What is the continuous Fourier transform of a periodic signal?

- We have extended Fourier series when the period $T_0 \to \infty$ for a periodic signal.
- **What happens when continuous Fourier transform is applied to a periodic signal?**

$$F(jw) = \int_{-\infty}^{\infty} f(t) e^{-jwt} dt$$

- Assume $f(t)$ is a periodic signal with period T_0.
- From Fourier series, we know $f(t)$ can be represented as

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}, \qquad w_0 = \frac{2\pi}{T_0}$$

Preliminary Property

- **When $\omega \neq 0$, the integral of $e^{j\omega t}$ over the whole range is zero:**

$$\int_{-\infty}^{\infty} e^{jwt} dt = 0$$

- It is because that

$$\int_{-\infty}^{\infty} e^{jwt} dt = \sum_{k=-\infty}^{\infty} \int_{(2\pi/w)k}^{(2\pi/w)(k+1)} e^{jwt} dt$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{jw} e^{jwt} \Big|_{t=(2\pi/w)k}^{t=(2\pi/w)(k+1)} = 0$$

What is the continuous Fourier transform of a periodic signal?

- **Continuous Fourier transform of a periodic signal becomes**

$$F(jw) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} e^{-jwt} dt$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} a_k e^{jkw_0 t} e^{-jwt} dt = \sum_{k=-\infty}^{\infty} a_k \left(\int_{-\infty}^{\infty} e^{j(kw_0 - w)t} dt \right)$$

$$= \begin{cases} 0, & w \neq kw_0 \\ a_k \int_{-\infty}^{\infty} 1 dt, & w = kw_0 \end{cases} \quad = \begin{cases} 0, & w \neq kw_0 \\ \infty, & w = kw_0 \end{cases}$$

Continuous Fourier Transform of a Periodic Signal

- Hence, we see that when ω is an integer multiple of the fundamental frequency ω_0 (i.e., $\omega = k\omega_0$, k is an integer), the continuous F. T. becomes infinity.

- Rigorously speaking, the continuous F. T. doesn't exist.

Continuous Fourier Transform of a Periodic Signal

- **So, what can we do?**
- Should we separate the signals into two types, one is periodic and the other is non-periodic for signal analysis?
- When we encounter a periodic signal, are we only allowed to use Fourier series?

- Moreover, even for very simple signals, such as $f(t) = 1$, its continuous F. T. doesn't exist according to a similar argument.

Impulse Function

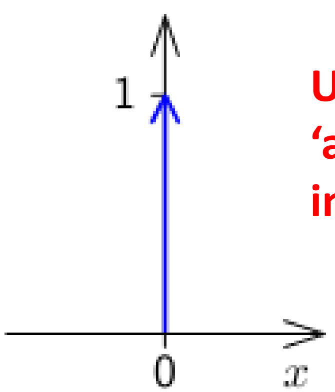
- To overcome this difficulty, and also to make continuous F. T. more applicable, the **Dirac's delta function** (**or** **unit impulse function**) is introduced.
- The impulse time-domain signal is the most concentrated time signal that we can have.

Dirac's delta function

The continuous equivalent of the impulse sequence $\{\delta_n\}$ is known as Dirac's delta function $\delta(x)$. It is a generalized function, defined such that

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$$



Using an 'arrow' to indicate it

What is it mathematically?

- Dirac's delta function can be thought of as the limit of function sequence such as

$$\delta(x) = \lim_{n \to \infty} \begin{cases} 0, & |x| \geq 1/n \\ n/2, & |x| < 1/n \end{cases}$$

or

$$\delta(x) = \lim_{n \to \infty} \frac{n}{\sqrt{\pi}} \mathrm{e}^{-n^2 x^2}$$

- The delta (or unit impulse) function is **mathematically speaking not a function**, but a **distribution**, that is in an expression that is **only defined when integrated**.
- In engineering, we don't care about many of the possible ways for its rigorous definition. We only care its property when computing integrals.

Some properties of Delta (Unit Impulse) Function

(1) $$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,\mathrm{d}x \;=\; f(a)$$

Remark: The inner product of a signal f and the delta function centered at the position a is just $f(a)$.

(2) Remember when $\omega \neq 0$, we have shown that

$$\int_{-\infty}^{\infty} e^{j\omega t} dt = 0.$$

By exchanging the role of ω and t, we thus have

$$\int_{-\infty}^{\infty} e^{j\omega t} d\omega = 0 \text{ when } t \neq 0,$$

Some properties of Delta (Unit Impulse) Function

Note that when $t = 0$, the integral is originally ∞.

(Also, by definition, $\delta(t) = \delta(-t)$)

Now, with the delta-function representation, we set

(3) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\pm \mathrm{j}\omega t}\,\mathrm{d}\omega \;=\; \delta(t)$$

Some properties of Delta (Unit Impulse) Function

- By the above properties, the Fourier transform pair of delta function is

Fourier transform:

$$\mathcal{F}\{\delta(t)\}(\omega) \quad = \quad \int_{-\infty}^{\infty} \delta(t) \cdot \mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t \quad = \quad \mathrm{e}^{0} \quad = \quad 1$$

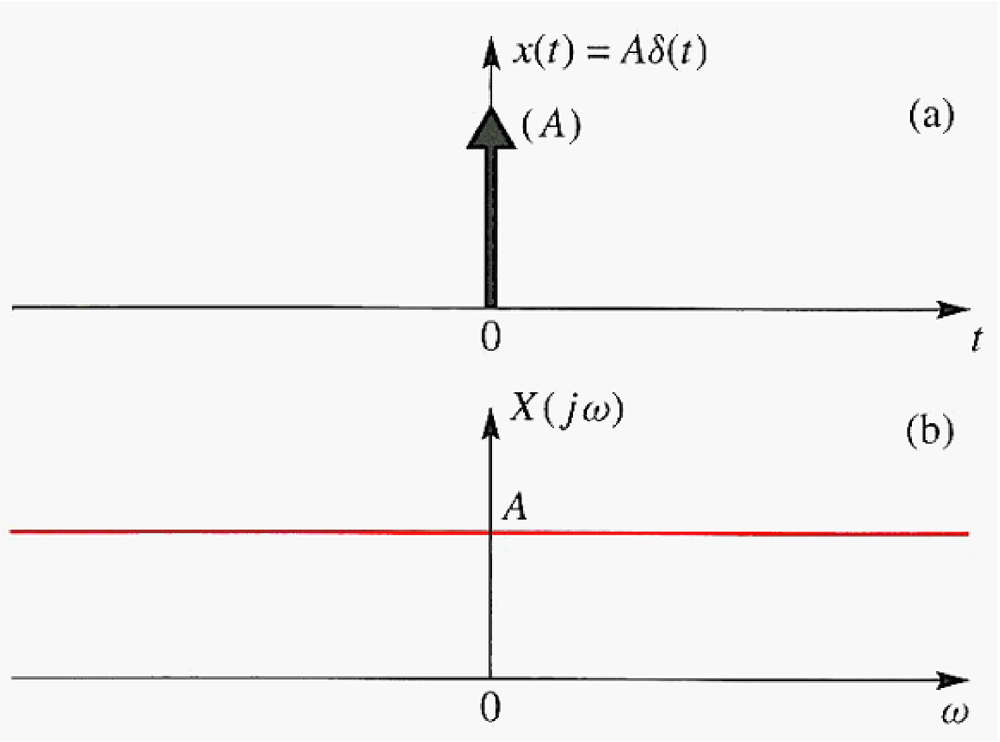
by using property (1)

$$\mathcal{F}^{-1}\{1\}(t) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot \mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\omega \quad = \quad \delta(t)$$

by using property (3)

Impulse can have magnitude

- **Remark**: the unit-impulse function is allowed to be multiplied by a constant (here, A) to reflect the 'magnitude' of an impulse.
- $A\delta(t)$: *impulse of magnitude* A



Further properties

- Exchange the role of ω and t in (3), we also have

(4) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm jwt}dt = \delta(w)$$

Time-Domain		***Frequency-Domain***
1	$\xleftrightarrow{\mathcal{F}}$	$2\pi\delta(\omega)$

$2\pi\delta(\omega)$: *impulse of magnitude* 2π

Continuous Fourier Transform of a Periodic Signal

- By introducing the impulse function, let's go back to the continuous Fourier transform of a periodic signal.
- Now,

$$F(jw) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} e^{-jwt} dt$$

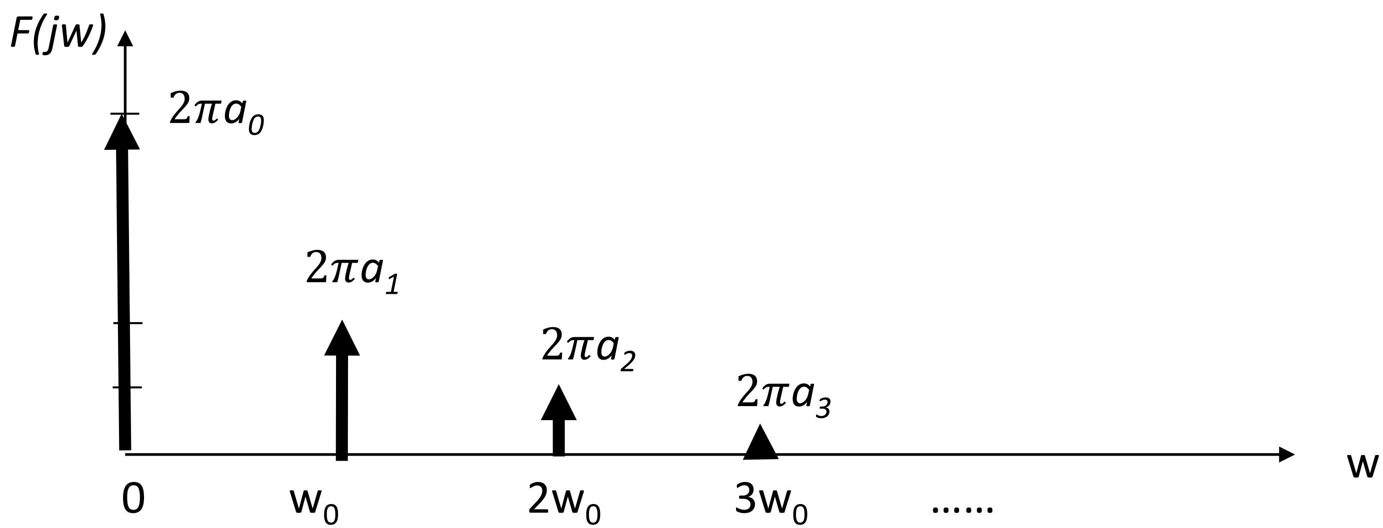
$$= \sum_{k=-\infty}^{\infty} a_k \left(\int_{-\infty}^{\infty} e^{j(kw_0 - w)t} dt \right) = \sum_{k=-\infty}^{\infty} a_k \delta(w - kw_0) 2\pi$$

by using property (4)

- The unit-impulse function is multiplied by a constant (here, $2\pi a_k$) to reflect the 'magnitude' of an impulse.

Continuous Fourier Transform of a Periodic Signal

- Hence, the continuous Fourier transform of a periodic signal is an impulse sequence, with each unit impulse weighted by $2\pi a_k$.



- Note that $\boldsymbol{a_k}$ **is just the spectrum of Fourier series**. Hence, the magnitudes of the impulse functions are **proportional to** those computed from **Fourier series**.

Continuous Fourier Transform

A general Fourier Transform for Spectrum Representation

- With the unit-impulse function incorporated, the continuous Fourier transform can represent a broad range of continuous-time signals.
- It is the most general F. T., including Fourier series as its special case.
- It can also takes discrete-time signal as input. We will investigate it later.
- It **provides a unified and general definition of 'spectrum.'** When mention the spectrum of a signal, we mean the continuous Fourier transform of the signal.

Continuous Fourier Transform

- We have introduced the continuous Fourier transform (and now allows the delta-function representation)

Forward

$$F(jw) = \int_{-\infty}^{\infty} f(t) e^{-jwt} dt$$

Backward

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(jw) e^{jwt} dw$$

- The continuous Fourier transform defines completely and exactly the frequency domain, where the frequency domain is continuous and range un-limited.

Examples of Fourier Transform Pairs

- Rectangular function (rectangular pulse signal)

$$x(t) = \begin{cases} 1 & -\frac{1}{2}T \le t < \frac{1}{2}T \\ 0 & \text{otherwise} \end{cases}$$

Time-Domain		*Frequency-Domain*
$\left[u(t+\frac{1}{2}T) - u(t-\frac{1}{2}T)\right]$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$\dfrac{\sin(\omega T/2)}{\omega/2}$

A real-valued function in frequency domain (sinc function)

- Real exponential ($a > 0$)

Time-Domain		*Frequency-Domain*
$e^{-at}u(t)$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$\dfrac{1}{a+j\omega}$

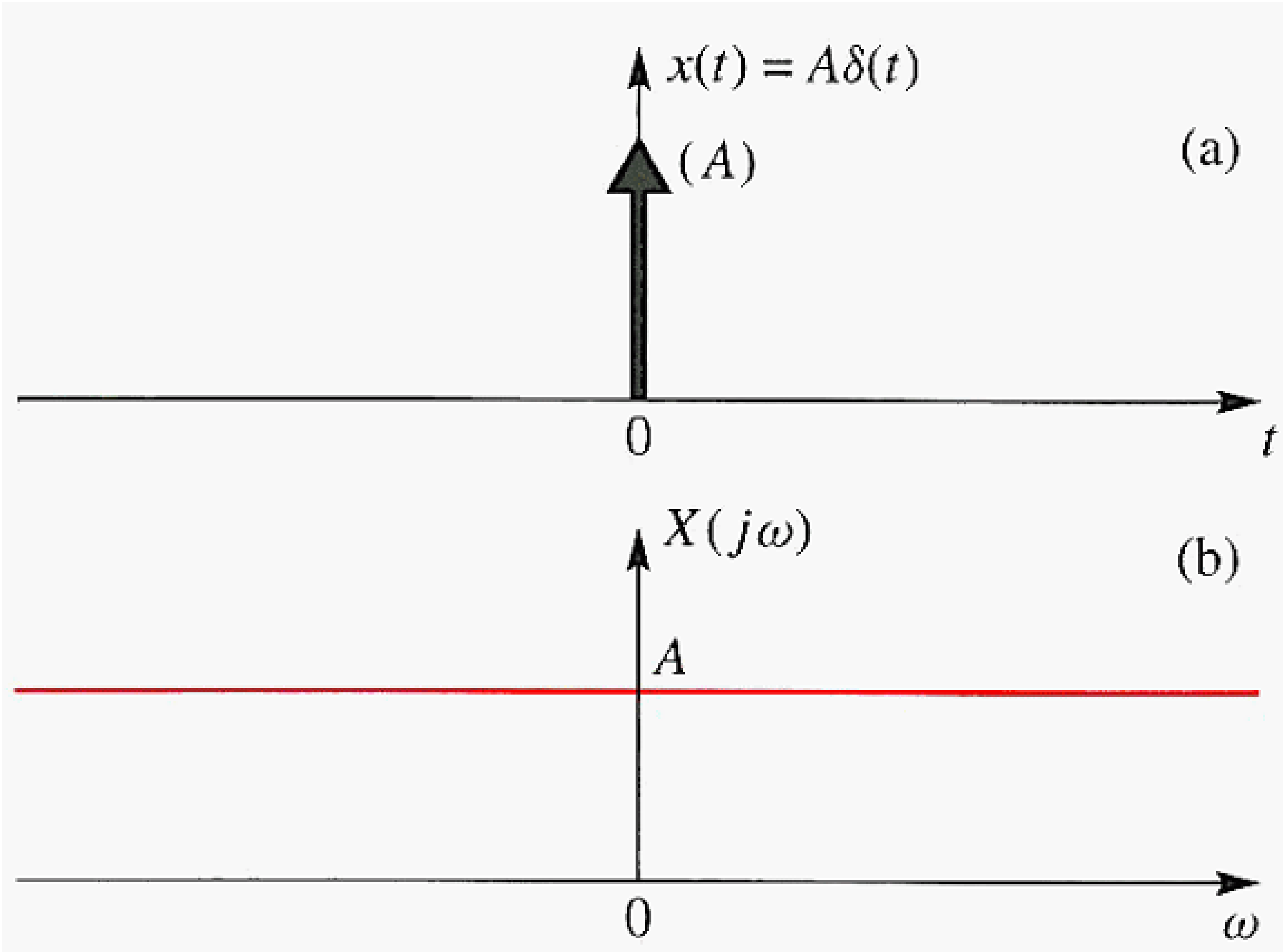
A complex-valued function in frequency domain

Impulse in Time and Frequency

Time-Domain		***Frequency-Domain***
$A\delta(t)$	$\overset{\mathcal{F}}{\longleftrightarrow}$	A

- Derivation:

$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} A\delta(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} Ae^{-j\omega 0}\delta(t)dt \\
&= A\int_{-\infty}^{\infty} \delta(t)dt \\
&= A
\end{aligned}
$$



$x(t) = A\delta(t)$
(A)
(a)
0
t
$X(j\omega)$
(b)
A
0
ω

Impulse in Time and Frequency

- By the duality (or symmetry) between time and frequency domain

Time-Domain		***Frequency-Domain***
1	$\overset{\mathcal{F}}{\longleftrightarrow}$	$2\pi\delta(\omega)$

- **Intuitive interpretation**: the constant signal $x(t) = 1$ for all t has only one frequency, namely DC, and we see that its transform is an impulse concentrated at ω=0.

Complex Exponential

Time-Domain		**Frequency-Domain**
$e^{j\omega_0 t}$	$\stackrel{\mathcal{F}}{\longleftrightarrow}$	$2\pi\delta(\omega-\omega_0)$

- It says that a complex exponential signal of frequency ω_0 has a Fourier transform that is nonzero at only the frequency ω_0.
- Linear Property:

Time-Domain		**Frequency-Domain**
$ax_1(t)+bx_2(t)$	$\stackrel{\mathcal{F}}{\longleftrightarrow}$	$aX_1(j\omega)+bX_2(j\omega)$

Sinusoids

Time-Domain	Frequency-Domain
$A\cos(\omega_0 t + \phi) \overset{\mathcal{F}}{\longleftrightarrow}$	
	$\pi A e^{j\phi}\delta(\omega - \omega_0) + \pi A e^{-j\phi}\delta(\omega + \omega_0)$

- **Derivation**: Since

$$x(t) = \tfrac{1}{2}Ae^{j\phi}e^{j\omega_0 t} + \tfrac{1}{2}Ae^{-j\phi}e^{-j\omega_0 t}$$

- By the linear property, we have

$$\begin{aligned} X(j\omega) =& 2\pi\delta(\omega - \omega_0)(\tfrac{1}{2})Ae^{j\phi} \\ &+ 2\pi\delta(\omega + \omega_0)(\tfrac{1}{2})Ae^{-j\phi} \end{aligned}$$

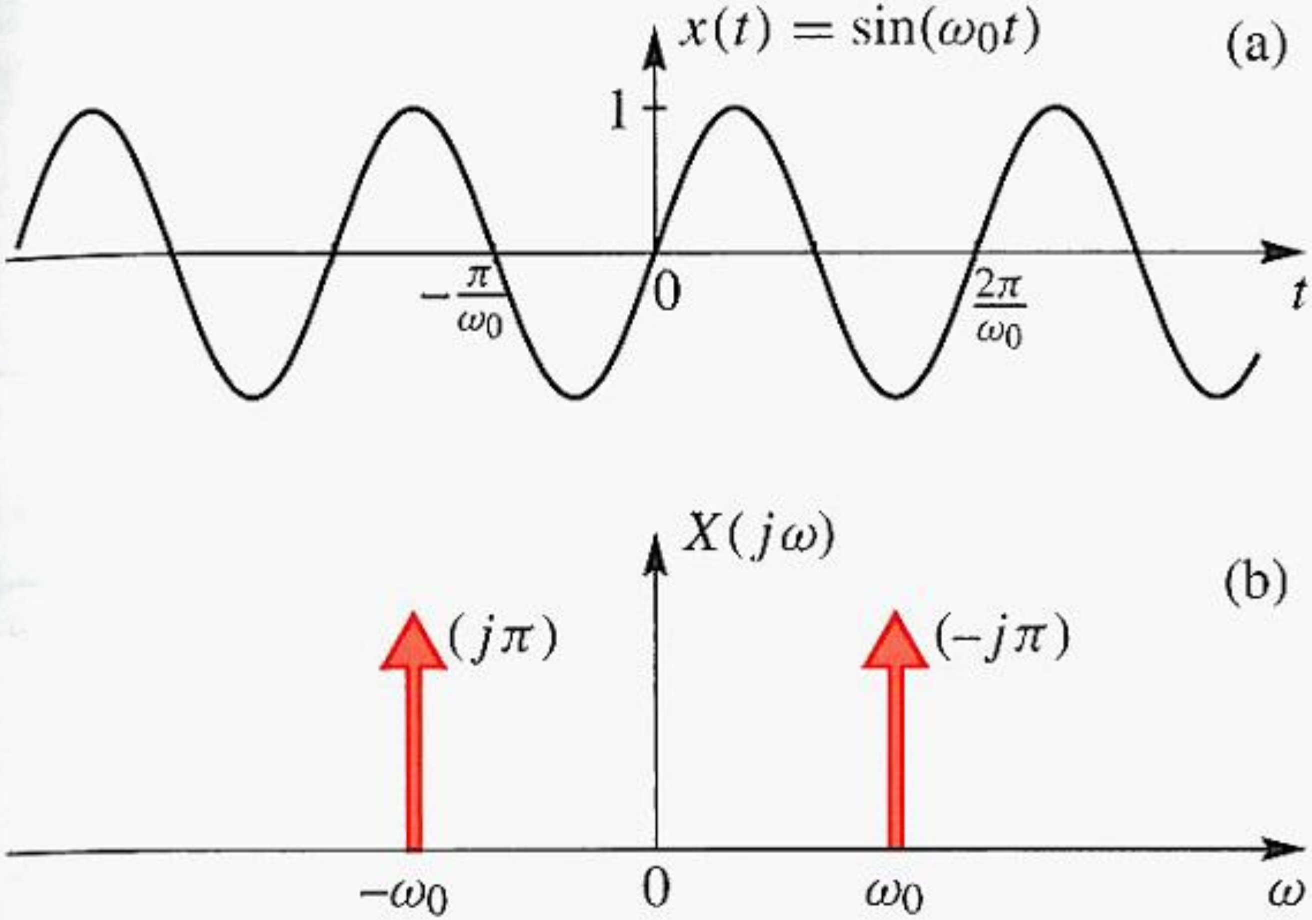


Figure 11-7: Fourier transform of a sinusoid: (a) Time function $x(t) = \sin(\omega_0 t) = \cos(\omega_0 t - \pi/2)$, and (b) corresponding Fourier transform $X(j\omega) = -j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)$.

Periodic Signals

Time-Domain | **Frequency-Domain**

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \stackrel{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Represented as a Fourier series

- Example: time domain a periodic square wave

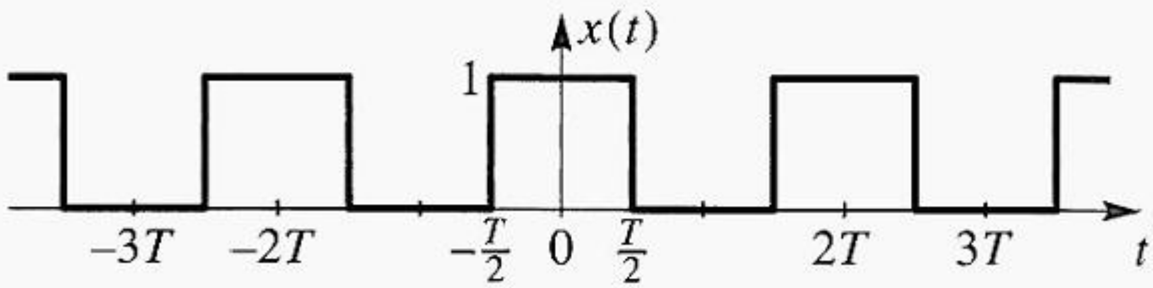


Figure 11-9: Signal $x(t)$ is a 50% duty cycle square wave whose period is $T_0 = 2T$. Its transform is shown

Frequency domain of squared wave

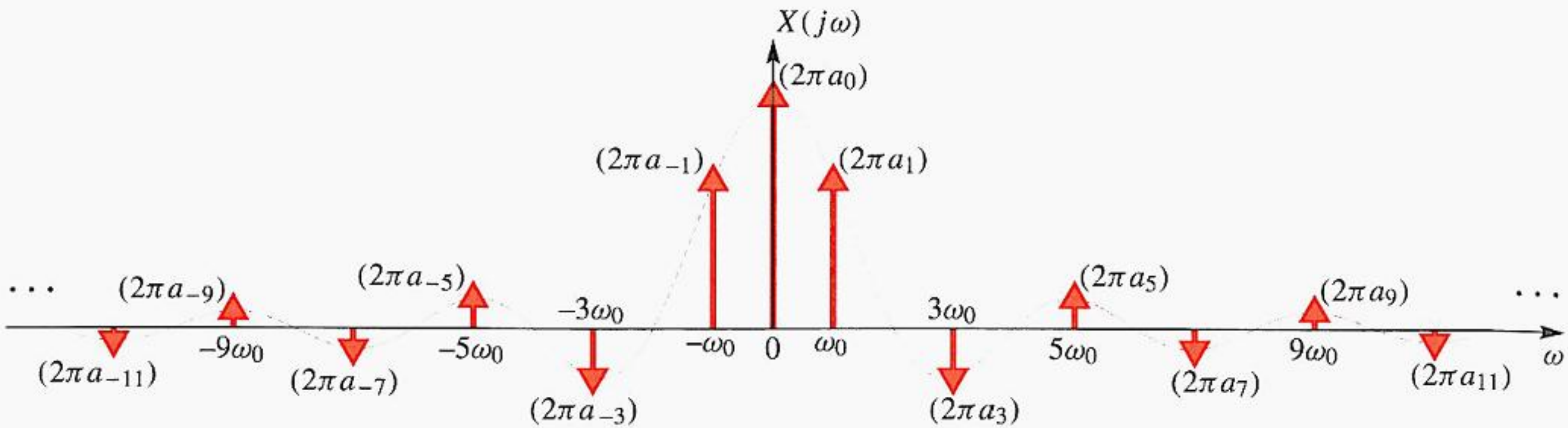


Figure 11-8: Fourier transform of the square wave shown in Fig. 11-9. The transform $X(j\omega)$ has regularly spaced impulses at $\omega = 2\pi k/T_0$.

Basic Fourier Transform Pairs

Table of Fourier Transform Pairs	
Time-Domain: $x(t)$	*Frequency-Domain:* $X(j\omega)$
$e^{-at}u(t) \quad (a > 0)$	$\dfrac{1}{a + j\omega}$
$e^{bt}u(-t) \quad (b > 0)$	$\dfrac{1}{b - j\omega}$
$u(t + \frac{1}{2}T) - u(t - \frac{1}{2}T)$	$\dfrac{\sin(\omega T/2)}{\omega/2}$
$\dfrac{\sin(\omega_b t)}{\pi t}$	$[u(\omega + \omega_b) - u(\omega - \omega_b)]$
$\delta(t)$	1
$\delta(t - t_d)$	$e^{-j\omega t_d}$

Basic Fourier Transform Pairs

$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$A\cos(\omega_0 t + \phi)$	$\pi A e^{j\phi}\delta(\omega - \omega_0) + \pi A e^{-j\phi}\delta(\omega + \omega_0)$
$\cos(\omega_0 t)$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
$\sin(\omega_0 t)$	$-j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)$
$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$\sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T}k)$

Properties of Fourier Transform Pairs

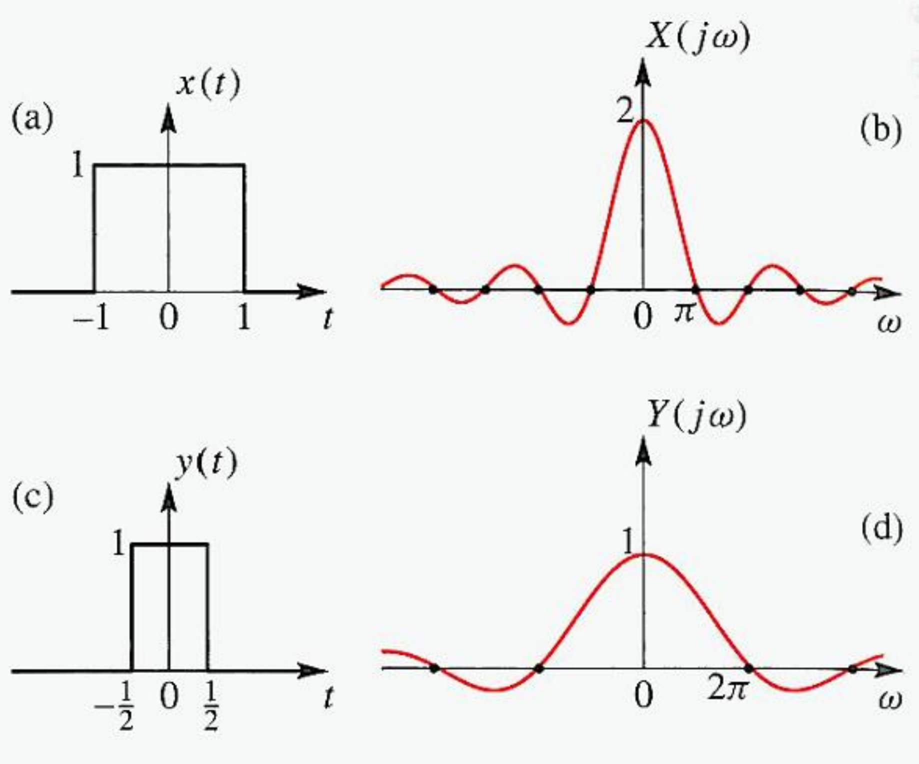
- Scaling Property

Time-Domain	***Frequency-Domain***

$$y(t) = x(at) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = \frac{1}{|a|} X(j\omega/a)$$

Stretching a time signal will compress its Fourier transform.

Compressing a time signal will stretch its Fourier transform.



Properties of Fourier Transform Pairs

- Flip Property

Time-Domain		**Frequency-Domain**
$x(-t)$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$X(-j\omega)$

- Derivation: from the scaling property, we have

$$Y(j\omega) = \frac{1}{|-1|} X(j(\omega/(-1))) = X(-j\omega)$$

Properties of Fourier Transform Pairs

- Time delay property

Time-Domain		Frequency-Domain
$x(t - t_d)$	$\stackrel{\mathcal{F}}{\longleftrightarrow}$	$e^{-j\omega t_d} X(j\omega)$

- Differentiation property

Time-Domain		Frequency-Domain
$\dfrac{d^k x(t)}{dt^k}$	$\stackrel{\mathcal{F}}{\longleftrightarrow}$	$(j\omega)^k X(j\omega)$

Symmetry Properties of Fourier Transform Pairs

- If we take complex conjugate of the spectrum, we obtain

$$X^*(-j\omega) = \left(\int_{-\infty}^{\infty} x(t) e^{-(-j\omega)t} dt \right)^* = \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt$$

- Hence, $X^*(-jw)$ is the Fourier transform of $x^*(t)$

Time-Domain		Frequency-Domain
$x^*(t)$	$\overset{\mathcal{F}}{\longleftrightarrow}$	$X^*(-j\omega)$

Symmetry Properties of Fourier Transform Pairs

- Therefore, if $x(t)$ is a **real-valued function**, i.e., $x(t) = x^*(t)$, the above property reveals that $X(jw) = X^*(-jw)$. Hence, we have

$$\Re e\{X(j\omega)\} = \Re e\{X^*(-j\omega)\}$$
$$\Im m\{X(j\omega)\} = \Im m\{X^*(-j\omega)\}$$

- Then, we can conduct that

$$\Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\}$$
$$\Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\}$$

Real part of $X(jw)$ is an even function

Imaginary part of $X(jw)$ is an odd function

Symmetry Properties of Fourier Transform Pairs

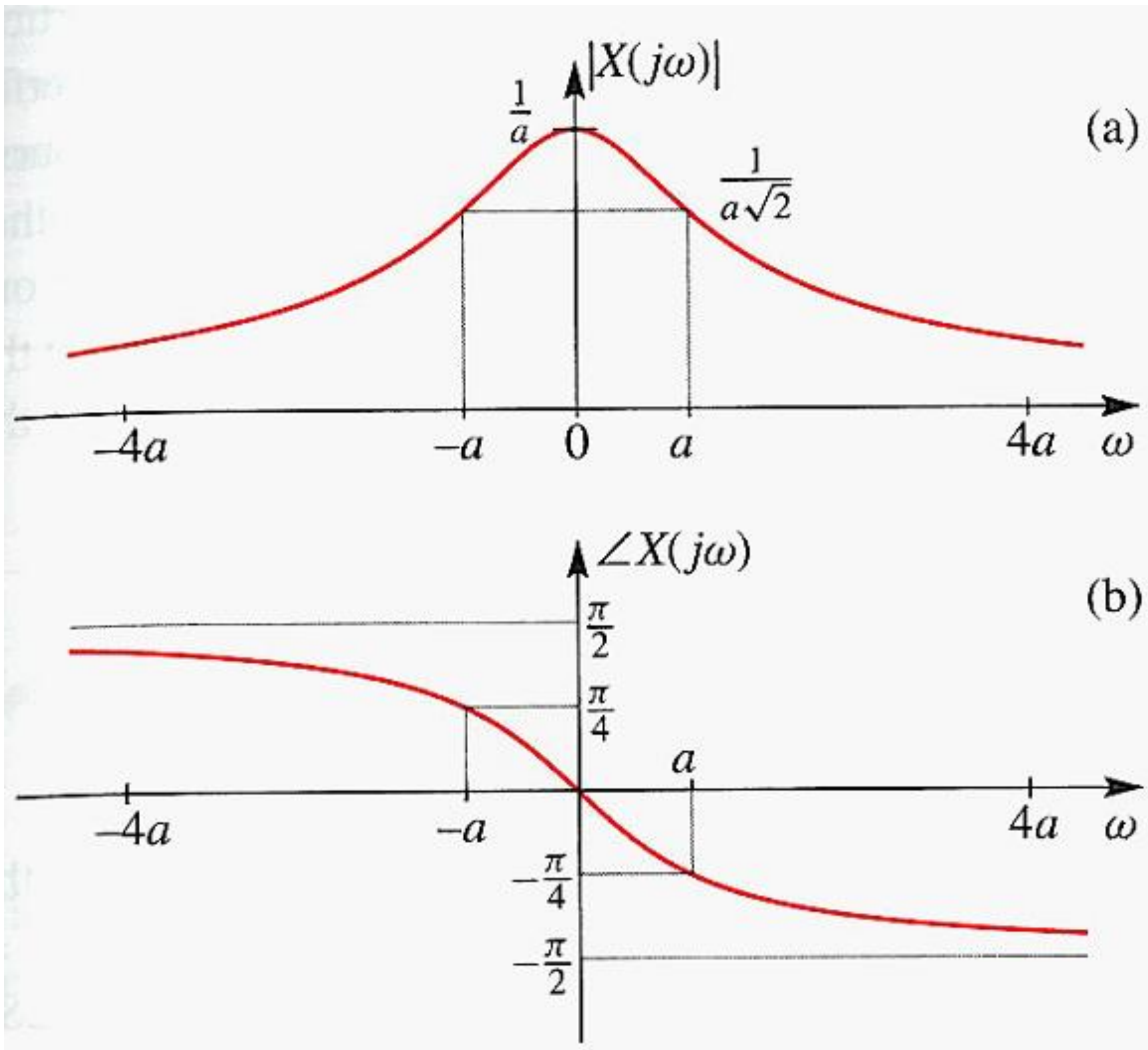
- In other words, when $x(t)$ is real-valued, the real part of its Fourier transform $X(jw)$ is even, and the imaginary part is odd.
- Similarly, we also have a symmetric property for magnitude (amplitude) and phase in polar form when $x(t)$ is real

$$|X(j\omega)| = |X(-j\omega)|$$

$$\angle X(j\omega) = -\angle X(-j\omega)$$

- **That is, for a real-valued signal,**
  - **its magnitude spectrum is an even function**
  - **its phase spectrum is an odd function.**

Example: magnitude and phase spectra of a real signal

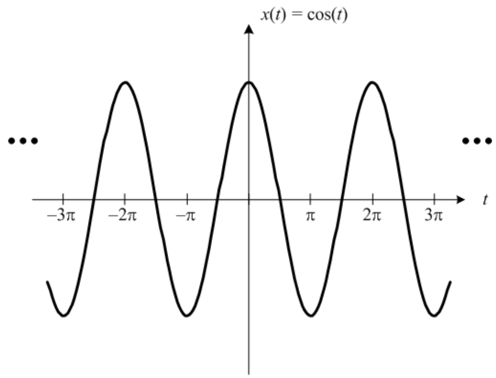


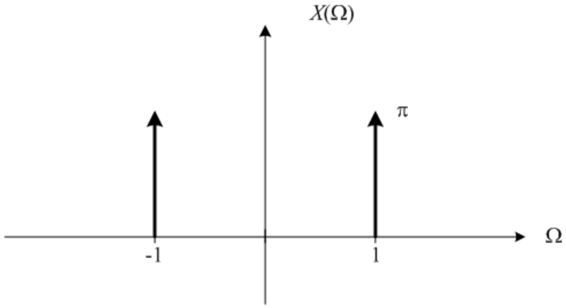
Even function

Odd function

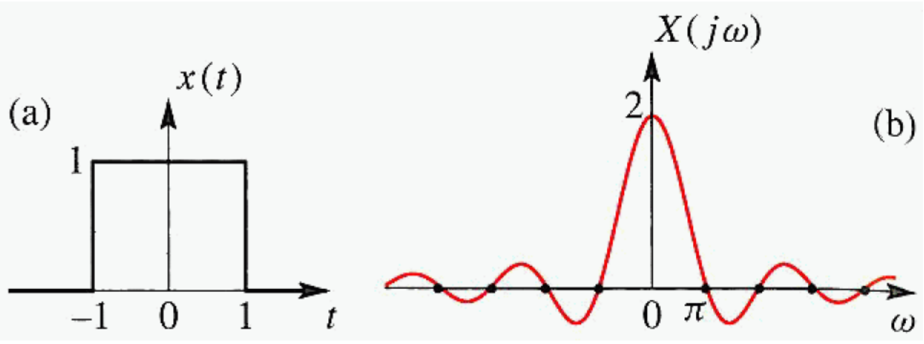
Signals that are both real and even

- For a **real-valued even function**, its Fourier transform (i.e., its spectrum) is **both real and even** too.
- Examples:
  - Cosine function





  - Rectangular function



Parseval theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(jw)|^2 dw$$

- **Energy preserving**: the time-domain energy is equivalent to the frequency domain energy.
- All orthogonal-bases representations have this property.

Uncertainty Principle of CFT

- For a signal $x(t)$, let

$$P_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(jw)|^2 dw$$

(from Parseval's theorem, the equality holds)

- Define the *centroid* of $x(t)$ to be

$$C_x = \frac{1}{P_x} \int_{-\infty}^{\infty} t|x(t)|^2 dt$$

(similar to the definition of *mean* of a probability distribution)

Uncertainty Principle of CFT (cont.)

- Likewise, define the *centroid* of $X(jw)$ to be

$$C_X = \frac{1}{P_x}\int_{-\infty}^{\infty} w|X(jw)|^2 dw$$

in the frequency domain.

Uncertainty Principle of CFT (cont.)

- The *effective width* of a signal $x(t)$ is defined as

$$W_x = \left(\frac{1}{P_x} \int_{-\infty}^{\infty} (t - C_x)^2 |x(t)|^2 dt \right)^{1/2}$$

- The *effective width* of the spectrum $X(jw)$ is defined as

$$W_X = \left(\frac{1}{P_x} \int_{-\infty}^{\infty} (w - C_X)^2 |X(jw)|^2 dw \right)^{1/2}$$

(they are similar to the definition of *standard deviation* of a probability distribution)

Uncertainty Principle of CFT (cont.)

- Uncertainty Principle: It an be proven that

$$W_x W_X \geq 0.5$$

for any CFT transform pair $x(t)$ and $X(jw)$.

Uncertainty Principle

- The product of the effective width of a signal and the effective width of its continuous Fourier transform is not smaller than 0.5.

Uncertainty Principle of CFT (cont.)

$$W_x W_X \geq 0.5$$

- When will the equality hold? i.e. What are the signals satisfying the property $W_x W_X = 0.5$?

- Gaussian functions are the only functions for making the equality hold:

$$x(t) = e^{-t^2/(2\sigma^2)}, X(jw) = \sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$$