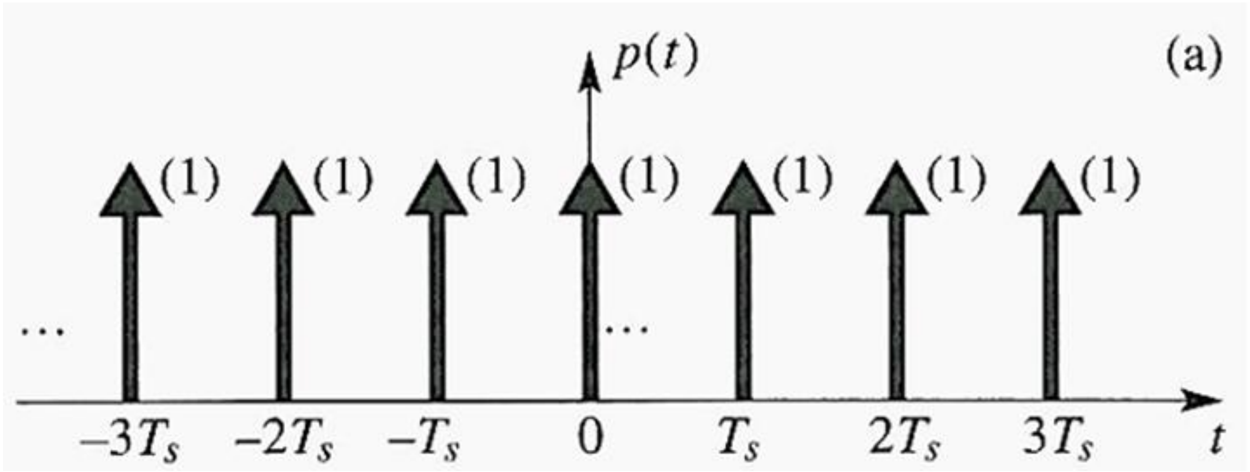
Impulse Train



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

- What is the continuous Fourier Transform of an impulse train $p(t)$?

Impulse Train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

- Derivation: because $p(t)$ is a periodic signal, it can be expressed by Fourier series:

$$p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

where $\omega_s = 2\pi / T_s$.

Impulse Train

- What is a_k in this case?
- To determine the coefficients a_k of Fourier series, we evaluate the Fourier series integral over one period $[-T_s/2,\ \ T_s/2]$.

Impulse Train

- Since $p(t)$ is the impulse training with period T_s, there is only a single delta function, $\delta(t)$, within the period $[-T_s/2, \;\; T_s/2]$.
- Hence,

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-jk\omega_s t} dt$$

Impulse Train

- Remember that we have the property about the inner product of an impulse function and an arbitrary signal:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,\mathrm{d}x \;=\; f(a)$$

- Hence, when $a = 0$ and $f(x) = e^{-jk\omega_s t}$

$$a_k = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)e^{-jk\omega_s t}dt \quad = \frac{1}{T_s}\int_{-\infty}^{\infty} \delta(t)e^{-jkw_s t}dt$$

$$= \frac{1}{T_s}e^{-j0} \quad = \frac{1}{T_s}$$

Impulse Train

- Remember that the continuous Fourier transform of a periodic signal is discrete (i.e., sum of delta functions) centered at integer multiples of ω_s, where a_k are the Fourier series coefficients:

$$P(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_s)$$

- Hence, the Fourier transform of the impulse train $p(t)$ is another **impulse train**

$$P(j\omega) = \sum_{k=-\infty}^{\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - k\omega_s)$$

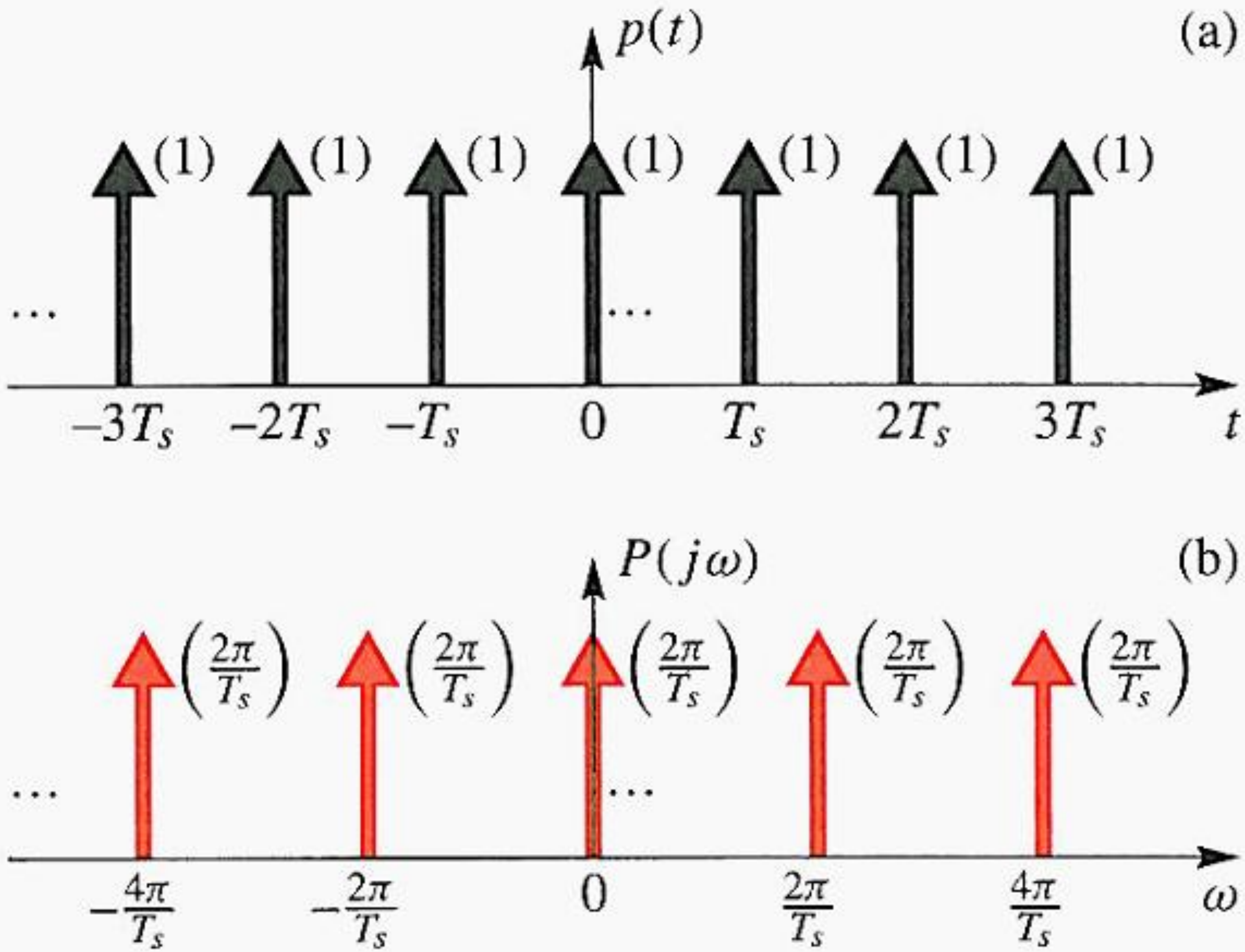


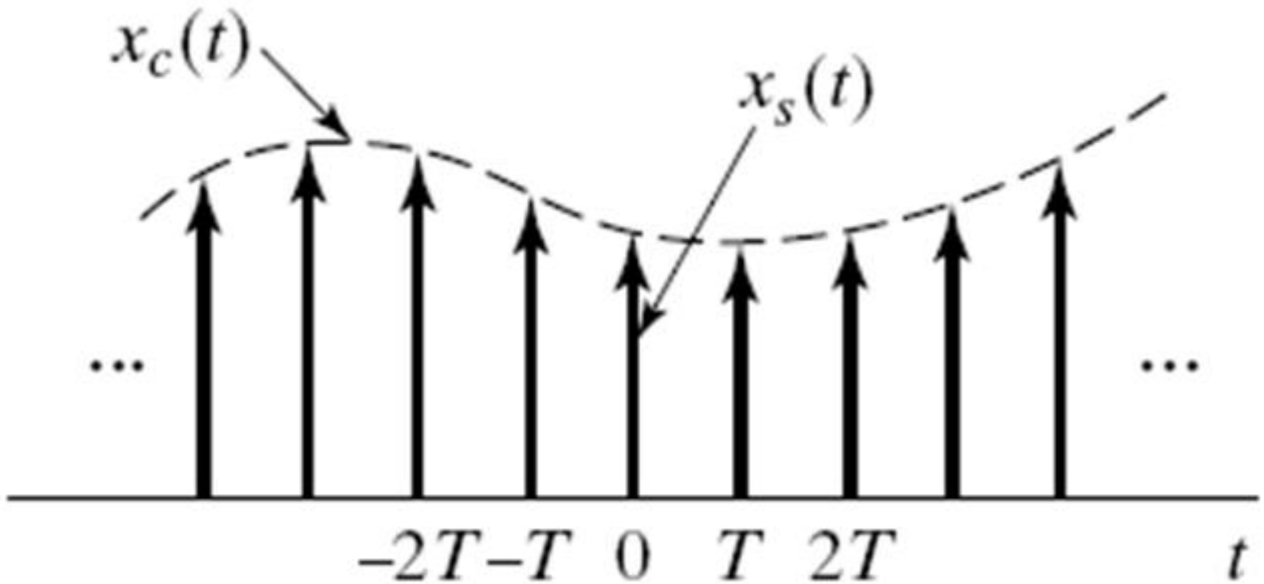
Figure 11-10: Periodic impulse train: (a) Time-domain signal $p(t)$; and (b) Fourier transform $P(j\omega)$. Regular spacing in the frequency-domain is $\omega_s = 2\pi/T_s$

Discrete-time Signals in Continuous Domain

- Let us consider discrete-time signals hereafter.
- How to represent a discrete-time signal in the time domain for continuous Fourier transform?
- A discrete-time signal can be represented as a sequence of impulse functions (a.k.a., an **impulse sequence**) occurred at equally spaced time instances, in the continuous-functional domain.

Discrete-time Signals

- A common way to obtain a discrete-time signal is to sample a continuous-time signal at equally spaced time instances.
- This is referred to as uniform sampling:



Impulse Sequence

Continuous Fourier Transform of an Impulse Sequence

When the frequency domain spectrum is an equally-spaced impulse sequence, what is the time domain signal?

Analogically, as time and frequency are dual in CFT,

- The time-domain signal is a **periodic function**.

Continuous Fourier Transform of an Impulse Sequence

Note that

- Continuous Fourier transform is time-frequency analogous.

Forward

$$F(jw) = \int_{-\infty}^{\infty} f(t) e^{-jwt} dt$$

Backward

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(jw) e^{jwt} dw$$

Continuous Fourier Transform of an Impulse Sequence

- If we (purposively) perform the forward transform for the complex conjugate of the frequency domain spectrum $F^*(j\omega)$ and exchange the roles of ω and t, we have the following symmetry property of the Fourier transform:

$$\boxed{\int_{-\infty}^{\infty} F^*(jw)e^{-jwt}dw = \left(\int_{-\infty}^{\infty} F(jw)e^{jwt}dw\right)^* = 2\pi f^*(t)}$$

Forward transform of $F^*(j\omega)$

the conjugate of $f(t)$

Continuous Fourier Transform of an Impulse Sequence

- By exchanging the variables w and t, we have

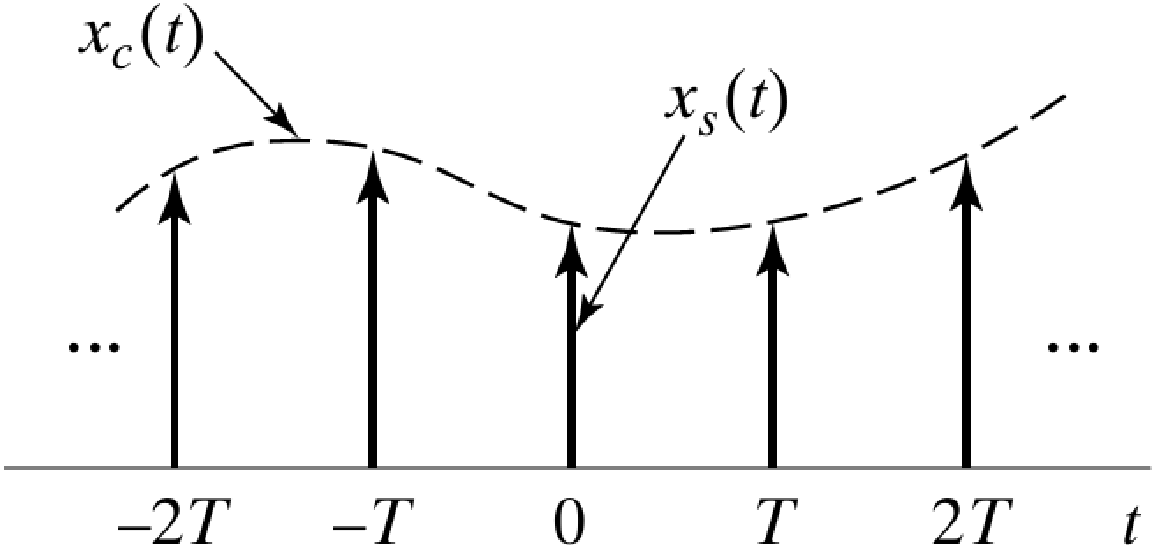
$$\boxed{\int_{-\infty}^{\infty} F^*(jt)e^{-jtw}dt = 2\pi f^*(w)}$$

- That is, when $F^*(jt)$ is a time-domain signal, its frequency domain spectrum is $2\pi f^*(\omega)$.

Continuous Fourier Transform of an Impulse Sequence

- Taking the complex conjugate of $f(t)$ or $F(j\omega)$ does not affect their property of being a periodic function or an impulse sequence.
- So, when the time domain signal is an equally-spaced impulse sequence, the frequency domain spectrum is a **periodic function.**

Sampling a Continuous Function



- **What is the relationship of the continuous-time signal and the sampled signal in the frequency domain?**

Sampling Theorem

- Given an analog (i.e., continuous-time) signal $x_a(t)$
- Let us assume that a discrete-time signal $x_a(nT)$ (n is an integer) is uniformly sampled from the analog signal $x_a(t)$ with the time step T.

Sampled Signals

- **How to represent the sampled signal?**
- The sampled signal can be represented as the **multiplication** of the continuous-time signal $x_a(t)$ and the **impulse train** signal

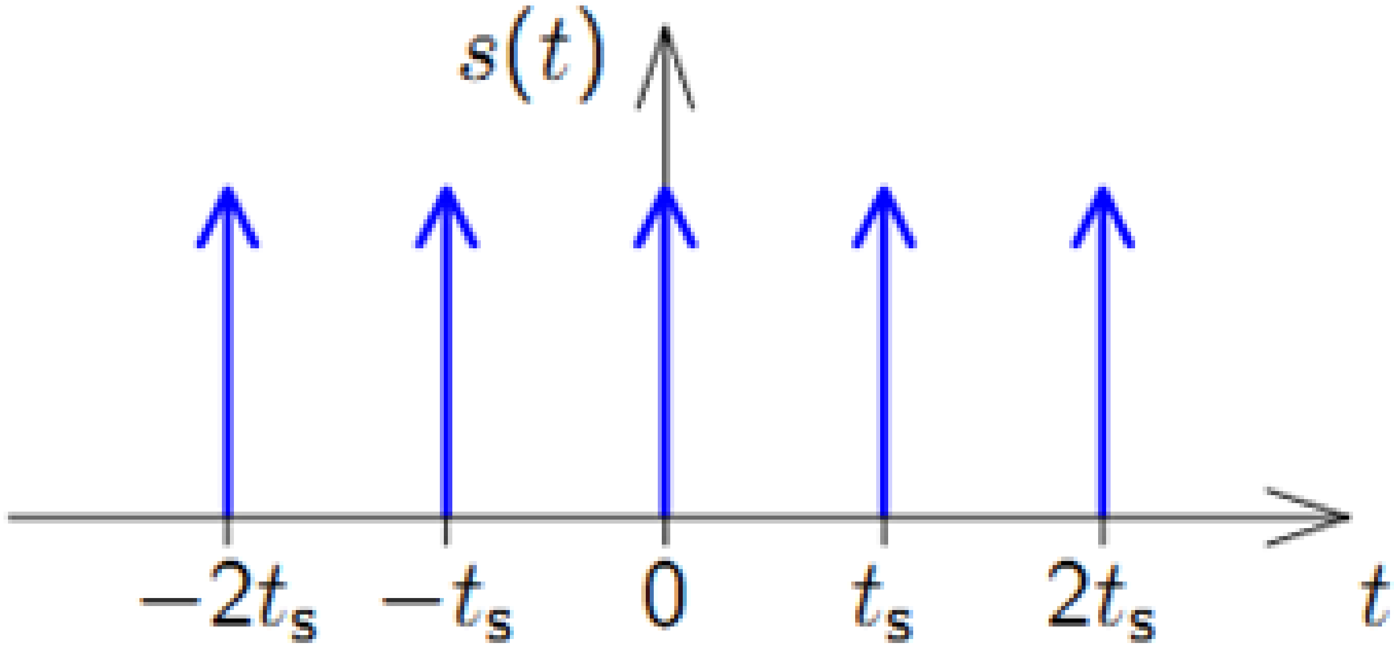
$$s(t) = \sum_{-\infty}^{\infty} \delta(t - nT)$$

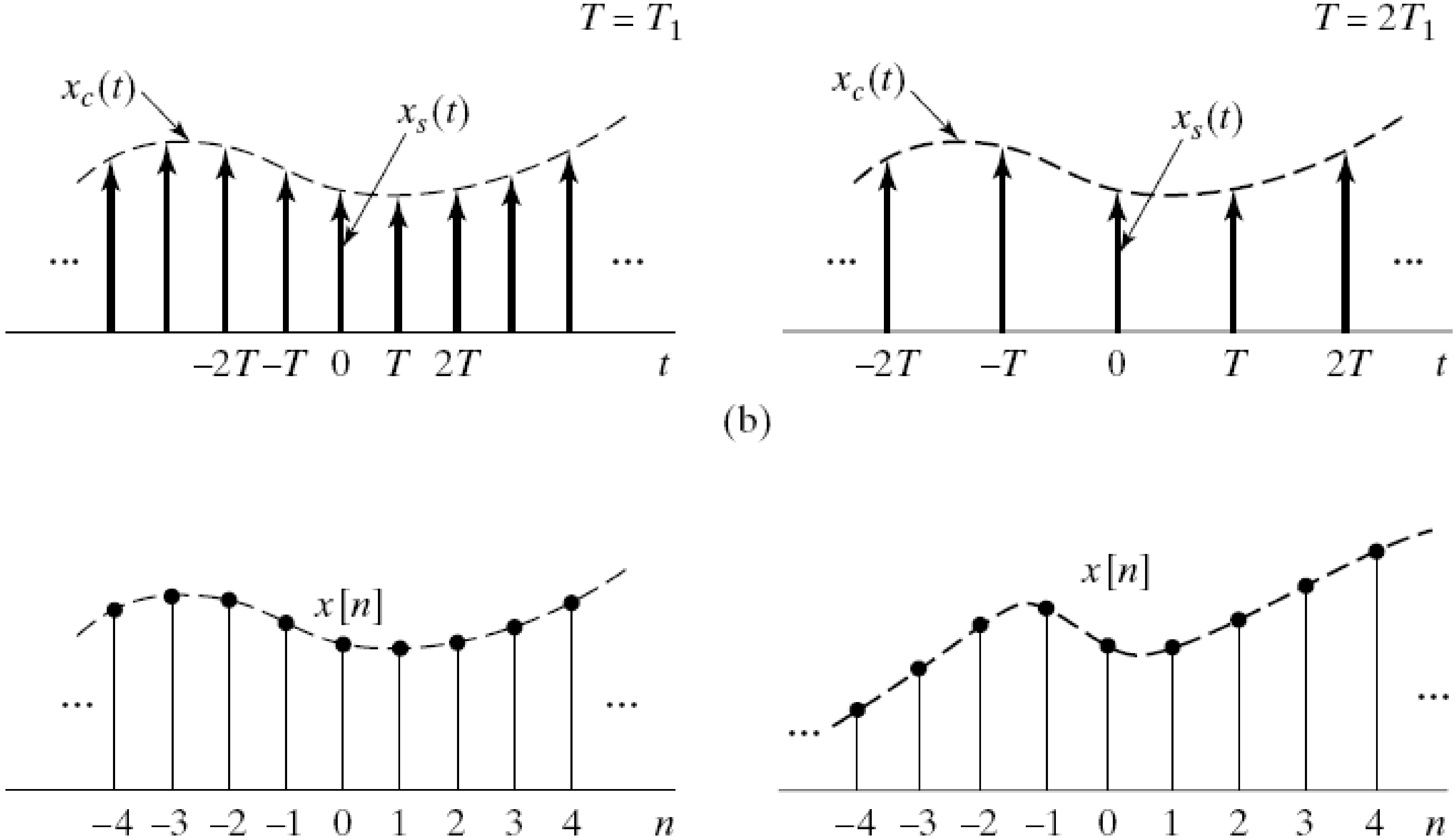
- That is

$$x_s(t) = x_a(t)\, s(t) = x_a(t) \sum_{-\infty}^{\infty} \delta(t - nT)$$

Recall of impulse train signal

- Impulse train: an impulse sequence with an equal height 1.





(b)

Examples of $x_s(t)$ for two sampling rates

Using Convolution

- Let us now consider the continuous Fourier transform of $x_s(t)$.
- Since $x_s(t)$ is a product of $x_a(t)$ and $s(t)$, its frequency (or spectrum) domain corresponds to the **convolution** of $X_a(jw)$ and $S(jw)$ (divided by the scaling factor 2π),

Remark:

	Time domain	**Frequency domain**
Multiplication	$x(t)p(t)$	$\frac{1}{2\pi}X(j\omega) * P(j\omega)$

That is, from $x_s(t) = x_a(t) \sum_{-\infty}^{\infty} \delta(t - nT)$

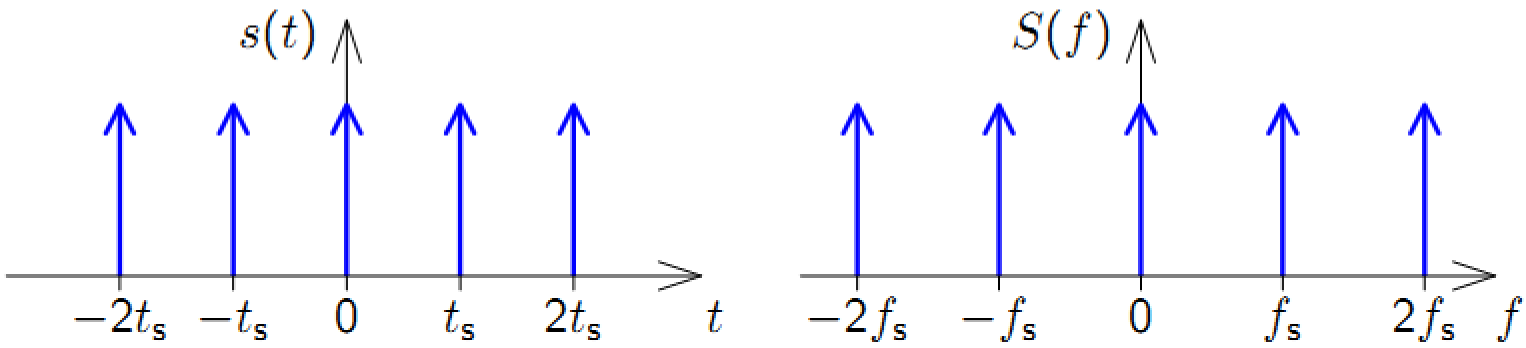
we have $X_s(jw) = \dfrac{1}{2\pi} X_a(jw) * S(jw)$

Hence, it follows that in the frequency domain

$$\boxed{X_s(jw) = \frac{1}{2\pi} X_a(jw) * S(jw)}$$

Recall: the continuous Fourier transform of a periodic impulse train $s(t)$ is still a periodic impulse train with the period $w_s = 2\pi/T$,

$$S(jw) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(w - kw_s)$$



$T = t_s$ in this figure.

where $w_s = 2\pi/T$ (or $f_s = 1/T$) is the sampling frequency in radians/s.

Property

- Convolution of a signal $f(t)$ with a delta function $\delta(t-a)$ will **shift** that signal to the location a.

$$
\begin{aligned}
&f(t) * \delta(t-a) \\
&= \int_{-\infty}^{\infty} f(\tau)\delta((t-a)-\tau)d\tau \\
&= \int_{-\infty}^{\infty} f(\tau)\delta(\tau-(t-a))d\tau \\
&= f(t-a)
\end{aligned}
$$

$$\frac{1}{2\pi} X_a(jw) * S(jw)$$

$$= \frac{1}{2\pi} X_a(j\omega) * \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(j\omega) * \delta(\omega - kw_s)$$

$$\boxed{= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(j(\omega - k\omega_s))}$$

Shifting $X_a(j\omega)$ to the locations $k\omega_s, k \in Z$

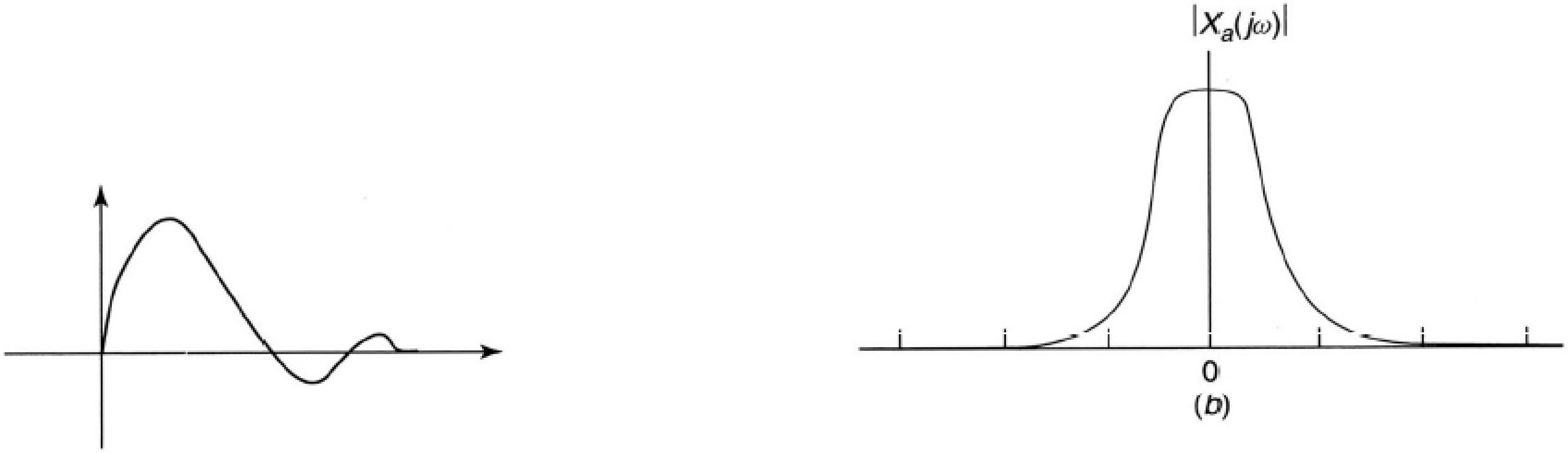
Hence, we see that the copies of $X_a(j\omega)$ are shifted by integer multiples of the sampling frequency, and then added to produce a periodic function in the Fourier transform domain.

$$\boxed{X_s(jw) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(j\left(w - kw_s\right)\right)}$$

Uniformly Sampling a Continuous Function

$$X_s(jw) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(j\left(w - kw_s\right)\right)$$

- **What does it mean?**
- Recall that $X_a(j\omega)$ is the spectrum of the analog signal $x_a(t)$
- $X_s(j\omega)$ is the spectrum of the uniformly sampled signal $x_s(t)$
- The equation means that $X_s(j\omega)$ is a periodic duplication of the continuous Fourier transform $X_a(j\omega)$ a period $2\pi/T = \omega_s$ and scaled by T.



An analog signal $x_a(t)$ and the magnitude of its Fourier transform $X(j\omega)$.

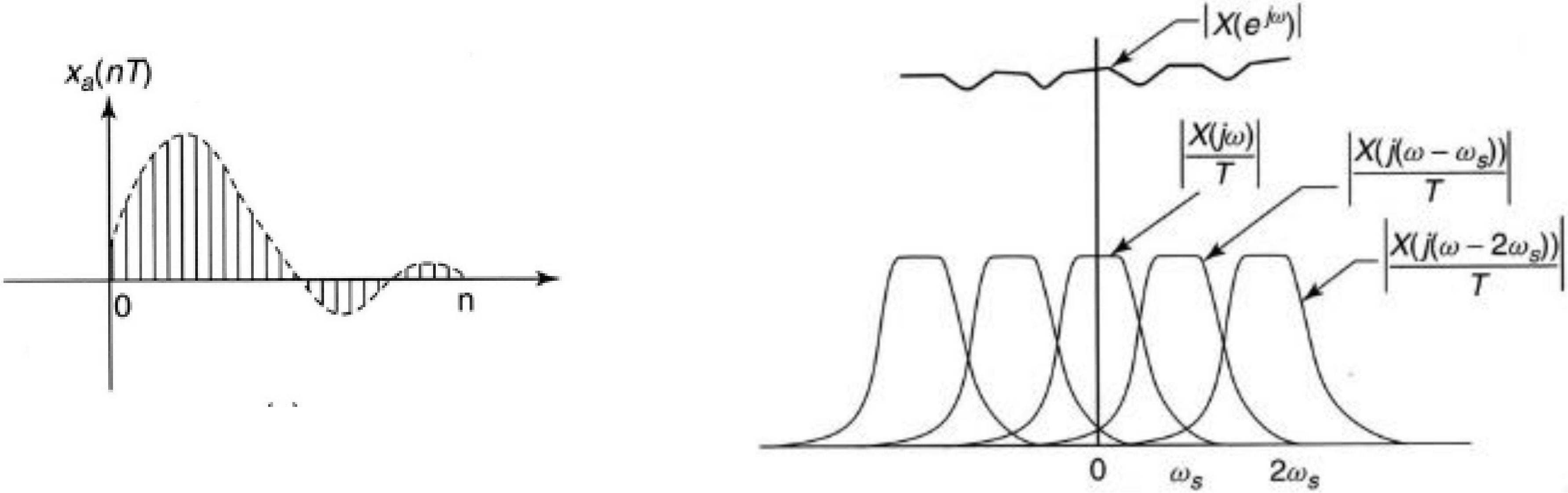
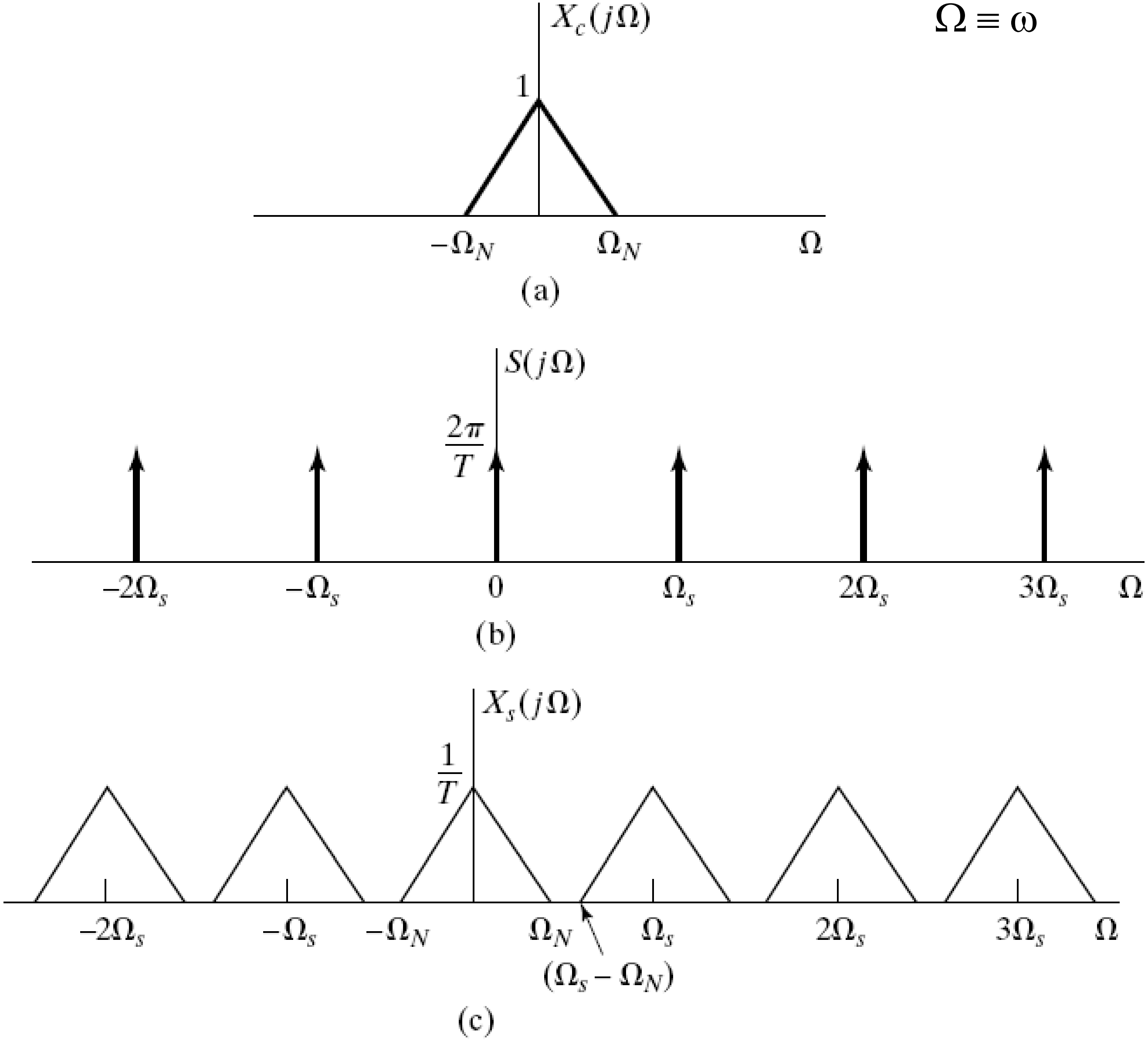


Figure 3.2 The discrete-time signal $x_a(nT)$ obtained from the analog signal $x_a(t)$ and the discrete-time Fourier transform $H(e^{j\omega})$.

Illustration: Frequency domain convolution

$\Omega \equiv \omega$

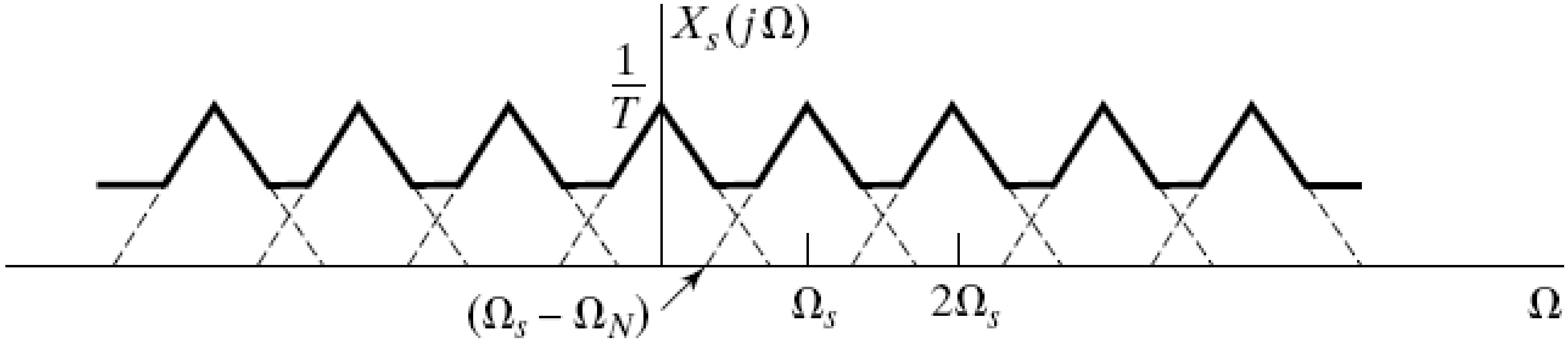


This is a case of non-overlapping

Illustration: Frequency domain convolution

The above shows the case without overlapping.

When Ω_N is getting larger, there will be overlapping between the duplicates of $X_c(j\Omega)$



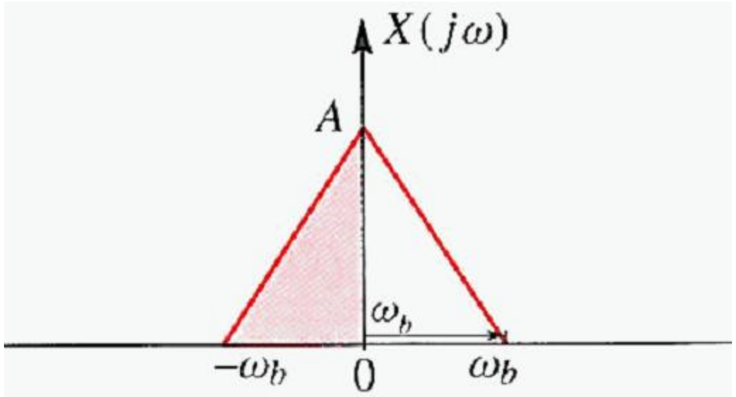
(d)

Aliasing Effect

- Because of the overlapping effect, more commonly known as **aliasing**, there is no way of retrieving $X_a(j\omega)$ from $X_s(j\omega)$
- In other words, we have lost the information contained in the analog function $x_a(t)$ if the aliasing occurs when performing uniform sampling on it.

Aliasing Effect

- **How to avoid aliasing?**
- **Band-limited** signal: the continuous signal $x_a(t)$ is band limited if its Fourier transform $X_a(j\omega) = 0$ for $|\omega| > \omega_b$, where ω_b is a frequency bound.



- For a signal that is not band-limited, there is no chance to avoid aliasing.

Illustration for Avoid Aliasing

- A case without aliasing:

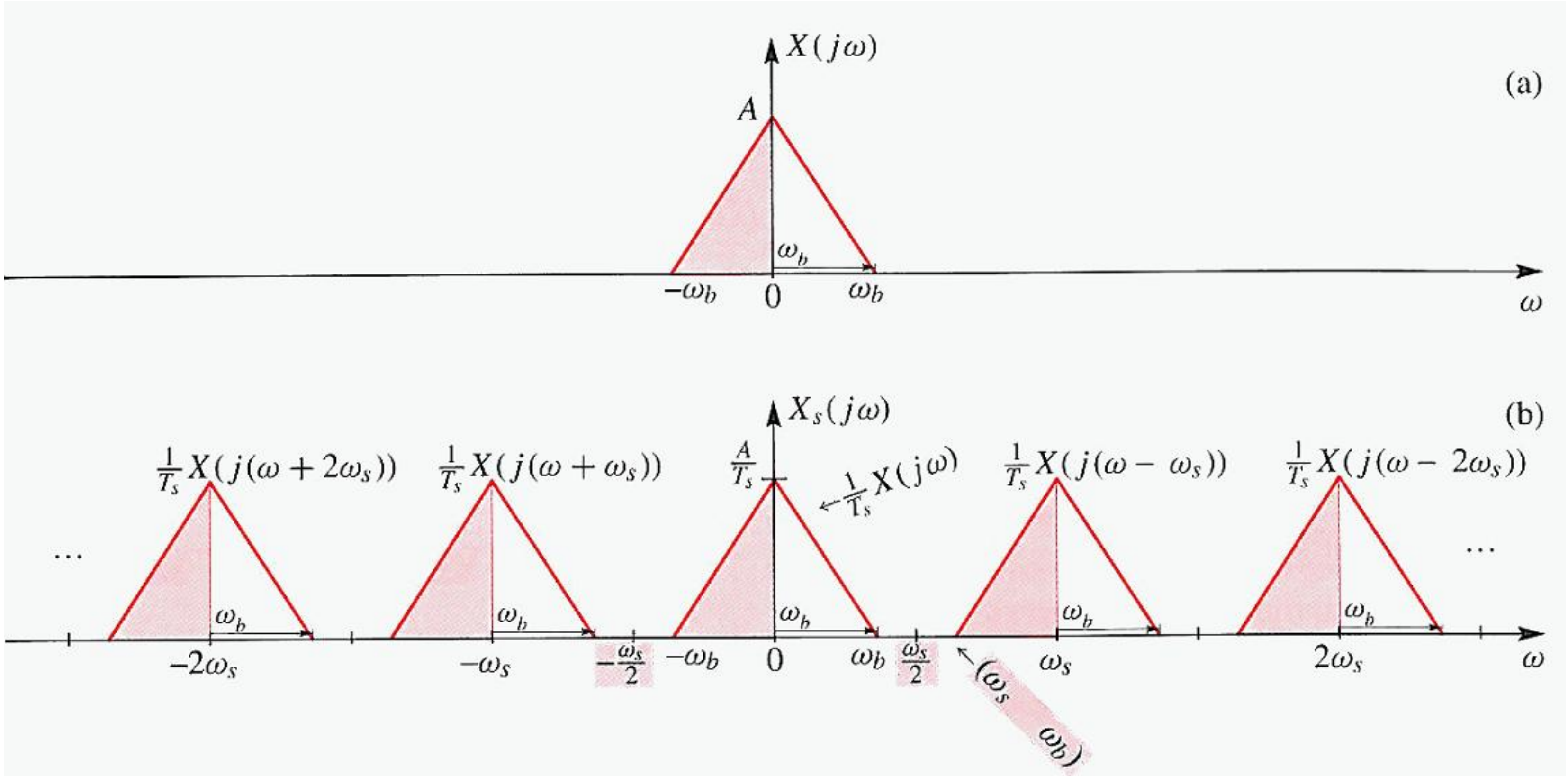
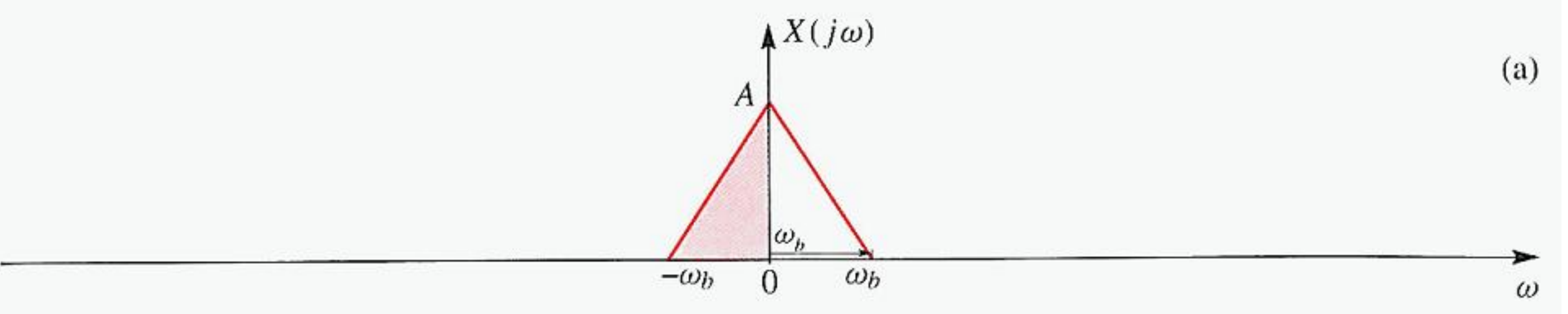
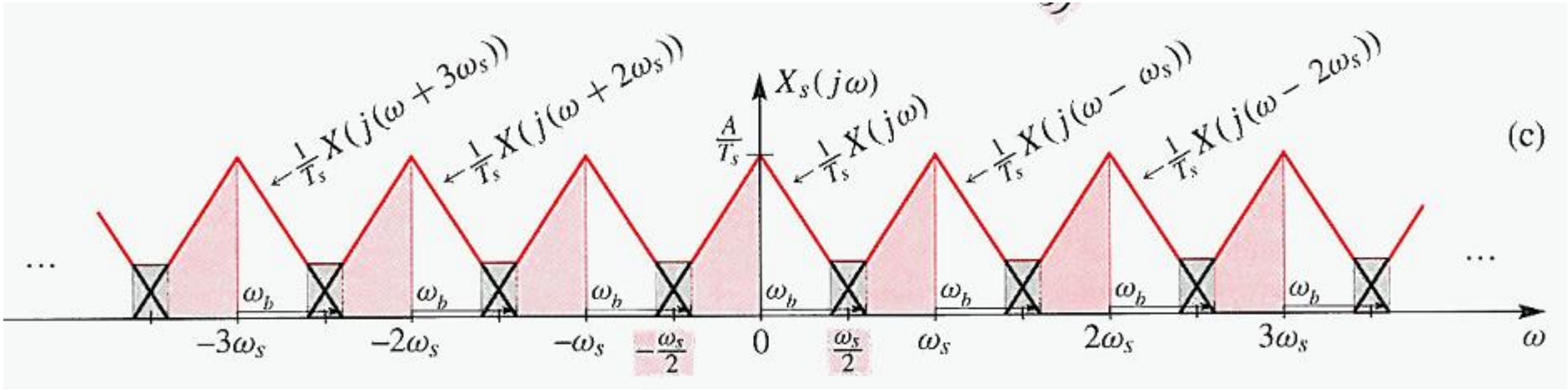


Illustration for Avoid Aliasing

- In this case, aliasing occurs:





Avoid Aliasing

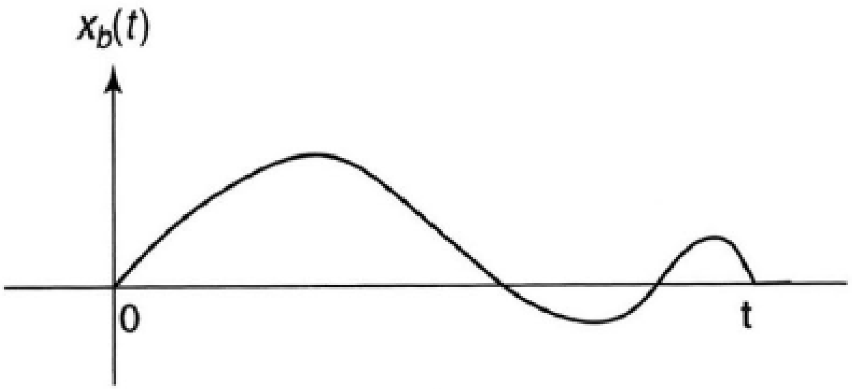
- From the above illustrations, we can see that
- If the continuous signal $x_a(t)$ is a band-limited signal with the bound w_b, then, the aliasing can be avoided when the sampling frequency is chosen such that $\boldsymbol{\omega_s > 2\omega_b}$.
- When there is no aliasing, we can reconstruct the continuous signal from its uniform samples.

Sampling Theorem

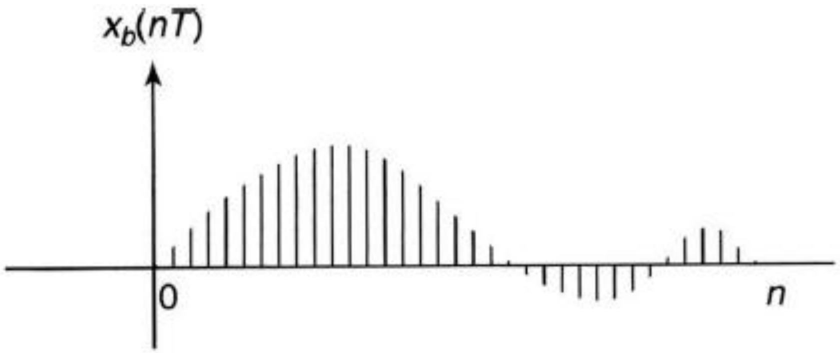
- To avoid the situation of aliasing, the sampling frequency shall be larger than twice of the highest frequency of the continuous signal.

- Nyquist sampling theorem: $f_b(=\frac{\omega_b}{2\pi})$ is called the Nyquist frequency, and $2f_b$ is called the Nyquist rate.

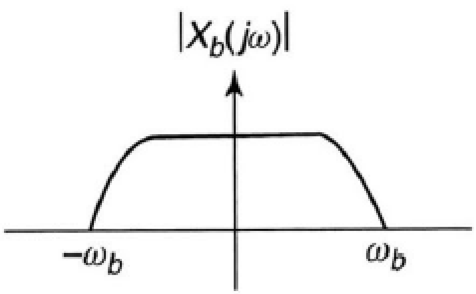
Further illustration



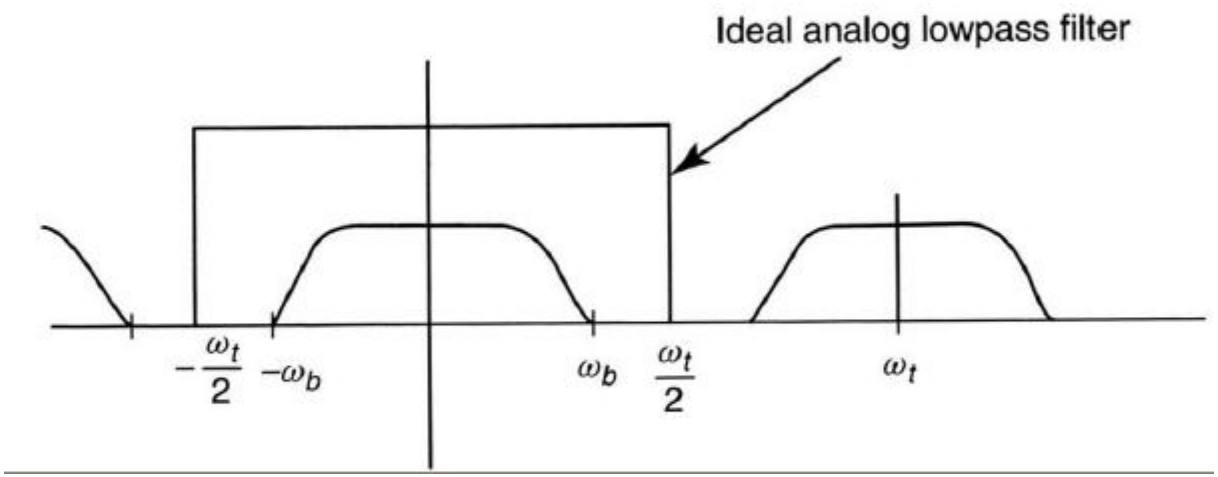
Continuous signal



Sampled signal



The magnitude spectrum of the continuous signal



The spectrum of the sampled signal

Practical situation (over-sampling)

- In practice, a signal could never be ideally band limited but ‘approximately’ band limited with its spectrum tail degenerate to zero.
- Though sampling theorem suggests 2x. Usually choose higher than 2x (eg., 4x, 8x) for sampling, so that only a small portion of high-frequency noises will be folded back to the limited band.
  - Payoff: the increasing of computation, power and memory usages.

Time-domain Example of Aliasing

- In the above, the aliasing is analyzed in frequency domain. What does it happen in time domain?
- From the sampling theorem, we see that aliasing occurs when the sampling rate ω_s is not high enough.
- Use sinusoids as an example: when we sample a sinusoidal signal with a low frequency, we can see the aliasing effect in time domain.

Time Domain Example: sampling with sufficiently large rates.

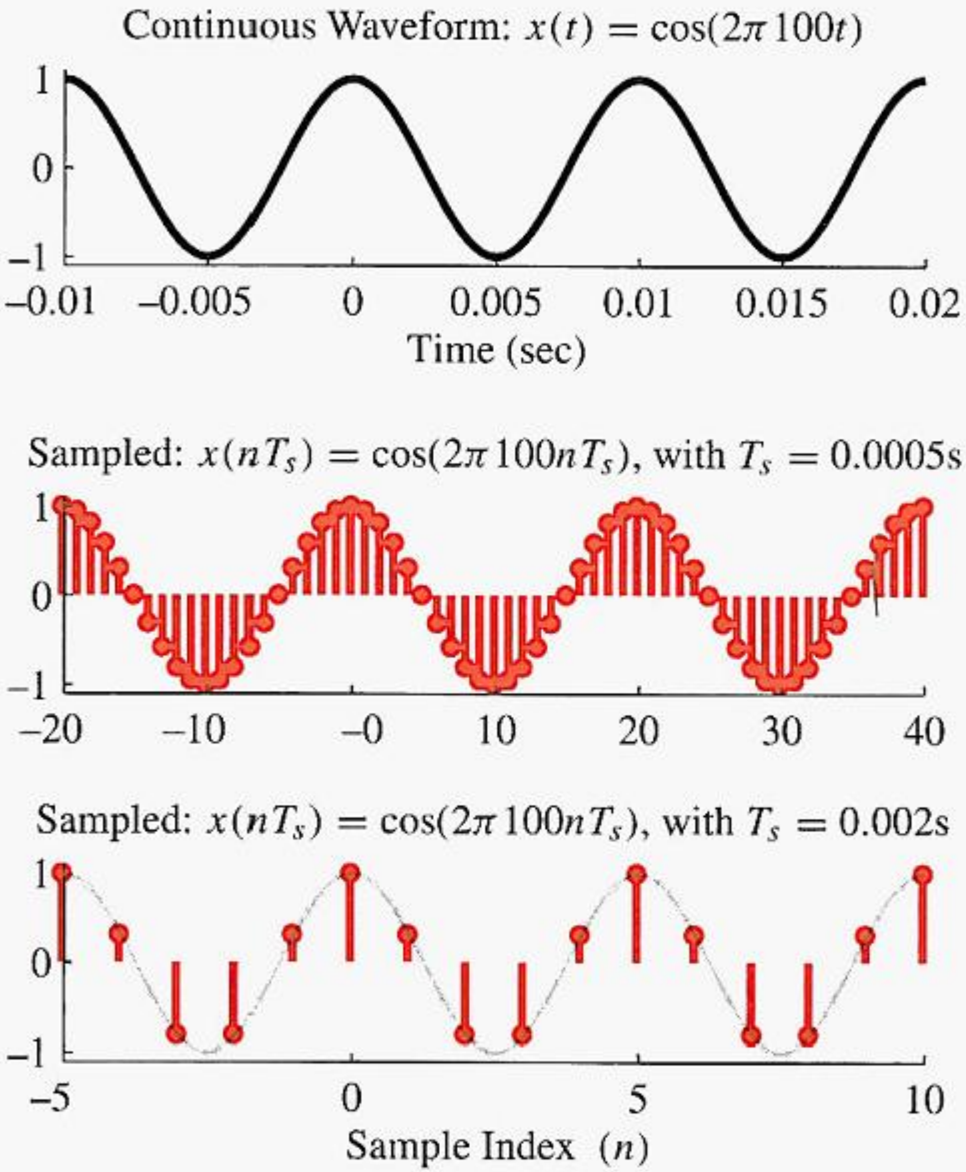


Figure 4-3: A continuous-time 100-Hz sinusoid (top) and two discrete-time sinusoids formed by sampling at $f_s =$ 2000 samples/sec (middle) and at $f_s = 500$ samples/sec (bottom).

A 100 Hz sinusoid sampled at

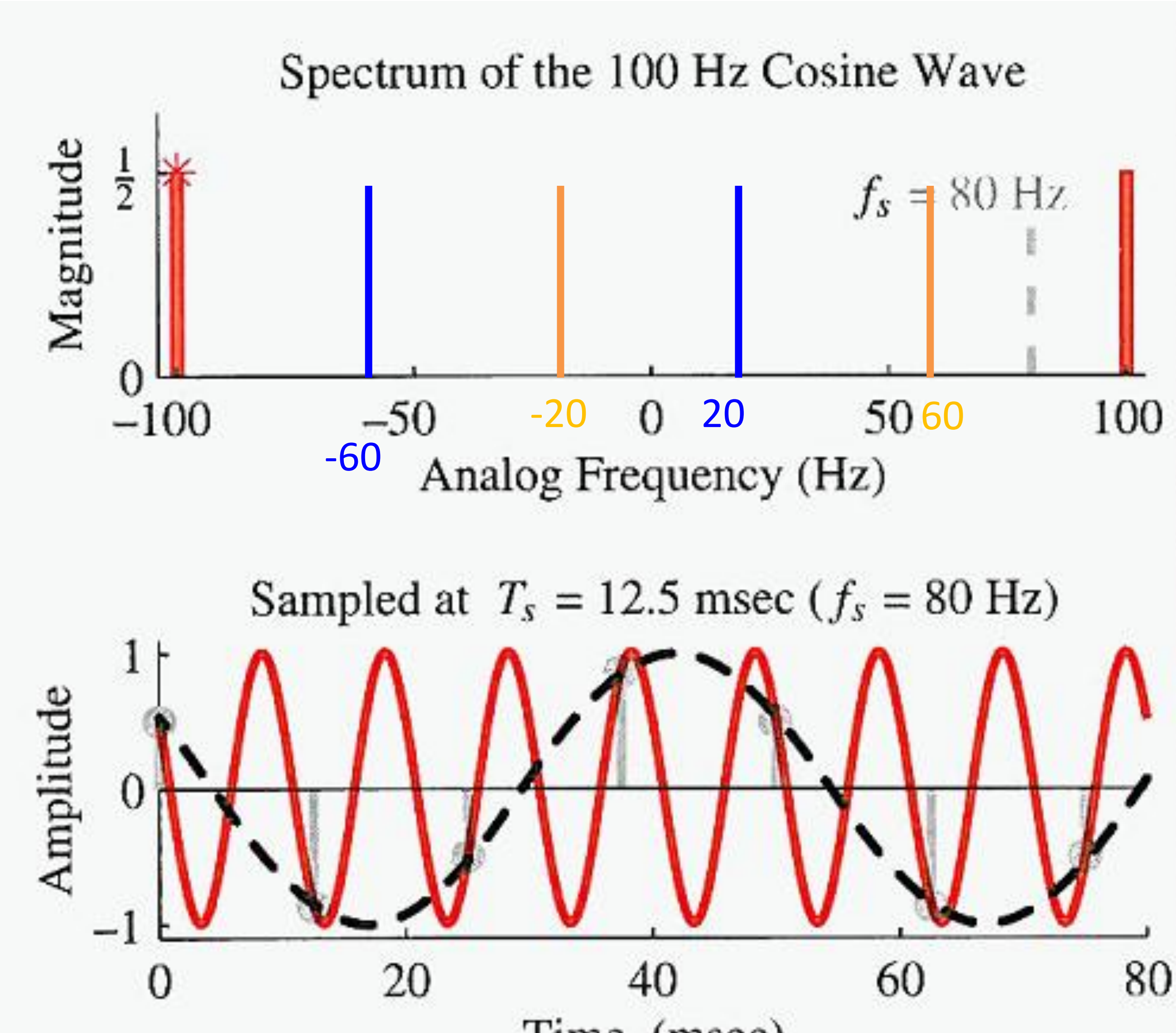
$$f_s = 2000$$

and

$$f_s = 500$$

samples/sec.

Eg., sampling of insufficient rate: a 100 Hz sinusoid sampled at $f_s = 80$ samples/sec.



Shifted to left
$$100 \rightarrow 20 \rightarrow -60$$

Shifted To right
$$-100 \rightarrow -20 \rightarrow 60$$

Dashed Black signal: 20 Hz (We tend to interpret the signal by the smoothest solution)