Detailed Investigation of DTFT (c.f. Shenoi, 2006)

Slight different notations

Forward Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Inverse Transform

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

DTFT Properties (with proofs)

- **Time shifting**

If $x(n)$ has a DTFT $X(e^{j\omega})$, then $x(n-k)$ has a DTFT equal to $e^{-j\omega k}X(e^{j\omega})$, where k is an integer. This is known as the *time-shifting property* and it is easily proved as follows: DTFT of $x(n-k) = \sum_{n=-\infty}^{\infty} x(n-k)e^{-j\omega n} = e^{-j\omega k}\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = e^{-j\omega k}X(e^{j\omega})$. So we denote this property by

$$\boxed{x(n-k) \Leftrightarrow e^{-j\omega k}X(e^{j\omega})}$$

Frequency shifting

If $x(n) \Leftrightarrow X(e^{j\omega})$, then

$$\boxed{e^{j\omega_0 n}x(n) \Leftrightarrow X(e^{j(\omega-\omega_0)})}$$

This is known as the *frequency-shifting property*, and it is easily proved as follows:

$$\sum_{n=-\infty}^{\infty} x(n)e^{j\omega_0 n}e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega-\omega_0)n} = X(e^{j(\omega-\omega_0)})$$

Time reversal

$$\boxed{\begin{array}{c}\text{if } x(n) \Leftrightarrow X(e^{j\omega}) \text{ then} \\ x(-n) \Leftrightarrow X(e^{-j\omega})\end{array}}$$

Proof: DTFT of $x(-n) = \sum_{n=-\infty}^{\infty} x(-n)e^{-j\omega n}$. We substitute $(-n) = m$, and we get $\sum_{n=-\infty}^{\infty} x(-n)e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x(m)e^{j\omega m} = \sum_{m=-\infty}^{\infty} x(m)e^{-j(-\omega)m} = X(e^{-j\omega})$.

➢ DTFT of impulse function $\delta(n)$

$$\boxed{\sum_{n=-\infty}^{n} \delta(n) e^{-jwn} = e^{jw0} = 1}$$

➢ DTFT of $\delta(n+k) + \delta(n-k)$

According to the time-shifting property,

$$\text{DTFT of } \delta(n+k) \text{ is } e^{jwk}, \text{ DTFT of } \delta(n-k) \text{ is } e^{-jwk}$$

Hence

$$\boxed{\text{DTFT of } \delta(n+k)+\delta(n-k) \text{ is } e^{jwk} + e^{-jwk} = 2\cos(wk)}$$

➢ DTFT of $x(n) = 1$ (for all n)

$x(n)$ **can be represented as** $x(n) = \sum_{k=-\infty}^{\infty} \delta(n-k)$

We prove that its DTFT is $2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$

Continuous delta function

Proof: The inverse DTFT of $2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$ is evaluated as

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \right] e^{j\omega n} \, d\omega$$

$$= \int_{-\pi}^{\pi} \left[\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \right] e^{j\omega n} \, d\omega$$

From the sifting property we get

$$\left[\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \right] e^{j\omega n} = \left[\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \right] e^{j2\pi kn}$$

$$= \left[\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \right]$$

where we have used $e^{j2\pi kn} = 1$ for all n. When we integrate the sequence of impulses from $-\pi$ to π, we have only the impulse at $\omega = 0$.

Hence

$$\int_{-\pi}^{\pi}\left[\sum_{k=-\infty}^{\infty}\delta(w-2\pi k)\right]e^{jwn}dw$$

$$=\int_{-\pi}^{\pi}\left[\sum_{k=-\infty}^{\infty}\delta(w-2\pi k)\right]dw$$

$$=\int_{-\pi}^{\pi}\delta(w)dw = 1 \qquad \text{for all } n$$

➢ DTFT of $a^n u(n)$ $(|a| < 1)$

let $x_1(n) = a^n u(n)$

then $$X_1(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n$$

Geometric series to infinity

This infinite sequence converges to $1/(1 - ae^{-j\omega}) = e^{j\omega}/(e^{j\omega} - a)$ when $|a| < 1$.

➢ DTFT of **Unit Step Sequence**

Note that $a^n u(n) \Leftrightarrow 1/(1 - ae^{-j\omega}) = e^{j\omega}/(e^{j\omega} - a)$ is valid only when $|a| < 1$.
When $a = 1$, we get the unit step sequence $u(n)$

We express the unit step function as the sum of two functions

$$u(n) = u_1(n) + u_2(n)$$

where

$$u_1(n) = \tfrac{1}{2} \quad \text{for} \quad -\infty < n < \infty$$

and

$$u_2(n) = \begin{cases} \tfrac{1}{2} & \text{for } n \geq 0 \\ -\tfrac{1}{2} & \text{for } n < 0 \end{cases}$$

Therefore we express $\delta(n) = u_2(n) - u_2(n-1)$. Using $\delta(n) \Leftrightarrow 1$ and $u_2(n) - u_2(n-1) \Leftrightarrow U_2(e^{j\omega}) - e^{-j\omega}U_2(e^{j\omega}) = U_2(e^{j\omega})[1 - e^{-j\omega}]$, and equating the two results, we get

$$1 = U_2(e^{j\omega})[1 - e^{-j\omega}]$$

Therefore

$$U_2(e^{j\omega}) = \frac{1}{[1 - e^{-j\omega}]}$$

We know that the DTFT of $u_1(n) = \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) = U_1(e^{j\omega})$.

Adding these two results, we have the final result

Continuous delta function

$$\boxed{u(n) \Leftrightarrow \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) + \frac{1}{(1 - e^{-j\omega})}}$$

This gives us the DTFT of the unit step function $u(n)$, which is unique.

Differentiation Property

$$\boxed{\text{To prove that } nx(n) \Leftrightarrow j[dX(e^{j\omega})]/d\omega,}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

differentiate both sides to get

$$[dX(e^{j\omega})]/d\omega = \sum_{n=-\infty}^{\infty} x(n)(-jn)e^{-j\omega n}$$

multiplying both sides by j, we get

$$j[dX(e^{j\omega})]/d\omega = \sum_{n=-\infty}^{\infty} nx(n)e^{-j\omega n}.$$

DTFT of a **rectangular pulse**

Consider a rectangular pulse

$$x_r(n) = \begin{cases} 1 & |n| \leq N \\ 0 & |n| > N \end{cases}$$

Its DTFT is derived as follows:

$$X_r(e^{j\omega}) = \sum_{n=-N}^{N} e^{-j\omega n}$$

To simplify this summation, we use the identity[5]

$$\sum_{n=-N}^{N} r^n = \frac{r^{N+1} - r^{-N}}{r - 1}; \quad r \neq 1$$

$$= 2N + 1; \quad r = 1$$

and get

$$X_r(e^{j\omega}) = \frac{e^{-j(N+1)\omega} - e^{jN\omega}}{e^{-j\omega} - 1}$$

$$= \frac{e^{-j0.5\omega}\left(e^{-j(N+0.5)\omega} - e^{j(N+0.5)\omega}\right)}{e^{-j0.5\omega}(e^{-j0.5\omega} - e^{j0.5\omega})}$$

$$\boxed{= \begin{cases} \dfrac{\sin[(N+0.5)\omega]}{\sin[0.5\omega]} & \omega \neq 0 \\ 2N+1 & \omega = 0 \end{cases}}$$

$\sin(\boldsymbol{ax})/\sin(\boldsymbol{x})$: called Dirichlet Kernel

$\sin(\boldsymbol{ax})/\boldsymbol{x}$: called sinc function