DSP2025 Homework 8 Due date: 12:00 noon, May. 29, 2025

生醫電資所
R13945041
王巘霳

1. Given two length-4 sequences $x[n]=[2,0,1,0]$ and $y[n]=[1,-1,0,0]$, for n=0,1,2,3.
(a) (1%) Compute the length-4 circular convolution of $x[n]$ and $y[n]$.
(b) (2%) Find $X[k]$ and $Y[k]$, the DFT of $x[n]$ and $y[n]$, respectively.
(c) (1%) Find $Z[k]=X[k]Y[k]$, the multiplication of $X[k]$ and $Y[k]$.
(d) (1%) Find the inverse DFT of $Z[k]$.

<sol> DFT $Z[n]=\sum_{m=0}^{N-1} x[m]\, y[(n-m) \bmod N]$

(a) $Z[n]=x[n] \circledast y[n]=[2,0,1,0]\circledast[1,-1,0,0] \longrightarrow Z[n]=\sum_{m=0}^{3} x[m]\cdot y[(n-m) \bmod 4]$

$Z[0]=x[0]y[0]+x[1]y[3]+x[2]y[2]+x[3]y[1]$
$= 2\times1+0\times0+1\times0+0\times-1=\underline{2}$

$Z[1]=x[0]y[1]+x[1]y[0]+x[2]y[3]+x[3]y[2]$
$=2\times-1+0\times1+1\times0+0\times0=\underline{-2}$

$Z[2]=x[0]y[2]+x[1]y[1]+x[2]y[0]+x[3]y[3]$
$=2\times0+0\times-1+1\times1+0\times0=\underline{1}$

$Z[3]=x[0]y[3]+x[1]y[2]+x[2]y[1]+x[3]y[0]$
$=2\times0+0\times0+1\times-1+0\times1=\underline{-1}$

$Z[n]=[2,-2,1,-1]$ #

(b) DFT: $X[k]=\sum_{n=0}^{3} x[n]\cdot W^{kn},\ W=e^{-j\frac{2\pi}{N}}=e^{-j\frac{\pi}{2}}=\cos(\frac{\pi}{2})-j\sin(\frac{\pi}{2})=-j$

$$\text{DFT}\quad \begin{bmatrix} X[0]\\ X[1]\\ X[2]\\ X[3]\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&-j&-1&j\\1&-1&1&-1\\1&j&-1&-j\end{bmatrix}\begin{bmatrix}x[0]\\x[1]\\x[2]\\x[3]\end{bmatrix} \longrightarrow \begin{bmatrix}1&1&1&1\\1&-j&-1&j\\1&-1&1&-1\\1&j&-1&-j\end{bmatrix}$$

$$=\begin{bmatrix}1&1&1&1\\1&-j&-1&j\\1&-1&1&-1\\1&j&-1&-j\end{bmatrix}\begin{bmatrix}2\\0\\1\\0\end{bmatrix}=\begin{bmatrix}3\\1\\3\\1\end{bmatrix}$$

$\underline{X[k]=[3,1,3,1]}$ #

$$\begin{bmatrix} Y[0]\\ Y[1]\\ Y[2]\\ Y[3]\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&-j&-1&j\\1&-1&1&-1\\1&j&-1&-j\end{bmatrix}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\1+j\\2\\1-j\end{bmatrix}$$

$\underline{Y[k]=[0,1+j,2,1-j]}$ #

(c)

$$Z[k]=X[k]Y[k]$$

$$Z[0]=X[0]Y[0]=3\times 0=0$$

$$Z[1]=X[1]Y[1]=1\times(1+j)=1+j$$

$$Z[2]=X[2]Y[2]=3\times 2=6$$

$$Z[3]=X[3]Y[3]=1\times(1-j)=1-j$$

$$Z[k]=[0,1+j,6,1-j]$$

#

(d) ☆ IDFT : $z[n]=\frac{1}{N}\sum_{k=0}^{N-1} Z[k]\cdot W^{-kn}$

$$Z[k]=[0,1+j,6,1-j] \quad N=4$$

$$W^{-kn}=e^{\frac{j2\pi kn}{4}} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$z[n]=\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 0 \\ 1+j \\ 6 \\ 1-j \end{bmatrix}=\begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix}$$

(法2)

$X[k]Y[k]$ 的 inverse DFT

$= x[n]$ circular convolution $y[n]$

$$= x[n] \circledast y[n] \xrightarrow{\text{同(a)}} [2,-2,1,-1]$$

2. (5%) Suppose that $v[n]$ is a real-number sequence of length $2N$. Let $g[n]$ and $h[n]$ be the even and odd parts of $v[n]$, respectively, i.e.,

$$g[n] = v[2n], \qquad h[n] = v[2n+1], \qquad 0 \le n < N.$$

Let $V[k]$ be the DFT of $v[n]$, then $V[k]$ can be computed via the DFT of $g[n]$ and $h[n]$ by the following equation:

$$V[k] = G[k \bmod N] + f[k]H[k \bmod N], \qquad 0 \le k \le 2N-1,$$

with $G[k]$ and $H[k]$ the DFT of $g[n]$ and $h[n]$, respectively. **Question**: what is $f[k]$?

$$W_{2N} = e^{-j\frac{2\pi}{2N}}, \quad W_N = e^{-j\frac{2\pi}{N}}$$

$$\because V[k] = \sum_{n=0}^{2N-1} v[n] W_{2N}^{nk}$$

$$= \sum_{m=0}^{N-1} v[2m] W_{2N}^{2mk} + \sum_{m=0}^{N-1} v[2m+1] W_{2N}^{(2m+1)k}$$

$$= \sum_{m=0}^{N-1} g(m) (W_{2N}^{k})^{2m} + \sum_{m=0}^{N-1} h[m] (W_{2N}^{k})^{2m} \cdot W_{2N}^{k}$$

$$\longrightarrow (W_{2N}^{k})^{2m} = e^{-j\frac{2\pi k}{2N}\cdot 2m} = e^{-j\frac{2\pi km}{N}} = W_N^{km}$$

$$\therefore V[k] = \sum_{m=0}^{N-1} g(m) W_N^{km} + W_{2N}^{k} \sum_{m=0}^{N-1} h[m] W_N^{km}$$

$$= G[k \bmod N] + W_{2N}^{k} H[k \bmod N]$$

$$f[k] = W_{2N}^{k} = e^{-j\frac{2\pi}{2N}k}.$$

#