Pattern Recognition - Fall 2024

Homework 2
Due date: Tuesday, December 10, 2024

1. (20 %) **Maximum Likelihood Estimation**
Given the samples $x_1, x_2, \ldots, x_n$ drawn independently from a Bernoulli experiment with

$$\begin{aligned} P(x=1) &= \theta, \\ P(x=0) &= 1-\theta. \end{aligned}$$

   (a) Find the maximum likelihood estimate of θ.

   (b) Is the maximum likelihood estimate unbiased? Please justify your answer.

2. (20 %) **Bayesian Parameter Estimation**
The form of an unknown density is assumed to be exponential, i.e.,

$$p(x|\theta) = \begin{cases} \theta \, e^{-\theta x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where the value of θ is not known exactly. The prior density $p(\theta)$ is

$$p(\theta) = \begin{cases} \lambda \, e^{-\lambda\theta} & \theta \geq 0 \\ 0 & \theta < 0 \end{cases}$$

Given the samples $x_1, x_2, \ldots, x_n$ drawn independently from $p(x|\theta)$, what is the maximum a posteriori estimator of θ?

3. (30 %) **Principal Component Analysis**
Table 1 shows three-dimensional data sampled from three categories. Project these three-dimensional data onto a two-dimensional subspace using Principal Component Analysis (PCA).

   (a) Determine the scatter matrix.

   (b) Determine the two largest eigenvalues of the scatter matrix and the corresponding eigenvectors.

   (c) Plot the projected data points on the two-dimensional subspace.

   (d) Assume that the prior probabilities are equal and the class-conditional probability density functions are normal. For each category, the mean vector and the covariance matrix are given by

$$\begin{aligned} \boldsymbol{\mu} &= \frac{1}{10}\sum_{k=1}^{10} \mathbf{x}_k \\ \boldsymbol{\Sigma} &= \frac{1}{10}\sum_{k=1}^{10} (\mathbf{x}_k - \boldsymbol{\mu})(\mathbf{x}_k - \boldsymbol{\mu})^t \end{aligned}$$

   where $\mathbf{x}_k$ denotes the k-th samples in that category. Calculate the percentage of misclassified samples in the two-dimensional subspace.

4. (30 %) **Fisher Linear Discriminant**

Table 1 shows three-dimensional data sampled from three categories. Project these three-dimensional data onto a two-dimensional subspace using Fisher Linear Discriminant.

(a) Determine the within-class scatter matrix $\mathbf{S}_W$ and the between-class scatter matrix $\mathbf{S}_B$.

(b) Determine the two largest eigenvalues of $\mathbf{S}_W^{-1}\mathbf{S}_B$ and the corresponding eigenvectors.

(c) Plot the projected data points on the two-dimensional subspace.

(d) Assume that the prior probabilities are equal and the class-conditional probability density functions are normal. For each category, the mean vector and the covariance matrix are given by

$$\boldsymbol{\mu} = \frac{1}{10}\sum_{k=1}^{10}\mathbf{x}_k$$

$$\boldsymbol{\Sigma} = \frac{1}{10}\sum_{k=1}^{10}(\mathbf{x}_k-\boldsymbol{\mu})(\mathbf{x}_k-\boldsymbol{\mu})^t$$

where $\mathbf{x}_k$ denotes the k-th samples in that category. Calculate the percentage of misclassified samples in the two-dimensional subspace.

Table 1: Three-dimensional data sampled from three categories.

	ω_1			ω_2			ω_3		
sample	x_1	x_2	x_3	x_1	x_2	x_3	x_1	x_2	x_3
1	0.42	-0.087	0.58	-0.4	0.58	0.089	0.83	1.6	-0.014
2	-0.2	-3.3	-3.4	-0.31	0.27	-0.04	1.1	1.6	0.48
3	1.3	-0.32	1.7	0.38	0.055	-0.035	-0.44	-0.41	0.32
4	0.39	0.71	0.23	-0.15	0.53	0.011	0.047	-0.45	1.4
5	-1.6	-5.3	-0.15	-0.35	0.47	0.034	0.28	0.35	3.1
6	-0.029	0.89	-4.7	0.17	0.69	0.1	-0.39	-0.48	0.11
7	-0.23	1.9	2.2	-0.011	0.55	-0.18	0.34	-0.079	0.14
8	0.27	-0.3	-0.87	-0.27	0.61	0.12	-0.3	-0.22	2.2
9	-1.9	0.76	-2.1	-0.065	0.49	0.0012	1.1	1.2	-0.46
10	0.87	-1.0	-2.6	-0.12	0.054	-0.063	0.18	-0.11	-0.49