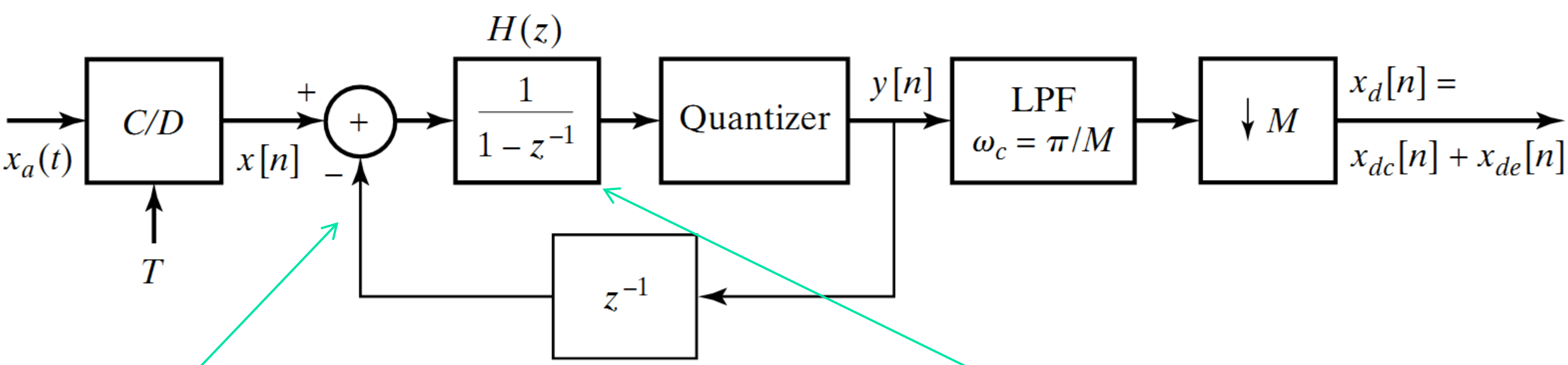
# Further reduction of quantization error

# Oversampling and Noise Shaping

- Previously, we have shown that oversampling and decimation can improve the signal-to-quantization-noise ratio.
- The result is remarkable, but if we want to make a significant reduction, we need very large sampling ratios.
  - Eg., to reduce the number of bits from 16 to 12 would require $M=4^4=256$.
- The basic concept in **noise shaping** is to modify the A/D conversion procedure so that the power density spectrum of the quantization noise is no longer uniform.

# Oversampled Quantizer with Noise Shaping

- Can be represented by the discrete-time equivalent system as follows:
  - Discrete-time form




Minus the delayed feedback: delta

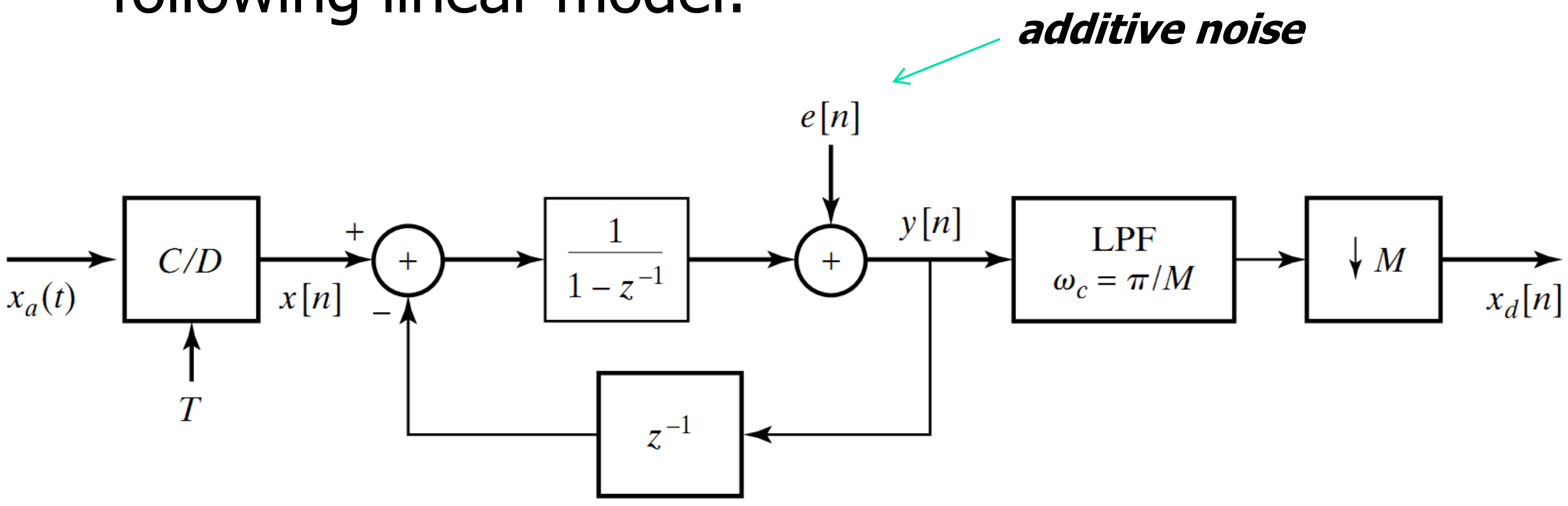
Accumulator (like an integrator): Sigma

# What is $\frac{1}{1-z^{-1}}$ ?

- $Y(z) = \frac{1}{1-z^{-1}} X(z)$, so $Y(z) = X(z) + z^{-1} Y(z)$
- Hence, $y[n] = x[n] + y[n-1]$.
- That is, the system $\frac{1}{1-z^{-1}}$ is the accumulator $y[n] = \sum_{k=-\infty}^{n} x[n]$. This is why it is named "Sigma."

# Modeling the quantization error

- As before, we model the quantization error as an additive noise source.
- Hence, the above figure can be replaced by the following linear model:



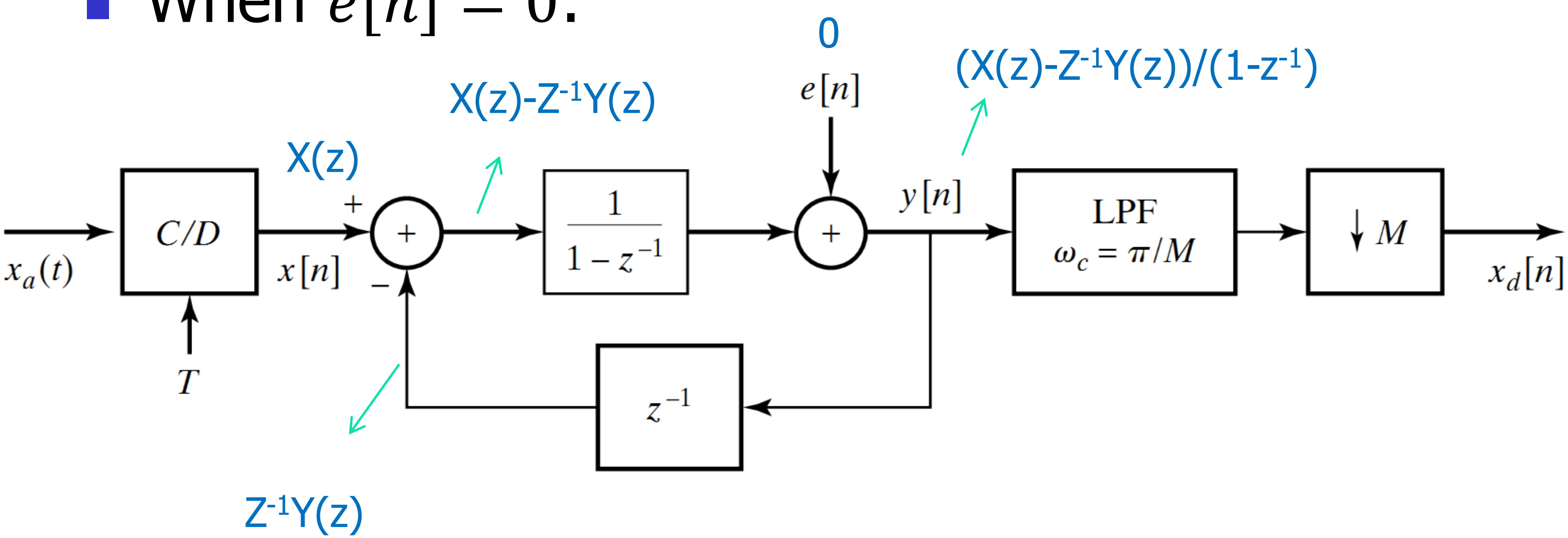
# Output of a linear system

- This linear system has two inputs, $x[n]$ and $e[n]$. According to the linearity, we can get the output $y[n]$ by

  1. set $x[n]$=0, find the output $y[n]$ w.r.t. $e[n]$
  2. set $e[n]$ =0, find the output $y[n]$ w.r.t. $x[n]$
  3. add the above two outputs.

# Transfer functions

- Consider the output in the z-domain. We denote the transfer function from $x[n]$ to $y[n]$ as $H_x(z)$ and from $e[n]$ to $y[n]$ as $H_e(z)$.
- When $e[n] = 0$:



# Output when $e[n] = 0$

- We have

$$Y[z] = \frac{X[z] - z^{-1}Y(z)}{1 - z^{-1}}$$

- So $Y[z] - z^{-1}Y[z] = X[z] - z^{-1}Y(z)$

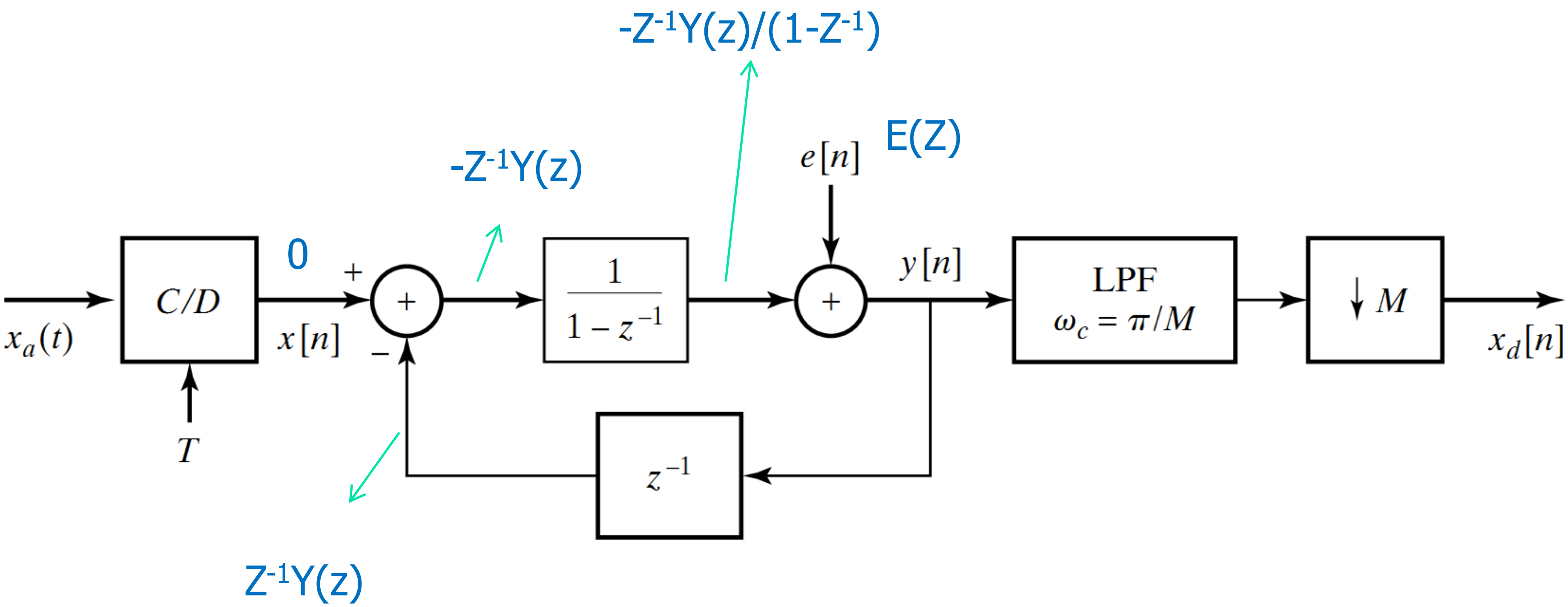
- That is

$$\boxed{Y[z] = X[z]}$$

when $E[z]$ is zero.

# Transfer functions when $x[n] = 0$

- When $x[n] = 0$:



# Output when $x[n] = 0$

- We have

$$Y[z] = E(z) - \frac{z^{-1}Y(z)}{1-z^{-1}}$$

- So $$Y[z] - z^{-1}Y[z] = E[z] - z^{-1}E[z] - z^{-1}Y(z)$$

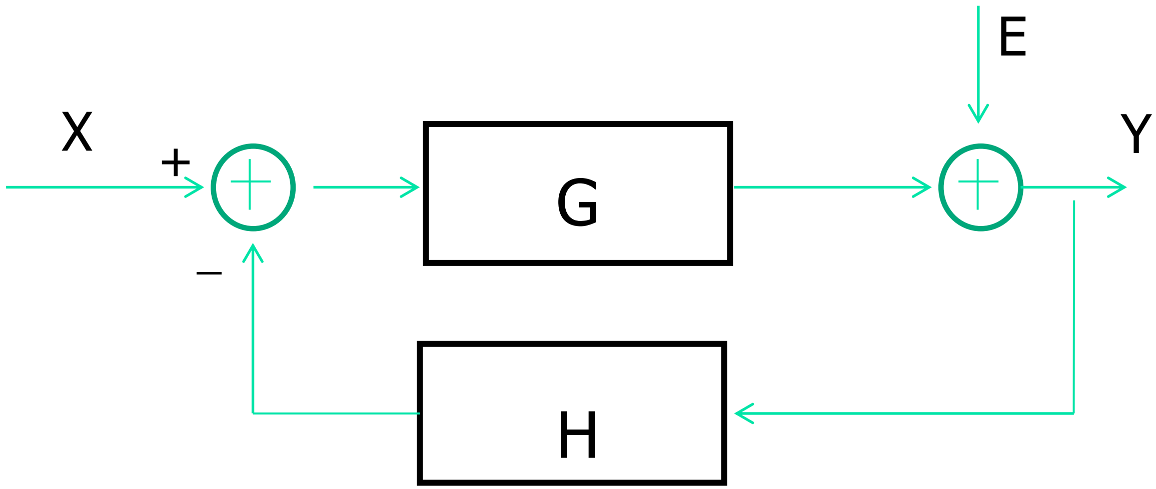
- That is

$$Y[z] = (1-z^{-1})E[z]$$

when $X[z]$ is zero.

# Remark: feedback system

- In fact, feedback systems have been widely used (serve as a fundamental architecture) in control engineering.
- Generally:




- Formula: $\dfrac{Y(z)}{X(z)} = \dfrac{G(z)}{1+G(z)H(z)}; \quad \dfrac{Y(z)}{E(z)} = \dfrac{1}{1+G(z)H(z)}$

# Another way of derivation

- From the feedback system formula, we can also obtain that

$$H_x(z) = \frac{Y(z)}{X(z)} = \frac{G(z)}{1+G(z)H(z)} = \frac{\dfrac{1}{1-Z^{-1}}}{1+\dfrac{Z^{-1}}{1-Z^{-1}}} = 1$$

$$H_e(z) = \frac{Y(z)}{E(z)} = \frac{1}{1+G(z)H(z)} = \frac{1}{1+\dfrac{Z^{-1}}{1-Z^{-1}}} = 1-Z^{-1}$$

# Time domain relation

- Hence, in the time domain, we have

$$y_x[n] = x[n]$$

$$y_e[n] = \hat{e}[n] = e[n] - e[n-1]$$

- The output y[n] can be represented as

$$y[n] = y_x[n] + y_e[n]$$

- The quantization noise $e[n]$ has been modified as $\hat{e}[n]$

# Power spectral density of the modified noise

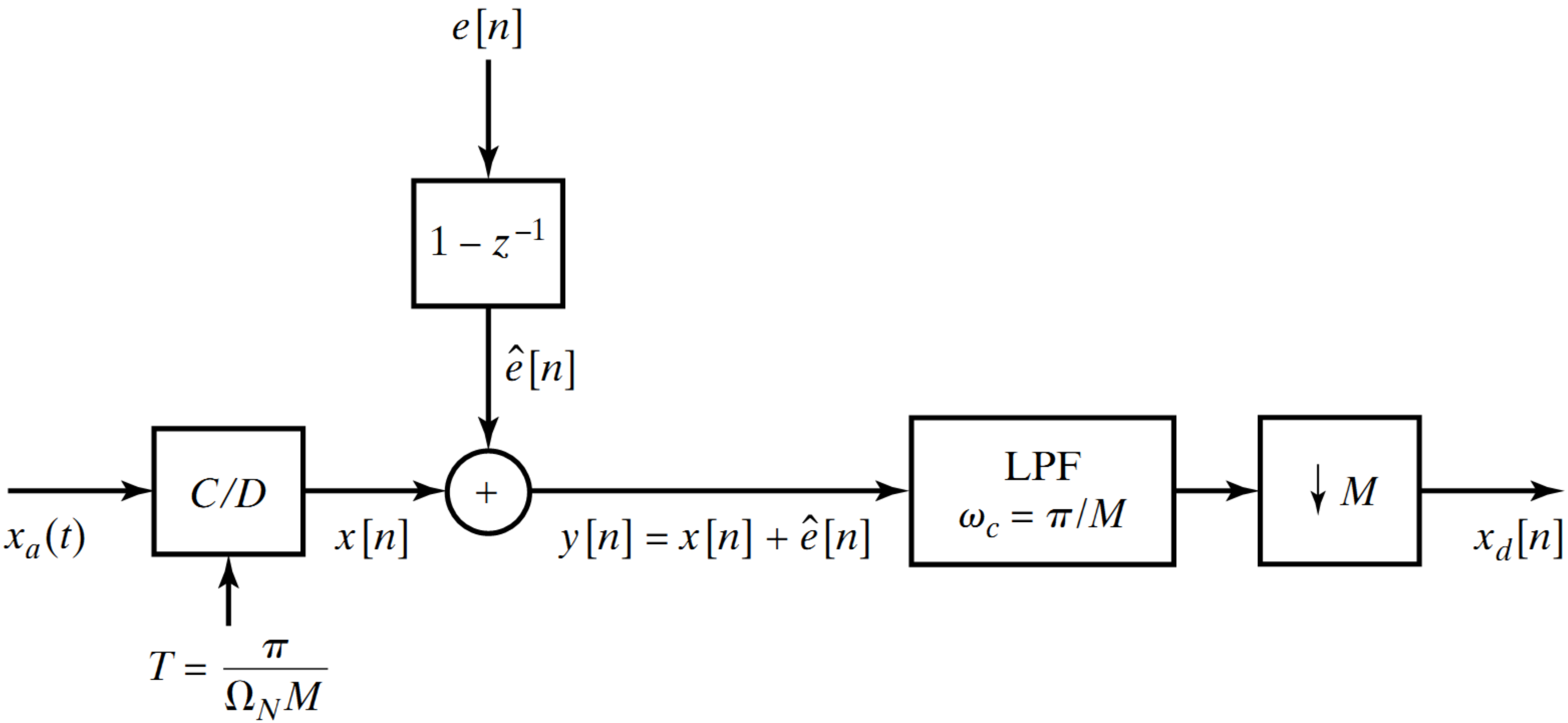
- To show the reduction of the quantization noise, let's consider the power spectral density of $\hat{e}[n]$
- Since we have the input-output relationship between $e[n]$ and $\hat{e}[n]$ as

$$Y_e(z) = (1 - Z^{-1})E(z)$$

- In the frequency domain, we have

$$\hat{E}(e^{jw}) = Y_e(e^{jw}) = (1 - e^{-jw})E(e^{jw})$$

# Equivalent system



# Power spectral density of the modified noise

- The power-spectral-density relation between the modified and original quantization noises is thus

$$
\begin{aligned}
\Phi_{\hat{e}\hat{e}}(e^{jw}) &= \| 1 - e^{-jw} \|^2 \, \Phi_{ee}(e^{jw}) \\
&= \| 1 - e^{-jw} \|^2 \, \sigma_e^2 \\
&= (1 - e^{-jw})(1 - e^{jw})\sigma_e^2 = (1 - e^{-jw} - e^{-jw} + 1)\sigma_e^2 \\
&= (2 - (e^{-jw} + e^{-jw}))\sigma_e^2 = (2 - 2\cos(w))\sigma_e^2 \\
&= \underbrace{(2\sin(w/2))^2}_{\text{p.s.d. of the modified noise}} \sigma_e^2
\end{aligned}
$$

Model the original quantization error as white noise with this variance

p.s.d. of the original noise

# Quantization-noise power

- Remember that the downsampler does not remove any of the signal power, the signal power in $x_{da}[n]$ is

$$P_{da} = \varepsilon\{x_{da}^2[n]\} = \varepsilon\{x^2[n]\} = \varepsilon\{x_a^2(t)\}$$

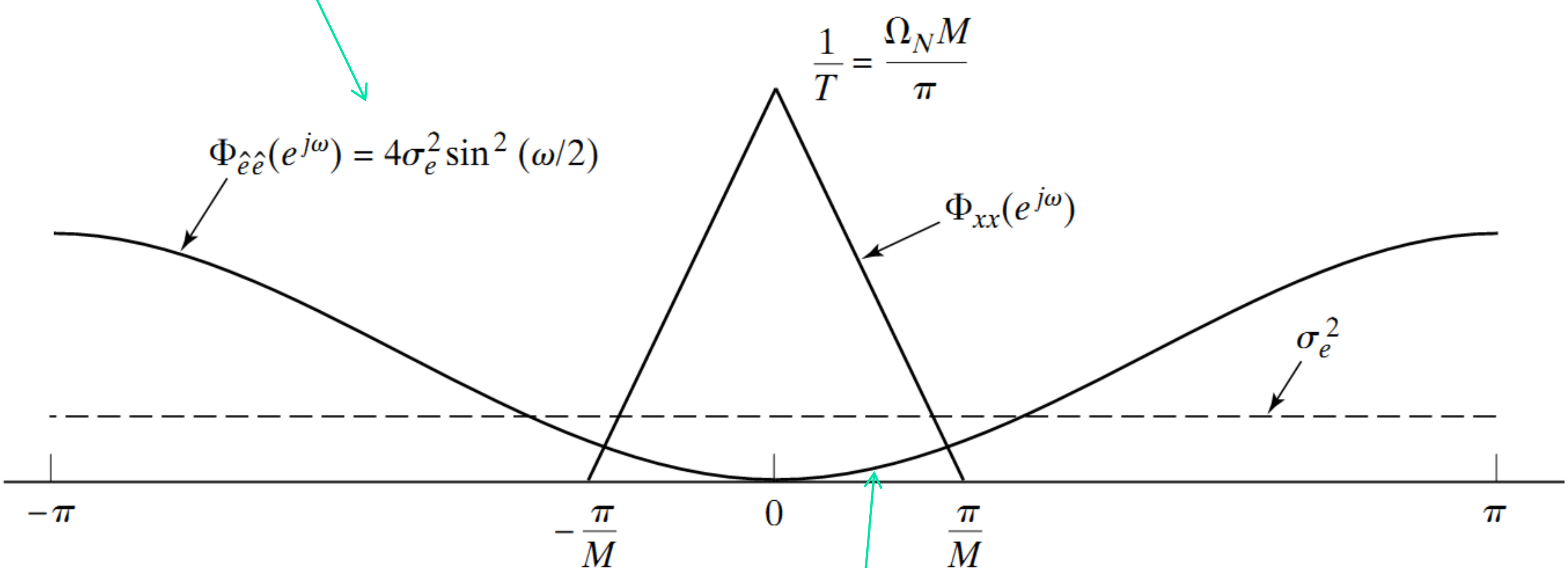
- The quantization-noise power in the final output is

$$P_{de} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{x_{de}x_{de}}(e^{jw})$$

  - See the following illustration for its computation

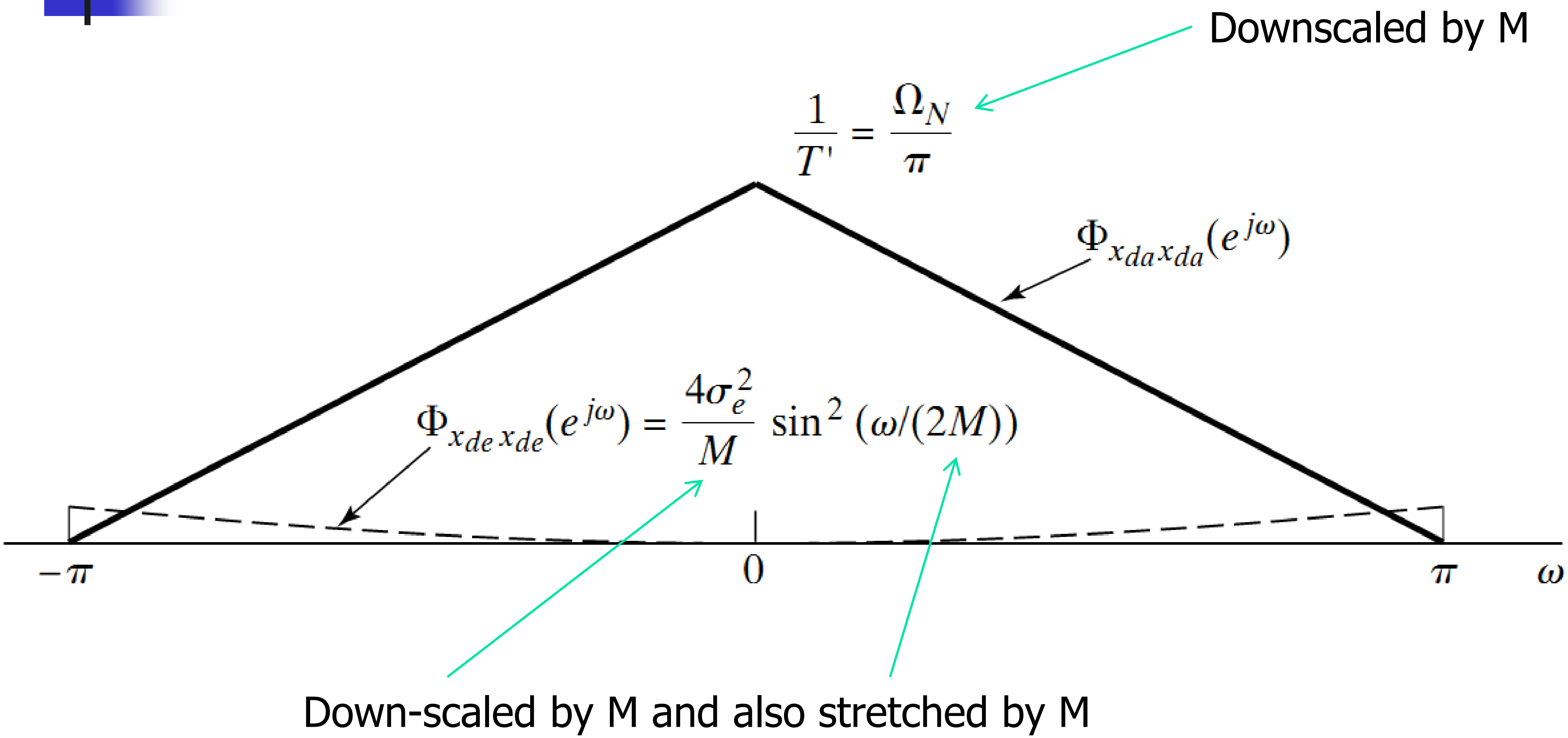
# Before decimation

Power spectral density
of modified noise




The modified noise density is non-uniform and lower in the effective band region

# After decimation



# Quantization-noise power

- Hence, the quantization-noise power in the final output is

$$P_{de} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{x_{de}x_{de}}(e^{jw}) = \frac{1}{2\pi}\frac{4\Delta^2}{12M}\int_{-\pi}^{\pi} \sin(\frac{w}{2M})^2 dw$$

$$= \frac{1}{2\pi}\frac{\Delta^2}{12M}\int_{-\pi}^{\pi} (2\sin(\frac{w}{2M}))^2 dw$$

- Assume that M is sufficiently large, we can approximate that

$$= \sin(\frac{w}{2M}) \approx \frac{w}{2M}$$

# Bits and quantization tradeoff in noise shaping

- With this approximation,

$$P_{de} = \frac{1}{36}\frac{\Delta^2\pi^2}{M^3}$$

- For a (B+1)-bit quantizer and maximum input signal level between plus and minus $X_m$, $\Delta = X_m/2^B$. To achieve a given quantization-noise power $P_{de}$, we have

$$B = -\frac{3}{2}\log_2 M + \frac{1}{2}\log_2(\pi/6) - \frac{1}{2}\log_2 P_{de} + \log_2 X_m$$

- We see that, whereas with direct quantization a doubling of the oversampling ratio $M$ gained ½ bit in quantization, the use of noise shaping results in a gain of 1.5 bits.

**TABLE 4.1** EQUIVALENT SAVINGS IN QUANTIZER BITS RELATIVE TO $M = 1$ FOR DIRECT QUANTIZATION AND FIRST-ORDER NOISE SHAPING

| M | Direct quantization | Noise shaping |
|---|---|---|
| 4 | 1 | 2.2 |
| 8 | 1.5 | 3.7 |
| 16 | 2 | 5.1 |
| 32 | 2.5 | 6.6 |
| 64 | 3 | 8.1 |

# Second-order noise shaping

- The noise-shaping strategy can be extended by incorporating a second stage of accumulation, as shown in the following:

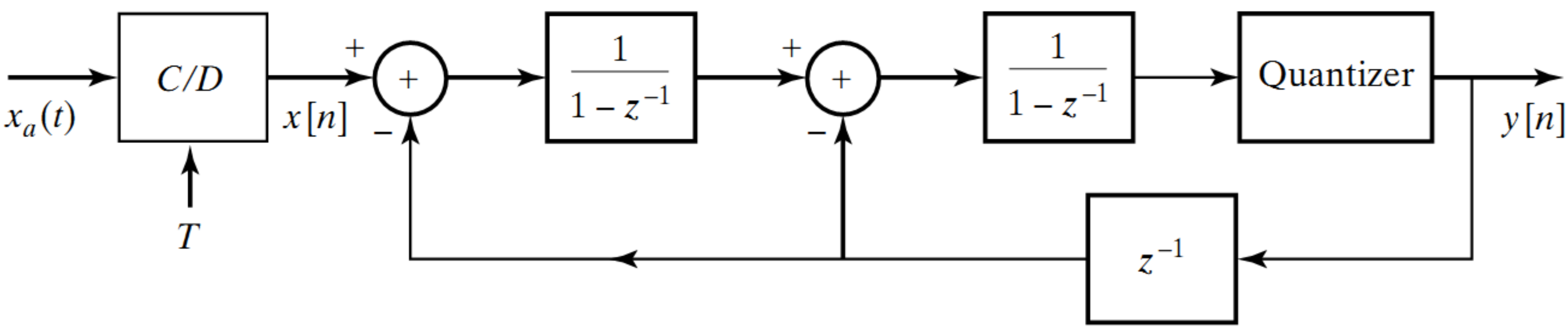



**Figure 4.66** Oversampled quantizer with second-order noise shaping.

# Second-order (i.e., 2-stage) noise shaping

- In the two-stage case, it can be derived that

$$H_e(z) = (1 - z^{-1})^2$$

$$\phi_{\hat{e}\hat{e}}(e^{jw}) = \sigma_e^2 (2\sin(w/2))^4$$

- In general, if we extend the case to p-stages, the corresponding noise shaping is given by

$$\phi_{\hat{e}\hat{e}}(e^{jw}) = \sigma_e^2 (2\sin(w/2))^{2p}$$

- By evaluation, with p=2 and M=64, we obtain almost 13 bits of increase in accuracy, suggesting that a 1-bit quantizer could achieve about 14-bit accuracy at the output of the decimator.

- Noise shaping can be employed for audio signal recording.

- Although multiple feedback loops promise greatly increased quantization-noise reduction, for large values of p, there is an increased potential for instability and oscillations to occur.