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DSP2025 Homework 2 Due date: 12:00 noon, March 27, 2025

1. Consider an aperiodic signal $x(t)$ with its CFT being $X(jw)$, and a periodic signal $y(t)$ with fundamental frequency T_0 and Fourier coefficients c_k $(k \in Z)$. It is apparent that their product $z(t) = x(t)y(t)$ is an aperiodic signal. **Question**: Derive the CFT of $z(t)$.

Ans:

aperiodic signal : $x(t) \xrightarrow{CFT} X(jw)$

periodic signal : $y(t)$, $T = T_0$, Fourier coefficients $= C_k$

$\longrightarrow y(t) = \sum_{k=-\infty}^{\infty} C_k e^{jkw_0 t}$, $w = \frac{2\pi}{T_0}$ $\xrightarrow[\text{轉換}]{\text{傅立葉}} Y(jw) = 2\pi \sum_{k=-\infty}^{\infty} C_k \delta(w - kw_0)$

delta functions.

$$\boxed{☆ \quad x(t) \cdot y(t) \xrightarrow{\mathcal{F}} \frac{1}{2\pi} X(jw) Y(jw)}$$

$$z(t) = x(t) y(t) \xrightarrow{\mathcal{F}} \frac{1}{2\pi} X(jw) \cdot 2\pi \sum_{k=-\infty}^{\infty} C_k \delta(w - kw_0)$$

$$= \sum_{k=-\infty}^{\infty} C_k X(j(w - kw_0))$$

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$$z(t) = x(t) \cdot \sum_{k=-\infty}^{\infty} C_k e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} C_k x(t) e^{jkw_0 t}$$

$$\mathcal{F}\{x(t) e^{jkw_0 t}\} = X(j(w - kw_0))$$

$$Z(jw) = \sum_{k=-\infty}^{\infty} C_k X(j(w - kw_0))$$

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2. Let $x(t) = \frac{2\sin(20\pi t)}{\pi t}$, $h(t) = \frac{5\sin(10\pi t)}{\pi t}$ be two sinc functions. Derive the convolution of them, $y(t) = x(t) * h(t)$. **Hint**: Using convolution theorem.

Ans:

☆ $y(t) = x(t) \cdot h(t) \xrightarrow{\mathcal{F}} Y(j\omega) = X(j\omega) \cdot H(j\omega)$

☆ $\frac{\sin(\pi t)}{\pi t} = \text{sinc}(t)$, $\frac{\sin(at)}{\pi t} = \frac{a}{\pi}\text{sinc}\left(\frac{at}{\pi}\right)$

$$x(t) = \frac{2\sin(20\pi t)}{\pi t} = 2 \cdot \left(\frac{20\pi}{\pi} \cdot \text{sinc}(20t)\right) = 40 \cdot \text{sinc}(20t)$$

$$h(t) = \frac{5\sin(10\pi t)}{\pi t} = 5 \cdot \left(\frac{10\pi}{\pi} \cdot \text{sinc}(10t)\right) = 50 \cdot \text{sinc}(10t)$$

☆ $\text{sinc}(t) \xleftrightarrow{\mathcal{F}} \text{rect}\left(\frac{\omega}{2\pi}\right)$, $\text{sinc}(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|}\text{rect}\left(\frac{\omega}{2\pi a}\right)$

$$x(t) = 40 \cdot \text{sinc}(20t) \Leftrightarrow 40 \times \frac{1}{20} \times \text{rect}\left(\frac{\omega}{2\pi \times 20}\right) = 2\,\text{rect}\left(\frac{\omega}{40\pi}\right) = X(j\omega)$$

$$h(t) = 50 \cdot \text{sinc}(10t) \Leftrightarrow 50 \times \frac{1}{10} \times \text{rect}\left(\frac{\omega}{2\pi \times 10}\right) = 5\,\text{rect}\left(\frac{\omega}{20\pi}\right) = H(j\omega)$$

$$Y(j\omega) = X(j\omega) \cdot H(j\omega) = \Big[\underbrace{2\,\text{rect}\left(\tfrac{\omega}{40\pi}\right)}_{\text{寬度}\pm 20\pi}\Big] \times \Big[\underbrace{5\,\text{rect}\left(\tfrac{\omega}{20\pi}\right)}_{\text{寬度}\pm 10\pi}\Big]$$

$$= 10\,\text{rect}\left(\frac{\omega}{20\pi}\right)$$

重疊區為較窄那部分.

$$Y(j\omega) = 10\,\text{rect}\left(\frac{\omega}{20\pi}\right) \xleftrightarrow{\mathcal{F}^{-1}} 100 \times \frac{1}{10} \times \text{rect}\left(\frac{\omega}{2\pi \times 10}\right)$$

$$= 100 \cdot \text{sinc}(10t) = y(t)$$

$$y(t) = 100 \cdot \text{sinc}(10t) = \frac{10 \cdot \sin(10\pi t)}{\pi t}$$

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3. Let the DTFT of the following sequence be $R(e^{j\omega})$

$$r[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

(a) Consider the sequence

$$w[n] = \begin{cases} \frac{1}{2}\left[1 - \cos\frac{2\pi n}{M}\right], & 0 \le n \le M \\ 0, & otherwise \end{cases}$$

Sketch $w[n]$ and express $W(e^{j\omega})$, the DTFT of $w[n]$, in terms of $R(e^{j\omega})$, the DTFT of $r[n]$.

(b) Sketch the magnitude of $R(e^{j\omega})$ and $W(e^{j\omega})$ for the case when $M = 4$.

Ans: (a)

$$r[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases} \longrightarrow R(e^{j\omega}) = \sum_{n=0}^{M} e^{-j\omega n}$$

$$w[n] = \begin{cases} \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{M}\right)\right], & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$w[n] = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{M}\right)\right] = \frac{1}{2} r[n] - \frac{1}{2} r[n] \cos\left(\frac{2\pi n}{M}\right)$$

☆ r[n] 只有在 $0 \le n \le M$ 為1.

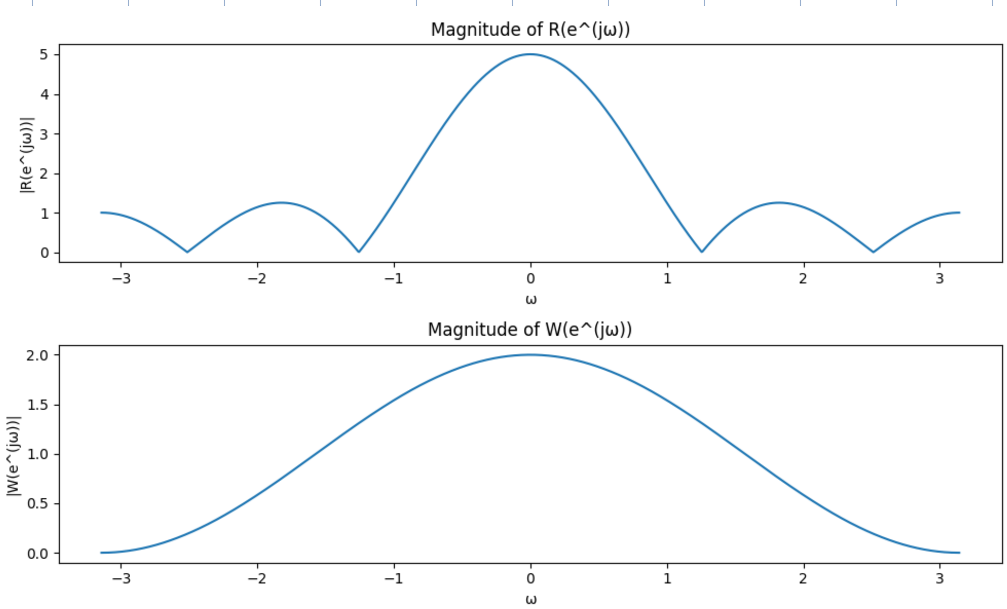
$$\longrightarrow W(e^{j\omega}) = \frac{1}{2} R(e^{j\omega}) - \frac{1}{2} \mathcal{F}\left\{r[n] \cos\left(\frac{2\pi n}{M}\right)\right\}$$

☆ $\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$

$$\longrightarrow \mathcal{F}\left\{r[n]\cos\left(\frac{2\pi n}{M}\right)\right\} = \frac{1}{2} R\left(e^{j(\omega - \frac{2\pi}{M})}\right) + \frac{1}{2} R\left(e^{j(\omega + \frac{2\pi}{M})}\right)$$

$$\Rightarrow W(e^{j\omega}) = \frac{1}{2} R(e^{j\omega}) - \frac{1}{4} R\left(e^{j(\omega - \frac{2\pi}{M})}\right) - \frac{1}{4} R\left(e^{j(\omega + \frac{2\pi}{M})}\right)$$ #

(b)



4. Suppose that a discrete-time signal $x[n]$ is given by the formula

$$x[n] = 2.2\cos(0.3\pi n - \frac{\pi}{3})$$

and that it was obtained by sampling a continuous-time signal

$$x(t) = 2.2\cos(2\pi f_0 t - \frac{\pi}{3})$$

at a sampling rate of f_s = 6000 samples/sec. Suppose the absolute value of f_0 is less than 8 kHz, i.e., $|f_0| < 8000$. What are the three values of f_0 that could have produced $x[n]$ under the sampling rate f_s = 6000?

已知 $W_d = 2\pi \cdot \frac{f_0}{f_s}$, $W_d = 0.3\pi = 2\pi \cdot \frac{f_0}{6000}$

$\rightarrow f_0 = \underline{900\,Hz}$

考慮 Aliasing , $W_d = 2\pi \cdot \frac{f_0 + kf_s}{f_s}$ $(k \in \mathbb{Z})$

$k=1$, $f_0 = 900 - 6000 = \underline{-5100\,Hz} < 8000$

$k=-1$, $f_0 = 900 + 6000 = \underline{6900\,Hz} < 8000$

$k = \pm 2$, $|f_0| > 8000$

所以可能答案為 900, −5100, 6900 #.