Spectrum Representation

- **Frequency-domain representation of a signal.**
- As we know, a signal can be decomposed into a linear combination of zero-phase complex-exponential basis functions.
- When decomposing, the coefficients obtained are referred to as the **spectrum** of the signal.
  - Review of **inverse Euler formula**:

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

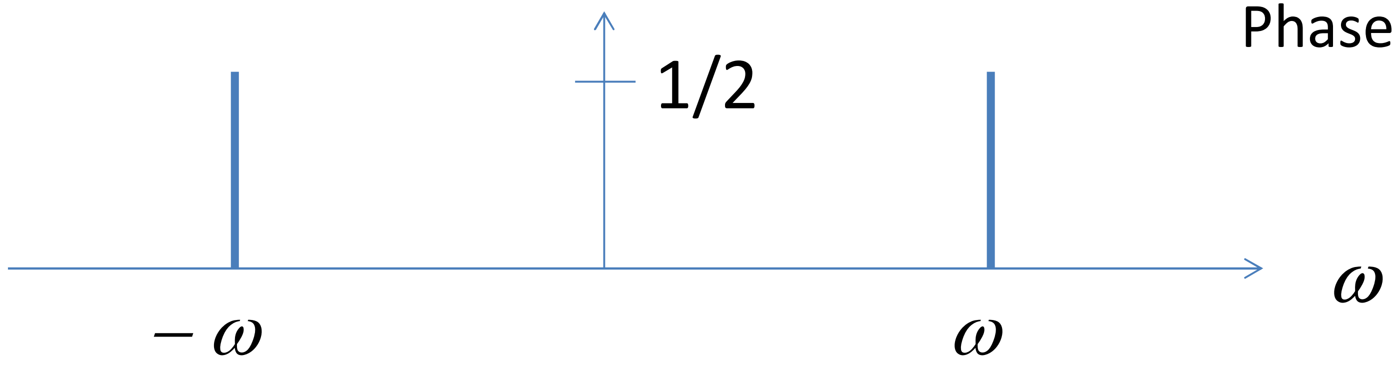
$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

Spectrum of a single sinusoid

- **What is the spectrum of a single cosine function?**
- Note that we employ complex exponential as bases.
- Because

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

the spectrum is (ω,1/2), ($-\omega$, 1/2), containing both positive and negative frequencies:



Example

- **Even summing the complex exponentials, we still get a real-value signal**

SPECTRUM Interpretation

- Cosine = sum of 2 complex exponentials:

$$A\cos(7t) = \frac{A}{2}e^{j7t} + \frac{A}{2}e^{-j7t}$$

One has a positive frequency
The other has negative freq.
Amplitude of each is half as big

Example of sine function

$$\boxed{\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})}$$

SPECTRUM of SINE

- Sine = sum of 2 complex exponentials:

$$A\sin(7t) = \frac{A}{2j}e^{j7t} - \frac{A}{2j}e^{-j7t}$$

$$= \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

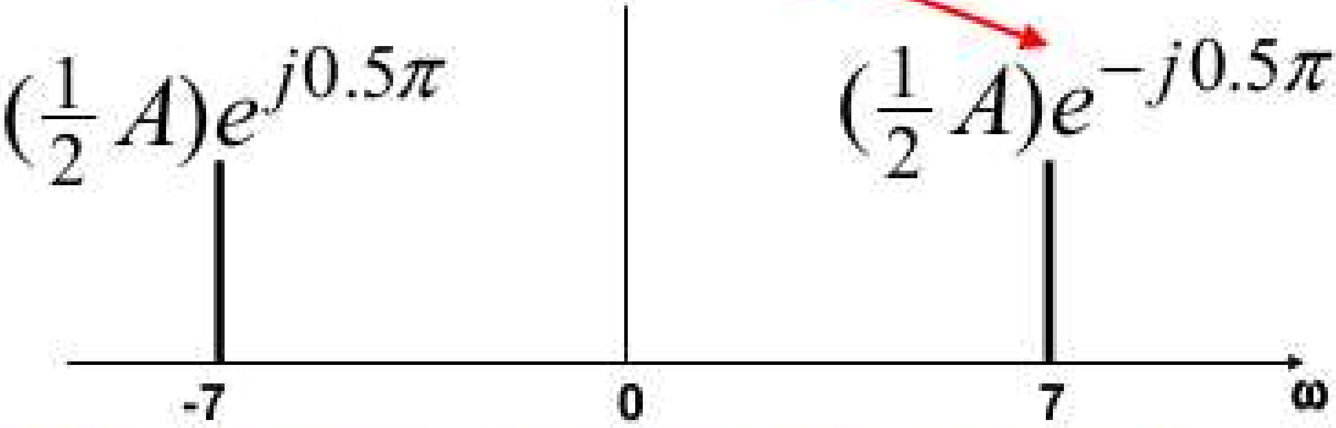
$$\boxed{\frac{-1}{j} = j = e^{j0.5\pi}}$$

- Positive freq. has phase = -0.5π
- Negative freq. has phase = +0.5π

GRAPHICAL SPECTRUM

EXAMPLE of SINE

$$\boxed{A\sin(7t) = \tfrac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \tfrac{1}{2}Ae^{j0.5\pi}e^{-j7t}}$$



AMPLITUDE, PHASE & FREQUENCY are shown

Spectrum Representation

- The most straightforward way of viewing and understanding a spectrum: adding N sinusoids of different frequencies:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

Spectrum Representation

- By the inverse Euler formula

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

- It gives a way to represent $x(t)$ in the alternative form: (What are X_k and X_k^*?)

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t} \right\}$$

Spectrum Representation

- $x(t)$ is composed of $2N$+1 complex amplitudes corresponding to the $2N$+1 frequencies:

$$\left\{ (0, X_0)\,,\ \left(f_1, \tfrac{1}{2}X_1\right),\ \left(-f_1, \tfrac{1}{2}X_1^*\right), \ldots \right.$$
$$\left. \left(f_k, \tfrac{1}{2}X_k\right),\ \left(-f_k, \tfrac{1}{2}X_k^*\right), \ldots \right\}$$

- We call it as the frequency-domain representation of the signal $x(t)$.

Example

$$x(t) = 10 + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2)$$

- Apply the inverse Euler formula

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

- The spectrum:

$$\left\{(0,\ 10),\ (100,\ 7e^{-j\pi/3}),\ (-100,\ 7e^{j\pi/3}),\ (250,\ 4e^{j\pi/2}),\ (-250,\ 4e^{-j\pi/2})\right\}$$

Example

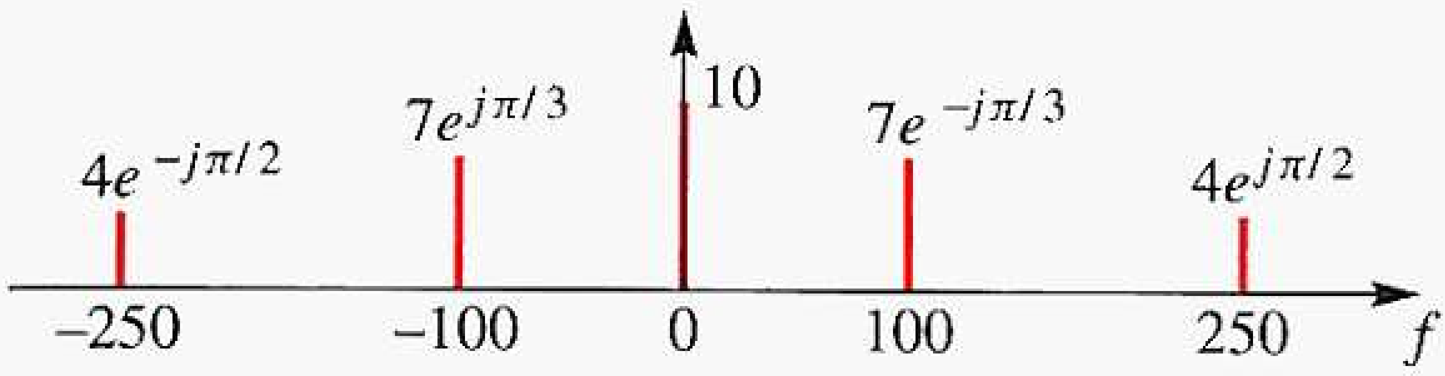


Figure 3-1: Spectrum plot for the signal $x(t) = 10 + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2)$. Units of frequency ($f$) are Hz. Negative frequency components should be included for completeness even though they are conjugates of the corresponding positive frequency components.

- They are called the frequency components.

DC Component

- The constant component (corresponding the zero frequency) is referred to as the DC component.
- In the above example, the DC component is 10.
- We can separate the frequency components into the amplitude (magnitude) and phase components.

Gather (A,ω,ϕ) information

Frequencies:	Amplitude & Phase	
-250 Hz	4	$-\pi/2$
-100 Hz	7	$+\pi/3$
0 Hz	10	**0**
100 Hz	7	$-\pi/3$
250 Hz	4	$+\pi/2$

Note the **conjugate phase**

DC is another name for zero-freq component

DC component always has $\phi=0$ or π (for real $\boldsymbol{x(t)}$)

Synthetic Sound Example

- A periodic signal could be synthesized as the sum of complex exponentials

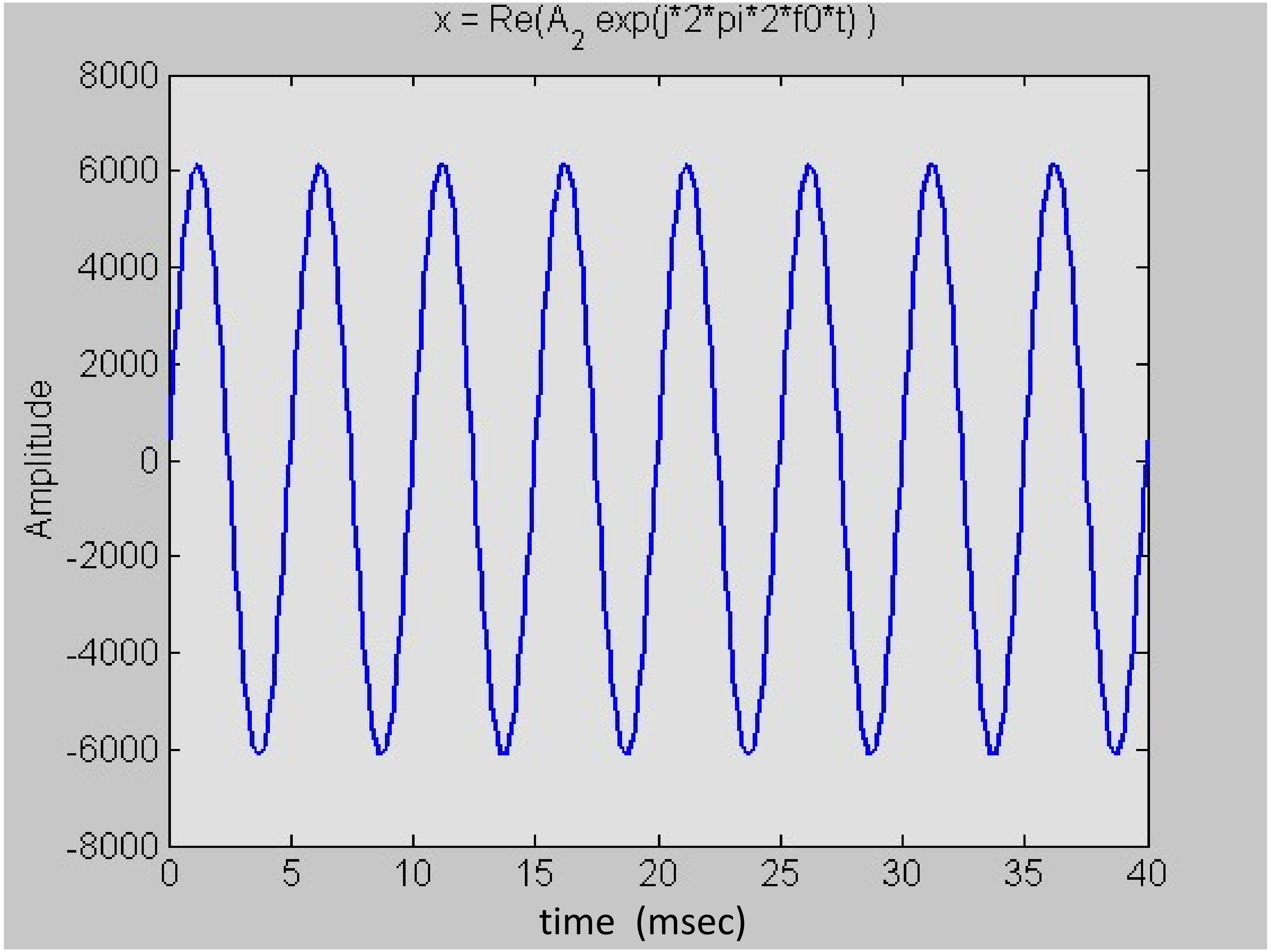
$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}$$

$$= a_0 + 2\Re e\left\{\sum_{k=1}^{N} a_k e^{j2\pi k f_0 t}\right\}$$

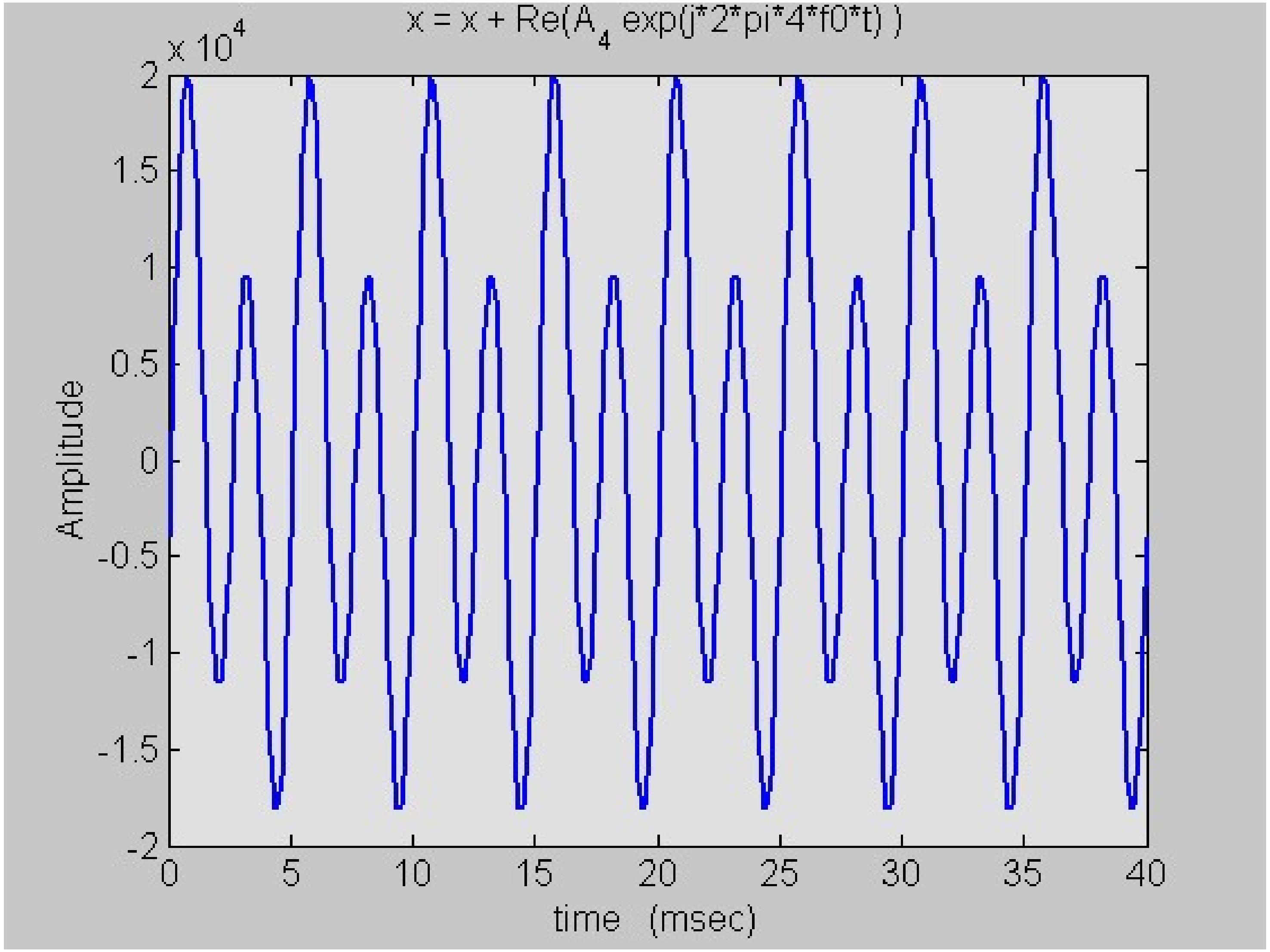
- How is it sounds like: consider a signal containing nonzero terms for only

$$\{a_{\pm 2},\ a_{\pm 4},\ a_{\pm 5},\ a_{\pm 16},\ a_{\pm 17}\}$$

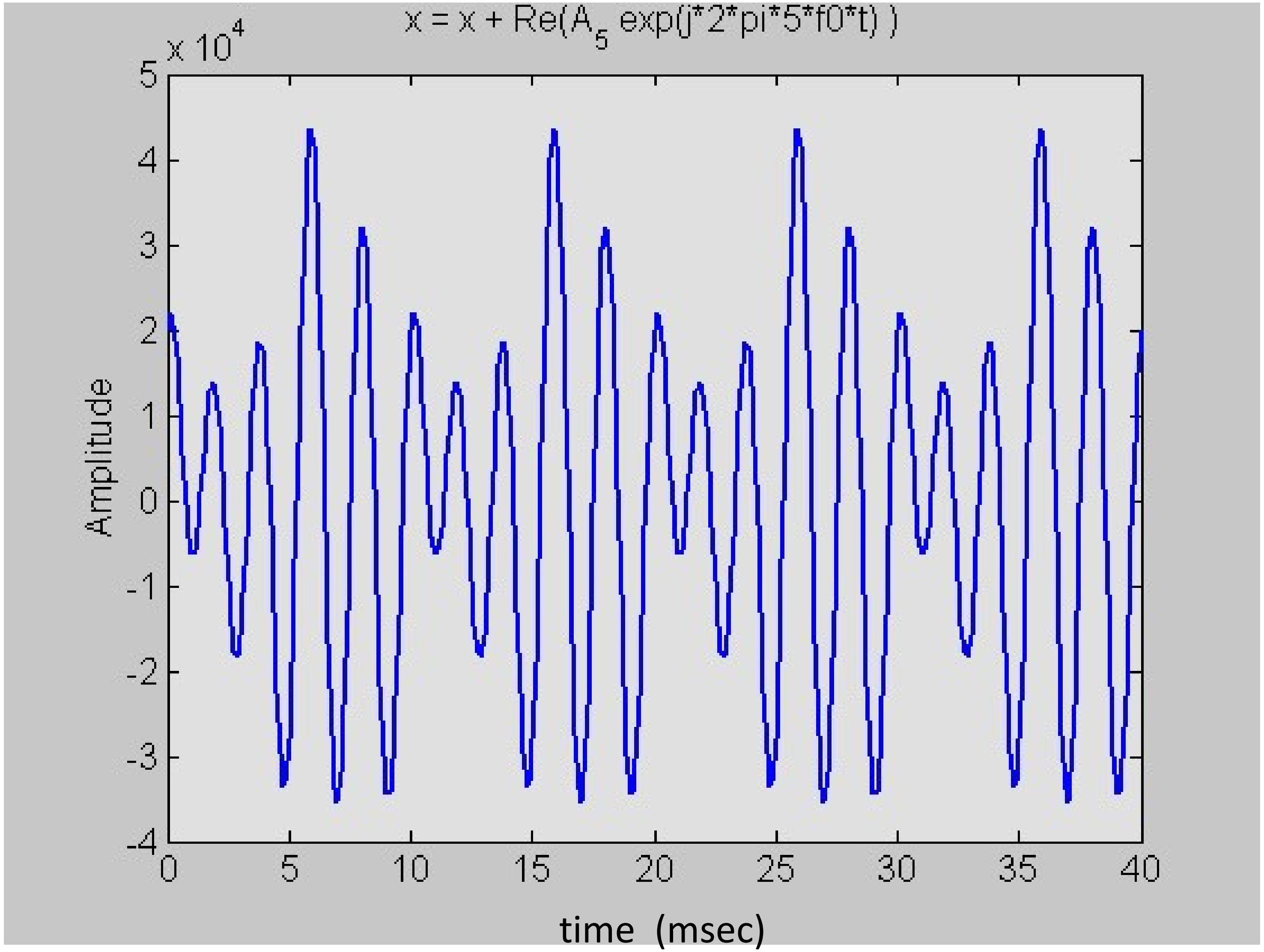
Vowel Example: Single component *a2*



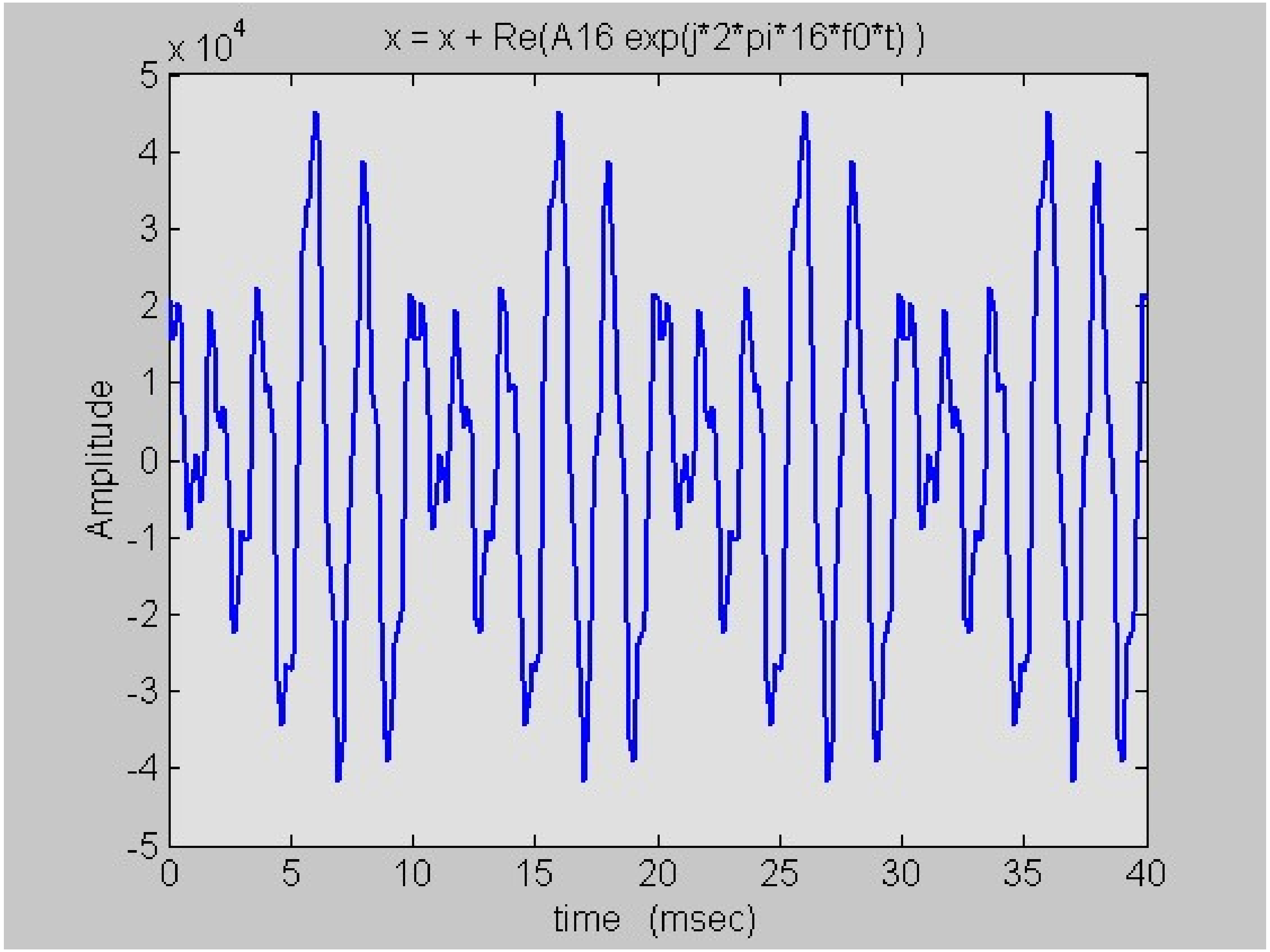
Vowel: Two components: *a2+a4*



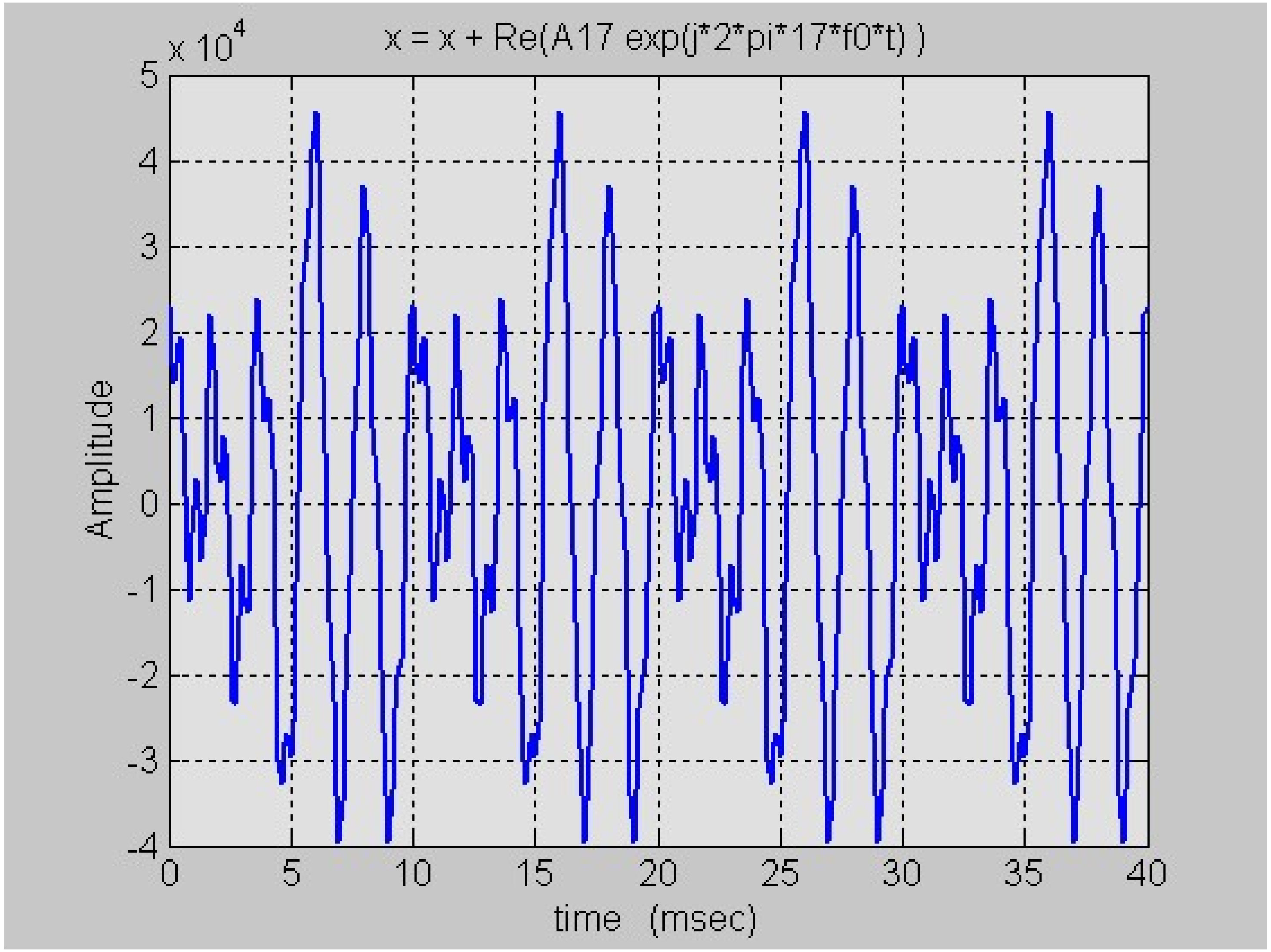
Vowel: Three components: *a2+a4+a5*



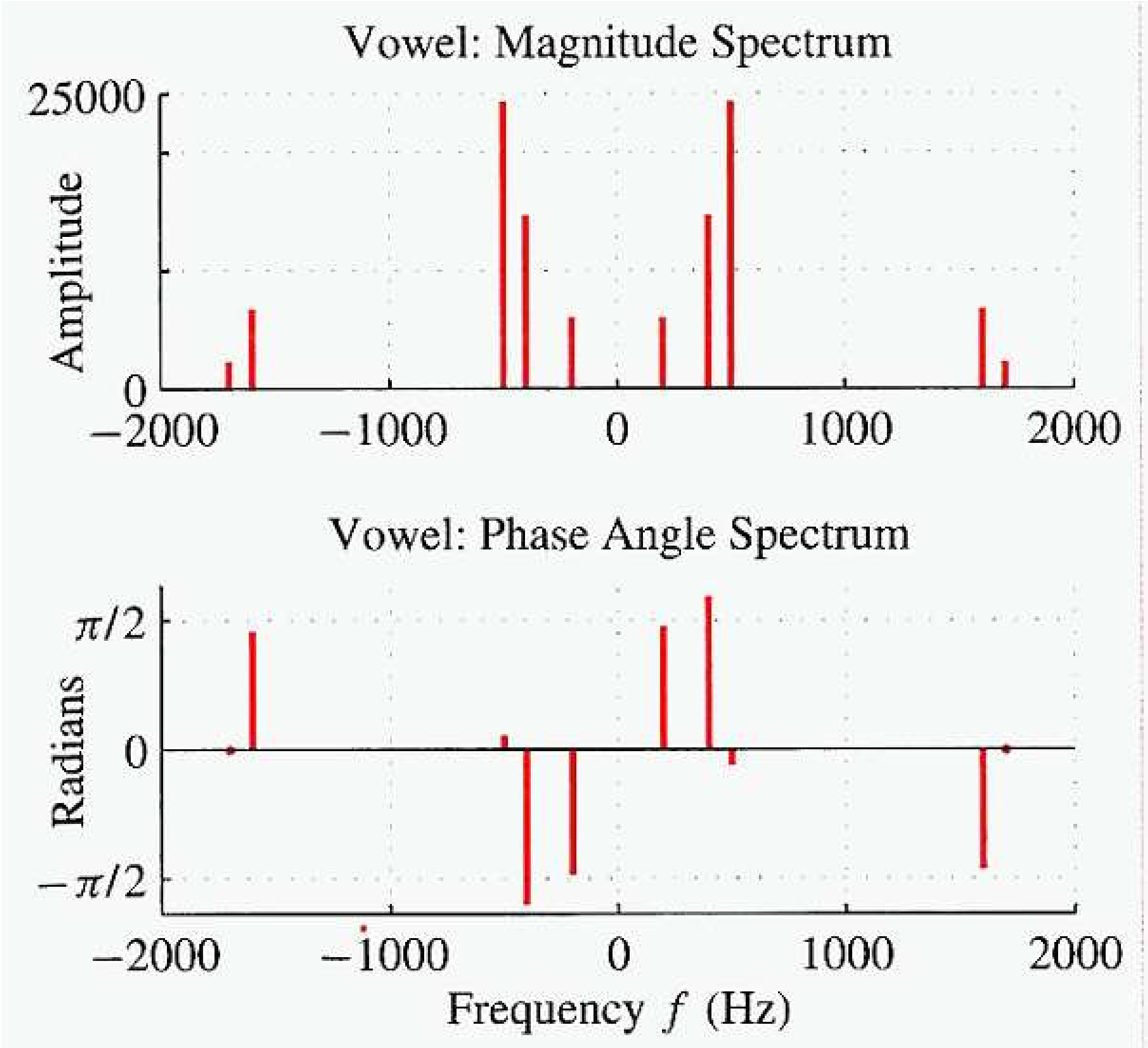
Vowel: Four components: *a2+a4+a5+a16*



Vowel: Five components
a2+a4+a5+a16+a17



Vowel signal: Frequency Domain



Periodic Waveforms

- In the above, the number of frequency components are finite.
- It is possible the number of frequency components is infinite.
- Consider a general periodic signal,
  - A periodic signal satisfies the condition that $x(t + T_0) = x(t)$ for all t.
  - The time interval T_0 is called the period of $x(t)$.

Fourier Series

- **Fourier series: Any periodic signal of period T_0 can be synthesized by sum of complex exponentials of the frequencies of integer multiples of $2\pi/T_0$.**
- **The sum may need a infinite number of terms.**
- This is the mathematical theory of Fourier series: if $x(t+T_0) = x(t)$ for all t, then $x(t)$ can be represented as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{j(2\pi/T_0)kt}$$

Fourier Series

- **How do we derive the coefficients a_k in Fourier Series?**
- Previously, we have shown the bases

$$v_{k(t)} = e^{jwkt}$$

with the fundamental frequency $w = 2\pi/T_0$ satisifying the orthogonality property,

$$\int_0^{T_0} v_k(t) v_\ell^*(t)\, dt = \begin{cases} 0 & \text{if } k \neq \ell \\ T_0 & \text{if } k = \ell \end{cases} \qquad v_k(t) = e^{j(2\pi/T_0)kt}$$

- Hence, to derive a_k, we need simply to project the signal $x(t)$ onto the orthogonal basis $v_k(t)$ by inner product.

Fourier Series

- So, we can obtain a_k by the inner product of $x(t)$ and $v_k(t)$:

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} \, dt$$

  - Note: remember that the inner product of two complex signals uses complex conjugates for the right-hand terms.

Proof of it

- From the equation, we can verify that for any $l \in Z$,

$$a_l T_0 = \int_0^{T_0} x(t) e^{-j(2\pi/T_0)\ell t}\, dt =$$

$$\int_0^{T_0} \left(\sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \right) e^{-j(2\pi/T_0)\ell t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \left(\int_0^{T_0} e^{j(2\pi/T_0)(k-\ell)t}\, dt \right) = a_\ell T_0$$

Fourier Series

- In particular, from the above the DC component is obtained by

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt$$

- That is, the *DC component* is simply the **average value** (or **mean**) in one period.

Transform pair of Fourier Series

- In sum, we have the following **transform pair** that can be used for the analysis of periodic signals:

Fourier Analysis Equation

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}\,dt$$

transform pair

Fourier Synthesis Equation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

- The left, from $x(t)$ to a_k, is called the forward transform, which transform the signal $x(t)$ to the frequency domain, and a_k are called *frequencies*, *frequency components*, or *spectrum*.
- There could be infinite frequency components, $a_k, k \in Z$.
- The right, from a_k to $x(t)$, is called the inverse transform.

Integral over a period

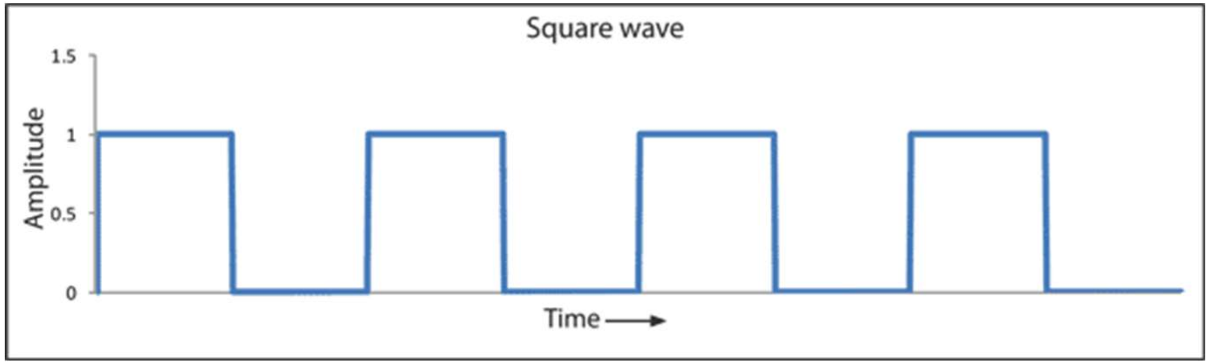
- Note that integrals over a period $[0, T_0]$ and $[-T_0/2, T_0/2]$ are the same for a periodic function.
- Hence, the forward transform of Fourier series **can also be written as**

$$a_k = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-j(2\pi/T_0)kt} \, dt$$

- There are infinite spectral components, but are discrete. (i.e., the spectrum of a periodic signal is discrete)

Illustration Example

- What is the Fourier series of a squared wave?



Squared wave

- Consider a finite-duration signal $x(t)$ at first,

$$x(t) = \begin{cases} 1 & -\frac{1}{2}T < t < \frac{1}{2}T \\ 0 & \text{otherwise} \end{cases}$$

Illustration Example: build a period signal from the finite-duration signal

- A convenient way to express the above periodic replication process is to write an infinite sum of time-shifted copies of $x(t)$

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} x(t - nT_0)$$

$T < T_0$

Illustration Example: build a period signal from the finite-duration signal

Assume the period is T_0,
$T_0 > T$

Let $x_{T_0}(t)$ be the periodic signal obtained by repeating copies of $x(t)$ every T_0 seconds.

Eg., the examples of square waves of $T_0 = 2T, 4T, 8T$.

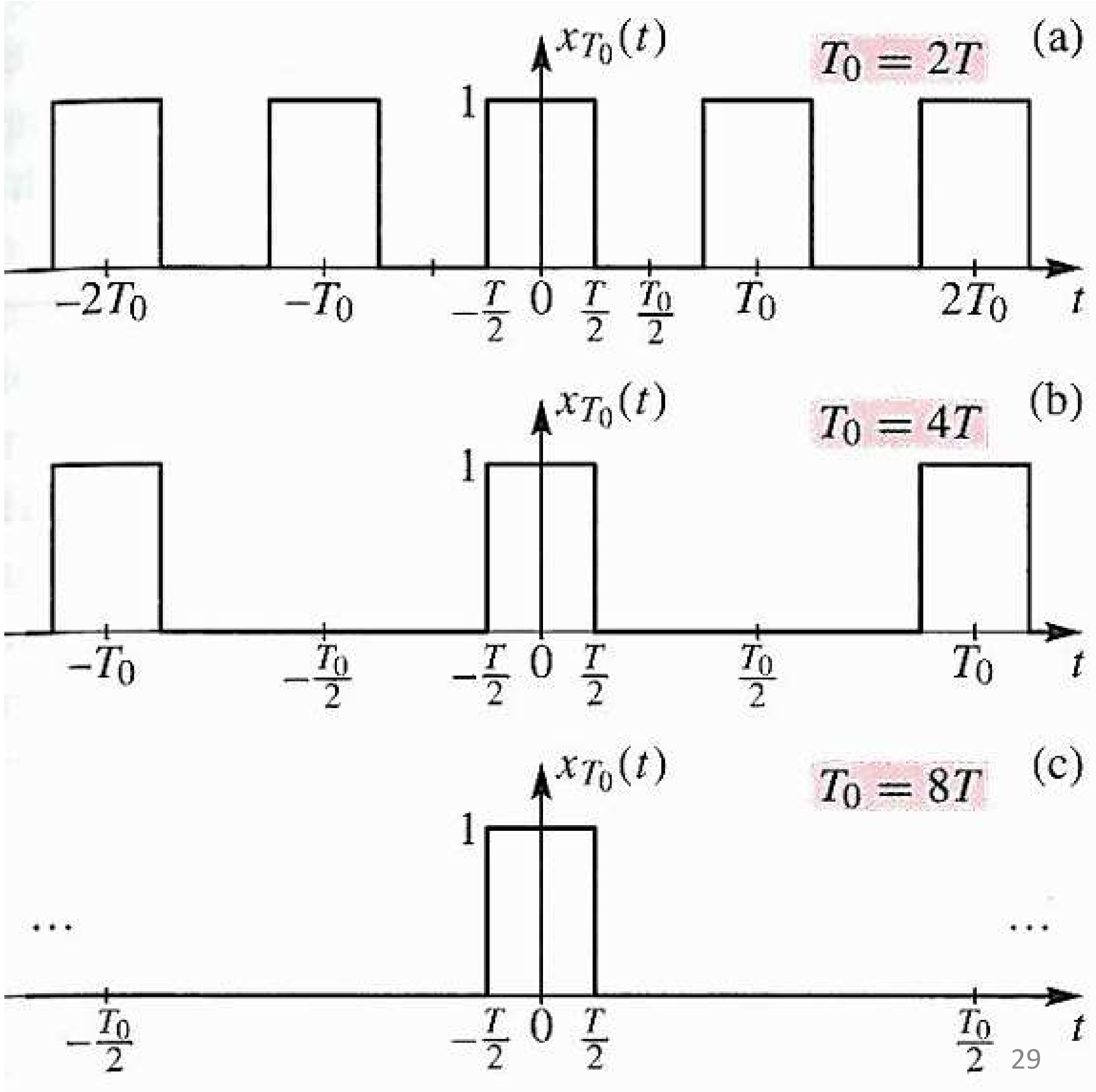


Illustration Example: Fourier series of a squared wave

- By forward transform of the Fourier series, the integral of a single period T_0 is as follows:

$$a_k T_0 = \int_{-T/2}^{T/2} e^{-jk\omega_0 t} dt$$

Integrals over a period $[-T_0/2, T_0/2]$

$$= -\frac{1}{jk\omega_0} e^{-jk\omega_0 t} \Big|_{-T/2}^{T/2} = -\frac{e^{-jk\omega_0 T/2} - e^{jk\omega_0 T/2}}{jk\omega_0}$$

$$= \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

Function of the form $\frac{sin(x)}{x}$ is called the sinc function.

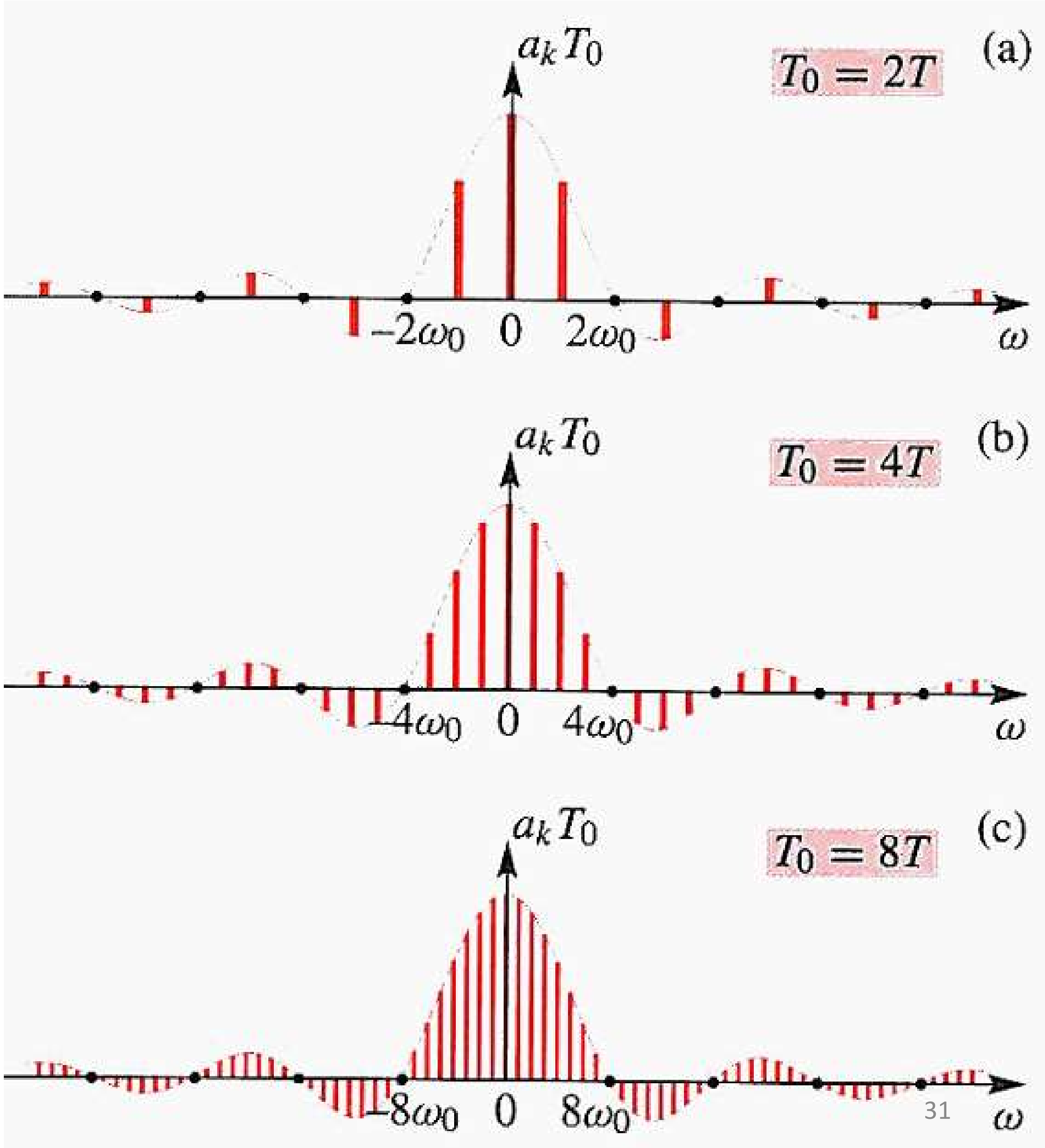
Thus, the spectrum of a squared wave is a (discrete) sinc function.

Figures of the spectra of the periodic signal:

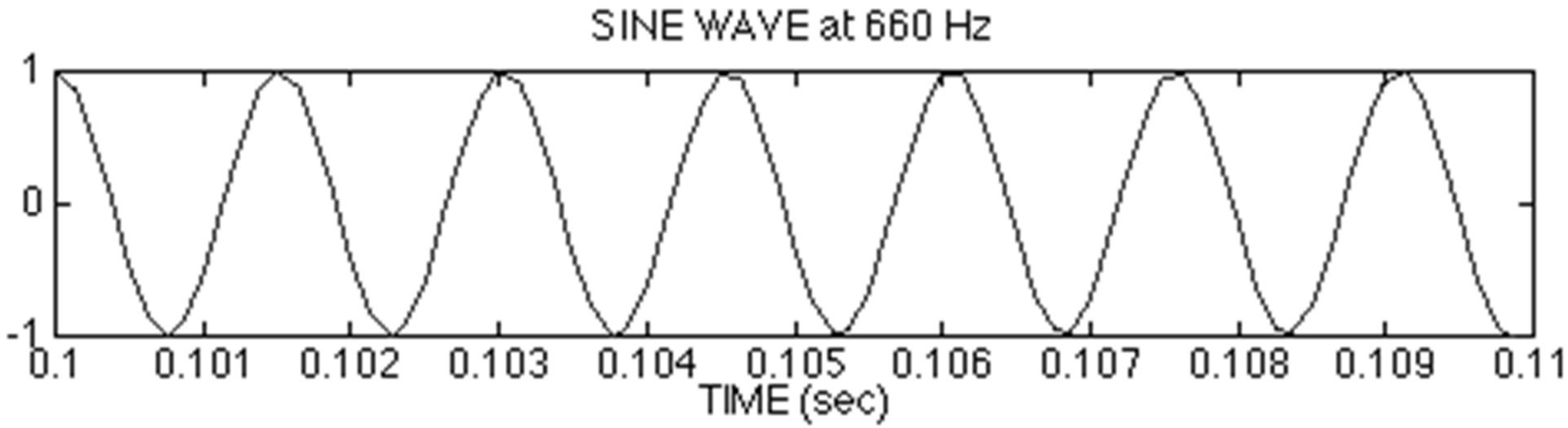
(a) $T_0 = 2T$

(b) $T_0 = 4T$

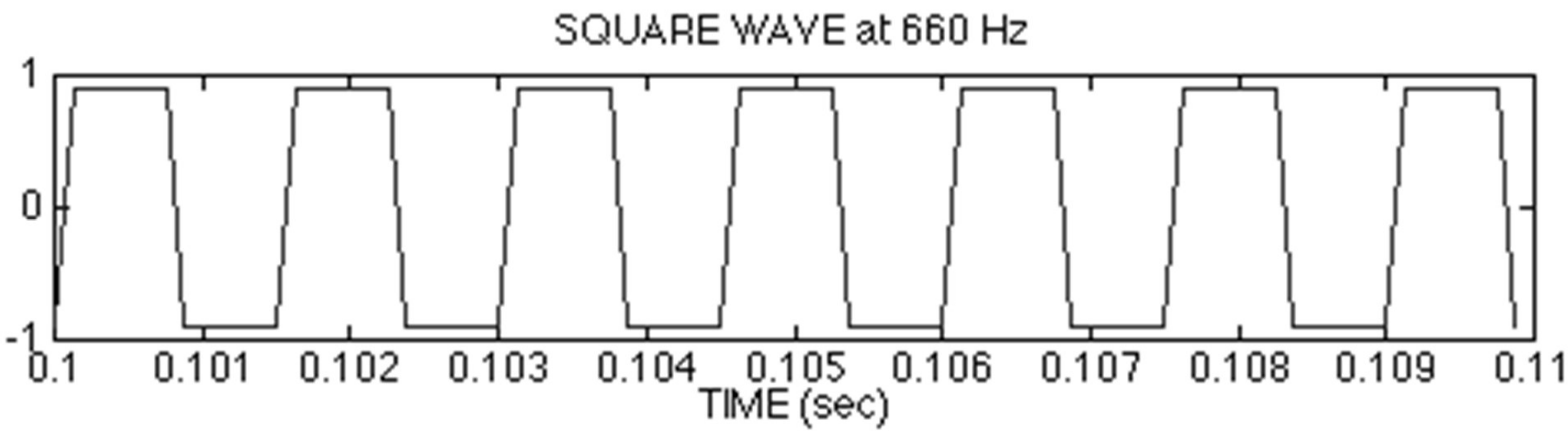
(c) $T_0 = 8T$.



Sound Example of Periodic Signals: Sine Wave



Sound Example of Periodic Signals: Square Wave



Sound Example of Periodic Signals: SAW Wave