Circular Convolution (for DFT)

➢ Time-domain convolution implies frequency domain multiplication.

➢ This property is valid for continuous Fourier transform, Fourier series, and DTFT, **but is not exactly true for DFT**.

➢ The DFT pair considered here (following Openheim's book, where the $1/N$ is put on the inverse-transform side):

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad k = 0,1\ldots, N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \qquad n = 0,1\ldots, N-1$$

where $W_N = e^{-j2\pi/N}$ is a root of the polynomial equation $\boldsymbol{W}^N = 1$.

Circular Convolution

- For DFT, time domain **circular convolution** implies frequency domain multiplication, and vice versa.

  - Consider a periodic sequence. Its DTFT is both periodic and discrete in frequency (if we extend the DTFT spectrum to all periods)

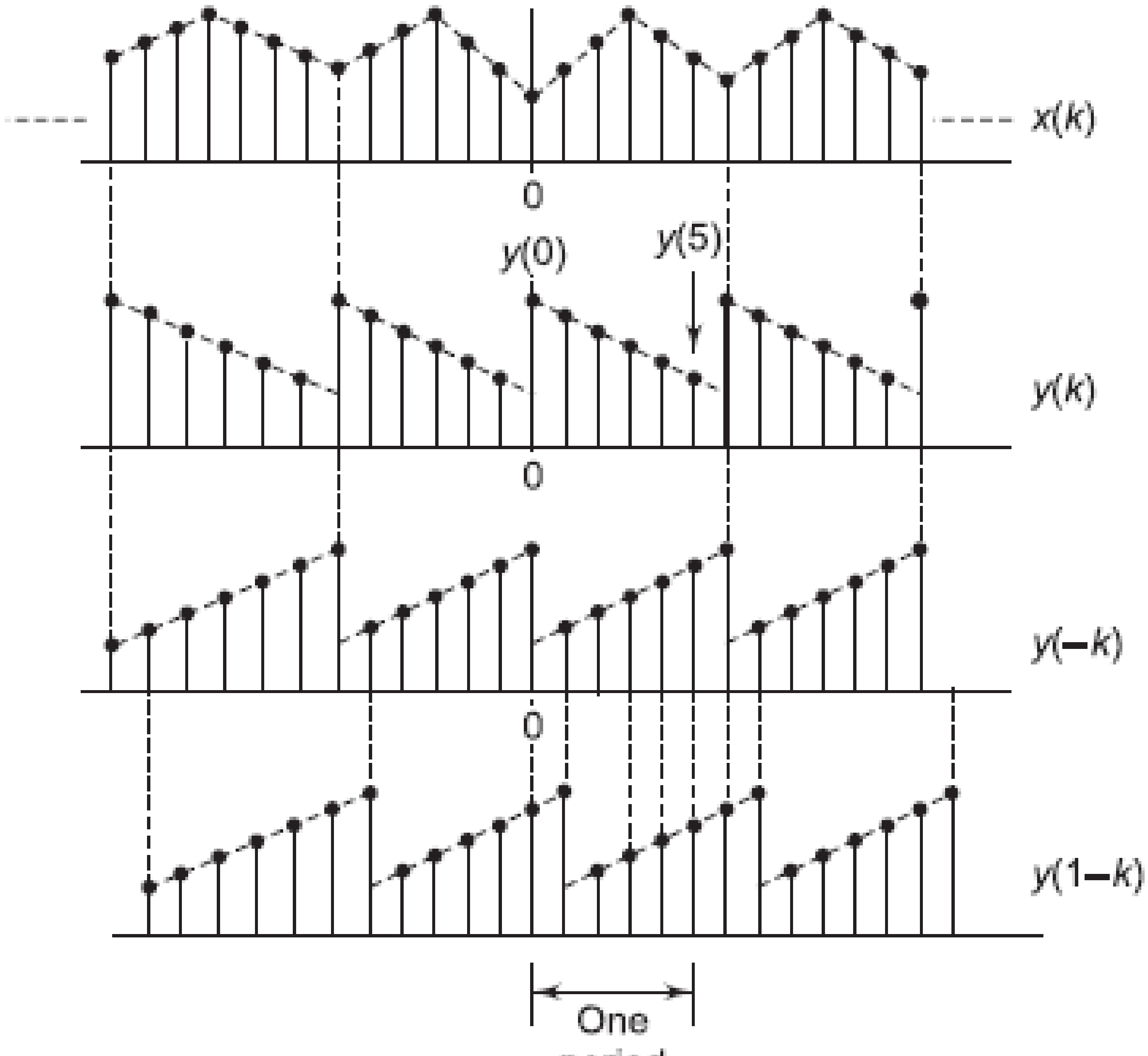
  - Multiplication in the frequency domain results in a convolution of the two periodic sequences in time domain.

Circular Convolution

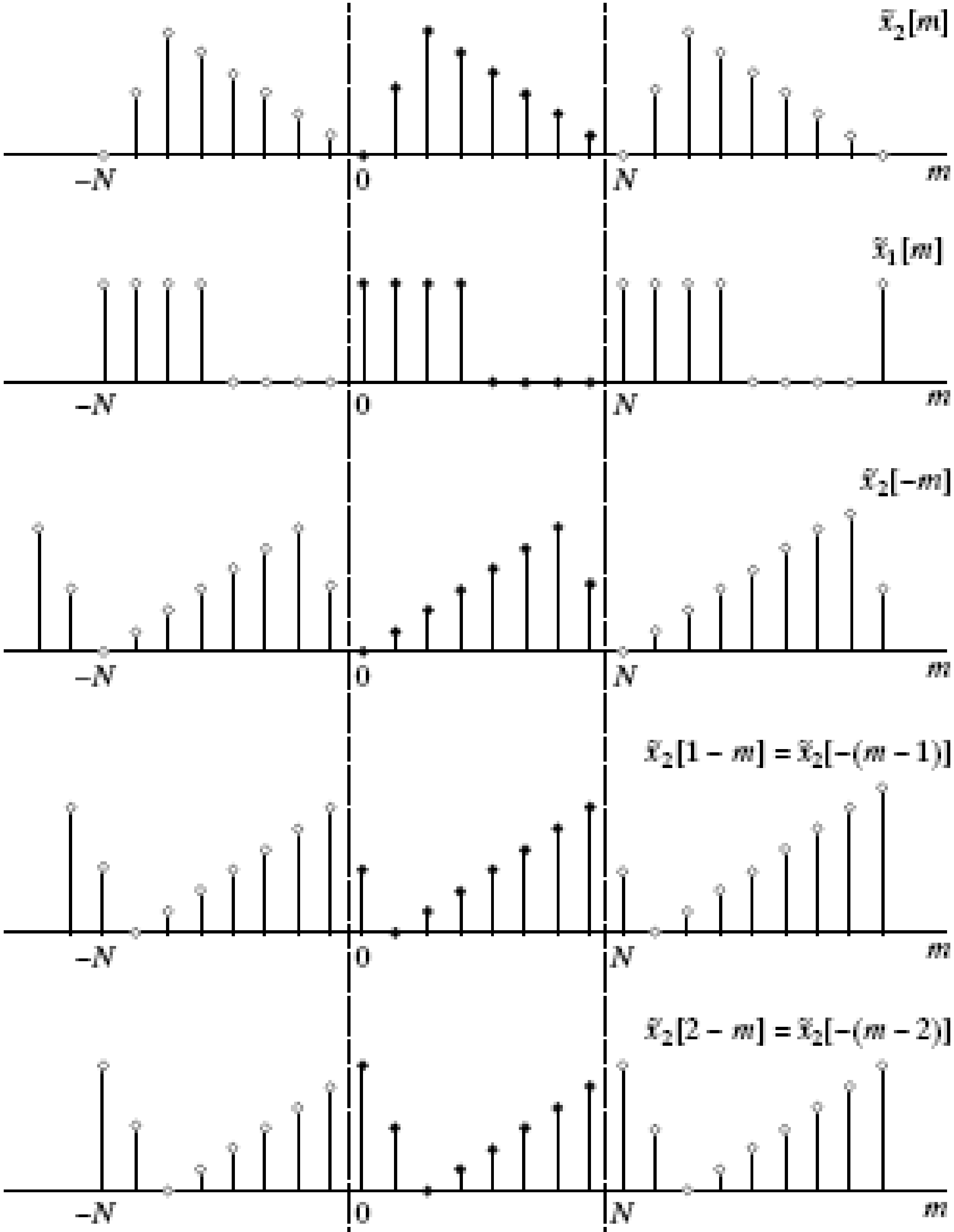
- Now let's consider a **single period** of the resulted sequence.

- Since the two sequences are both periodic, the convolution appears as 'folding' the rear of a sequence to the front one by one, and summing the inner products so obtained, in a single period.

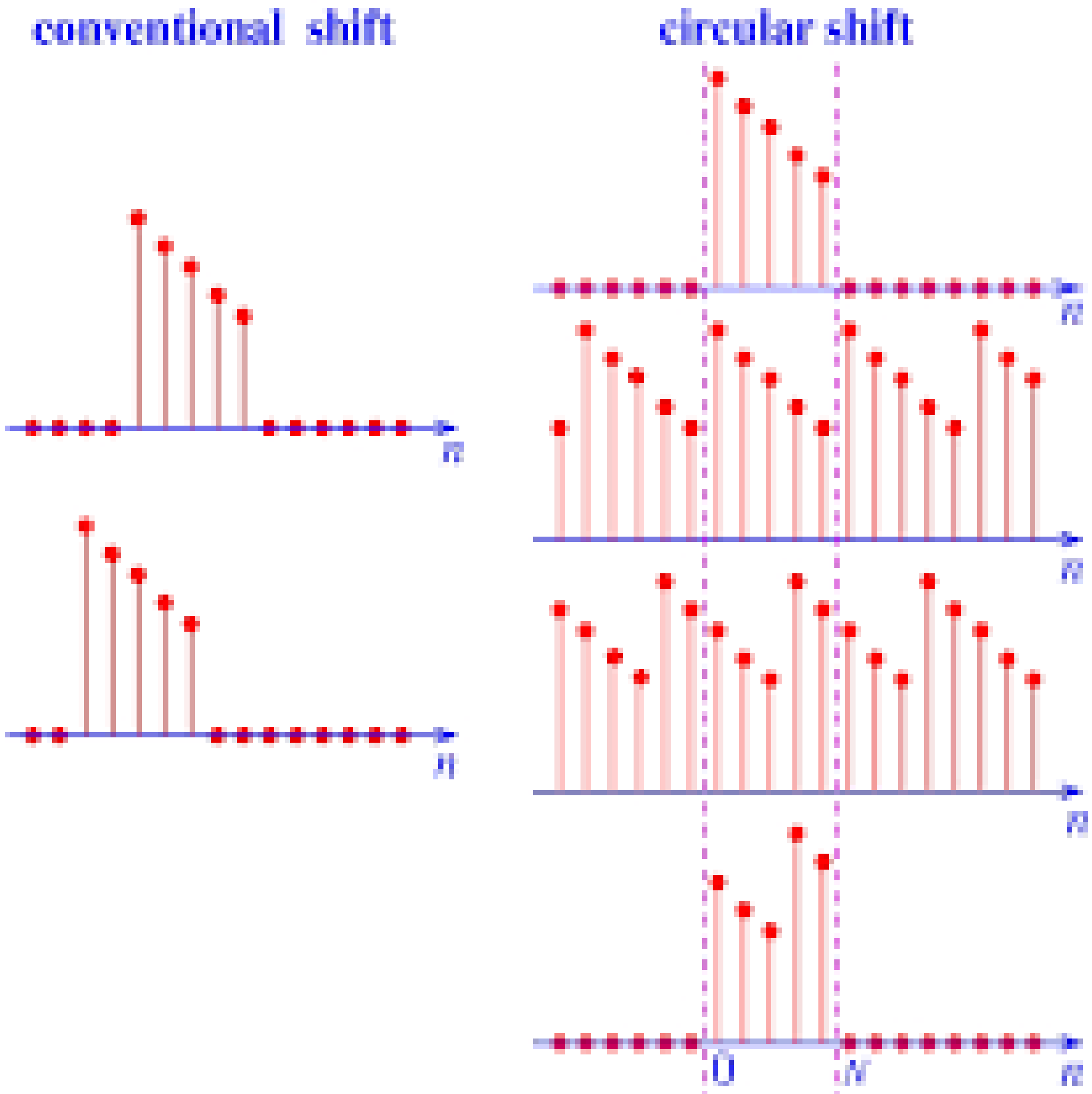
Convolution of Periodic Signals



Convolution of Periodic Signals -- Circular Shift in a Period



Convolution of Periodic Signals
Circular Shift



Circular convolution (definition)

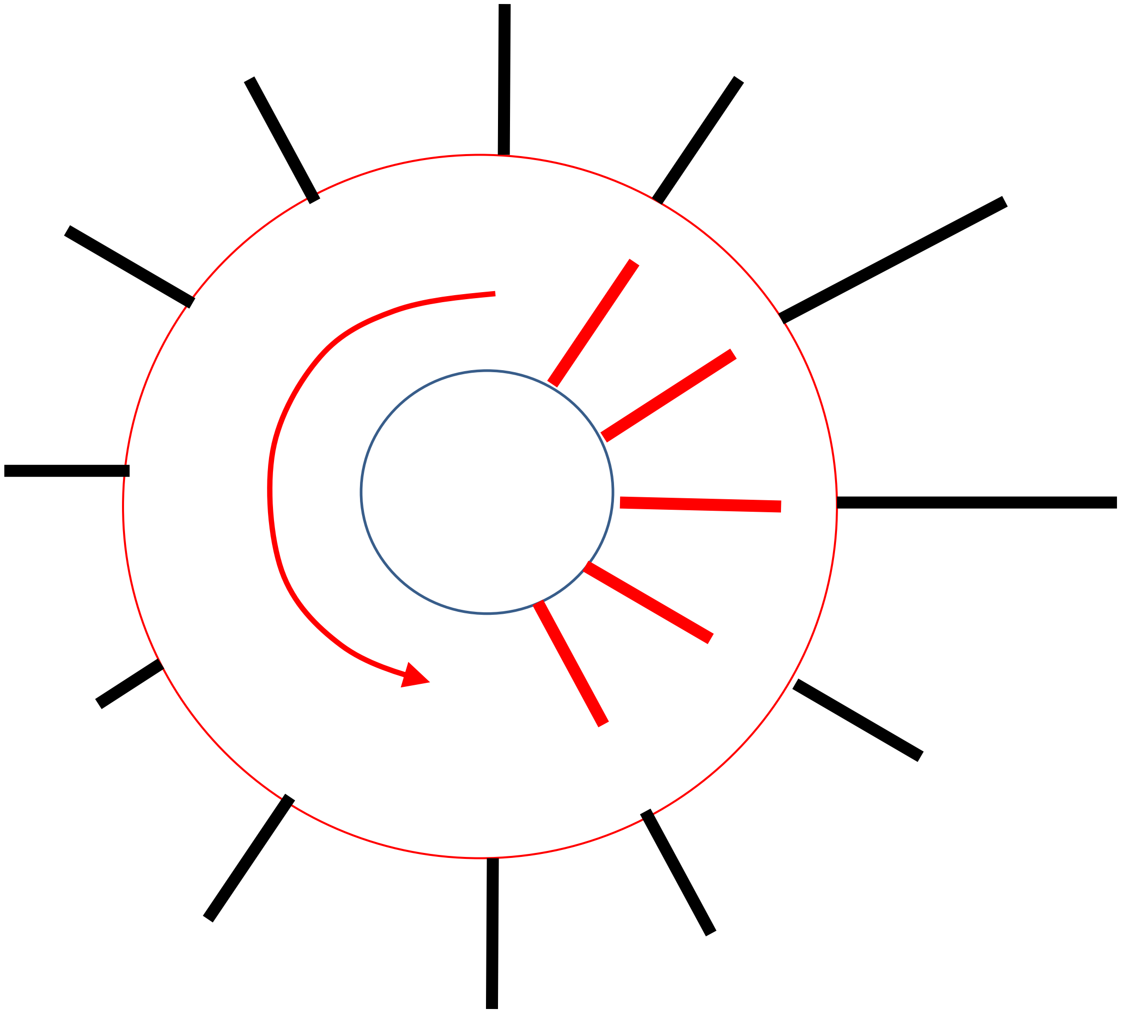
➢ Let $x_1[n]$ and $x_2[n]$ be length-N signals, Their circular convolution $x_3[n]$ is also a length-N signal defined below

$$x_3[n] = x_1[n] \otimes x_2[n] = x_2[n] \otimes x_1[n]$$

$$\equiv \sum_{m=0}^{N-1} x_2[m] x_1[((n-m))_N] = \underbrace{\sum_{m=0}^{N-1} x_2[m] x_1[(n-m) \bmod N]}_{\text{circular convolution}}$$

➢ Symbol for representing circular convolution: $\otimes$ or Ⓝ.

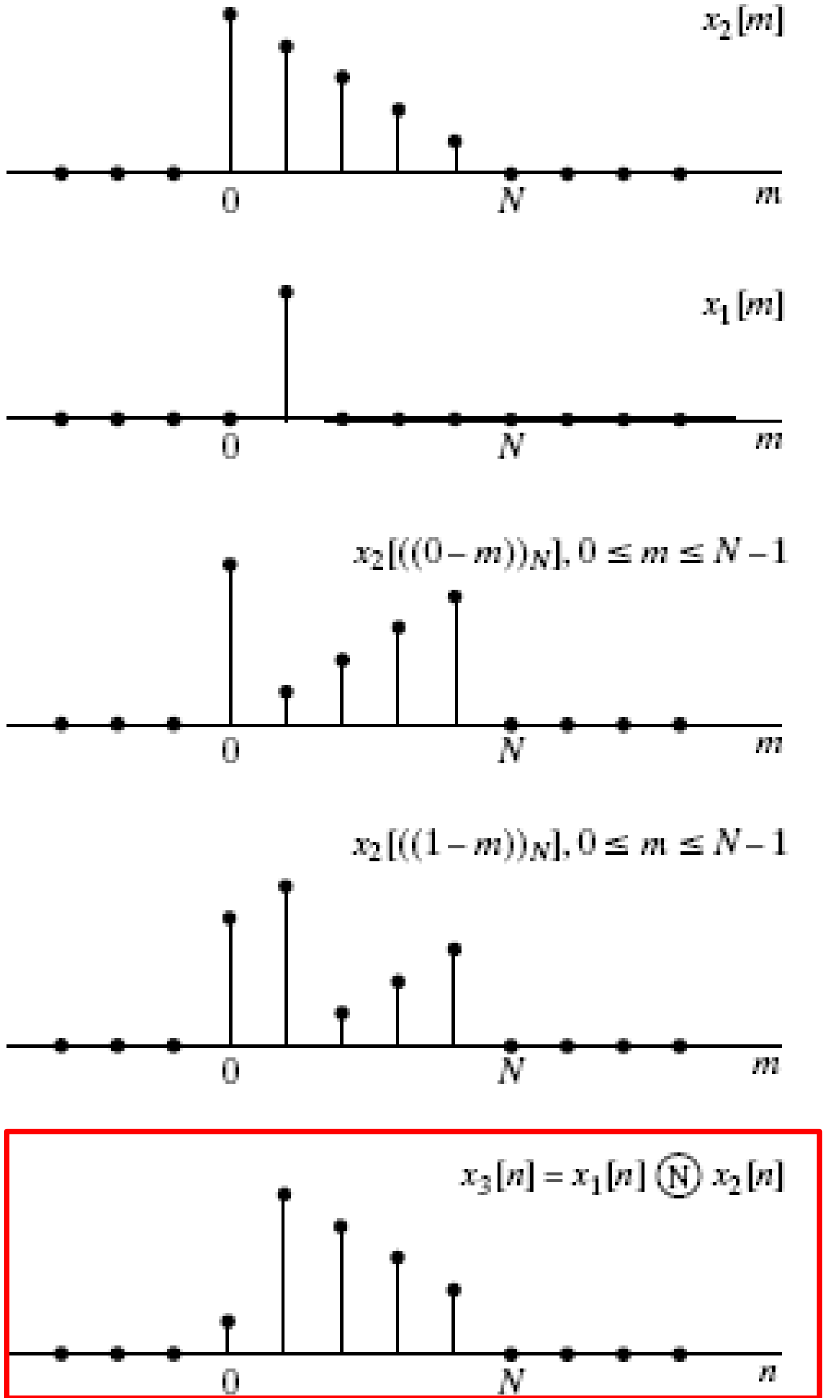
Imaging circular convolution as
‘convolution on a circle’



Example: circular convolution of $x_2[m]$ with a delayed impulse sequence

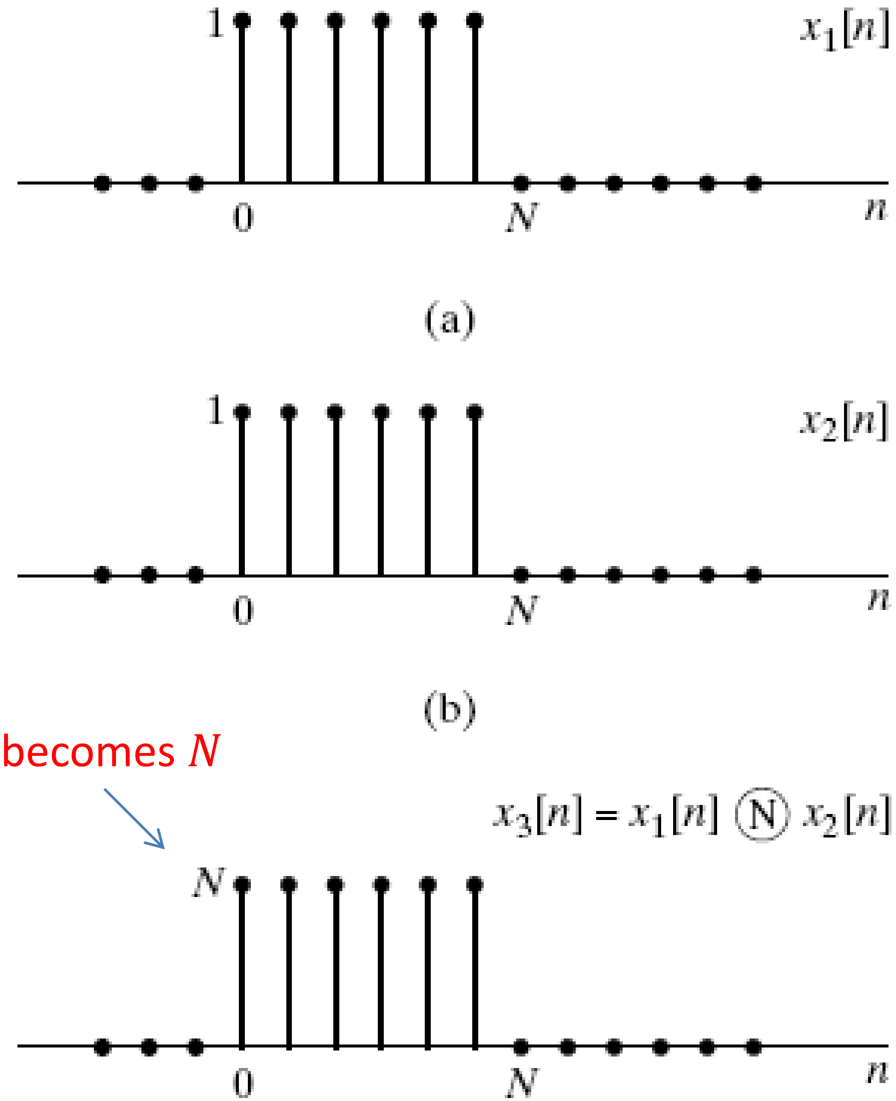
$$x_1[m] = \delta[m-1]$$

(need to trace it step by step)



Example: circular convolution of two rectangular pulses

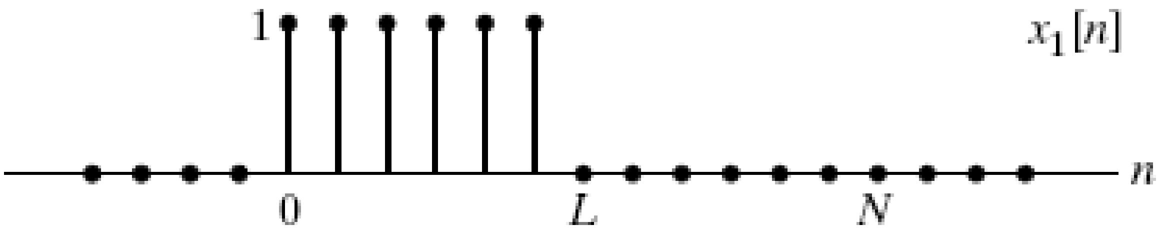
Magnitude becomes N



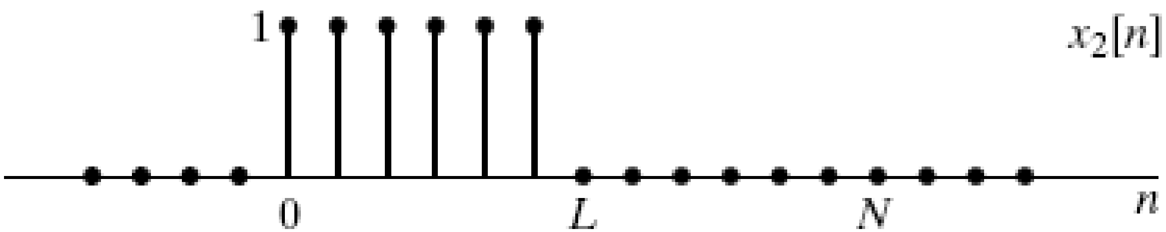
N-point circular convolution of two sequences of length N.

Circular convolution of two rectangular pulses with zero padding

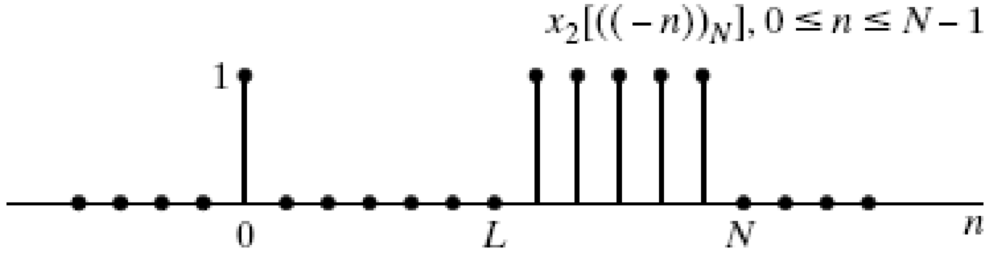
Given two sequences of length L, assume that we add L zeros on its end, making an $N = 2L$ point sequence, referred to as zero padding



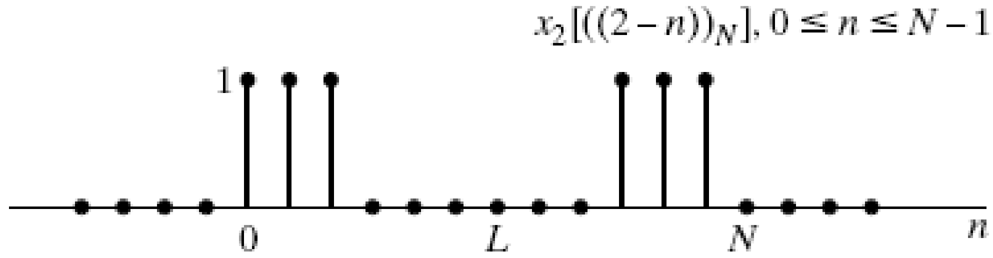
(a)



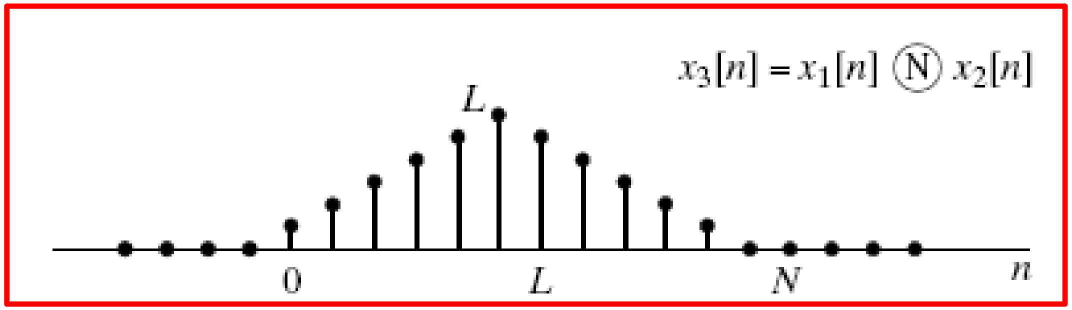
N-point circular convolution of two sequences of length L, where $N = 2L$.



(c)



(d)



N-point circular convolution of two sequences of length L, where N=2L (continue).

Property: As can be seen in the figures, by zero padding, we can use circular convolution to compute linear convolution of two finite length signals.

Circular Convolution vs. DFT

For DFT

- Time domain circular convolution implies DFT frequency domain multiplication:

$$x_3[n] = x_1[n] \otimes x_2[n] \leftrightarrow X_3[k] = X_1[k] X_2[k]$$

- Time domain multiplication implies DFT frequency domain circular convolution (with $1/N$ amplitude reduction):

$$x_3[n] = x_1[n] x_2[n] \leftrightarrow X_3[k] = \frac{1}{N} X_1[k] \otimes X_2[k]$$

Some other properties involving **circulation**:

- Time domain circular shift implies frequency domain linear phase change:

$$x[((n-m))_N], \quad 0 \le n \le N-1 \;\leftrightarrow\; e^{-j(2\pi k/N)m} X[k] = W^{km} X[k]$$

Duality property of DFT

- Since DFT and IDFT has very similar form, we have a duality property for DFT:

If $$x[n] \overset{\text{DFT}}{\leftrightarrow} X[k]$$

Then $$X[n] \overset{\text{DFT}}{\leftrightarrow} Nx[((-k))_N], \qquad 0 \le k \le N-1$$

DFT Properties:

Finite-Length Sequence (Length N)	N-point DFT (Length N)
1. $x[n]$	$X[k]$
2. $x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3. $ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4. $X[n]$	$Nx[((-k))_N]$
5. $x[((n-m))_N]$	$W_N^{km}X[k]$
6. $W_N^{-\ell n}x[n]$	$X[((k-\ell))_N]$
7. $\sum_{m=0}^{N-1} x_1(m)x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8. $x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1(\ell)X_2[((k-\ell))_N]$