Several ways of processing the frequencies of signals

- Upsampling or downsampling the time-domain signals.
- Filtering: Processing the time-domain signals using an LTI system, and the frequency response of the system is easily obtainable via substituting $z = e^{j\omega}$ in the system function of the LTI system.
- **Directly transform using DTFT**: transform the time-domain signal into the frequency domain using DTFT.
  - However, we can only handle the finite-length signal in practice.
  - We'll illustrate the effect of doing this.

What is Spectrum in general?

- There are different kinds of Fourier transforms.
- What is the one that **defines generally the concept of "spectrum?"**
- Answer: The continuous Fourier transform (CFT) defines the spectrum in general.
- CFT pair

$$\boxed{F(jw) = \int_{-\infty}^{\infty} f(t) e^{-jwt} dt}$$

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(jw) e^{jwt} dw}$$

Recall: Sampling for Processing

- In DSP, we have to sample continuous-time (analog) signals into discrete-time signals for processing.
- **Sampling in time domain**: remember that if we perform sampling on an analog signal x_a,

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t - nT)$$

- **In frequency domain**: the spectrum becomes the sum of infinite many shifted copies of the original spectrum,

$$\boxed{X_s(jw) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_a\left(jw + j\frac{2\pi r}{T}\right)}$$

Recall: Aliasing and Sampling Theorem

- Hence, if the analog signal is band-limited with the frequency bound w_b:

$$X_a(j\omega) = 0 \text{ for } |\omega| > \omega_b,$$

- and the sampling rate satisfies the Nyquist sampling theorem that $\omega_s > 2\omega_b$.
- Then, we know that the aliasing effect can be avoided, and the analog signal x_a can be reconstructed by applying an ideal low-pass filter with the cutoff frequency ω_s.

Recall: Why using DTFT

- Since we have to handle discrete-time signals in DSP, we have defined a Fourier transform, DTFT, particularly for discrete-time signal processing.

- DTFT pair:

$$X\left(e^{jw}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-jwn} \qquad x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{jw}\right)e^{jwn}dw$$

- The spectrum of DTFT defined in [-π, π] implicitly assumes that the discrete-time signals are sampled satisfying the Nyquist rate.

Recall: How DTFT approximates CFT?

- **Exact recovery:**

When $X_a(j\omega)$ is band-limited and the sampling rate is high enough satisfying the sampling theorem, $X_a(j\omega)$ can be exactly recovered from the DTFT $X(e^{j\omega T})$ (by investigating only the range only in $\omega \in [-\pi/T, \pi/T\,]$).

Recall: How DTFT approximates CFT?

- **Approximation**: When the sampling rate is not high enough or $X_a(j\omega)$ is not band-limited, $X(e^{j\omega T})$ is only an approximation of $X_a(j\omega)$ because of the aliasing effect.
  - Some high-frequency part will be folded to the range $[-\pi/T, \pi/T]$.

- **How DTFT approximates CFT** can be completely characterized **by** sampling theorem and aliasing effect.

How to compute the spectrum?

- After converting an analog signal to discrete-time samples, another practical problem is how to compute the CFT spectrum (so that we can transform the signal to the frequency domain).
- Although DTFT can be used to recover the exact spectrum for band-limited signal under high-enough sampling rate, it requires summing from $n=-\infty$ to ∞.
- This is still infeasible in practice since we cannot compute the sum for an infinite-long signal.

Approximation by Finite-duration Signals

- **So, what can we do?**
- A practical way commonly employed is to use a finite range $t \in [-\boldsymbol{T}/2, \boldsymbol{T}/2]$ of the analog signal $x_a(t)$, and see how it can approximate the spectrum of the entire signal defined in $t \in (-\infty, \infty)$.
- After sampling with $x[n]=x_a(nT)$, there are $N=\boldsymbol{T}/T$ samples in the range $t \in [-\boldsymbol{T}/2, \boldsymbol{T}/2]$, resulting a finite-duration discrete-time signal $y[n]$ from $x[n]$,

$$y[n] = \begin{cases} x[n], & 0 \le n < N-1 \\ 0, & otherwise \end{cases}$$

Approximation by Finite-duration Signals

- Compute the DTFT for the finite-duration signal $y[n]$ (now feasible in practice), and see how it can approximate the DTFT of $x[n]$.
- We can expect that, the larger is N (or equivalently, the larger is the range $\boldsymbol{T}$), the better is the approximated spectrum.

Approximation by Finite-duration Signals

- In the above, there are two main factors affecting the approximation: (1) sampling, and (2) applying only a finite range of the signal.
- We have already seen how it approximates the waveform in the spectral domain by sampling in time domain)
- Now we focus on the other factor: **what is the approximation** if we **employ only a finite range of the signal?**

Rectangular Windowing

- Employing a finite range $t \in [\boldsymbol{T}/2, \boldsymbol{T}/2]$ of the analog signal $x_a(t)$, is equivalent to multiplying the original signal $x_a(t)$ with a rectangular window:

$$\boxed{w_R(t) = \begin{cases} 1, & t \in [-T/2, T/2] \\ 0, & otherwise \end{cases}}$$

- **Time domain multiplication $\leftrightarrow$ Frequency domain convolution** (up to a scale)
- So, in the frequency domain, the spectrum $X_a(j\omega)$ is convolved with the CFT $W_R(j\omega)$ that is a sinc function

$$\frac{\sin(\omega T/2)}{\omega/2}$$

Recall: Basic CFT Properties

Modulation	$x(t)\cos(\omega_0 t)$	$\frac{1}{2}X(j(\omega-\omega_0))+\frac{1}{2}X(j(\omega+\omega_0))$
Differentiation	$\frac{d^k x(t)}{dt^k}$	$(j\omega)^k X(j\omega)$
Convolution	$x(t)*h(t)$	$X(j\omega)H(j\omega)$
Multiplication	$x(t)p(t)$	$\frac{1}{2\pi}X(j\omega)*P(j\omega)$

Note that there is a scale ½π.

Multiplication with a rectangular window in the analog domain

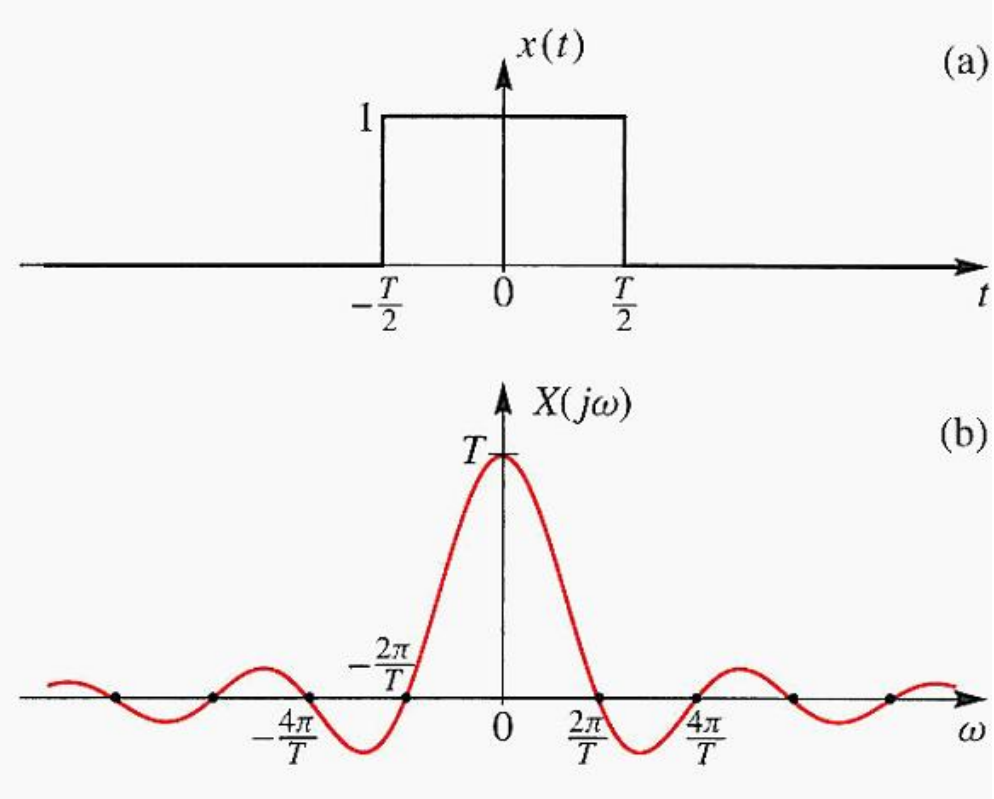


Figure 11-4: Fourier transform of a rectangular pulse. (a) Time function $x(t) = u(t + \frac{1}{2}T) - u(t - \frac{1}{2}T)$, and (b) Corresponding Fourier transform $X(j\omega)$ is a sinc function.

Multiplication with the rectangular window,

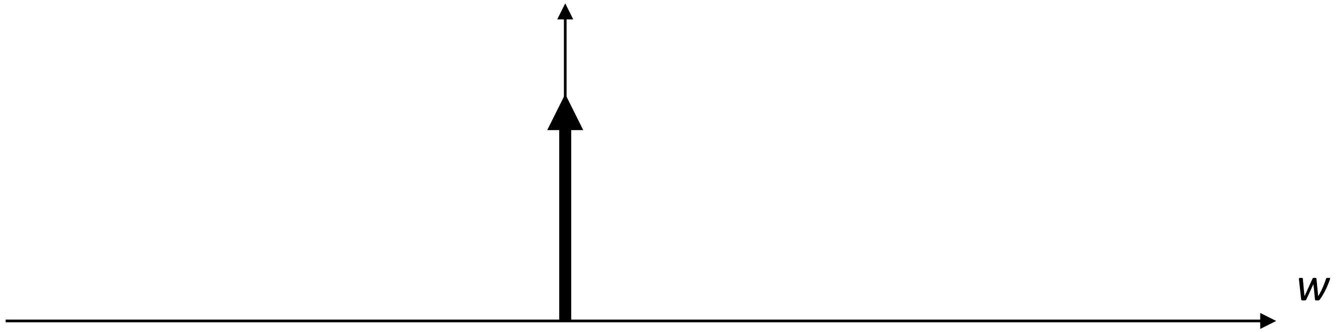
$w_R(t)$

Convolution with the following sinc function then divided by 2π,

$$\frac{\sin(\omega T/2)}{\omega/2}$$

Convolution with Sinc Function

- What is the effect of convolution with a sinc function?
- Note that when $\boldsymbol{T} \rightarrow \infty$ (that is, $w_R(t) \rightarrow 1$), the sinc function approaches to the delta function $2\pi\delta(\omega)$.



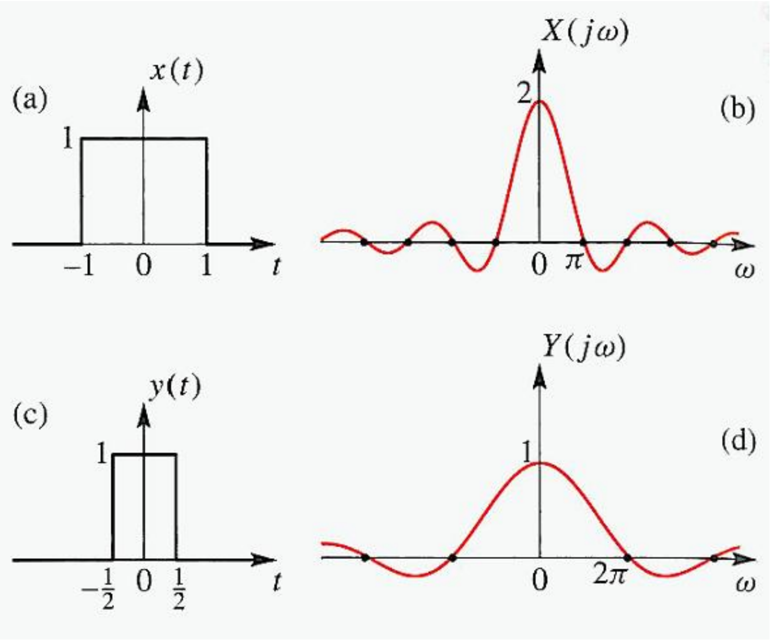
- In the frequency domain, convolution with a delta function and then divided by 2π recovers exactly the original spectrum.

Convolution with Sinc Function

- In general, when the window is wider, the sinc function is narrower, and vice versa.
- It is easy to see that convolution with a narrower sinc function approximates the original spectrum better.

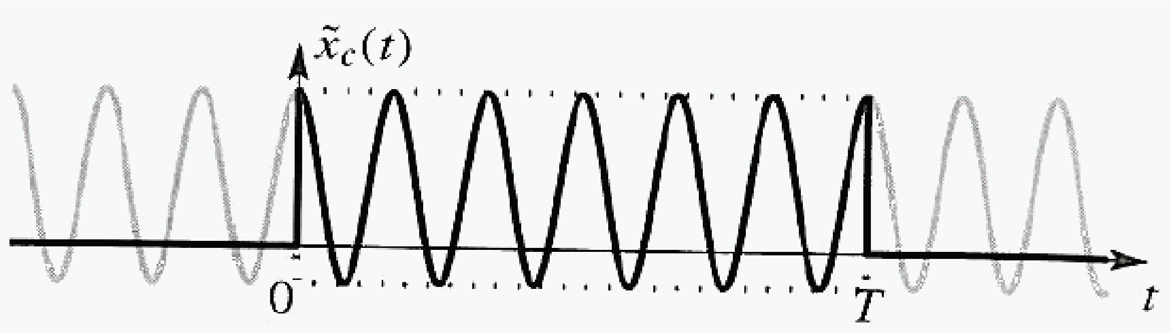
Stretching a time signal will compress its Fourier transform.

Compressing a time signal will stretch its Fourier transform.

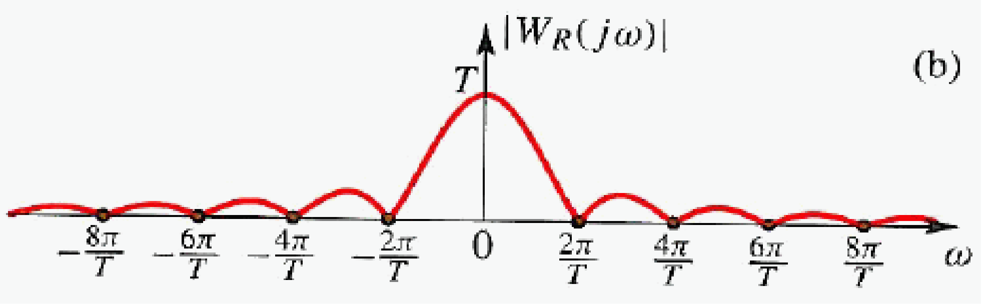


Example: approximation for a single sinusoid

- Assume that there is a single sinusoidal signal applied by the rectangular window in time domain:

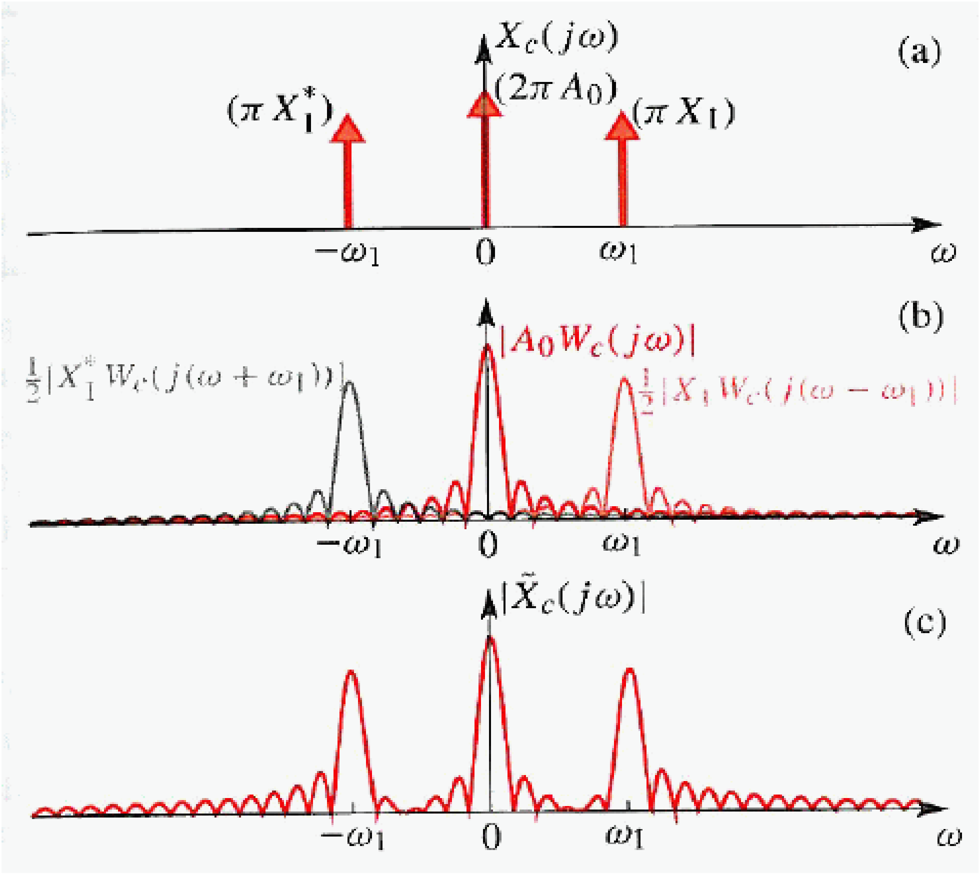


- Let us consider the magnitude response of the sinc function:



Example: approximation for a single sinusoid

- Then, convolution of a spectrum of a single sinusoid with the sinc function looks like



Continuous FT of a cosine signal

The frequency magnitude of the three terms in different colors

The frequency magnitude of the cosine signal

Approximation by rectangular windowing in the Analog domain

- It can be seen that, by using a finite-length portion of an analog signal, the approximation can be analyzed by convolving a sinc function in the frequency domain.
- This convolution disturbs the original spectrum: the spectrum is blurred and somewhat oscillated.
- The narrower is the sinc function (i.e., the longer is the window), the more accurate is the approximation.