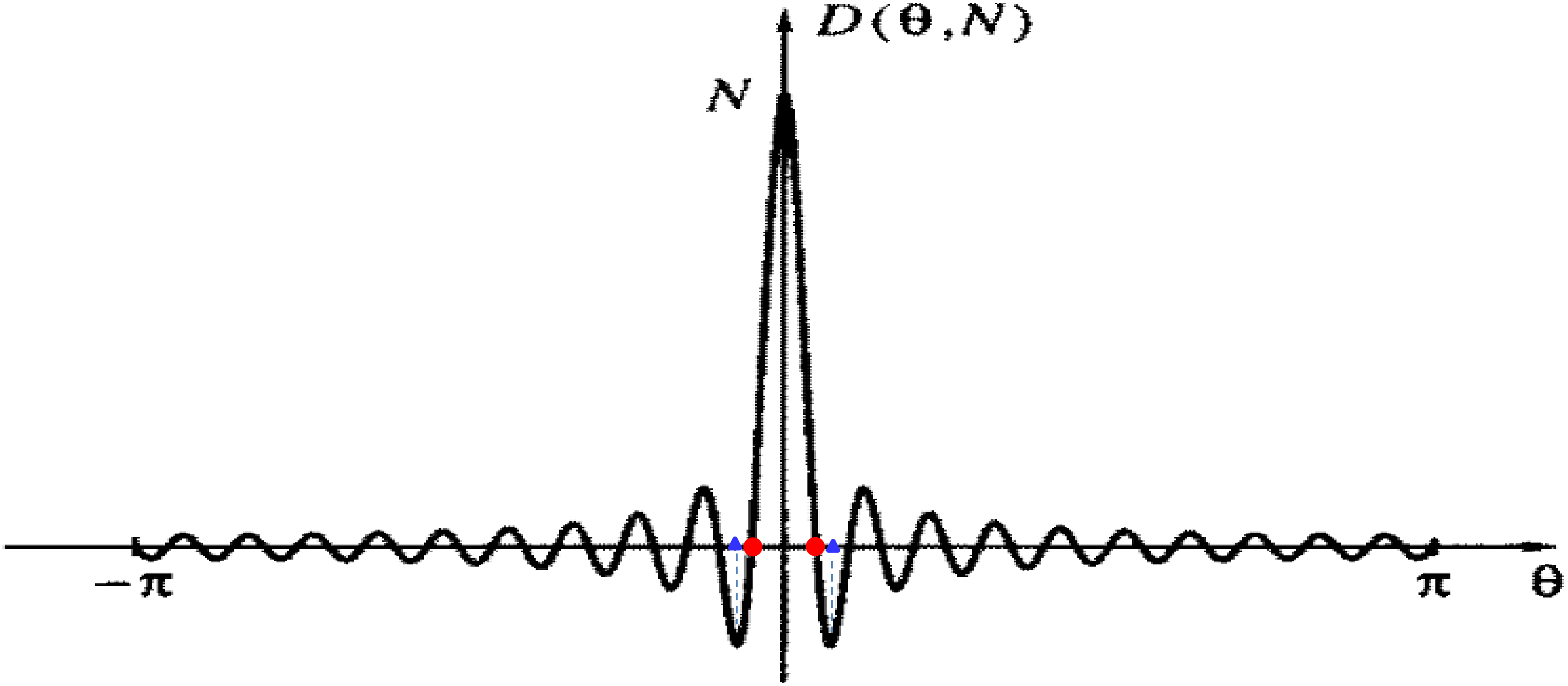


Figure 6.6 The Dirichlet kernel for $N = 40$.

Main property of the Dirichlet kernel are as follows

- Its maximum value is N, occurring at $\theta = 0$.
- Its zeros nearest to the origin (the two red points in the figure) are at $\theta = \pm 2\pi/N$. The region between these two zeros is called the **main lobe** of the Dirichlet kemel. The **main-lobe width** is thus $\boldsymbol{4\pi/N}$.
- There are additional zeros at $\{\theta = 2m\pi/N,\ m = \pm 2, \pm 3, \ldots\}$.

Main property of the Dirichlet kernel (cont.)

- Between every pair of adjacent zeros there is a local maximum or a local minimum, approximately at $\theta = (2m + 1)\pi/N$.

- The regions between these zeros are called side lobes.

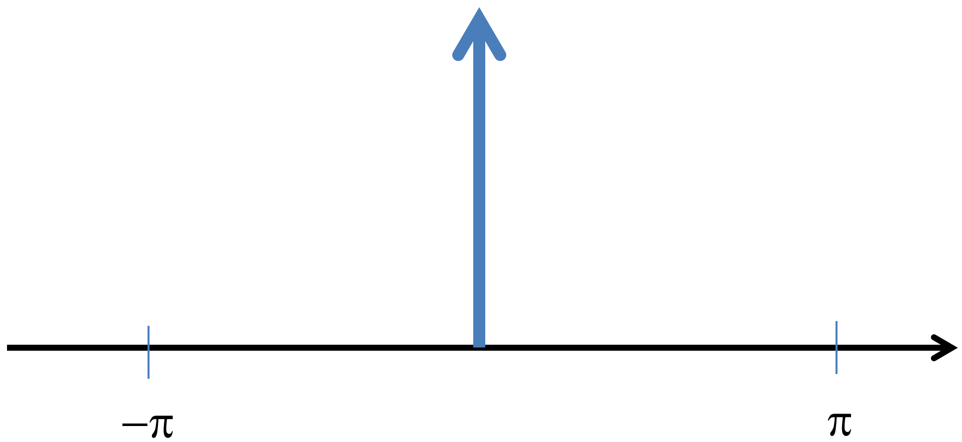
- The highest-value side lobe (in absolute value) occurs at $\theta = \pm 3\pi/N$ ((the two blue triangles in the figure)) and its value (for large N) is approximately $2N/3\pi$.

- The ratio (in the log domain) of the highest side lobe to the main lobe is about –13.5 dB.
  - **decibel (*dB*) is a *logarithmic unit* of the ratio**:

$$20 \log_{10} \left(\frac{\left| X^{\mathrm{f}}(\theta_{HighestSideLobe}) \right|}{\left| X^{\mathrm{f}}(\theta_{MainLobe}) \right|} \right)$$

What is the ideal kernel in frequency domain?

- Its shall be an impulse function between a single period $[-\pi, \pi]$.



- i.e., it should be an impulse train in the DTFT domain, implying the ideal case that the time domain is 1 for all $n \in [-\infty, \infty]$.

- Choosing a window $w[n]$ is always a tradeoff between the width of the main lobe and the level of the side lobes.
- In general, the narrower the main lobe, the higher the side lobes, and vice versa.

Property of rectangular window:

- the rectangular window has the narrowest possible main lobe of all windows of the same length, but its side lobes are the highest.
- The side-lobe level of the rectangular window, -13.5 dB, is undesired in most applications.

- The frequency is often drawn via its magnitude in the log domain.
- For rectangular window:

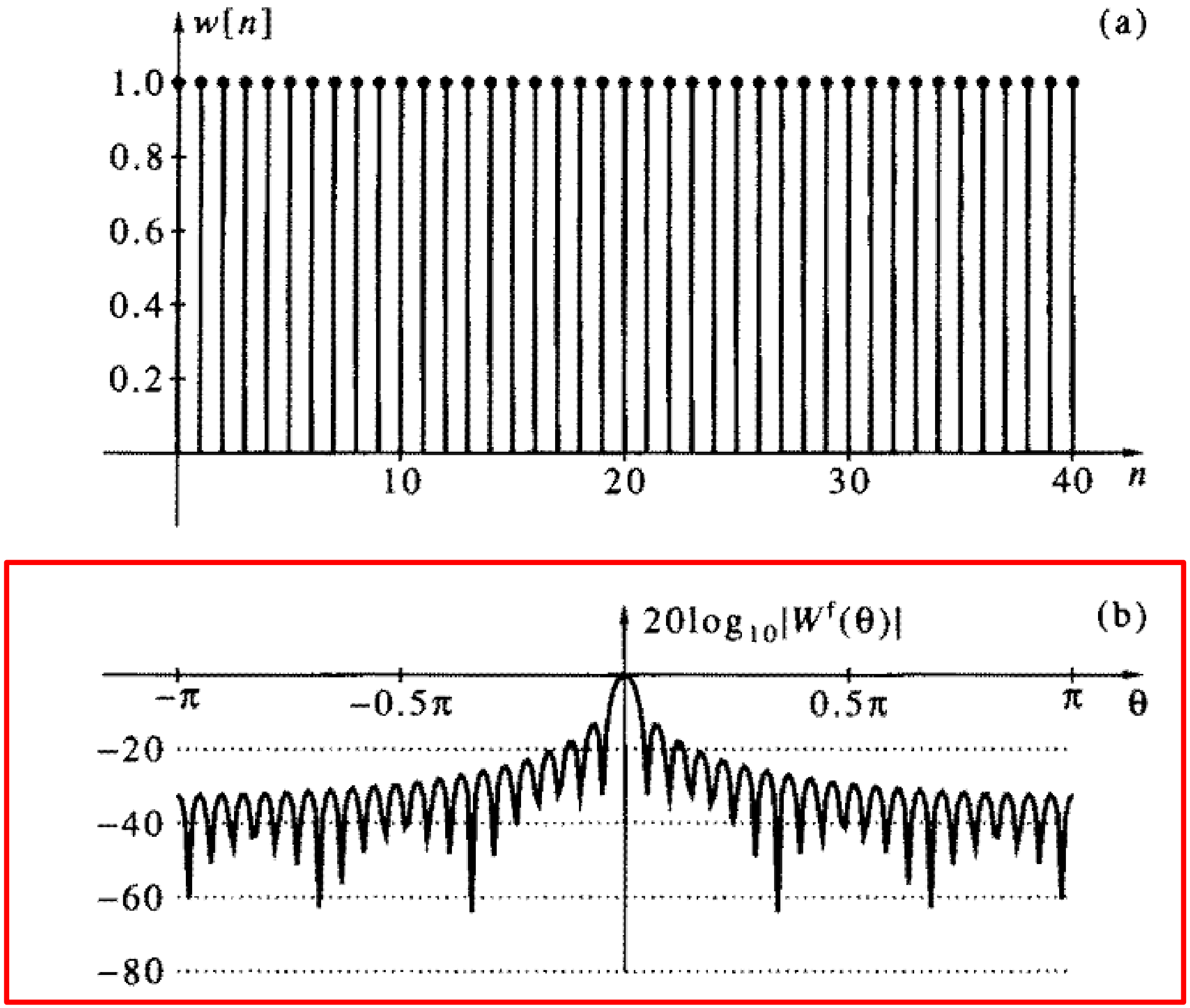


Figure 6.8 A rectangular window, $N = 41$: (a) time-domain plot; (b) frequency-domain magnitude plot.

Bartlett Window (triangle-shaped)

- Suppose the desired window length N is an odd number.
- Let $w_r[n]$ be a rectangular window of length $(N+1)/2$.

- Define the Bartlett window $w_t[n]$ from the rectangular window as

$$w_t[n] = \frac{2}{N+1}\{w_r * w_r\}[n] = 1 - \frac{|2n - N + 1|}{N+1}, \quad 0 \le n \le N-1.$$

- Since convolving a rectangular window with itself in the time domain is equivalent to squaring in the frequency domain, the corresponding kernel function is then

$$W_t^f(\theta) = \frac{2}{N+1} D^2(\theta, 0.5(N+1)) e^{-j0.5\theta(N-1)} = \frac{2\sin^2[0.25\theta(N+1)]}{(N+1)\sin^2(0.5\theta)} e^{-j0.5\theta(N-1)}. \tag{6.11}$$

- Resulting in a triangle-shaped window in the time domain.

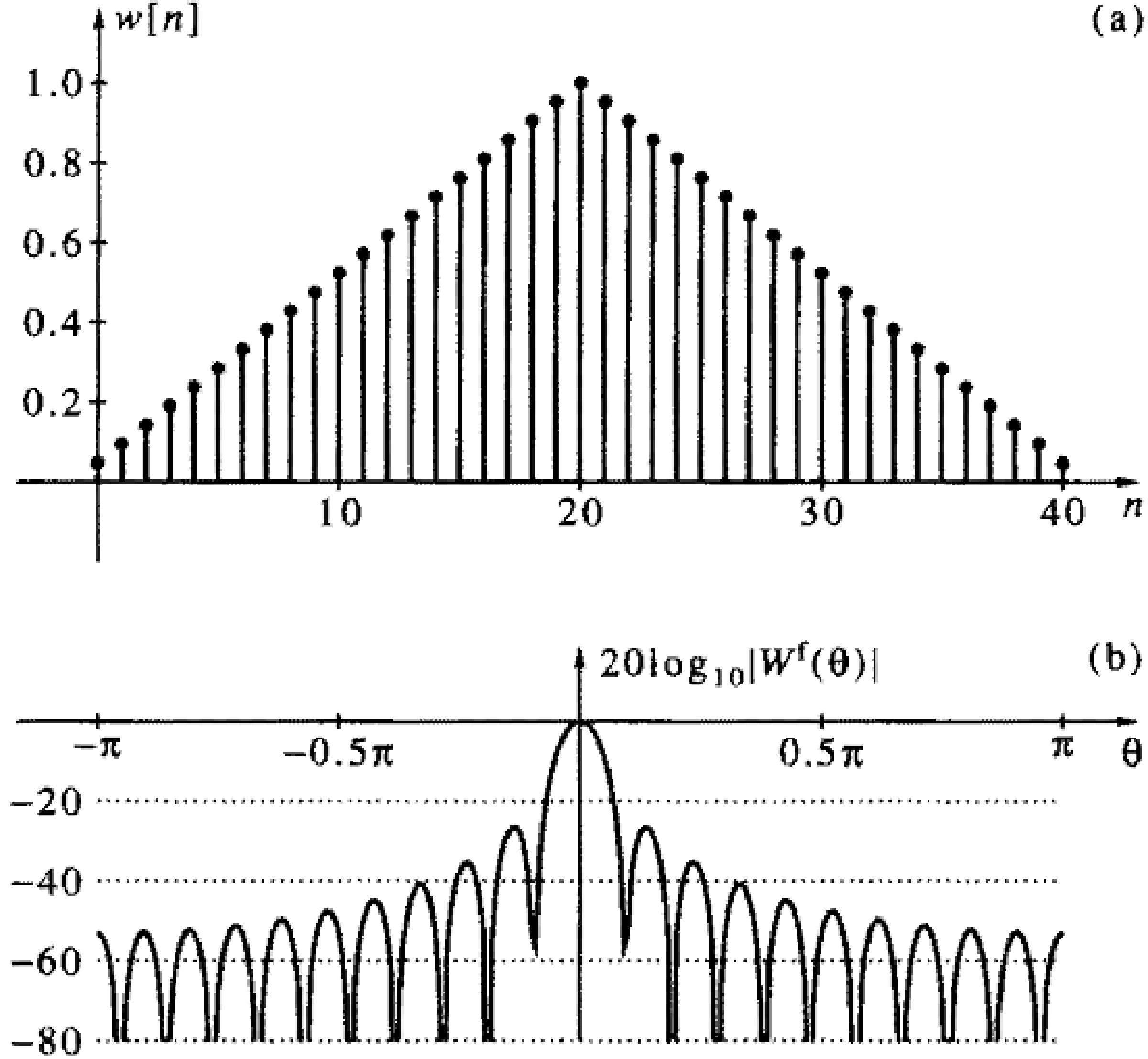


Figure 6.9 Bartlett window, $N = 41$: (a) time-domain plot; (b) frequency-domain magnitude plot.

- This window is called the Bartlett window (after its discoverer) or a triangular window (owing to its shape).
- As from its construction, the side-lobe level of the Bartlett window is -27 dB. The width of the main lobe is $\frac{8\pi}{N+1}$, which is nearly twice that of a rectangular window of the same length.

Hann Window (or Hanning window; cosine-shaped)

- Whereas the Bartlett window achieves side-lobe level reduction by squaring, the Hann window achieves a similar effect by summation.
- The kernel function of the Hann window is obtained by adding three Dirichlet kernels, shifted in frequency so as to yield partial cancellation of their side lobes.

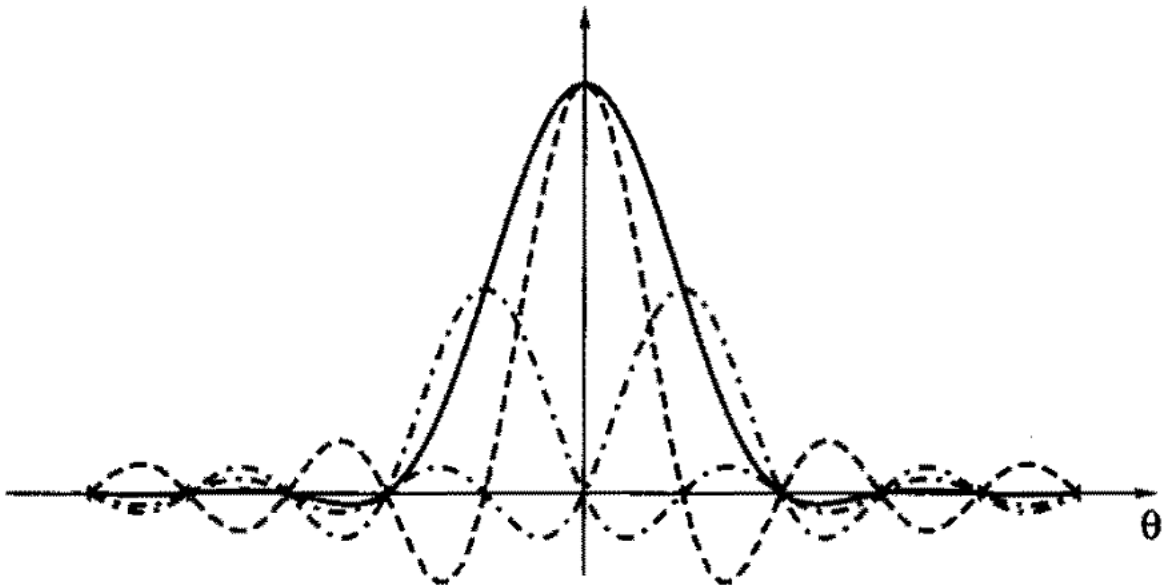


Figure 6.10 Construction of the Hann window from three Dirichlet kernels. Dashed line: center kernel; dot-dashed lines: shifted kernels; solid line: the sum.

- The kernel function of Hann window in the frequency is given by

$$W_{\mathrm{hn}}^{\mathrm{f}}(\theta) = \left[0.5D(\theta, N) + 0.25D\left(\theta - \frac{2\pi}{N-1}, N\right) + 0.25D\left(\theta + \frac{2\pi}{N-1}, N\right)\right] e^{-j0.5\theta(N-1)}$$
$$= 0.5W_{\mathrm{r}}^{\mathrm{f}}(\theta) - 0.25W_{\mathrm{r}}^{\mathrm{f}}\left(\theta - \frac{2\pi}{N-1}\right) - 0.25W_{\mathrm{r}}^{\mathrm{f}}\left(\theta + \frac{2\pi}{N-1}\right). \qquad (6.13)$$

- Its window function in time domain is

$$w_{\mathrm{hn}}[n] = 0.5 - 0.25\exp\left(\frac{j2\pi n}{N-1}\right) - 0.25\exp\left(-\frac{j2\pi n}{N-1}\right)$$
$$= 0.5\left[1 - \cos\left(\frac{2\pi n}{N-1}\right)\right], \quad 0 \le n \le N-1.$$

- The Hann window is also called the cosine window.
- The side-lobe level of this window is –32 dB and the width of the main lobe is $\frac{8\pi}{N}$.

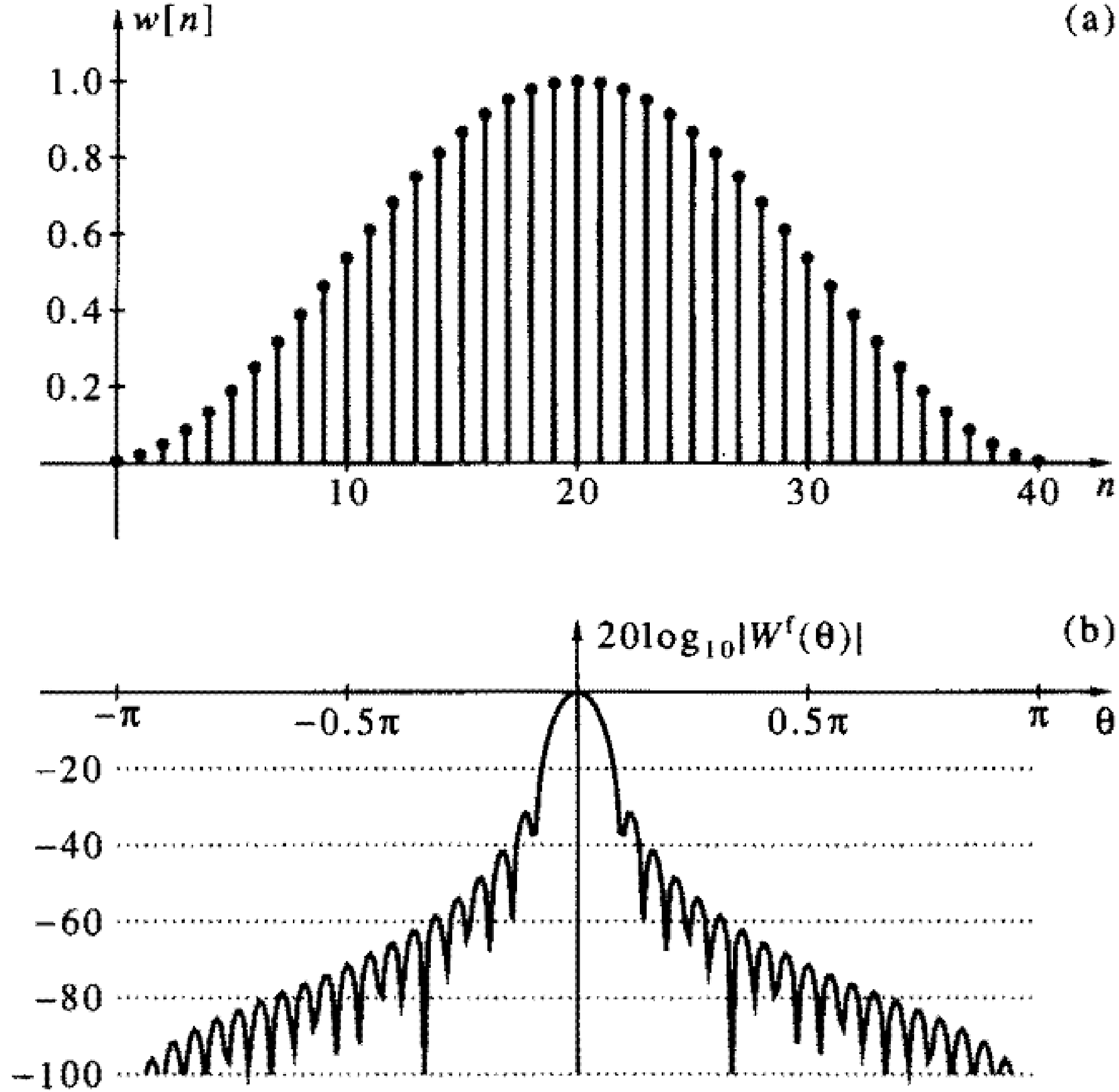


Figure 6.11 Hann window, $N = 41$: (a) time-domain plot; (b) frequency-domain magnitude plot.

- The Hann window has a peculiar property: Its two end points are zero.

- When applied to a signal $y[n]$, it effectively deletes the points $y[0]$ and $y[N-1]$.

- This suggests increasing the window length by 2 with respect to the desired N and delete the two end points.

- The modified Hann window thus obtained is

$$w_{\mathrm{hn}}[n] = 0.5\left\{1-\cos\left[\frac{2\pi(n+1)}{N+1}\right]\right\}, \quad 0 \le n \le N-1;$$

Hamming Window (cosine-shaped)

- Obtained by a slight modification of the Hann window, which amounts to choosing different magnitudes for the three Dirichlet kernels.

$$w_{\mathrm{hm}}[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \le n \le N-1,$$

$$W_{\mathrm{hm}}^{\mathrm{f}}(\theta) = 0.54W_{\mathrm{r}}^{\mathrm{f}}(\theta) - 0.23W_{\mathrm{r}}^{\mathrm{f}}\left(\theta - \frac{2\pi}{N-1}\right) - 0.23W_{\mathrm{r}}^{\mathrm{f}}\left(\theta + \frac{2\pi}{N-1}\right).$$

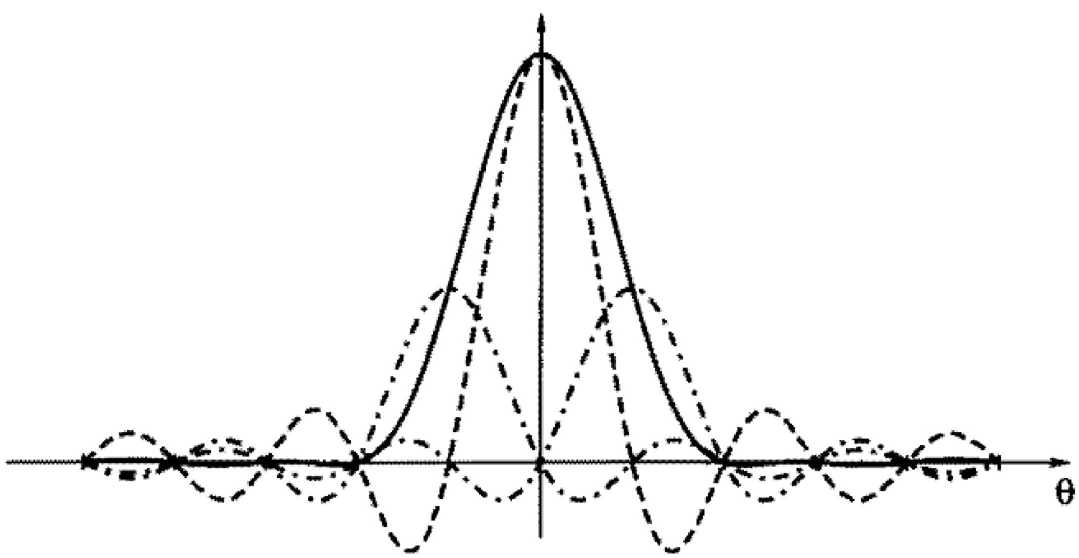


Figure 6.12 Construction of the Hamming window from three Dirichlet kernels. Dashed line: center kernel; dot-dashed lines: shifted kernels; solid line: the sum.

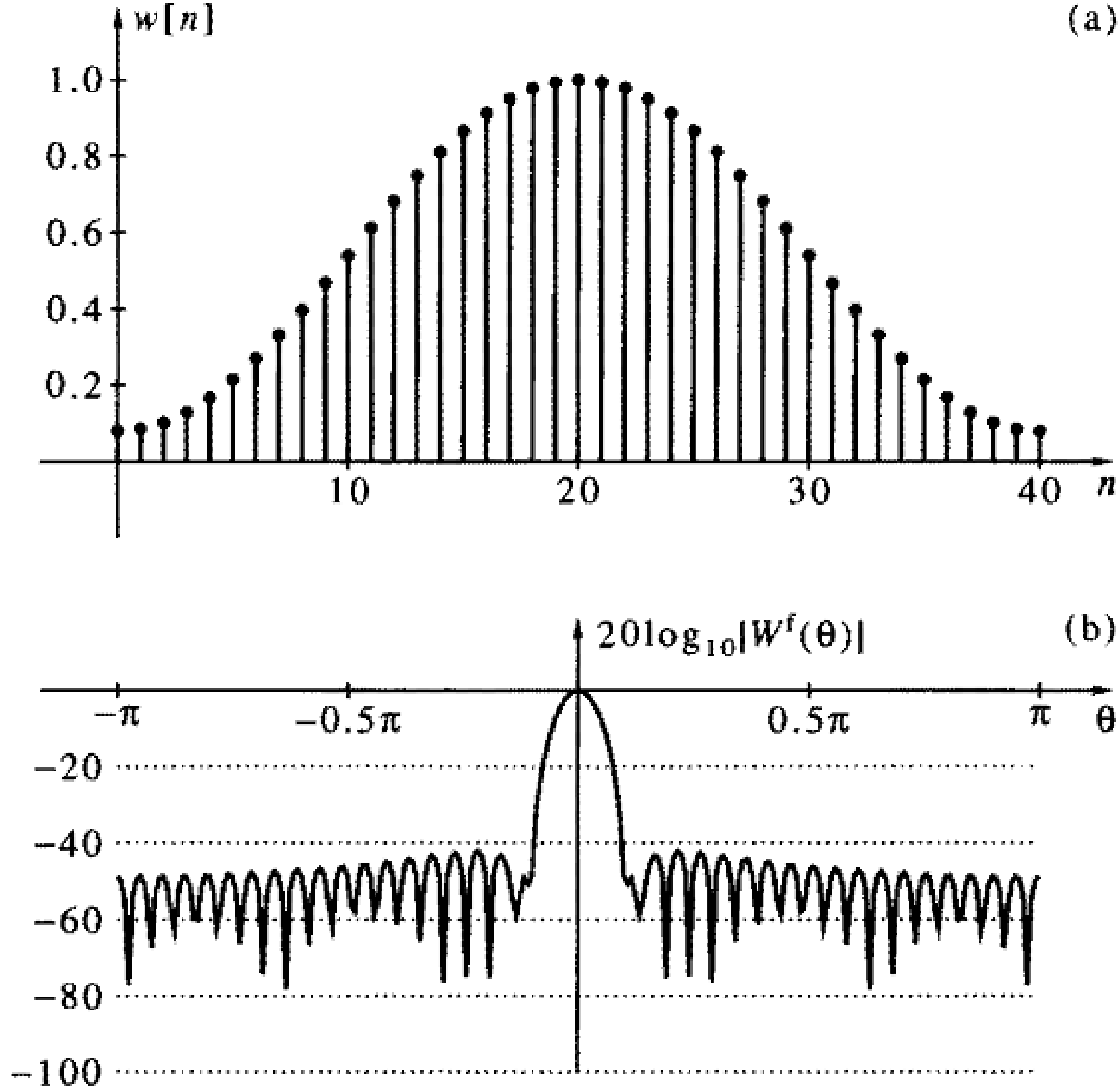


Figure 6.13 Hamming window, $N = 41$: (a) time-domain plot; (b) frequency-domain magnitude plot.

- As we see, the side lobes of the sum are lower than those of the Hann window.

- Hamming got the numbers 0.54 and 0.46 by trial and error, seeking to minimize the amplitude of the highest side lobe.

- The side-lobe level of this window is –43 dB, and the width of the main lobe is $\frac{8\pi}{N}$.

- The Hamming window is also called the raised-cosine window.

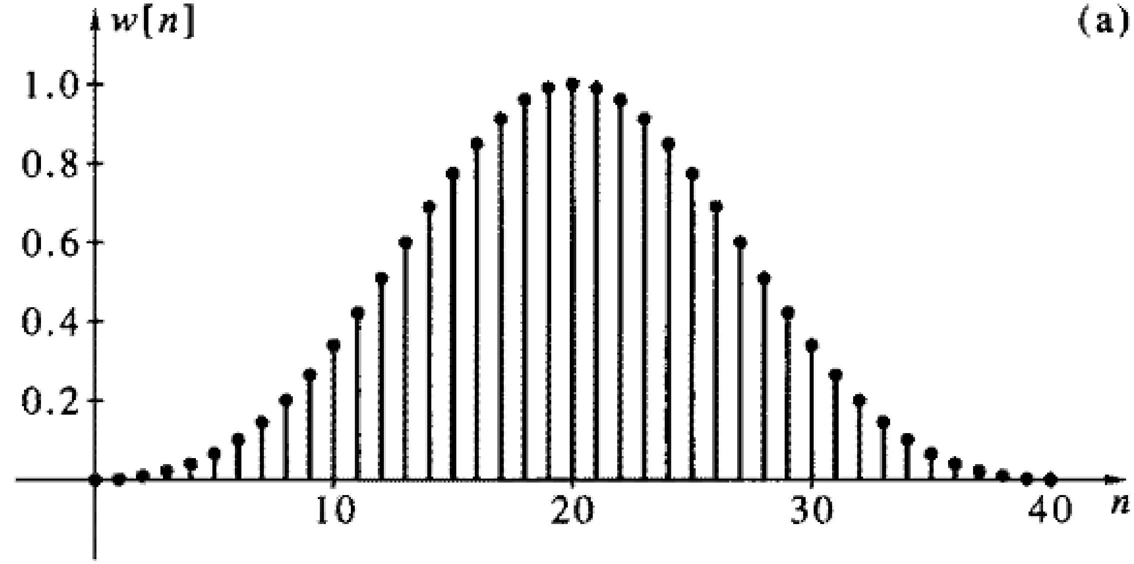
Blackman Window (sum of two cosine-shaped)

- Hamming window has the lowest possible side-lobe level among all windows based on three Dirichlet kernels.
- The Blackman window uses five Dirichlet kernels, thus reducing the side-lobe level still further.

$$w_b[n] = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right), \quad 0 \le n \le N-1,$$

$$W_b^f(\theta) = 0.42W_r^f(\theta) - 0.25W_r^f\left(\theta + \frac{2\pi}{N-1}\right) - 0.25W_r^f\left(\theta - \frac{2\pi}{N-1}\right) + 0.04W_r^f\left(\theta + \frac{4\pi}{N-1}\right) + 0.04W_r^f\left(\theta - \frac{4\pi}{N-1}\right).$$

- The side-lobe level of the Blackman window is –57 dB and the width of the main lobe is $\frac{12\pi}{N}$.
- As in the case of the Hann window, the two end points of the Blackman window are zero, so in practice we can increase N by 2 and remove the two end points.



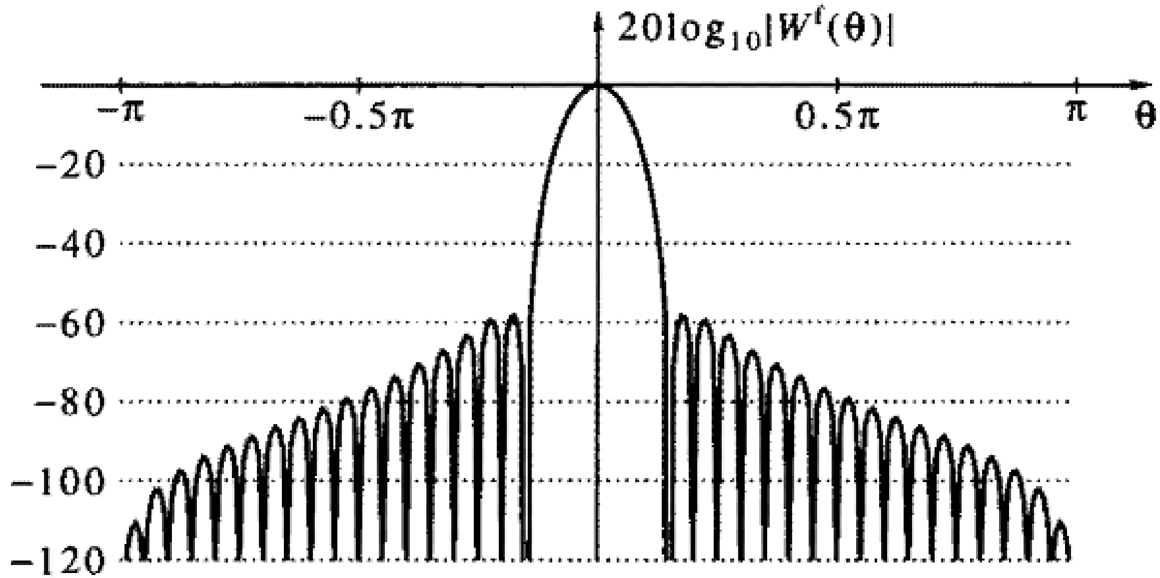


Figure 6.14 Blackman window, $N = 41$: (a) time-domain plot; (b) frequency-domain magnitude plot.

Kaiser Window

- The windows described so far are considered as classical.
- They have been derived based on intuition and educated guesses.

- Modern windows are based on optimality criteria; they aim to be best in certain respect, while meeting certain constraints.

- Different optimality criteria give rise to different windows.

- Dolph's criterion: Minimize the width of the main lobe, under the constraint that the window length is fixed and the side-lobe level does not exceed a given maximum value.

- Kaiser criterion: Minimize the width of the main lobe, under the constraint that the window length is fixed and the energy in the side lobes does not exceed a given percentage of the total energy.

  where the energy in the side lobes is defined as the integral of the square magnitude of the kernel function over the range [$-\pi$, π], excluding the interval of the main lobe.

- Kaiser window:
  - Of the windows based on these two criteria, the Kaisor window is much more popular than Dolph window.
  - Kaiser criterion gives rise to a family of windows that has become, perhaps, the most widely used for modern digital signal processing.

- The solution of Kaiser's optimization problem is described in terms of the modified Bessel function of order zero. This function is given by the infinite power series

$$I_0(x) = \sum_{k=0}^{\infty} \left(\frac{x^k}{2^k k!} \right)^2 .$$

- Using this function, the Kaiser window is given by

$$w_k[n] = \frac{I_0 \left[\alpha \sqrt{1 - \left(\frac{|2n-N+1|}{N-1} \right)^2} \right]}{I_0[\alpha]}, \quad 0 \le n \le N-1.$$

- The parameter α is used for controlling the tradeoff between main-lobe width and the side-lobe level.
- Higher α leads to a wider main lobe and lower side lobes.

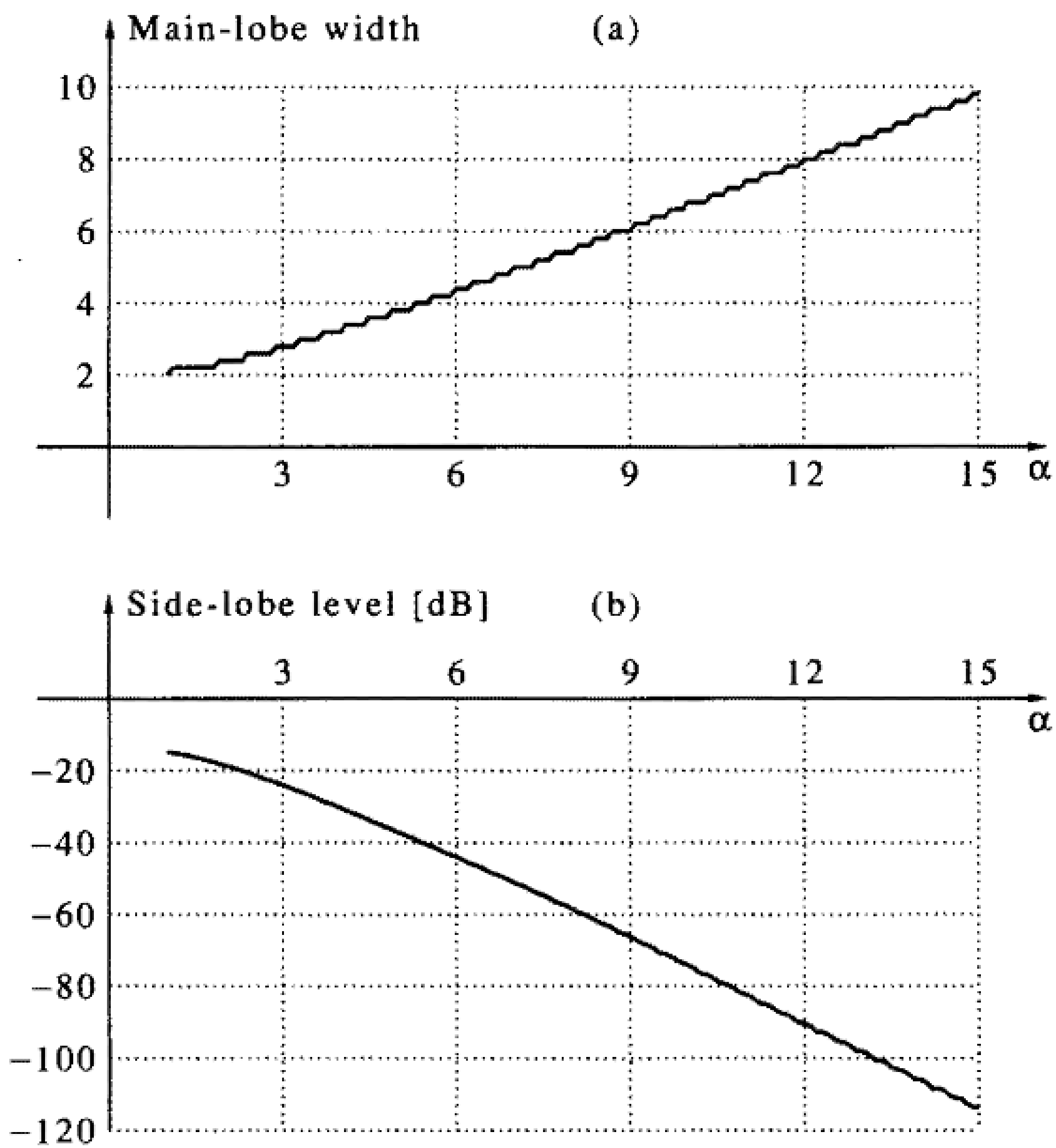


Figure 6.15 Properties of the Kaiser window as a function of the parameter α: (a) main-lobe width, as a multiple of $2\pi/N$; (b) side-lobe level.

- The following figure depicts the Kaisor window for N=41 and α=12. In this case the main-lobe width is $16\pi/N$ and the side-lobe level is –90dB.

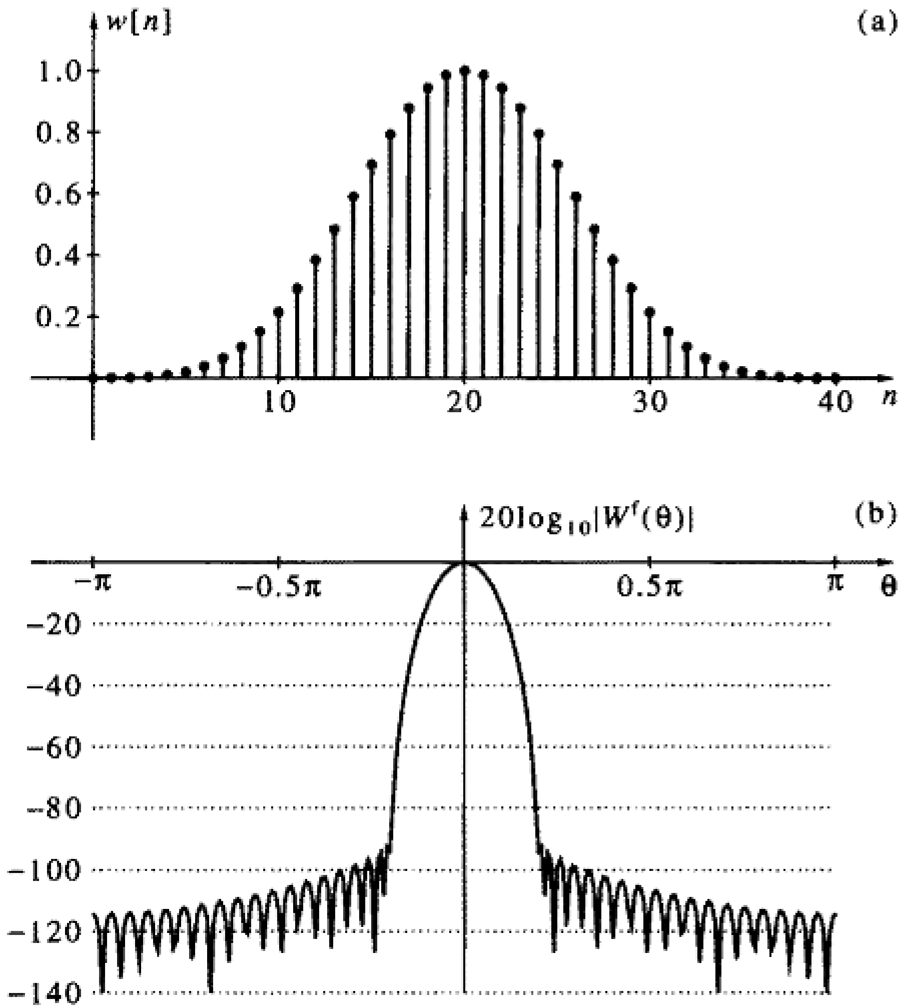


Figure 6.16 Kaiser window, $N = 41$, $\alpha = 12$: (a) time-domain plot; (b) frequency-domain magnitude plot.

Short-time Fourier Transform (STFT)

- A general extension of spectrum is the short-time Fourier transform (STFT), where the window slides at every time sites.
- The signal is multiplied by a window function.

$$\mathbf{STFT}\{x(t)\}(\tau,\omega) \equiv X(\tau,\omega) = \int_{-\infty}^{\infty} x(t)w(t-\tau)e^{-j\omega t}\,dt$$

- The CFT of the resulting signal is taken as the window is slid along the time axis, resulting in a two-dimensional representation of the signal.