Correlation

- In addition to convolution, there is another operation called **correlation**.
- Given a pair of sequences $x[n]$ and $y[n]$, their **cross correlation** sequence is $r_{xy}[l]$ is defined as

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l]$$

  for all integer l.

- The cross correlation sequence can help measure similarities between two signals.

- Cross correlation is very similar to convolution, unless the indices changes from $l - n$ to $n - l$.
- Relation between cross correlation and convolution:

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[-(l-n)] = x(l) * y[-l]$$

- If both signals are the same, the cross correlation becomes autocorrelation.

- Autocorrelation of a signal $x[n]$ is defined as

$$r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]x[n-l]$$

Properties of correlation

- Consider the following non-negative expression:

$$\sum_{n=-\infty}^{\infty}\left(ax[n]+y[n-l]\right)^2 = a^2\sum_{n=-\infty}^{\infty}x^2[n]+2a\sum_{n=-\infty}^{\infty}x[n]y[n-l]+\sum_{n=-\infty}^{\infty}y^2[n-l]$$

$$= a^2 r_{xx}[0]+2ar_{xy}[l]+r_{yy}[0]\geq 0$$

That is, $\begin{bmatrix} a & 1 \end{bmatrix}\begin{bmatrix} r_{xx}[0] & r_{xy}[l] \\ r_{xy}[l] & r_{yy}[0] \end{bmatrix}\begin{bmatrix} a \\ 1 \end{bmatrix}\geq 0 \quad \text{for all } a$

- Thus, the matrix $\begin{bmatrix} r_{xx}[0] & r_{xy}[l] \\ r_{xy}[l] & r_{yy}[0] \end{bmatrix}$ is positive semi-definite.
  - Its determinate is nonnegative.

- The determinant is $r_{xx}[0]r_{yy}[0] - r_{xy}{}^2[l] \geq 0$. Hence, For all signals $x[n]$ and $y[n]$, the following properties hold for all $l \in Z$,

$$r_{xx}[0]r_{yy}[0] \geq r_{xy}^2[l]$$

- This property can also be explained by Schwartz inequality.
  - Consider x and y two infinite-long vectors. Then $r_{xx}[0]$ and $r_{yy}[0]$ are the squared length of x and y, respectively.
  - $r_{xy}[l]$ is the inner product between $x[n]$ and $y[n-l]$, where $y[n-l]$ and $y[n]$ are of the same squared length.

- The property also implies the following inequality when $x = y$,

$$r_{xx}^2[0] \geq r_{xx}^2[l],$$

which means that the autocorrelation $r_{xx}[l]$ is maximized when l=0.

Normalized cross correlation and autocorrelation

- Normalized cross correlation and normalized autocorrelation are defined as

$$\boxed{\rho_{xx}[l] = \frac{r_{xx}[l]}{r_{xx}[0]} \qquad\qquad \rho_{xy}[l] = \frac{r_{xy}[l]}{\sqrt{r_{xx}[0]\, r_{yy}[0]}}}$$

Properties:

The above results imply that $|\rho_{xx}[l]| \leq 1$ and $|\rho_{xy}[l]| \leq 1$.

Application of autocorrelation

- Autocorrelation is quite often used for **finding the period of a periodical signal.** By definition, autocorrelation peaks at the integer multiples of the period.

Autorrelation in the frequency domain

Property: Given a real-valued sequence $x[l]$, The DTFT of the autocorrelation signal $r_{xx}[l]$ is the squared magnitude of the DTFT of $x[l]$. That is

$$DTFT(r_{xx}) = \left\|X\left(e^{j\omega}\right)\right\|^2$$

- **Proof**

by the relationship between correlation and convolution,

$$r_{xx}[l] \;=\; x[l] * x[-l].$$

Time domain convolution implies frequency domain multiplication. According to the time-reversal property and $x[-l]$ is real,

$$x[-l] \leftrightarrow X(e^{-j\omega}) = X^*(e^{j\omega}).$$

Taking DTFT on both sides, we have

$$\mathrm{DTFT}(r_{xx}) = X(e^{j\omega})X^*(e^{j\omega}) = |X(e^{j\omega})|^2$$

- Hence, signals with the same autocorrelation sequence share the same magnitude spectrum in the frequency domain, albeit their phase spectrum are different.

    - An example of describing the property of a class of signals.

Correlation is useful in random signal modeling and processing.