Z-Transform

- Z-transform: polynomial representation of a sequence

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- Z-transform can be used to solve difference equation systems when the initial-rest condition is satisfied.
- Z-transform can also be used to identify whether an LTI system is stable.
  - An IIR filter is recursive, which is a **dynamic system** with its current output depending on both the current input and the previous output.
  - An FIR filter is always stable. However, for a dynamic system such as IIR, we must consider whether and when it is stable (i.e., the output is bounded or not divergent to infinity).

Convergence Region of Z-transform

The sum of a series may not be converge for all $z \in Z$.

- Region of convergence (ROC)
  - Eg., $x[n] = u[n]$ is absolutely summable if $|z|>1$. This means that the Z-transform for the unit step signal exists with ROC $|z|>1$.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \le \sum_{n=-\infty}^{\infty} |x[n]| \left| z^{-n} \right|$$

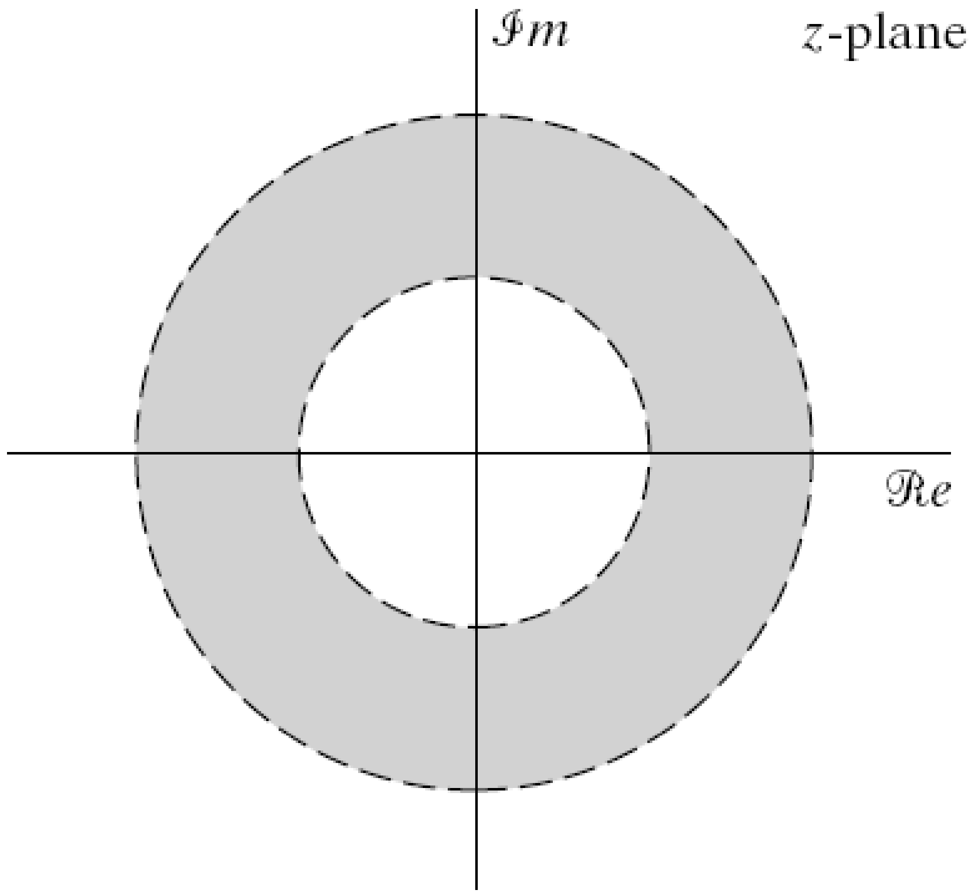
ROC of Z-transform

- Property: convergence of the power series $X(z)$ depends only on $|z|$.

$$\sum_{n=-\infty}^{\infty} \left| x[n] \right| |z^{-n}| < \infty$$

- If some value of z, say $z = z_1$, is in the ROC, then all values of z on the circle defined by $|z| = |z_1|$ will also be in the ROC.
- Thus the ROC will consist of a ring in the z-plane.

ROC of Z-transform – Ring Shape



Poles and Zeros

- Pole:
  - The *pole* of a z-transform $X(z)$ are the values of z for which $X(z)=\infty$.
- Zero:
  - The *zero* of a z-transform $X(z)$ are the values of z for which $X(z)=0$.
- When $X(z) = P(z)/Q(z)$ is a rational form, and both $P(z)$ and $Q(z)$ are polynomials of z, the poles of are the roots of $Q(z)$, and the zeros are the roots of $P(z)$, respectively.

Examples

- Zeros of a system function
  - The system function of the FIR system $y[n] = 6x[n] - 5x[n-1] + x[n-2]$ has been shown as

$$H(z) = 6\frac{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}{z^2} = \frac{P(z)}{Q(z)}$$

- The zeros of this system are $1/3$ and $1/2$, and the pole is 0.
- Since 0 and 0 are double roots of $Q(z)$, the pole is a second-order pole.

Example: Finite-length Sequence (FIR System)

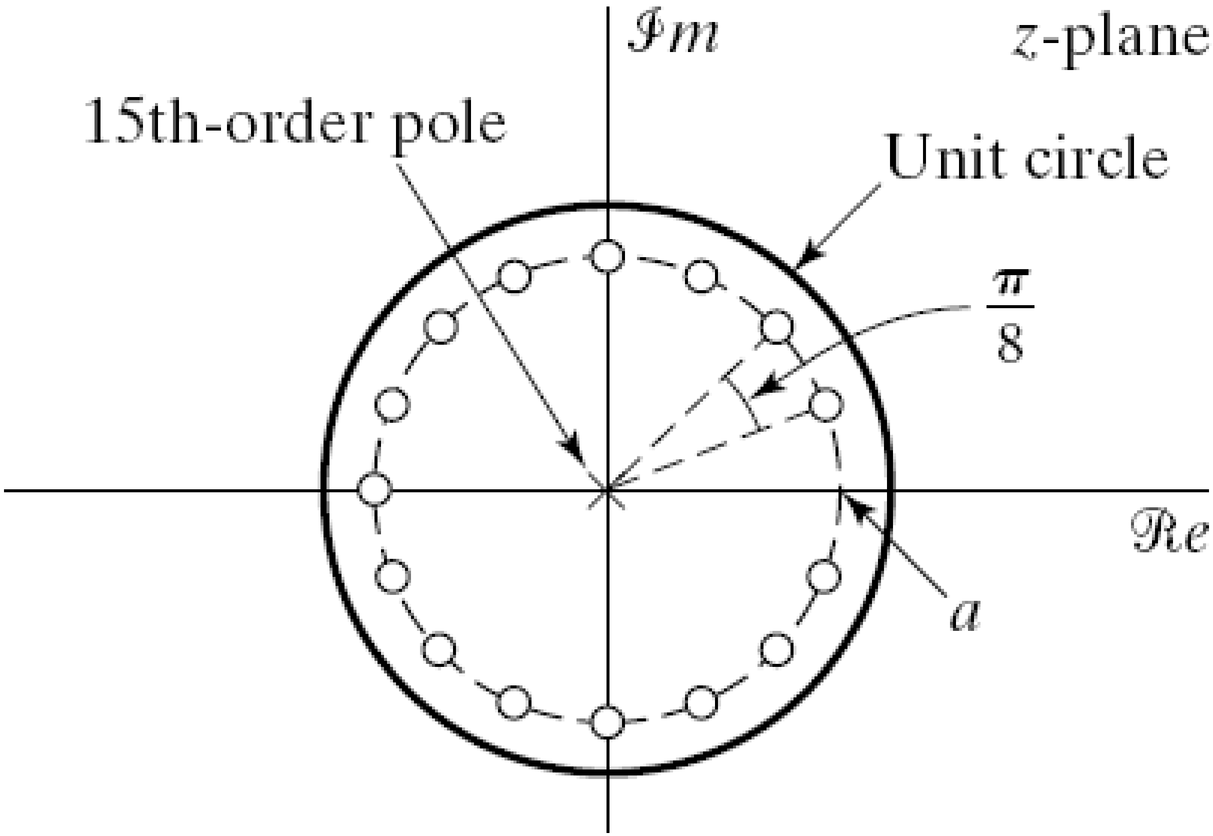
Given $$x(n) = \begin{cases} a^n & 0 \le n \le N-1 \\ 0 & otherwise \end{cases}$$

Then $$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n = \frac{1-\left(az^{-1}\right)^N}{1-az^{-1}}$$

$$= \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

There are the N roots of $z^N = a^N$: $z_k = ae^{j2\pi k/N}$.
The root of $k = 0$ cancels the pole at $z=a$.
Thus there are $N-1$ zeros, $z_k = ae^{j2\pi k/N}$, $k = 1 \ldots N$, and an $(N-1)$-th order pole at zero.

Pole-zero Plot of the above FIR system



Inverse Z-transform

Identify a sequence from its Z-transform

- To uniquely identify a sequence from its Z-transform, we have to specify additionally the ROC of the Z-transform.
- Example: Right-sided sequence:
  - A discrete-time signal is right-sided if it is nonzero only for $n \geq 0$.
- Consider the signal $x[n] = a^n u[n]$.

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

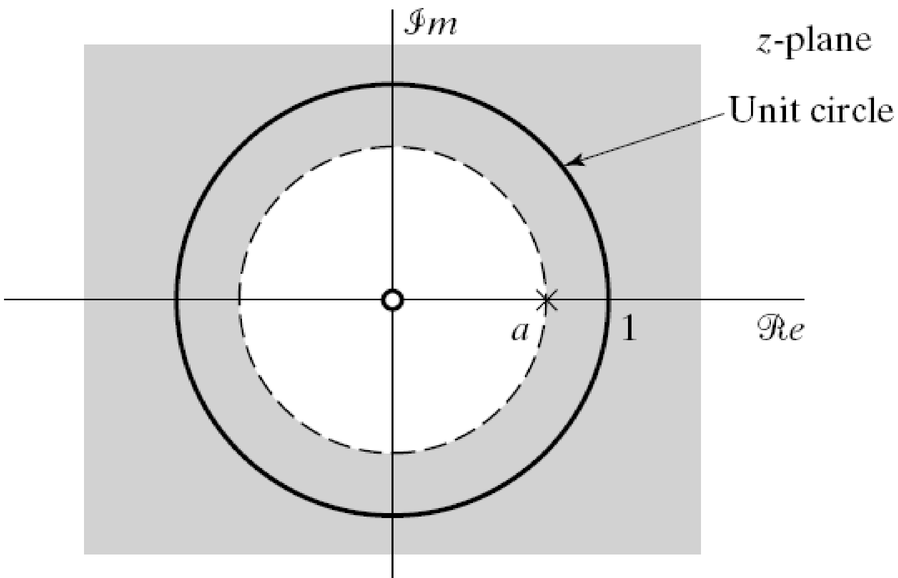
- For convergent $X(z)$, we need $\sum_{n=0}^{\infty} \left(az^{-1}\right)^n < \infty$
  - Thus, the ROC is the range of values of z for which $|az^{-1}| < 1$ or, equivalently, $|z| > a$.

Example: Right-sided Exponential Sequence (continue)

- By sum of power series,

$$X(z)=\sum_{n=0}^{\infty}\left(az^{-1}\right)^{n}=\frac{1}{1-az^{-1}}=\boxed{\frac{z}{z-a},\quad |z|>|a|}$$

- There is one zero, at $z=0$, and one pole, at $z=a$.



○ : zeros

× : poles

Gray region: ROC

Example: Left-sided Exponential Sequence

- Left-sided sequence:
  - A discrete-time signal is left-sided if it is nonzero only for $n \leq -1$.
- Consider the signal $x[n] = -a^n u[-n-1]$.

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= \sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} \left(a^{-1} z\right)^n$$

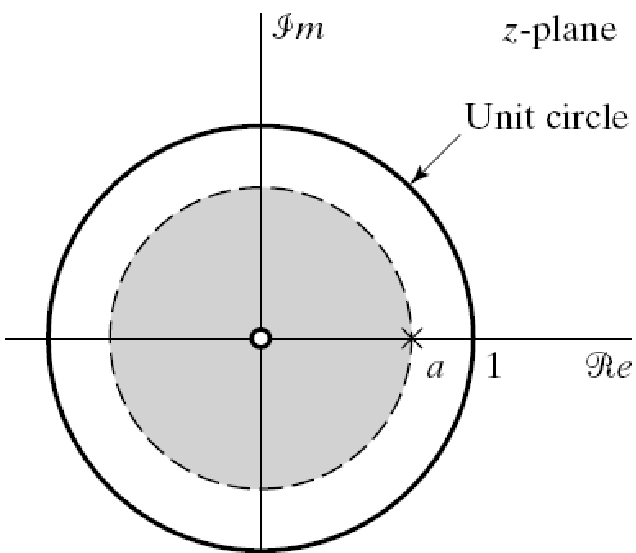
  - If $|az^{-1}| < 1$ or, equivalently, $|z| < a$, the sum converges.

Example: Left-sided Exponential Sequence (continue)

- By sum of power series,

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z} = \boxed{\frac{z}{z-a}, \quad |z| < |a|}$$

- There is one zero, at $z=0$, and one pole, at $z=a$.



The pole-zero plot and the algebraic expression of the system functions are the same as those in the previous right-sided sequence example, but the ROC is different.

Inverse Z-transform

To uniquely identify a sequence from its Z-transform, we have to specify additionally the ROC of the Z-transform.

Another Example

Given $x(n)=\left(\frac{1}{2}\right)^n u(n)+\left(-\frac{1}{3}\right)^n u(n)$

Then $X(z)=\sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^n u(n)z^{-n}+\sum_{n=-\infty}^{\infty}\left(-\frac{1}{3}\right)^n u(n)z^{-n}$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n}+\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n z^{-n}$$

$$=\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{1}{1+\frac{1}{3}z^{-1}}=\boxed{\frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}} \quad (1)$$

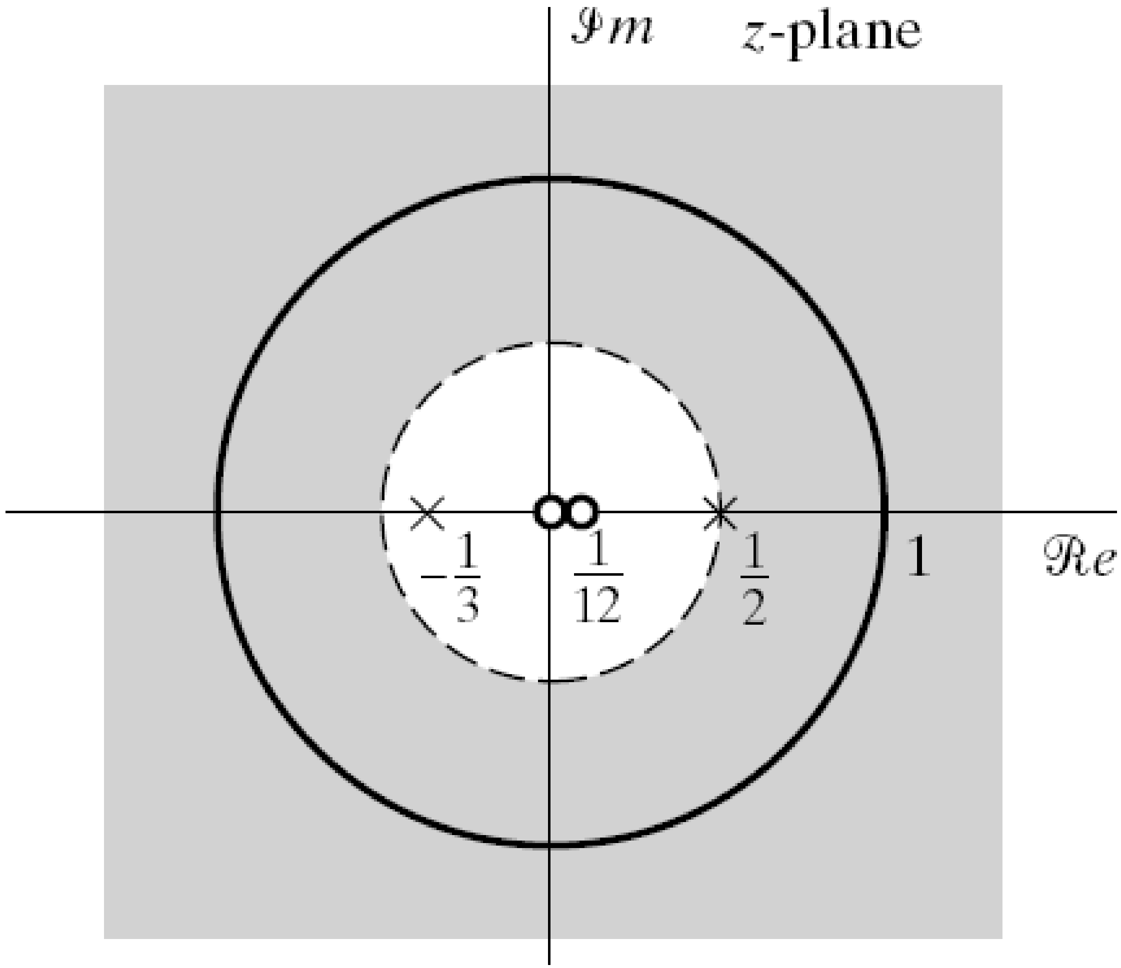
$$\left(\frac{1}{2}\right)^n u(n) \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}$$

$$\left(-\frac{1}{3}\right)^n u(n) \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3}$$

Thus

$$\left(\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{3}\right)^n u(n) \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{2}$$

ROC



- However, when consider another two-sided exponential sequence,

$$x(n) = \left(-\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)$$

Since $\left(-\frac{1}{3}\right)^n u(n) \overset{z}{\leftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3}$

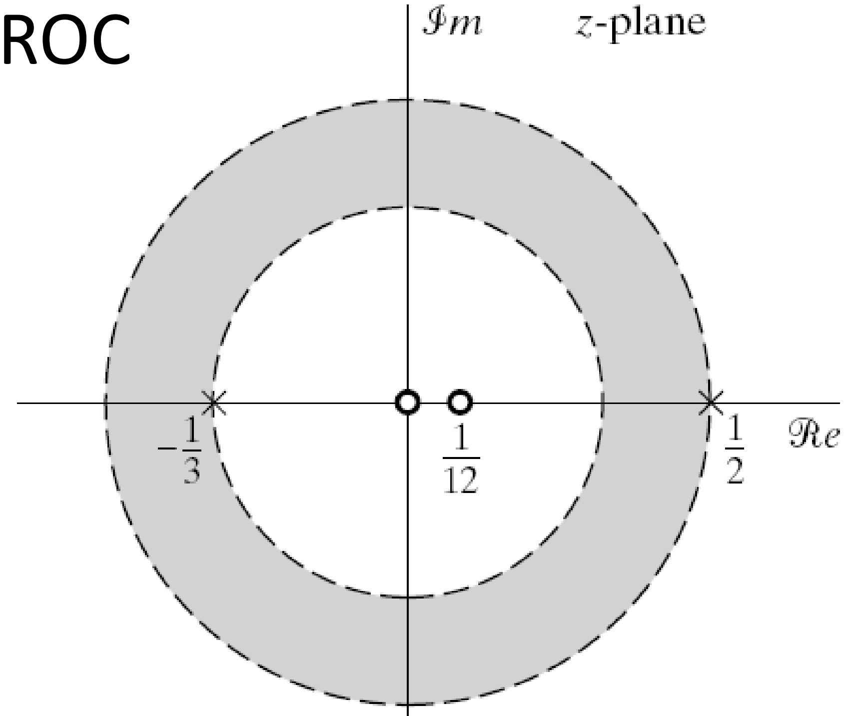
and by the left-sided sequence example

$$-\left(\frac{1}{2}\right)^n u(-n-1) \overset{z}{\leftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| < \frac{1}{2}$$

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \boxed{\frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}}$$

The same as (1)

ROC



The system function, poles and zeros are the same as the previous example, but the ROC is not.

Stability of causal LTI System

- **Bounded**: A signal $x[n]$ is called *bounded* if there is a finite value B such that $x[n] < B, \forall n$

- **Stability of a system**: **BIBO** (bounded in bounded out)
  - If a system is BIBO stable, then the output will be bounded for every input to the system that is bounded.

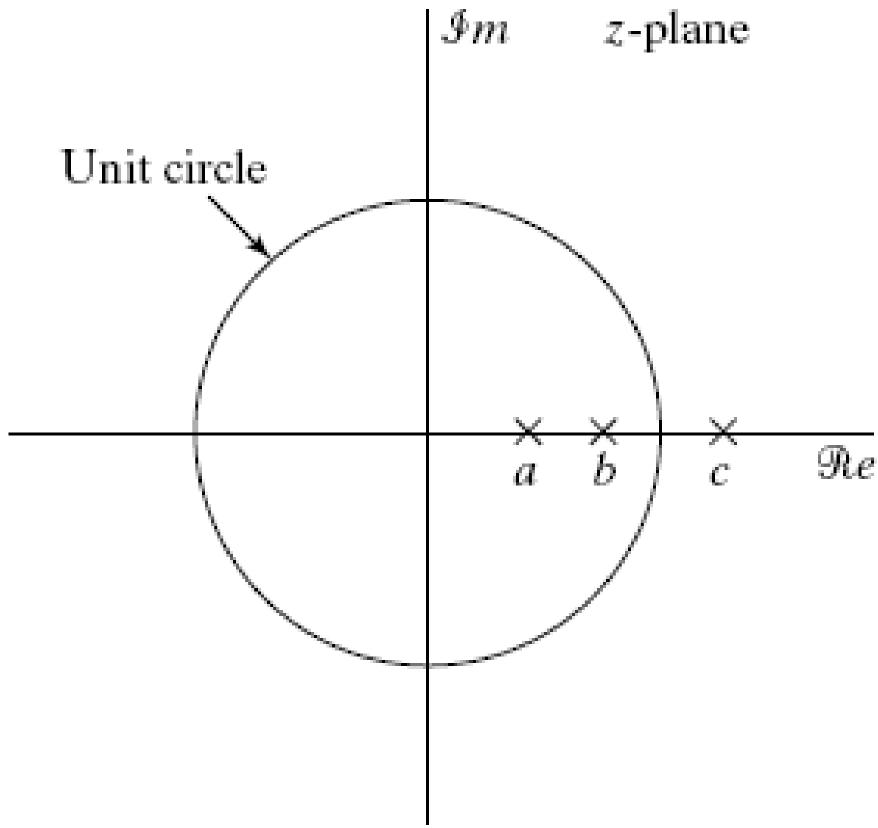
Properties of the ROC

- The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \le r_R < |z| \le r_L \le \infty$.
- The ROC cannot contain any poles.
- If $x[n]$ is a finite-length sequence (i.e., FIR), then the ROC is the entire z-plane except possible $z = 0$ or $z = \infty$.
- If $x[n]$ is a right-sided sequence (i.e., causal), the ROC extends outward from the outermost (i.e., largest magnitude) finite pole in $X(z)$ to $z=\infty$.

Properties of the ROC (continue)

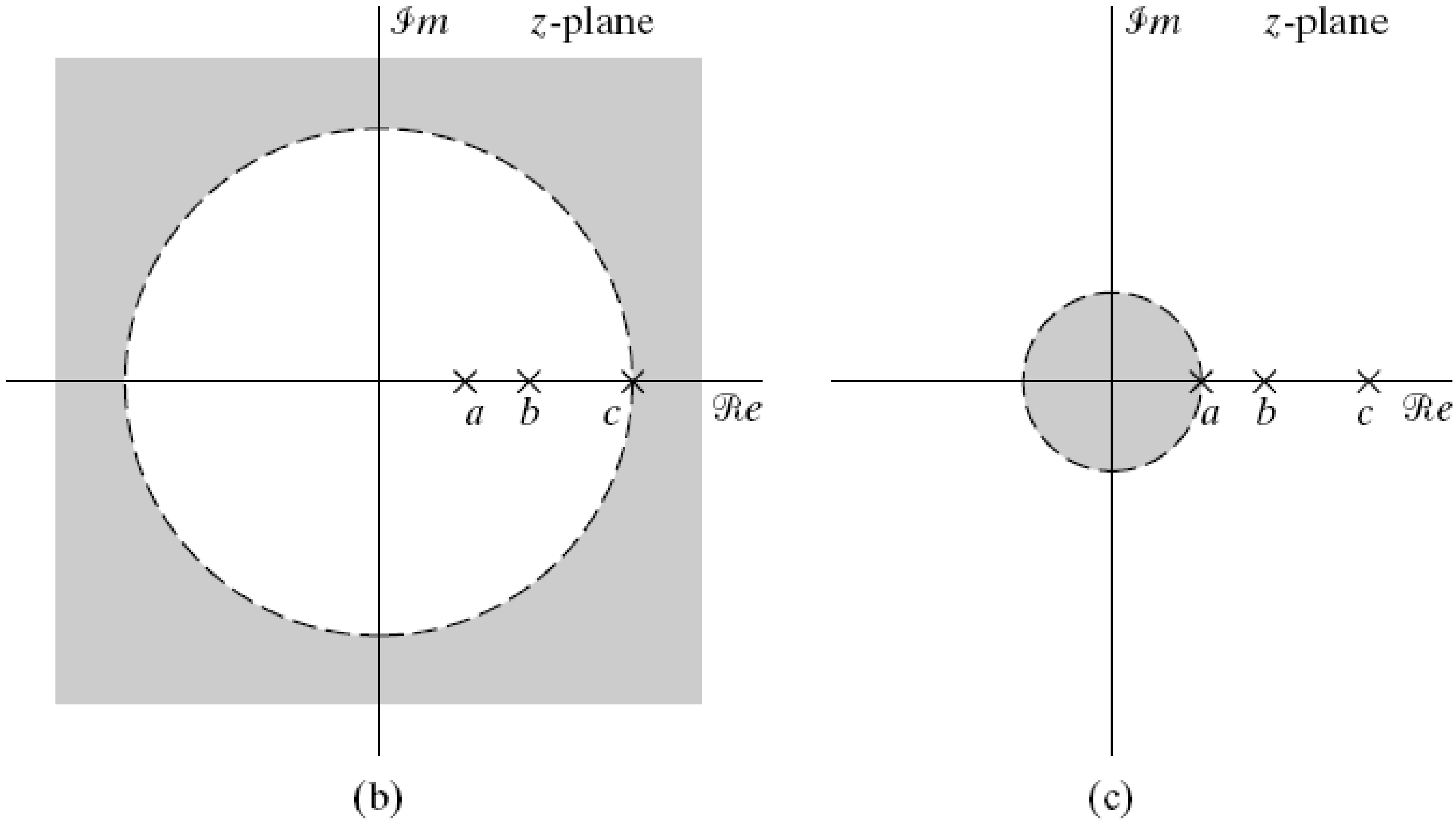
- If $x[n]$ is a left-sided sequence, the ROC extends inward from the innermost (i.e., smallest magnitude) nonzero pole in $X(z)$ to (and possibly include) $z = 0$.
- A two-sided sequence $x[n]$ is an infinite-duration sequence that is neither right nor left sided. The ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole, but not containing any poles.

Example

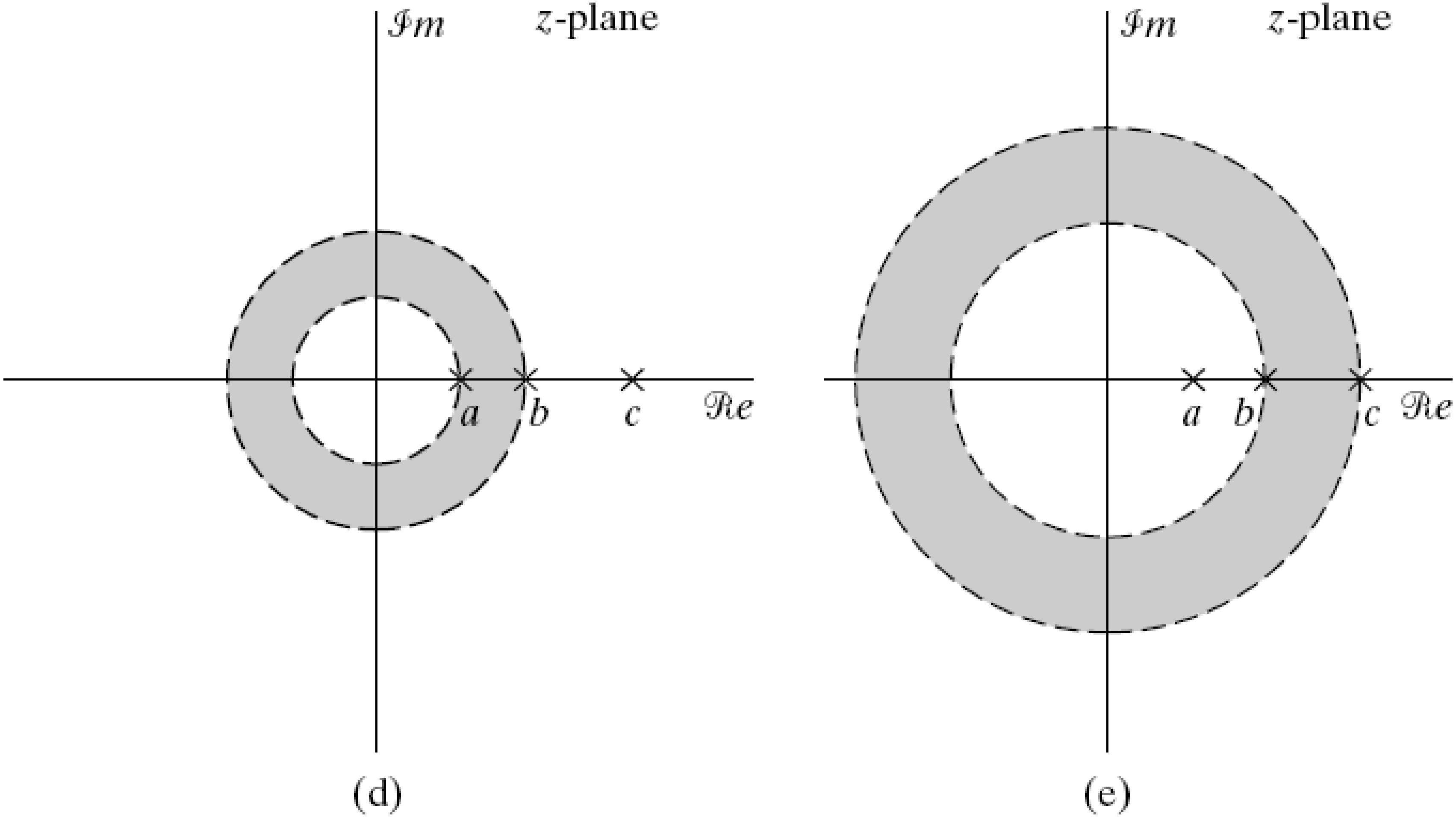


(a)

A system with three poles



Different possibilities of the ROC. (b) ROC to a right-sided sequence. (c) ROC to a left-handed sequence.

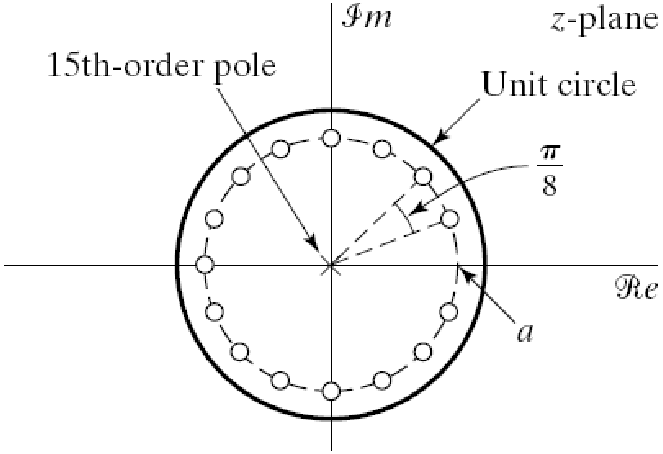


Different possibilities of the ROC. (d) ROC to a two-sided sequence. (e) ROC to another two-sided sequence.

Stability of causal LTI System (cont.)

- Property: A causal LTI system is BIBO stable iff **the poles are all inside the unit circle** (defined by the Z-transform)
  - In the continuous domain, this is analogous that a continuous causal LTI system is stable iff the poles are on the left-side plane (defined by the Laplace transform).

- FIR systems are always stable
  - Because the poles of a FIR system are at the origin.



Inverse Z-transform

- Given $X(z)$, find the sequence $x[n]$ that has $X(z)$ as its z-transform.
- We need to specify both algebraic expression and ROC to make the inverse Z-transform unique.
- Techniques for finding the inverse z-transform:
  - Investigation method:
    - By inspect certain transform pairs.
    - Eg. If we need to find the inverse z-transform of

$$X(z) = \frac{1}{1 - 0.5z^{-1}}$$

From the transform pair we see that $x[n] = 0.5^n u[n]$.

Inverse Z-transform by Partial Fraction Expansion

- If $X(z)$ is the rational form with

$$X(z) = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- An equivalent expression is

$$X(z) = \frac{z^{-M} \sum_{m=0}^{M} b_m z^{M-m}}{z^{-N} \sum_{k=0}^{N} a_k z^{N-k}} = \frac{z^{N} \sum_{m=0}^{M} b_m z^{M-m}}{z^{M} \sum_{k=0}^{N} a_k z^{N-k}}$$

Inverse Z-transform by Partial Fraction Expansion (continue)

- There will be M zeros and N poles at nonzero locations in the z-plane.
- Note that $X(z)$ could be expressed in the form

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{m=1}^{M} \left(1 - c_m z^{-1}\right)}{\prod_{m=1}^{N} \left(1 - d_k z^{-1}\right)}$$

where c_k's and d_k's are the nonzero zeros and poles, respectively.

Inverse Z-transform by Partial Fraction Expansion (continue)

- Then $X(z)$ can be expressed as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \qquad \text{（部分分式）}$$

Multiplying both sides of the above equation by $1 - d_k z^{-1}$ and evaluating for $z = d_k$ shows that

$$A_k = \left(1 - d_k z^{-1}\right) X(z)\Big|_{z=d_k}$$

Example

- Find the inverse z-transform of

$$X(z) = \frac{1}{\left(1-(1/4)z^{-1}\right)\left(1-(1/2)z^{-1}\right)} \qquad |z| > \frac{1}{2}$$

$X(z)$ can be decomposed as

$$X(z) = \frac{A_1}{\left(1-(1/4)z^{-1}\right)} + \frac{A_2}{\left(1-(1/2)z^{-1}\right)}$$

Then

$$A_1 = \left(1-(1/4)z^{-1}\right)X(z)\Big|_{z=1/4} = -1$$

$$A_2 = \left(1-(1/2)z^{-1}\right)X(z)\Big|_{z=1/2} = 2$$

Example (continue)

- Thus

$$X(z) = \frac{-1}{\left(1-(1/4)z^{-1}\right)} + \frac{2}{\left(1-(1/2)z^{-1}\right)}$$

From the ROC if we have a right-hand sequence,

$$x[n] = 2\left(\frac{1}{2}\right)^{n} u[n] - \left(\frac{1}{4}\right)^{n} u[n]$$

Another Example

- Find the inverse z-transform of

$$X(z)=\frac{\left(1+z^{-1}\right)^2}{\left(1-(1/2)z^{-1}\right)\left(1-z^{-1}\right)} \qquad |z|>1$$

Since both the numerator and denominator are of degree 2, a constant term exists.

$$X(z)=B_0+\frac{A_1}{\left(1-(1/2)z^{-1}\right)}+\frac{A_2}{\left(1-z^{-1}\right)}$$

B_0 can be determined by the fraction of the coefficients of z^{-2}, $B_0 = 1/(1/2) = 2$.

Another Example (continue)

$$X(z) = 2 + \frac{A_1}{\left(1 - (1/2)z^{-1}\right)} + \frac{A_2}{\left(1 - z^{-1}\right)}$$

$$A_1 = 2 + \frac{-1 + 5z^{-1}}{\left(1 - (1/2)z^{-1}\right)\left(1 - z^{-1}\right)} \left(1 - (1/2)z^{-1}\right)\Big|_{z=1/2} = 9$$

$$A_2 = 2 + \frac{-1 + 5z^{-1}}{\left(1 - (1/2)z^{-1}\right)\left(1 - z^{-1}\right)} \left(1 - z^{-1}\right)\Big|_{z=1} = 8$$

From the ROC, the solution is right-handed. So

$$X(z) = 2 - \frac{9}{\left(1 - (1/2)z^{-1}\right)} + \frac{8}{\left(1 - z^{-1}\right)}$$

$$x[n] = 2\delta[n] - 9(1/2)^n u[n] + 8u[n]$$

Example: Finite-length Sequence

- Find the inverse z-transform of

$$X(z) = z^2\left(1 - 0.5z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right)$$

By directly expand $X(z)$, we have

$$X(z) = z^2 - 0.5z - 1 + 0.5z^{-1}$$

Thus,

$$x[n] = \delta[n+2] - 0.5\delta[n+1] - \delta[n] + 0.5\delta[n-1]$$

Inverse z-transform formula

- General formula of inverse z-transform:

$$\mathcal{Z}[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z^{-1}\{X(z)\} = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

(For your reference only)

Find the inverse z-transform by contour Integral in the complex domain (need the foundation or prior course of Function of Complex Variables; 複變數函數論)

Example of solving difference equations by using inverse Z-transform

- Consider the difference equation

$$y[n] - 0.75y[n-1] + 0.125y[n-2] = 2x[n-1]$$

Assume that the initial rest condition is satisfied. Determine the impulse response of this LTI system.

Remark of initial-rest condition: If the input $x[n]$ is zero for n less than some time n_0, the output $y[n]$ is also zero for n less than n_0.

Sol: The system is causal because the output does not rely on the future input. Hence, the impulse response is a right-sided sequence.

Taking the z-transform on both sides, we have

$$Y(z) - 0.75z^{-1}Y(z) + 0.125z^{-2}Y(z) = 2z^{-1}X(z).$$

When $x[n] = \delta(n)$, we have

$$Y(z) - 0.75z^{-1}Y(z) + 0.125z^{-2}Y(z) = 2z^{-1}.$$

So, $Y(z) = \frac{2z^{-1}}{1-0.75z^{-1}+0.125z^{-2}}$

Example of solving difference equations using Z-transform (cont.)

Suppose $Y(z) = \frac{2z^{-1}}{1-0.75z^{-1}+0.125z^{-2}} = \frac{2z^{-1}}{(1-a_1z^{-1})(1-a_2z^{-1})}$,

We have $a_1 + a_2 = 0.75,\ a_1a_2 = 0.125.$

By solving this quadratic equation system, we have

$a_1 = 0.5,\ a_2 = 0.25.$

Hence, $Y(z) = \frac{2z^{-1}}{(1-0.5z^{-1})(1-0.25z^{-1})}$

Assume that $Y(z) = \left(\frac{b_1}{(1-0.5z^{-1})} + \frac{b_2}{(1-0.25z^{-1})}\right).$

Then, $b_1 + b_2 = 0,\ -0.25b_1 - 0.5b_2 = 2.$

So, $b_1 = 8,\ b_2 = -8.$

$\therefore\ y[n] = 8 \times 0.5^n u[n] - 8 \times 0.25^n\ u[n].$ (due to right-sided sequence)

Example of solving difference equations using Z-transform (cont.)

Further question: Is this system stable?

Sol: The system function is

$$\frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

$$= \frac{z^{-1}}{(1 - 0.5z^{-1})(1 - 0.25z^{-1})} = \frac{z}{(z - 0.5)(z - 0.25)}$$

In terms of z. The two poles are $z = 0.5, 0.25$ are both inside the unit circle in the Z-plane. So, the system is stable. That is, for any input sequence $x[n]$ with all its entries bounded by some value, the output sequence $y[n]$ also has all its entries bounded by another value (so, won't going to infinity).

A naïve solution (Iterative replacement)

see how difficult it is if we don't use inverse z-transform

- Z-transform provides an efficient way to solve the difference equations. For comparison, we show the effort if you use iterative replacement to solve the problem

$$y[n] = 0.75y[n-1] - 0.125y[n-2] + 2x[n-1]. \qquad (1)$$

Let $x[n]$ be the delta function $x[n] = \delta[n]$. The initially rest condition holds, i.e., $y[-1] = y[-2] = 0$.

To perform iterative replacement, rewrite the difference equation of (1) in a matrix form (referred to as the state-space form),

$$\begin{bmatrix} y[n] \\ y[n-1] \end{bmatrix} = \begin{bmatrix} 0.75 & -0.125 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y[n-1] \\ y[n-2] \end{bmatrix} + 2 \begin{bmatrix} x[n-1] \\ 0 \end{bmatrix}, \qquad (2)$$

A naïve solution (Iterative replacement)

Let $x[n] = \delta[n]$. We have

$$t_n = At_{n-1} + 2v_{n-1}, \qquad (3)$$

where

$$t_n = \begin{bmatrix} y[n] \\ y[n-1] \end{bmatrix},\ A = \begin{bmatrix} 0.75 & -0.125 \\ 1 & 0 \end{bmatrix},\ v_{n-1} = \begin{bmatrix} \delta[n-1] \\ 0 \end{bmatrix}.$$

Note that $v_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ when $n \neq 0$, and $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Via iteration of equation (3), we obtain

$$t_1 = At_0 + 2v_0 = A\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

$$t_2 = At_1 + 2v_1 = At_1 = A\begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

$$t_3 = At_2 + 2v_2 = At_2 = A^2\begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

$$\vdots$$

Thus,

$$t_n = A^{n-1}\begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

A naïve solution (Iterative replacement)

To find A^{n-1}, let's perform eigen-decomposition, $A = PDP^{-1}$ (D is diagonal, P is nonsingular):

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} 2 & -0.5 \\ -1 & 0.5 \end{bmatrix},$$

where $P = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 2 & -0.5 \\ -1 & 0.5 \end{bmatrix}$ and $D = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.25 \end{bmatrix}$.

Thus $A^{n-1} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0.5^{n-1} & 0 \\ 0 & 0.25^{n-1} \end{bmatrix} \begin{bmatrix} 2 & -0.5 \\ -1 & 0.5 \end{bmatrix}$

$$= \begin{bmatrix} 0.5^{n-1} & 0.25^{n-1} \\ 0.5^{n-2} & 0.25^{n-2} \end{bmatrix} \begin{bmatrix} 2 & -0.5 \\ -1 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5^{n-2} - 0.25^{n-1} & \cdots \\ \cdots & \cdots \end{bmatrix}.$$

($\cdots$ means "don't care.").

A naïve solution (Iterative replacement)

Then, we get

$$\begin{bmatrix} y[n] \\ y[n-1] \end{bmatrix} = t_n = A^{n-1}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5^{n-3} - 2 \times 0.25^{n-1} \\ \cdots \end{bmatrix}$$

$$= \begin{bmatrix} (0.5)^{-3} \times 0.5^{n} - 2 \times (0.25)^{-1} \times 0.25^{n} \\ \cdots \end{bmatrix}$$

Hence, $y[n] = (8 \times 0.5^{n} - 8 \times 0.25^{n})u[n]$, where $u[n]$ is the unit-step function.

Z-transform Properties

- Suppose

$$x[n] \overset{z}{\leftrightarrow} X(z) \qquad \mathrm{ROC} = R_x$$

$$x_1[n] \overset{z}{\leftrightarrow} X_1(z) \qquad \mathrm{ROC} = R_{x_1}$$

$$x_2[n] \overset{z}{\leftrightarrow} X_2(z) \qquad \mathrm{ROC} = R_{x_2}$$

- Linearity

$$ax_1[n] + b\,x_2[n] \overset{z}{\leftrightarrow} aX_1(z) + bX_2(z) \qquad \mathrm{ROC} = R_{x_1} \cap R_{x_2}$$

Z-transform Properties (continue)

- Time shifting

$$x[n-n_0] \overset{z}{\leftrightarrow} z^{-n_0} X(z) \qquad \mathrm{ROC} = R_x$$ (except for the possible addition or deletion of $z=0$ or $z=\infty$.)

- Multiplication by an exponential sequence

$$z_0^n x[n] \overset{z}{\leftrightarrow} X(z / z_0) \qquad \mathrm{ROC} = |z_0| R_x$$

Z-transform Properties (continue)

- Differentiation of $X(z)$

$$nx[n] \overset{z}{\leftrightarrow} -z\frac{dX(z)}{dz} \qquad \mathrm{ROC} = R_x$$

- Conjugation of a complex sequence

$$x^*[n] \overset{z}{\leftrightarrow} X^*(z^*) \qquad \mathrm{ROC} = R_x$$

Z-transform Properties (continue)

- Time reversal

$$x^*[-n] \overset{z}{\leftrightarrow} X^*(1/z^*) \qquad \text{ROC} = \frac{1}{R_x}$$

If the sequence is real, the result becomes

$$x[-n] \overset{z}{\leftrightarrow} X(1/z) \qquad \text{ROC} = \frac{1}{R_x}$$

- Convolution

$$x_1[n] * x_2[n] \overset{z}{\leftrightarrow} X_1(z)X_2(z) \qquad \text{ROC contains } R_{x_1} \cap R_{x_2}$$

Z-transform Properties (continue)

- Initial-value theorem: If $x[n]$ is zero for $n<0$ (i.e., if $x[n]$ is causal), then

$$x[0] = \lim_{z \to \infty} X(z)$$

Some Common Z-transform Pairs

$$\delta[n] \leftrightarrow 1 \qquad \text{ROC: all } z.$$

$$u[n] \leftrightarrow \frac{1}{1-z^{-1}} \qquad \text{ROC: } |z| > 1.$$

$$-u[-n-1] \leftrightarrow \frac{1}{1-z^{-1}} \qquad \text{ROC: } |z| < 1.$$

$$\delta[n-m] \leftrightarrow z^{-m} \qquad \text{ROC: all } z \text{ except } 0 \text{ (if } m>0\text{) or } \infty \text{ (if } m<0\text{)}.$$

$$a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}} \qquad \text{ROC: } |z| > |a|.$$

$$-a^n u[-n-1] \leftrightarrow \frac{1}{1-az^{-1}} \qquad \text{ROC: } |z| < |a|.$$

$$na^n u[n] \leftrightarrow \frac{az^{-1}}{\left(1-az^{-1}\right)^2} \qquad \text{ROC: } |z| > |a|.$$

Some Common Z-transform Pairs (continue)

$$-na^n u[-n-1] \leftrightarrow \frac{az^{-1}}{\left(1-az^{-1}\right)^2} \qquad \text{ROC} : |z| < |a|.$$

$$[\boldsymbol{cos}\, w_0 n]u[n] \leftrightarrow \frac{1-[\boldsymbol{cos}\, w_0]z^{-1}}{1-[2\,\boldsymbol{cos}\, w_0]z^{-1}+z^{-2}} \qquad \text{ROC} : |z| > 1.$$

$$[\boldsymbol{sin}\, w_0 n]u[n] \leftrightarrow \frac{[\boldsymbol{sin}\, w_0]z^{-1}}{1-[2\,\boldsymbol{cos}\, w_0]z^{-1}+z^{-2}} \qquad \text{ROC} : |z| > 1.$$

$$[r^n\, \boldsymbol{cos}\, w_0 n]u[n] \leftrightarrow \frac{1-[r\,\boldsymbol{cos}\, w_0]z^{-1}}{1-[2r\,\boldsymbol{cos}\, w_0]z^{-1}+r^2 z^{-2}} \qquad \text{ROC} : |z| > r.$$

$$[r^n\, \boldsymbol{sin}\, w_0 n]u[n] \leftrightarrow \frac{[r\,\boldsymbol{sin}\, w_0]z^{-1}}{1-[2r\,\boldsymbol{cos}\, w_0]z^{-1}+r^2 z^{-2}} \qquad \text{ROC} : |z| > r.$$

$$\begin{cases} a^n & 0 \le n \le N-1 \\ 0 & otherwise \end{cases} \leftrightarrow \frac{1-a^N z^{-N}}{1-az^{-1}} \qquad \text{ROC} : |z| > 0.$$