Discrete-time Fourier Transform (DTFT)

- The DTFT pair is defined as follows: Let $x[n]$ be a discrete-time signal, $n \in Z$.

Forward DTFT:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Inverse DTFT:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

Convolution of Discrete-time Signals

- Definition of **discrete-time convolution**:

$$y[n] \equiv x[n] * h[n]$$

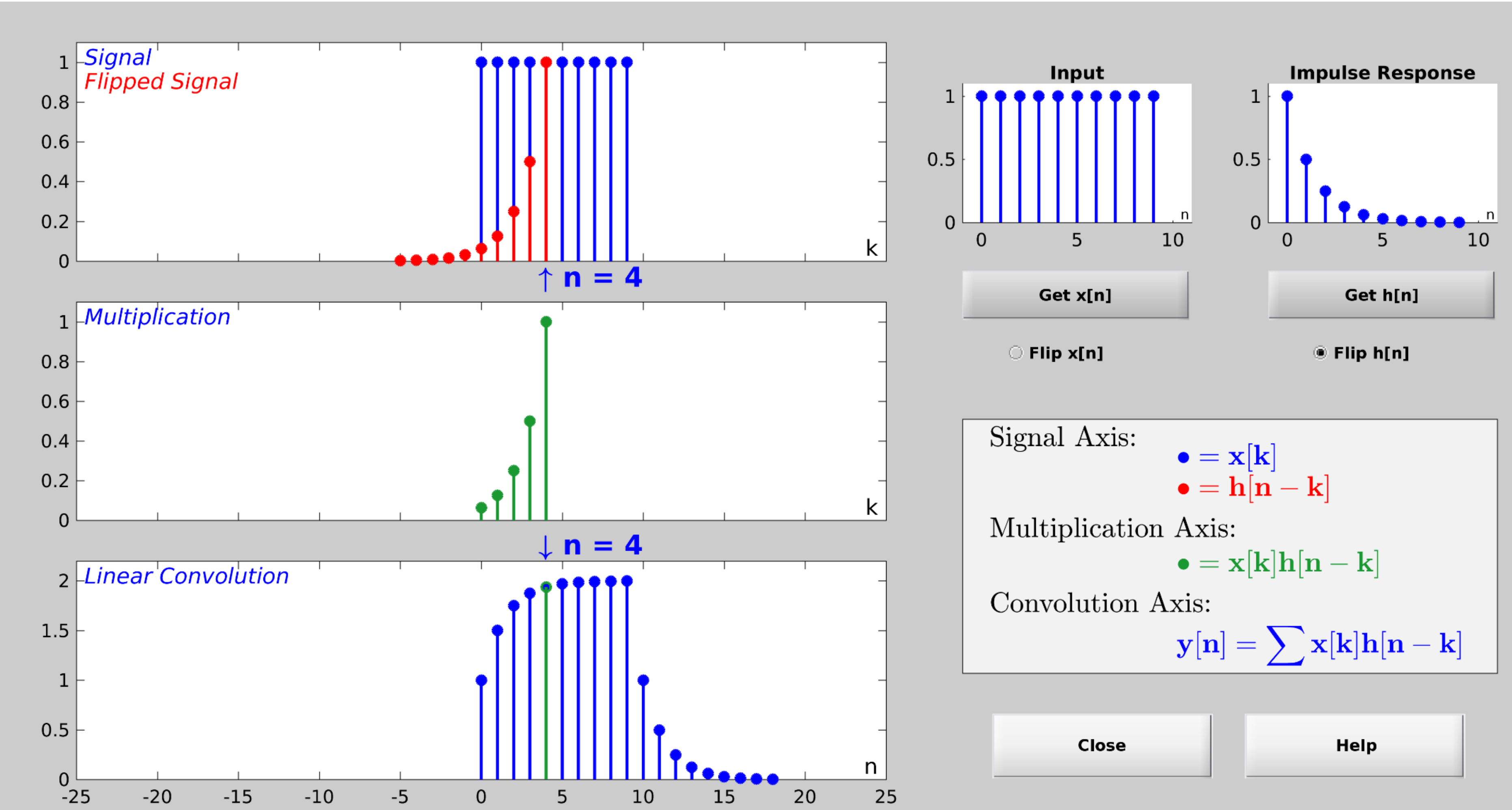
$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

- Convolution is commutative

$$y[n] \equiv x[n] * h[n] = h[n] * x[n]$$

- Similar to the definition for the continuous-time signals, but summation is used instead of integration.

Example



Signal
Flipped Signal
k
↑ n = 4
Multiplication
k
↓ n = 4
Linear Convolution
n
Input
Impulse Response
Get x[n]
Get h[n]
Flip x[n]
Flip h[n]
Signal Axis:
● = x[k]
● = h[n − k]
Multiplication Axis:
● = x[k]h[n − k]
Convolution Axis:
y[n] = ∑ x[k]h[n − k]
Close
Help

Example

- $h_1[n] * h_2[n]$

n	$n<0$	0	1	2	3	4	5	6	$n>6$
$h_1[n]$	0	1	1	1	1	0	0	0	0
$h_2[n]$	0	0	1	1	1				
$h_2[0]h_1[n]$	0	0	0	0	**0**	0	0	0	0
$h_2[1]h_1[n-1]$	0	0	1	1	**1**	1	0	0	0
$h_2[2]h_1[n-2]$	0	0	0	1	**1**	1	1	0	0
$h_2[3]h_1[n-3]$	0	0	0	0	**1**	1	1	1	0
$h[n]$	0	0	1	2	**3**	3	2	1	0

Analogous to the arithmetic multiplication

- In fact, convolution has an intuitive explanation: it is just the “arithmetic multiplication”
- Eg.,

$x[n] = 0., 0, 5, 2, 3, 0, 0\ldots$

$h[n] = 0., 0, 1, 4, 3, 0, 0\ldots$

- $x[n] * h[n]$:

$$
\begin{array}{r}
0.,\ 0,\ 5,\ 2,\ 3,\ 0,\ 0,\ldots \\
*)\ 0.,\ 0,\ 1,\ 4,\ 3,\ 0,\ 0,\ldots \\
\hline
\ldots.0,\ 0,\ 5,\ 2,\ 3,\ 0,\ \ 0,\ 0,\ldots\ldots. \\
\ldots\ldots0,\ 0, 20, 8, 12,\ 0,\ 0, 0,\ldots.. \\
\ldots\ldots\ldots0,\ 0, 15,\ 6,\ 9,\ 0, 0, 0,\ldots\ldots. \\
\hline
0,\ 0,\ 5, 22, 26, 18, 9,\ 0, 0, 0,\ldots..
\end{array}
$$

Convolution result

Interpreted as Polynomial Multiplication

- Convolution can also be explained as **polynomial multiplication**:
- Consider $f[n] * h[n]$

3. Let $f(z) = \sum_{n=0}^{\infty} f[n]z^n$

$$h(z) = \sum_{n=0}^{\infty} h[n]z^n$$

then $g[n]$ is the coefficient of z^n for the polynomial $g(z)$ obtained by $g(z) = f(z) \times h(z)$

4.

	f_0	f_1	f_2	f_3	f_4	f_5
h_0	f_0h_0	f_1h_0	f_2h_0	f_3h_0	f_4h_0	f_5h_0
h_1	f_0h_1	f_1h_1	f_2h_1	f_3h_1	f_4h_1	f_5h_1
h_2	f_0h_2	f_1h_2	f_2h_2	f_3h_2	f_4h_2	f_5h_2
h_3	f_0h_3	f_1h_3	f_2h_3	f_3h_3	f_4h_3	f_5h_3
h_4	f_0h_4	f_1h_4	f_2h_4	f_3h_4	f_4h_4	f_5h_4

g_0 g_1 g_2 g_3 g_4 … g_9

- Summing up along the arrow gives the g[n]

DTFT Theorems Reviews (From Oppenheim)

- **Linearity**

$$x_1[n] \leftrightarrow X_1(e^{jw}) \text{ and } x_2[n] \leftrightarrow X_2(e^{jw})$$

implies that

$$a_1x_1[n] \ + \ a_2x_2[n] \leftrightarrow a_1X_1(e^{jw}) \ + \ a_2X_2(e^{jw})$$

- **Time shifting**

$$x[n] \leftrightarrow X(e^{jw})$$

implies that

$$x\left[n - n_d\right] \leftrightarrow e^{-jwn_d} X\left(e^{jw}\right)$$

DTFT Theorems Reviews (continue)

- **Frequency shifting**

$$x[n] \leftrightarrow X(e^{jw})$$

implies that

$$e^{jw_0 n} x[n] \leftrightarrow X\left(e^{j(w-w_0)}\right)$$

- **Time reversal**

$$x[n] \leftrightarrow X(e^{jw})$$

If the sequence is time reversed as being $x[-n]$,

then $x[-n] \leftrightarrow X(e^{-jw})$

DTFT Theorems Reviews (continue)

- **Differentiation in frequency**

$$x[n] \leftrightarrow X(e^{jw})$$

implies that

$$nx[n] \leftrightarrow j\frac{dX\left(e^{jw}\right)}{dw}$$

DTFT Theorems (continue)

- **Parseval's theorem**

$$x[n] \leftrightarrow X(e^{jw})$$

implies that

$$E = \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X\left(e^{jw}\right) \right|^2 dw$$

DTFT Theorems (continue)

- **The convolution theorem**

$$x[n] \leftrightarrow X(e^{jw}) \text{ and } h[n] \leftrightarrow H(e^{jw}),$$

and if $y[n] = x[n] * h[n]$, then

$$Y(e^{jw}) = X(e^{jw})H(e^{jw})$$

- **The modulation or windowing theorem**

$$x[n] \leftrightarrow X(e^{jw}) \text{ and } w[n] \leftrightarrow W(e^{jw}),$$

and if $y[n] = x[n]w[n]$, then

$$Y\left(e^{jw}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right)W\left(e^{j(w-\theta)}\right)d\theta$$

Periodic convolution,
$Y(e^{jw}) = Y(e^{j(w+2\pi)})$

DTFT Theorems (continue)

- Hence, unlike continuous Fourier transform, DTFT's time and frequency domains are not that analogous.
- For example, convolution in the time domain implies multiplication in the frequency domain.
- But multiplication in the time domain implies periodic-convolution in the frequency domain.

DTFT Pairs

$$\delta[n] \leftrightarrow 1$$

$$\delta[n-n_0] \leftrightarrow e^{-jwn_0}$$

$$1 \ (-\infty < n < \infty) \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta[w+2\pi k]$$

DTFT Pairs (continue)

Real Exponentials

$$a^n u[n] \quad (|a|<1) \leftrightarrow \frac{1}{1-ae^{-jw}}$$

$$u[n] \leftrightarrow \frac{1}{1-e^{-jw}} + \sum_{k=-\infty}^{\infty} \pi\delta[w+2\pi k]$$

$$(n+1)a^n u[n] \quad (|a|<1) \leftrightarrow \frac{1}{\left(1-ae^{-jw}\right)^2}$$

Example: Determining a Fourier Transform Pair-wise Formulas

- Suppose we wish to find the DTFT of

$x[n] = a^n u[n-5], \ |a| < 1$

$$x_1[n] = a^n u[n] \leftrightarrow \frac{1}{1-ae^{-jw}} = X_1\left(e^{jw}\right)$$

$$x_2[n] = x_1[n-5]$$

$$\text{Thus, } X_2\left(e^{jw}\right) = e^{-j5w} X_1\left(e^{jw}\right) = \frac{e^{-j5w}}{1-ae^{-jw}}$$

$$x[n] = a^5 x_2[n], \text{ so } X\left(e^{jw}\right) = \frac{a^5 e^{-j5w}}{1-ae^{-jw}}$$

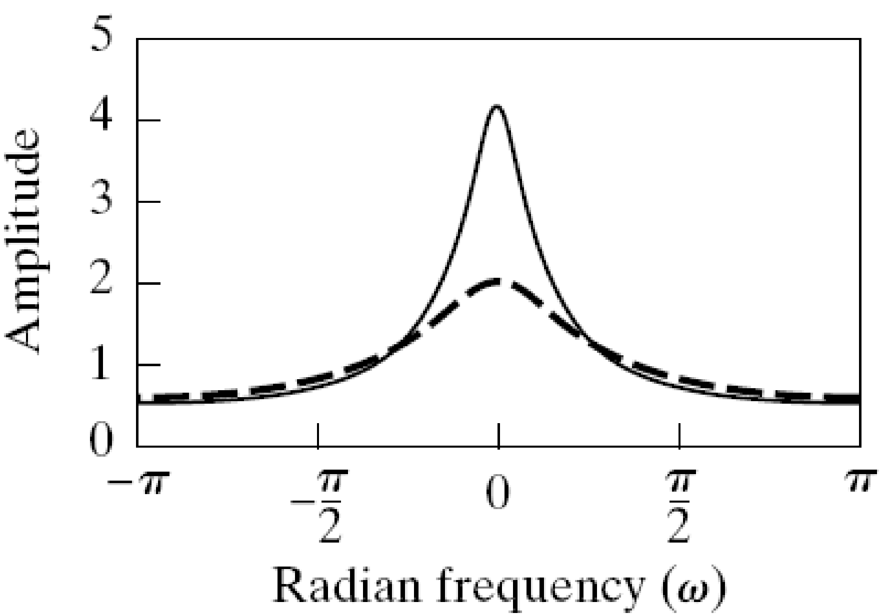
Symmetry Property

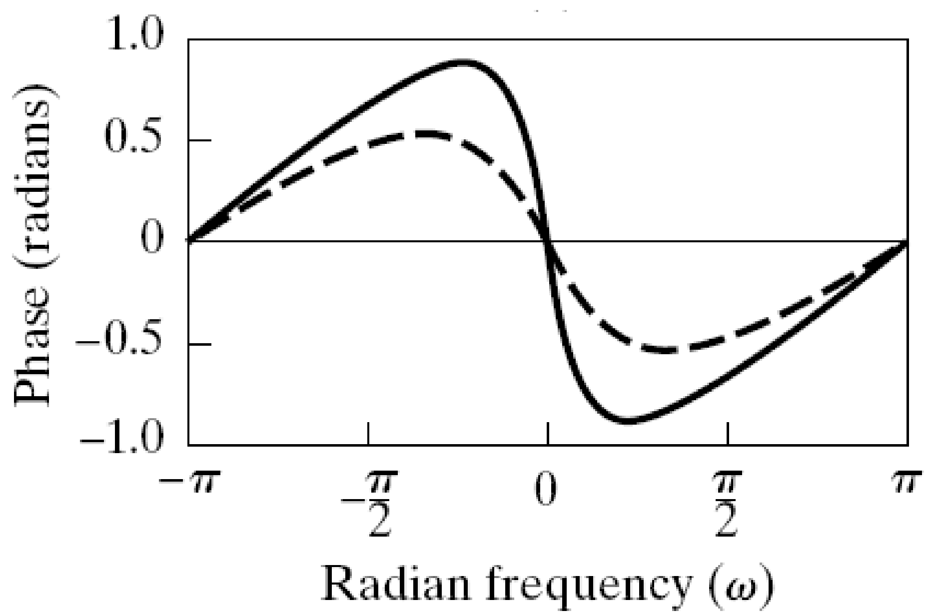
- Even function: $f[n] = f[-n]$
- Odd function: : $f[n] = -f[-n]$

- If $x[n]$ is a real-valued sequence (i.e., $x[n]$ is real) then
  - $|X(e^{jw})| = |X(e^{-jw})|$ (magnitude is an even function)
  - $\angle X(e^{jw}) = -\angle X(e^{-jw})$ (phase is an odd function)

Example of Symmetry Properties

- The Fourier transform of the real sequence $x[n] = a^n u[n]$ for $a < 1$ is $X\left(e^{jw}\right) = \dfrac{1}{1 - ae^{-jw}}$
  - Its magnitude is an even function, and phase is





Symmetry Property (cont.)

- If $x[n]$ is both **real and even**, then $X(e^{jw})$ is also both **real and even**.
  - Eg., rectangular function, cosine function.

More detailed table for your ref.

DTFT Theorems and Properties

Property	Time Domain	Frequency Domain
Notation:	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_1(\omega)$
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting:	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution:	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Multiplication:	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{2\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Correlation:	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$
		$= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ real]
Frequency Differentiation:	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Wiener-Khintchine:	$r_{xx}(l) = x(l) * x(-l)$	$S_{xx}(\omega) = \lvert X(\omega)\rvert^2$

In this table, $X(e^{j\omega})$ is abbreviated as $X(\omega)$

More detailed table for your ref.

DTFT Symmetry Properties

Time Sequence	DTFT
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x(-n)$	$X(-\omega)$
$x_R(n)$	$X_e(\omega) = \frac{1}{2}[X(\omega) + X^*(-\omega)]$
$jx_I(n)$	$X_o(\omega) = \frac{1}{2}[X(\omega) - X^*(-\omega)]$
$x(n)$ real	$X(\omega) = X^*(-\omega)$ $X_R(\omega) = X_R(-\omega)$ $X_I(\omega) = -X_I(-\omega)$ $\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$ $\angle X(\omega) = -\angle X(-\omega)$
$x'_e(n) = \frac{1}{2}[x(n) + x^*(-n)]$	$X_R(\omega)$
$x'_o(n) = \frac{1}{2}[x(n) - x^*(-n)]$	$jX_I(\omega)$

In this table, $X(e^{j\omega})$ is abbreviated as $X(\omega)$