FIR Filter Design

- As we see, an ideal low-pass filter is required in many situations, such as frequency band selection, down-sampling and up-sampling.
- However, ideal filter is non-achievable. Instead, we can only design a filter that approximates the ideal low-pass filter.
- How to design a filter? Two kinds of filters: FIR and IIR.
  - FIR filters (i.e., convolution with a finite-length sequence) are almost restricted to discrete-time implementations.
  - IIR filters are usually conducted from the continuous-time counterparts.
- We focus on FIR filter in the following. Especially, we focus on the windowing method for filter design.

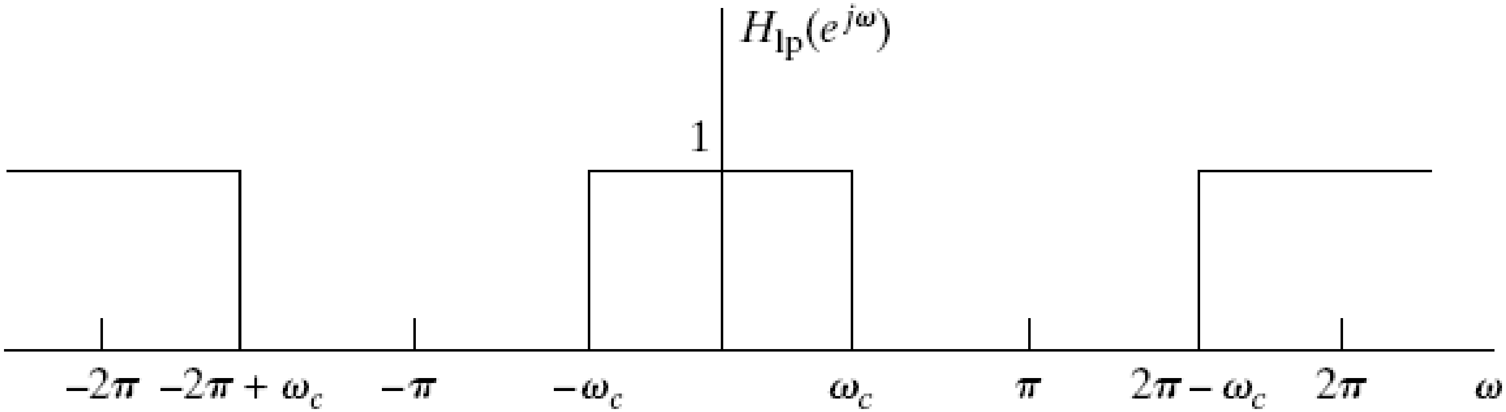
Recall: Discrete-time Ideal Low-pass Filter

- Ideal low-pass filter in DTFT domain

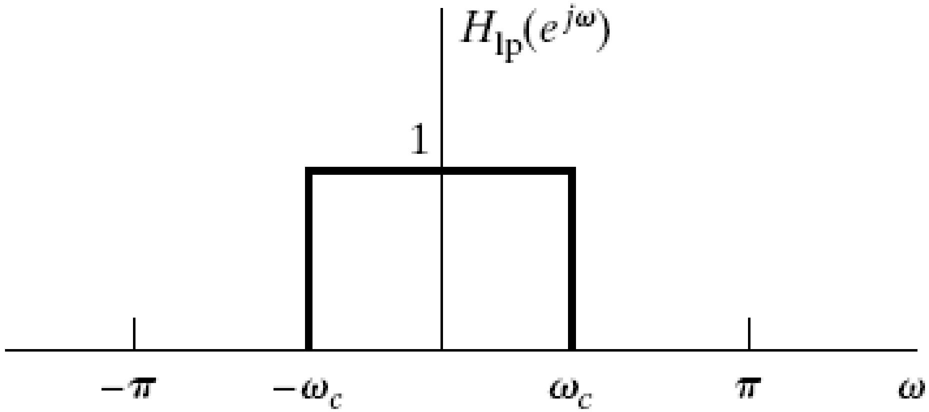
$$H_{lowpass}(e^{jw}) = \begin{cases} 1, & |w| < w_c \\ 0, & w_c < |w| < \pi \end{cases}$$

- Note that we depict the frequency response in the range [-π,π] only for discrete-time signals. The "low frequencies" are frequencies close to zero, while the "high frequencies" are those close to ±π.

Recall: Ideal Low-pass Filter's frequency response



(a)



Discrete-time Ideal Low-pass Filter

- As for a rectangular window (ideal low-pass filter) in the frequency domain, its time domain sequence $h_{lowpass}[n]$ can be found by inverse DTFT:

$$h_{lowpass}[n] = \frac{1}{2\pi}\int_{-w_c}^{w_c} e^{jwn} dw$$

$$= \frac{1}{2\pi jn} e^{jwn} \Big|_{-w_c}^{w_c} = \frac{1}{2\pi jn}(e^{jw_c n} - e^{-jw_c n})$$

$$= \frac{\sin w_c n}{\pi n}$$

Sampled Sinc function

Recall: Approximation of Discrete-time Ideal Low-pass Filter

- In other words, the DTFT of the sampled sync function is the ideal low-pass filter:

$$H_{lowpass}(e^{jw}) = \sum_{n=-\infty}^{\infty} \frac{\sin w_c n}{\pi n} e^{-jwn}$$

- However, the sampled sync function cannot be realized because its length is infinite.
- To **approximate the ideal low-pass filter**, a practical method is to use the **partial sum** instead

$$H_M(e^{jw}) = \sum_{n=-M}^{M} \frac{\sin w_c n}{\pi n} e^{-jwn}$$

Filter design by windowing – general principle

- Windowing method for approximation: approximate the ideal filter by truncating $h_{lowpass}[n]$ in time domain.

$$h[n] = h_{lowpass}[n]w[n]$$

with $w[n]$ a rectangular window: $w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & otherwise \end{cases}$

- In frequency domain, the response becomes

$$H(e^{j\omega}) = H_{\mathrm{lowpass}}(e^{j\omega}) * W(e^{j\omega}),$$

as time domain multiplication implies frequency domain convolution.

Causal low-pass FIR Filter design by windowing

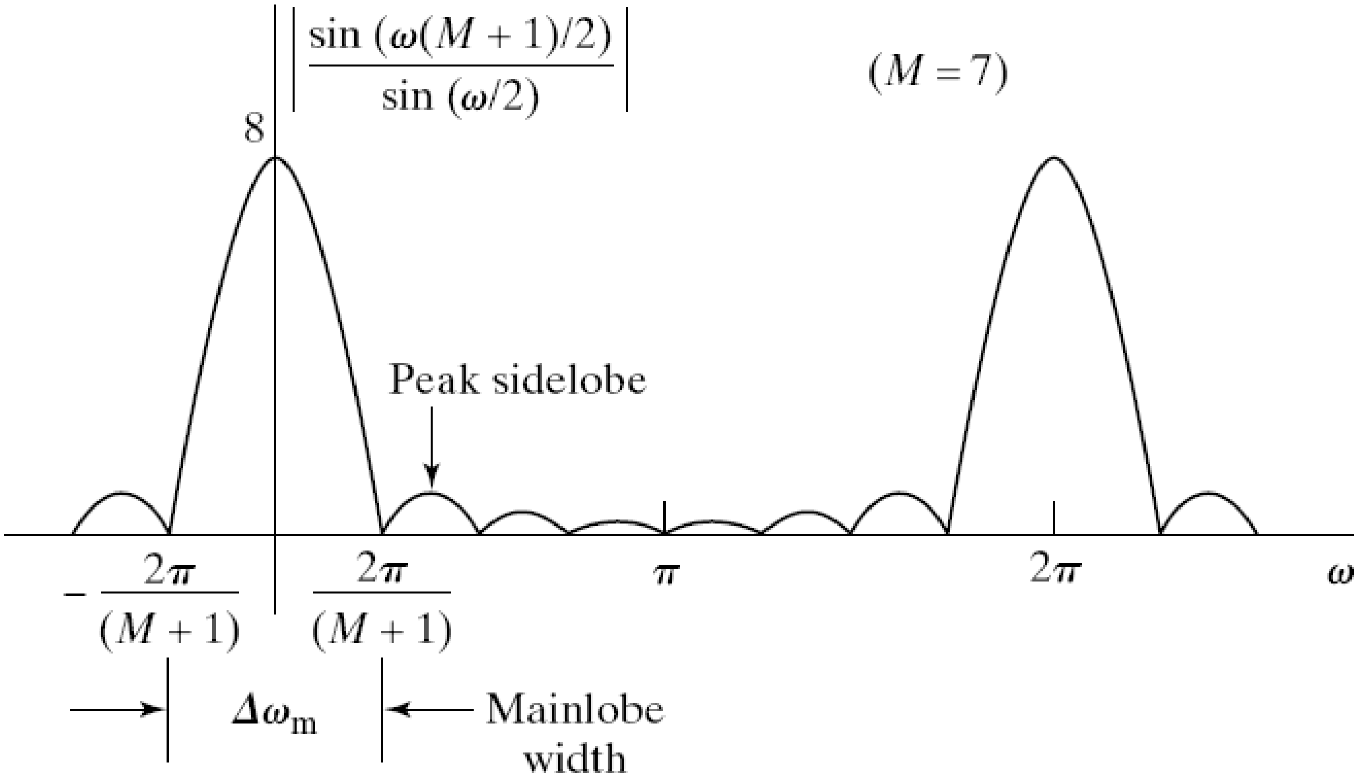
- What is $W\left(e^{j\omega}\right)$: Recall that DTFT's frequency domain of a rectangular window $w[n]$ is a Dirichlet kernel:

$$W(e^{jw}) = \sum_{n=0}^{M} e^{-jwn} = \frac{1-e^{-jw(M+1)}}{1-e^{-jw}} = e^{-jwM/2}\frac{\sin[w(M+1)/2]}{\sin(w/2)}$$

- Frequency domain: convolution of the ideal low-pass response and the Dirichlet kernel,

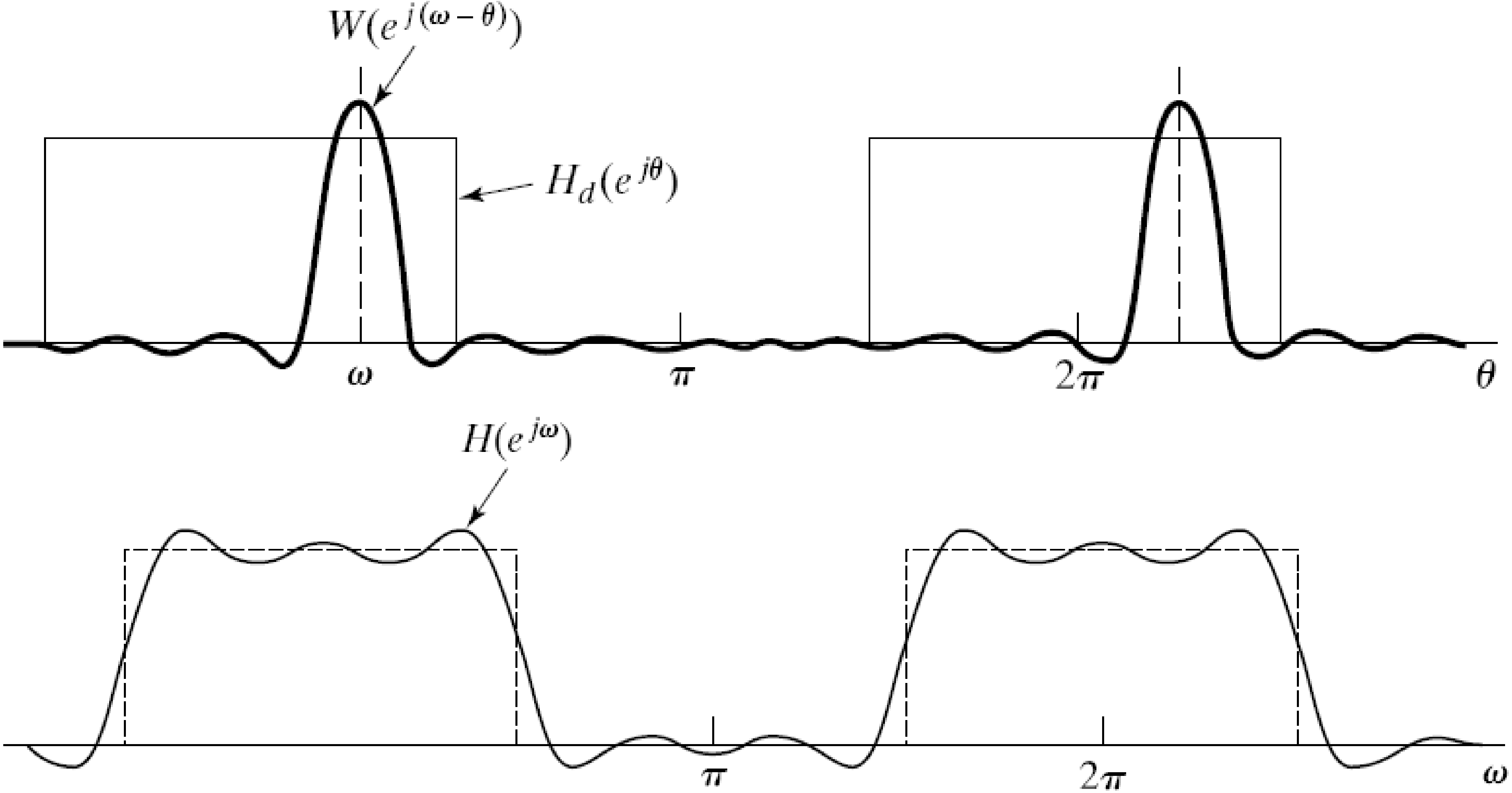
$$H(e^{jw}) = (1/2\pi)\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(w-\theta)})d\theta$$

Recall: Dirichlet kernel



As M increases, the main lobe width of the Dirichlet kernel decreases, and so the shape approximates more to impulses.

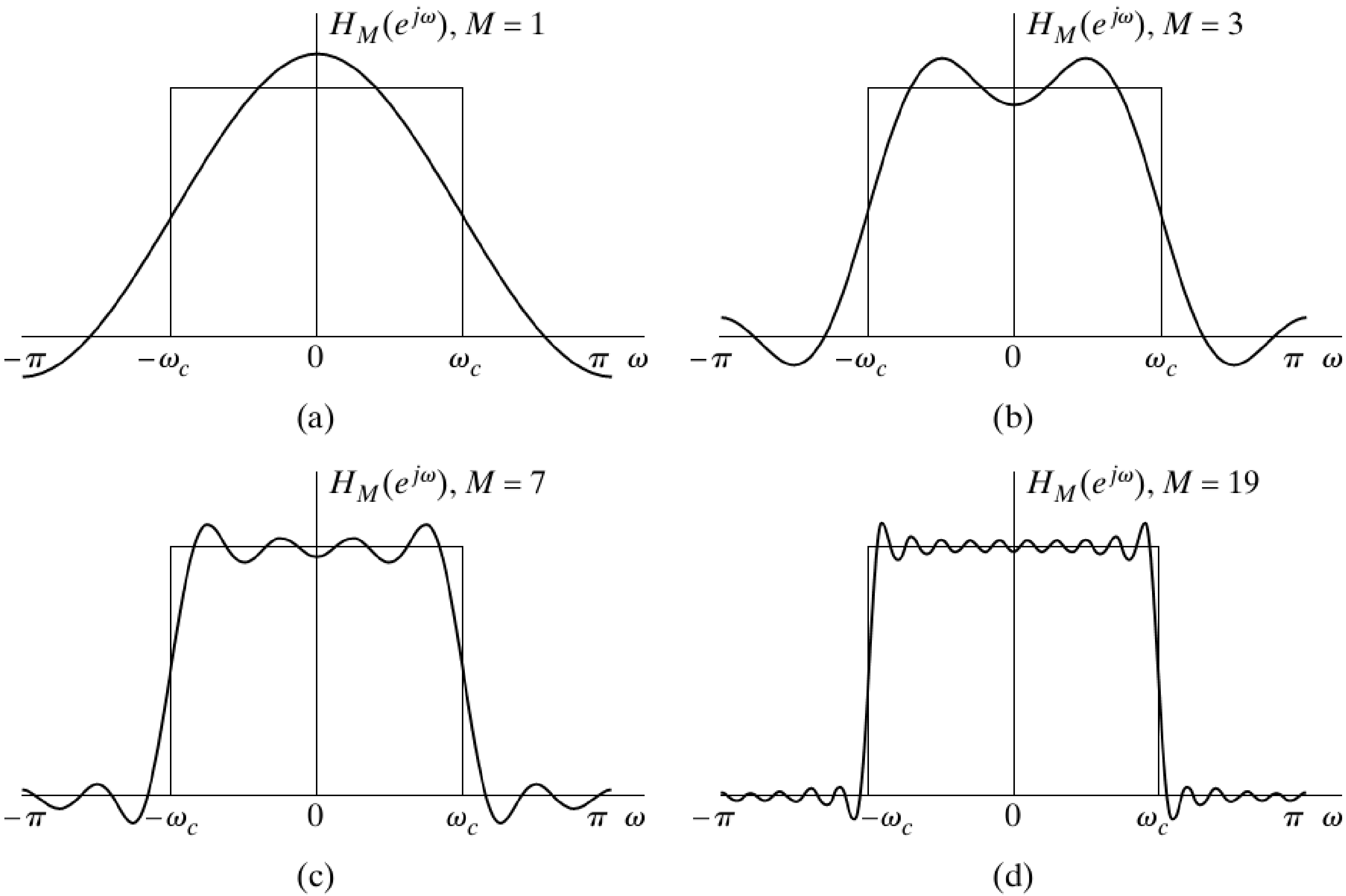
Convolution of a rectangular window and a Dirichlet kernel in the DTFT domain: An illustration



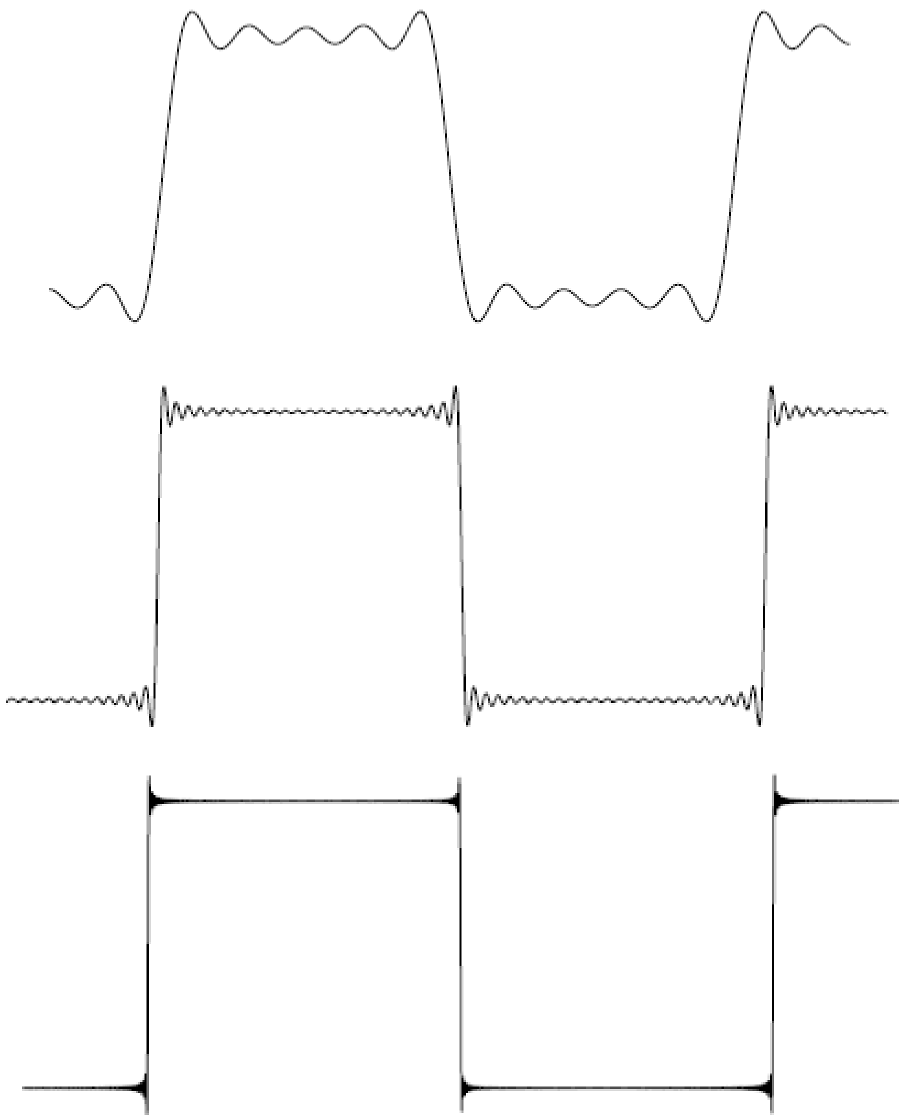
The above shows a typical approximation of the ideal lowpass filter by using rectangular-window.

Overview of Approximation of Discrete-time Ideal Low-pass Filter

- Examples of M=1, 3, 7, 19



■ **Gibbs phenomenon of the rectangular window:** as M increases, the **maximum amplitude of the oscillation does not approach zero** when the rectangular window is used.



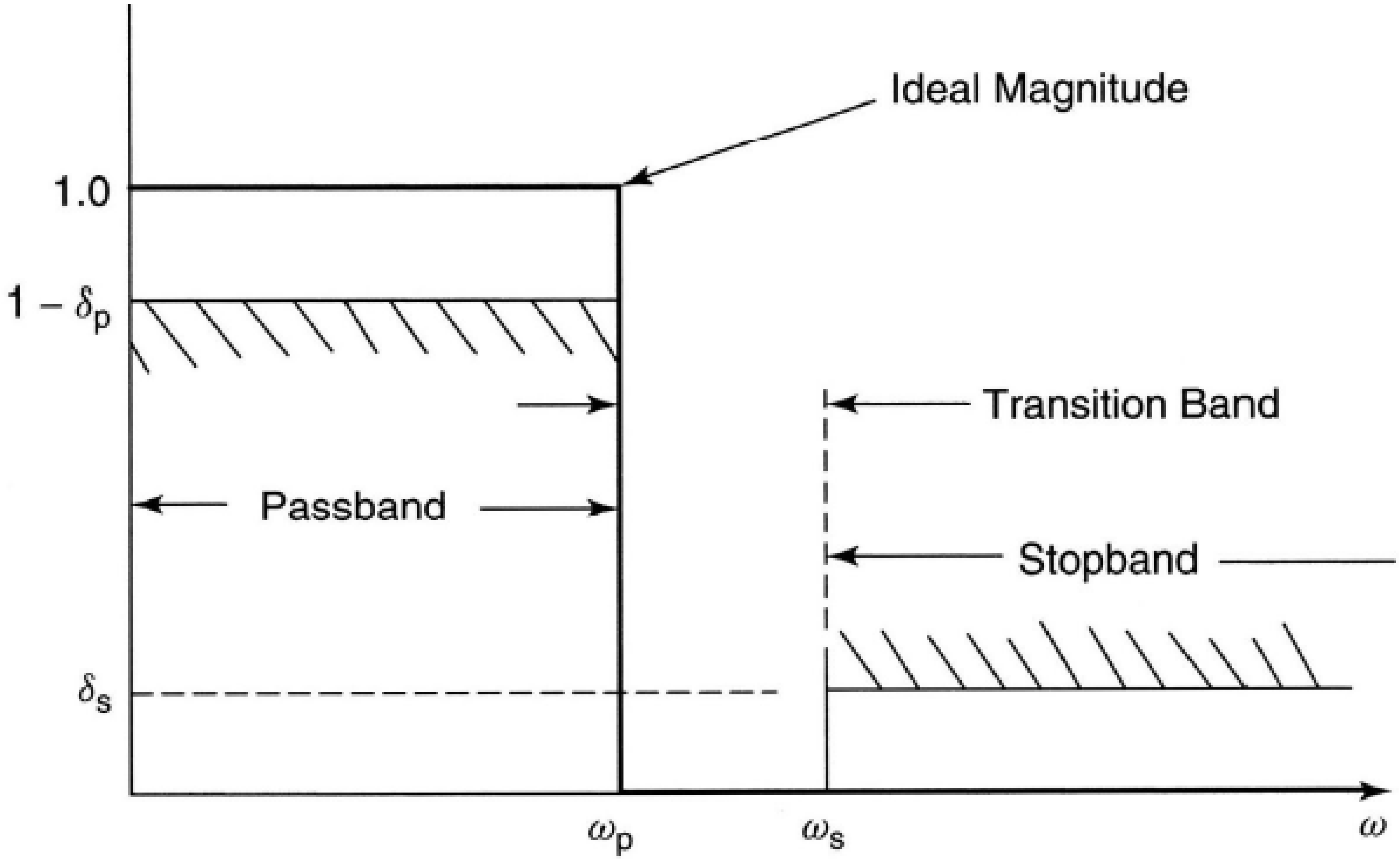
Functional approximation of square wave using 5 harmonics

using 25 harmonics

using 125 harmonics

Design criterion: Passband and stopband

- Magnitude response of a low-pass analog filter with tolerances:



- **passband**: $[0, w_p]$, **transition region**: $[w_p, w_s]$, **stopband**: $[w_s, \infty]$, passband tolerance: δ_p, stopband tolerance: δ_s.

Using other windows

- For the rectangular window, the width of the main lobe decreases as M increases. The transition region gets smaller but the ripple remains because of the Gibbs phenomenon.
- **Solution to sharp discontinuity**: Use windows with no abrupt discontinuity in their time domain.
    - Hence, like the case of spectrogram construction, **other window functions with different main lobes and side lobe gains are used** as $w[n]$ to avoid Gibbs phenomenon.
    - When using other window functions, the reduced ripple comes at the expense of a **wider transition region** (remember that rectangular window has the smallest main-lobe width among all windows of the same length). However, this can be compensated for by increasing the length of the filter.

- Thus, in FIR filter design, we often use other window functions.
- Some **commonly used windows** used in spectrogram are also used here:

Bartlett (triangular):

$$w[n] = \begin{cases} 2n/M & 0 \le n \le M/2 \\ 2 - 2n/M & M/2 \le n \le M \\ 0 & otherwise \end{cases}$$

Hanning:

$$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M) & 0 \le n \le M \\ 0 & otherwise \end{cases}$$

Hamming:

$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M) & 0 \le n \le M \\ 0 & otherwise \end{cases}$$

Kaiser window

Kaiser found a near-optimal window defined as

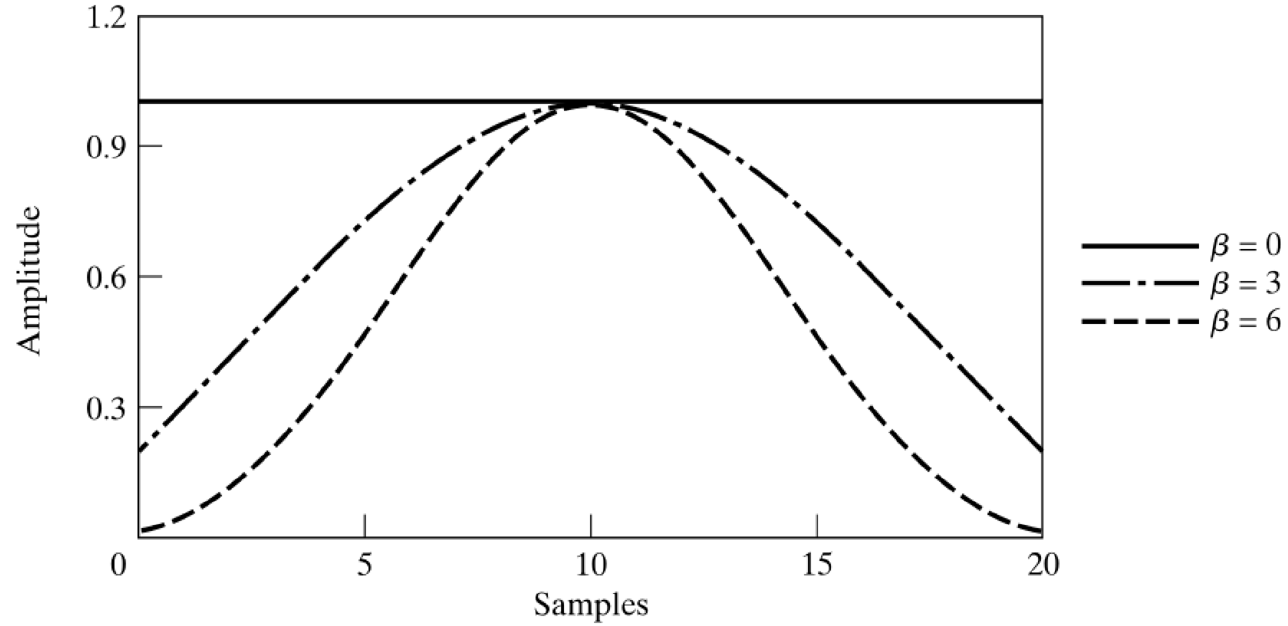
$$w[n] = \begin{cases} I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}] & 0 \le n \le M \\ 0 & otherwise \end{cases}$$

where $\alpha = M/2$ and $I_0(\cdot)$ represents the zeroth-order modified Bessel function of the first kind.

- In contrast to the other windows, the Kaiser window has two parameters: the length $M+1$ and a shape parameter β.

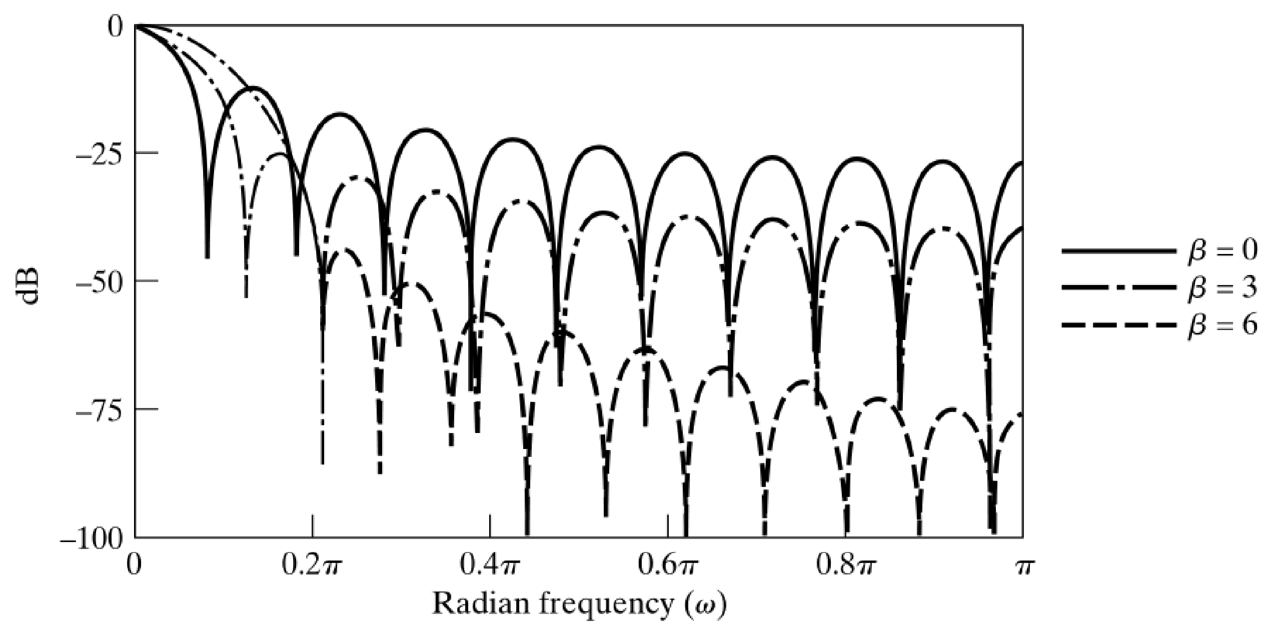
Kaiser window review

Note that when $\beta = 0$, Kaiser window reduces to the rectangular window.



Kaiser window for $M = 20$; $\beta = 0, 3, 6$

Viewing the side-lob level in the log-scaled frequency domain (in dB)



Summary

In sum, like the case of spectrogram, using different windows allows us to have a trade-off between the following two factors:

- transition region

(influenced by the main-lobe width of the window)

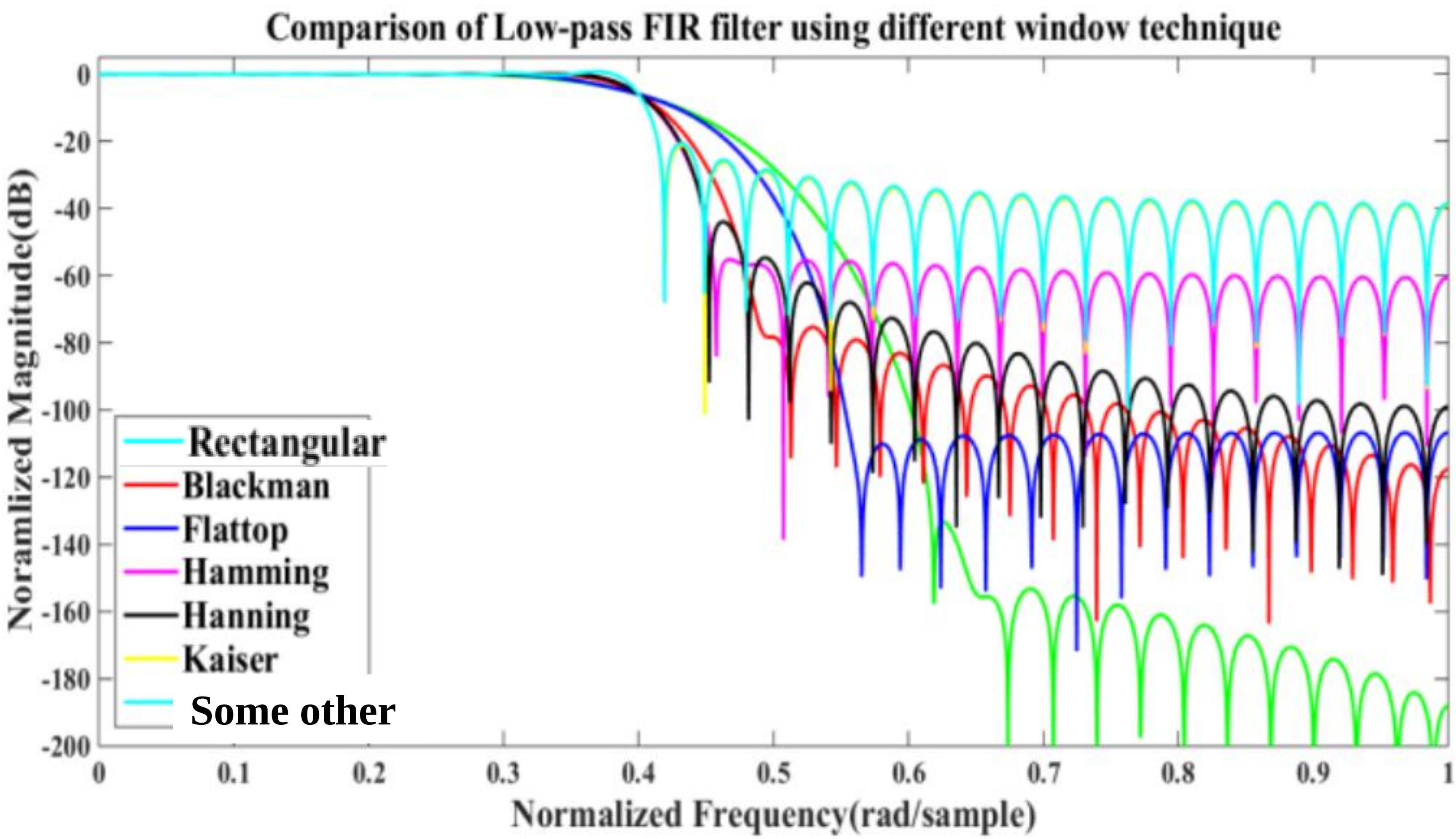
versus

- maximum oscillation

(influenced by the side-lobe gain of the window).

Illustrated figure

viewing the approximated low-pass filter in log-scale (dB)



Tradeoff between the transition region and ripple magnitude