Discrete-time signals

- Definition:
  - A discrete-time signal is a function from $Z \rightarrow C$
  - Eg., $x[n], n \in Z$,
    $\dots\dots, x[-2], x[-1], x[0], x[1], x[2], x[3], \dots\dots$
  - i.e., a discrete-time signal is a discrete series of complex numbers

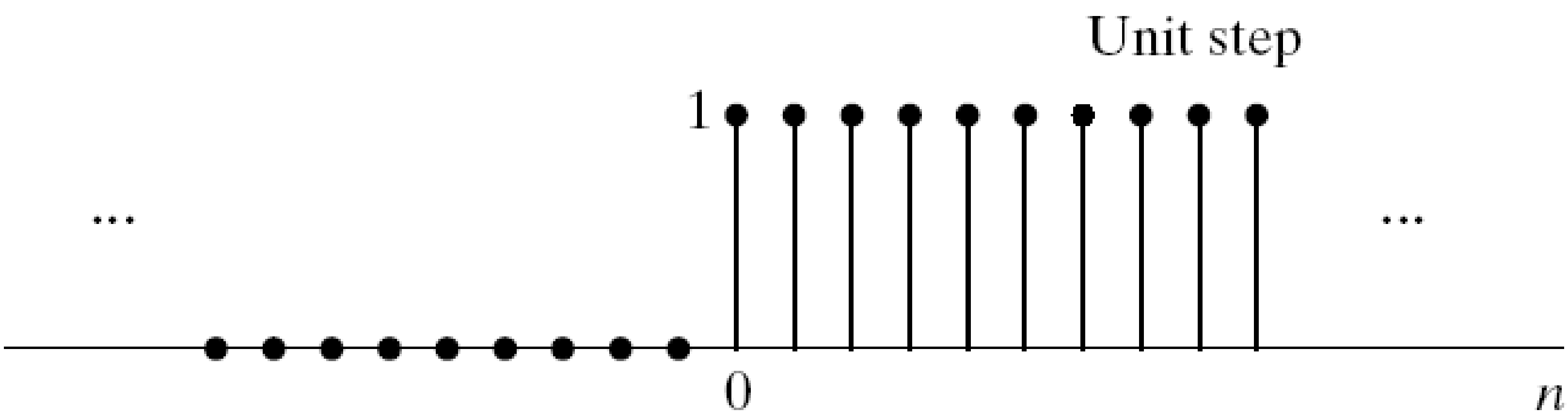
Discrete-time signals

- We often assume that the discrete-time signal is sampled from a band-limited continuous-time signal following the sampling theorem.
  - That is, samples from a "smooth" function (because a band-limited signal is within a limited frequency range $[-\omega_b, \omega_b]$). Low frequency $\Rightarrow$ signal is smooth.
  - In this case, we often adopt the simplest interpretation (Occam's razor) of the discrete-time signal. That is, the signal can be smoothly interpolated as a continuous function.
- There could also be signals that are discrete-time by nature (not necessarily the samples of a continuous function).

Some signal examples

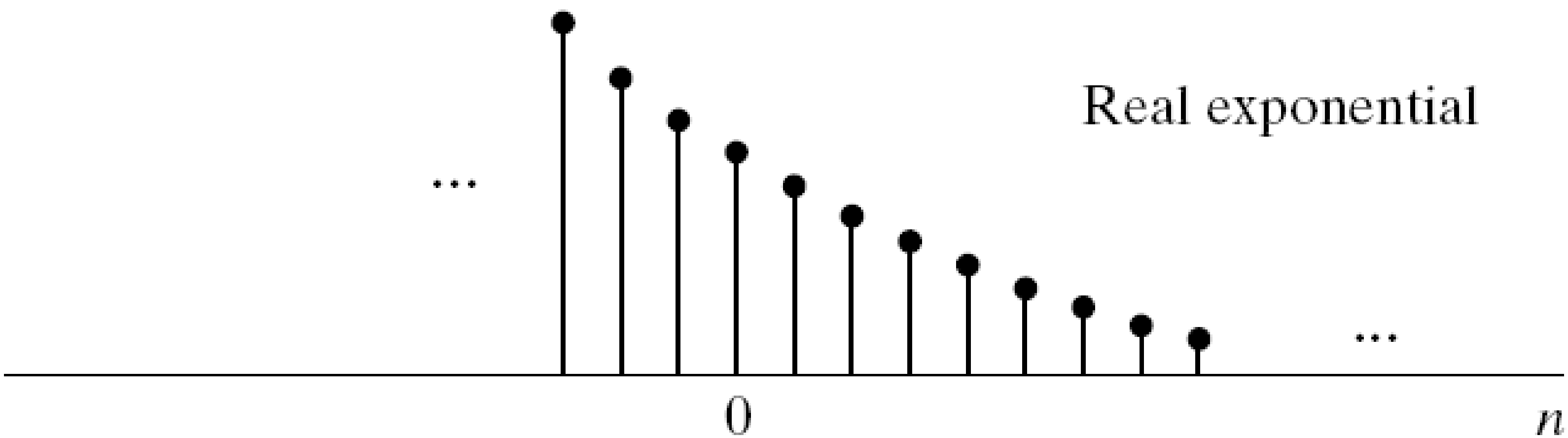
- Unit step sequence

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Some Signal Examples (cont.)

- Real exponential sequence $$x[n] = A\alpha^n$$



$$y[n] = \begin{cases} A\alpha^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

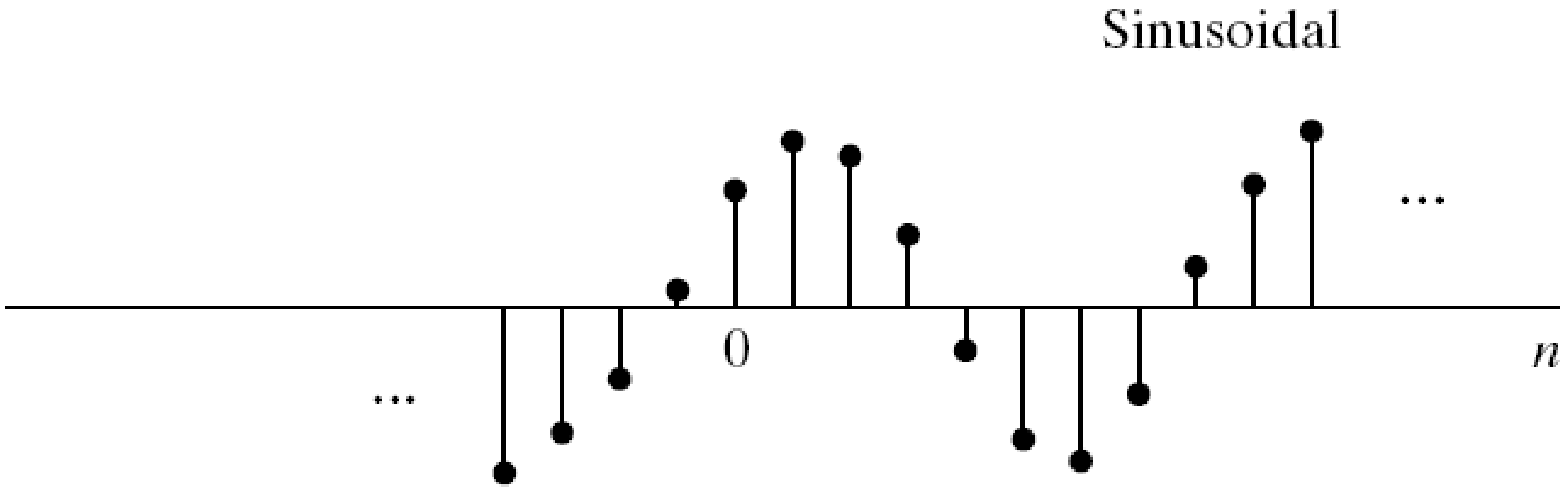
- $y[n]$ can be represented as

$$y[n] = A\alpha^n u[n]$$

Some Signal Examples (cont.)

- Sinusoidal sequence

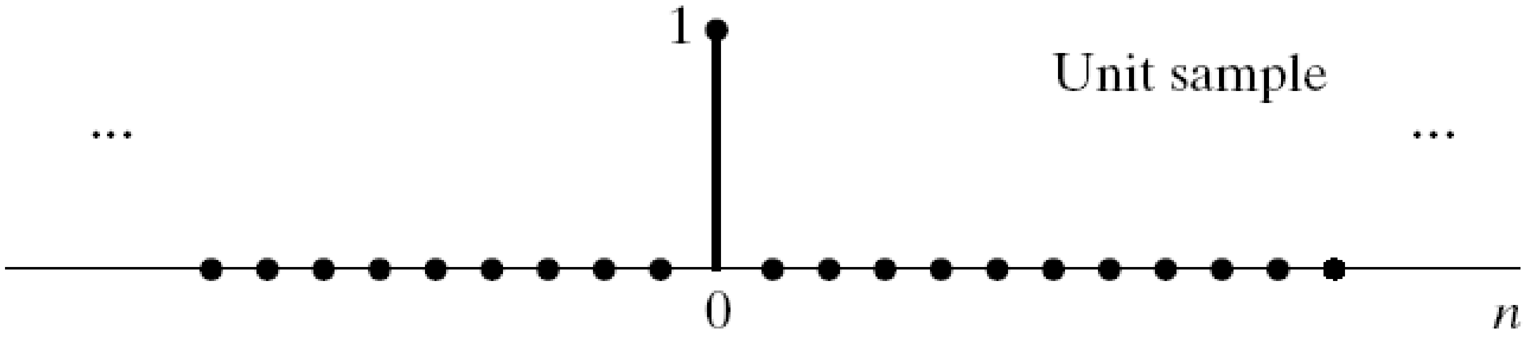
$$x[n] = A\cos(w_0 n + \phi)$$



A Particular Signal

- Unit sample sequence
  - Unit impulse function, Dirac delta function, impulse

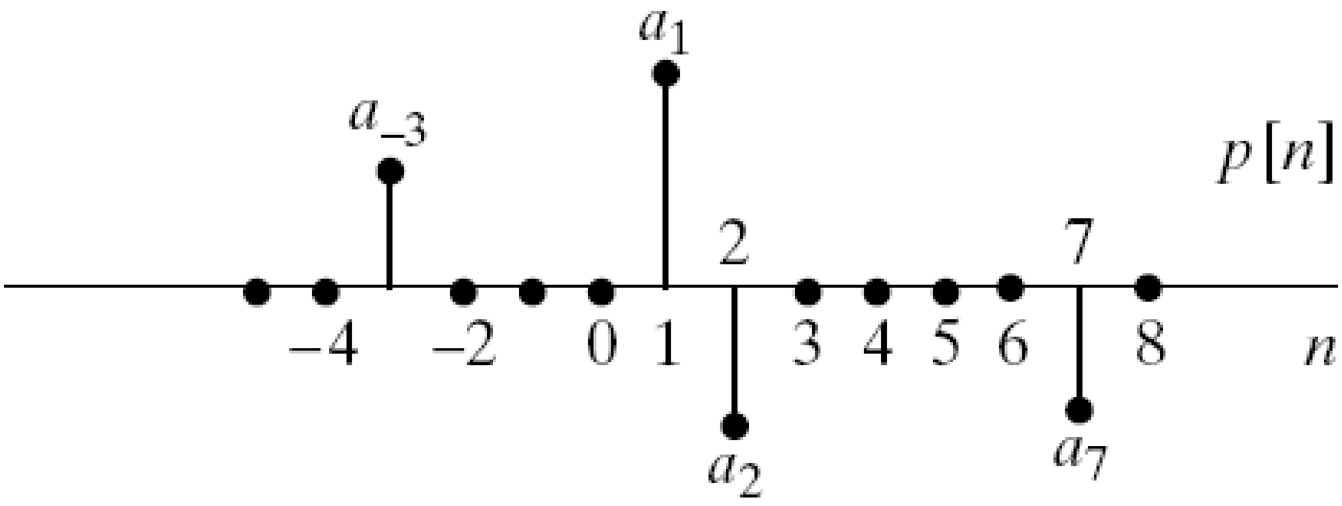
$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$



Signal Representation

- An arbitrary sequence can be represented as a sum of scaled, delayed impulses.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$



Fourier Transform of Discrete-time Signals

- From continuous Fourier transform, when the **time domain is discrete, the frequency domain is periodic**.
  - Specifically, the discrete samples can be expressed as a series of impulses in the continuous domain.
  - **The continuous Fourier transform of such a signal is a periodic function with the period $2\pi/T$.**

Fourier Transform of Discrete-time Signals

- We usually take the smoothest interpretation of the discrete-time signal.
  - Occam's Razor: two explanations that account for all the facts, the simpler one is more likely to be correct.
- That is, only consider the period that contains zero frequency in the frequency domain, $[-\pi/T, \pi/T]$, (or the signal has been low-pass filtered in advance).

Spectrum of discrete-time signal

- More specifically, the spectrum is only informative within a single period [$-\pi/T$, π/T] if we assume that the signal is pre-**low-pass-filtered** (i.e., **smoothed**) and sampled following the sampling theorem.
  - In this case, we can use the Sinc function to interpolate and reconstruct the continuous-time signal.

Discrete-time Fourier Transform

- **Discrete time Fourier transform (DTFT)**: a Fourier transform dealing with the case where time domain is a discrete-time signal …, $x(-2T), x(-T), x(0), x(T), x(2T)$, … ($T$ is the time step)
- DTFT is introduced for the spectrum representation of the discrete-time signals, just like that Fourier series is particular for periodic signal.

Discrete-time Fourier Transform (DTFT)

- The DTFT pair is defined as follows: Let $x[n]$ be a discrete-time signal, $n \in Z$.

Forward DTFT:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Inverse DTFT:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- Note that we do not incorporate the sampling period T in the transform. It can be treated as the normalized case of $T = 1$.

DTFT (cont.)

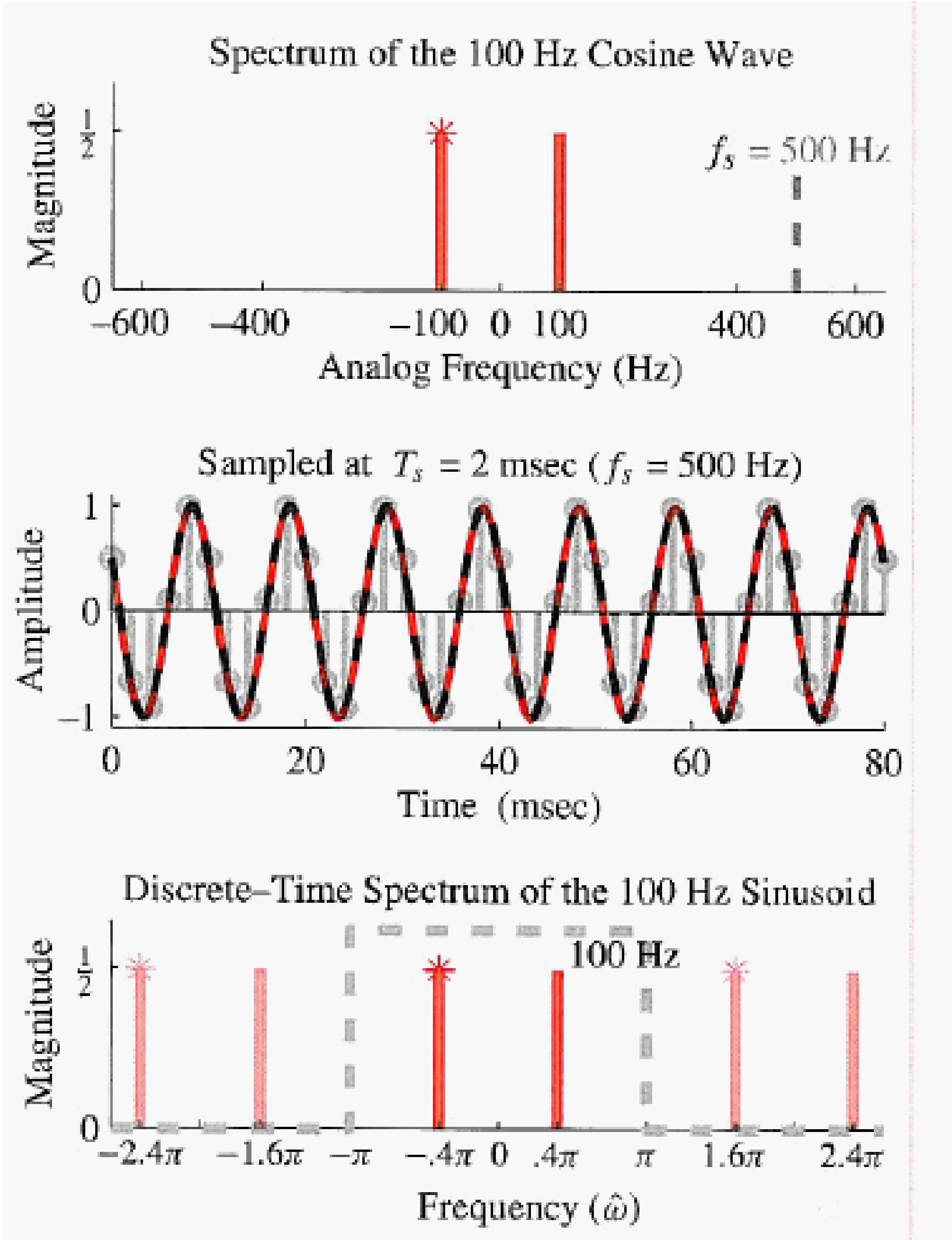
- In digital signal processing, discrete-time signals are of the main interest, and DTFT is a main tool for analyzing the spectrum of them.
- In general, DTFT has a periodic spectrum in the frequency domain.
- In practice, when using DTFT, we usually assume the discrete-time signal is sampled from a band-limited analog signal following the sampling theorem. Therefore, we often consider a finite-duration $[-\pi, \pi]$.

DTFT

- **Frequency range of DTFT:** $[-\pi, \pi]$ (assuming a smoothed signal)
  - **High frequency region**: The frequency **nearing $-\pi$ or π.**
  - **Low frequency region**: The frequency nearing 0.
- When the discrete-time signal is sampled from a continuous signal and we know the sampling period T, the DTFT frequency π corresponds to the analog frequency π/T.

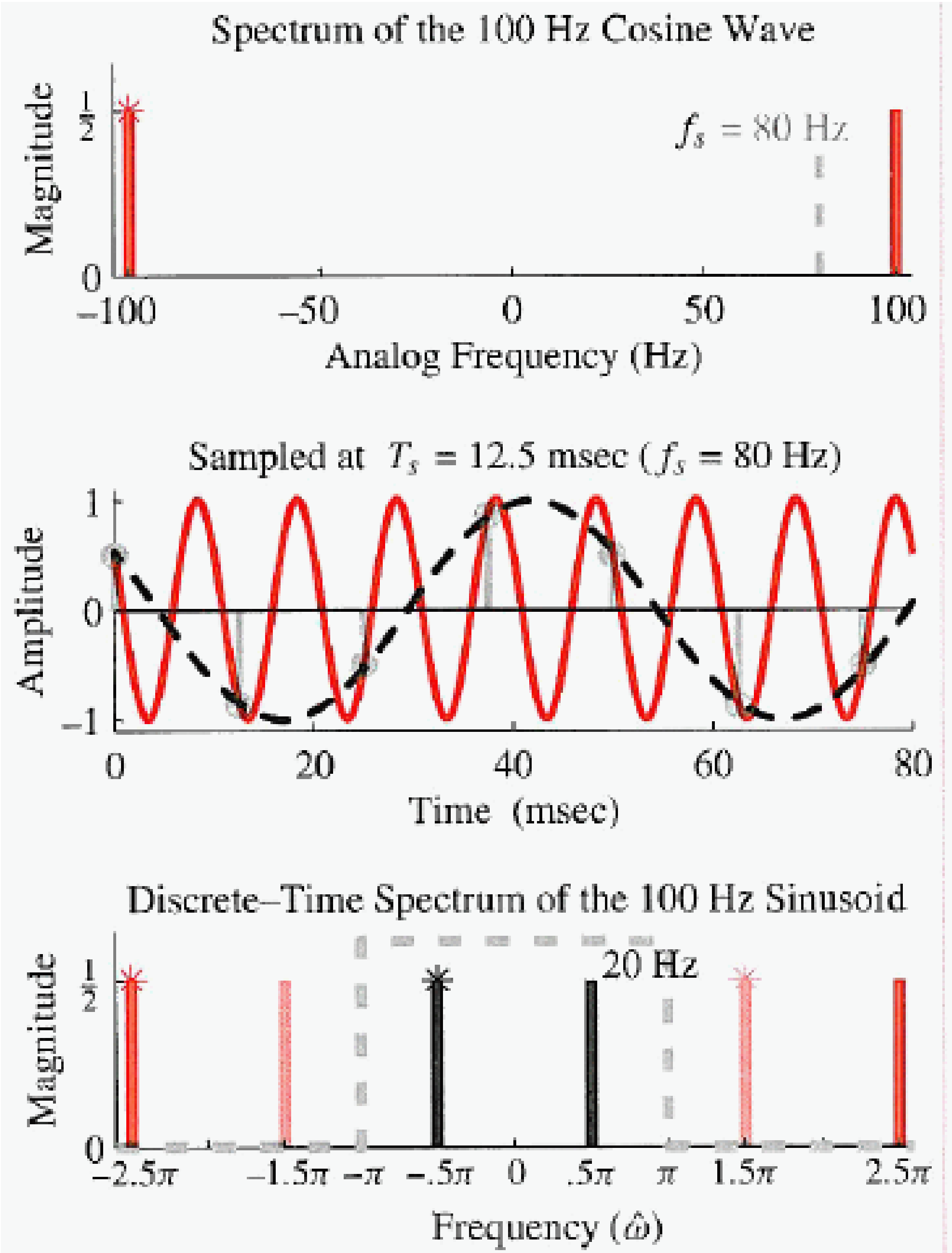
DTFT (cont.)

- When the discrete-time signal is sampled from a continuous function of the sampling period T; $f_s = 1/T$ is the sampling frequency.
- The DTFT frequency π corresponds to the analog frequency π/T (in radian) or $1/2T$ (in hertz). They are also represented as πf_s (in radian) or $f_s/2$ (in hertz).



Sampling interpretation in DTFT (discrete-time spectrum)

- Sampling a 100 Hz signal with rate f_s=500 Hz.
- In DTFT , π is corresponding to $f_s/2 = 250$ Hz in this case.



Aliasing example in DTFT (discrete-time spectrum)

- In DTFT , π is corresponding to $\frac{f_s}{2} = 40\ Hz$ in this case.

DC Component

- When $\omega = 0$, the complex exponential $e^{-j\omega}$ becomes a constant signal; the frequency response $X(e^{j\omega})$ is often called the DC component.
  - The term DC stands for direct current, which is a constant current.