Multiplication in the frequency domain

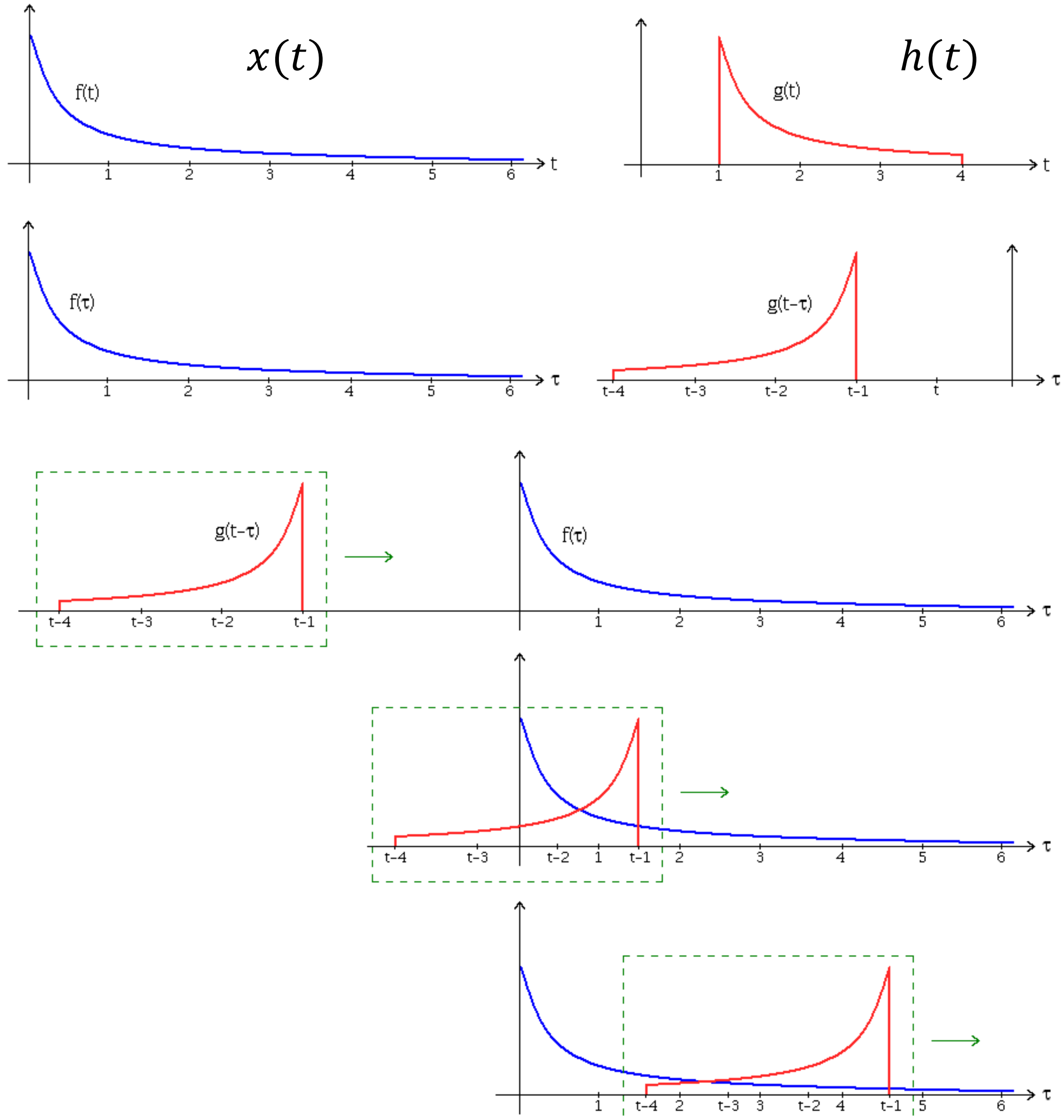
- **When multiplying two signals in the frequency domain, what will be obtained in the time domain?**
- **Convolution**: a moving average operation:
- Convolution of two continuous-time signals $x(t)$ and $h(t)$ is defined as

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

- It can be written in short by

$$y(t) = x(t) * h(t)$$

Illustration example of convolution



Convolution

- Convolution is commutative:

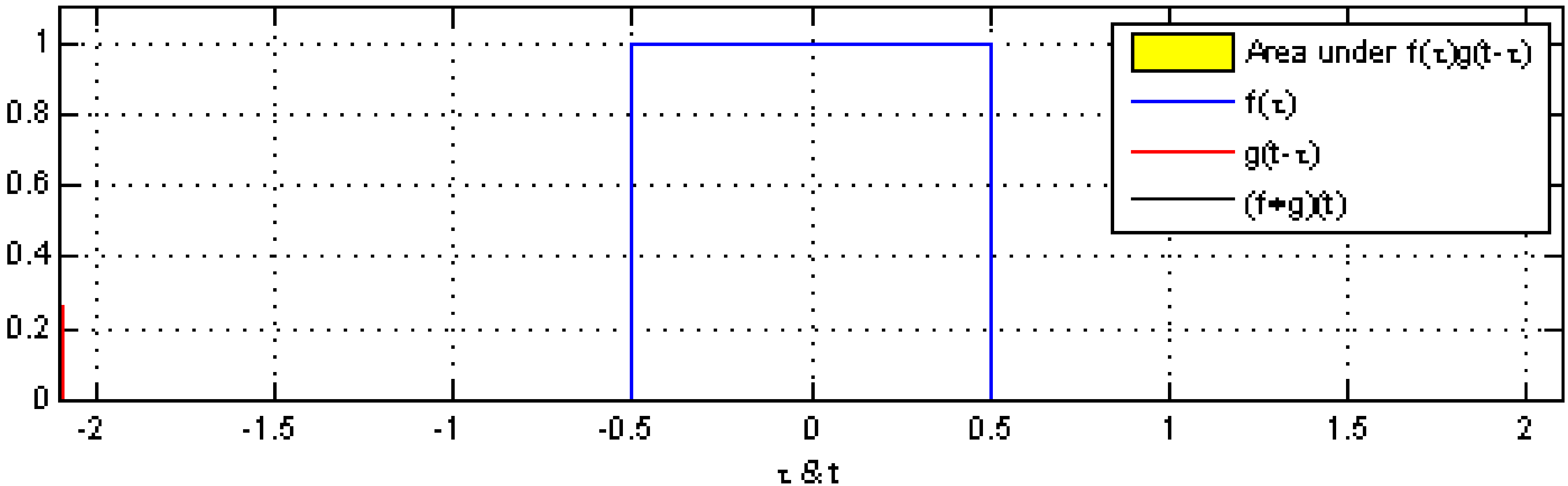
$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- Let $s = t - \tau$, then we have

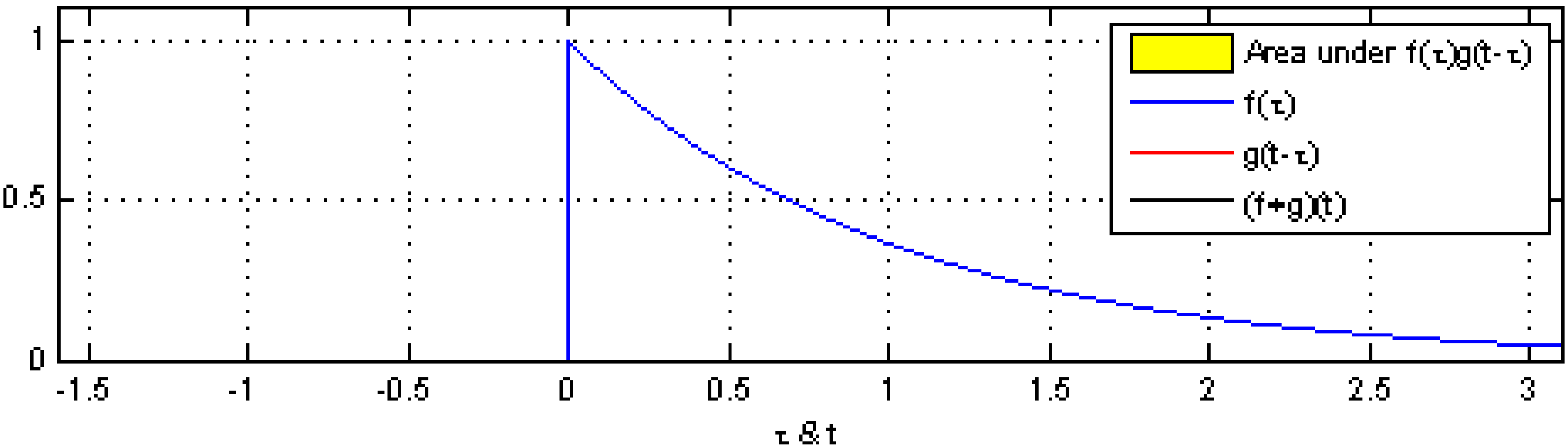
$$x(t) * h(t) = \int_{\infty}^{-\infty} h(s)x(t-s)ds$$

$$= h(t) * x(t)$$

Animation example of convolution
when $h(t)$ is even



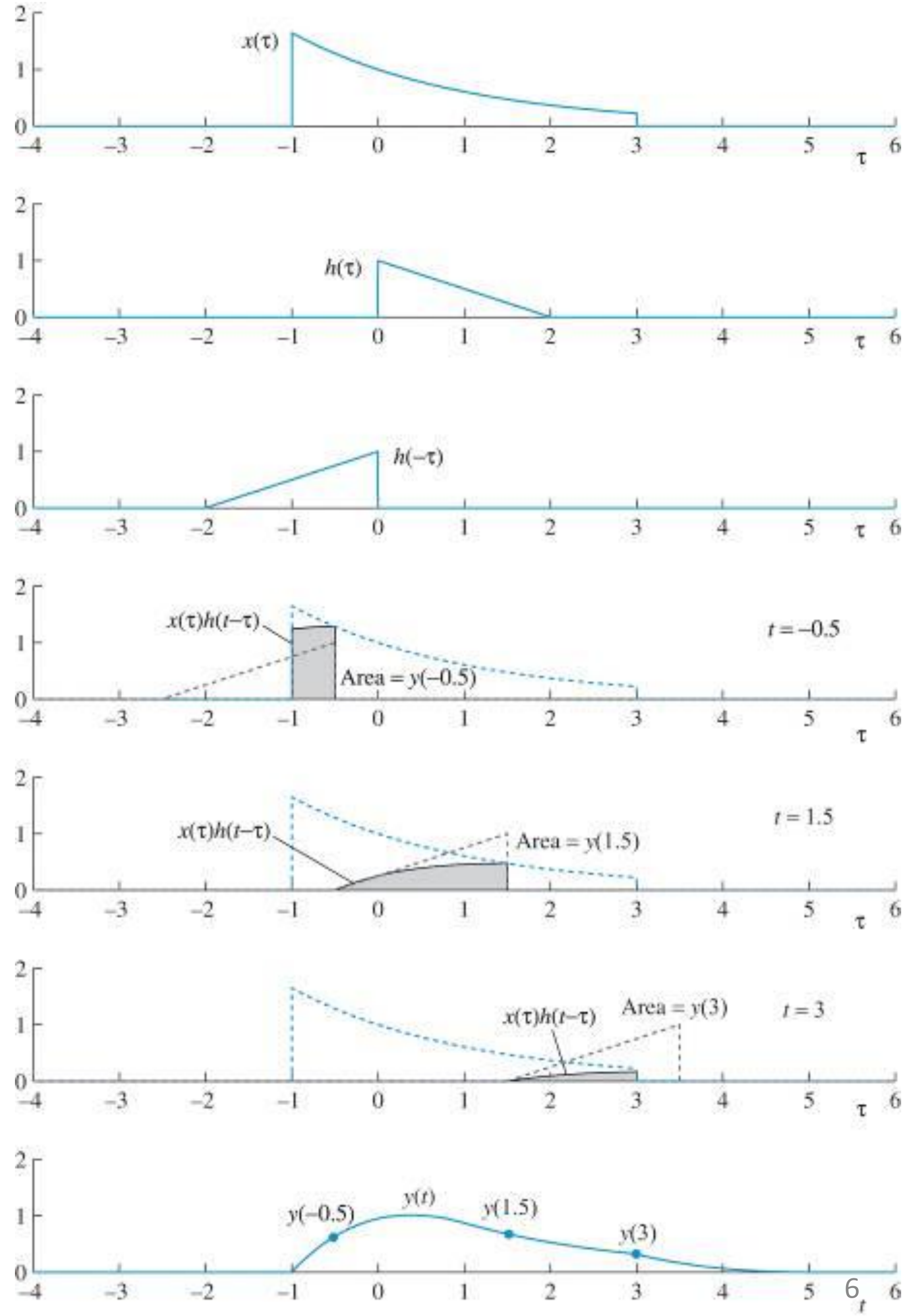
Animation example of convolution
when $h(t)$ is even



- Illustration of convolution

$$x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$



Convolution in time domain

Time-Domain	***Frequency-Domain***

$$y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{F}} Y(j\omega) = X(j\omega)H(j\omega)$$

Convolution in the time-domain corresponds to multiplication in the frequency-domain.

- Derivation

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \right) e^{-j\omega t}dt$$

Convolution Property Derivation

- Interchange the order of integrals

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) \underbrace{\left(\int_{-\infty}^{\infty} h(t-\tau) e^{-j\omega t} dt \right)} d\tau$$

- Let $\sigma = t - \tau$

$$\left(\int_{-\infty}^{\infty} h(t-\tau) e^{-j\omega t} dt \right) = \left(\int_{-\infty}^{\infty} h(\sigma) e^{-j\omega\sigma} d\sigma \right) e^{-j\omega\tau}$$

$$= H(j\omega) e^{-j\omega\tau} \qquad (11.65)$$

Convolution Property Derivation

- By substitution back,

$$
\begin{aligned}
Y(j\omega) &= \int_{-\infty}^{\infty} x(\tau)\left(H(j\omega)e^{-j\omega\tau}\right)d\tau \\
&= H(j\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \\
&= H(j\omega)X(j\omega)
\end{aligned}
$$

- **Convolution property is one of the most important properties in Fourier transform.**

Convolution Property Concept

- Due to the duality of frequency and time domains, we also have the property that multiplication in the time domain corresponds to the convolution in the frequency domain:

$$\mathcal{F}\{(f * g)(t)\} = \mathcal{F}\{f(t)\} \cdot \mathcal{F}\{g(t)\}$$

$$\mathcal{F}\{f(t) \cdot g(t)\} = \mathcal{F}\{f(t)\} * \mathcal{F}\{g(t)\}$$

Convolution in the time domain is equivalent to (complex) scalar multiplication in the frequency domain.

Convolution in the frequency domain corresponds to scalar multiplication in the time domain.

The above is only a conceptual property. When using the radian (ω), for different definitions of Continuous Fourier transform, the convolution property is a slight different according to how to decompose the scaling $1/2\pi$.

Convolution Property

- **For the continuous-time Fourier transform pair defined below,**

Forward

$$\boxed{F(jw) = \int_{-\infty}^{\infty} f(t) e^{-jwt}\, dt}$$

Backward

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(jw) e^{jwt}\, dw}$$

- The convolution property is

Time domain	**Frequency domain**
$x(t) * h(t)$	$X(j\omega)H(j\omega)$
$x(t)p(t)$	$\frac{1}{2\pi} X(j\omega) * P(j\omega)$

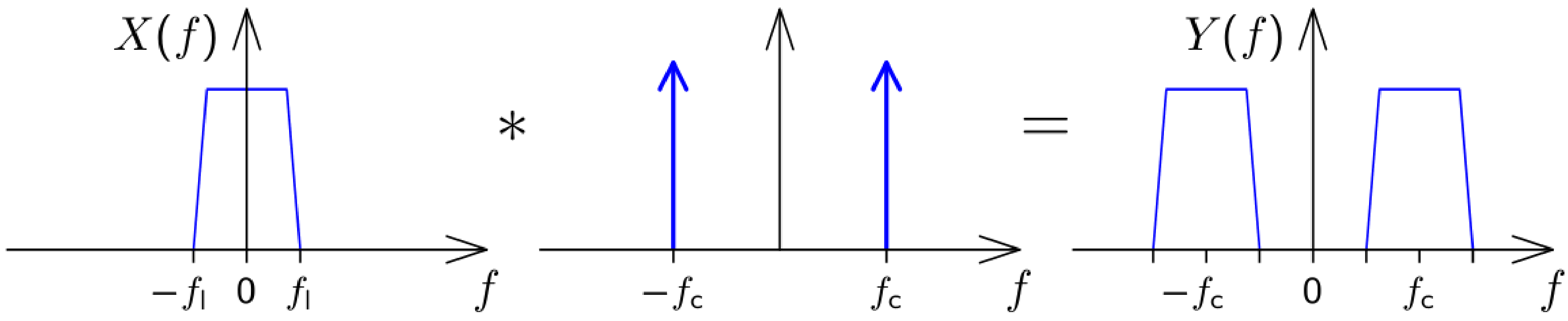
There is a scaling factor

Example: AM

Amplitude modulation (AM):

Time domain multiplication

$$y(t) = A \cdot \cos(2\pi t f_c) \cdot x(t)$$



Frequency domain convolution

Basic Fourier Transform Properties

Table of Fourier Transform Properties		
Property Name	***Time-Domain:*** $x(t)$	***Frequency-Domain:*** $X(j\omega)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(j\omega) + bX_2(j\omega)$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time-Reversal	$x(-t)$	$X(-j\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X(j(\omega/a))$
Delay	$x(t - t_d)$	$e^{-j\omega t_d} X(j\omega)$
Modulation	$x(t)e^{j\omega_0 t}$	$X(j(\omega - \omega_0))$

Basic Fourier Transform Properties (cont.)

Modulation	$x(t)\cos(\omega_0 t)$	$\frac{1}{2}X(j(\omega-\omega_0))+\frac{1}{2}X(j(\omega+\omega_0))$
Differentiation	$\frac{d^k x(t)}{dt^k}$	$(j\omega)^k X(j\omega)$
Convolution	$x(t) * h(t)$	$X(j\omega)H(j\omega)$
Multiplication	$x(t)p(t)$	$\frac{1}{2\pi}X(j\omega) * P(j\omega)$