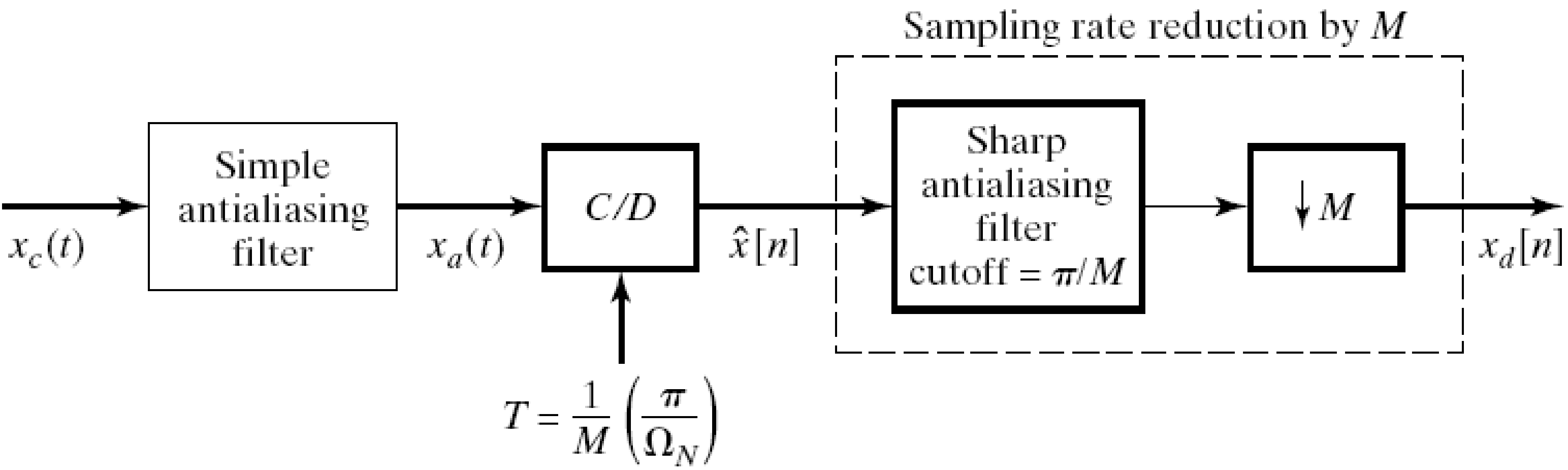
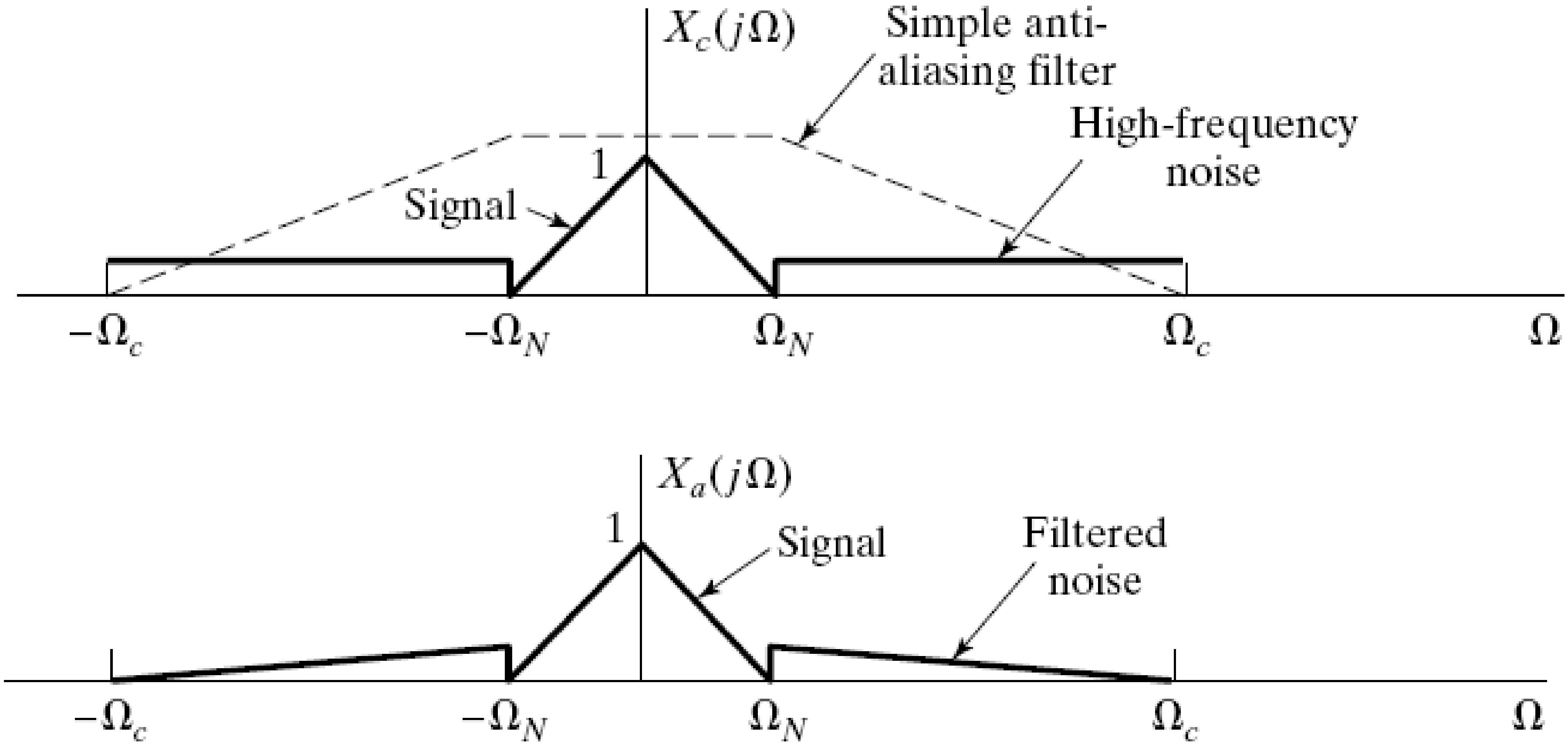
Over-sampled A/D conversion

- The anti-aliasing filter is an analog filter.
- However, in applications involving powerful, but inexpensive, digital processors, these continuous-time filters may account for a major part of the cost of a system.
- Let Ω_N be the highest frequency of the analog signal. Instead, we first apply a very simple anti-aliasing filter (in the analog domain) that has a gradual cutoff (instead of a sharp cutoff) with significant attenuation at $M\Omega_N$. Next, implement the continuous-to-discrete (C/D) conversion at the sampling rate higher than $2M\Omega_N$.
- After that, sampling rate reduction by a factor of M that includes sharp anti-aliasing filtering is implemented in the discrete-time domain.

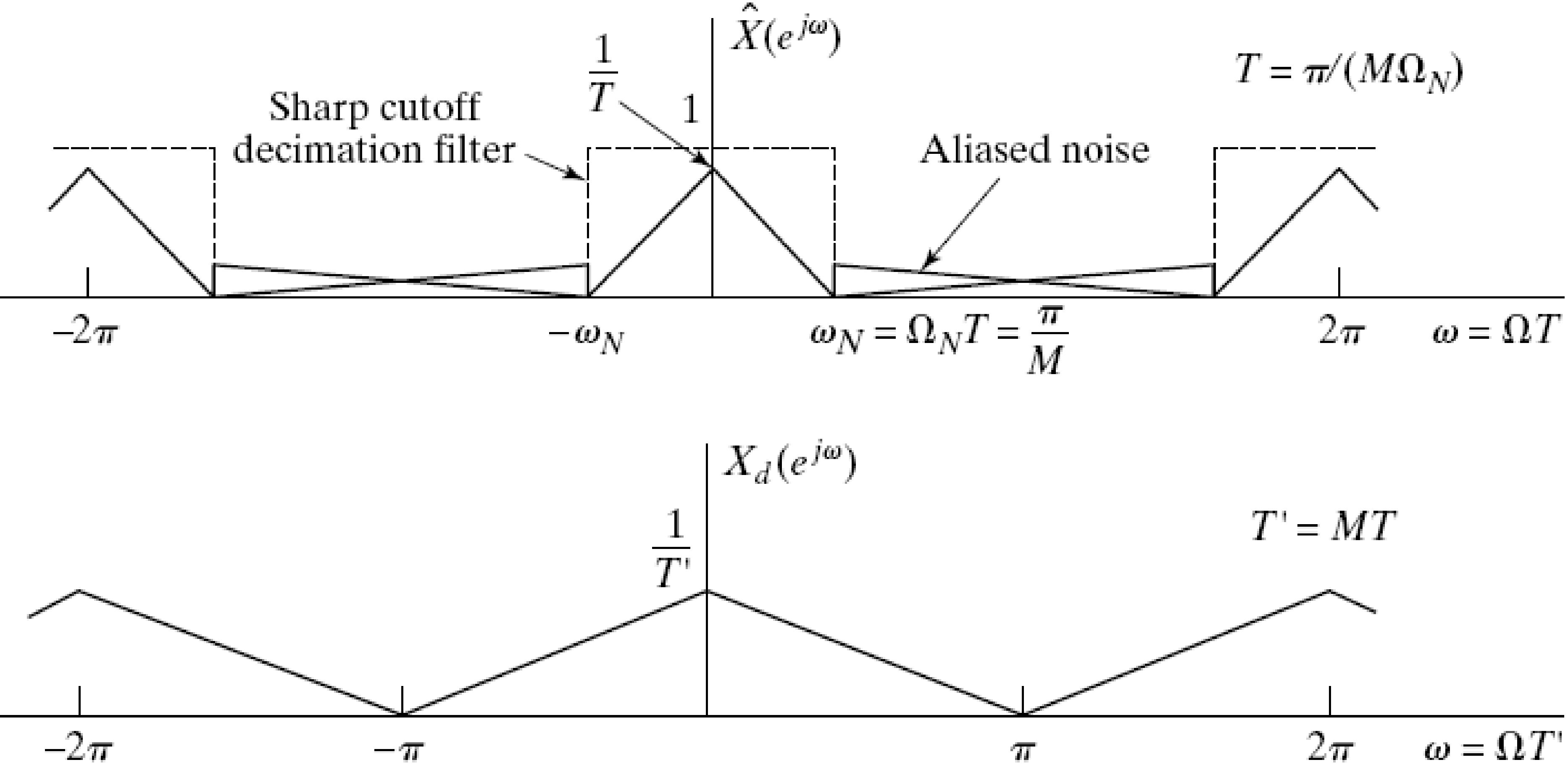
Using over-sampled A/D conversion to simplify a continuous-time anti-aliasing filter



Example of over-sampled A/D conversion (analog domain)



Example of over-sampled A/D conversion (discrete-time domain)



Oversampling vs. quantization
(Oppenheim, Chap. 4)

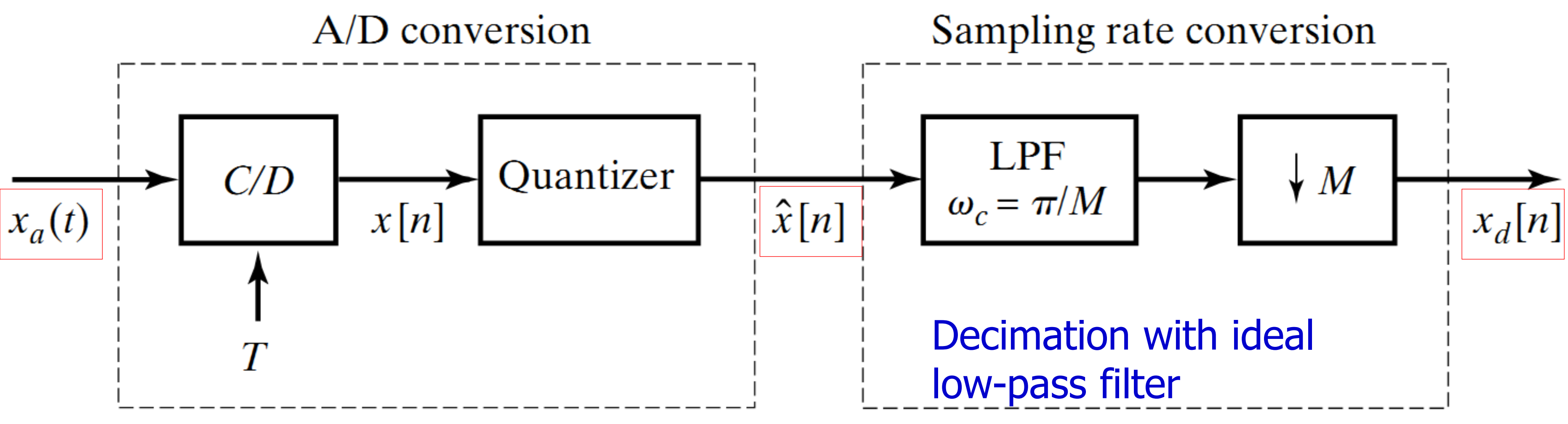
- We consider the analog signal $x_a(t)$ as wide-sense-stationary, random process with power-spectral density denoted by $\Phi_{x_a x_a}(e^{jw})$ and the autocorrelation function by $\phi_{x_a x_a}(\tau)$.

- To simplify our discussion, assume that $x_a(t)$ is already bandlimited to Ω_N, i.e.,

$$\boxed{\Phi_{x_a x_a}(j\Omega) = 0, \quad |\Omega| \geq \Omega_N,}$$

Oversampling

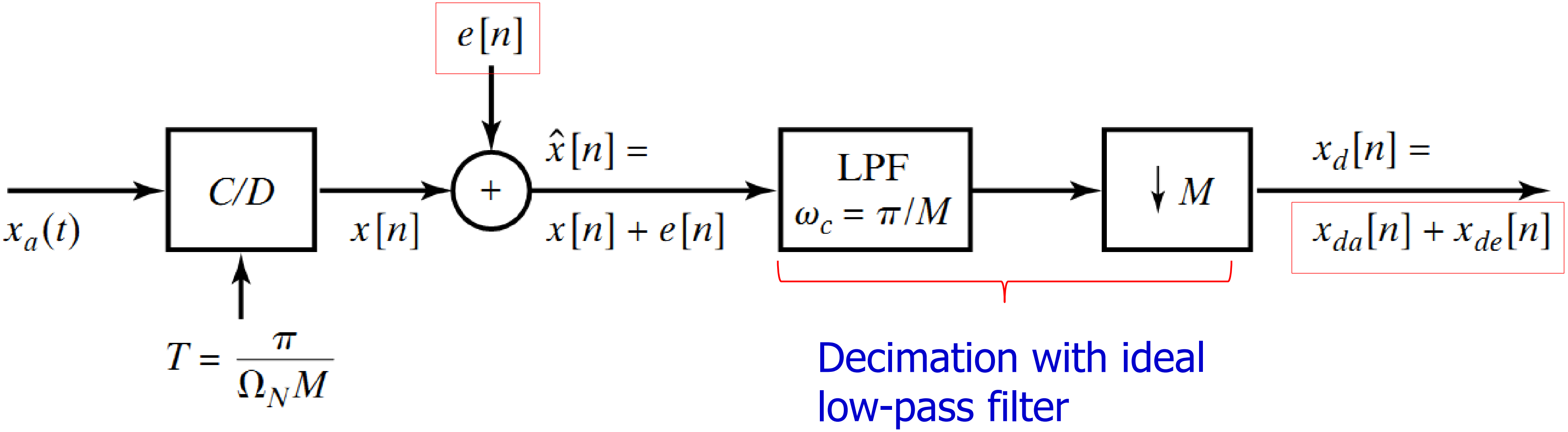
- Oversampling: We assume that $2\pi/T = 2M\Omega_N$.
- M is an integer, called the oversampling ratio.



Oversampled A/D conversion with simple quantization and down-sampling

Additive noise model

- Using the additive noise model, the system can be replaced by



- Its output $x_d[n]$ has two components, one from the signal input $x_a(t)$ and the other from the quantization noise input $e[n]$. Denote them as $x_{da}[n]$ and $x_{de}[n]$, respectively.

Signal component (assume $e[n] = 0$)

- Goal: determine the signal-to noise ratio of signal power $\varepsilon\{x_{da}^2\}$ to the quantization-noise power $\varepsilon\{x_{de}^2\}$. ($\varepsilon\{.\}$ denotes the **expectation value**.)
- As $x_a(t)$ is converted into $x[n]$, and then $x_{da}[n]$, we focus on the power of $x[n]$ first.
- Let us analyze this in the time domain. Denote $\phi_{xx}[n]$ and $\Phi_{xx}(e^{jw})$ to be the autocorrelation and power spectral density of $x[n]$, respectively.
- By definition, $\phi_{xx}[m] = \varepsilon\{x[n+m]x[n]\}$.

Power of $x[n]$ (assume $e[n] = 0$)

- Since $x[n] = x_a(nT)$, it is easy to see that

$$\begin{aligned}\phi_{xx}[m] &= \varepsilon\{x[n+m]x[n]\} \\ &= \varepsilon\{x_a((n+m)T)x_a(nT)\} \\ &= \phi_{x_a x_a}(mT)\end{aligned}$$

- That is, the autocorrelation function of the sequence of samples is a sampled version of the autocorrelation function.
- The wide-sense-stationary assumption implies that $\varepsilon\{x_a^2(t)\}$ is a constant independent of t. It then follows that

$$\varepsilon\{x^2[n]\} = \varepsilon\{x_a^2(nT)\} = \varepsilon\{x_a^2(t)\}$$

for all n or t.

Power of $x_{da}[n]$ (assume $e[n] = 0$)

- Since the decimation filter is an ideal lowpass filter with cutoff frequency $w_c = \pi/M$, the signal $x[n]$ passes unaltered through the filter.
- Therefore, the downsampled signal component at the output, $x_{da}[n] = x[nM] = x_a(nMT)$, also has the same power.
- In sum, the above analyses show that

$$\varepsilon\{x_{da}^2[n]\} = \varepsilon\{x^2[n]\} = \varepsilon\{x_a^2(t)\}$$

which shows that the power of the signal component **stays the same** as it traverses the entire system from the input $x_a(t)$ to the corresponding output component $x_{da}[n]$.

Power of the noise component

- According to previous studies, let us assume that $e[n]$ is a wide-sense-stationary white-noise process with zero mean and variance

$$\sigma_e^2 = \frac{\Delta^2}{12}$$

- Consequently, the autocorrelation function and power density spectrum for $e[n]$ are,

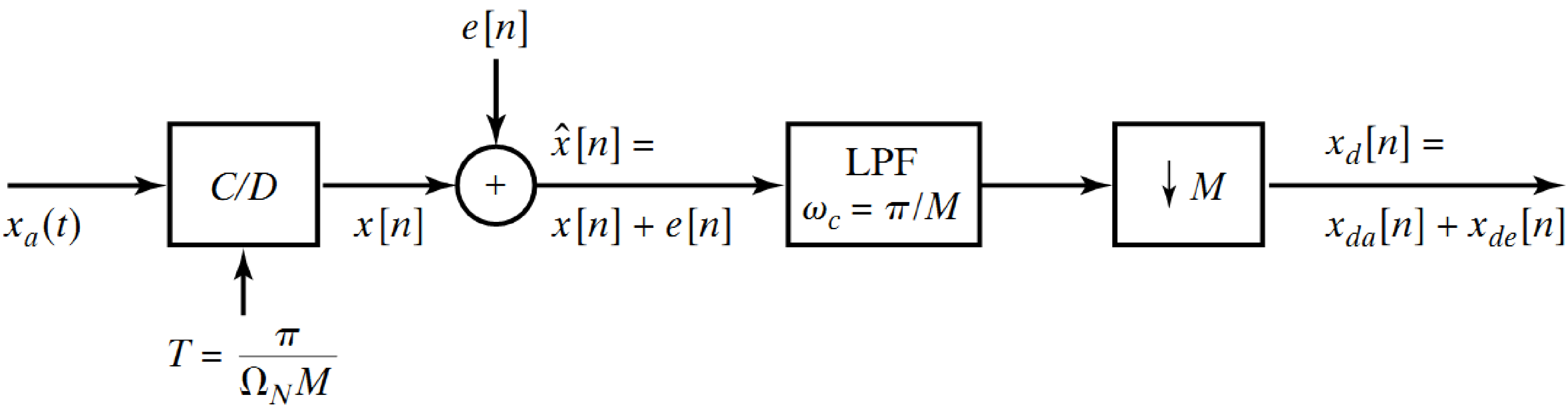
white noise

$$\phi_{ee}[n] = \sigma_e^2 \delta[n]$$

- The power spectral density is the DTFT of the autocorrelation function. So,

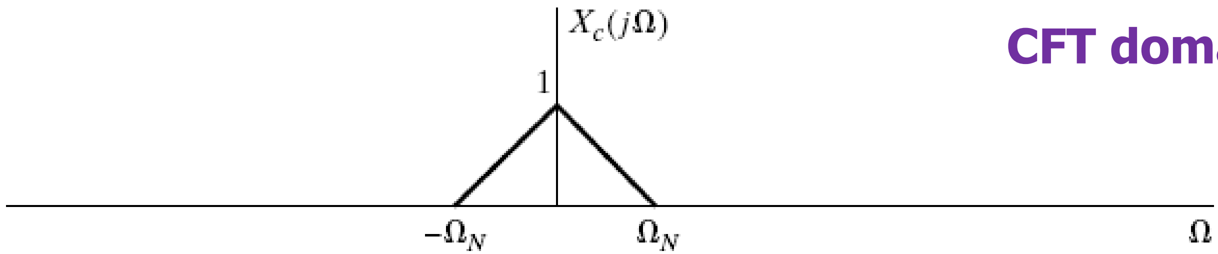
$$\Phi_{ee}(e^{jw}) = \sigma_e^2, \qquad -\pi < w < \pi$$

Power of the noise component (assume $x_a(t)=0$)



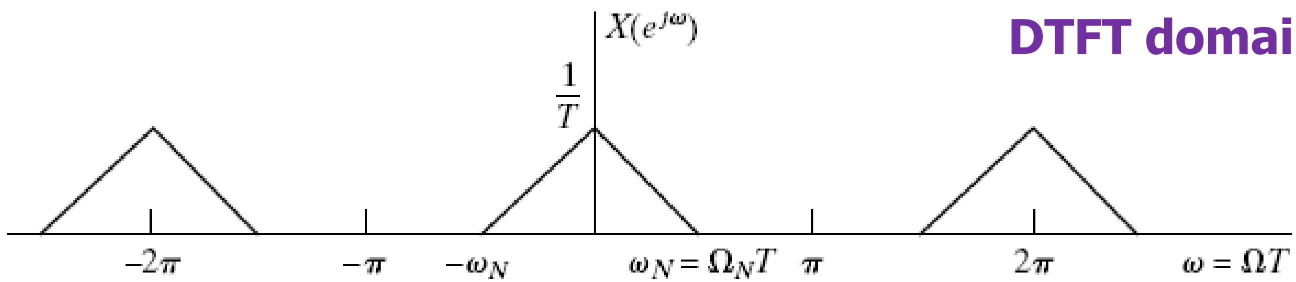
- Although we have shown that the power in $x_{da}[n]$ does not depend on M, we will show that the noise component $x_{de}[n]$ does not keep the same noise power.
  - It is because that, as the oversampling ratio M increases, less of the quantization noise spectrum overlaps with the signal spectrum, as shown below.

Review of Downsampling in the Frequency domain (without aliasing)



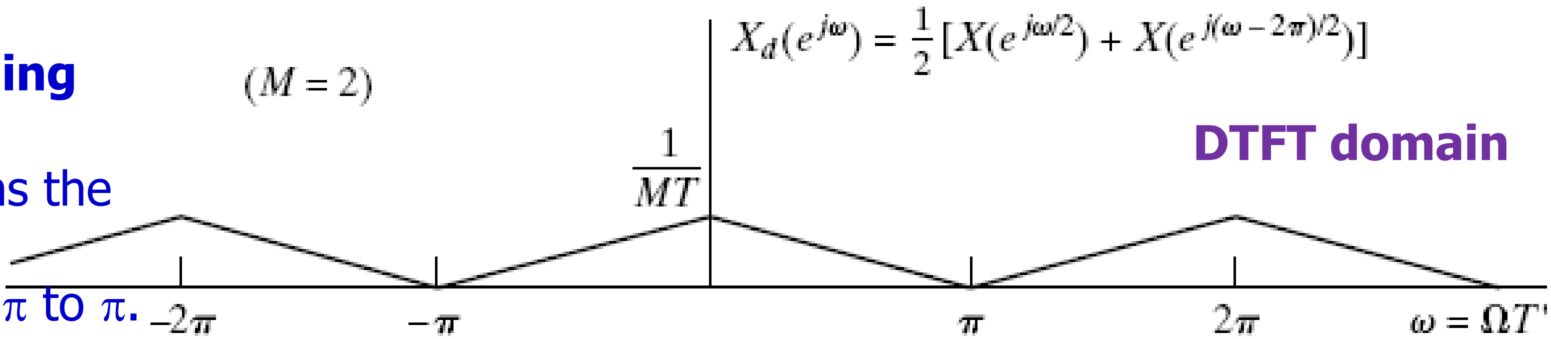
CFT domain

(over) sampling



DTFT domain

Down-sampling



DTFT domain

(power remains the same for the integral from -π to π.

Illustration of frequency and amplitude scaling

- So, when oversampled by M, the power spectrum of $x_a(t)$ and $x[n]$ in the frequency domain are illustrated as follows.

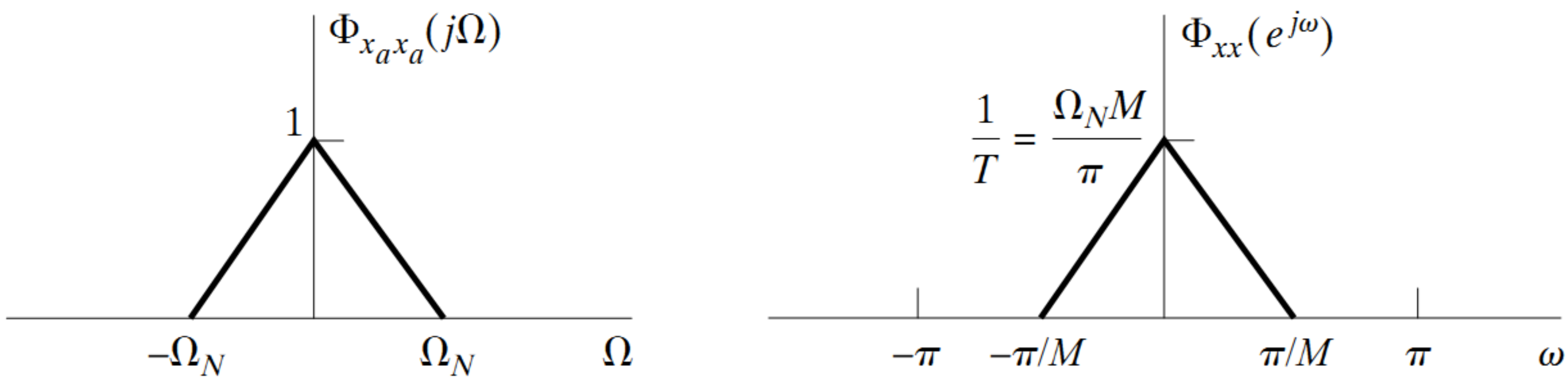
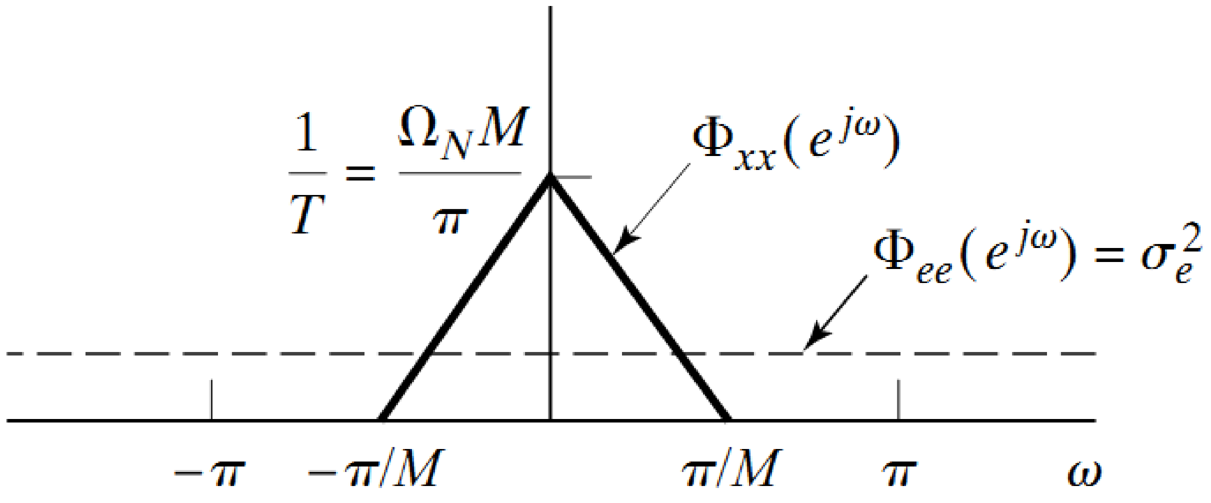
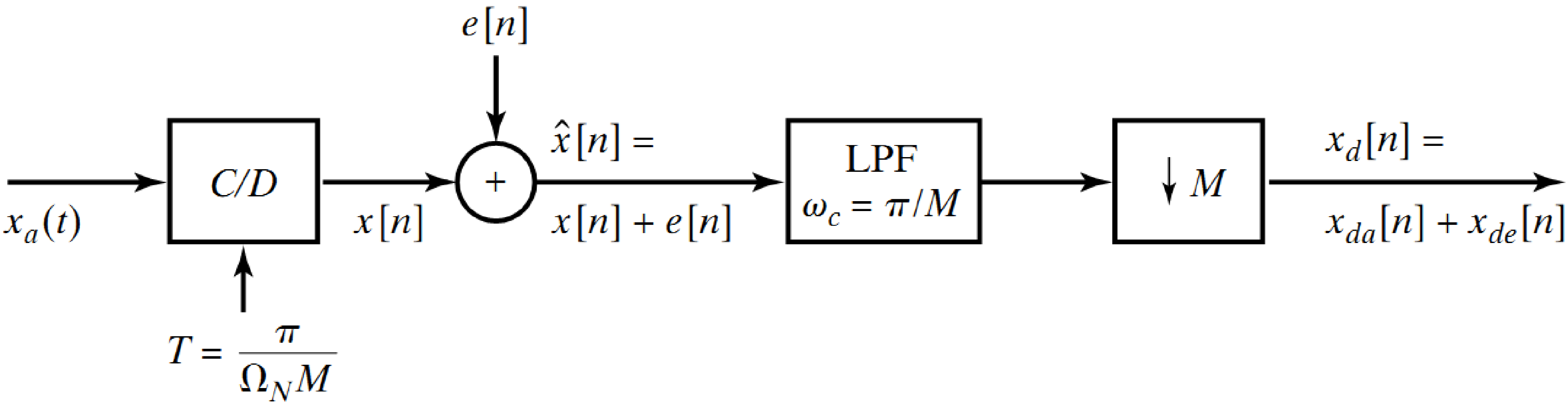


Illustration of frequency for noise

- By considering both the signal and the quantization noise, the power spectra of $x[n]$ and $e[n]$ in the frequency domain are illustrated as



Noise component power

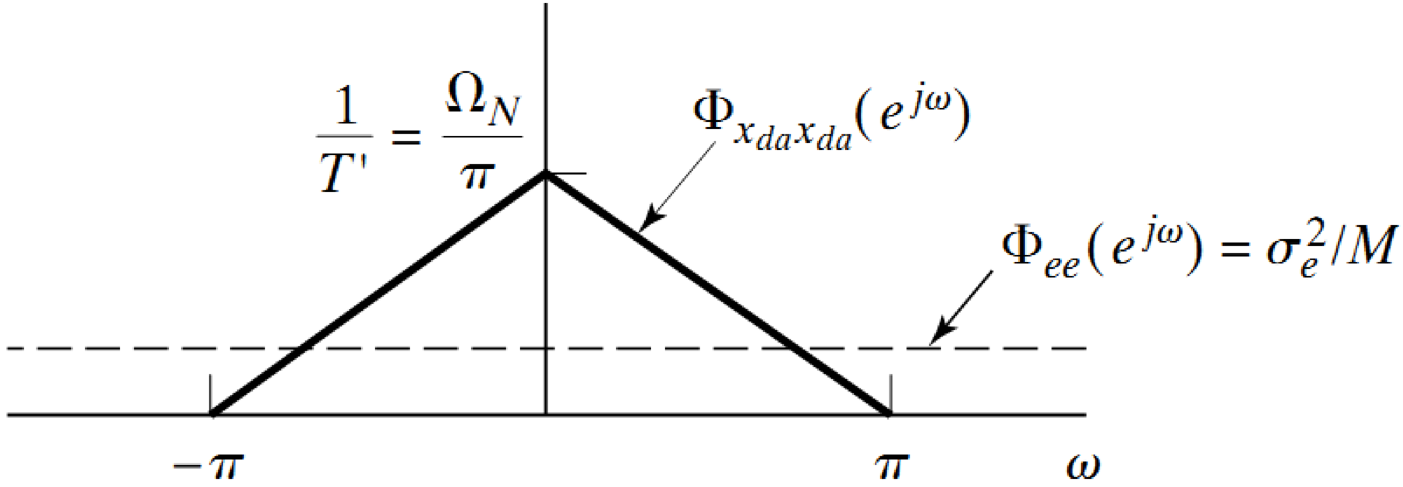


- Then, by ideal low pass with cutoff $w_c = \pi/M$ in the decimation, the noise power at the output becomes

$$\varepsilon\{e^2[n]\} = \frac{1}{2\pi}\int_{-\pi/M}^{\pi/M} \sigma_e^2 dw = \frac{\sigma_e^2}{M}$$

Powers after downsampling

- Next, the lowpass filtered signal is downsampled, and as we have seen, the signal power remains the same. Hence, the power spectrum of $x_{da}[n]$ and $x_{de}[n]$ in the frequency domain are illustrated as follows:



Noise power reduction

- Conclusion: The quantization-noise power $\varepsilon\{x_{de}^2\}$ has been reduced by a factor of M through the decimation (low-pass filtering + downsampling), while the signal power has remained the same.

$$\varepsilon\{x_{de}^2\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sigma_e^2}{M}dw = \frac{\sigma_e^2}{M} = \frac{\Delta^2}{12M}$$

- For a given quantization noise power, there is a clear tradeoff between the oversampling factor M and the quantization step Δ.

Oversampling for noise power reduction

- Remember that $\Delta = \dfrac{X_m}{2^B}$
- Therefore $\varepsilon\{x_{de}^2\} = \dfrac{1}{12M}(\dfrac{X_m}{2^B})^2$
- The above equation shows that for a fixed quantizer, the noise power can be decreased by increasing the oversampling ratio M.
- Since the signal power is independent of M, increasing M will increase the signal-to-quantization-noise ratio.

Tradeoff between oversampling and quantization bits

- Alternatively, for a fixed quantization noise power,

$$P_{de} = \varepsilon\{x_{de}^2\} = \frac{1}{12M}\left(\frac{X_m}{2^B}\right)^2$$

the required value for B is

$$B = -\frac{1}{2}\log_2 M - \frac{1}{2}\log_2 12 - \frac{1}{2}\log_2 P_{de} + \log_2 X_m$$

- From the equation, every doubling of the oversampling ratio M, we need ½ bit less to achieve a given signal-to-quantization-noise ratio.
- In other words, if we oversample by a factor $M = 4$, we need one less bit to achieve a desired accuracy in representing the signal.