Discrete-time Systems

- We have introduced the **signals** and their frequency domains. Now, we proceed to the **systems**.
- System: A transformation or operator $T\{\cdot\}$ that **maps an input signal $x[n]$ into an output signal $y[n]$.**

$$y[n] = T\{x[n]\}$$

$$x[n] \longrightarrow \boxed{T\{\cdot\}} \longrightarrow y[n]$$

Linear System

- A **linear system** is referred to as a system $T\{\cdot\}$, where $y[n] = T\{x[n]\}$ implies the following property for all α, β:

$$x[n] = \alpha x_1[n] + \beta x_2[n] \quad \overset{T}{\longmapsto} \quad y[n] = \alpha y_1[n] + \beta y_2[n]$$

Example of linear Systems

- A linear system constructed by a **linear combination of the current and finite previous inputs** is referred to as a kind of (Finite Impulse Response) **FIR system** or **FIR filter**.
- General form of the FIR system:

$$y[n] = \sum_{k=0}^{M} b_k\, x[n-k]$$

Example of FIR filter: running average filters

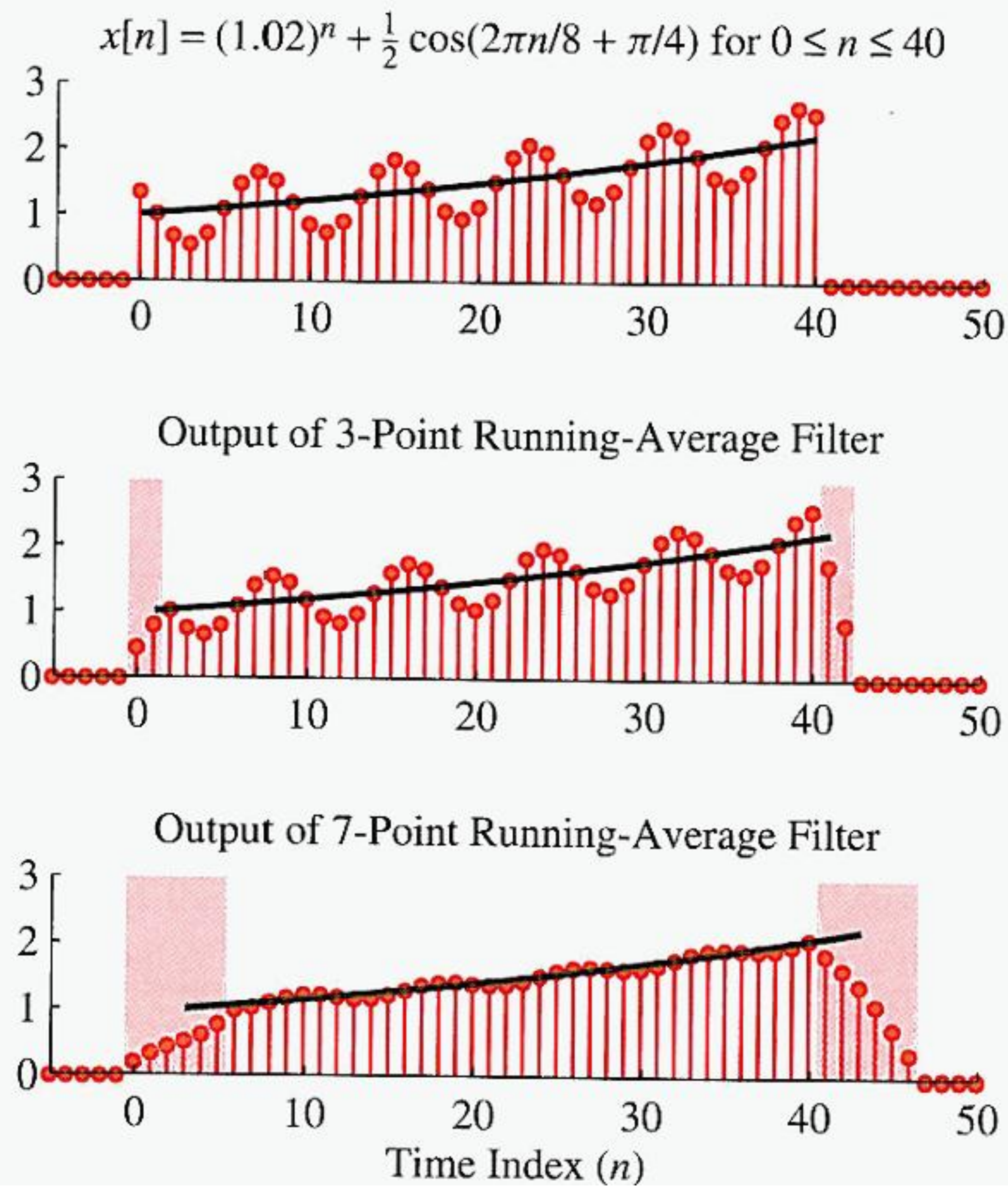
- 3-point running average filter:

$$y[n] = \tfrac{1}{3}(x[n] + x[n-1] + x[n-2])$$

- 7-point running average filter:

$$y_7[n] = \frac{1}{7}\left(\sum_{k=0}^{6} x[n-k]\right)$$

- Both of the above are FIR filters.



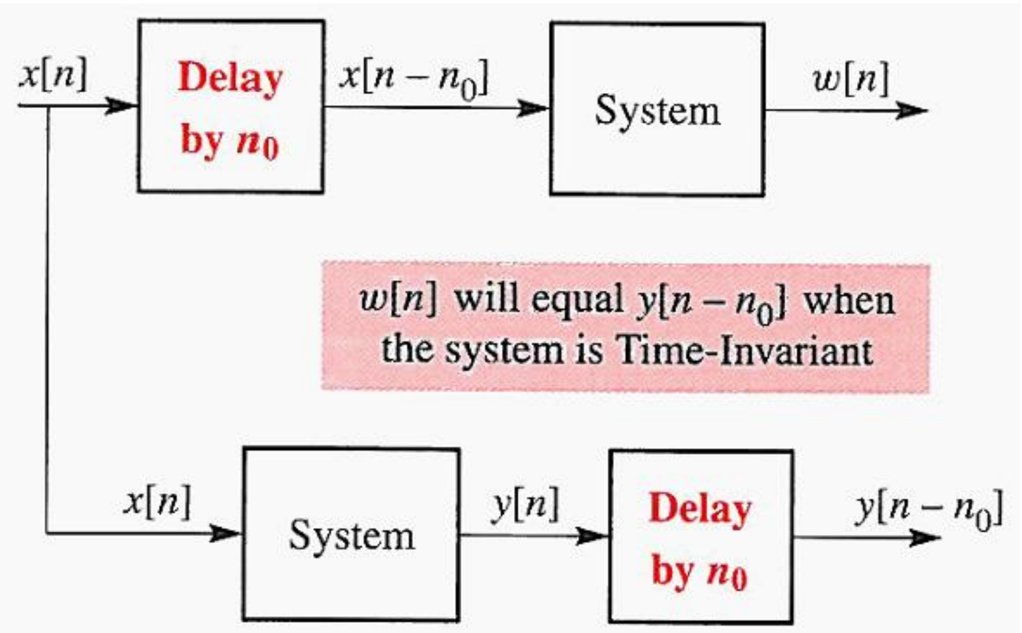
$x[n] = (1.02)^n + \frac{1}{2}\cos(2\pi n/8 + \pi/4)$ for $0 \le n \le 40$
Output of 3-Point Running-Average Filter
Output of 7-Point Running-Average Filter
Time Index (n)

Time-invariant System

- **time-invariant system** is referred to as a system $T\{\cdot\}$, where $y[n] = T\{x[n]\}$ implies that

$$x[n-n_0] \stackrel{T}{\longmapsto} y[n-n_0]$$

for all n_0. That is, when the input is delayed (shifted) by n_0, the output is delayed by the same amount.



Example of time-invariant systems

- It can be easily verified that the FIR filter of the form

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

is also a time-invariant system.

Linear Time-invariant (LTI) System

- A system that is both linear and time-invariant is called an **LTI system**.
- So, FIR filter is an LTI system.

IIR System or IIR Filter

- **IIR system (or IIR filter):** A system constructed by **linear combination of both of the current and finite previous inputs as well as the previous outputs.**
- General form of IIR filter:

$$y[n] = \sum_{\ell=1}^{N} a_\ell y[n-\ell] + \sum_{k=0}^{M} b_k x[n-k]$$

- **Recursion**: In IIR filter, previous outputs have been used recursively.

General form of LTI system

- It can be easily verified that IIR filter is also an LTI system.
- We have the definition of LTI system, but **what is the general form of an LTI system?**
- To answer this question, let us investigate **what happens when we input the simplest signal, delta function, into an LTI system.**
- Recall: delta function (unit impulse) in discrete-time domain:

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

n	...	-2	-1	0	1	2	3	4	5	6	...
$\delta[n]$	0	0	0	1	0	0	0	0	0	0	0
$\delta[n-2]$	0	0	0	0	0	1	0	0	0	0	0

Impulse Response

- **Impulse response**: When taking the unit impulse $\delta[n]$ as input to an LTI system, the output $h[n]$ is called the **impulse response** of this LTI system.
- Remember that any discrete-time signal $x[n]$ can be represented as the linear combination of delayed unit-impulse functions:

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

Impulse Response

- Then, we have the following property for the impulse response of an LTI system:

$$y[n] = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k] T\{\delta[n-k]\}$$

by Linearity

- So

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}$$

by Time-invariance

Impulse Response

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}$$

- The operation is convolution. Hence, when we know the impulse response $h[n]$ of an LTI system, then the output signal can be completely determined by the input signal and the impulse response via the convolution operation.

Knowing an LTI system $\xleftrightarrow{\text{equivalent to}}$ **Knowing its impulse response**

Response of LTI System

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- **Property**
- **Output of an LTI system** is the convolution of the input sequence and the impulse response of the LTI system.
- Hence, we can use the impulse response to fully describe an LTI system.

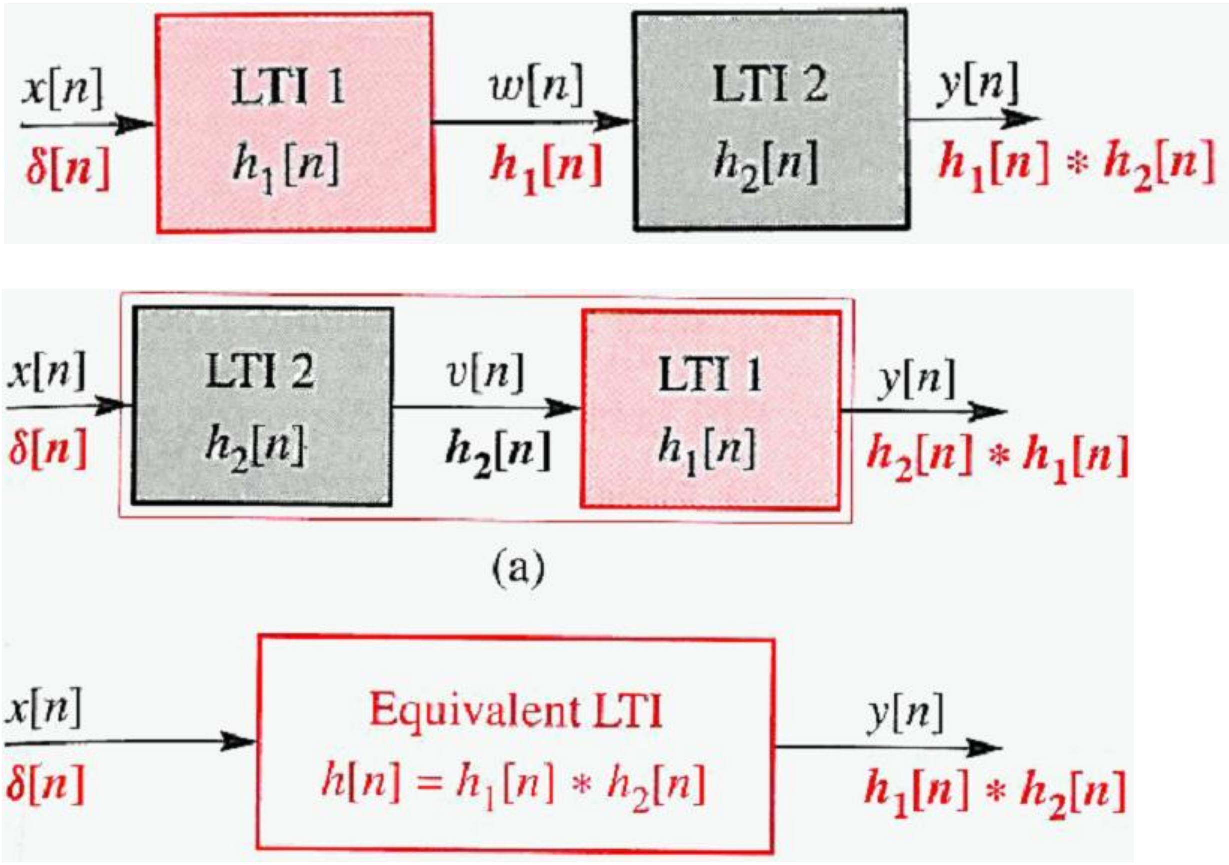
Convolution properties

- **Convolution** is **commutative** and **associative**:

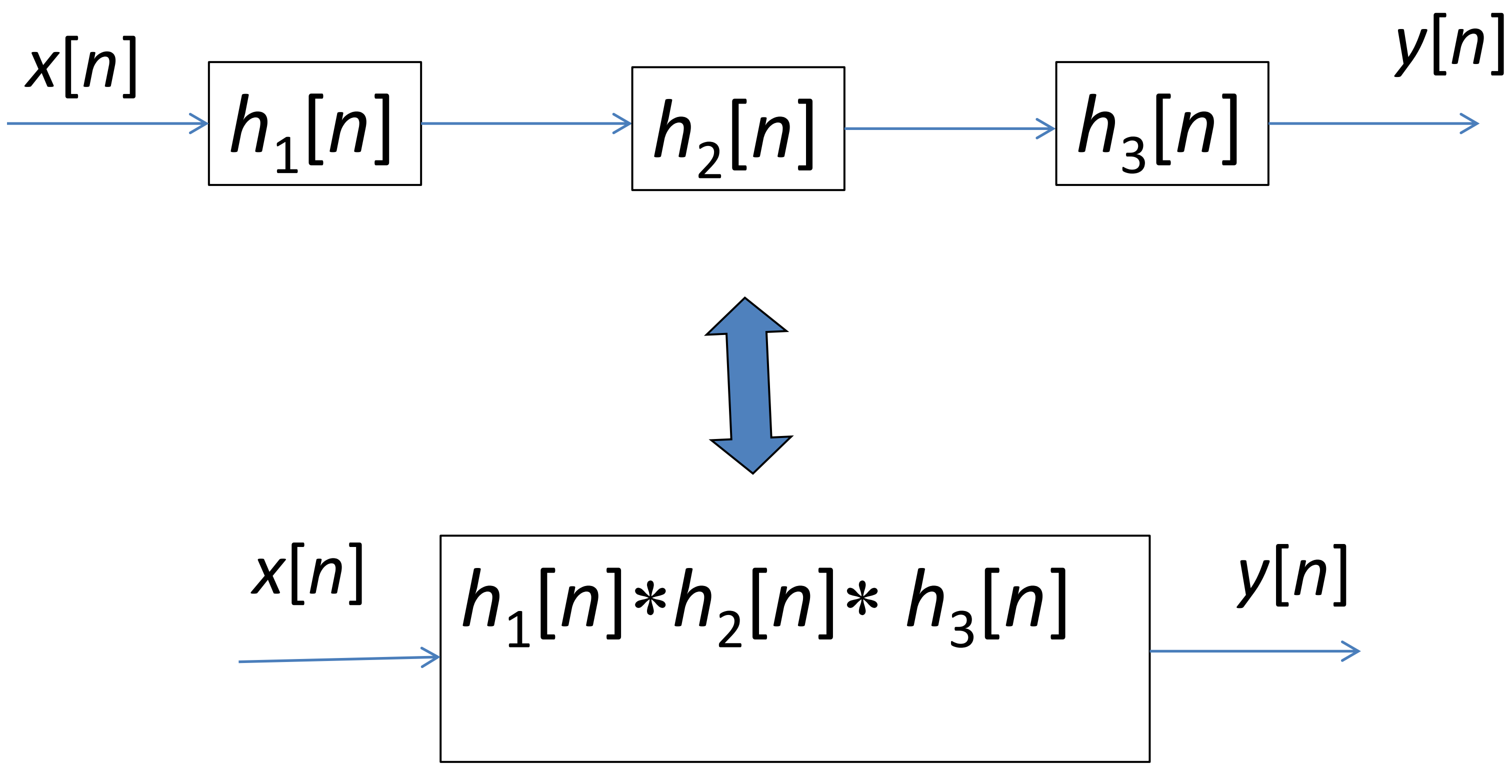
$$
\begin{aligned}
y[n] &= (x[n] * h_1[n]) * h_2[n] \\
&= x[n] * (h_1[n] * h_2[n]) \\
&= x[n] * (h_2[n] * h_1[n]) \\
&= (x[n] * h_2[n]) * h_1[n]
\end{aligned}
$$

Equivalent Systems

- **Hence, we have the following equivalent systems:**



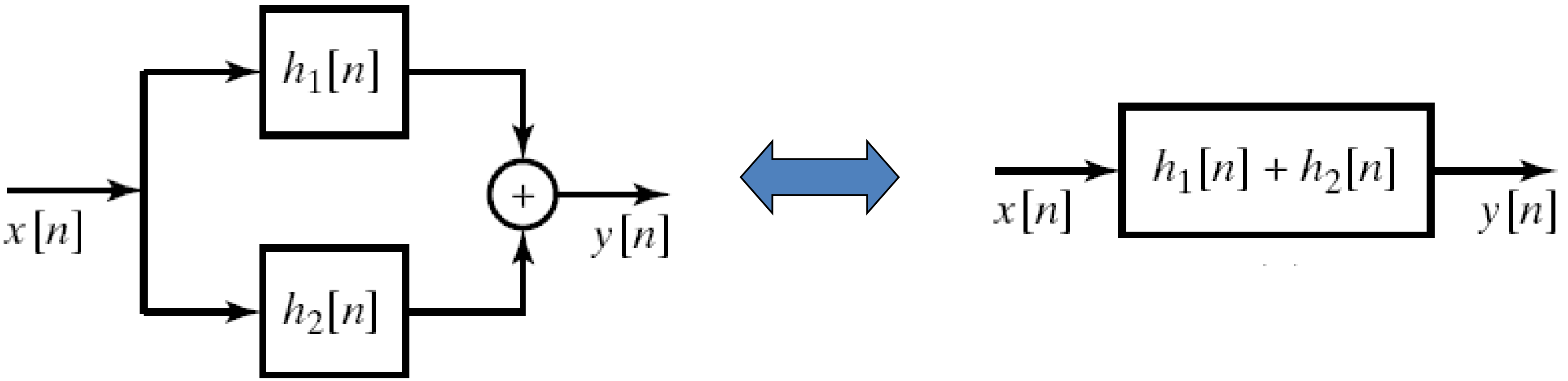
Equivalent Systems



Convolution and Equivalent System

- **Property**
- **Convolution** is also **distributive over addition**:

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$



Impulse Responses of Some LTI Systems

- Ideal delay $y[n] = x[n - n_d]$
  - Impulse response: $h[n] = \delta[n\text{-}n_d]$

- Moving average $y[n] = \frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{k=M_2} x[n-k]$
  - Impulse response:

$$h[n] = \begin{cases} \dfrac{1}{M_1 + M_2 + 1} & -M_1 \le n \le M_2 \\ 0 & \text{otherwise} \end{cases}$$

Example

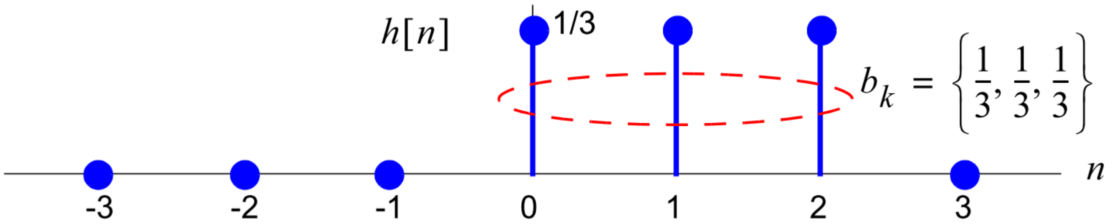
- Moving average when $M_1 = 0, M_2 = 2$

For this filter $b_k = \{1/3, 1/3, 1/3\}$

$$h[n] = \sum_{k=0}^{2} b_k \delta[n-k]$$

$$= \frac{1}{3} \sum_{k=0}^{2} \delta[n-k]$$

$$= \frac{1}{3}(\delta[n] + \delta[n-1] + \delta[n-2])$$



Impulse Responses of Some LTI Systems

- Accumulator $y[n] = \sum_{k=0}^{k=\infty} x[n-k]$
  - Impulse response

$$h[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

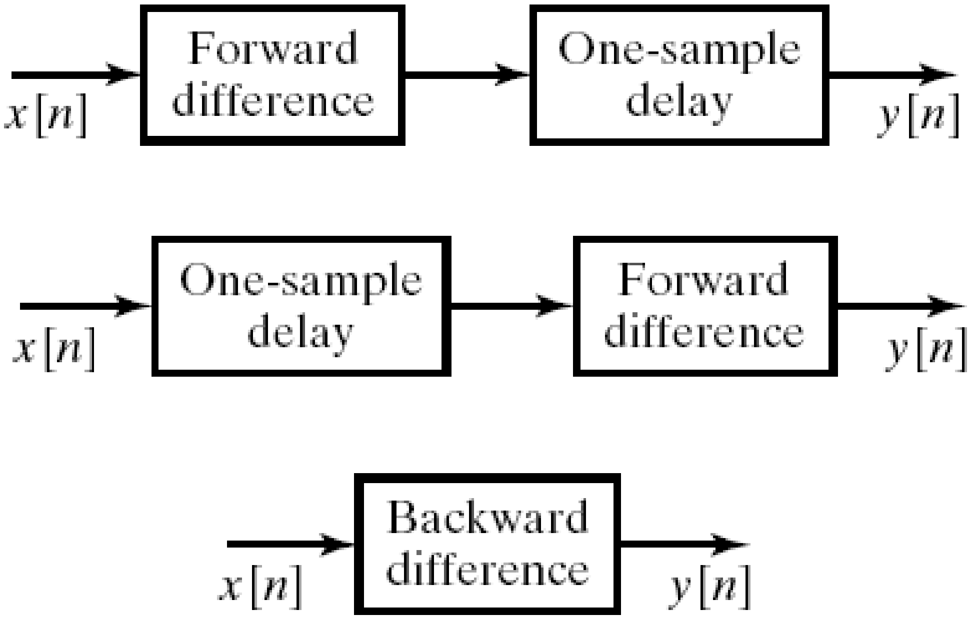
i.e., $h[n] = u[n]$, the unit-step signal

- Forward difference $y[n] = x[n+1] - x[n]$
  - Impulse response: $h[n] = \delta[n+1] - \delta[n]$

- Backward difference $y[n] = x[n] - x[n-1]$
  - Impulse response: $h[n] = \delta[n] - \delta[n-1]$

Cascading Systems

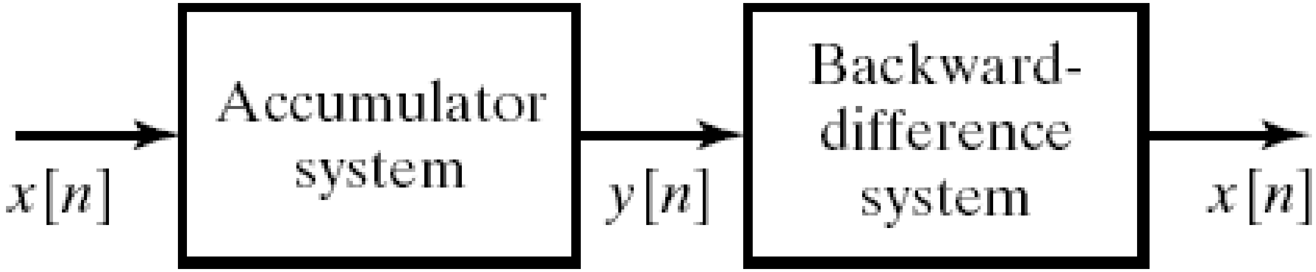
- **An LTI system can be realized in different ways** by separating it into different subsystems.
- The following systems are equivalent:



$$
\begin{aligned}
h[n] &= (\delta[n+1] - \delta[n]) * \delta[n-1] \\
&= \delta[n-1] * (\delta[n+1] - \delta[n]) \\
&= \delta[n] - \delta[n-1]
\end{aligned}
$$

Equivalent Cascading Systems

- Another example of cascading systems – inverse system.



$$h[n] = u[n] * (\delta[n] - \delta[n-1]) = u[n] - u[n-1] = \delta[n]$$

(where $u[n]$ is the unit-step signal)

Causality

- **Causal system**: A system is causal if it does not dependent on future inputs.
- That is, if $x_1[n] = x_2[n]$ when $n < n_0$, then the output $y_1[n] = y_2[n]$ when $n < n_0$ for all n_0.
  - or equivalently, the output $y[n_0]$ depends only the input sequence values for $n \le n_0$.
- What is the property of the impulse response of a causal system?

- **Property: An LTI system is causal if and only if**

$$\boldsymbol{h[n] = 0 \text{ for all } n < 0.}$$

General Form of LTI Systems

General LTI System
(The followings are its Special cases)

$$y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Causal LTI System

$$y[n]=\sum_{k=0}^{\infty} h[k]x[n-k]$$

Causal FIR Filter
(namely, **F**inite **I**mpulse **R**esponse)

$$y[n]=\sum_{k=0}^{M} h[k]x[n-k]$$

IIR Filter
(namely, **I**nfinite **I**mpulse **R**esponse)

$$y[n]=\sum_{k=0}^{M} b[k]x[n-k]+\sum_{k=1}^{N} a[k]y[n-k]$$

Difference Equation

- From the above, we can find that both FIR and IIR filters defined above are causal LTI Systems.
- They are both **difference equations**.
- **General form of difference equation**: (variables are functions.

  y: unknown function; x: given)

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- Both FIR and IIR filters are solutions of difference equations.
- IIR filter is the case when $a_0 = 1$. **Difference equation is merely a more general form**.

Illustration: (infinite-long) Matrix × Vector

- A General LTI system is of the form:

$$\boxed{y[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]}$$

- Can be explained as a infinite-dimensional matrix/vector product

$$\begin{bmatrix} \vdots \\ y[-3] \\ y[-2] \\ y[-1] \\ y[0] \\ y[1] \\ y[2] \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & h[0] & h[-1] & h[-2] & h[-3] & h[-4] & h[-5] & \cdots \\ \cdots & h[1] & h[0] & h[-1] & h[-2] & h[-3] & h[-4] & \cdots \\ \cdots & h[2] & h[1] & h[0] & h[-1] & h[-2] & h[-3] & \cdots \\ \cdots & h[3] & h[2] & h[1] & h[0] & h[-1] & h[-2] & \cdots \\ \cdots & h[4] & h[3] & h[2] & h[1] & h[0] & h[-1] & \cdots \\ \cdots & h[5] & h[4] & h[3] & h[2] & h[1] & h[0] & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x[-3] \\ x[-2] \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \\ \vdots \end{bmatrix}$$

What Happens in **Frequency Domain**?

- What is the influence in Frequency domain when input a discrete-time signal $x[n]$ to an LTI system of impulse response $h[n]$?
- To investigate the influence, consider that an LTI system can be characterized by an infinite-dimensional matrix as illustrated above.
- Conceptually, this matrix should have "infinite-long" eigenvectors.
- **Eigen function**: A discrete-time signal is called eigenfunction if it satisfies the following property: When applying the function as input to a system, the output is the same function multiplied by a (complex) constant (i.e., eigenvalue).

Eigenfunction of LTI System

- **Property**: $x[n] = e^{j\omega n}$ are the eigenfunctions of all LTI systems ($\omega \in R$)
- *Pf*: Let $h[n]$ be the impulse response of an LTI system, when $e^{j\omega n}$ is applied as the input,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{jw(n-k)} = e^{jwn} \sum_{k=-\infty}^{\infty} h[k] e^{-jwk}$$

- Let $\boxed{H(e^{jw}) = \sum_{k=-\infty}^{\infty} h[k] e^{-jwk}}$

we can see $\boxed{y[n] = H(e^{jw}) e^{jwn}}$

Frequency Response

- Hence, $e^{j\omega n}$ is the eigenfunction of the LTI system;
- The associated "eigenvalue" is $H(e^{j\omega})$.
- We call $H(e^{j\omega})$ the LTI system's **frequency response**
  - The frequency response is a complex function consisting of the real and imaginary parts, $H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$
- **Physical meaning**: When input is a signal of a single frequency w (i.e., $e^{j\omega n}$)), the output is a signal of the same frequency ω, but the magnitude and phase could be changed (characterized by the complex number $H(e^{j\omega})$)

Frequency Response

Another explanation of frequency response

- Recall the discrete-time Fourier transform (DTFT) pair:

- Forward DTFT $$X(e^{j\omega}) = \sum_{n=0}^{\infty} x(n)e^{-j\omega n}$$
- Inverse DTFT $$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

Frequency Response

- By definition, the **frequency response**:

$$H\left(e^{jw}\right) = \sum_{k=-\infty}^{\infty} h[k] e^{-jwk}$$

is just the **DTFT of the impulse response $h[n]$.**

$$\boxed{\textbf{Impulse Response}} \underset{\text{Inverse DTFT}}{\overset{\text{DTFT}}{\rightleftarrows}} \boxed{\textbf{Frequency Response}}$$

- This is an important reason why we need DTFT for discrete-time signal analysis in discrete-time LTI systems, besides the use of continuous Fourier transform for the continuous-time LTI systems.

LTI System and Frequency Response

- Now, let's go back to the problem: **When input a signal $x[n]$ to an LTI system, what happens in the frequency domain?**
- Remember that the output of an LTI system is $y[n] = x[n] * h[n]$, the convolution of $x[n]$ and $h[n]$.
- In DTFT, we still have the convolution theorem: time domain convolution is equivalent to frequency domain multiplication.
- Hence, the **output in the frequency domain is the multiplication**,

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Multiplication in Frequency Domain

- In sum, the output sequence's spectrum is the multiplication of the input spectrum and the frequency response.

Knowing an LTI system

equivalent to

Knowing its impulse response

equivalent to

equivalent to

Knowing its frequency response

Example: Frequency Response of Time-delay System

- The time-delay system is a simple FIR filter by the difference equation

$$y[n] = x[n - n_0]$$

  - The impulse response of the system is $h[n] = \delta[n - n_0]$.
  - The frequency response of the system is $H\left(e^{j\omega}\right) = e^{-j\omega n_0}$.

Example: Frequency Response of Time-delay System (cont.)

- The magnitude response of the time-delay system is a constant 1 because $|e^{-j\omega n_0}| = 1$.
- The phase response of the system (eg. $n_0 = 2$ is shown below (linear phase).

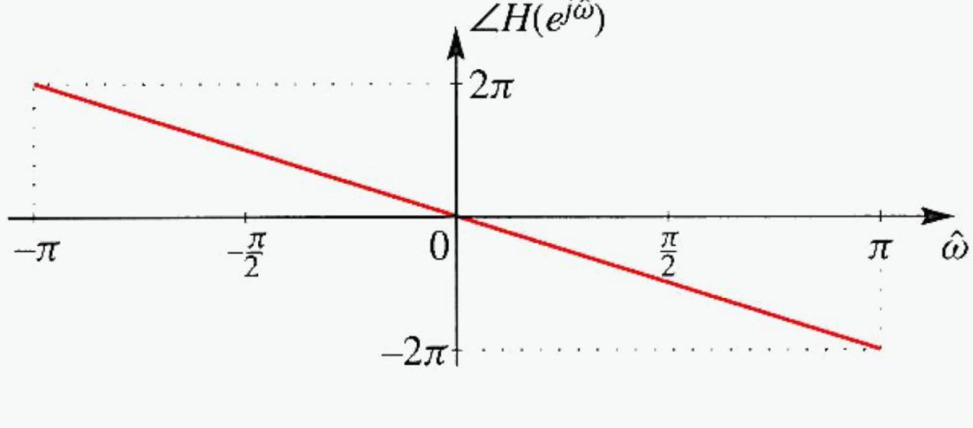


Figure 6-2: Phase response of pure delay ($n_0 = 2$) system, $H(e^{j\hat{\omega}}) = e^{-j2\hat{\omega}}$.

Example: Frequency Response of Backward-difference System

- The backward-difference system is a FIR filter of the first-difference equation

$$y[n] = x[n] - x[n-1]$$

  - The impulse response of the system is $h[n] = \delta[\mathrm{n}] - \delta[n-1]$.
  - The frequency response of the system is $H\left(e^{j\omega}\right) = 1 - e^{-j\omega} = 1 - \cos(\omega) + jsin(\omega)$.

Example: Frequency Response of Backward-difference System (cont.)

- A further investigation of the frequency response:
  - The real part of the frequency response is

$$Re\{\mathrm{H}(\mathrm{e}^{\mathrm{j}\omega})\} = 1 - \cos(\omega)$$

  - The imagery part of the frequency response is

$$Im\{\mathrm{H}(\mathrm{e}^{\mathrm{j}\omega})\} = sin(\omega)$$

Example: Frequency Response of Backward-difference System (cont.)

- A further investigation of the frequency response:
  - The magnitude response of the system is

$$
\begin{aligned}
\left|\mathrm{H}\left(\mathrm{e}^{\mathrm{j}\omega}\right)\right| &= [\left(1-\cos(\omega)\right)^2 + \sin^2(\omega)]^{1/2} \\
&= [\left(1-\cos(\omega)\right)^2 + \sin^2(\omega)]^{1/2} \\
&= [2\left(1-\cos(\omega)\right)]^{1/2} = 2\sin^2(\omega/2)
\end{aligned}
$$

  - The phase response of the system is
$\angle H\left(e^{j\omega}\right) = \arctan\left(\frac{\sin(\omega)}{1-\cos(\omega)}\right).$

Example: Frequency Response of Backward-difference System (cont.)

- We have known that the magnitude of the response is $2\sin^2(\omega/2)$.
- We hope to write the frequency response in the form: $H(e^{j\omega}) = Ae^{j\phi}$, where we have known $A = 2\, sin^2(\omega/2) = -j(\mathrm{e}^{j\omega/2} - e^{-j\omega/2})$ (magnitude)
- Because $H(e^{j\omega}) = 1 - e^{-j\omega}$, we have

$$H(e^{j\omega}) = 1 - e^{-j\omega} = (\mathrm{e}^{j\omega/2} - e^{-j\omega/2})e^{-j\omega/2}.$$

- Hence, $H(e^{j\omega}) = 2j\sin^2(\omega/2)\, e^{-j\omega/2}$

$$= 2\sin^2(\omega/2)e^{j(\pi/2-\omega/2)}$$

  - That is, $\phi = \pi/2 - \omega/2$ (Phase)

Example: Frequency Response of Backward-difference System (cont.)

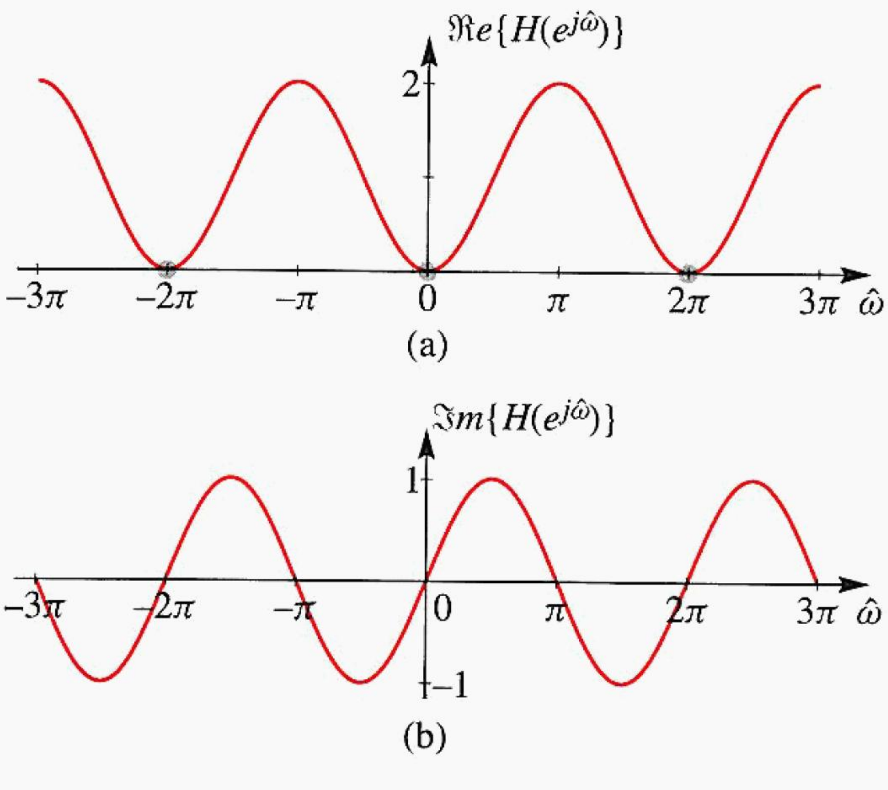


Figure 6-3: (a) Real and (b) imaginary parts for $H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}}$ over three periods showing periodicity and conjugate symmetry of $H(e^{j\hat{\omega}})$.

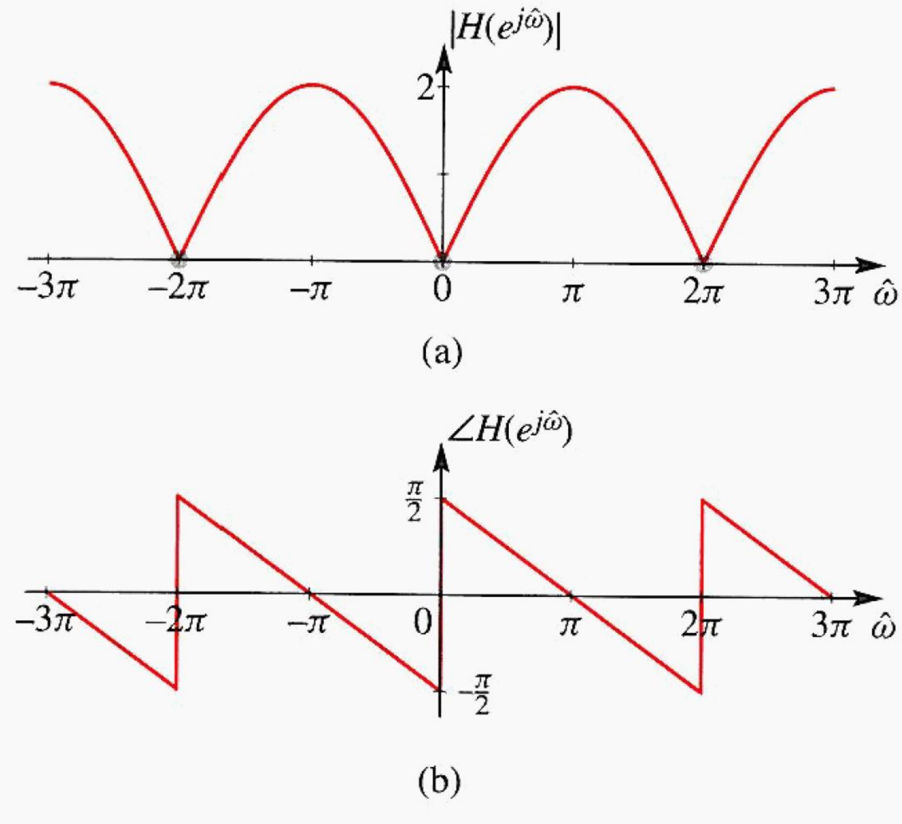


Figure 6-4: (a) Magnitude and (b) phase for $H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}}$ over three periods showing periodicity and conjugate symmetry of $H(e^{j\hat{\omega}})$.

Real and **Imaginary** parts

Magnitude and **Phase response**

Example: Input a signal to the backward-difference system

- Suppose that we input the following signal to the system: $x[n] = 4 + 2\cos(0.3\pi n - \pi/4)$
- How to compute the output $y[n]$ given the input $x[n]$ to the backward-difference system?
  - **Solution 1** (time domain) compute $x[n] * (\delta[n] - \delta[n-1])$, convolution of the input and the impulse response.
  - **Solution 2** (frequency domain) compute the DTFT of $x[n]$, obtain $X(e^{j\omega})$; multiply $X(e^{j\omega})$ and the frequency response $H(e^{j\omega})$, $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$; Finally, compute the inverse DTFT of $Y(e^{j\omega})$.

Example: Input a signal to the backward-difference system (cont.)

- Suppose that we input the following signal to the system: $x[n] = 4 + 2\cos(0.3\pi n - \pi/4)$
- How to compute the frequency domain output $Y(e^{j\omega})$, given the input $x[n]$ to the backward-difference system?
  - **Solution 1**: compute $y[n] = x[n] * (\delta[n] - \delta[n-1])$; then compute the DTFT of $y[n]$ to obtain $Y(e^{j\omega})$.
  - **Solution 2**: compute the DTFT of $x[n]$, then obtain $Y(e^{j\omega})$ via $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$.

Example: Frequency Response of the Moving-average System

- Impulse response of the moving-average system is

$$h[n]=\begin{cases}\dfrac{1}{M_1+M_2+1} & -M_1 \le n \le M_2 \\ 0 & \text{otherwise}\end{cases}$$

  - Therefore, by definition, the frequency response is the DTFT of $h[n]$, i.e.,

$$H\left[e^{jw}\right]=\frac{1}{M_1+M_2+1}\sum_{n=-M_1}^{M_2} e^{-jwn}$$

Geometric Series Formula

- The following is the geometric series formula:

$$\sum_{k=0}^{L} \alpha^k = \frac{1-\alpha^{L+1}}{1-\alpha},$$

- So

$$\sum_{k=n}^{m} \alpha^k = \alpha^n \sum_{k=0}^{m-n} \alpha^k$$

$$= \frac{\alpha^n - \alpha^{m+1}}{1-\alpha}, \quad m > n$$

Frequency response of the moving-average system (further derivation)

$$H\left[e^{jw}\right] = \frac{1}{M_1 + M_2 + 1} \sum_{n=-M_1}^{M_2} e^{-jwn}$$

$$= \frac{1}{M_1 + M_2 + 1} \frac{e^{jwM_1} - e^{-jw(M_2+1)}}{1 - e^{-jw}}$$

$$= \frac{1}{M_1 + M_2 + 1} \frac{e^{jw(M_1+M_2+1)/2} - e^{-jw(M_1+M_2+1)/2}}{1 - e^{-jw}} e^{-jw(M_2-M_1+1)/2}$$

$$= \frac{1}{M_1 + M_2 + 1} \frac{e^{jw(M_1+M_2+1)/2} - e^{-jw(M_1+M_2+1)/2}}{e^{jw/2} - e^{-jw/2}} e^{-jw(M_2-M_1)/2}$$

Frequency Response of the Moving-average System (cont.)

- Further evaluation

$$H\left[e^{jw}\right]=\frac{1}{M_1+M_2+1}\frac{e^{jw(M_1+M_2+1)/2}-e^{-jw(M_1+M_2+1)/2}}{e^{jw/2}-e^{-jw/2}}e^{-jw(M_2-M_1)/2}$$

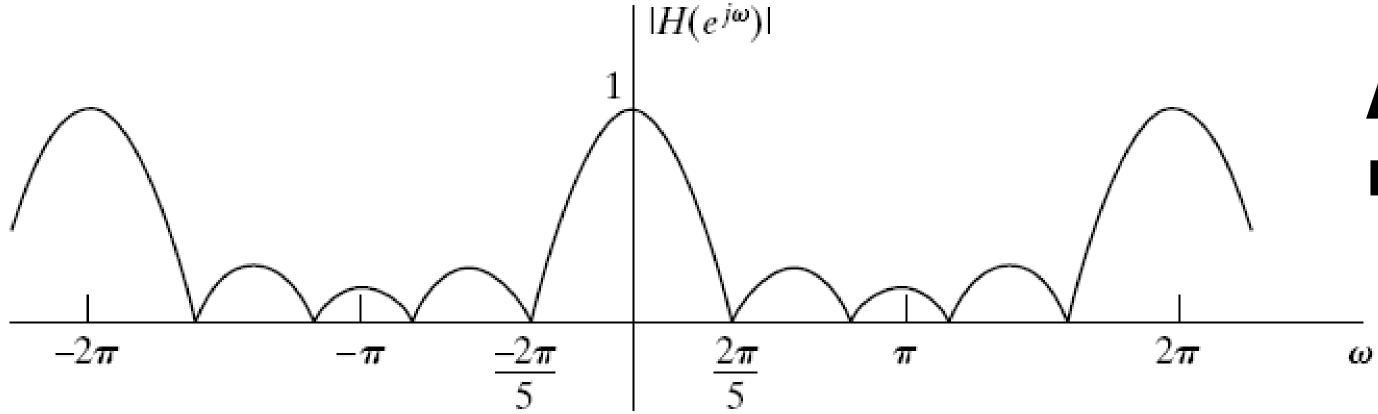
$$=\frac{1}{M_1+M_2+1}\frac{\sin\left[w\left(M_1+M_2+1\right)/2\right]}{\sin\left[w/2\right]}e^{-jw(M_2-M_1)/2}$$

$$=\left|H\left(e^{jw}\right)\right|\exp^{j\angle H\left(e^{jw}\right)}$$

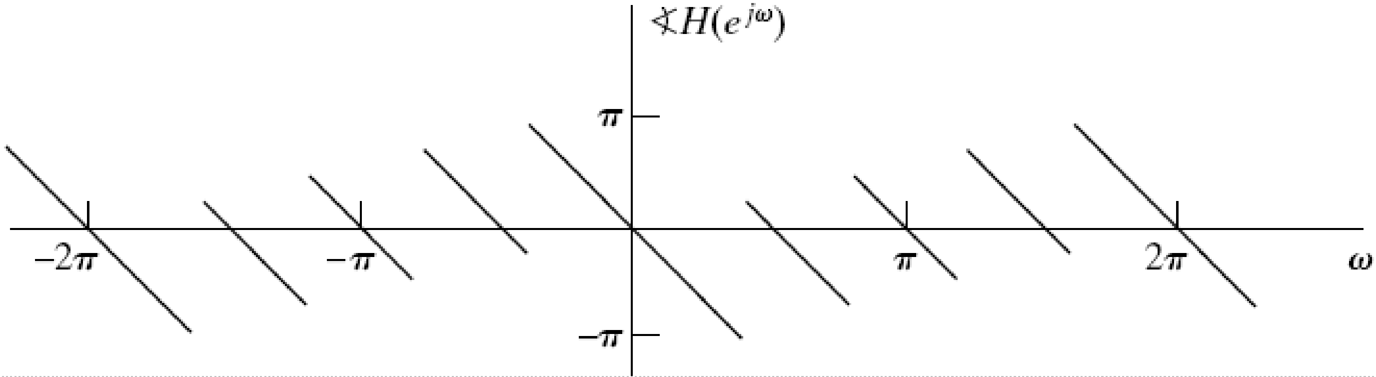
(magnitude and phase)

Spectrum of the Moving-average System ($M_1 = 0$ and $M_2 = 4$)

- Recall that, in DTFT, the frequency response is repeated with period 2π. "High frequency" is close to $\pm\pi$.



Amplitude response



Phase response

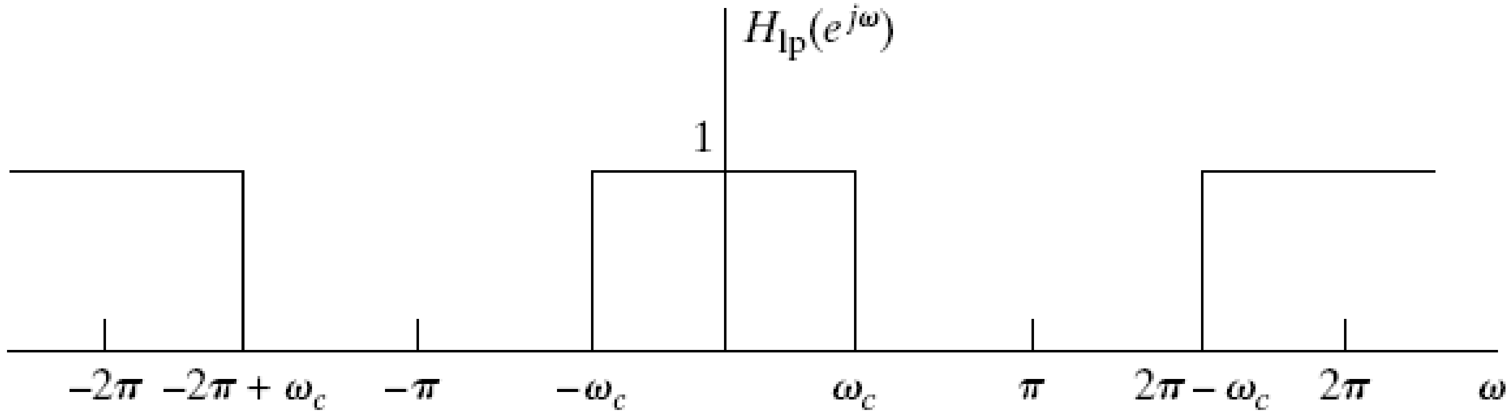
Example: **Discrete-time Ideal Low-pass Filter**

- Ideal low-pass filter in DTFT domain

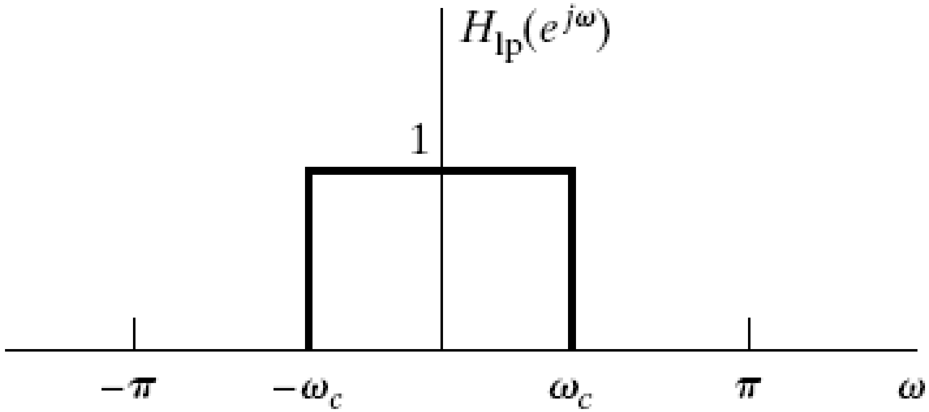
$$H_{lowpass}(e^{jw}) = \begin{cases} 1, & |w| < w_c \\ 0, & w_c < |w| < \pi \end{cases}$$

- Note that we depict the frequency response in the range [-π,π] only for discrete-time signals. The "low frequencies" are frequencies close to zero, while the "high frequencies" are those close to ±π.

Ideal Low-pass Filter in DTFT domain



(a)



Discrete-time Ideal Low-pass Filter

- The impulse response $h_{lowpass}[n]$ found by inverse Fourier transform is a uniformly sampled sync function:

$$h_{lowpass}[n] = \frac{1}{2\pi}\int_{-w_c}^{w_c} e^{jwn} dw$$

$$= \frac{1}{2\pi jn} e^{jwn} \Big|_{-w_c}^{w_c} = \frac{1}{2\pi jn}(e^{jw_c n} - e^{-jw_c n})$$

$$= \frac{\sin w_c n}{\pi n}$$

Uniform samples of Sinc function

Approximation of Discrete-time Ideal Low-pass Filter

- That is, the forward DTFT of the sampled sync function is the ideal low-pass filter:

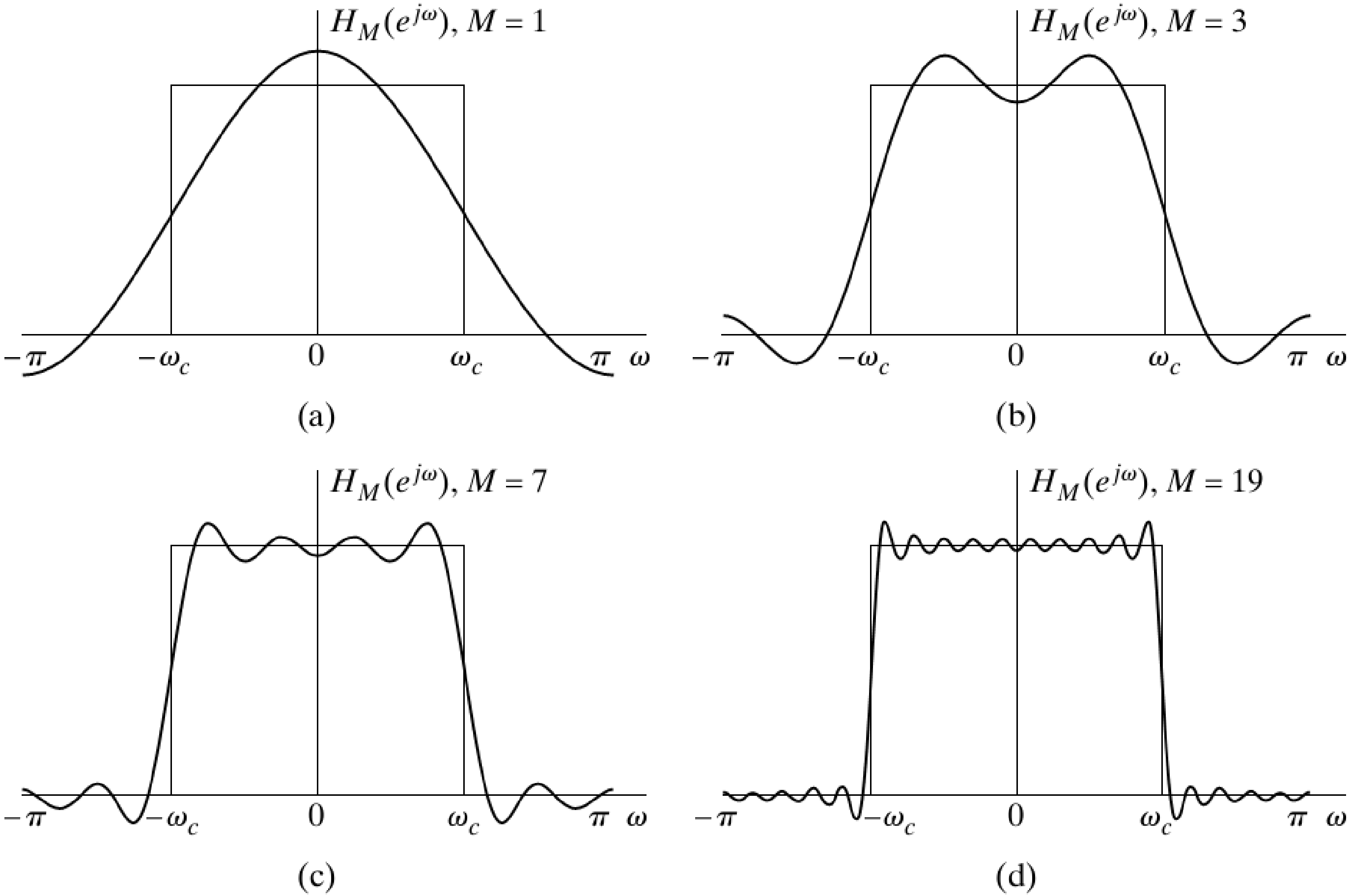
$$H_{lowpass}(e^{jw}) = \sum_{n=-\infty}^{\infty} \frac{\sin w_c n}{\pi n} e^{-jwn}$$

- As the sampled sync function cannot be realized by difference equation because it reaches infinity on both ends, to approximate the ideal low-pass filter, we often use the partial sum instead (a FIR filter)

$$H_M(e^{jw}) = \sum_{n=-M}^{M} \frac{\sin w_c n}{\pi n} e^{-jwn}$$

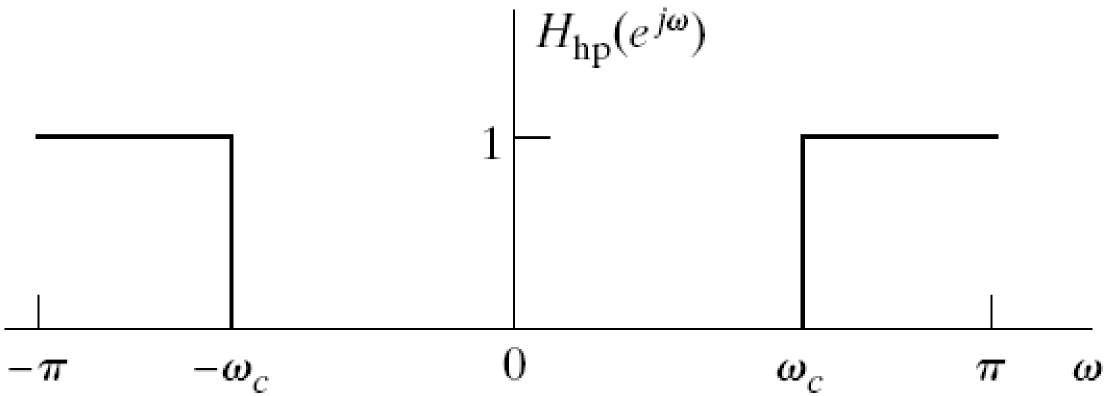
Approximation of Discrete-time Ideal Low-pass Filter by partial sum

- Examples of M=1, 3, 7, 19

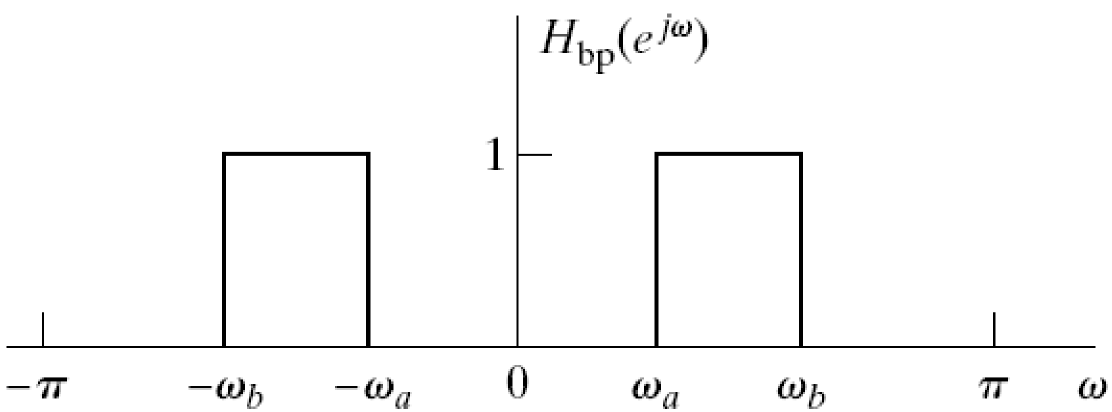


Ideal frequency-selection filters

- In addition to ideal low-pass filter, likewise, we can consider other ideal frequency-selection filters too.



Ideal high-pass filter



Ideal band-pass filter