Input W.S.S. random process to a LTI system

- We have seen that a random process is actually a collection of signals, instead of a single or unique signal.
- To apply a random process as input to a LTI system, we mean that each signal in this collection serves as input, and we obtain a collection of output signals.
- We want to characterize the output collection of signals. What are their ensemble properties?

Mean of the output process

- Consider a linear system with the impulse response $h[n]$.
- If $x[n]$ is a stationary random signal with mean m_x, then the output $y[n]$ is also a stationary random signal with mean m_y equaling to

$$m_y[n] = \varepsilon\{y[n]\} = \sum_{k=-\infty}^{\infty} h[k]\varepsilon\{x[n-k]\} = \sum_{k=-\infty}^{\infty} h[k]m_x[n-k]$$

- Since the input is stationary, $m_x[n-k] = m_x$, and consequently,

$$\boxed{m_y = m_x \sum_{k=-\infty}^{\infty} h[k] = H\left(e^{j0}\right)m_x}$$

Stationary and LTI System

- If $x[n]$ is a real and stationary random signal, the autocorrelation sequence of the output process is

$$\phi_{yy}[n, n+m] = \varepsilon\{y[n]y[n+m]\}$$

$$= \varepsilon\left\{\sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} h[k]h[r]x[n-k]x[n+m-r]\right\}$$

$$= \sum_{k=-\infty}^{\infty} h[k]\sum_{r=-\infty}^{\infty} h[r]\varepsilon\{x[n-k]x[n+m-r]\}$$

- Since $x[n]$ is stationary, $\varepsilon\{x[n-k]x[n+m-r]\}$ depends only on the time difference $m+k-r$.

Stationary and LTI System (continue)

- Therefore, $$\varphi_{yy}[n, n+m]$$
$$= \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r] \varphi_{xx}[m+k-r]$$
$$\equiv \varphi_{yy}[m]$$

Hence, **the autocorrelation of the output signal is also stationary**.

- Generally, for a LTI system having a **wide-sense stationary input**, the output is **also wide-sense stationary**.

Power Density Spectrum and LTI System

- Furthermore, by substituting $l = r - k$ in the above,

$$\varphi_{yy}[m] = \sum_{l=-\infty}^{\infty} \varphi_{xx}[m-l] \sum_{k=-\infty}^{\infty} h[k]h[l+k]$$

$$= \sum_{l=-\infty}^{\infty} \varphi_{xx}[m-l] c_{hh}(l)$$

where

$$\boxed{c_{hh}[l] = \sum_{k=-\infty}^{\infty} h[k]h[l+k]}$$

i.e., $c_{hh}[l]$ **is defined as the autocorrelation of the impulse response**

- $c_{hh}[l]$ is called a deterministic autocorrelation sequence of the system.

Power Density Spectrum and LTI System

- Hence, $$\boxed{\varphi_{yy}[m] = \sum_{l=-\infty}^{\infty} \varphi_{xx}[m-l] c_{hh}(l)}$$
- That is, the autocorrelation sequence of the output random signal is the **convolution** of $c_{hh}[l]$ and the autocorrelation sequence of the input random signal.
- So, in the **DTFT domain**,

$$\boxed{\Phi_{yy}\left(e^{jw}\right) = C_{hh}\left(e^{jw}\right)\Phi_{xx}\left(e^{jw}\right)}$$

where $C_{hh}(e^{jw})$ is defined as the DTFT of $c_{hh}[l]$.

What is on earth $C_{hh}\left(e^{jw}\right)$?

- For real $c_{hh}[l]$,

$$c_{hh}[l] = h[l] * h[-l]$$
$$C_{hh}\left(e^{jw}\right) = H\left(e^{jw}\right)H^*\left(e^{jw}\right)$$

Cross correlation of $a[n]$ and $b[n]$ is the convolution of $a[n]$ and $b[-n]$

- So

$$\boxed{C_{hh}\left(e^{jw}\right) = \left|H\left(e^{jw}\right)\right|^2}$$

$C_{hh}(e^{jw})$ is equal to the magnitude square of the frequency response

Power Density Spectrum and LTI System (continue)

- We then have **the relation of the input and the output power spectra** (in terms of autocorrelation) as follows:

$$\boxed{\Phi_{yy}\left(e^{jw}\right) = \left|H\left(e^{jw}\right)\right|^2 \Phi_{xx}\left(e^{jw}\right)}$$

Power Density Spectrum and LTI System (continue)

- **In sum**: when input a W.S.S. random process with the autocorrelation $\varphi_{xx}[n]$ to an LTI system of impulse response $h[n]$:

$$\boxed{\varphi_{yy}[m] = \sum_{l=-\infty}^{\infty} \varphi_{xx}[m-l]\, c_{hh}(l)}$$

Time domain response

$$\boxed{\Phi_{yy}\left(e^{jw}\right) = \left|H\left(e^{jw}\right)\right|^2 \Phi_{xx}\left(e^{jw}\right)}$$

Frequency domain response

Power Density Property

- We have seen that $P_{xx}(\omega) = \Phi_{xx}(e^{jw})$ can be viewed as "density."

- **Property**: The area over a band of frequencies, $w_a < |w| < w_b$, is proportional to the power of the signal in that band.
- To understand this, consider an ideal band-pass filter. Let $H_{bp}(e^{jw})$ be the frequency response of the ideal band pass filter for the band $w_a < |w| < w_b$.
- Also, note that **ideal band pass filter is an LTI system**.

$$H_{bp}\left(e^{jw}\right) = \begin{cases} 1 & w_a < |w| < w_b \\ 0 & \text{otherwise} \end{cases}$$

Power Density Property

- Consider the power of the output random signal y when the ideal band-pass filter is applied:

$$\varphi_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{yy}(e^{jw})\, dw$$

$$= \frac{1}{2\pi}\int_{-w_b}^{-w_a} \Phi_{xx}\left(e^{jw}\right) dw + \frac{1}{2\pi}\int_{w_a}^{w_b} \Phi_{xx}\left(e^{jw}\right) dw$$

is just equivalent to the power of the random signal x in the band w_a<$|w|$<w_b. (up to a scale factor $1/2\pi$)

- When w_a and w_b are getting closer so that $w_a \approx w_b$, we can image that the power degenerates to the "power density."

Review of white noise

- The power spectral density (i.e., DTFT of the autocorrelation) of a white noise is a **constant**

$$\Phi_{xx}\left(e^{j\omega}\right) = \sigma_x^2, \quad \text{for all } \omega$$

Modeling random signals by white signal

- White noise is useful in the representation of random signals whose power spectra are not constant in the frequency domain.
  - A stationary random signal $y[n]$ with the power spectral density $\phi_{yy}(e^{jw})$ below can be modeled as the output of an LTI system with a white-noise input.

$$\Phi_{yy}\left(e^{jw}\right)=\left|H\left(e^{jw}\right)\right|^2\sigma_x^2$$

    - That is, we can model a "colored signal" source as the output of an LTI system of the white signal input.

Time Averages

- For any single sample sequence $x[n]$, define their time average to be

$$\langle x[n] \rangle = \lim_{L \to \infty} \frac{1}{2L+1} \sum_{n=-L}^{L} x[n]$$

Defined by averaging all time indices for an arbitrary instance of the random process

- Similarly, time-average autocorrelation is

$$\left\langle x[n+m] x[n]^* \right\rangle = \lim_{L \to \infty} \frac{1}{2L+1} \sum_{n=-L}^{L} x[n+m] x^*[n]$$

Ergodic Process

- Note that the above time average is defined for a deterministic signal sampled from the random process.
- A stationary random process for which time averages equal ensemble averages is called an **ergodic process**:

$$\langle x[n] \rangle = m_x$$

$$\left\langle x[n+m]x[n]^* \right\rangle = \phi_{xx}[m]$$

Ergodic Process (continue)

- It is common to assume that a given sequence is a sample sequence of an ergodic random process, so that averages can be computed from only a single sequence.

■ In practice, we cannot compute with the limits, but instead the finite-sum quantities for approximation

$$\hat{m}_x = \frac{1}{L}\sum_{n=0}^{L-1} x[n]$$

$$\sigma_x^2 = \frac{1}{L}\sum_{n=0}^{L-1} \left(x[n] - \hat{m}_x\right)^2$$

$$\left\langle x[n+m]x^*[n]\right\rangle_L = \frac{1}{L}\sum_{n=0}^{L-1} x[n+m]x^*[n]$$

Power Spectral Density Estimation from Deterministic Signal

- When the ergodic property is available, we can realize more nature about the power density spectrum.

- Suppose we sample a deterministic signal y from the random process x.
- Remember the autocorrelation sequence defined for a deterministic signal is

$$r_{yy}[l] = \sum_{n=-\infty}^{\infty} y[n]\, y[n-l]$$

Power Spectral Density Estimation from Deterministic Signal

Applying the ergodic property:

- By the ergodic property, we can use the autocorrelation of an arbitrary signal y sampled from the signal source (random process x) to estimate the autocorrelation of the random process x.
- So, we can also use the DTFT of y's autocorrelation sequence, $r_{yy}[l]$ to estimate $P_{xx}(w)$, where $P_{xx}(w)$ is the DTFT of the autocorrelation of the random process x.
- Remember that the DTFT of $r_{yy}[l]$ is the squared magnitude of the DTFT of y.

$$DTFT(r_{yy}) = \left|Y(e^{jw})\right|^2$$

Power Spectral Density and Squared Magnitude of DTFT

- Hence, the power spectral density $P_{xx}(w)$ of a random process x is equal to the squared magnitude spectrum of any of its instance y when the ergodic assumption is hold.
- So, we can use a sample sequence (or a set of sample sequences) to estimate the power spectrum of a random signal.
- Computing the DTFT magnitude square of the sample sequence(s) then estimates the power spectrum.

Power Spectral Density and Squared Magnitude of DTFT

- Like the deterministic case, we cannot perform integration in [-∞,∞], and so can use only a finite range $[-T, T]$ instead. Window functions (such as Hamming, Kaiser) are also used.

- Thus, the power spectrum estimation process is the same as the spectrogram estimation process of a deterministic signal.

Power Spectral Density and Squared Magnitude of DTFT

- Note that the power spectral density is the squared magnitude frequency response.
- power spectral density does not contain phase information (phase is zero) and is always real and positive.