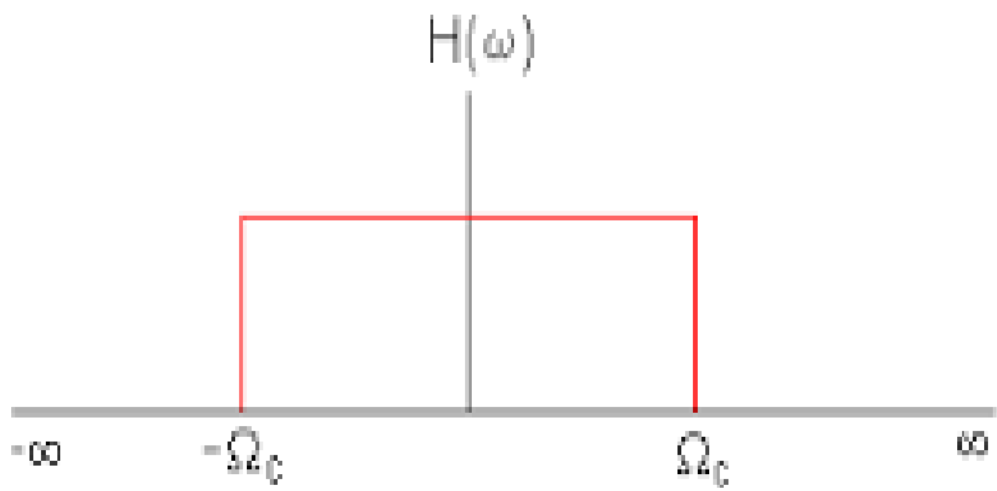
Reconstruct the continuous-time signal from the sampled signal

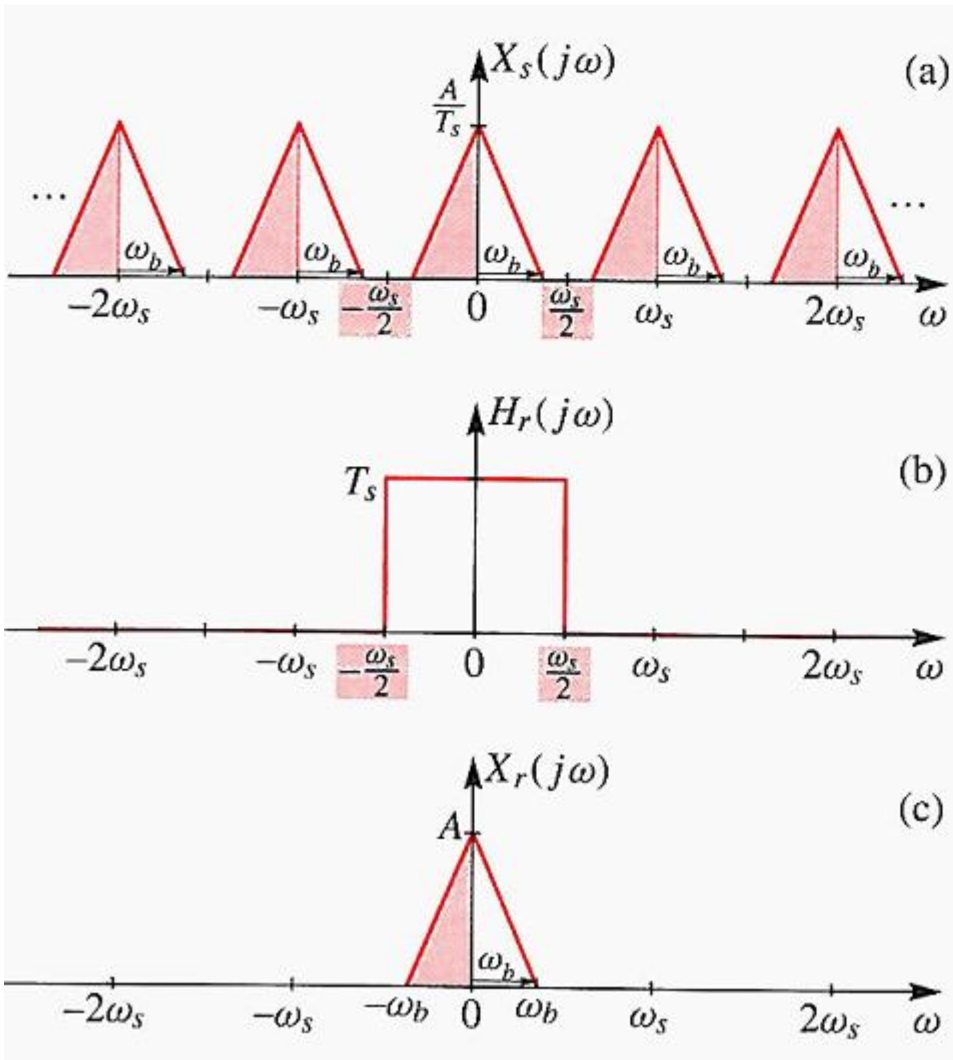
- If a band-limited signal is sampled with a frequency higher than the Nyquist rate, then it is possible to reconstruct the original continuous-time signal.

Reconstruct the continuous signal from the sampled signal with low-pass filter

- Ideal low-pass filter: a system for which the frequency response is constant over a low range of frequencies and is zero at the high frequencies.



Ideal low-pass filter: frequency domain



- In frequency domain: the effect is equivalent to multiply the spectrum of the input signal and the rectangular window.

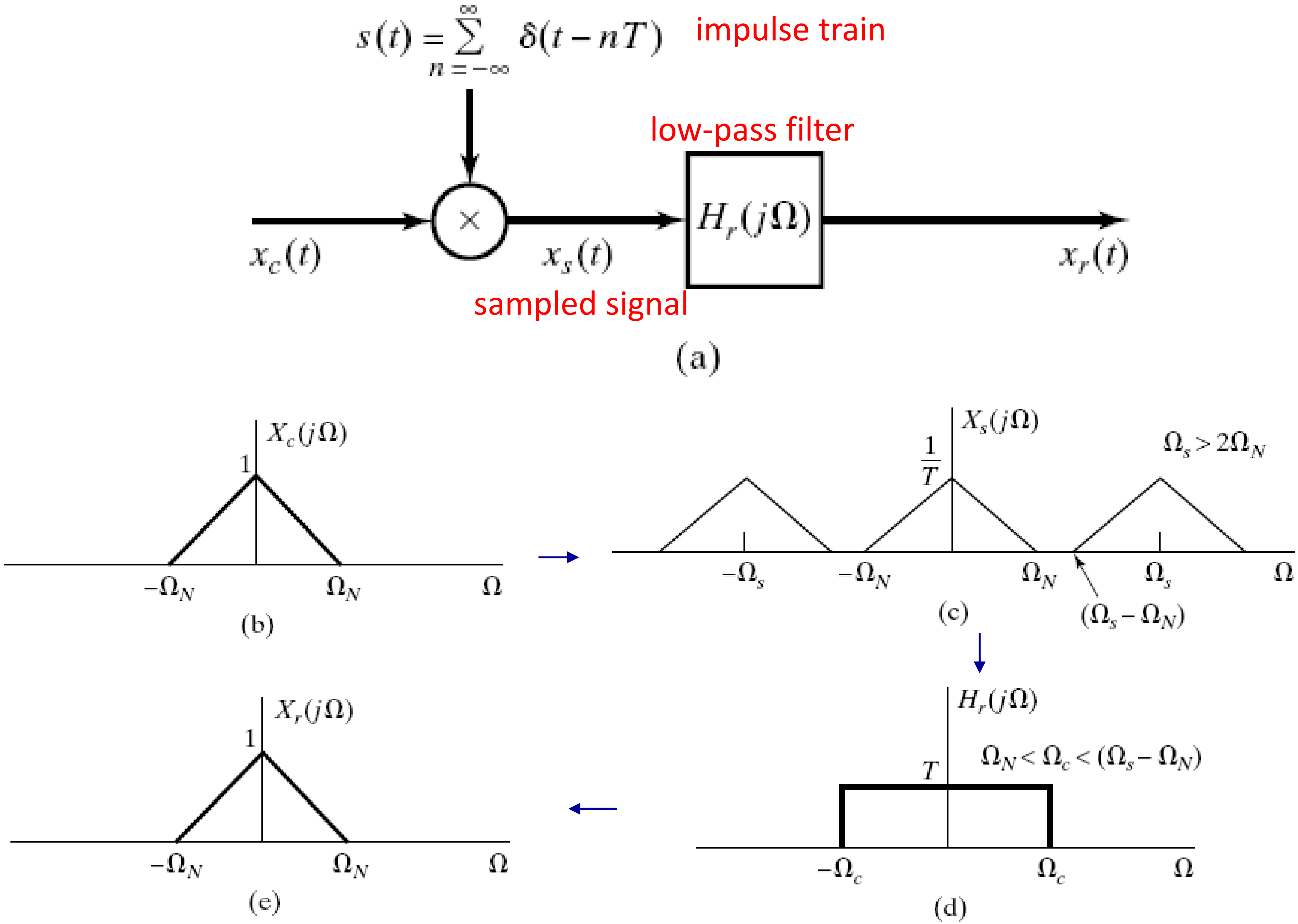
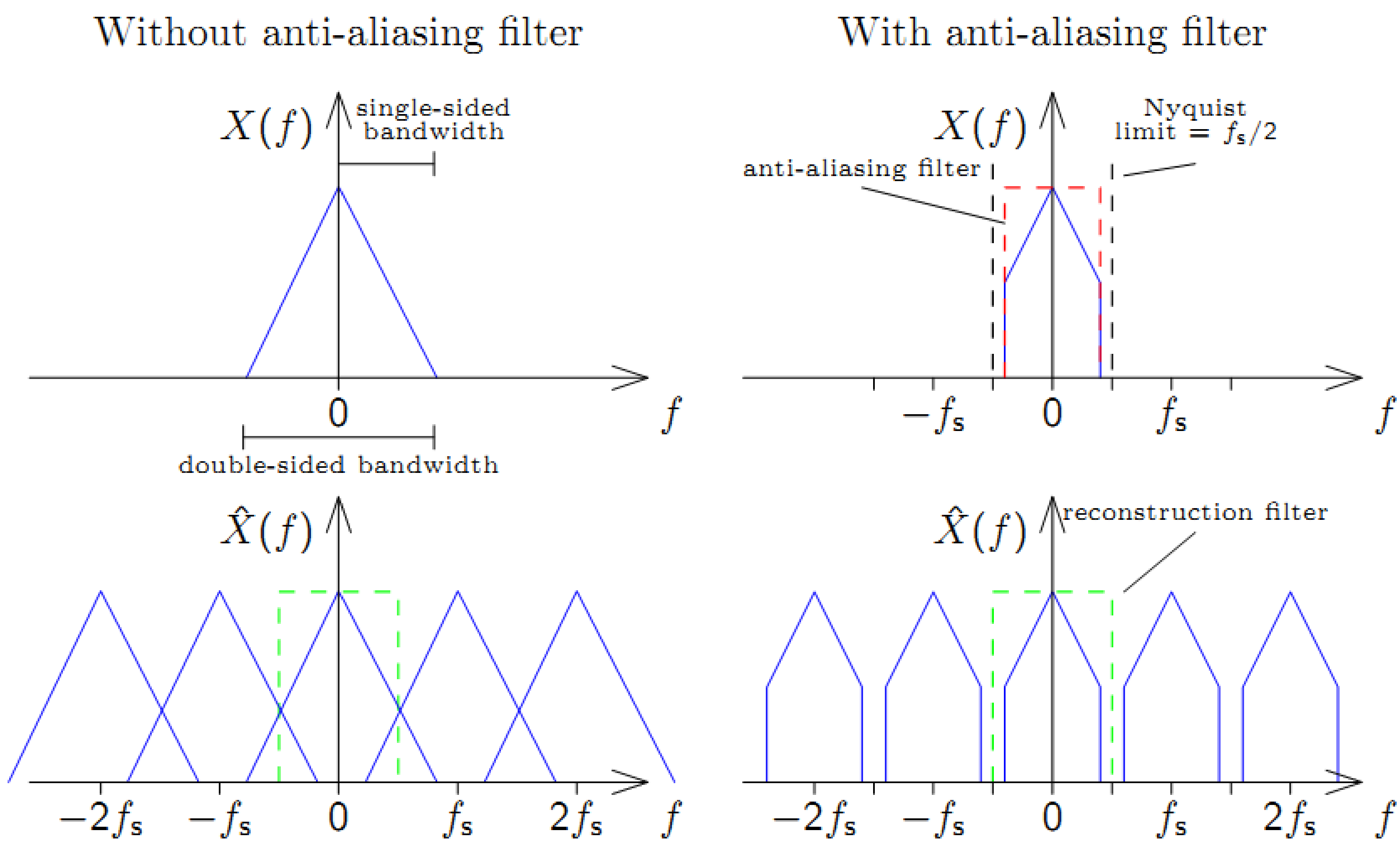


Figure 4.4 Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter.

Reconstruction and Anti-aliasing filter

- For a band-limited signal, if we sample it with a sufficiently high rate (higher than the Nyquist rate), then it can be recovered exactly by ideal low-pass filter.
- Ideal low pass filter acts as an anti-aliasing filter when the signal is not band-limited (which is common in practice).
  - In this way, some rear frequencies of the signal's spectrum are discarded or sacrificed, but the aliasing effect is avoided.

Nyquist limit and anti-aliasing filters



Anti-aliasing and reconstruction filters both suppress frequencies outside $|f| < f_s/2$.

Ideal low-pass filter: time domain

- In frequency domain, multiplication of a rectangle window is performed for an ideal low-pass filter.
- In time domain, convolution is performed.
- Recall the Fourier transform of a rectangular function is a sync function.

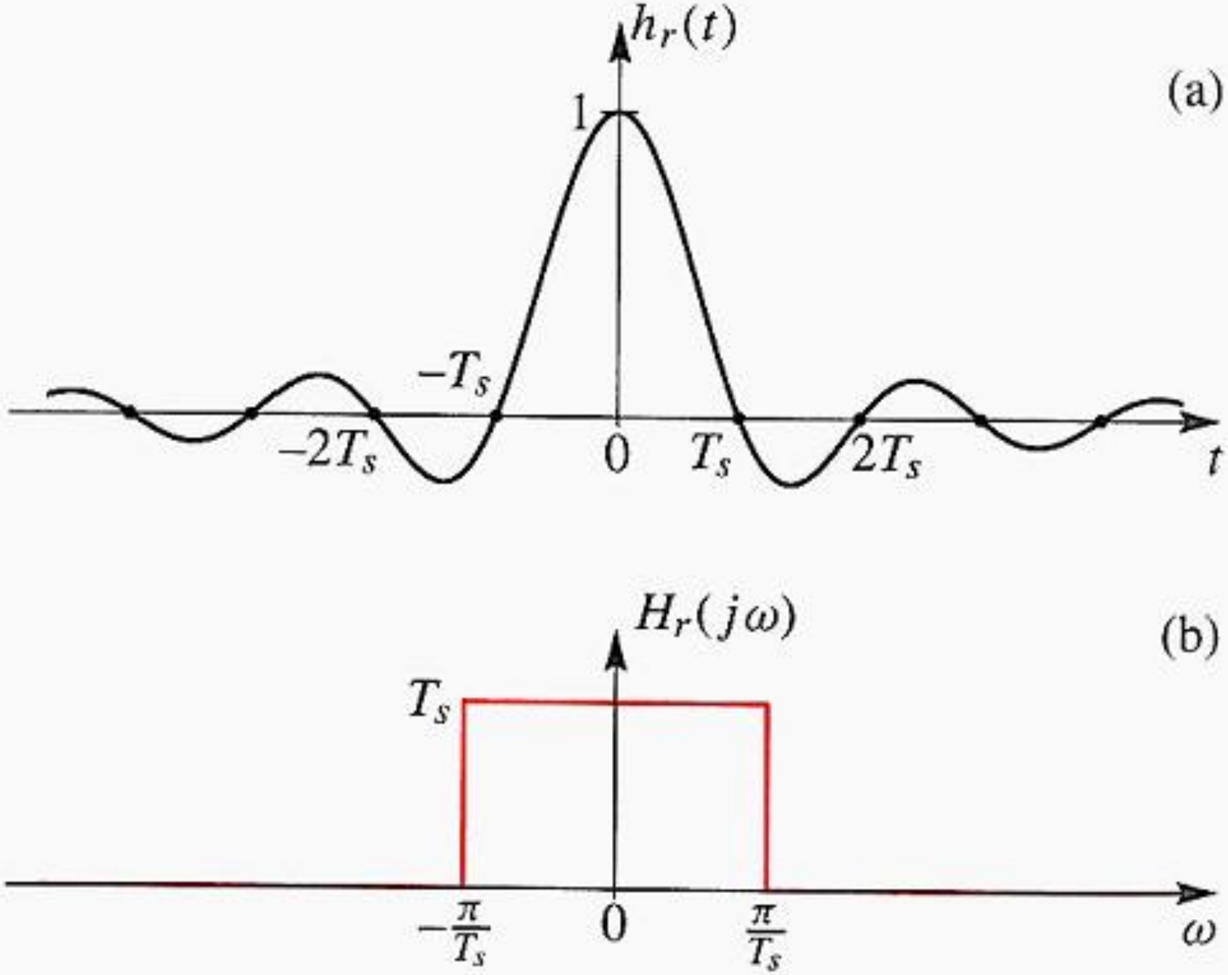
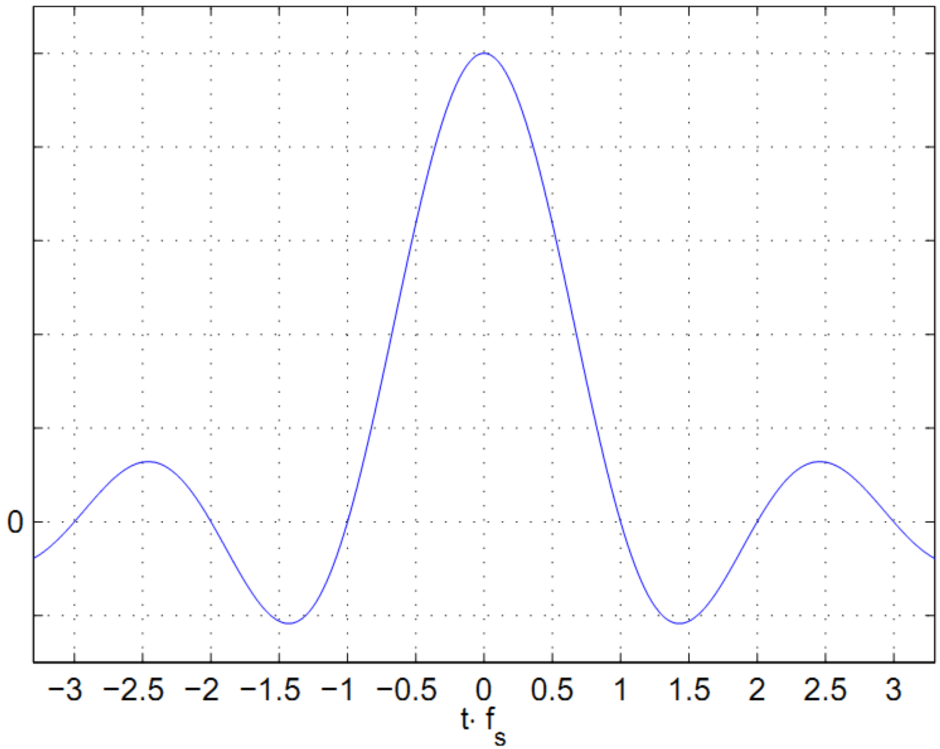


Figure 12-31: Ideal bandlimited reconstruction filter: (a) Impulse response $h_r(t)$, and (b) frequency response $H_r(j\omega)$.

Sync Function Interpolation

- Hence, in time domain, ideal low-pass filter is equivalent to the convolution with the sync function.

Impulse response of ideal low-pass filter with cut-off frequency $f_s/2$:



Reconstruction filter derivation (I)

- Let $x_a(t)$ be a band-limited signal with its frequencies inside the range $[-\omega_b, \omega_b]$. Suppose $x_a(t)$ is sampled with the frequency $\omega_s = 2w_b$ (without aliasing).
- Hence, $T = 2\pi/\omega_s = \pi/\omega_b$ is the sampling period. Remember that the sampled signal is the multiplication of $x_a(t)$ and the impulse train,

$$x_s(t) = x_a(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- To reconstruct $x_a(t)$ from $x_s(t)$, we use the low-pass filter (a rectangular window) with the cut-off frequency w_b in the frequency domain.

Reconstruction filter derivation (II)

- Recall that when the frequency domain is a rectangle window in $[-\omega_b, \omega_b]$, the time domain is the sync function $\frac{\sin(\omega_b t)}{\pi t}$.

Table of Fourier Transform Pairs	
Time-Domain: $x(t)$	*Frequency-Domain:* $X(j\omega)$
$e^{-at}u(t) \quad (a > 0)$	$\frac{1}{a + j\omega}$
$e^{bt}u(-t) \quad (b > 0)$	$\frac{1}{b - j\omega}$
$u(t + \frac{1}{2}T) - u(t - \frac{1}{2}T)$	$\frac{\sin(\omega T/2)}{\omega/2}$
$\frac{\sin(\omega_b t)}{\pi t}$	$[u(\omega + \omega_b) - u(\omega - \omega_b)]$

Reconstruction filter derivation (III)

- Ideal low-pass filtering: frequency domain multiplication with the rectangle window scaled by T.
- In time domain, idea low-pass filtering is equivalent to convolution with the sync function.

$$
\begin{aligned}
x_r(t) &= x_s(t) * \frac{\sin(\omega_b t)}{\pi t} T \\
&= \left(x_a(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \right) * \frac{\sin(\omega_b t)}{\pi t} T
\end{aligned}
$$

Reconstruction filter derivation (IV)

- So, $x_r(t) =$

$$\int_{-\infty}^{\infty} \left(x_a(\tau) \sum_{n=-\infty}^{\infty} \delta(\tau - nT) \right) \frac{\sin(\omega_b(t-\tau))}{\pi(t-\tau)} T d\tau$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x_a(\tau) \frac{\sin(\omega_b(t-\tau))}{\pi(t-\tau)} T\delta(\tau - nT) d\tau$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT) \frac{\sin(\omega_b(t-nT))}{\pi(t-nT)} T$$

Reconstruction filter derivation (V)

- Let $x[n] = x_a(nT)$ be the sampled signal, the reconstructed signal via ideal low-pass filtering is then

$$\boxed{x_r(t) = \frac{T}{\pi} \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\omega_b(t - nT))}{(t - nT)}}$$

Linear combination of the functions $\sin(\omega_b t)\,/t$ centered at the integer multiples of T.

Reconstruction filter example