Fourier Transform for Continuous Functions (or Signals)

- **How to define the frequency or spectrum for general continuous-time signals**?
- Central goal: representing a signal by a set of orthogonal bases corresponding to frequencies or spectrum.
- Fourier series allows to find the spectrum of only periodic functions.

From Periodic to Non-Periodic

- Fourier series transforms a periodic continuous signal into the frequency domain.
- What will happen when the continuous signal is not periodic?
- Consider the period of a signal with the fundamental frequency ω_0.
- T_0 specifies the fundamental period,

$$T_0 = \frac{2\pi}{w_0} = \frac{1}{f}$$

Review of Fourier Series

- Fourier series representation of a periodic signal $x_{T_0}(t)$ can be given by the pair of equations

Forward Transform

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-jk\omega_0 t} dt$$

Integrals over a period $[0, T_0]$ and $[-T_0/2,\ T_0/2]$ are the same

Inverse Transform

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T_0}$

Imaging $T_0 \to \infty$

- A non-periodic signal can be conceptually thought of as a periodic signal whose fundamental period T_0 is infinitely long,

$$T_0 \to \infty.$$

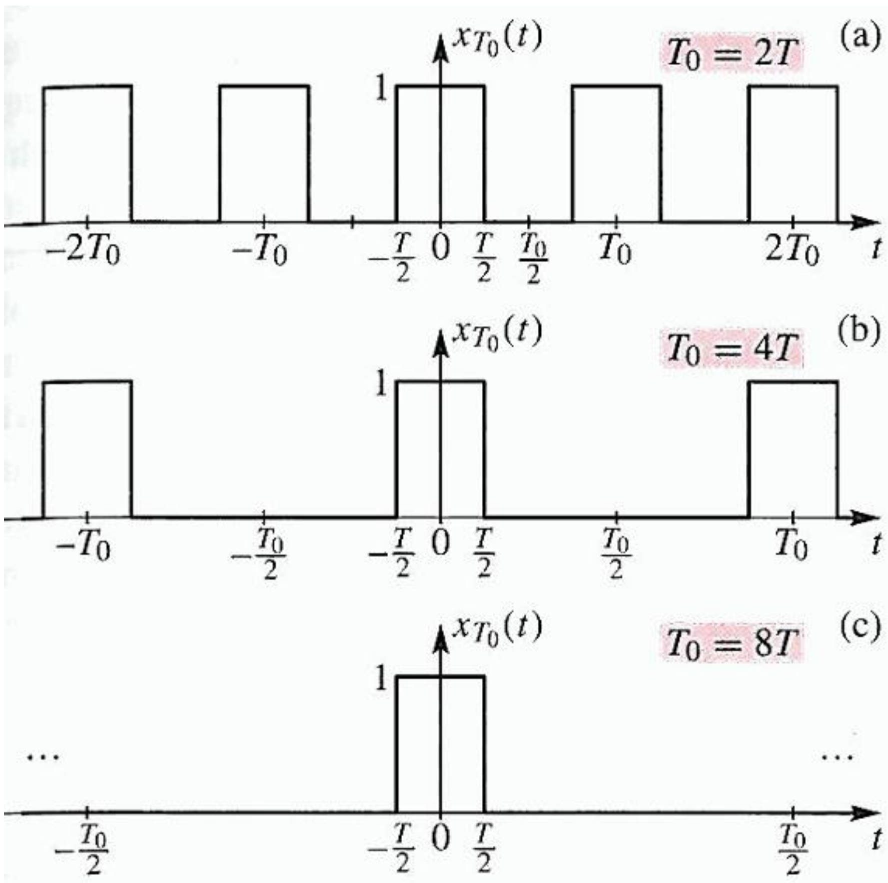
- In this case, the fundamental frequency

$$\omega_0 \to 0.$$

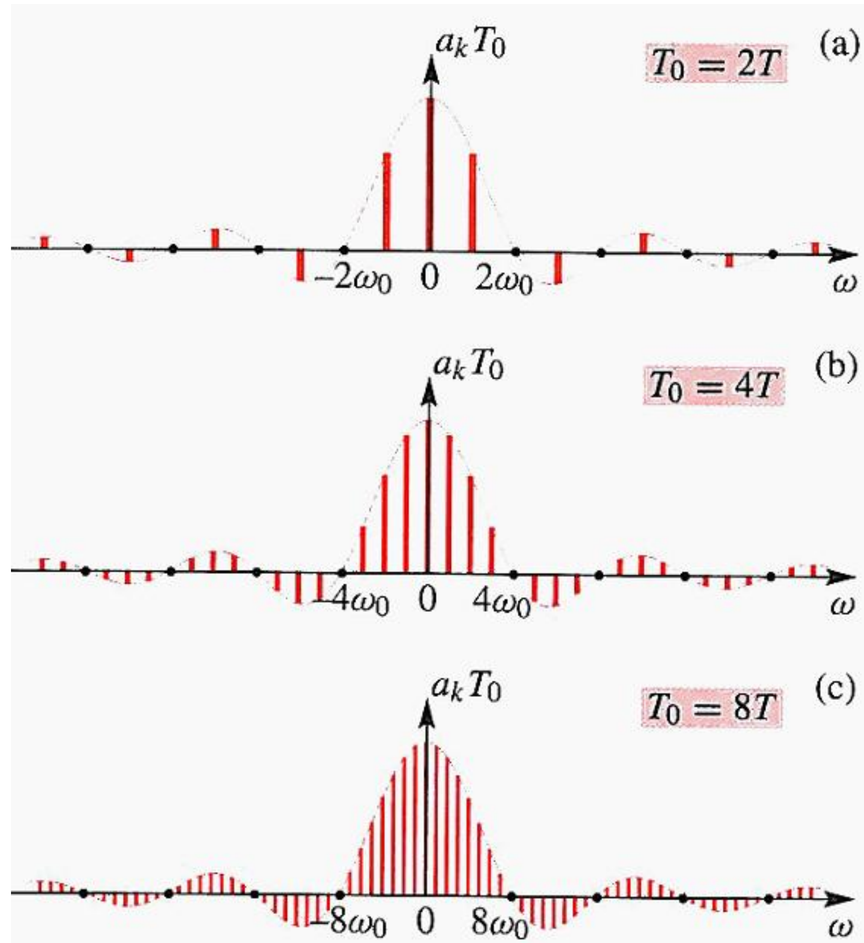
Interval between adjacent frequency

- Remember that the spectrum (in the frequency domain) of a periodic continuous signal is discrete, specified by $k\omega_0$ (k is an integer).
- Therefore, the interval between adjacent frequencies, $k\omega_0$ and $(k+1)\omega_0$, is just ω_0.

Example: recall the case of squared wave



Time domain



Fourier series domain

Notation change of the forward transform of Fourier series

- Because $\frac{1}{T_0} = \frac{w_0}{2\pi}$, let us re-denote $f(t) = x_{T_0}(t)$ and rewrite the forward transform of Fourier series as

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-jkw_0 t} dt$$

$$= \frac{w_0}{2\pi} \int_{-\pi/w_0}^{\pi/w_0} f(t) e^{-jkw_0 t} dt$$

Extreme Case of Fourier Series: $T_0 \to \infty$ (i.e., $\omega_0 \to 0$)

- **Further changing the notation:**
- Let $b_{k\omega_0} = a_k$, the forward transform becomes

$$b_{k\omega_0} = \frac{w_0}{2\pi} \int_{-\pi/w_0}^{\pi/w_0} f(t) e^{-jkw_0 t} dt$$

- The inverse transform becomes

$$f(t) = \sum_{k=-\infty}^{\infty} b_{k\omega_0} e^{jkw_0 t}$$

Extreme Case of Fourier Series: $\boldsymbol{\omega_0 \to 0}$

- When $\omega_0 \to 0$, we can *image* that the frequency becomes continuous:
- The interval ω_0 becomes $d\omega$
- Let $\omega = k\omega_0$. Then, when $\omega_0 \to 0$, the forward transform approaches

$$b(w) = \frac{dw}{2\pi} \int_{-\infty}^{\infty} f\left(t\right) e^{-jwt} dt$$

Extreme Case of Fourier Series

- Combining with the inverse transform,

$$f(t) = \sum_{k=-\infty}^{\infty} b(w)e^{jwt}$$

- we have

Define this as a transform

$$f(t) = \int_{-\infty}^{\infty} \underbrace{\frac{dw}{2\pi}\boxed{\left[\int_{-\infty}^{\infty} f(t)e^{-jwt}dt\right]}}_{b(\omega)} e^{jwt}$$

when $\omega_0 \rightarrow 0$, where the summation in the inverse transform becomes integral.

It becomes:
Continuous Fourier Transform

- Consider the above recovering equation of $f(t)$. Let us decompose the equation into forward transform and inverse transform:

$$f(t)=\underbrace{\int_{-\infty}^{\infty}\frac{1}{2\pi}\underbrace{\left[\int_{-\infty}^{\infty} f(t)e^{-jwt}dt\right]}_{\text{Forward Transform of } f(t) \text{ to the frequency domain } F(w)} e^{jwt}dw}_{\text{Inverse Transform of } F(w) \text{ to the time domain } f(t)}$$

Continuous Fourier Transform

a.k.a. Continuous-time Fourier Transform

- Forward Transform

$$F(jw) = \int_{-\infty}^{\infty} f(t) e^{-jwt} dt$$

- Inverse Transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(jw) e^{jwt} dw$$

- Remark: in the continuous domain, $e^{j\omega t}$ $(\omega \in R)$ still form a set of orthogonal bases (with the amount un-countable infinite), no matter whether ω is a multiple of an integer or continuous real value.

Continuous Fourier Transform

- Both time and frequency domains in continuous Fourier transform are continuous.
- The frequency waveform is also referred to as the ‘spectrum’.

Continuous Fourier Transform

- Continuous Fourier transform is the most fundamental Fourier transform. We will see later that other Fourier transforms (including Fourier series) are its special cases.
- It is also the most ‘standard’ Fourier transform. When there is no specification, we usually referred to Fourier Transform as this type.

Continuous Fourier Transform Pair

- **Transform pair**

Forward

$$F(jw)=\int_{-\infty}^{\infty} f(t)e^{-jwt}dt$$

Backward

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(jw)e^{jwt}dw$$

- Sometimes also written as

$$F(jw)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-jwt}dt$$

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(jw)e^{jwt}dw$$

depending on how we decompose the normalization constant $1/(2\pi)$.

Note that there are Many Variations of the Forms of Continuous F. T. (example from Kuhn's slides 2005)

Recall: Fourier transform

The Fourier integral transform and its inverse are defined as

$$\mathcal{F}\{g(t)\}(\omega) = G(\omega) = \alpha \int_{-\infty}^{\infty} g(t) \cdot \mathrm{e}^{\mp \mathrm{j}\omega t}\,\mathrm{d}t$$

$$\mathcal{F}^{-1}\{G(\omega)\}(t) = g(t) = \beta \int_{-\infty}^{\infty} G(\omega) \cdot \mathrm{e}^{\pm \mathrm{j}\omega t}\,\mathrm{d}\omega$$

where α and β are constants chosen such that $\alpha\beta = 1/(2\pi)$.

Many equivalent forms of the Fourier transform are used in the literature, and there is no strong consensus on whether the forward transform uses $\mathrm{e}^{-\mathrm{j}\omega t}$ and the backwards transform $\mathrm{e}^{\mathrm{j}\omega t}$, or vice versa. Some authors set $\alpha = 1$ and $\beta = 1/(2\pi)$, to keep the convolution theorem free of a constant prefactor; others use $\alpha = \beta = 1/\sqrt{2\pi}$, in the interest of symmetry.

The substitution $\omega = 2\pi f$ leads to a form without prefactors:

$$\mathcal{F}\{h(t)\}(f) = H(f) = \int_{-\infty}^{\infty} h(t) \cdot \mathrm{e}^{\mp 2\pi \mathrm{j} f t}\,\mathrm{d}t$$

$$\mathcal{F}^{-1}\{H(f)\}(t) = h(t) = \int_{-\infty}^{\infty} H(f) \cdot \mathrm{e}^{\pm 2\pi \mathrm{j} f t}\,\mathrm{d}f$$

Symmetry between Time and Frequency of Continuous Fourier Transform

- Unlike Fourier series, the continuous Fourier transform has **very similar forward and inverse transforms**.
- Except to the normalization constant, the only difference is that the forward uses $-j$ and the inverse uses j in the complex exponential basis.
- This suggests that **the roles of time and frequency can be exchanged**, and some properties are symmetric to each other.

Existence and Convergence of Fourier Transform

- We have ‘derived’ continuous Fourier transform as an extreme extension of Fourier series.
- To ensure the existence of continuous Fourier transform, we should consider the conditions where the integrals exist.
- A sufficient condition is

Sufficient Condition for Existence of $X(j\omega)$

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

Rational

$$|X(j\omega)| = \left| \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \right|$$

$$\leq \int_{-\infty}^{\infty} |x(t)e^{-j\omega t}|dt = \int_{-\infty}^{\infty} |x(t)|dt$$

- Hence, if $\int_{-\infty}^{\infty} |x(t)|dt < \infty$, or equivalently, the integral is

bounded, then the continuous Fourier transform is also bounded.

Existence and Convergence of Fourier Transform

- The above is **a sufficient condition** but not a necessary condition.
- Many functions that do not satisfy the above condition, but we can still obtain a useful Fourier transform representation particularly when the impulse signals are allowed to be used.
- The impulse signals will be introduced in the next course.
  - In engineering, we usually do not care much about the exact necessary and sufficient conditions, despite there are mathematically rigorous ways to specify these conditions.

Examples of Fourier Transform Pairs

- Rectangular function (rectangular pulse signal)

$$x(t) = \begin{cases} 1 & -\frac{1}{2}T \le t < \frac{1}{2}T \\ 0 & \text{otherwise} \end{cases}$$

- Derivation of its continuous F. T.

$$X(j\omega) = \int_{-T/2}^{T/2} e^{-j\omega t}\,dt$$

$$= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$= \frac{\sin(\omega T/2)}{\omega/2}$$

(Sinc function) (

Note that in this case, the Fourier transform is a real function. The phase is zero

Fourier transform of rectangular function

- Rectangular function can also be represented by the unit-pulse function *u(t)* as

$$u(t+\frac{T}{2})-u(t-\frac{T}{2})$$

where the unit-step function is defined as

$$u(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$$

- Hence, we have the Fourier transform pair:

Time-Domain		*Frequency-Domain*
$\left[u(t+\frac{1}{2}T)-u(t-\frac{1}{2}T)\right]$	$\xleftrightarrow{\mathcal{F}}$	$\dfrac{\sin(\omega T/2)}{\omega/2}$

A real-valued function in frequency domain (sinc function)

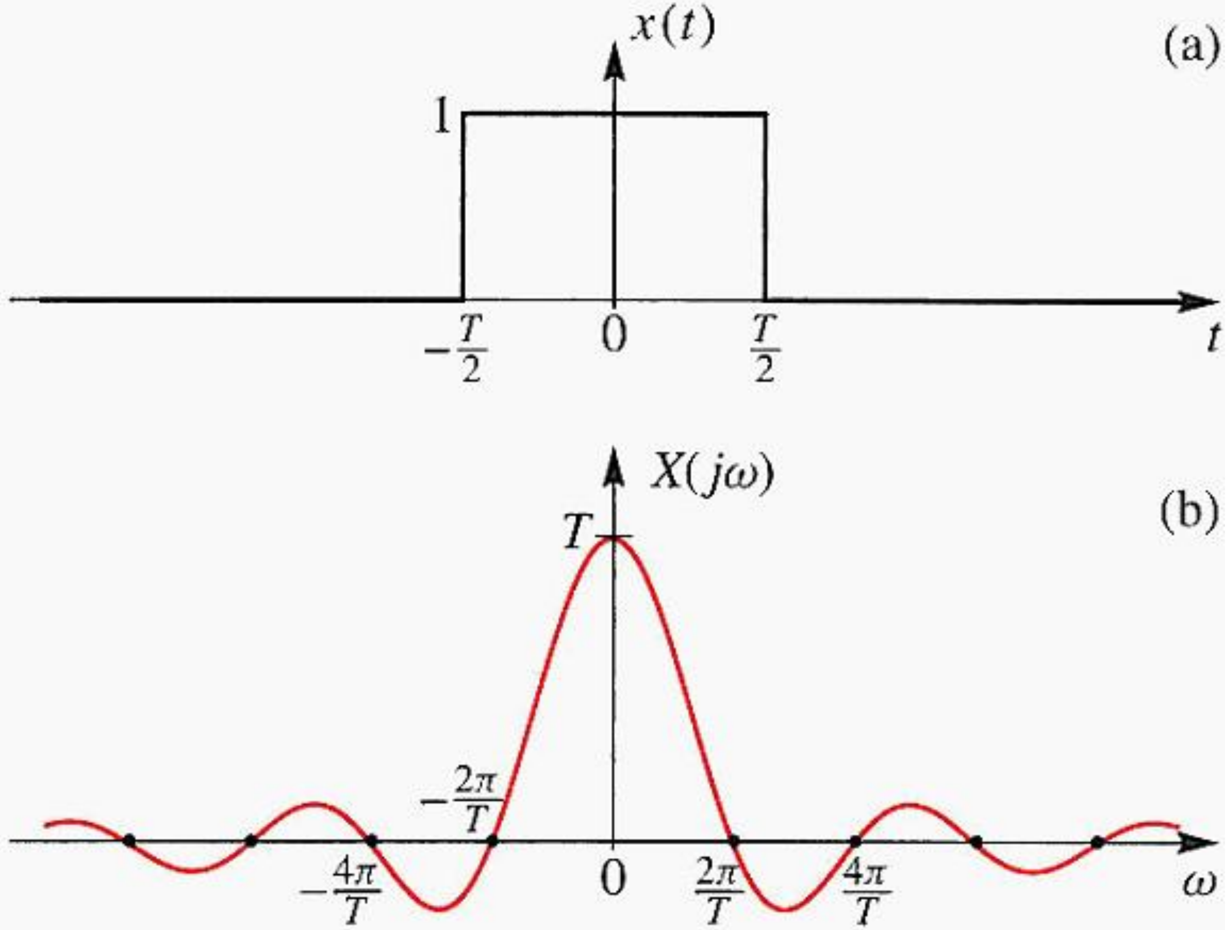


Figure 11-4: Fourier transform of a rectangular pulse. (a) Time function $x(t) = u(t + \frac{1}{2}T) - u(t - \frac{1}{2}T)$, and (b) Corresponding Fourier transform $X(j\omega)$ is a sinc function.

Time and Frequency domains are dual

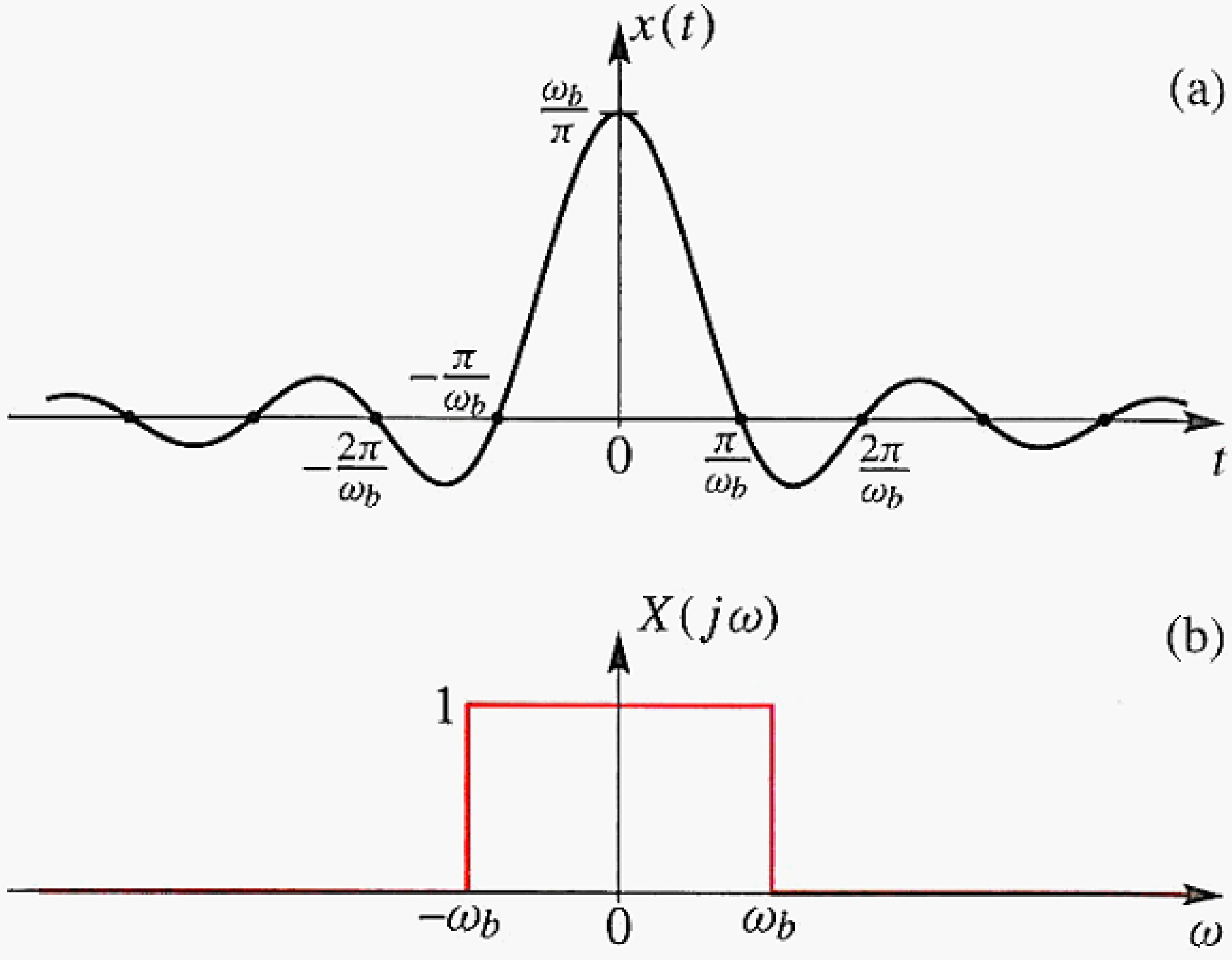


Figure 11-5: Fourier transform of sinc function: (a) Time function $x(t) = \sin(\omega_b t)/(\pi t)$, and (b) corresponding Fourier transform $X(j\omega) = u(\omega + \omega_b) - u(\omega - \omega_b)$.

Fourier transform of right-sided real-exponential signal

Time-Domain		Frequency-Domain
$a > 0 \quad e^{-at}\underbrace{u(t)}_{\text{Unit-step function}}$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$\dfrac{1}{a + j\omega}$

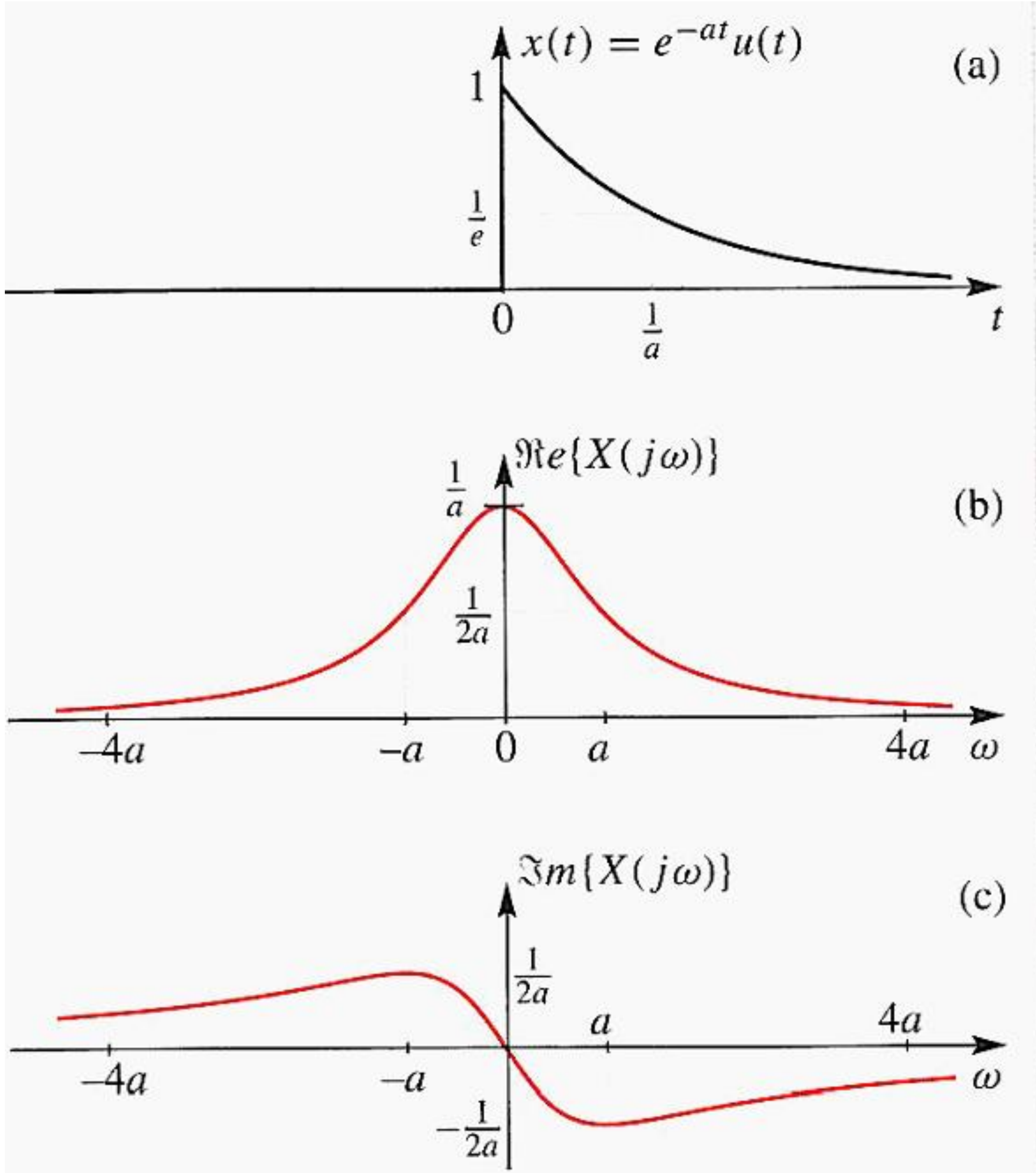
A complex function in frequency domain

- Since

$$X(j\omega) = \frac{1}{a + j\omega} = \frac{1}{a + j\omega}\left(\frac{a - j\omega}{a - j\omega}\right)$$
$$= \frac{a}{a^2 + \omega^2} + \frac{-j\omega}{a^2 + \omega^2}$$

- The real and imaginary parts are

$$\Re e\{X(j\omega)\} = \frac{a}{a^2 + \omega^2}$$
$$\Im m\{X(j\omega)\} = -\frac{\omega}{a^2 + \omega^2}$$



$x(t) = e^{-at}u(t)$
(a)
1
$\frac{1}{e}$
0
$\frac{1}{a}$
t
$\Re e\{X(j\omega)\}$
(b)
$\frac{1}{a}$
$\frac{1}{2a}$
$-4a$
$-a$
0
a
$4a$
ω
$\Im m\{X(j\omega)\}$
(c)
$\frac{1}{2a}$
a
$4a$
$-4a$
$-a$
ω
$-\frac{1}{2a}$

Continuous Fourier transform of a Gaussian Function

- Gaussian function: $e^{-t^2/(2\sigma^2)}$
- The CFT of a Gaussian function is also a Gaussian function (i.e., if time domain is Gaussian, the frequency domain is also Gaussian:

$$e^{-t^2/(2\sigma^2)} \quad \xleftrightarrow{CFT\ Pair} \quad \sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$$