Chapter 2
BAYESIAN DECISION THEORY

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2.1 Introduction

- Bayesian decision theory is a fundamental approach to the problem of classification.
- This approach is based on quantifying the tradeoffs between various decisions and the costs that accompany such decisions.
- It makes the assumption that all the relevant probability values are known.

- While we will give a quite general development of Bayesian decision theory, we begin our discussion with a specific example.

- We let ω denote the **state of nature**, with $\omega = \omega_1$ for sea bass and $\omega = \omega_2$ for salmon. discrete

- Because the state of nature is unpredictable, we consider ω to be a random variable.

- If the catch produced as much sea bass as salmon, we would say that the next fish is equally likely to be sea bass or salmon.

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- More generally, we assume that there is some **prior probability** $P(\omega_1)$ that the next fish is sea bass, and some prior probability $P(\omega_2)$ that it is salmon.
- If we assume there are no other types of fish relevant here, then $P(\omega_1)$ and $P(\omega_2)$ sum to one.

- Suppose that we were forced to make a decision about the type of fish that will appear next.
- If a decision must be made with so little information, it seems logical to use the following **decision rule**:

$$\text{Decide } \omega_1 \text{ if } P(\omega_1) > P(\omega_2)\text{; otherwise decide } \omega_2.$$

- In most circumstance we are not asked to make decisions with so little information.

- In our example, we might use a lightness measurement x to improve our classifier.

- We consider x to be a continuous random variable whose distribution depends on the state of nature and is expressed as $p(x|\omega)$[a].

特徵

discrete

continue

[a] We use an uppercase $P(\cdot)$ to denote a probability mass function and use a lowercase $p(\cdot)$ to denote a probability density function

- $p(x|\omega)$, the probability density function for x given that the sate of nature is ω, is called the **class-conditional probability density**.

- The difference between $p(x|\omega_1)$ and $p(x|\omega_2)$ describes the difference in lightness between populations of sea bass and salmon (Figure 1).

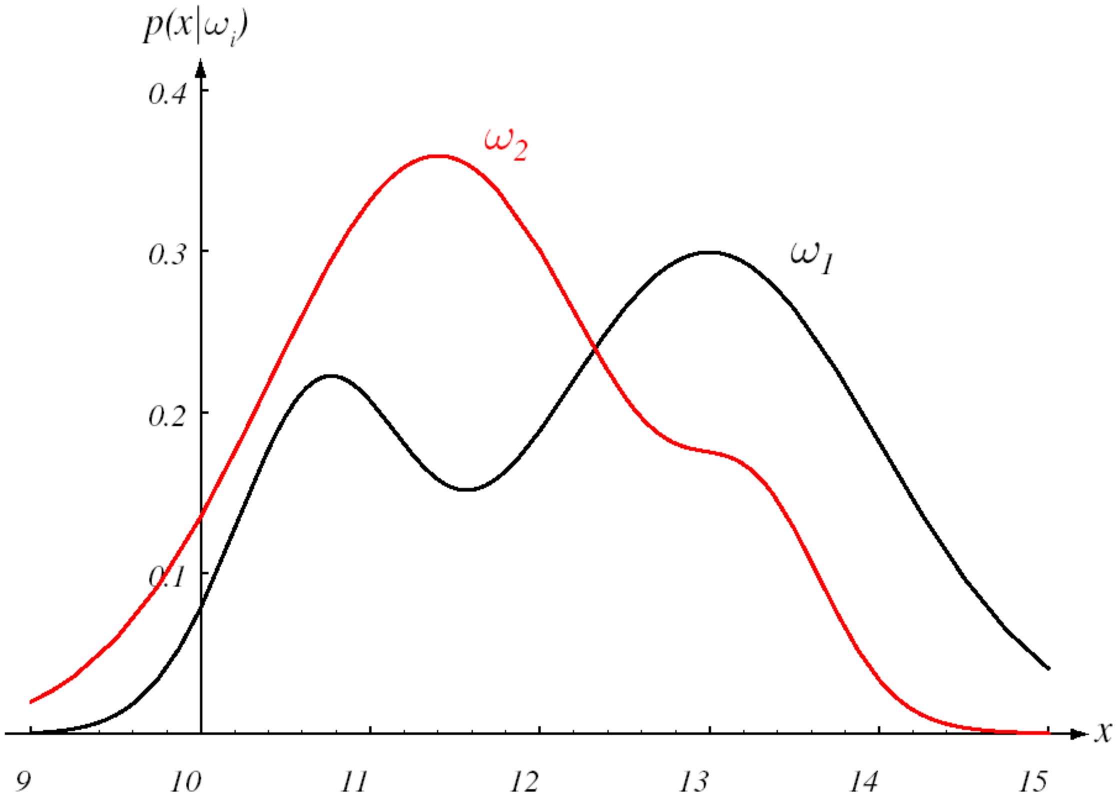


Figure 1: Class-conditional density functions show the distribution of x given the pattern is in category ω_i.

- Suppose that we know both the prior probabilities $P(\omega_j)$ and the conditional densities $p(x|\omega_j)$ for $j = 1, 2$.

- How does this change our attitude concerning the true state of nature?

- We note first that the joint probability density of a pattern that is in category ω_j and has feature value x can be written in two ways:

$$p(\omega_j, x) = \underline{P}(\overset{\text{discrete}}{\omega_j}|x)p(x) = \underline{p}(\overset{\text{continue}}{x}|\omega_j)P(\omega_j) \tag{1}$$

- Rearrange Equation (1) leads us to the answer to our question, which is called **Bayes formula**:

$$P(\omega_j|x) = \frac{p(x|\omega_j)P(\omega_j)}{p(x)} \tag{2}$$

where in this case of two categories

$$p(x) = \sum_{j=1}^{2} p(x|\omega_j)P(\omega_j) \tag{3}$$

- Bayes formula can be expressed formally in English by saying that

$$posterior = \frac{likelihood \times prior}{evidence} \tag{4}$$

- Bayes formula shows that by observing the value of x we can convert the prior probability $P(\omega_j)$ to the **a posteriori probability** (or **posterior**) $P(\omega_j|x)$, the probability of the state of nature being ω_j given that feature value x has been measured.

- We call $p(x|\omega_j)$ the **likelihood** of ω_j with respect to x, a term chosen to indicate that the category ω_j for which $p(x|\omega_j)$ is large is more likely to be the true category.
- Note that it is the product of the likelihood and the prior probability that is most important in determining the posterior probability.

- The **evidence**, $p(x)$, can be viewed as merely a scale factor that guarantees that the posterior probabilities sum to one.
- The variation of $P(\omega_j|x)$ with x is illustrated in Figure 2.

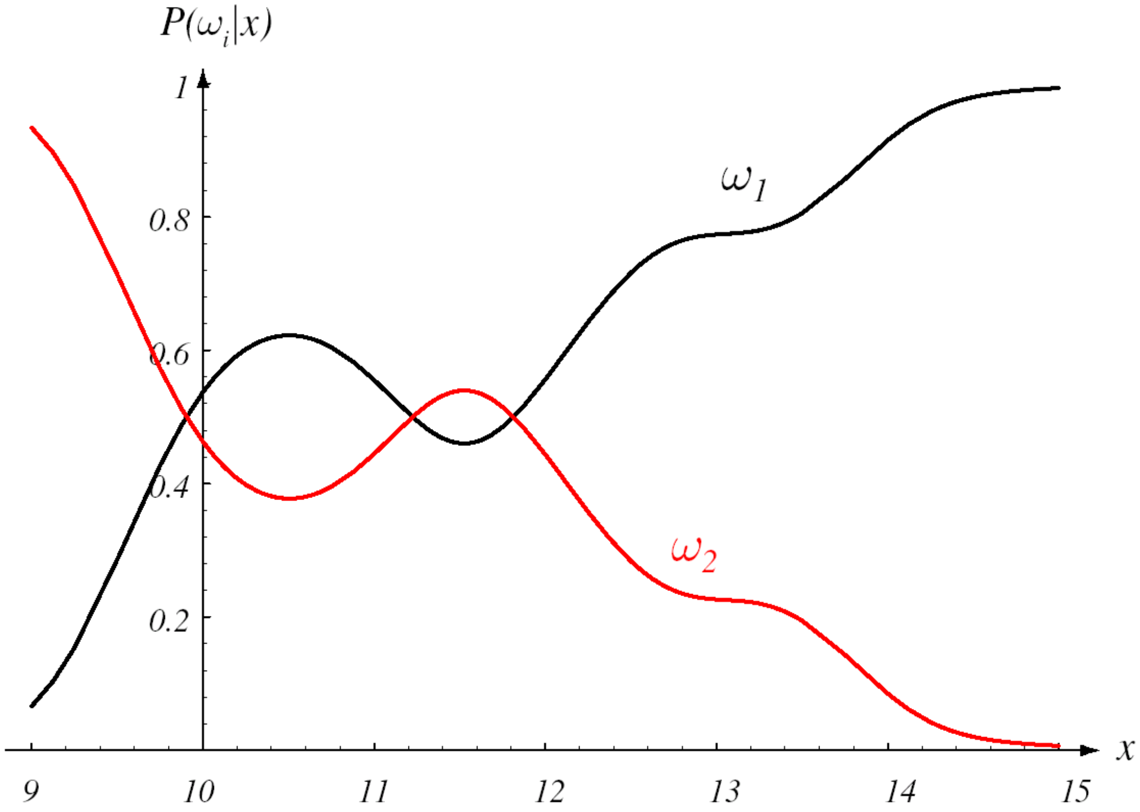
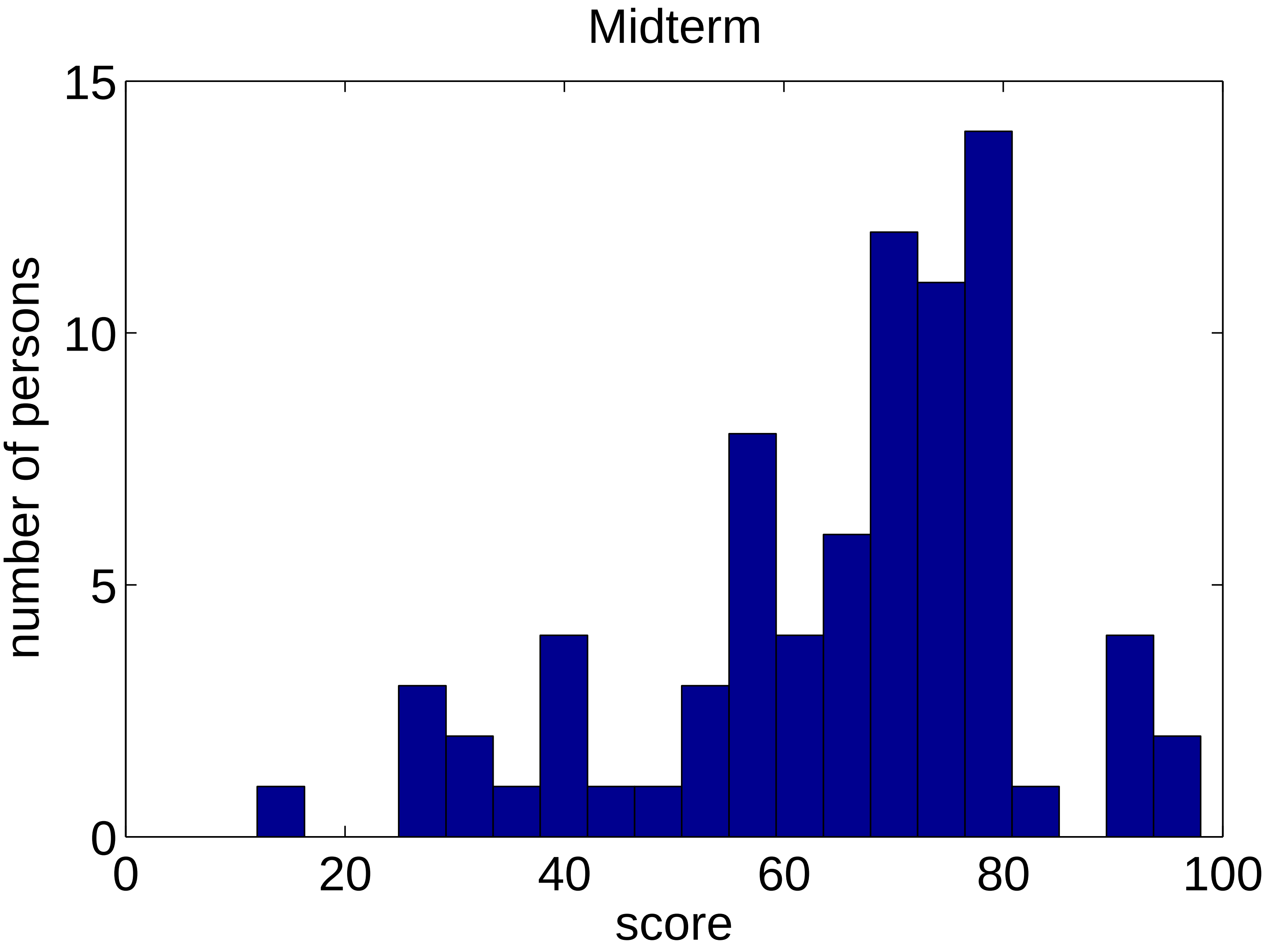
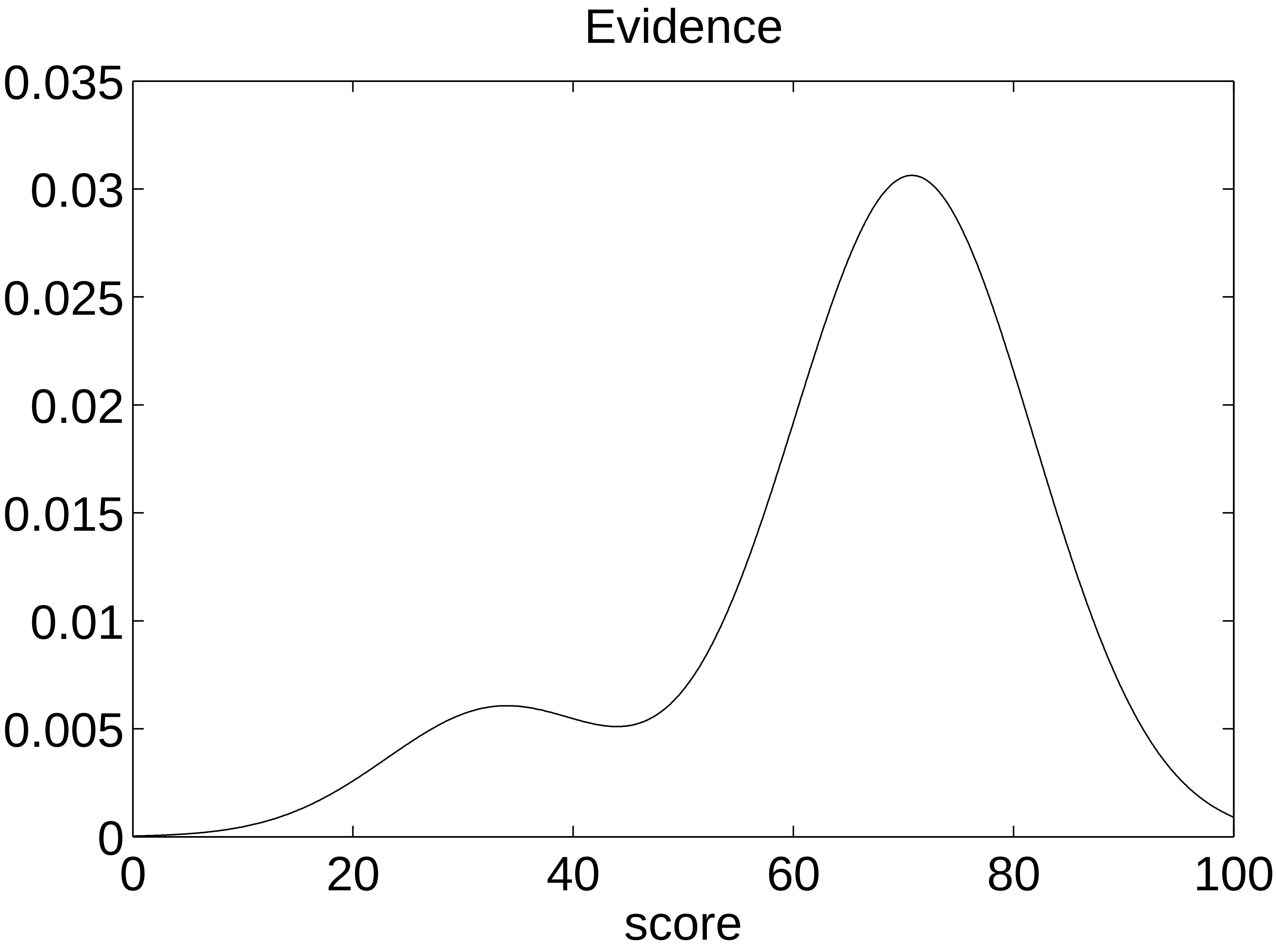
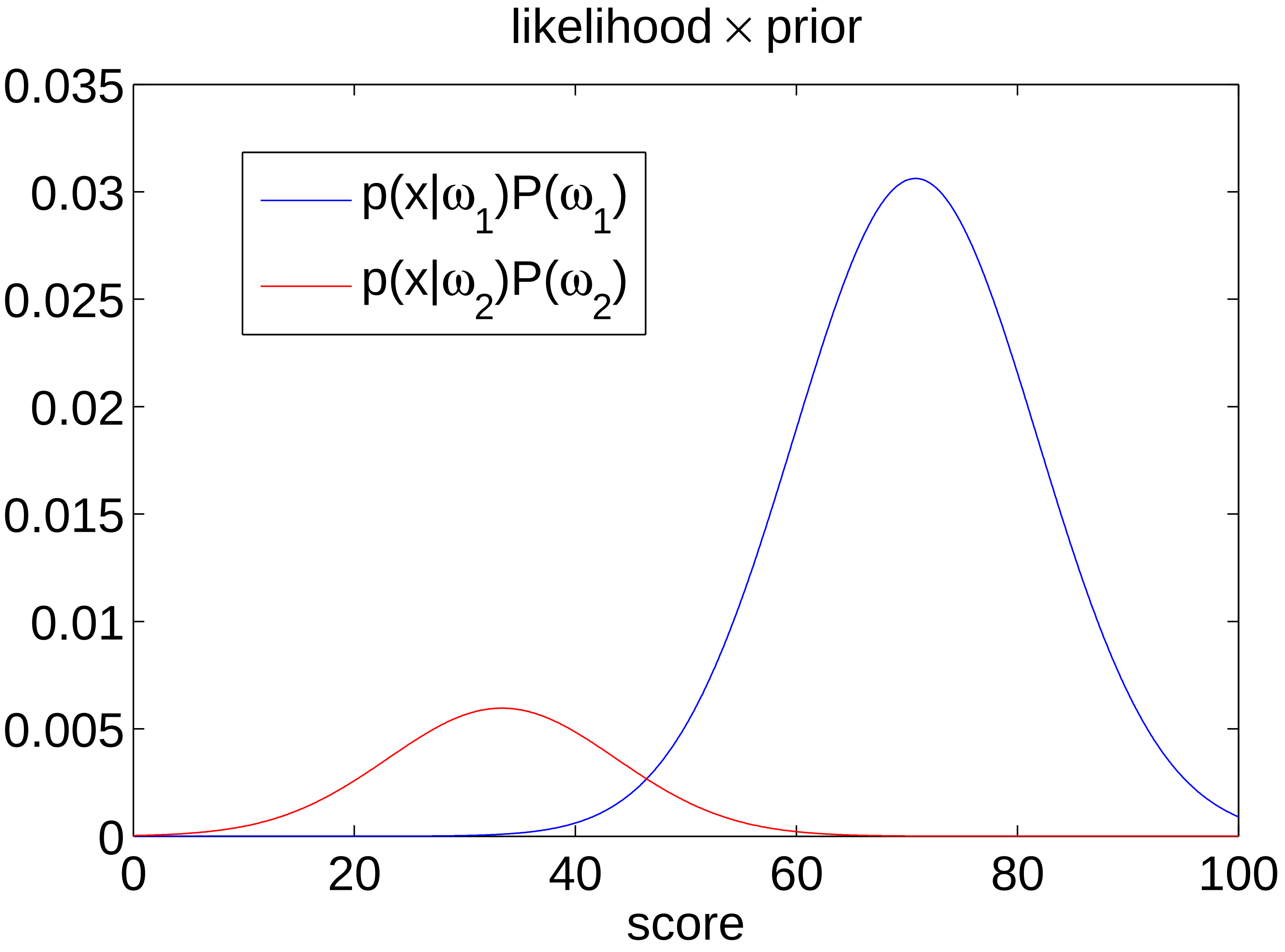


Figure 2: Posterior probabilities for $P(\omega_1) = 2/3$, $P(\omega_2) = 1/3$, and the class-conditional densities shown in Figure 1.


Midterm
number of persons
score
0
5
10
15
0
20
40
60
80
100


Evidence
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0
20
40
60
80
100
score


likelihood × prior
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0
20
40
60
80
100
score
$p(x|\omega_1)P(\omega_1)$
$p(x|\omega_2)P(\omega_2)$

- If we have an observation x for which $P(\omega_1|x)$ is greater than $P(\omega_2|x)$, we would naturally be inclined to decide that the true state of nature is ω_1.
- Conversely, if $P(\omega_2|x)$ is greater than $P(\omega_1|x)$, we would naturally be inclined to choose ω_2.
- To justify this decision procedure, let us calculate the probability of error whenever we make a decision.

- Whenever we observe a particular x, the probability of error is

$$P(error|x) = \begin{cases} P(\omega_1|x) & \text{if we decide } \omega_2 \\ P(\omega_2|x) & \text{if we decide } \omega_1. \end{cases} \tag{5}$$

- Clearly, for a given x we can minimize the probability of error by deciding ω_1 if $P(\omega_1|x) > P(\omega_2|x)$ and ω_2 otherwise.

- But, will this rule minimize the average probability of error?

- Yes, because the average probability of error is given by

$$P(error) = \int_{-\infty}^{\infty} P(error, x)dx = \int_{-\infty}^{\infty} P(error|x)p(x)dx \qquad (6)$$

- If for every x we ensure that $P(error|x)$ is as small as possible, then the integral must be as small as possible.

- Thus, we have justified the following **Bayes decision rule** for minimizing the probability of error:

$$\text{Decide } \omega_1 \text{ if } P(\omega_1|x) > P(\omega_2|x); \text{ otherwise decide } \omega_2.$$

- This form of decision rule emphaiszes the importance of the posterior probabilities.
- By using Eq. 2, we can express the rule in terms of the conditional densities and prior probabilities.

  Decide ω_1 if $p(x|\omega_1)P(\omega_1) > p(x|\omega_2)P(\omega_2)$; otherwise decide ω_2.

- Note that the evidence $p(x)$ in Eq. 2 is unimportant as far as making a decision is concerned.

- If for some x we have $p(x|\omega_1) = p(x|\omega_2)$, then that particular observation gives us no information about the state of nature.
  - In this case, the decision hinges entirely on the prior probabilities.
- On the other hand, if $p(\omega_1) = p(\omega_2)$, then the state of natures are equally probable.
  - In this case, the decision is based entirely on the likelihoods.
- In general, both of these factors are important and the Bayes decision rule combines them to achieve the minimum error rate.

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2.2 Bayesian Decision Theory

- We shall now generalize the ideas just considered in four ways:
  - By allowing the use of more than one feature
  - By allowing more than two states of nature
  - By allowing actions other than merely deciding the state of nature
  - By introducing a loss function

- Allowing the use of more than feature merely requires replacing the scalar x by the **feature vector** $\mathbf{x} \in \mathbb{R}^d$.
- Allowing more than two states of nature provides us with a useful generalization.
- Allowing actions other than classification primarily allows the possibility of rejection.
- The **loss function** states exactly how costly each action is.

- Let $\{\omega_1, \ldots, \omega_c\}$ be the set of c states of nature and let $\{\alpha_1, \ldots, \alpha_a\}$ be the set of a possible actions.
- The loss function $\lambda(\alpha_i|\omega_j)$ describes the loss incurred for taking action α_i when the state of nature is ω_j.
- Let $\mathbf{x}$ be a d-dimensional random vector and let $p(\mathbf{x}|\omega_j)$ be the state-conditional probability density function for $\mathbf{x}$.
- As before, $P(\omega_j)$ describes the prior probability that the state of nature is ω_j.

- The posterior probability $P(\omega_j|\mathbf{x})$ can be computed from $P(\mathbf{x}|\omega_j)$ by Bayes formula:

$$P(\omega_j|\mathbf{x}) = \frac{p(\mathbf{x}|\omega_j)P(\omega_j)}{p(\mathbf{x})} \tag{7}$$

where the evidence is

$$p(\mathbf{x}) = \sum_{j=1}^{c} p(\mathbf{x}|\omega_j)P(\omega_j) \tag{8}$$

- Suppose that we observe a particular $\mathbf{x}$ and that we contemplate taking action α_i.

- If the true state of nature is ω_j, we will incur the loss $\lambda(\alpha_i|\omega_j)$.

- The expected loss associated with taking action α_i is

$$R(\alpha_i|\mathbf{x}) = \sum_{j=1}^{c} \lambda(\alpha_i|\omega_j)P(\omega_j|\mathbf{x}) \tag{9}$$

- An expected loss is called a **risk**, and $R(\alpha_i|\mathbf{x})$ is called the **conditional risk**.

- We shall now show that the **Bayes decision rule** actually provides the optimal performance.

- Stated formally, our problem is to find a decision rule that minimizes overall risk.

- To be more specific, for every $\mathbf{x}$ the **decision rule** $\alpha(\mathbf{x})$ assumes one of the values $\alpha_1, \ldots, \alpha_a$.

- The overall risk R is the expected loss associated with a given decision rule.

$$R = \int R(\alpha(\mathbf{x})|\mathbf{x})p(\mathbf{x})d\mathbf{x} \tag{10}$$

where $d\mathbf{x}$ is the notation for a d-dimensional volume element.

- Clearly, if $\alpha(\mathbf{x})$ is chosen so that $R(\alpha(\mathbf{x})|\mathbf{x})$ is as small as possible for every $\mathbf{x}$, then the overall risk will be minimized.

- This justifies the Bayes decision rule: Compute the conditional risk $R(\alpha_i|\mathbf{x})$ for $i = 1, \ldots, a$ and then select the action with minimal conditional risk.

2.2.1 Two-Category Classification

- Let us consider the special case of two-category classification problems.
- Here, action α_1 corresponds to deciding that the true state of nature is ω_1.
- Similarly, action α_2 corresponds to deciding ω_2.
- For notational simplicity, let $\lambda_{ij} = \lambda(\alpha_i|\omega_j)$ be the loss incurred for deciding ω_i when the true state of nature is ω_j.

- The conditional risks are given by

$$
\begin{aligned}
R(\alpha_1|\mathbf{x}) &= \lambda_{11}P(\omega_1|\mathbf{x}) + \lambda_{12}P(\omega_2|\mathbf{x}) \qquad (11)\\
R(\alpha_2|\mathbf{x}) &= \lambda_{21}P(\omega_1|\mathbf{x}) + \lambda_{22}P(\omega_2|\mathbf{x}) \qquad (12)
\end{aligned}
$$

- Thus, the resulting decision rule is to decide ω_1 if $R(\alpha_1|\mathbf{x}) < R(\alpha_2|\mathbf{x})$.

- We can obtain an equivalent rule as follows:

$$\text{Decide } \omega_1 \text{ if } (\lambda_{21} - \lambda_{11})P(\omega_1|\mathbf{x}) > (\lambda_{12} - \lambda_{22})P(\omega_2|\mathbf{x}) \tag{13}$$

- Ordinarily, the loss incurred for making an error is greater than the loss incurred for being correct, i.e. $\lambda_{21} - \lambda_{11}$ and $\lambda_{12} - \lambda_{22}$ are positive.

- Under this assumption, another alternative is to decide ω_1 if

$$\frac{p(\mathbf{x}|\omega_1)}{p(\mathbf{x}|\omega_2)} > \frac{(\lambda_{12} - \lambda_{22})}{(\lambda_{21} - \lambda_{11})} \cdot \frac{P(\omega_2)}{P(\omega_1)} \tag{14}$$

- In other words, decide ω_1 if the **likelihood ratio** $p(\mathbf{x}|\omega_1)/p(\mathbf{x}|\omega_2)$ exceeds a threshold value that is independent of the observation $\mathbf{x}$.

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2.3 Minimum-Error-Rate Classification

- In classification problems, the action α_i is usually interpreted as the decision that the true state of nature is ω_i.
- If a decision α_i is taken and the true state of nature is ω_j, then the decision is wrong.
- It is natural to seek a decision rule that minimizes the probability of error.

- The loss function for this case is the so-called **zero-one loss function**.

$$\lambda(\alpha_i|\omega_j) = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases} \qquad i, j = 1, \cdots, c \tag{15}$$

- Thus, no loss is assigned to a correct decision and a unit loss is assigned to any error.

- The conditional risk is

$$\begin{aligned} R(\alpha_i|\mathbf{x}) &= \sum_{j=1}^{c} \lambda(\alpha_i|\omega_j) P(\omega_j|\mathbf{x}) \\ &= \sum_{j\neq i} P(\omega_j|\mathbf{x}) \qquad \text{(P.14, sum=1)} \\ &= 1 - P(\omega_i|\mathbf{x}) \qquad (16) \end{aligned}$$

- Thus, to minimize the average probability of error, we should select the i that maximizes the posterior probability $P(\omega_i|\mathbf{x})$.

- In other words, for **minimum error rate**:

$$\text{Decide } \omega_i \text{ if } P(\omega_i|\mathbf{x}) > P(\omega_j|\mathbf{x}) \qquad \text{for all } j \neq i$$

- The region where we decide ω_i is denoted $\mathcal{R}_i$; such a region need not to be simply connected (Figure 3).

- If we employ a zero-one loss, the decision regions are determined by θ_a.

- If we penalizes miscategorizing ω_2 as ω_1 more than the converse, we get the larger θ_b. Hence, $\mathcal{R}_1$ becomes smaller.

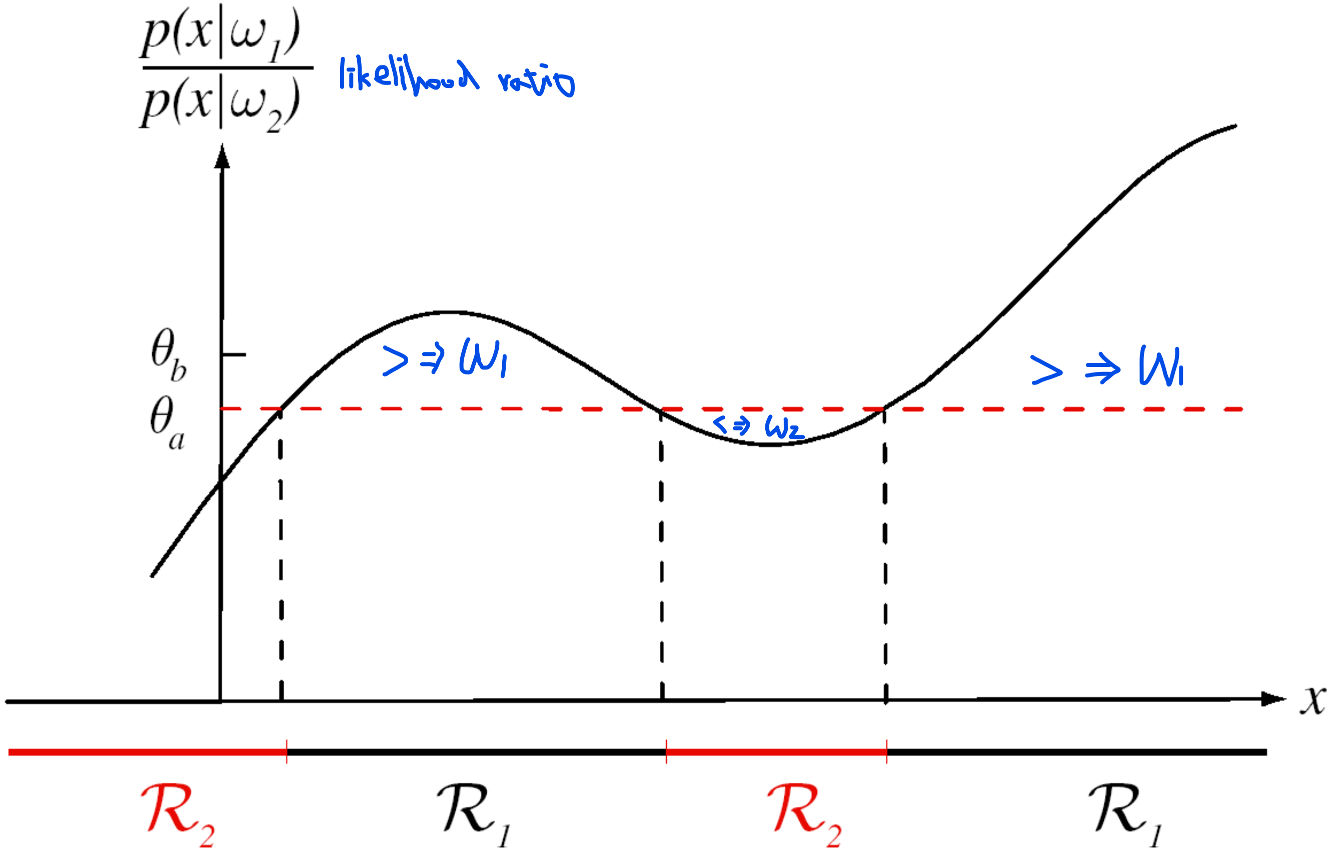


Figure 3: The likelihood ratio for the conditional-densities shown in Figure 1.

2.3.1 Minimax Criterion

- Sometimes we might expect a classifier to perform well over a range of prior probabilities.
- For instance, in the fish categorization problem, the prior probabilities might vary widely and in an unpredictable way.
- Or, we want to use a classifier in different plants where we do not know the prior probabilities.
- A reasonable approach is to design a classifier so that the worst overall risk for is as small as possible.

- Let $\mathcal{R}_1$ denote that region where a classifier decides ω_1 and likewise for $\mathcal{R}_2$ and ω_2.

- Then, we write the overall risk Eq. (10) in terms of conditional risk:

$$\begin{aligned} R &= \int_{\mathcal{R}_1} [\lambda_{11} P(\omega_1) p(\mathbf{x}|\omega_1) + \lambda_{12} P(\omega_2) p(\mathbf{x}|\omega_2)] d\mathbf{x} \\ &+ \int_{\mathcal{R}_2} [\lambda_{21} P(\omega_1) p(\mathbf{x}|\omega_1) + \lambda_{22} P(\omega_2) p(\mathbf{x}|\omega_2)] d\mathbf{x} \qquad (17) \end{aligned}$$

- We use the fact that $P(\omega_2) = 1 - P(\omega_1)$ and that $\displaystyle\int_{\mathcal{R}_1} p(\mathbf{x}|\omega_1)d\mathbf{x} = 1 - \int_{\mathcal{R}_2} p(\mathbf{x}|\omega_1)d\mathbf{x}$ to write the overall risk as:

$$\begin{aligned} R(P(\omega_1)) &= \lambda_{22} + (\lambda_{12} - \lambda_{22}) \int_{\mathcal{R}_1} p(\mathbf{x}|\omega_2)d\mathbf{x} \\ &+ P(\omega_1) \Bigg[(\lambda_{11} - \lambda_{22}) + (\lambda_{21} - \lambda_{11}) \int_{\mathcal{R}_2} p(\mathbf{x}|\omega_1)d\mathbf{x} \\ &- (\lambda_{12} - \lambda_{22}) \int_{\mathcal{R}_1} p(\mathbf{x}|\omega_2)d\mathbf{x} \Bigg] \end{aligned} \tag{18}$$

- The Eq. (18) shows that the overall risk is linear in $P(\omega_1)$.
- If we can find a decision rule such that the second term in Eq. (18) is zero, then the risk is independent of priors.
- The resulting **minimax risk**, R_{mm}, is:

$$\begin{aligned} R_{mm} &= \lambda_{22} + (\lambda_{12} - \lambda_{22}) \int_{R_1} p(\mathbf{x}|\omega_2) d\mathbf{x} \\ &= \lambda_{11} + (\lambda_{21} - \lambda_{11}) \int_{R_2} p(\mathbf{x}|\omega_1) d\mathbf{x} \qquad (19) \end{aligned}$$

- Figure 4 illustrates the approach.
- The curve at the bottom shows the overall error rate as a function of $P(\omega_1)$.
- For each value (e.g., $P(\omega_1) = 0.25$), there is a corresponding decision rule and associated overall error rate.
- For a fixed decision rule, if the priors are changed, the overall error rate will change as a linear function of $P(\omega_1)$ (shown by the dashed line).
- The maximum such error will occur at an extreme value of the prior, here at $P(\omega_1) = 1$.

- To minimize the maximum of such error, we should design a classifier for the maximum overall error rate, and thus the overall error will not change as a function of prior.
- Briefly stated, we search for the prior for which the Bayes risk is maximum, and the corresponding decision rule then gives the minimax solution.
- The value of the minimax risk, R_{mm}, is hence equal to the worst Bayes risk.

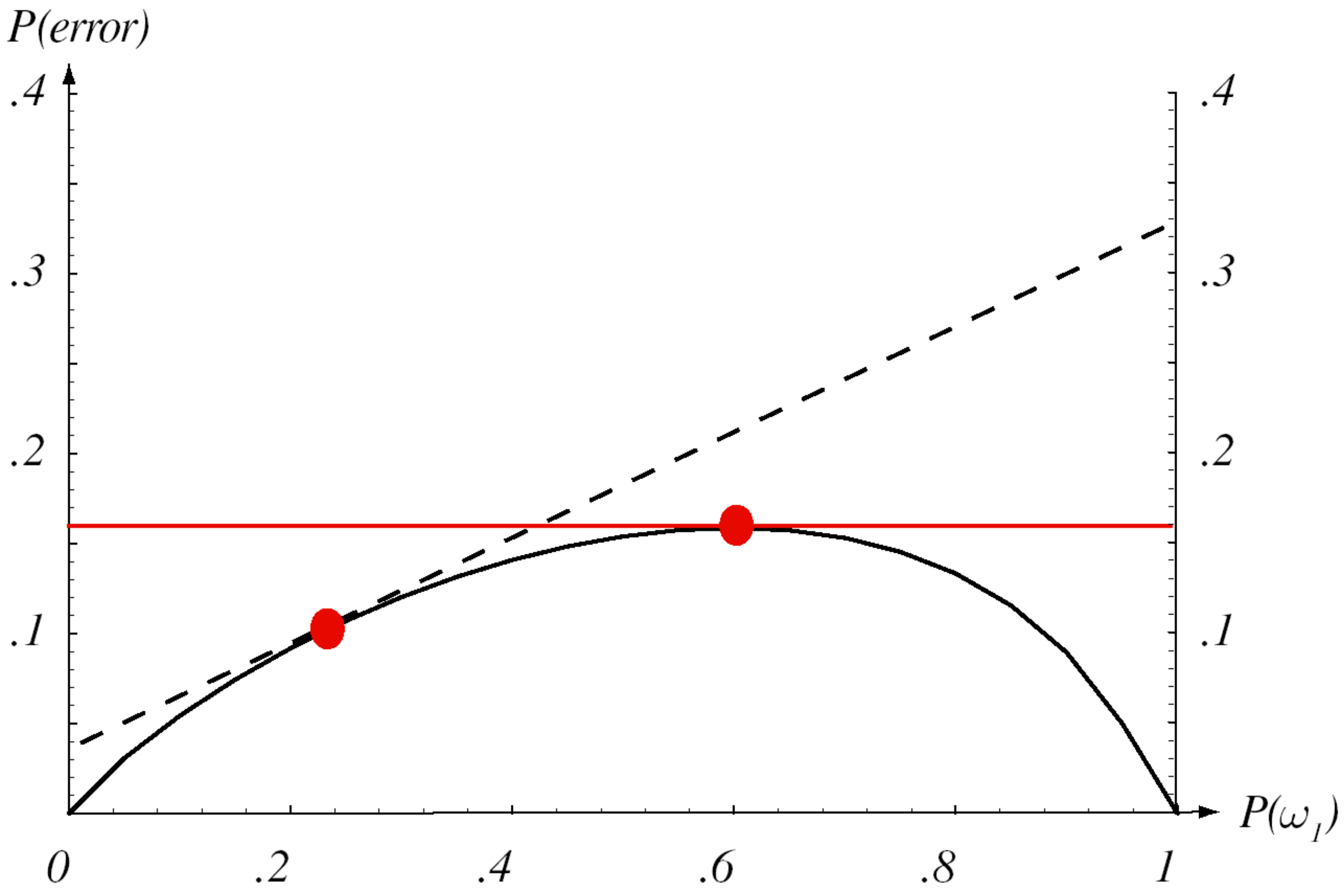


Figure 4: The overall error will not change as a function of prior, as shown by the red line.

- The minimax criterion finds greater use in game theory than it does in typical pattern recognition.
- In game theory, you have a hostile opponent who can be expected to take an action maximally detrimental to you.
- Thus, it makes great sense for you to take an action so that the worst case cost is minimized.

Example 1

Assume $p(x|\omega_1) \sim N(0, 1)$ and $p(x|\omega_2) \sim N(2, 1)$, under a zero-one loss. Please specify the decision regions based on minimax criterion. What is the resulting overall risk?

Answers:

(a) $\int_{R_2} p(x|w_1)dx = \int_{R_1} p(x|w_2)dx$

$$\Rightarrow \int_{\theta}^{\infty} \frac{1}{\sqrt{2\pi}} \exp[\frac{-1}{2}x^2]dx = \int_{-\infty}^{\theta} \frac{1}{\sqrt{2\pi}} \exp[\frac{-1}{2}(x-2)^2]dx$$

$$\Rightarrow \Phi(-\theta) = \Phi(\frac{\theta - 2}{1})$$

$$\Rightarrow -\theta = \theta - 2$$

$$\Rightarrow \theta = 1$$

(b) $$R = \int_{R_1} p(x|w_2)dx$$

$$= \int_{-\infty}^{1} p(x|w_2)dx$$

$$= \Phi(\frac{1-2}{1})$$

$$= \Phi(-1)$$

$$= 0.1587$$

Example 2

Consider two one-dimensional normal distributions $p(x|\omega_1) \sim N(-1,1)$ and $p(x|\omega_2) \sim N(1,1)$ with zero-one loss function.

(a) For each value of $P(\omega_1)$, there is a corresponding Bayes decision rule. The resulting overall risk is called the Bayes risk. Plot Bayes risk as a function of $P(\omega_1)$.

(b) Please state the Bayes decision rule for $P(\omega_1) = 0.25$. For this decision rule, if the priors are then changed, the overall risk will change as well. For this fixed classifier, plot the over risk as a function of $P(\omega_1)$.

(c) Please state the minimax decision rule and calculate the minimax risk. For the minimax classifier (fixed), plot the overall risk as a function of $P(\omega_1)$.

Answers:

(a)

$$\begin{aligned}
\frac{p(\theta|\omega_1)}{p(\theta|\omega_2)} &= \frac{(\lambda_{12}-\lambda_{22})}{(\lambda_{21}-\lambda_{11})}\cdot\frac{P(\omega_2)}{P(\omega_1)} \\
\frac{\frac{1}{\sqrt{2\pi}}\exp\left[\frac{-(\theta+1)^2}{2}\right]}{\frac{1}{\sqrt{2\pi}}\exp\left[\frac{-(\theta-1)^2}{2}\right]} &= \frac{(1-0)}{(1-0)}\cdot\frac{1-P(\omega_1)}{P(\omega_1)} \\
\exp\left[-2\theta\right] &= \frac{1-P(\omega_1)}{P(\omega_1)} \\
\theta &= \frac{-1}{2}\cdot\ln\frac{1-P(\omega_1)}{P(\omega_1)}
\end{aligned}$$

(a)

$$
\begin{aligned}
R(P(\omega_1)) &= \lambda_{22} + (\lambda_{12} - \lambda_{22}) \int_{\mathcal{R}_1} p(x|\omega_2)dx \\
&+ P(\omega_1) \left[(\lambda_{11} - \lambda_{22}) + (\lambda_{21} - \lambda_{11}) \int_{\mathcal{R}_2} p(x|\omega_1)dx \right. \\
&- \left. (\lambda_{12} - \lambda_{22}) \int_{\mathcal{R}_1} p(x|\omega_2)dx \right] \\
&= \int_{-\infty}^{\theta} p(x|\omega_2)dx + P(\omega_1) \left[\int_{\theta}^{\infty} p(x|\omega_1)dx \right. \\
&- \left. \int_{-\infty}^{\theta} p(x|\omega_2)dx \right]
\end{aligned}
$$

(a)

$$\begin{aligned} R(P(\omega_1)) &= \Phi(\frac{\theta-1}{1}) + P(\omega_1)[1-\Phi(\frac{\theta+1}{1}) - \Phi(\frac{\theta-1}{1})] \\ &= \Phi(\theta-1) + P(\omega_1)[1-\Phi(\theta+1)-\Phi(\theta-1)] \end{aligned}$$

- Note that the $X \sim N(m, \sigma^2)$ cdf can always be expressed using the standard normal cdf (See page 187 in [1]).

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\big[\frac{-1}{2}\big(\frac{\tau - m}{\sigma}\big)^2\big] d\tau = \Phi(\frac{x-m}{\sigma}).$$

(b)

$$\begin{aligned} \theta &= \frac{-1}{2}\cdot\ln\frac{1-0.25}{0.25} = -0.5493 \\ R(P(\omega_1)) &= \Phi(\theta-1) + P(\omega_1)[1-\Phi(\theta+1)-\Phi(\theta-1)] \end{aligned}$$

So, we decide ω_1 if $x < -0.5493$. Otherwise, decide ω_2.

(c) $\int_{R_2} p(x|w_1)dx = \int_{R_1} p(x|w_2)dx$

$$\Rightarrow \int_{\theta}^{\infty} \frac{1}{\sqrt{2\pi}} \exp[\frac{-1}{2}(x+1)^2]dx = \int_{-\infty}^{\theta} \frac{1}{\sqrt{2\pi}} \exp[\frac{-1}{2}(x-1)^2]dx$$

$$\Rightarrow 1 - \Phi(\frac{\theta+1}{1}) = \Phi(\frac{\theta-1}{1})$$

$$\Rightarrow \Phi(-\theta-1) = \Phi(\theta-1)$$

$$\Rightarrow \theta = 0$$

So, we decide ω_1 if $x < 0$. Otherwise, decide ω_2.

(c)

$$\begin{aligned} R_{mm} &= \lambda_{22} + (\lambda_{12} - \lambda_{22}) \int_{R_1} p(x|\omega_2)dx \\ &= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} \exp[\frac{-1}{2}(x-1)^2]dx \\ &= \Phi(0-1) = 0.1587 \end{aligned} \tag{20}$$

2.3.2 Neyman-Pearson Criterion

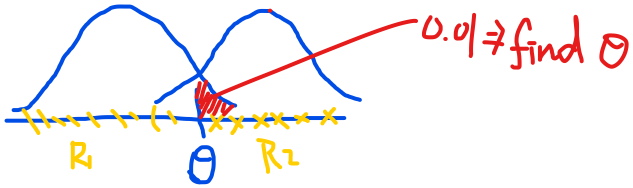
- In some problems, we might wish to minimize the total risk subject to a constraint $\int R(\alpha_i|\mathbf{x})d\mathbf{x} < constant$ for some particular i.
- Such a constraint might arise when there is a fixed resource that accompanies one particular action α_i.

- For instance, in an international airport, there might be a policy that we must **NOT** misclassify more than 0.1% of travelers as terrorists. We might then seek a decision rule that maximizes the chance of detecting a terrorist correctly subject to this constraint.

- We generally satisfy such a **Neyman-Pearson criterion** by adjusting decision boundaries numerically.

- For Gaussian and some other distributions, Neyman-Pearson solutions can be found analytically (Problem 6 and 7).

Example 3



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Consider the Neyman-Person criterion for two univariate normal distributions: $p(x|\omega_1) \sim N(-2, 2)$, $p(x|\omega_2) \sim N(2, 2)$ and $P(\omega_1) = P(\omega_2)$.

mean σ^2, variance

1. Suppose the maximum acceptable error rate for classifying a pattern that is actually in ω_1 as if it were in ω_2 is 0.01. What is the resulting single-point decision boundary?

2. For this boundary, what is the error rate for classifying ω_2 as ω_1?

3. What is the overall error rate? Also, compare your result to the Bayes error rate.

$$\frac{P(x,W_1)}{P(x,W_2)} = \frac{\overset{1}{\lambda_{12}} - \overset{0}{\lambda_{22}}}{\overset{1}{\lambda_{21}} - \overset{0}{\lambda_{11}}} \, \frac{\cancel{P(W_2)}}{\cancel{P(W_1)}}$$

Answers:

(a) $E_1 = 1 - \int_{-\infty}^{\theta} p(\mathbf{x}|w_1)d\mathbf{x}$

$$\Rightarrow E_1 = 1 - \overbrace{\Phi(\frac{\theta - \mu_1}{\sigma_1})}^{\text{standard normal distribution}}$$

$$\Rightarrow E_1 = 0.01 = 1 - \Phi(\frac{\theta + 2}{\sqrt{2}})$$

$$\Rightarrow \theta = \Phi^{-1}(0.99) \times \sqrt{2} - 2 = 2.33 \times \sqrt{2} - 2 = 1.295$$

(b) $E_2 = \int_{-\infty}^{\theta} p(\mathbf{x}|w_2) d\mathbf{x}$

$$= \Phi(\frac{\theta - \mu_2}{\sigma_2})$$

$$= \Phi(\frac{1.295 - 2}{\sqrt{2}})$$

$$= \Phi(-0.5)$$

$$= 0.3085$$

$$\Rightarrow \text{error rate} = E_2 \times P(w_2) = 0.3085 \times \frac{1}{2} = 0.15425$$

(c) Overall error rate of NP $= E_1 \times P(w_1) + E_2 \times P(w_2) = 0.15925$

For Bayes, $\theta = \frac{1}{2}(\mu_1 + \mu_2) = 0$

$$\Rightarrow E_1 = 1 - \Phi(\frac{0-(-2)}{\sqrt{2}}) = 0.0793$$

$$\Rightarrow E_2 = \Phi(\frac{0-2}{\sqrt{2}}) = 0.0793$$

Overall error rate of Bayes $= E_1 \times P(w_1) + E_2 \times P(w_2) = 0.0793$

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2.4.1 The Multicategory Case

- There are many different ways to represent pattern classifiers.
- One of the most useful is in terms of a set of discriminant functions $g_i(\mathbf{x}), i = 1, \ldots, c$.
- The classifier is said to assign a feature vector $\mathbf{x}$ to class ω_i if

$$g_i(\mathbf{x}) > g_j(\mathbf{x}) \qquad \text{for all } j \neq i \tag{21}$$

- A Bayes classifier is naturally represented in this way.
- For the general case, we can let $g_i(\mathbf{x}) = -R(\alpha_i|\mathbf{x})$ because the maximum discriminant function corresponds to the minimum conditional risk.
- For the minimum-error-rate case, we can simplify things by taking $g_i(\mathbf{x}) = P(\omega_i|\mathbf{x})$ so that the maximum discriminant function corresponds to the maximum posterior probability.

- Clearly, the choice of discriminant functions is not unique.
- In general, if we replace every $g_i(\mathbf{x})$ by $f(g_i(\mathbf{x}))$, where $f(\cdot)$ is a monotonically increasing function, the resulting classification is unchanged.
- This observation can lead to analytical and computational simplifications.

- For instance, any of the following choices gives identical classification results, but some can be much simpler to compute than others:

$$\begin{aligned} g_i(\mathbf{x}) &= P(\omega_i|\mathbf{x}) = \frac{p(\mathbf{x}|\omega_i)P(\omega_i)}{\sum_{j=1}^{c} p(\mathbf{x}|\omega_j)P(\omega_j)} && (22) \\ g_i(\mathbf{x}) &= p(\mathbf{x}|\omega_i)P(\omega_i) && (23) \\ g_i(\mathbf{x}) &= \ln p(\mathbf{x}|\omega_i) + \ln P(\omega_i) && (24) \end{aligned}$$

  where ln denotes natural logarithm.

- Even though the discriminant functions can be written in a variety of forms, the decision rules are equivalent.

- The effect of any decision rule is to divide the feature space into c **decision regions**, $\mathcal{R}_1, \ldots, \mathcal{R}_c$.

- If $g_i(\mathbf{x}) > g_j(\mathbf{x})$ for all $j \neq i$, then the decision rule calls for us to assign $\mathbf{x}$ to ω_i, i.e., $\mathbf{x}$ is in $\mathcal{R}_i$.

- The decision regions are separated by **decision boundaries** (Figure 5).

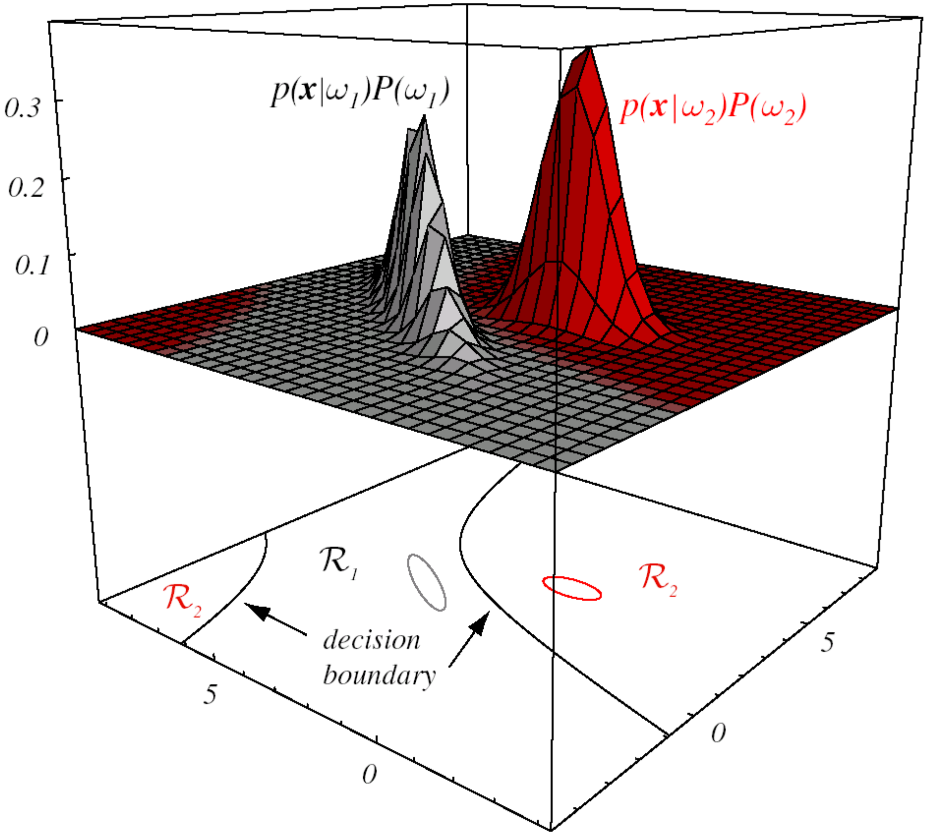

Figure 5: The decision boundaries consist of two hyperbolas, and thus the decision region $\mathcal{R}_2$ is not simply connected.

2.4.2 The Two-Category Case

- While the two-category problem is just a special case, it has traditionally received much more attentions.
- Instead of using two discriminant functions, it is more common to define a single discriminant function

$$g(\mathbf{x}) = g_1(\mathbf{x}) - g_2(\mathbf{x}) \tag{25}$$

and the decision rule is to

$$\text{Decide } \omega_1 \text{ if } g(\mathbf{x}) > 0\text{; otherwise decide } \omega_2.$$

- Of the various forms in which the minimum-error-rate discriminant function can be written, the following two (derived from Eqs. (22) and (24)) are particularly convenient:

$$g(\mathbf{x}) = P(\omega_1|\mathbf{x}) - P(\omega_2|\mathbf{x}) \tag{26}$$

$$g(\mathbf{x}) = \ln \frac{p(\mathbf{x}|\omega_1)}{p(\mathbf{x}|\omega_2)} + \ln \frac{P(\omega_1)}{P(\omega_2)} \tag{27}$$

Example 4

Consider a two-class classification problem. Given the class-conditional probability density functions

$$
\begin{aligned}
p(x|\omega_1) &\sim N(0,1) \\
p(x|\omega_2) &\sim N(0,2)
\end{aligned}
$$

with $P(\omega_1) = 0.3$ and $P(\omega_2) = 0.7$. Please specify the decision regions based on Bayesian decision theory.

Answers:

$$\frac{p(x|w_1)P(w_1)}{p(x|w_2)P(w_2)} = \frac{\sigma_2 P(w_1)e^{-\frac{1}{2}(\frac{x}{\sigma_1})^2}}{\sigma_1 P(w_2)e^{-\frac{1}{2}(\frac{x}{\sigma_2})^2}}$$

By taking logarithm, we have

$$\ln\{\frac{\sigma_2 P(w_1)}{\sigma_1 P(w_2)}\} - \frac{1}{2}x^2(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}) = -0.5007 - \frac{x^2}{4} < 0$$

$\Rightarrow$ We always decide w_2

$\Rightarrow R_2 = [-\infty, \infty],\ R_1 = \{\emptyset\}$

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2.5 The Normal Density

- Of the various density functions, none has received more attention than the multivariate normal or Gaussian density.
- To a large extent, this attention is due to its analytical tractability.
- In this section, we provide a brief exposition of the multivariate normal density.

- Recall the definition of the **expected value** of a scalar function $f(x)$, defined for some density $p(x)$:

$$\mathcal{E}[f(x)] = \int_{-\infty}^{\infty} f(x)p(x)dx \tag{28}$$

- If the values of x are restricted to points in a discrete set $\mathcal{D}$, we must sum over all samples as

$$\mathcal{E}[f(x)] = \sum_{x \in \mathcal{D}} f(x)P(x) \tag{29}$$

where $P(x)$ is a probability mass function.

2.5.1 Univariate Density

- We begin with the univariate normal density,

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \frac{(x-m)^2}{\sigma^2} \right] \tag{30}$$

- The expected value of x is

$$m = \mathcal{E}[x] = \int_{-\infty}^{\infty} x p(x) dx, \tag{31}$$

- The **variance** of x is

$$\sigma^2 = \mathcal{E}[(x-m)^2] = \int_{-\infty}^{\infty} (x-m)^2 p(x) dx, \tag{32}$$

- For simplicity, we often abbreviate Eq. (30) by writing $p(x) \sim N(m, \sigma^2)$ to say that x is distributed normally with mean m and variance σ^2.

- Samples from a normal distribution tend to cluster about the mean, with a spread related to the standard deviation σ (Figure 6).

- Moreover, as stated by the **Central Limit Theorem**, the sum of a large number of independent random variables will lead to a Gaussian distribution (computer exercise 5).

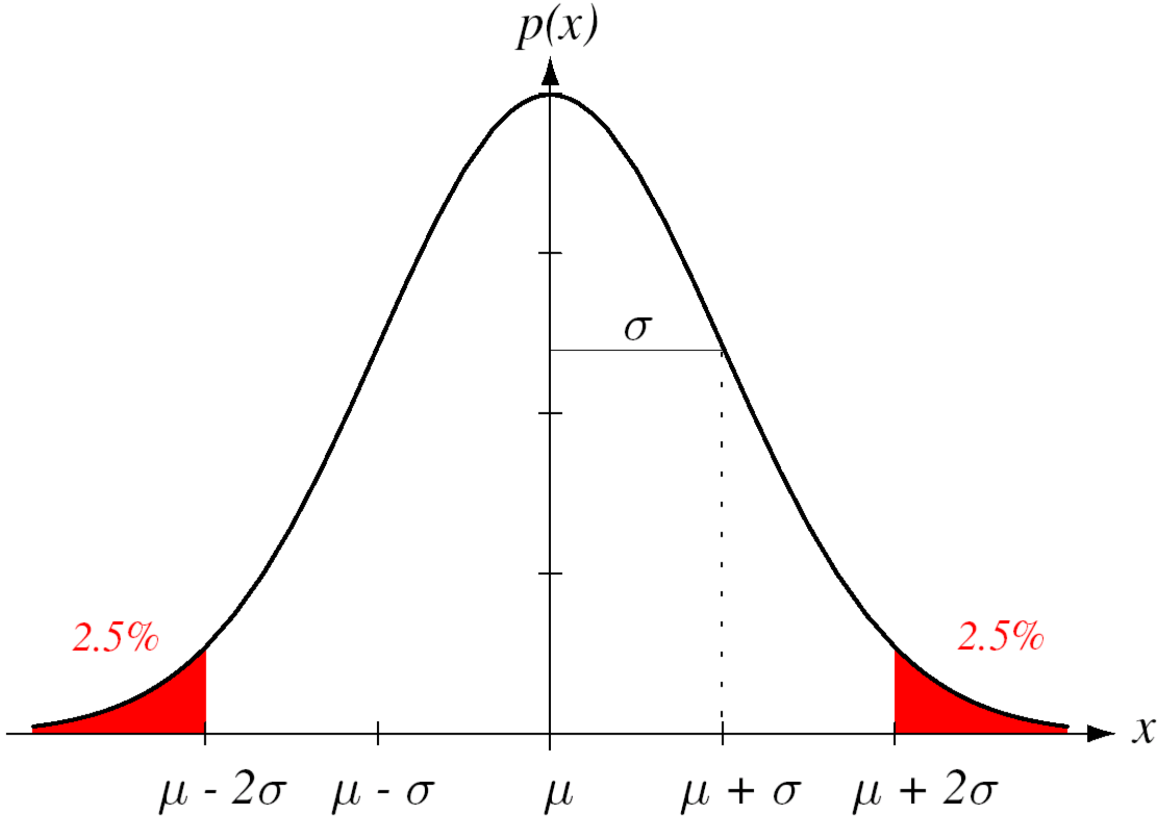

Figure 6: A univariate normal distribution has roughly 95% of its area in the range $|x - m| \leq 2\sigma$, as shown.

2.5.2 Multivariate Density

- The multivariate normal density is

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\mathbf{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right] \qquad (33)$$

where $\mathbf{x}$ is a d-component column vector, $\boldsymbol{\mu}$ is a d-component mean vector, $\mathbf{\Sigma}$ is a d-by-d **covariance matrix**, and $|\mathbf{\Sigma}|$ and $\mathbf{\Sigma}^{-1}$ are its determinant and inverse, respectively. Also, we let $(\mathbf{x}-\boldsymbol{\mu})^t$ denote the transpose of $\mathbf{x}-\boldsymbol{\mu}$.

- Formally, we have

$$\boldsymbol{\mu} = \mathcal{E}[\mathbf{x}] = \int \mathbf{x} p(\mathbf{x}) d\mathbf{x}, \tag{34}$$

and

$$\boldsymbol{\Sigma} = \mathcal{E}[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^t] = \int (\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^t p(\mathbf{x}) d\mathbf{x}, \tag{35}$$

where the expected value of a vector or a matrix is found by taking the expected values of its components.

- In other words, if x_i is the ith component of $\mathbf{x}$, m_i the ith component of $\boldsymbol{\mu}$, and σ_{ij} the ijth component of $\boldsymbol{\Sigma}$, then

$$m_i = \mathcal{E}[x_i] \tag{36}$$

and

$$\sigma_{ij} = \mathcal{E}[(x_i - m_i)(x_j - m_j)]. \tag{37}$$

- The diagonal elements σ_{ii} are the variances of the respective x_i and off-diagonal elements are the **covariances** of x_i and x_j.
- If x_i and x_j are **statistically independent**, then $\sigma_{ij} = 0$.

- Linear combination of jointly normally distributed random variables are normally distributed.

- In particular, if $p(\mathbf{x}) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\mathbf{A}$ is a d-by-k matrix and $\mathbf{y} = \mathbf{A}^t\mathbf{x}$, then $p(\mathbf{y}) \sim N(\mathbf{A}^t\boldsymbol{\mu}, \mathbf{A}^t\boldsymbol{\Sigma}\mathbf{A})$, as illustrated in Figure 7.

- In the special case where $k = 1$ and $\mathbf{A}$ is a unit-length vector $\mathbf{a}$, $y = \mathbf{a}^t\mathbf{x}$ is a scalar that represents the projection of $\mathbf{x}$ onto a line in the direction of $\mathbf{a}$; in that case $\mathbf{a}^t\boldsymbol{\Sigma}\mathbf{a}$ is the variance of the projection of $\mathbf{x}$.

- It is sometimes convenient to perform a coordinate transformation that converts an arbitrary multivariate normal distribution into a spherical one.

- If we define $\mathbf{\Phi}$ to be the matrix whose columns are the eigenvectors of $\mathbf{\Sigma}$, and $\mathbf{\Lambda}$ the diagonal matrix of the corresponding eigenvalues, then the **whitening transformation**

$$\mathbf{A}_w = \mathbf{\Phi}\mathbf{\Lambda}^{-1/2} \tag{38}$$

  ensures that the transformed distribution has covariance matrix equal to the identity matrix.

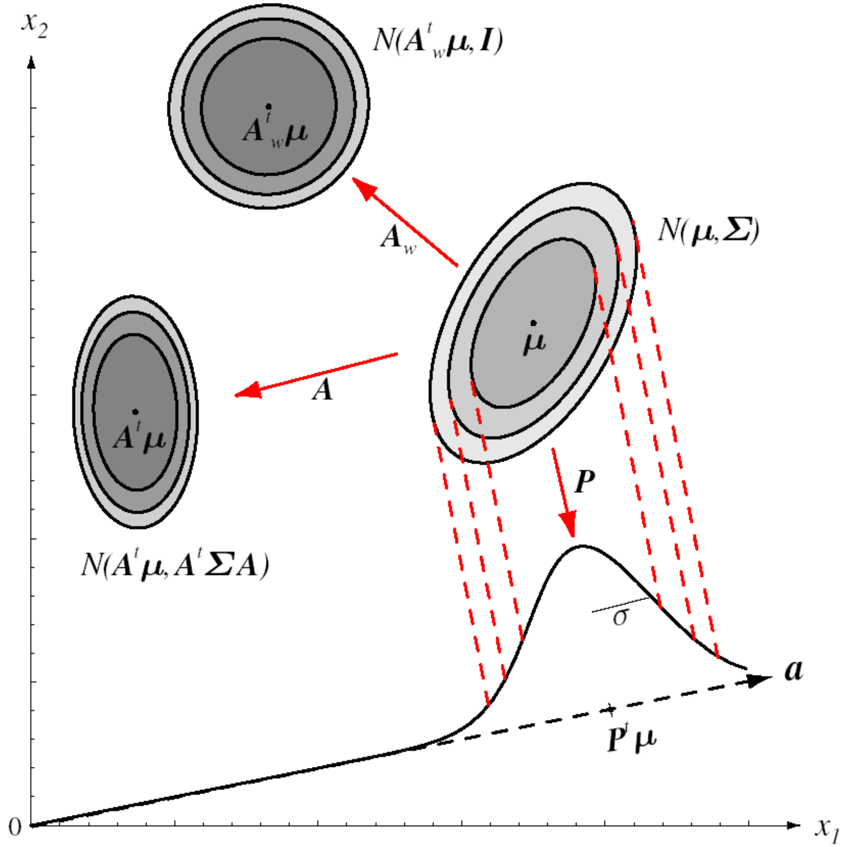
x_2
$N(\mathbf{A}_w^t\boldsymbol{\mu}, \mathbf{I})$
$\mathbf{A}_w^t\boldsymbol{\mu}$
$\mathbf{A}_w$
$N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
$\boldsymbol{\mu}$
$\mathbf{A}$
$\mathbf{A}^t\boldsymbol{\mu}$
$\mathbf{P}$
$N(\mathbf{A}^t\boldsymbol{\mu}, \mathbf{A}^t\boldsymbol{\Sigma}\mathbf{A})$
σ
$\mathbf{a}$
$\mathbf{P}^t\boldsymbol{\mu}$
0
x_1

Figure 7: A whitening transformation convert a normal distribution into a circularly symmetric Gaussian.

- Samples drawn from a normal population tend to fall in a single cloud (Figure 8).
- The center of the cloud is determined by the mean vector.
- The shape of the cloud is determined by the covariance matrix.
- The loci of points of constant density are hyperellipsoids for which $(\mathbf{x}-\boldsymbol{\mu})^t\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$ is constant.

- The principal axes of these hyperellipsoids are given by the eigenvectors of $\boldsymbol{\Sigma}$; the eigenvalues determine the lengths of these axes.

- The quantity

$$r^2 = (\mathbf{x} - \boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \tag{39}$$

  is called the squared **Mahalanobis distance** from $\mathbf{x}$ to $\boldsymbol{\mu}$.

- In Figure 8, an ellipse consists of points with an equal Mahalanobis distance from the mean $\boldsymbol{\mu}$.

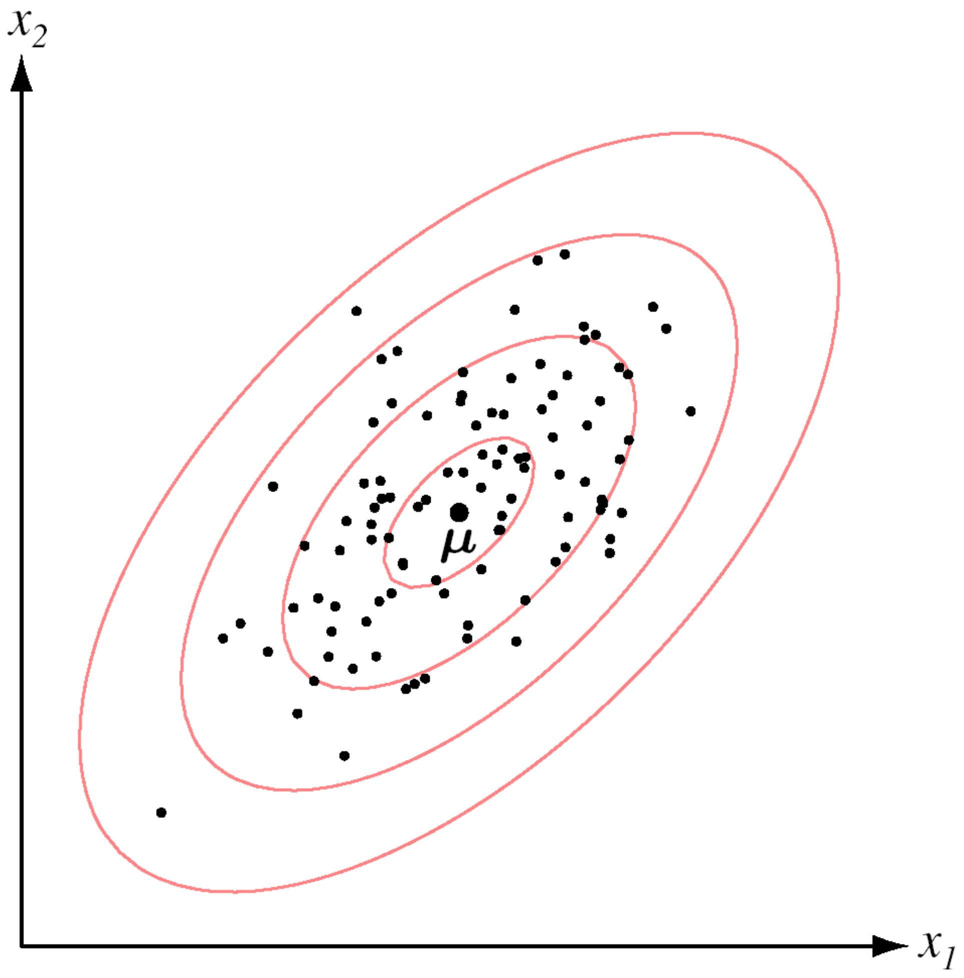

Figure 8: Samples drawn from a two-dimensional Gaussian density.

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2.6 Discriminant Functions for the Normal Density

- In section 2.4.1, the minimum-error-rate classification can be achieved by use of the discriminant functions

$$g_i(\mathbf{x}) = \ln p(\mathbf{x}|\omega_i) + \ln P(\omega_i) \tag{40}$$

- This expression can be evaluated if the densities are multivariate normal. Thus, we have

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_i)^t \boldsymbol{\Sigma}_i^{-1}(\mathbf{x} - \boldsymbol{\mu}_i) - \frac{d}{2}\ln 2\pi - \frac{1}{2}\ln|\boldsymbol{\Sigma}_i| + \ln P(\omega_i) \tag{41}$$

Let us examine this discriminant function for a number of special cases.

2.6.1 Case 1: $\boldsymbol{\Sigma}_i = \sigma^2\mathbf{I}$

- In this case, the covariance matrix is diagonal, being σ^2 times the identity matrix.
- Geometrically, this corresponds to equal-size hyperspherical clusters.

$$|\boldsymbol{\Sigma}_i| = \sigma^{2d} \quad \text{and} \quad \boldsymbol{\Sigma}_i^{-1} = \frac{1}{\sigma^2}\mathbf{I}. \tag{42}$$

- Both $|\boldsymbol{\Sigma}_i|$ and $(d/2)\ln 2\pi$ in Eq. (41) are constants that can be ignored.

- Thus, we obtain the simplified discriminant functions

$$g_i(\mathbf{x}) = -\frac{1}{2\sigma^2}\|\mathbf{x} - \boldsymbol{\mu}_i\|^2 + \ln P(\omega_i) \tag{43}$$

where $\|\cdot\|$ denotes the Euclidean norm, that is

$$\|\mathbf{x} - \boldsymbol{\mu}_i\|^2 = (\mathbf{x} - \boldsymbol{\mu}_i)^t(\mathbf{x} - \boldsymbol{\mu}_i) \tag{44}$$

- Furthermore, it is actually not necessary to compute norms.
- Note that the expansion of $(\mathbf{x} - \boldsymbol{\mu}_i)^t(\mathbf{x} - \boldsymbol{\mu}_i)$ yields

$$g_i(\mathbf{x}) = -\frac{1}{2\sigma^2}\left[\mathbf{x}^t\mathbf{x} - 2\boldsymbol{\mu}_i^t\mathbf{x} + \boldsymbol{\mu}_i^t\boldsymbol{\mu}_i\right] + \ln P(\omega_i) \qquad (45)$$

- The term $\mathbf{x}^t\mathbf{x}$ is the same for all i, making it an ignorable term.

- Thus, we obtain the equivalent **linear discriminant functions**

$$g_i(\mathbf{x}) = \mathbf{w}_i^t \mathbf{x} + w_{i0} \tag{46}$$

where

$$\mathbf{w}_i = \frac{1}{\sigma^2} \boldsymbol{\mu}_i \tag{47}$$

and

$$w_{i0} = \frac{-1}{2\sigma^2} \boldsymbol{\mu}_i^t \boldsymbol{\mu}_i + \ln P(\omega_i) \tag{48}$$

- We call w_{i0} the **threshold** or **bias** for ith category.

- The decision boundaries are pieces of hyperplanes defined by the equations $g_i(\mathbf{x}) = g_j(\mathbf{x})$ for two categories with the highest posterior probabilities.

- For this particular case, the equation can be written as

$$\mathbf{w}^t(\mathbf{x} - \mathbf{x}_0) = 0 \tag{49}$$

where

$$\mathbf{w} = \boldsymbol{\mu}_i - \boldsymbol{\mu}_j \tag{50}$$

and

$$\mathbf{x}_0 = \frac{1}{2}(\boldsymbol{\mu}_i + \boldsymbol{\mu}_j) - \frac{\sigma^2}{\|\boldsymbol{\mu}_i - \boldsymbol{\mu}_j\|^2} \ln \frac{P(\omega_i)}{P(\omega_j)}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j) \tag{51}$$

- These equations define a hyperplane through the point $\mathbf{x}_0$ and orthogonal to the vector $\mathbf{w}$.

- If $P(\omega_i) = P(\omega_j)$, the second term on the right of Eq. (51) vanishes. Thus, the point $\mathbf{x}_0$ is halfway between the means (Figure 9, 11, and 13).

- If $P(\omega_i) \neq P(\omega_j)$, the point $\mathbf{x}_0$ shifts away from the more likely mean (Figure 10, 12, and 14).

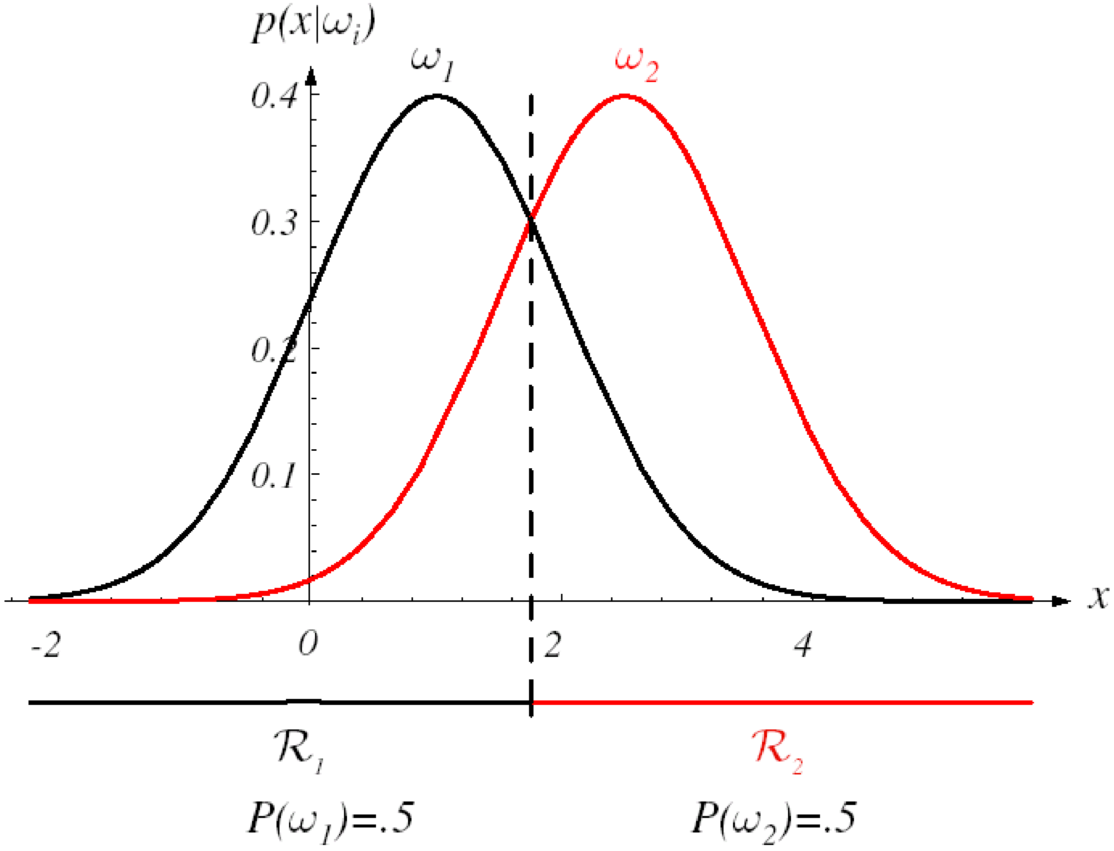

Figure 9: An one-dimensional example

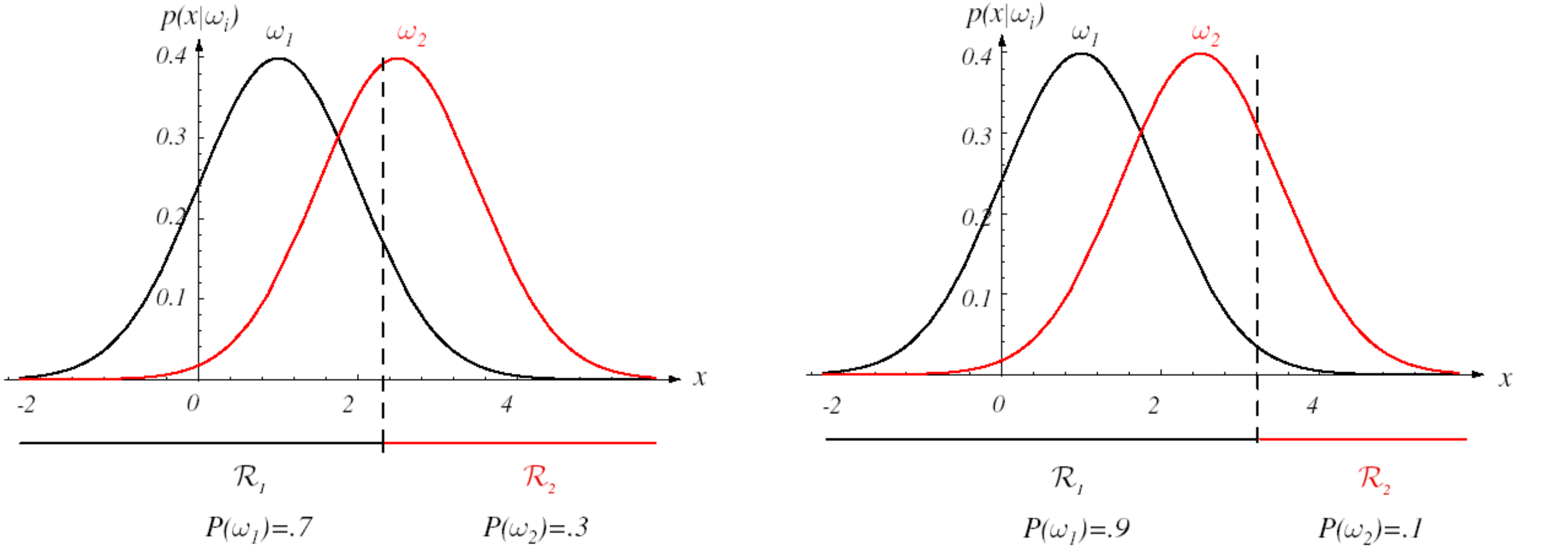

Figure 10: As priors are changed, the decision boundary shifts.

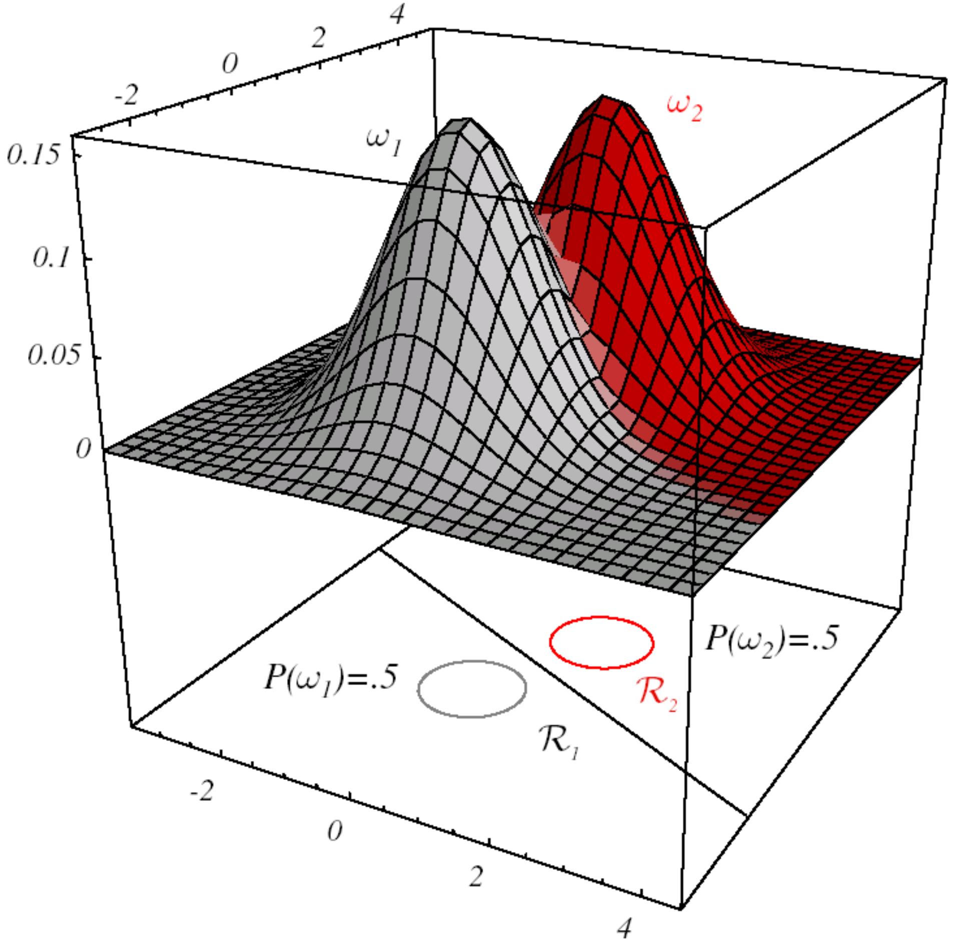

Figure 11: A two-dimensional example

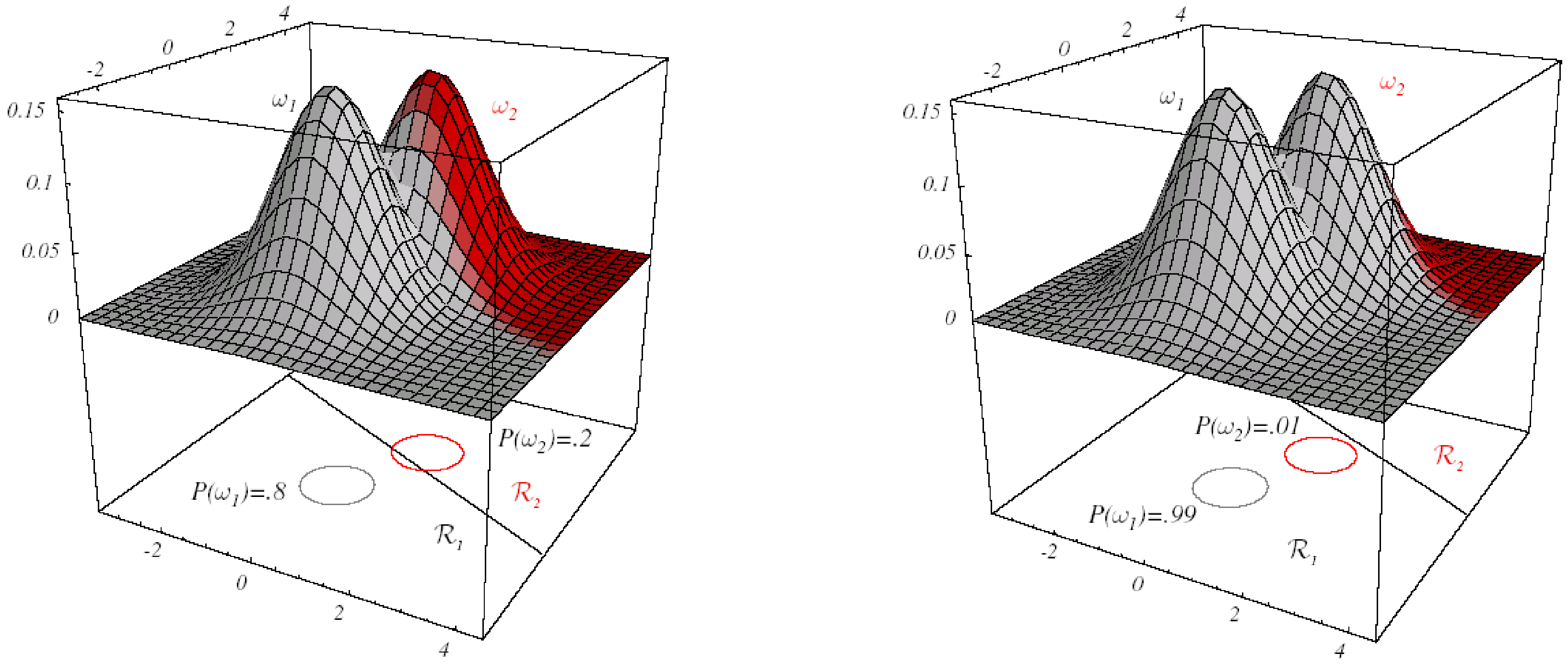

Figure 12: As priors are changed, the decision boundary shifts.

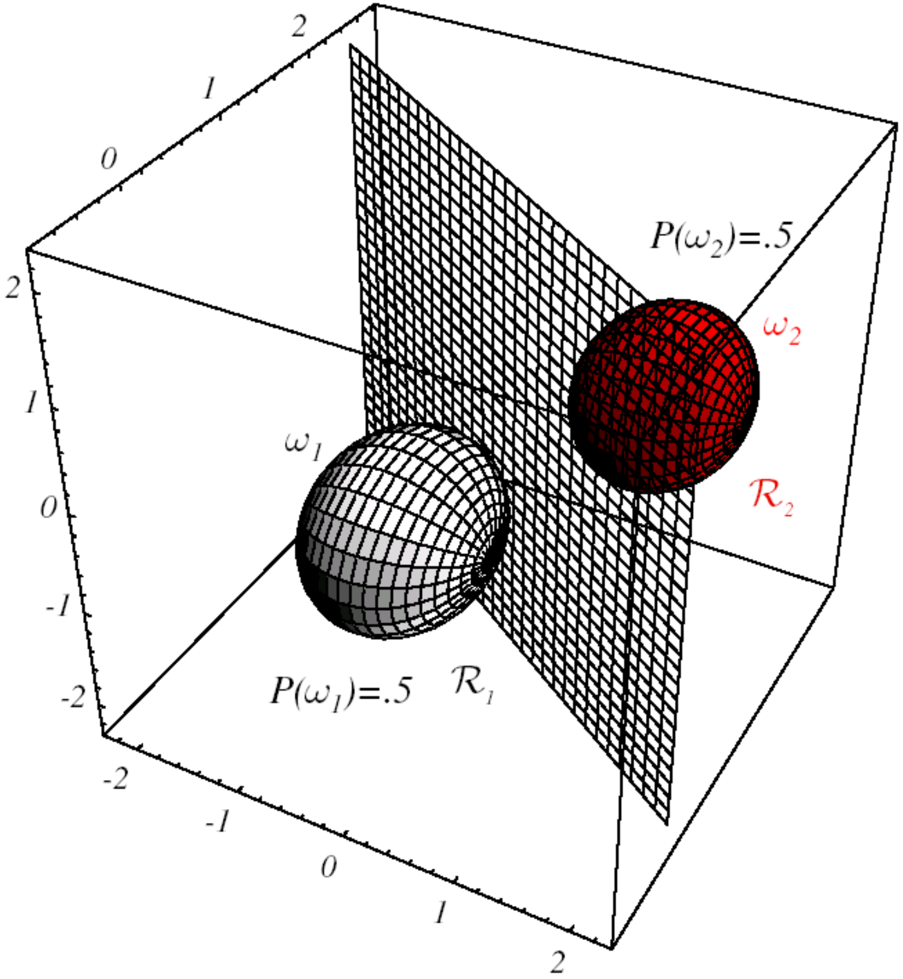

Figure 13: A three-dimensional example

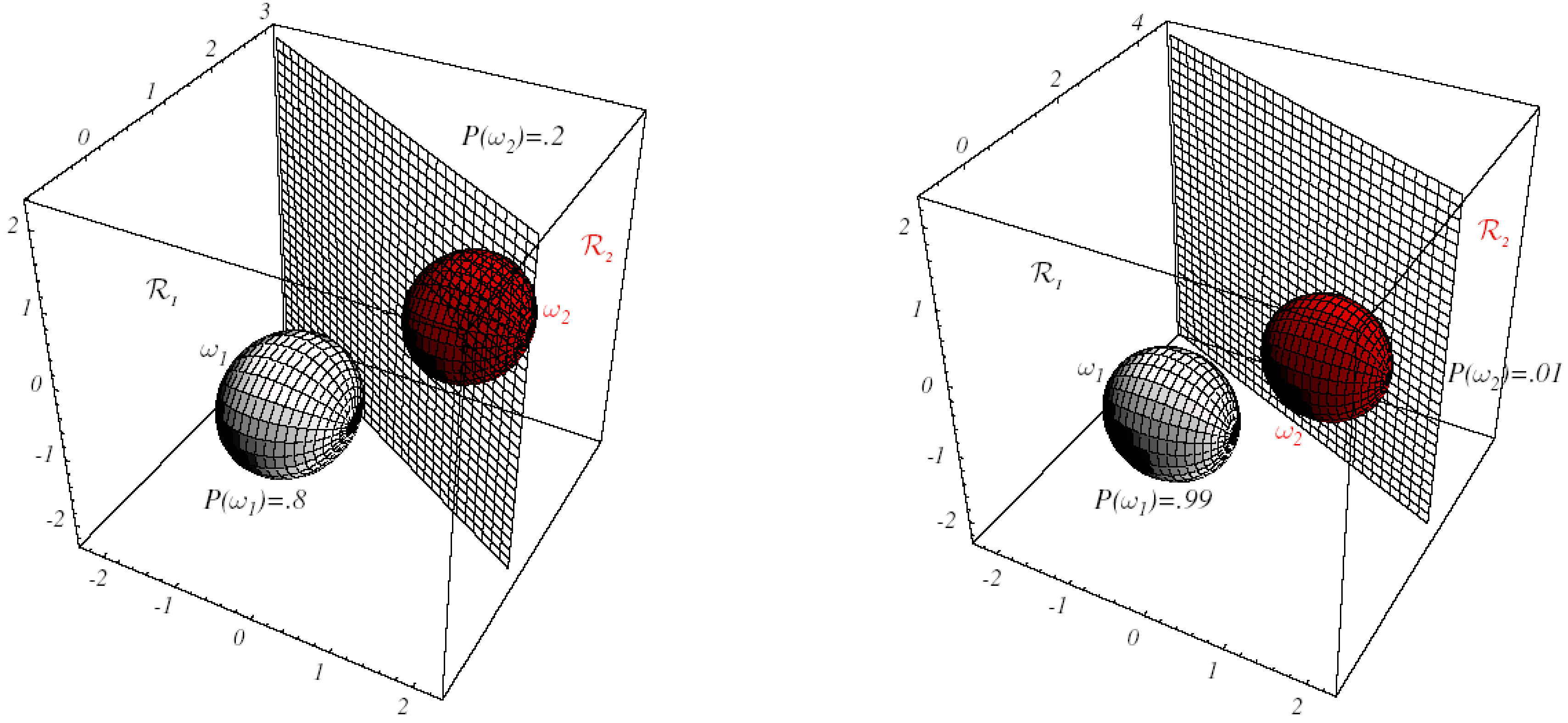

Figure 14: As priors are changed, the decision boundary shifts.

- If the priors are the same for all categories, the $\ln P(\omega_i)$ becomes unimportant terms.
- When this happens, the optimal decision rule can be stated very easily:

  > Compute the squared Euclidean distances $(\mathbf{x} - \boldsymbol{\mu}_i)^t(\mathbf{x} - \boldsymbol{\mu}_i)$ and assign a feature vector $\mathbf{x}$ to the category of the nearest mean.

- Such a classifier is called a **minimum-distance classifier**.

2.6.2 Case 2: $\boldsymbol{\Sigma}_i = \boldsymbol{\Sigma}$

- Another simple case arises when the covariance matrices for all categories are identical.
- Geometrically, this corresponds to the situation in which all samples fall in hyperellipsoidal clusters of equal size and shape.
- The terms, $|\boldsymbol{\Sigma}_i|$ and $(d/2)\ln 2\pi$, in Eq. 41 can be ignored because they are independent of i.

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_i)^t \boldsymbol{\Sigma}_i^{-1}(\mathbf{x} - \boldsymbol{\mu}_i) + \ln P(\omega_i) \qquad (52)$$

- If the priors are the same for all categories, the term $\ln P(\omega_i)$ can also be ignored.

- When this happens, the optimal decision rule can once again be stated very easily:

  > Compute the squared Mahalanobis distances $(\mathbf{x} - \boldsymbol{\mu}_i)^t \boldsymbol{\Sigma}_i^{-1} (\mathbf{x} - \boldsymbol{\mu}_i)$ and assign a feature vector $\mathbf{x}$ to the category of the nearest mean.

- Expansion of the form $(\mathbf{x}-\boldsymbol{\mu}_i)^t\boldsymbol{\Sigma}_i^{-1}(\mathbf{x}-\boldsymbol{\mu}_i)$ results in a term $\mathbf{x}^t\boldsymbol{\Sigma}_i^{-1}\mathbf{x}$ which is independent of i.
- After this term is dropped, the resulting discriminant functions are again linear:

$$g_i(\mathbf{x}) = \mathbf{w}_i^t\mathbf{x} + w_{i0} \tag{53}$$

where

$$\mathbf{w}_i = \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_i \tag{54}$$

and

$$w_{i0} = \frac{-1}{2}\boldsymbol{\mu}_i^t\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_i + \ln P(\omega_i) \tag{55}$$

- Because the discriminants are linear, the resulting decision boundaries are again hyperplanes (Figure 15 and 16).

- If $\mathcal{R}_i$ and $\mathcal{R}_j$ are contiguous, the boundary between them has the equation

$$\mathbf{w}^t(\mathbf{x} - \mathbf{x}_0) = 0 \tag{56}$$

where

$$\mathbf{w} = \mathbf{\Sigma}^{-1}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j) \tag{57}$$

and

$$\mathbf{x}_0 = \frac{1}{2}(\boldsymbol{\mu}_i + \boldsymbol{\mu}_j) - \frac{\ln\left[P(\omega_i)/P(\omega_j)\right]}{(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^t\mathbf{\Sigma}^{-1}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j) \tag{58}$$

- Because $\mathbf{\Sigma}^{-1}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)$ is generally not in the direction of $\boldsymbol{\mu}_i - \boldsymbol{\mu}_j$, the decision hyperplane is generally not orthogonal to the line connecting the means.

- If the priors are equal, $\mathbf{x}_0$ is halfway between the means.

- If $P(\omega_i) \neq P(\omega_j)$, the decision hyperplane is shifted away from the more likely mean (Figure 15 and 16).

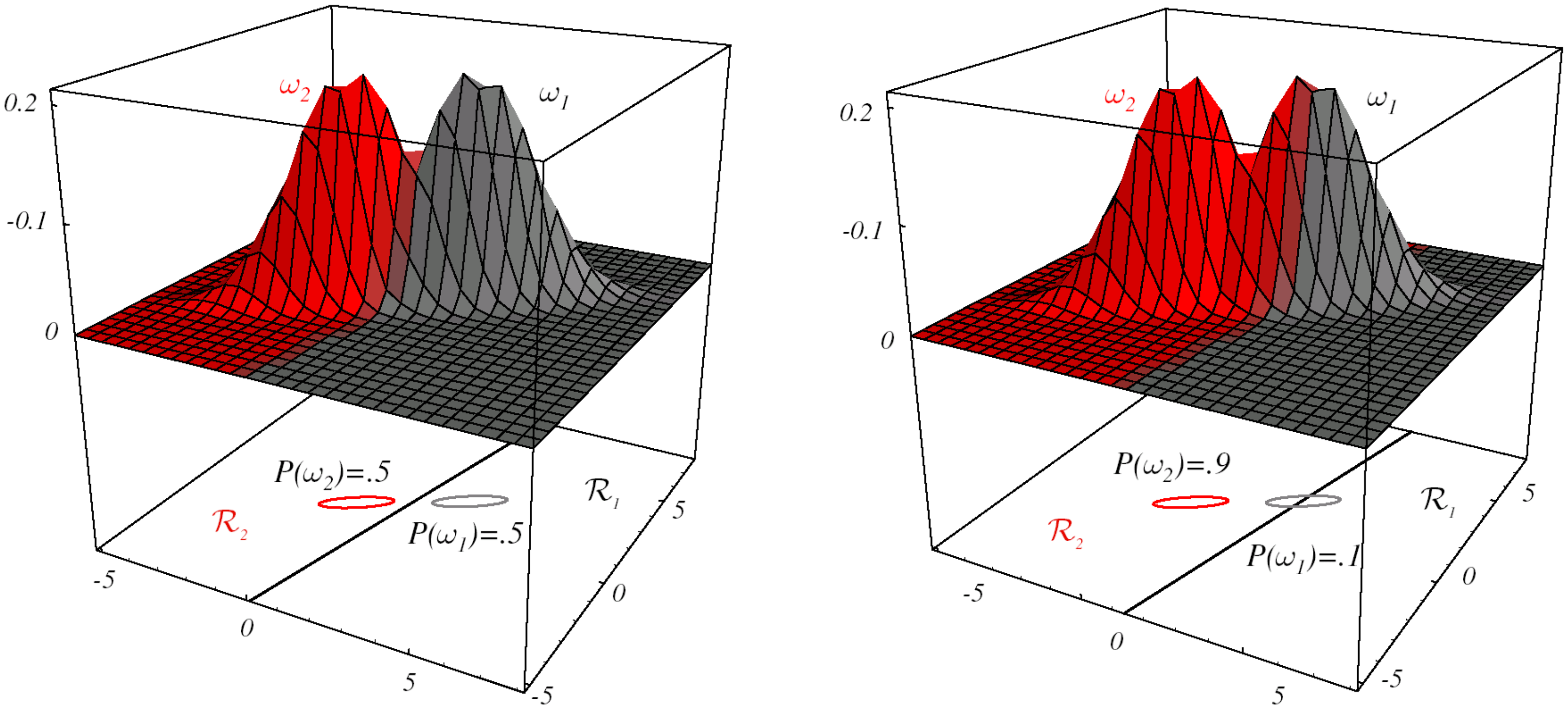

Figure 15: For equal but asymmetric Gaussian distributions, the decision hyperplanes are not perpendicular to the line connecting the means in general.

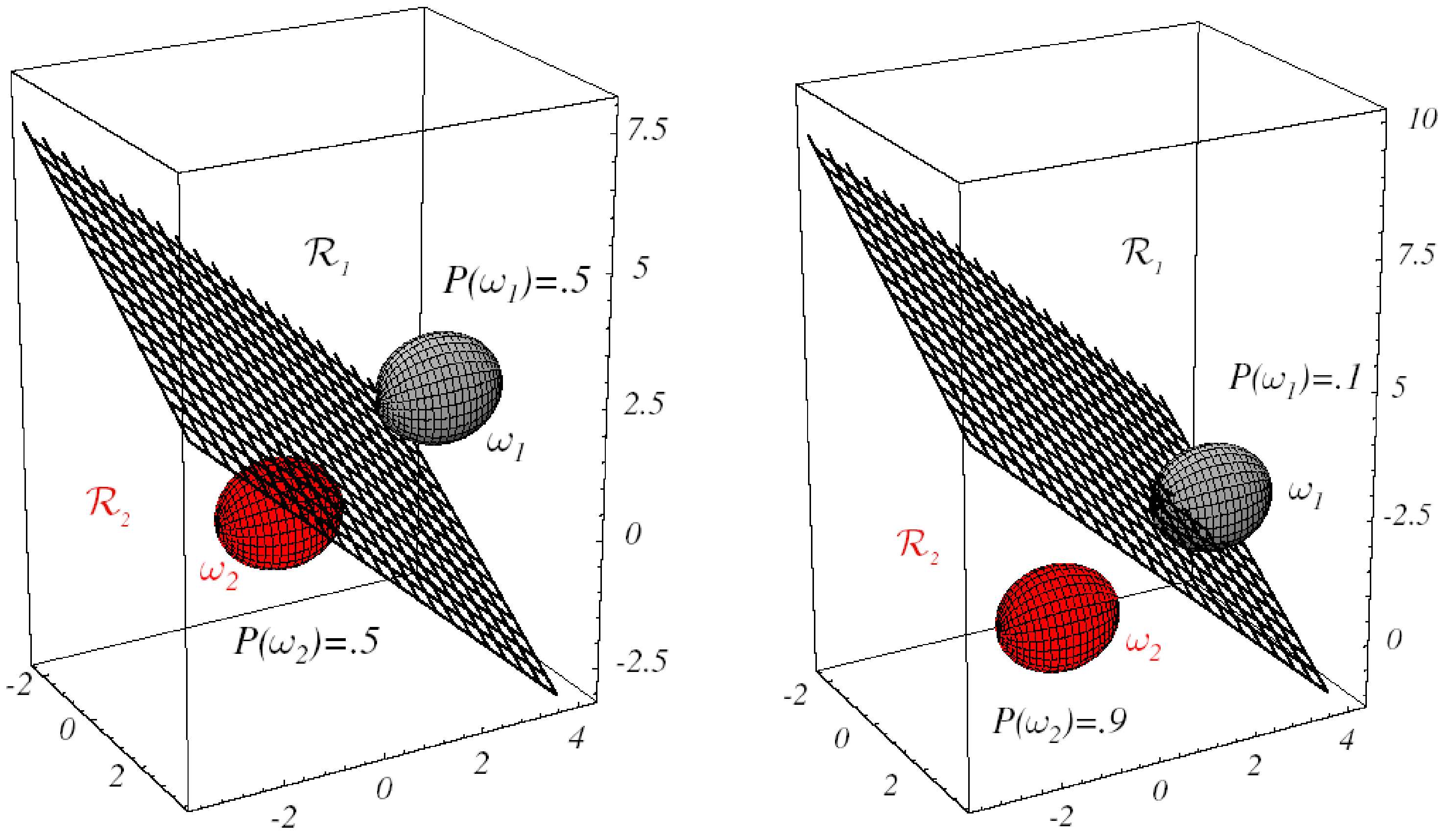

Figure 16: For equal but asymmetric Gaussian distributions, the decision hyperplanes are not perpendicular to the line connecting the means in general.

2.6.3 Case 3: $\boldsymbol{\Sigma}_i$ = arbitrary

- The only term that can be ignored from Eq. 41 is the $(d/2)\ln 2\pi$, and the resulting discriminants are inherently quadratic:

$$g_i(\mathbf{x}) = \mathbf{x}^t\mathbf{W}_i\mathbf{x} + \mathbf{w}_i^t\mathbf{x} + w_{i0} \tag{59}$$

where

$$\mathbf{W}_i = \frac{-1}{2}\boldsymbol{\Sigma}_i^{-1} \tag{60}$$

$$\mathbf{w}_i = \boldsymbol{\Sigma}_i^{-1}\boldsymbol{\mu}_i \tag{61}$$

and

$$w_{i0} = \frac{-1}{2}\boldsymbol{\mu}_i^t\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\mu}_i - \frac{1}{2}\ln|\boldsymbol{\Sigma}_i| + \ln P(\omega_i) \tag{62}$$

- In the two-category case, the decision boundaries are **hyperquadrics**.

- They can assume any of the general forms: hyperplanes, pair of hyperplanes, hyperellipsoids, and hyperhyperboloids of various types.

- Even in one dimension, for arbitrary variance the decision regions need not to be simply connected.

- Two-dimensional examples are shown in Figs 18, 19, and 20.

- Three-dimensional examples are shown in Figs 21, 22, and 23.

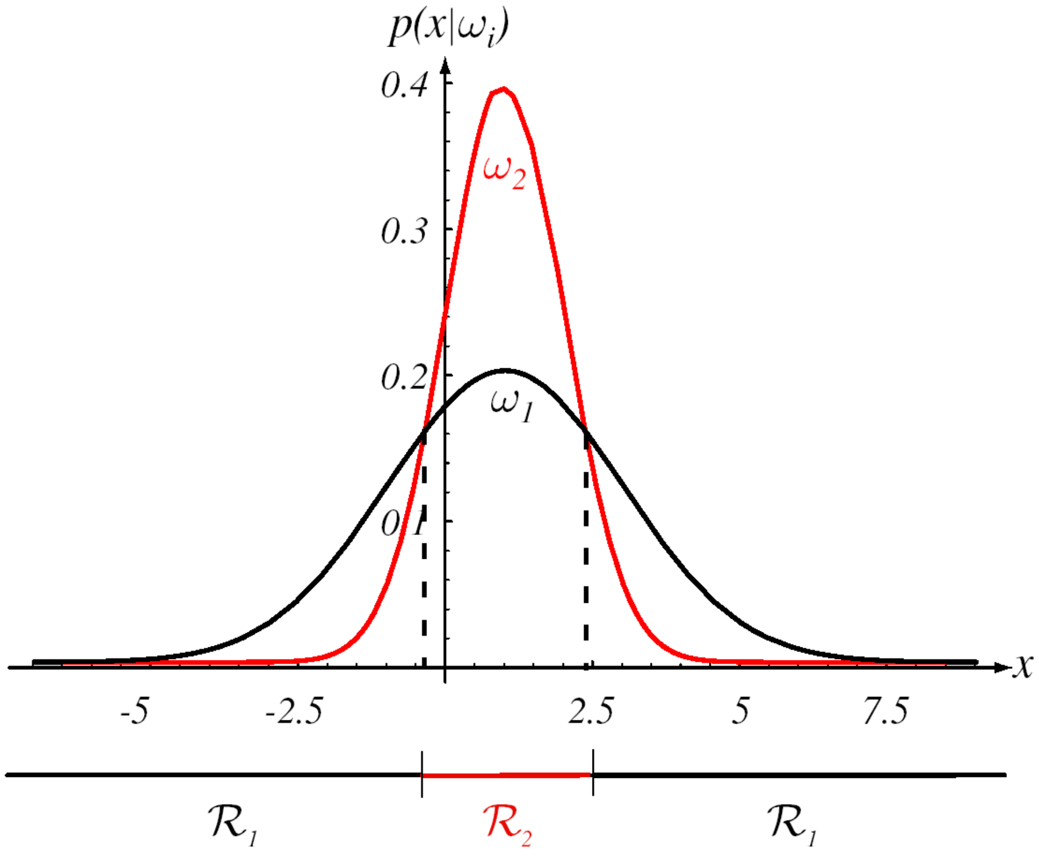

Figure 17: Non-simply connected decision regions can arise for Gaussian having unequal variance.

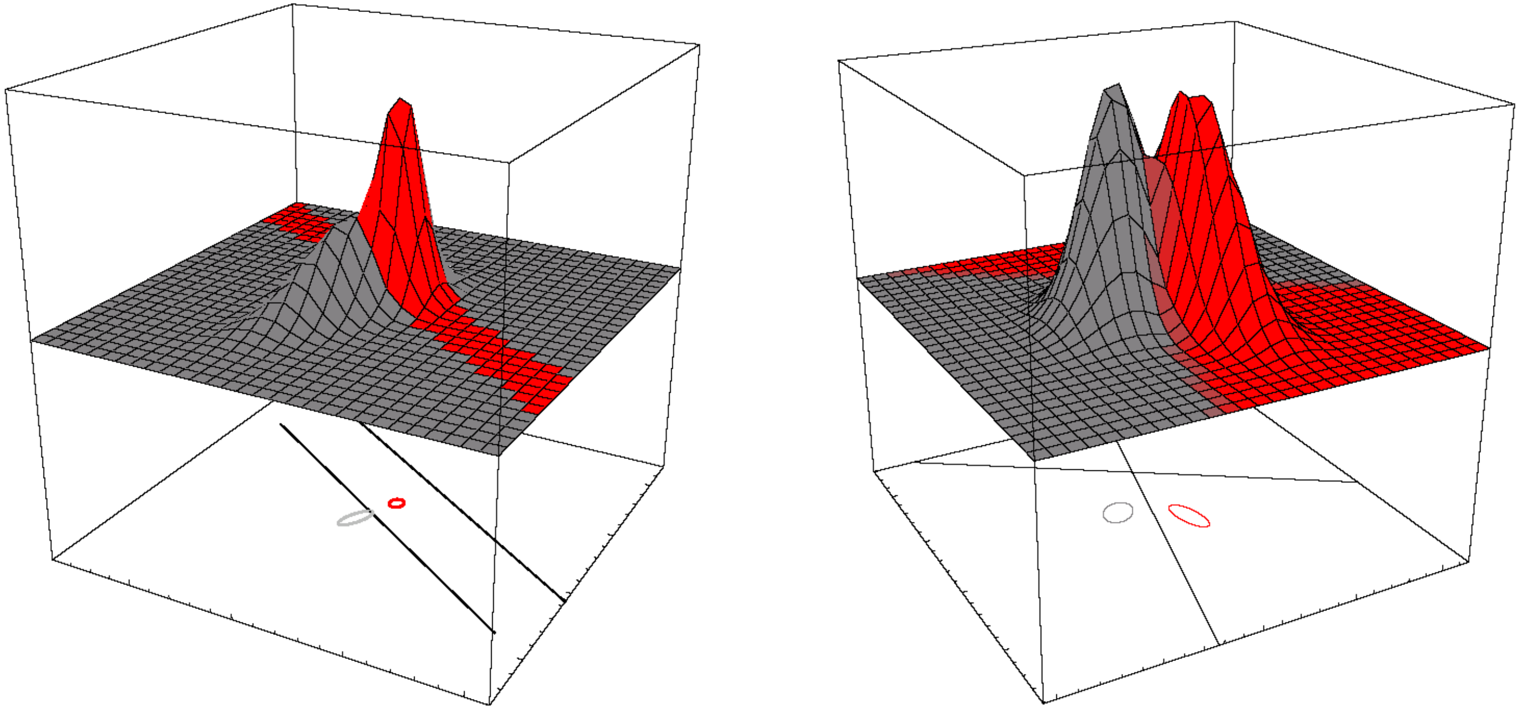

Figure 18: Arbitrary Gaussian distributions lead decision boundaries that are generally hyperquadratics.

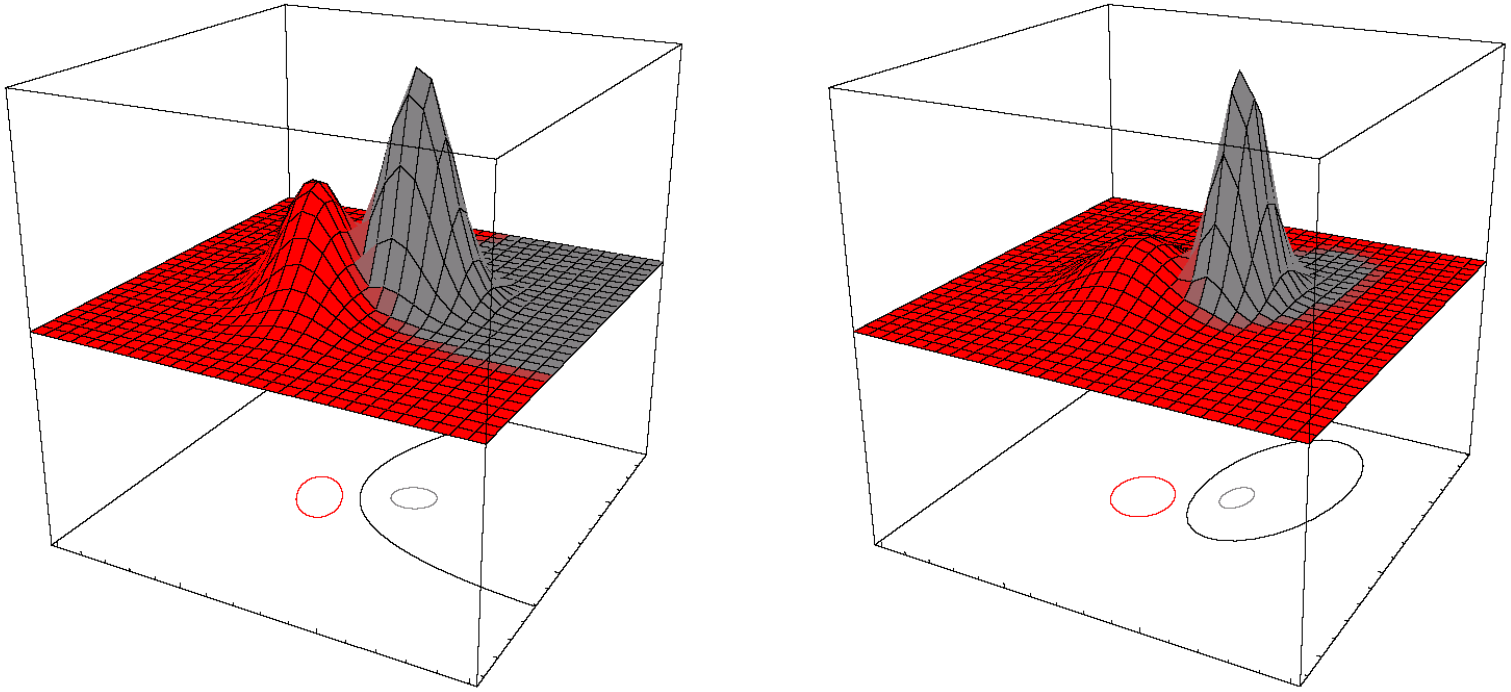

Figure 19: Arbitrary Gaussian distributions lead decision boundaries that are generally hyperquadratics.

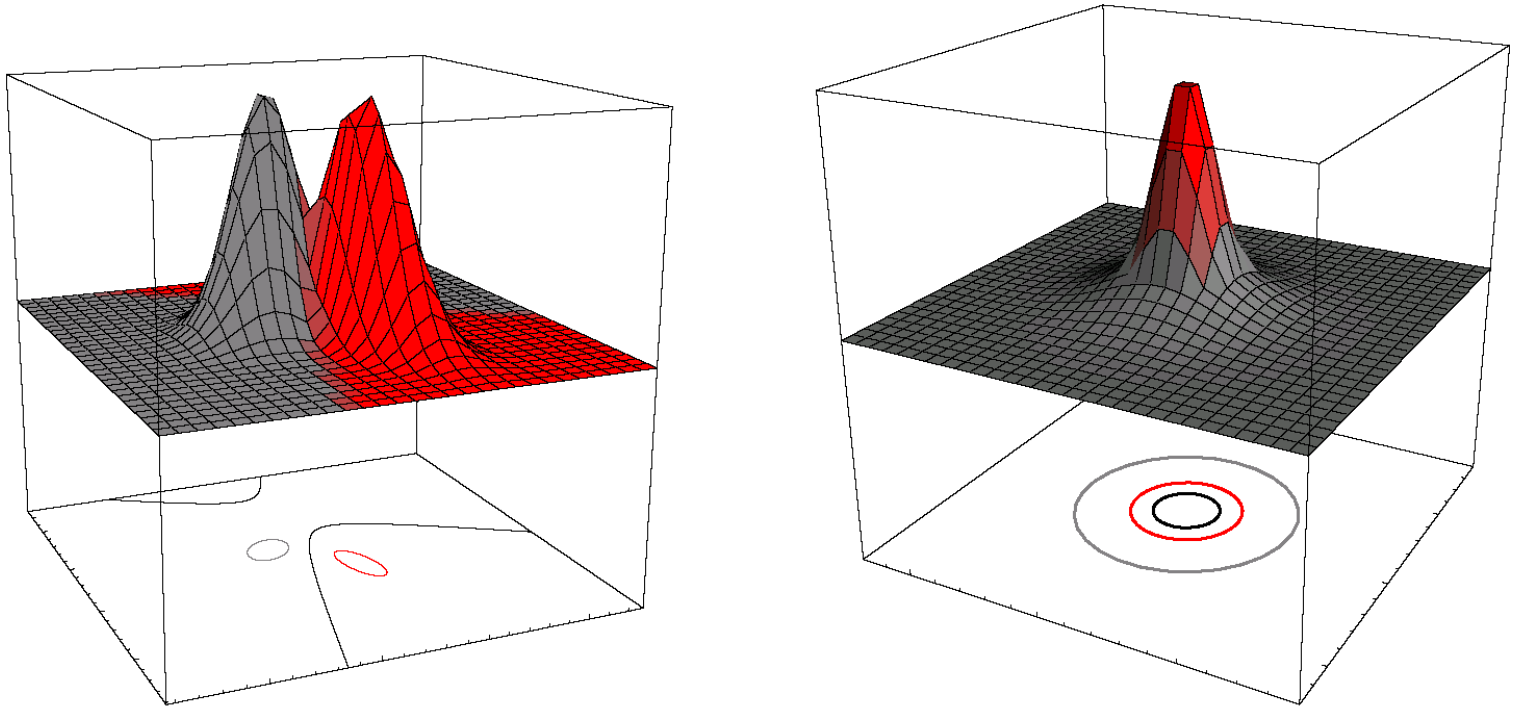

Figure 20: Arbitrary Gaussian distributions lead decision boundaries that are generally hyperquadratics.

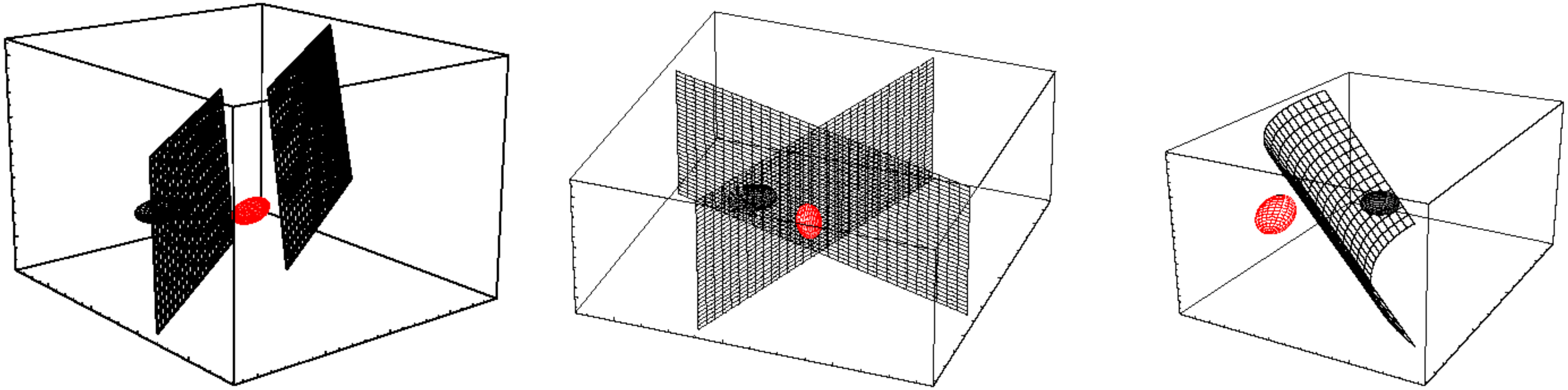

Figure 21: Arbitrary three-dimensional Gaussian distributions yield decision boundaries that are two-dimensional hyperquadratics.

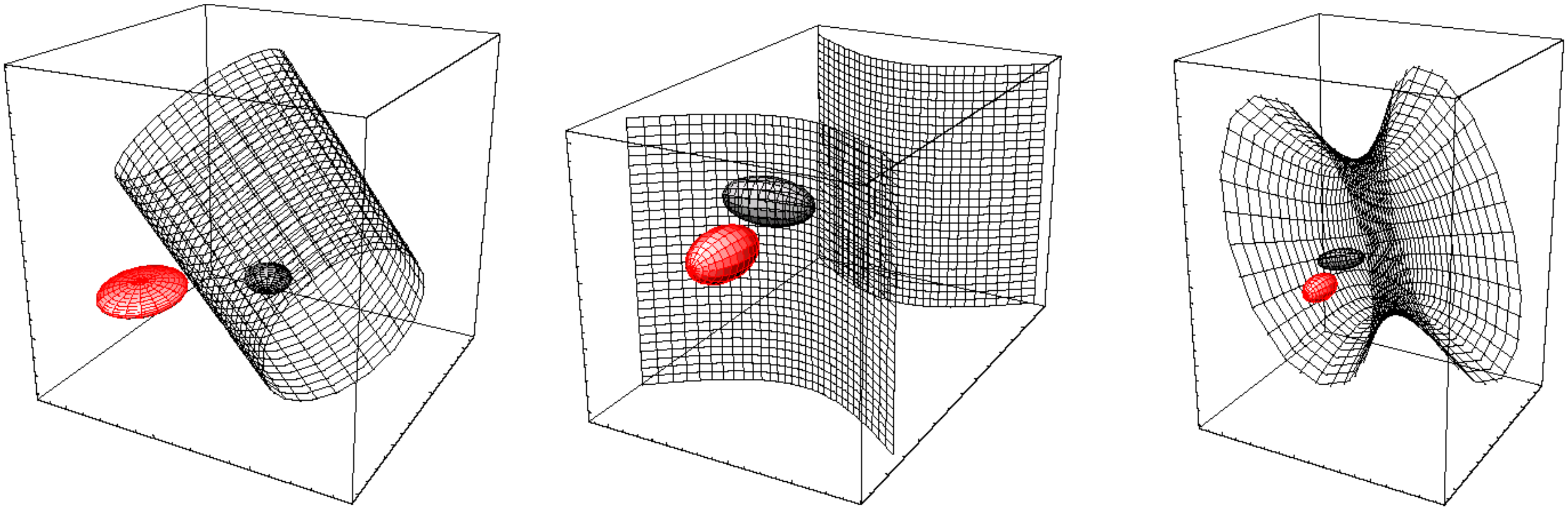

Figure 22: Arbitrary three-dimensional Gaussian distributions yield decision boundaries that are two-dimensional hyperquadratics.

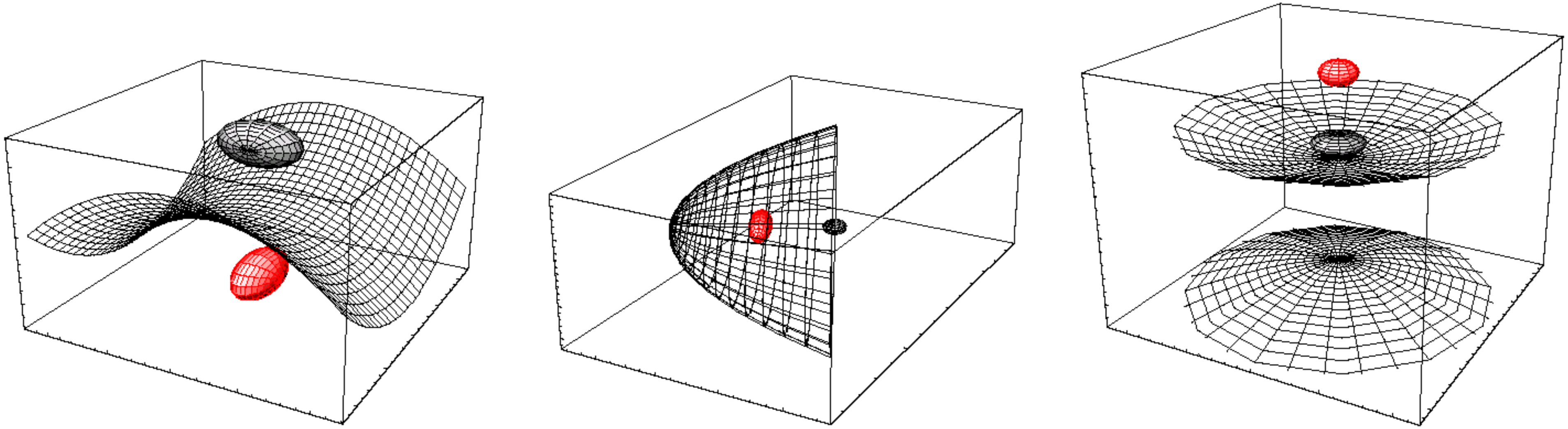

Figure 23: Arbitrary three-dimensional Gaussian distributions yield decision boundaries that are two-dimensional hyperquadratics.

- The extension to more than two categories is straightforward.
- Figure 24 shows the decision boundaries for a four-category made up of Gaussian distributions.
- Of course, if the distributions are more complicated, the decision regions can be more complex.

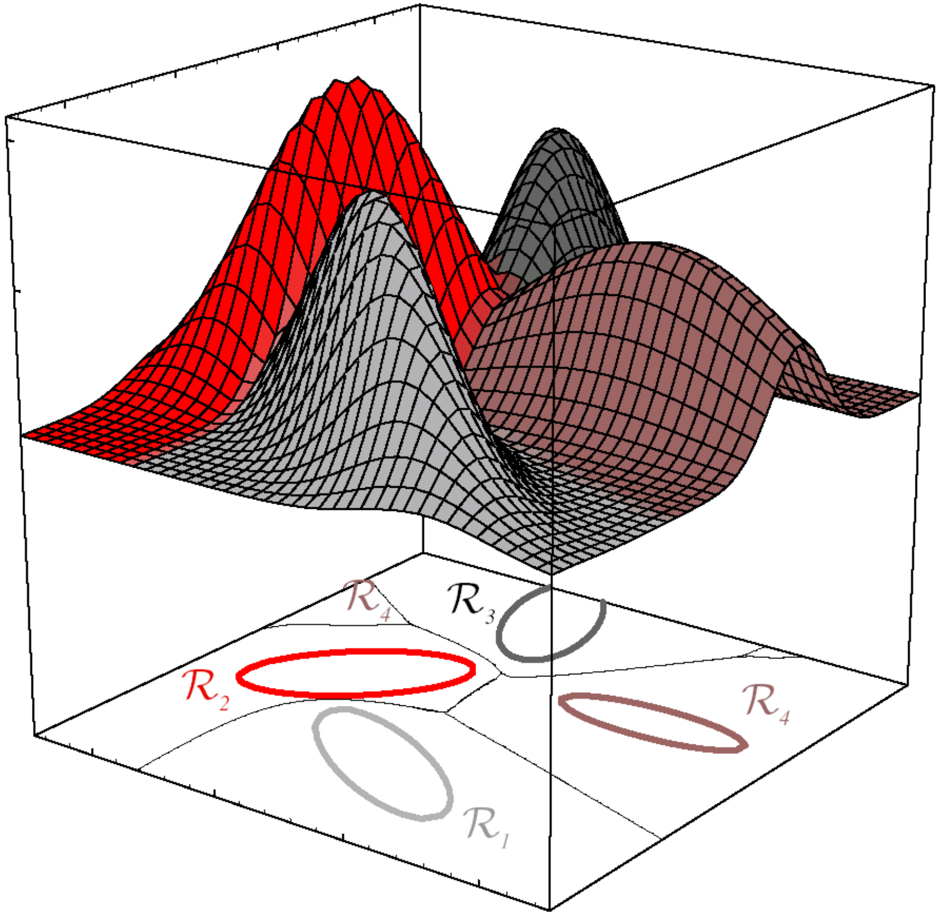

Figure 24: The decision regions for four normal distributions.

Example 5

Decision Regions for Two-Dimensional Gaussian Data

To illustrate these ideas, we calculate the decision boundary for the data in the Figure 25.

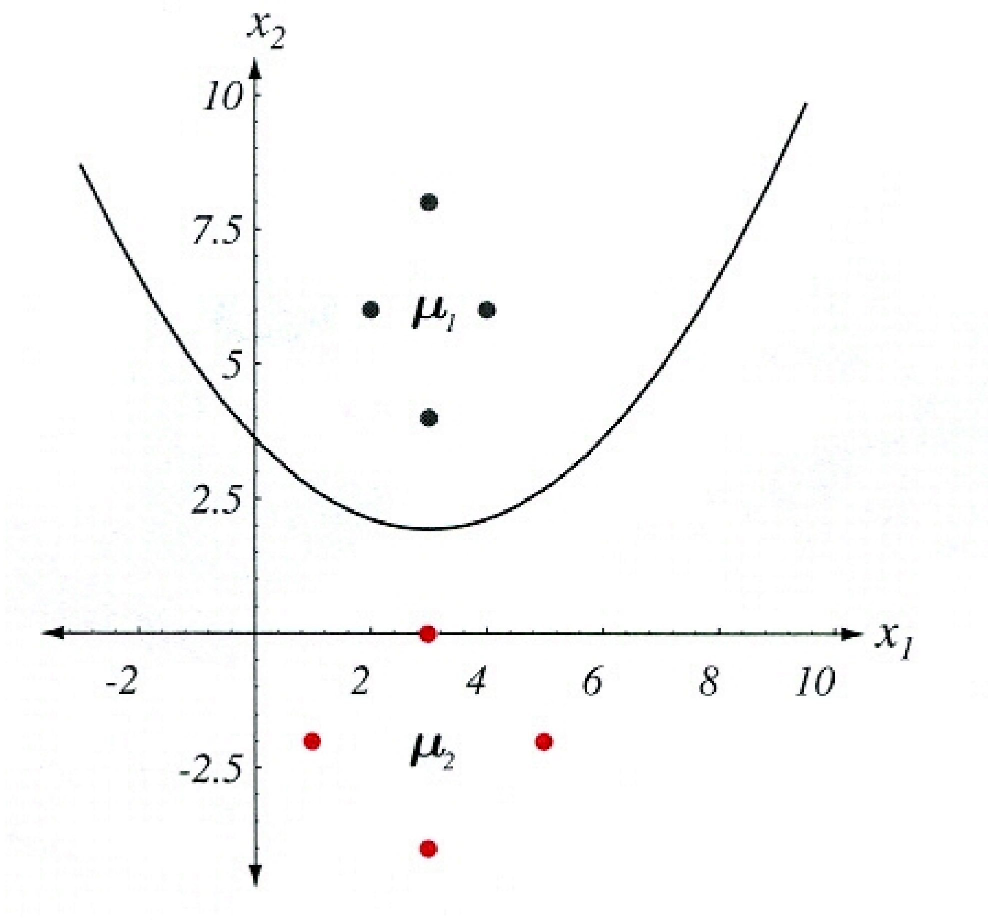

Figure 25: Two-category data and the computed decision boundary

- Let ω_1 be the set of black points, and ω_2 the red points.
- Although we will spend much of the next chapter on the parameter estimation, for now we simply calculate the means and covariances by

$$
\begin{aligned}
\boldsymbol{\mu}_1 &= \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \boldsymbol{\Sigma}_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \\
\boldsymbol{\mu}_2 &= \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \boldsymbol{\Sigma}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}
\end{aligned}
$$

- Thus, the inverse matrices are

$$\mathbf{\Sigma}_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \quad \mathbf{\Sigma}_2^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

- We assume equal priors and substitute these into the Eqs (59) -(62).

- By setting $g_1(\mathbf{x}) = g_2(\mathbf{x})$, we obtain the following decision boundary.

$$x_2 = 3.514 - 1.125x_1 + 0.1875x_1^2 \tag{63}$$

Some observations

- The decision boundary is a parabola with vertex at $\begin{bmatrix} 3 & 1.83 \end{bmatrix}^t$.
- The decision boundary does not pass through the point $\begin{bmatrix} 3 & 2 \end{bmatrix}^t$, the midway between two means, as we might have naively guessed.
- This is because for ω_1, the distribution is squeezed more in the x_1 direction. Hence, the ω_1 distribution is increased along the x_2 direction.
- The resulting decision boundary lies slightly lower than the midway point.

2.1 Introduction

2.2 Bayesian Decision Theory

2.3 Minimum-Error-Rate Classification

2.4 Classifiers, Discriminant Functions, and Decision Surfaces

2.5 The Normal Density

2.6 Discriminant Functions for the Normal Density

2.8.3 **Signal Detection Theory and Operating Characteristics**

2.8.3 Signal Detection Theory and Operating Characteristics

- Suppose we are interested in detecting a weak signal, such as a radar reflection.
- Because of random noise, the observation x is a random variable.
- Our model is that the observation x has mean μ_2 when the signal of interest is present, and mean μ_1 when it is not present (Figure 26).

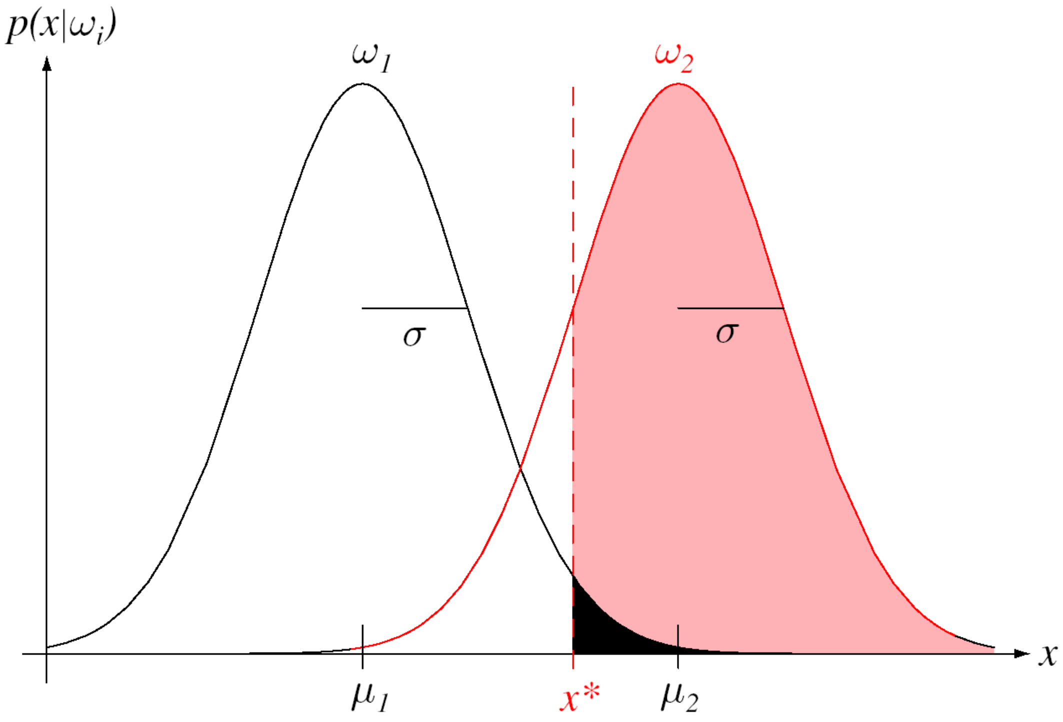

Figure 26: When the signal of interest is not present, the density is $p(x|\omega_1) \sim N(\mu_1, \sigma^2)$; when the signal is present, the density is $p(x|\omega_2) \sim N(\mu_2, \sigma^2)$. The decision threshold x^* will determine the probabilities of hit and of false alarm.

- A detector (classifier) employs a threshold values x^* for determining whether the signal of interest is present.
- Sometimes, one might seek to find some measure of the ease of detection (making decision), in a form independent of the choice of x^*.
- Such a measure is called **discriminability** , which completely depends on the underlying distributions, but not on a decision strategy.
- A popular choice for discriminability is

$$d' = \frac{|\mu_2 - \mu_1|}{\sigma} \tag{64}$$

A high d' is of course desirable.

Let us consider the following four probabilities:

- $P(x > x^*|x \in \omega_2)$: a **hit** – the probability that an observation x is above x^* given that the signal of interest is present.
- $P(x > x^*|x \in \omega_1)$: a **false alarm** – the probability that an observation x is above x^* given that the signal of interest is not present.
- $P(x < x^*|x \in \omega_2)$: a **miss** – the probability that an observation x is below x^* given that the signal of interest is present.
- $P(x < x^*|x \in \omega_1)$: a **correct rejection** – the probability that an observation x is below x^* given that the signal of interest is not present.

- If the densities are fixed but the threshold x^* is changed, then our hit and false alarm rates will also change.

- The pair of hit and false alarm rates will move along a curve – a **receiver operating characteristic** or **ROC** curve (Figure 27).

- It is a simple matter to generalize the above discussion to multidimensional distributions.

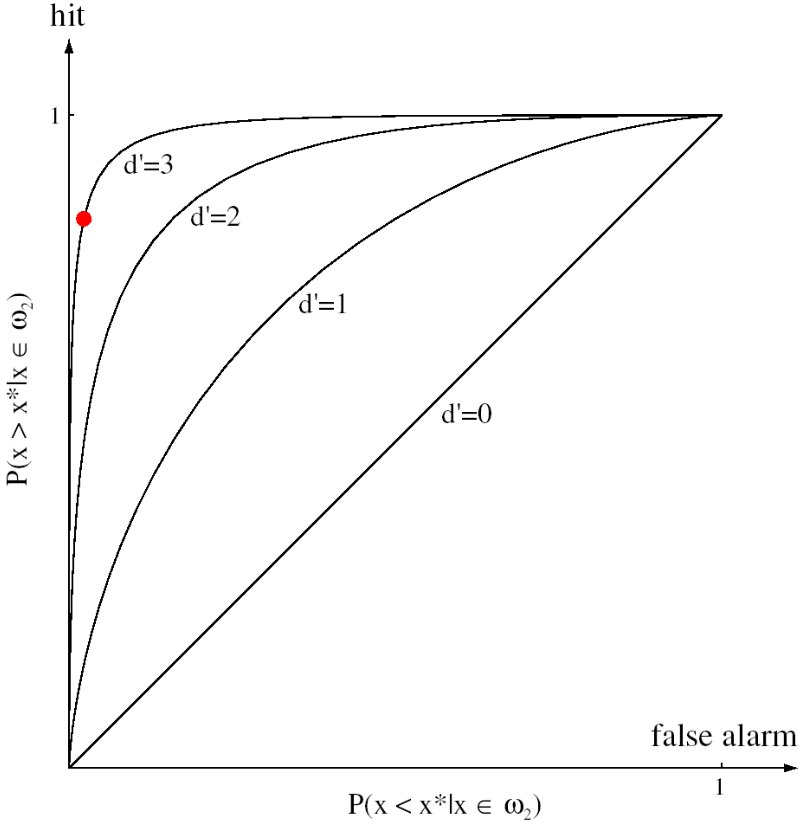

Figure 27: In an ROC curve, the abscissa is the probability of false alarm and the ordinate is the probability of hit.

Summary

- For practical applications, the main problem in applying Bayesian decision theory is that the conditional densities are not known.
- In some cases, we may know the form these densities, but we may not know the parameter values.
- For instance, the densities are known to be, or assumed to be, multivariate normal, but the values of means and covariance matrices are not known.
- Most of the following chapter will be devoted to various approaches for dealing with such problems.

References

[1] J. Gubner, *Probability and Random Processes for Electrical and Computer Engineers*. Cambridge University Press, 2006.