Decimation-in-frequency FFT algorithm

The decimation-in-time FFT algorithm is based on structuring the DFT computation by forming smaller and smaller subsequences of the input sequence x[n]. Alternatively, we can consider dividing the output sequence X[k] into smaller and smaller subsequences in the same manner.

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} \qquad k = 0,1,\ldots,N-1$$

The even-numbered frequency samples are

$$X[2r] = \sum_{n=0}^{N-1} x[n] W_N^{n(2r)} = \sum_{n=0}^{(N/2)-1} x[n] W_N^{n(2r)} + \sum_{n=(N/2)}^{N-1} x[n] W_N^{n(2r)}$$

So, $$X[2r] = \sum_{n=0}^{(N/2)-1} x[n] W_N^{2nr} + \sum_{n=0}^{(N/2)-1} x[n+(N/2)] W_N^{2r(n+(N/2))}$$

Since $W_N^{2r[n+(N/2)]} = W_N^{2nr} W_N^{rN} = W_N^{2nr}$

and $\boxed{W_N^2 = W_{N/2}}$, we have $W_N^{2nr} = W_{N/2}^{nr}$.

Then, both the first half and last half share the same multiplication term, $W_{N/2}^{nr}$, and so

$$X[2r] = \sum_{n=0}^{(N/2)-1} (x[n] + x[n+(N/2)]) W_{N/2}^{nr} \qquad r = 0,1,\ldots,(N/2)-1$$

The above equation is the (N/2)-point DFT of the (N/2)-point sequence obtained by adding the first and the last half of the input sequence.

Rational: Adding the two halves of the input sequence represents time aliasing, consistent with the fact that in computing only the even-number frequency samples, we are sub-sampling the Fourier transform of x[n].

We now consider obtaining the odd-numbered frequency points:

$$X[2r+1]=\sum_{n=0}^{N-1}x[n]W_N^{n(2r+1)}=\sum_{n=0}^{(N/2)-1}x[n]W_N^{n(2r+1)}+\sum_{n=(N/2)}^{N-1}x[n]W_N^{n(2r+1)}$$

Since

$$\sum_{n=N/2}^{N-1}x[n]W_N^{n(2r+1)}=\sum_{n=0}^{(N/2)-1}x[n+(N/2)]W_N^{(n+N/2)(2r+1)}$$

$$=W_N^{(N/2)(2r+1)}\sum_{n=0}^{(N/2)-1}x[n+(N/2)]W_N^{n(2r+1)}$$

$$=W_N^{Nr+N/2}\sum_{n=0}^{(N/2)-1}x[n+(N/2)]W_N^{n(2r+1)}$$

$$=W_N^{N/2}\sum_{n=0}^{(N/2)-1}x[n+(N/2)]W_N^{n(2r+1)}$$

$$=-\sum_{n=0}^{(N/2)-1}x[n+(N/2)]W_N^{n(2r+1)}$$

We obtain

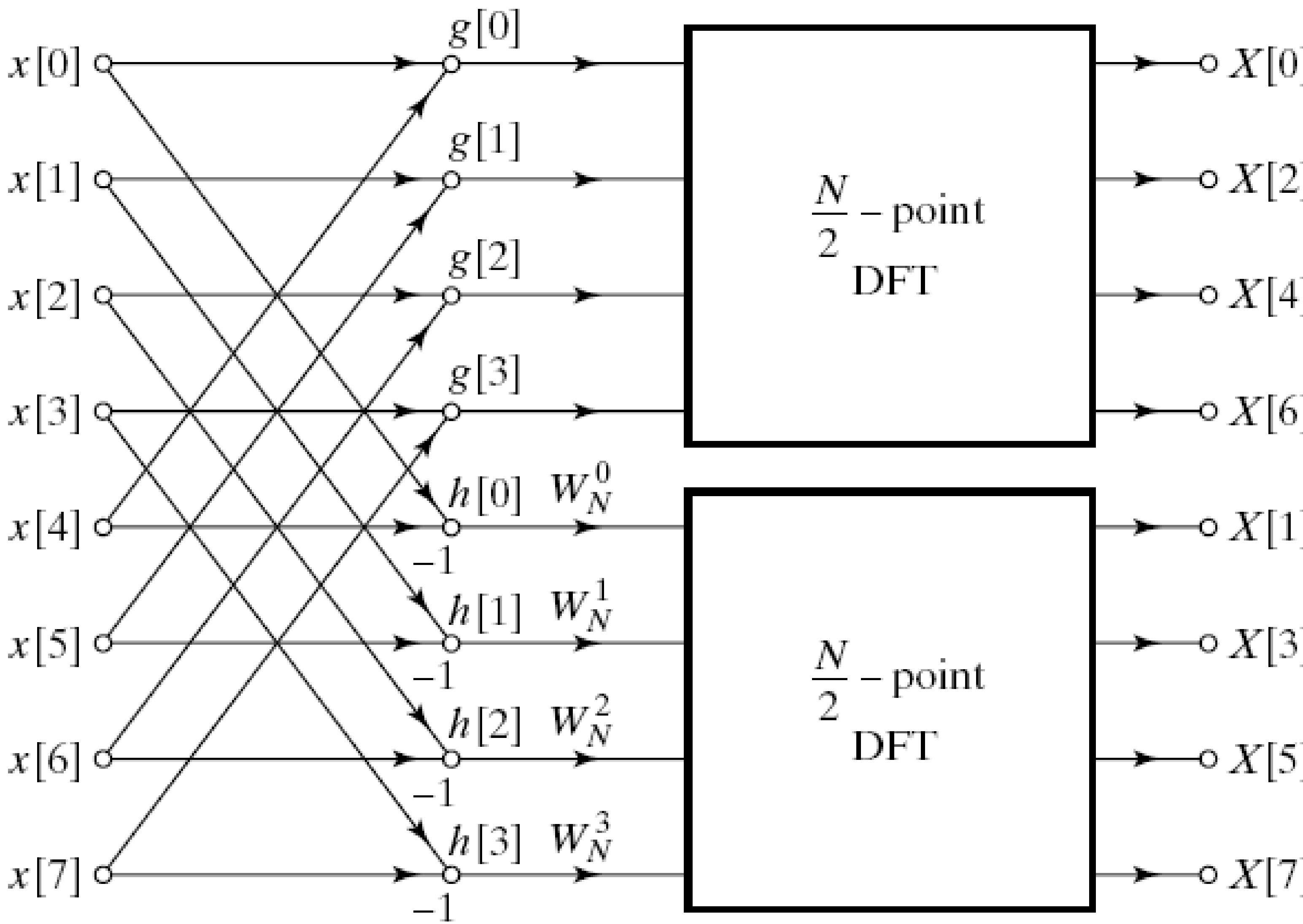
$$X[2r+1] = \sum_{n=0}^{(N/2)-1} (x[n] - x[n+N/2]) W_N^{n(2r+1)}$$

$$= \sum_{n=0}^{(N/2)-1} (x[n] - x[n+N/2]) W_N^n W_{N/2}^{nr} \qquad r = 0,1,\ldots,(N/2)-1$$

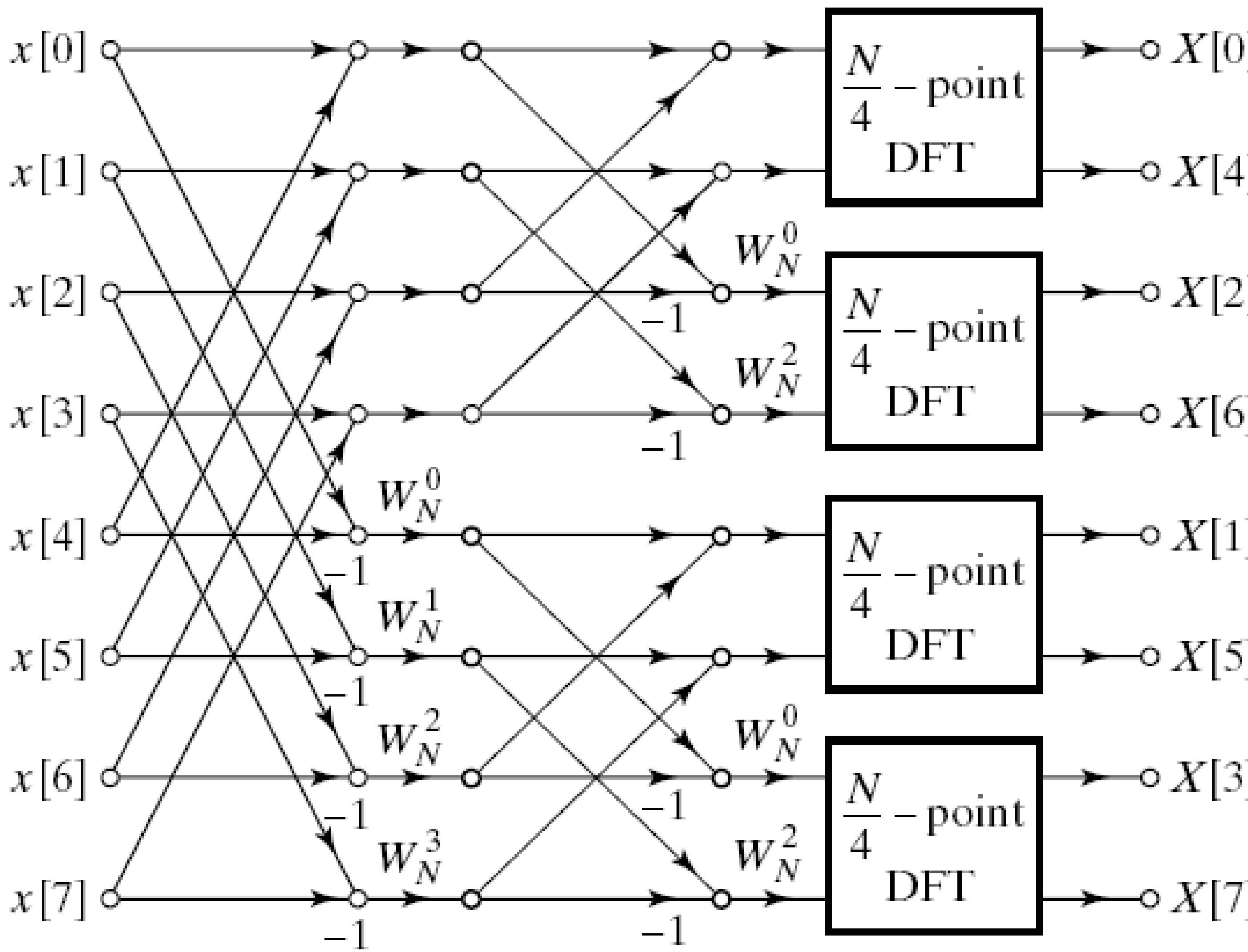
The above equation is the (N/2)-point DFT of the sequence obtained by subtracting the second half of the input sequence from the first half and multiplying the resulting sequence by W_N^n.

Let $g[n] = x[n]+x[n+N/2]$ and $h[n] = x[n]-x[x+N/2]$, the DFT can be computed by forming the sequences g[n] and h[n], then computing $h[n]W_N^n$, and finally computing the (N/2)-point DFTs of these two sequences.

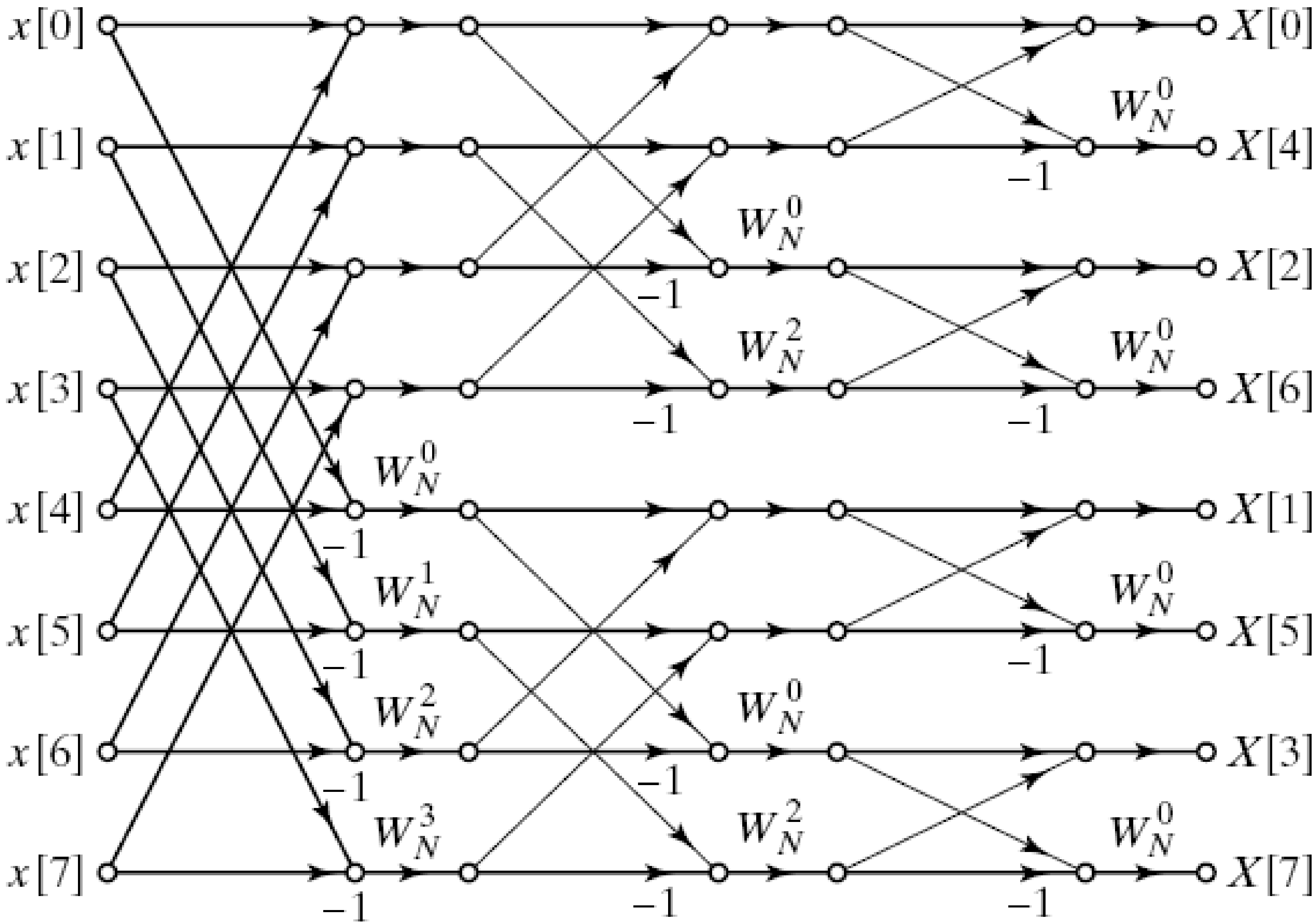
Flow graph of decimation-in-frequency decomposition of an N-point DFT (N=8).



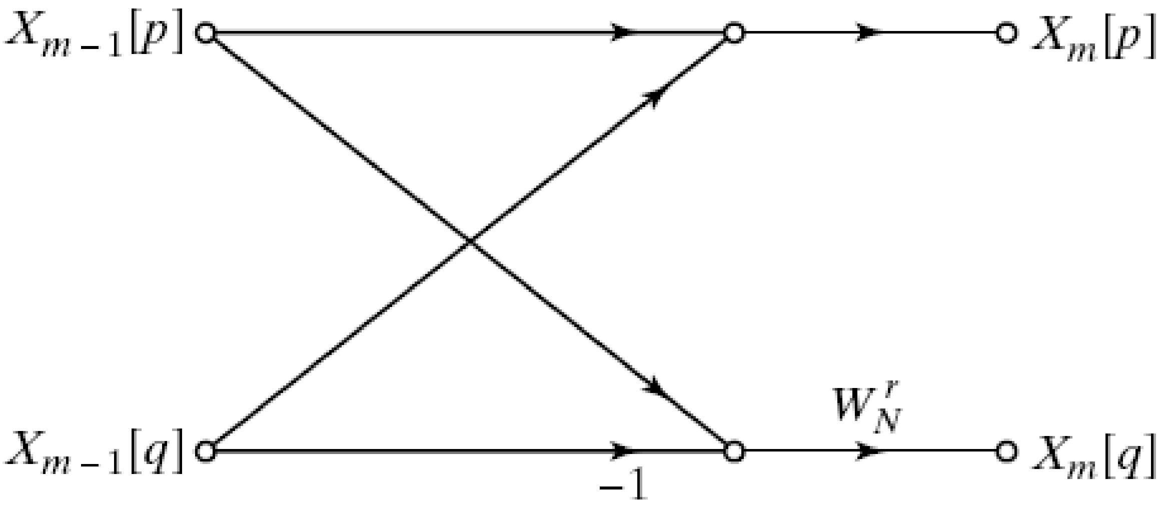
Recursively, we can further decompose the (N/2)-point DFT into smaller substructures:



Finally, we have



Butterfly structure for decimation-in-frequency FFT algorithm:



The decimation-in-frequency FFT algorithm also has the computation complexity of $O(N \log_2 N)$