Unsupervised Learning & Generative AI

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Machine Learning

Outline

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Unsupervised Learning

- Dataset: $\mathbb{X} = \{\boldsymbol{x}^{(i)}\}_i$, where $\boldsymbol{x}^{(i)}$'s are i.i.d. samples of $\mathbf{x}$
  - No supervision $\boldsymbol{y}^{(i)}$ (labels)
- What can we learn without labels?

Unsupervised Learning

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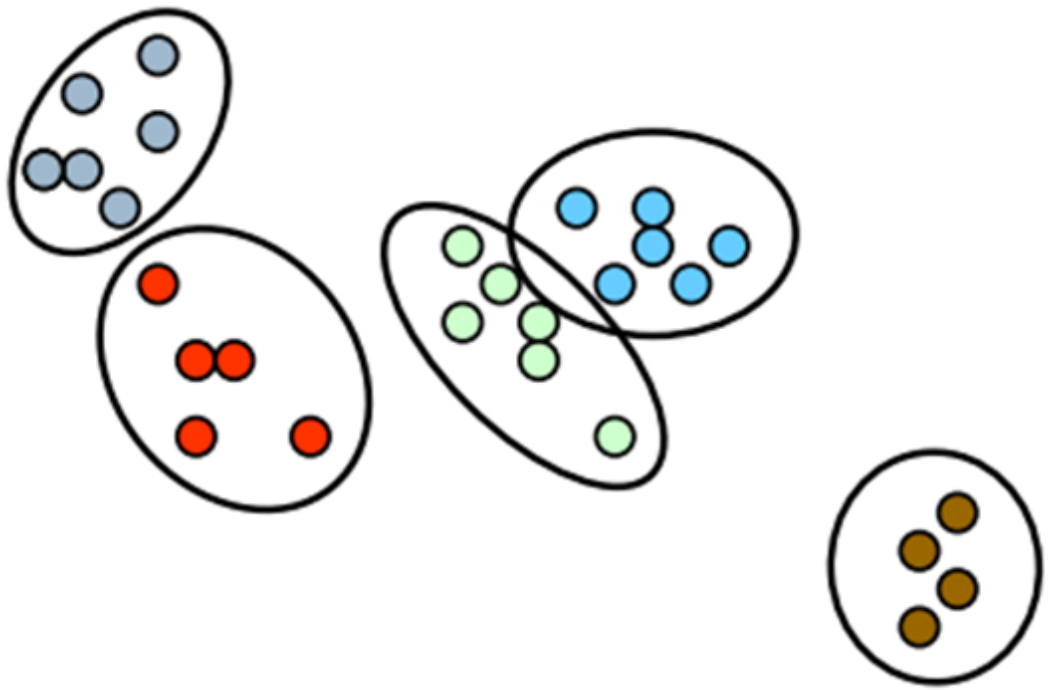
  - No supervision $\boldsymbol{y}^{(i)}$ (labels)
- What can we learn without labels? The ***structures*** in $\mathbb{X}$
  - Inter-sample structures
  - Intra-sample structures

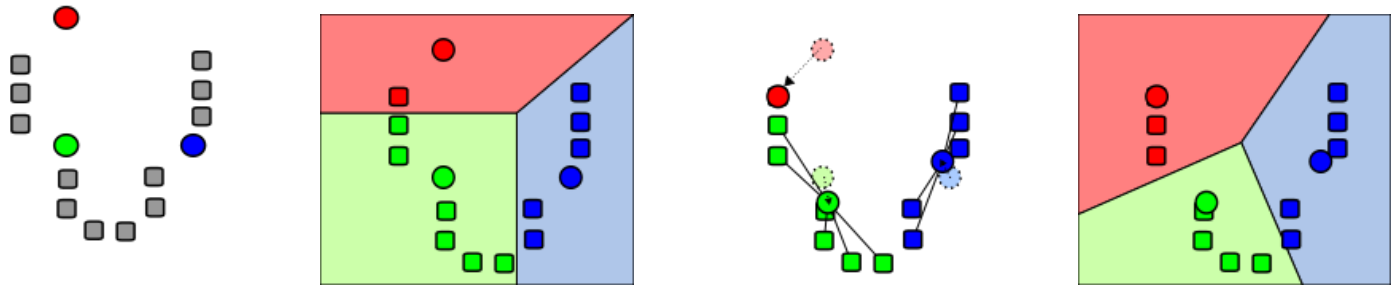
Clustering I

- Goal: to divide $\boldsymbol{x}^{(i)}$'s into K groups/***clusters***
  - Based on some similarity/distance measure between $\boldsymbol{x}^{(i)}$ and $\boldsymbol{x}^{(j)}$



Clustering II

- K-means algorithm (K fixed): iteratively move clusterheads until convergence



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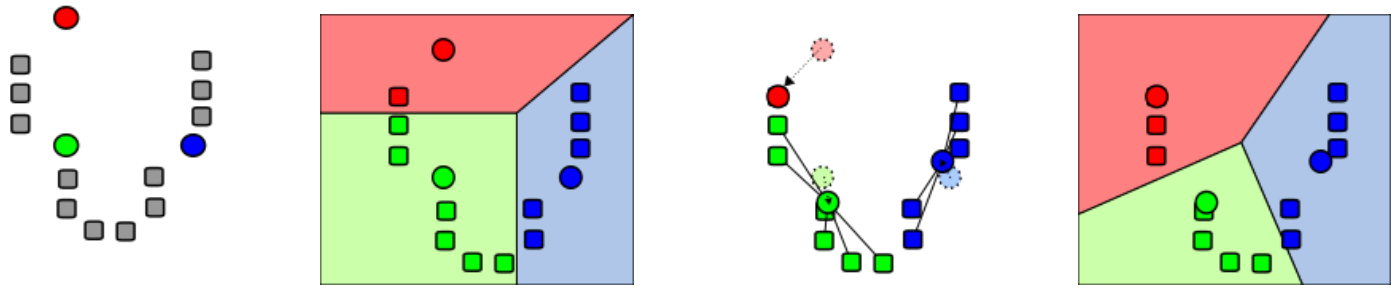
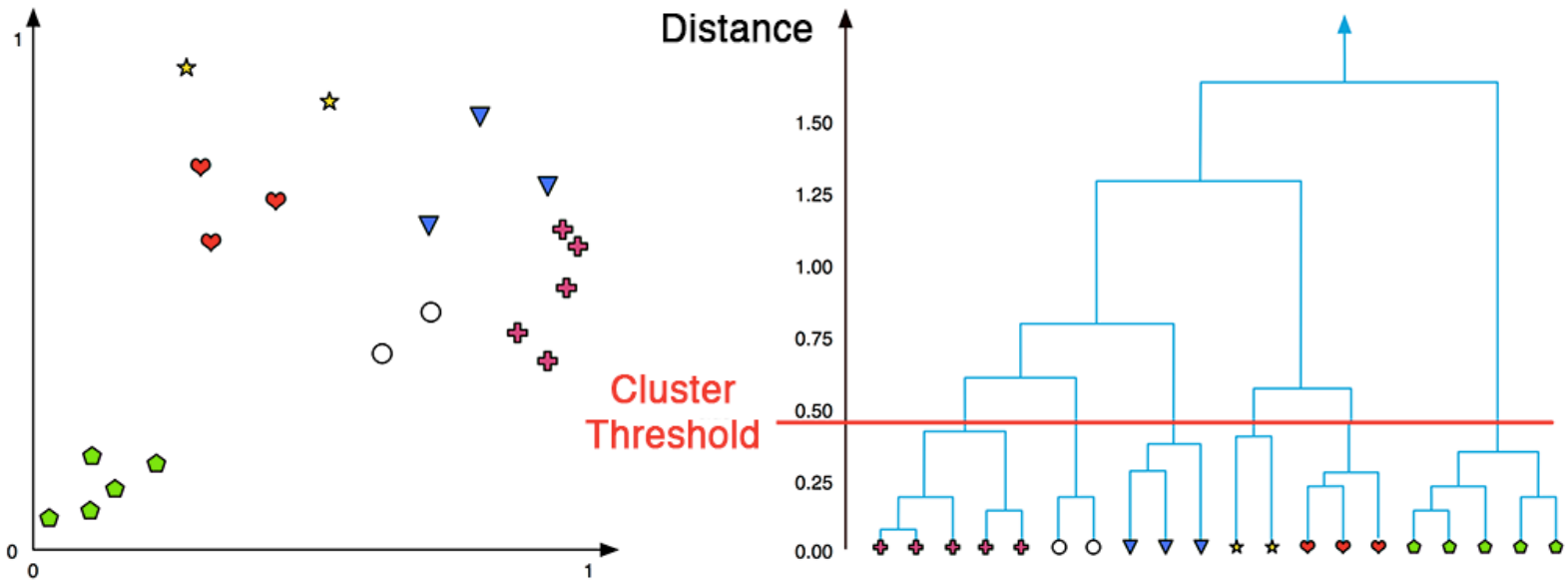
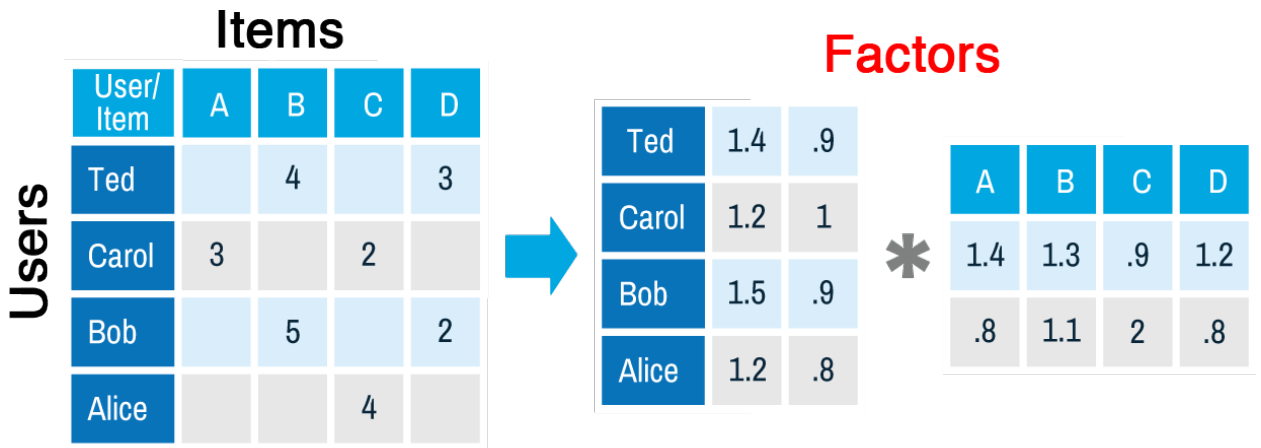


- Hierarchical clustering (variable K): iteratively merge two points/groups,then cut



Factorization for Tabular $\mathbb{X}$

- Let $\boldsymbol{X}$ be a rating matrix where $\boldsymbol{X}_{i,:} = \boldsymbol{x}^{(i)}$



- Goal: to approximate $\boldsymbol{X}$ with a dense matrix $\hat{\boldsymbol{X}} = \boldsymbol{WH}$
  - To make recommendations based on $\hat{\boldsymbol{X}}_{i,:}$ for user i, known as ***collaborative filtering***
- How?

Factorization for Tabular $\mathbb{X}$

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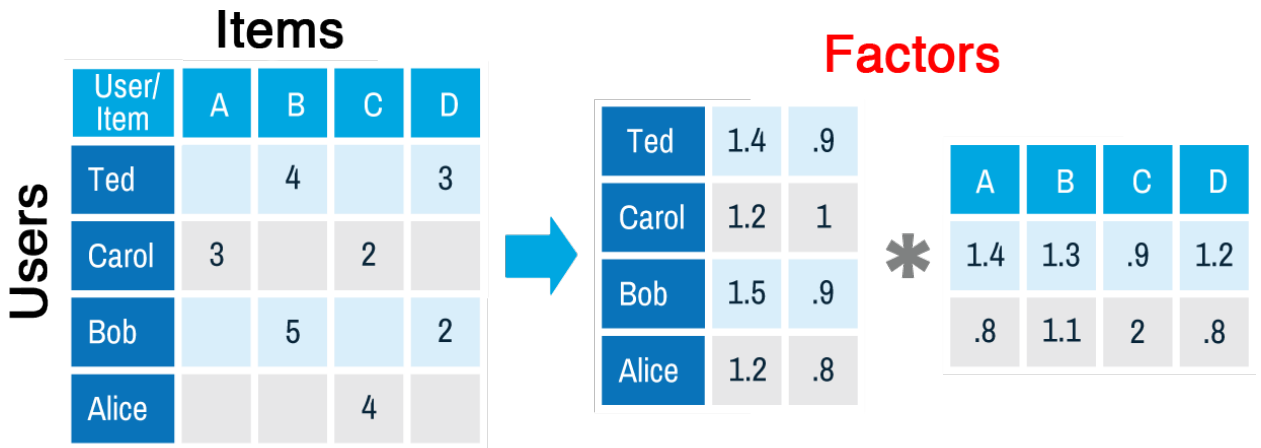


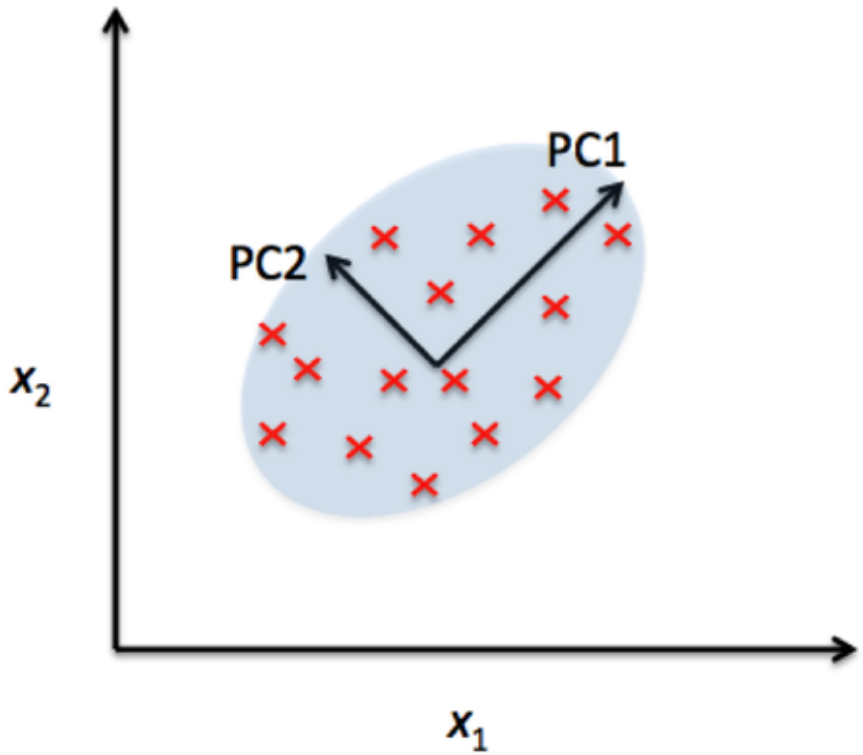
- Goal: to approximate $\boldsymbol{X}$ with a dense matrix $\hat{\boldsymbol{X}} = \boldsymbol{WH}$
  - To make recommendations based on $\hat{\boldsymbol{X}}_{i,:}$ for user i, known as ***collaborative filtering***
- How? Non-negative matrix factorization (NMF) [11, 12]:

$$\arg \min_{\boldsymbol{W} \geq \boldsymbol{O}, \boldsymbol{H} \geq \boldsymbol{O}} \|\boldsymbol{X} - \boldsymbol{WH}\|_F$$

  - So, positive elements in $\boldsymbol{W}$ and $\boldsymbol{H}$ can be seen as ***factor*** degrees

Dimension Reduction

- Goal: to learn a low dimensional representation $\mathbf{z}$ of $\mathbf{x}$
  - E.g., PCA



Self-Supervised Learning

- Goal: to learn a model that is able to "fill in the blanks"

$\boldsymbol{x}^{(i)}$  $\boldsymbol{y}^{(i)}$

Self-Supervised Learning

- Goal: to learn a model that is able to "fill in the blanks"
- Links unsupervised tasks with supervised models
  - Much more training data for models
  - Representation learning with deep networks

$\boldsymbol{x}^{(i)}$  $\boldsymbol{y}^{(i)}$

Example I: Word2Vec

- Goal: to learn a model for blank filling

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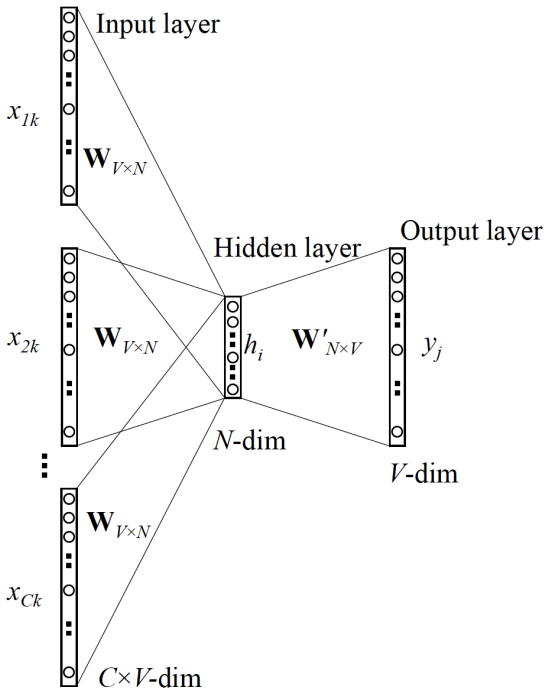
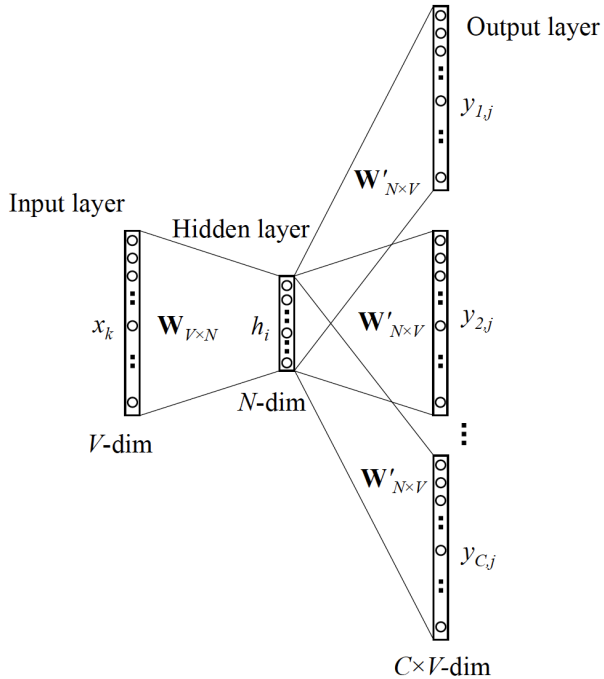
  - E.g., word2vec [17, 16]: "*... the cat sat **on**...*"



CBOW



Skip Gram

Example I: Word2Vec

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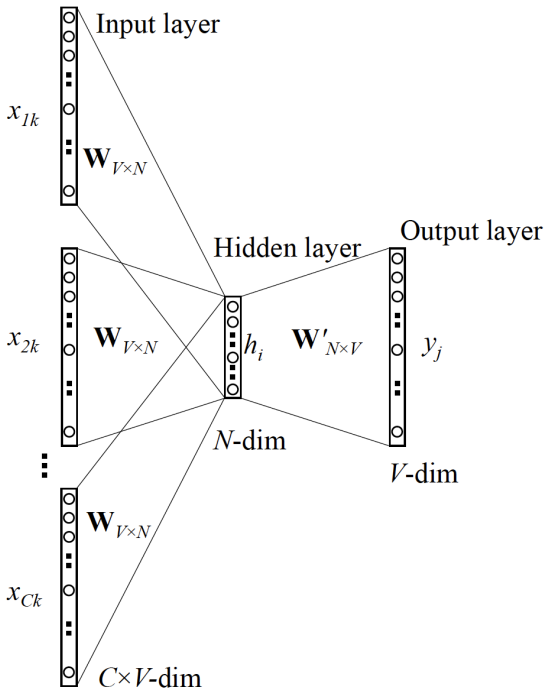
  - E.g., word2vec [17, 16]: "*... the cat sat **on**...*"



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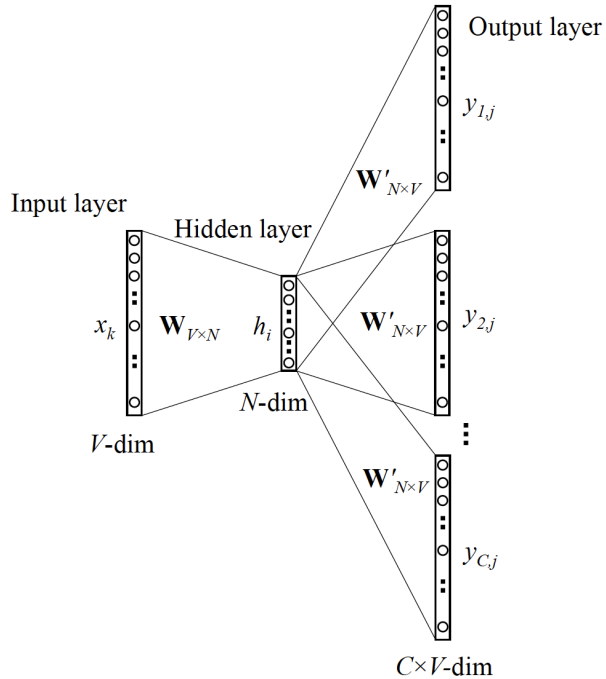

Skip Gram

- Latent representation $\mathbf{h}$ encodes the ***semantics*** of a word
  - No need for synonym dictionary; big data tell that already

Example II: Doc2Vec

- How to encode a document?

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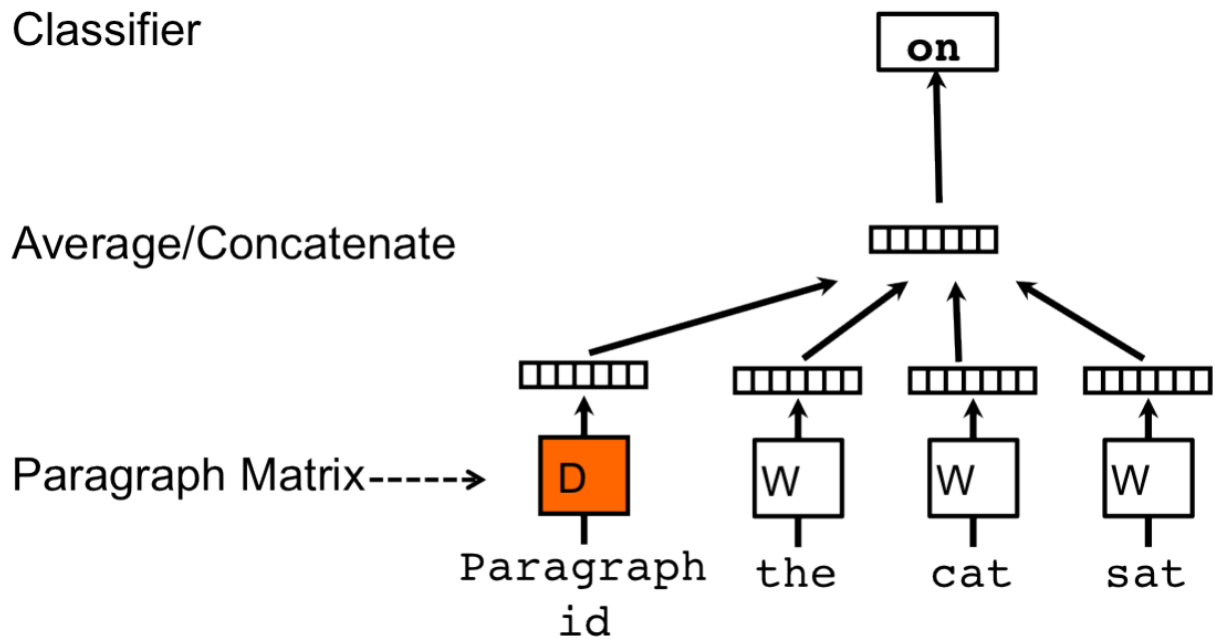
- How to encode a document?
- Bag of words (TF-IDF), average word2vec, etc.
  - Do ***not*** capture the semantics due to sentence/paragraph/doc structure
    - "*John likes* Mary" $\neq$ "*Mary likes* John"

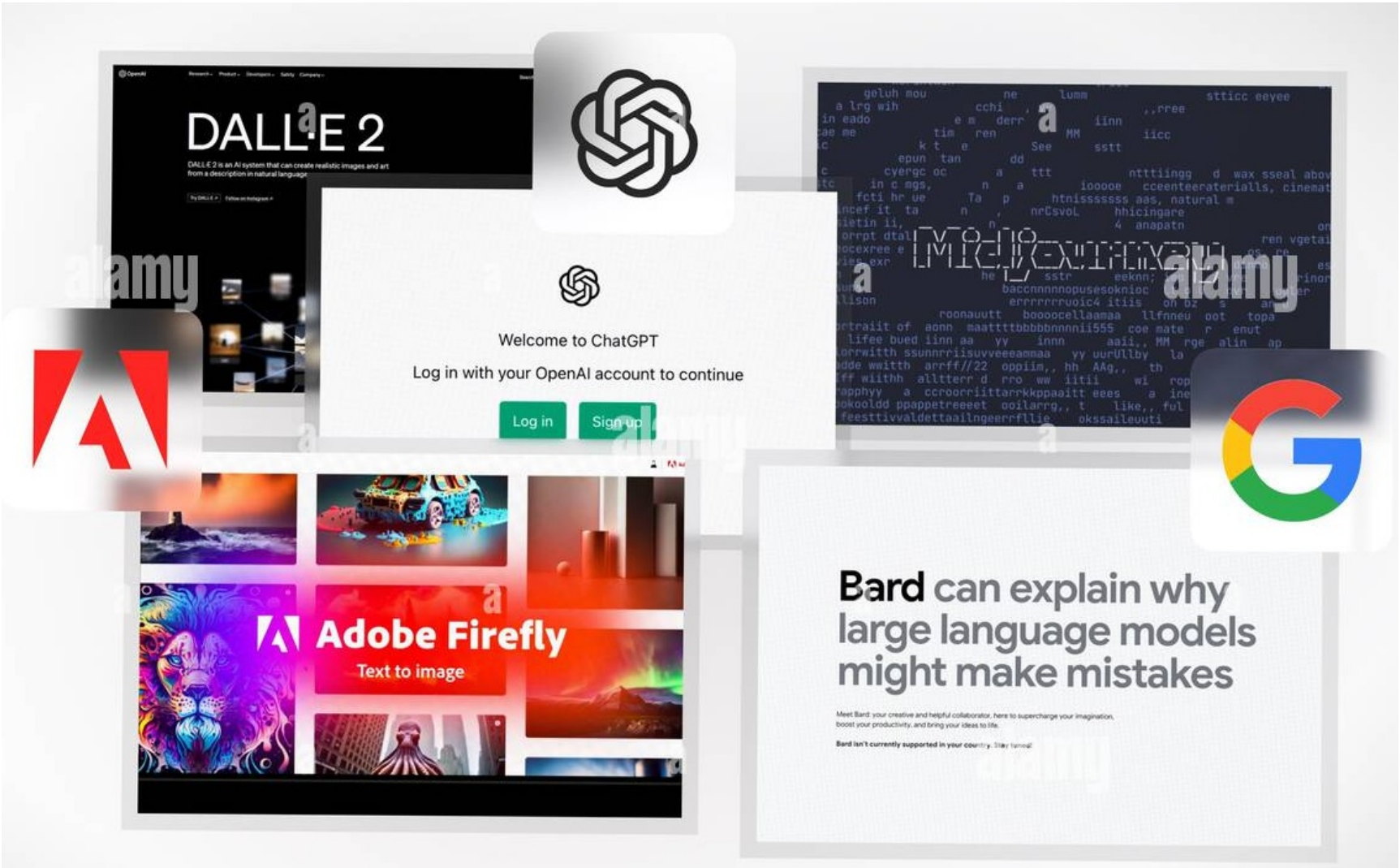
Example II: Doc2Vec

- How to encode a document?
- Bag of words (TF-IDF), average word2vec, etc.
  - Do ***not*** capture the semantics due to sentence/paragraph/doc structure
    - "*John likes* Mary" $\neq$ "*Mary likes* John"
- Why not apply self-supervised learning to docs?
  - Doc2vec [10]: to capture the ***context*** not explained by words
  - ***Transductive*** rather than inductive; does not work with unseen docs



Generative Models

- Goal: to generate new samples of $\mathbf{x}$
  - Can be conditioned on instructions (input)
- Largely based on self-supervised learning



Outline

Generating Text

- With large (self-supervised) training data, ***transformers*** [33] have shown to perform better than RNNs and CNNs
  - Mainly due to the $O(1)$ point distance
- Two common text models based on transformer:
  - BERT [4]: non-autoregressive
  - GPT [21]: autoregressive

BERT [4]

- Massively trained ***encoder*** of the original transformer
  - Non-autoregressive
- Pre-training tasks:
  - Masked language model ("Cloze" task)
    - "A quick brown [MASK] jumps over the lazy dog" → "fox" 11%, "ant" 5%, ...
  - Next sentence prediction
    - "[MASK] go to store [SEP] to buy a [MASK] of milk"→ True 93%, False 7%

One Pre-training, Multiple Fine-tuning Tasks

- Special input to identify downstream task: [CLS] token

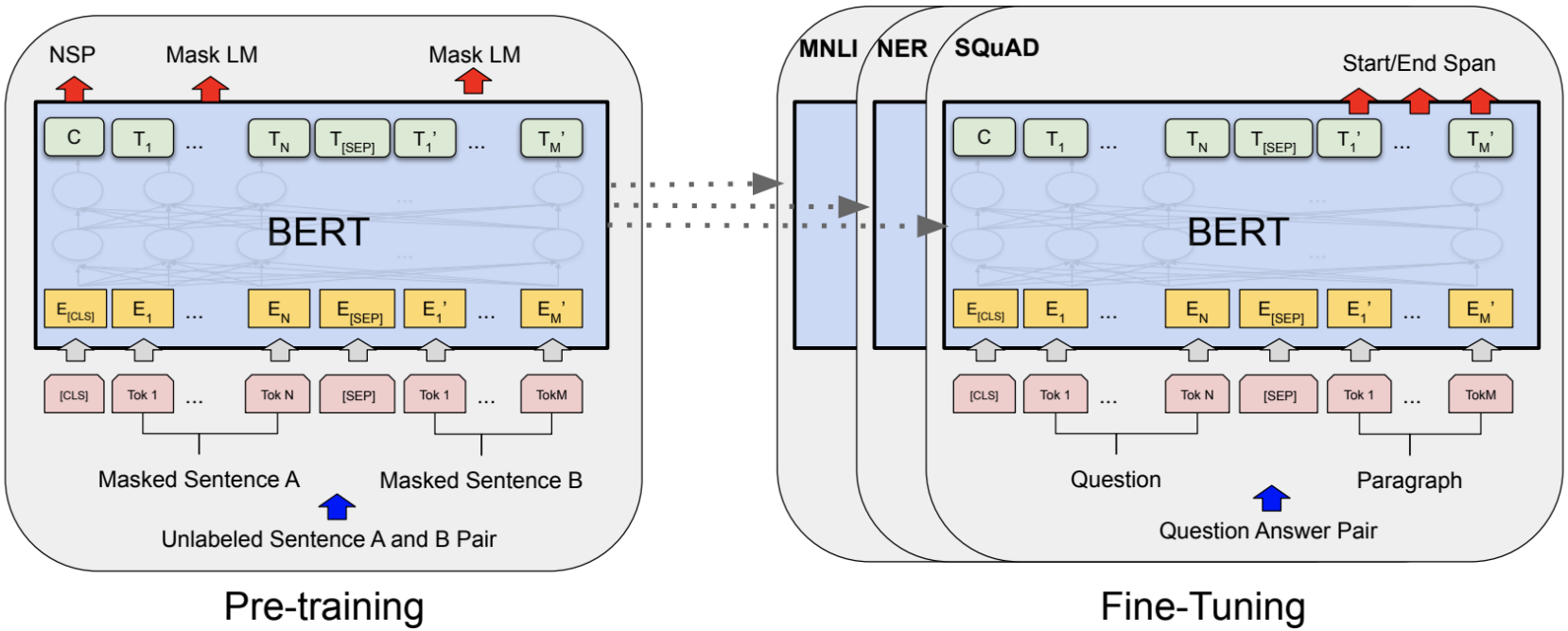
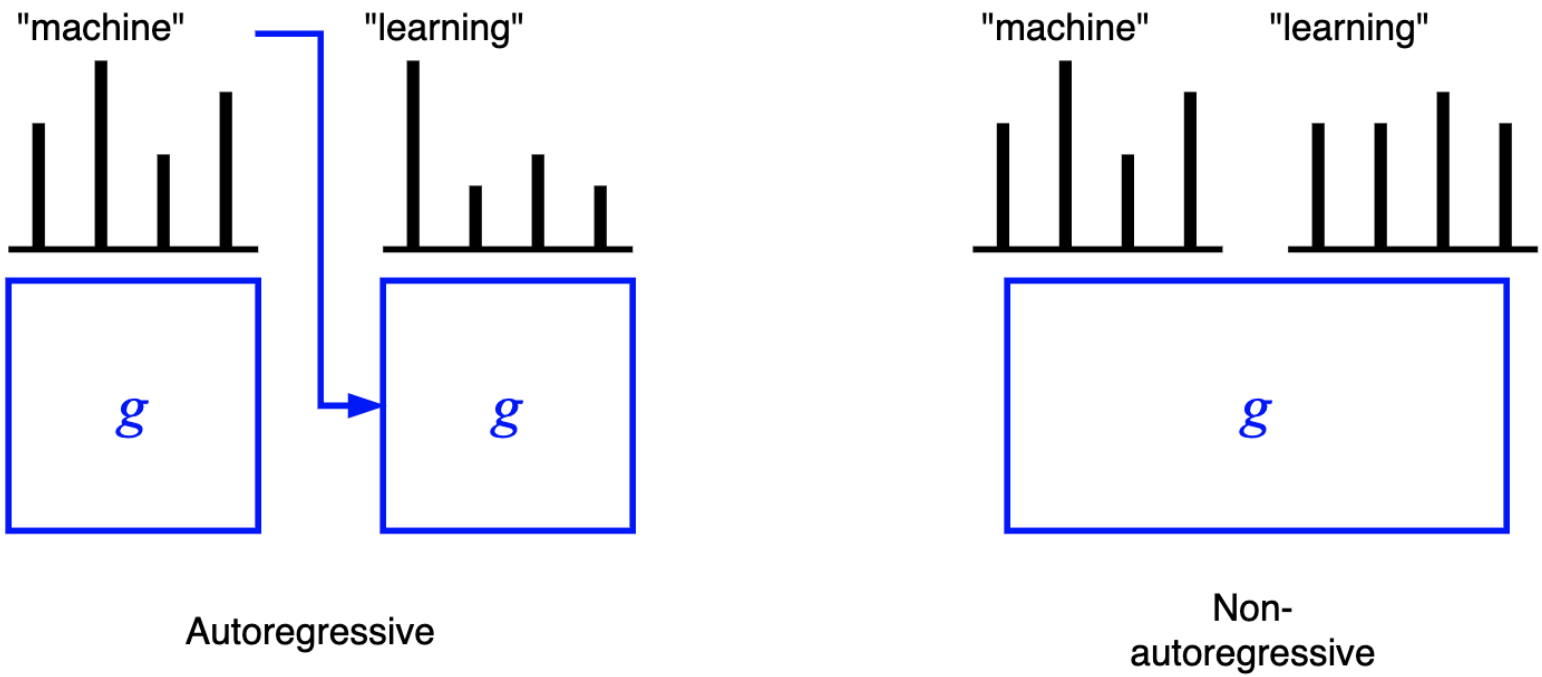


Figure 1: Overall pre-training and fine-tuning procedures for BERT. Apart from output layers, the same architectures are used in both pre-training and fine-tuning. The same pre-trained model parameters are used to initialize models for different down-stream tasks. During fine-tuning, all parameters are fine-tuned. `[CLS]` is a special symbol added in front of every input example, and `[SEP]` is a special separator token (e.g. separating questions/answers).

GPT [21]

- Massively trained ***decoder*** of the original transformer
  - Autoregressive
- Multitask pre-training by maximizing $\Pr(\text{output}|\text{input}, \text{task})$:
  - Translation: $\Pr(\text{french text}|\text{en text}, \text{translation})$
  - Question answering: $\Pr(\text{answer}|\text{question}, \text{qa})$
  - Reading comprehension: $\Pr(\text{answer}|\text{document}, \text{question}, \text{reading})$
- Usage for downstream task: fine-tuning or ***prompting***

Autoregressive or Not?

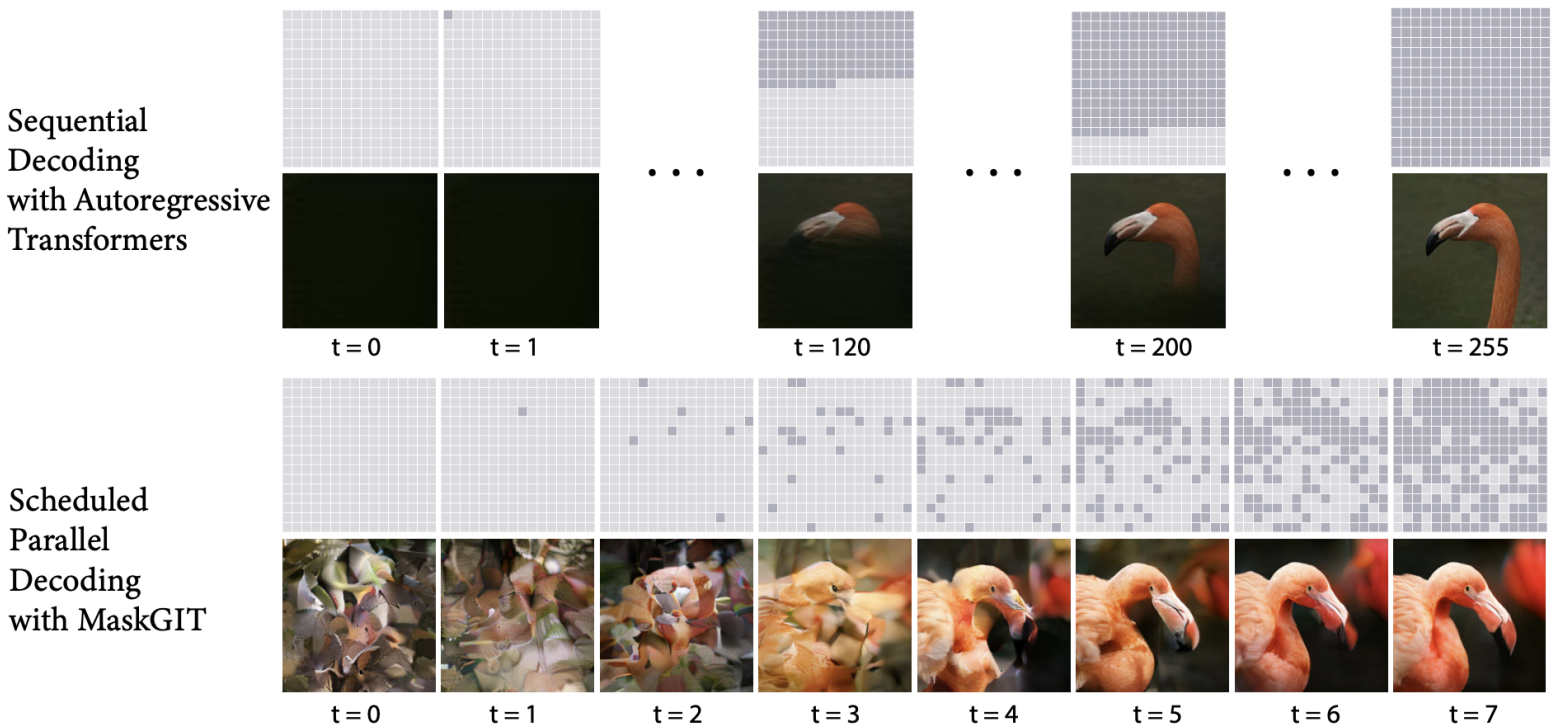


- It's easier for an autoregressive model to generate coherent text
- In this lecture, we focus on GPT and its variants

Outline

Autoregressive or Not?

- Autoregressive models can still give good (if not better) performance
  - E.g., PixelRNN [32], PixelCNN [31], or MaskGIT [3]
- But unlike in text domain, generating images pixel-by-pixel is ***very slow***
- Speed-up?
  - Scheduled ***parallel-pixel*** generation (e.g., MaskGIT)
  - ***Stepwise*** generation of ***whole*** images (e.g., flow-based and diffusion models)



Whole-Image Generation I

- Goal: given $\mathbb{X}$, to learn a ***generator function*** g such that $\hat{\boldsymbol{x}} = g(\boldsymbol{c}; \Theta_g)$ looks like a real image in $\mathbb{X}$
  - $\boldsymbol{c}$ is a code or condition
  - Θ_g represents parameters of g
- Objective from Information Theory perspective:

$$\arg\min_g \mathrm{D}(\mathrm{P}_{\mathsf{data}} \| \mathrm{P}_g)$$

  - $\mathrm{P}_{\mathsf{data}}$ is the distribution of real data in the ground truth
  - P_g is the distribution of generated data
  - D is a divergence measure, e.g., D_{KL}

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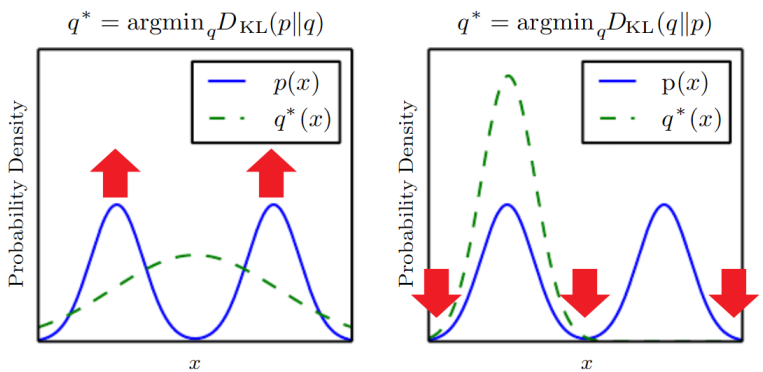
  - D is a divergence measure, e.g., D_{KL}
- Why not $\mathrm{D}_{\mathsf{KL}}(\mathrm{P}_g \| \mathrm{P}_{\mathsf{data}})$?



Whole-Image Generation II

- Minimizing $\mathrm{D}_{\mathsf{KL}}(\mathrm{P}_{\mathsf{data}} \| \mathrm{P}_g)$ amounts to maximizing $\mathrm{P}(\mathbb{X}|\Theta_g)$, the log likelihood of Θ_g:

$$
\begin{aligned}
g^* &= \arg\min_g \mathrm{D}_{\mathsf{KL}}(\mathrm{P}_{\mathsf{data}} \| \mathrm{P}_g) \\
&= \arg\max_g \mathrm{E}_{\mathbf{x}\sim\mathrm{P}_{\mathsf{data}}}[\log \mathrm{P}_g(\mathbf{x})] + \mathrm{H}(\mathbf{x} \sim \mathrm{P}_{\mathsf{data}}) \\
&= \arg\max_g \mathrm{E}_{\mathbf{x}\sim\mathrm{P}_{\mathsf{data}}}[\log \mathrm{P}_g(\mathbf{x})] \\
&\approx \arg\max_g \textstyle\sum_{\boldsymbol{x}^{(i)}\in\mathbb{X}} \log \mathrm{P}_g(\boldsymbol{x}^{(i)}) \\
&= \arg\max_g \log \textstyle\prod_{\boldsymbol{x}^{(i)}\in\mathbb{X}} \mathrm{P}_g(\boldsymbol{x}^{(i)})
\end{aligned}
$$

$$
\begin{aligned}
\Theta_g^* &= \arg\max_{\Theta_g} \log \textstyle\prod_{\boldsymbol{x}^{(i)}\in\mathbb{X}} \mathrm{P}(\boldsymbol{x}^{(i)}|\Theta_g) \\
&= \arg\max_{\Theta_g} \log \mathrm{P}(\mathbb{X}|\Theta_g)
\end{aligned}
$$

- Other divergence measures can lead to similar results

Common Whole-Image Generation Methods

- Autoencoder (single-step: $\hat{\boldsymbol{x}} = g(\boldsymbol{c})$)
  - Pros: easy
  - Cons: but no creativity, blurry images
- Variational Autoencoder (single-step: $\hat{\boldsymbol{x}} = g(\boldsymbol{c})$)
  - Pros: creative
  - Cons: only maximizes a lower bound of $\mathrm{P}(\mathbb{X}|\Theta_g)$; blurry images
- Flow-based methods (multi-step: $\hat{\boldsymbol{x}} = g^{(T)}(\cdots g^{(2)}(g^{(1)}(\boldsymbol{c})))$)
  - Pros: maximizes $\mathrm{P}(\mathbb{X}|\Theta_g)$ directly
  - Cons: limited expressiveness of g for invertibility; slow training and inference
- GANs (single-step: $\hat{\boldsymbol{x}} = g(\boldsymbol{c})$)
  - Pros: good image quality (sharp and coherent)
  - Cons: difficult to train (convergence issue, mode collapse, vanishing gradients, etc.)
- Diffusion models (multi-step: $\hat{\boldsymbol{x}} = g^{(T)}(\cdots g^{(2)}(g^{(1)}(\boldsymbol{c})))$)
  - Pros: good image quality; efficient & stable to train, can be made conditional easily
  - Cons: slow inference

Outline

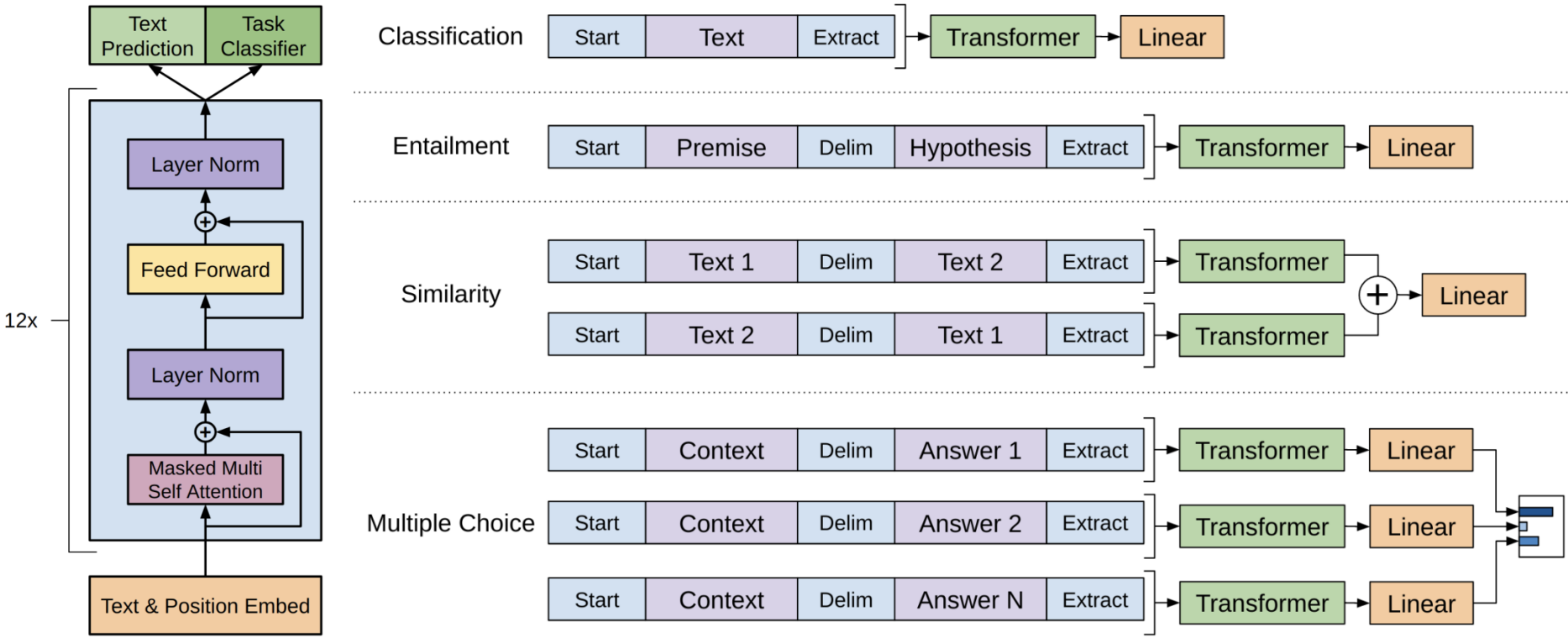
Evolutions

- 2018 GPTv1 [21]
  - Self-supervised pre-training
- 2019 GPTv2 [22]
  - ***Multitask*** pre-training
- 2020 GPTv3 [2]
  - ***Few-shot*** & ***in-context*** learning
- 2022 GPTv3.5 [20]
  - ***Alignment*** using (supervised) instruction tuning + reinforcement learning from human feedback (RLHF)
- 2023 GPT4: mixture of experts

GPTv1 [21]

- Aims at 2-step training process:
  1. Self-supervised pre-training on unlabeled data
     - To predict next word in a sentence (language model)
  2. Discriminative fine-tuning on labeled data in downstream tasks
     - Here, fine-tuning could also mean training a new model based on extracted features

Example Fine-tuning Tasks



Does Self-supervised Pre-training Help?

Table 5: Analysis of various model ablations on different tasks. Avg. score is a unweighted average of all the results. (*mc*= Mathews correlation, *acc*=Accuracy, *pc*=Pearson correlation)

Method	Avg. Score	CoLA (mc)	SST2 (acc)	MRPC (F1)	STSB (pc)	QQP (F1)	MNLI (acc)	QNLI (acc)	RTE (acc)
Transformer w/ aux LM (full)	74.7	45.4	91.3	82.3	82.0	**70.3**	**81.8**	**88.1**	**56.0**
Transformer w/o pre-training	59.9	18.9	84.0	79.4	30.9	65.5	75.7	71.2	53.8
Transformer w/o aux LM	**75.0**	**47.9**	**92.0**	**84.9**	**83.2**	69.8	81.1	86.9	54.4
LSTM w/ aux LM	69.1	30.3	90.5	83.2	71.8	68.1	73.7	81.1	54.6

GPTv2 [22]

- Many NLP tasks can be formulated as the problem of maximizing $\Pr(\text{output}|\text{input}, \text{task})$
  - Translation: $\Pr(\text{french text}|\text{en text}, \text{translation})$
  - Question answering: $\Pr(\text{answer}|\text{question}, \text{qa})$
  - Reading comprehension: $\Pr(\text{answer}|\text{document}, \text{question}, \text{reading})$

GPTv3 [2]

- The model learns from few shots (examples), even ***in context***
  - Enables ***prompting*** techniques for downstream tasks
  - E.g.,few shots, chain of thought (CoT), or simply "Let's work this out step-by-step to ensure the answer is correct"

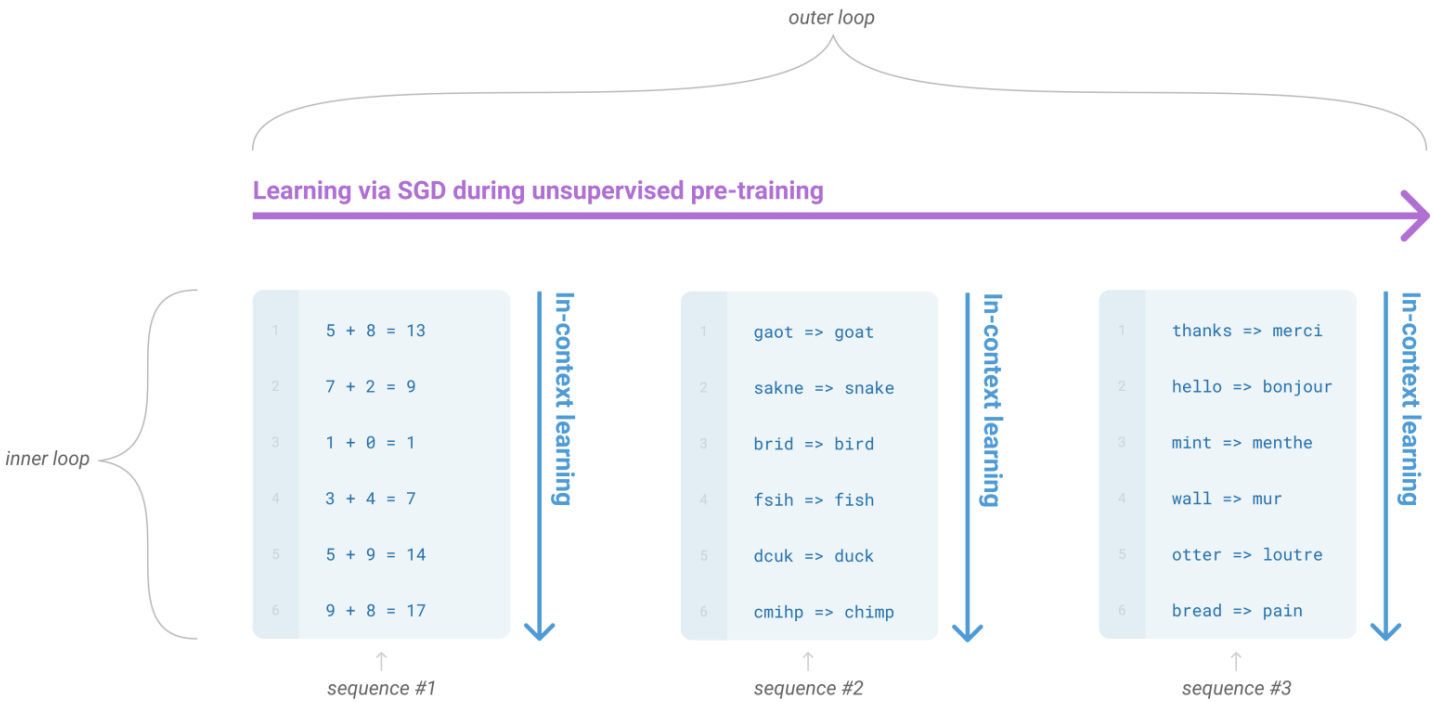


Figure 1.1: Language model meta-learning. During unsupervised pre-training, a language model develops a broad set of skills and pattern recognition abilities. It then uses these abilities at inference time to rapidly adapt to or recognize the desired task. We use the term "in-context learning" to describe the inner loop of this process, which occurs within the forward-pass upon each sequence. The sequences in this diagram are not intended to be representative of the data a model would see during pre-training, but are intended to show that there are sometimes repeated sub-tasks embedded within a single sequence.

Zero/Few Shot Prompting vs. Fine-tuning

Zero-shot

The model predicts the answer given only a natural language description of the task. No gradient updates are performed.

```
1  Translate English to French:        ← task description
2  cheese =>                           ← prompt
```

One-shot

In addition to the task description, the model sees a single example of the task. No gradient updates are performed.

```
1  Translate English to French:        ← task description
2  sea otter => loutre de mer          ← example
3  cheese =>                           ← prompt
```

Few-shot

In addition to the task description, the model sees a few examples of the task. No gradient updates are performed.

```
1  Translate English to French:        ← task description
2  sea otter => loutre de mer          ← examples
3  peppermint => menthe poivrée
4  plush girafe => girafe peluche
5  cheese =>                           ← prompt
```

Fine-tuning

The model is trained via repeated gradient updates using a large corpus of example tasks.

Performance vs. Model Size

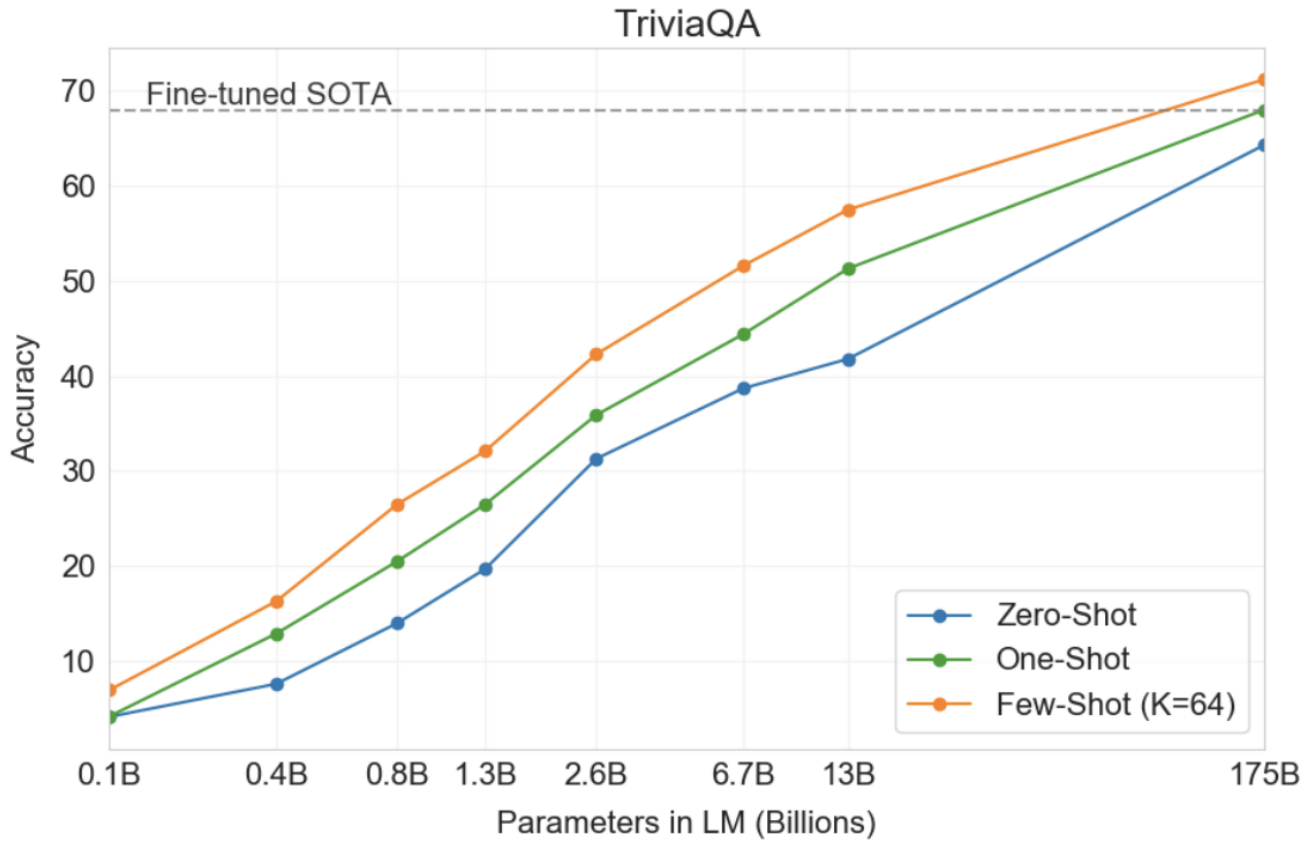
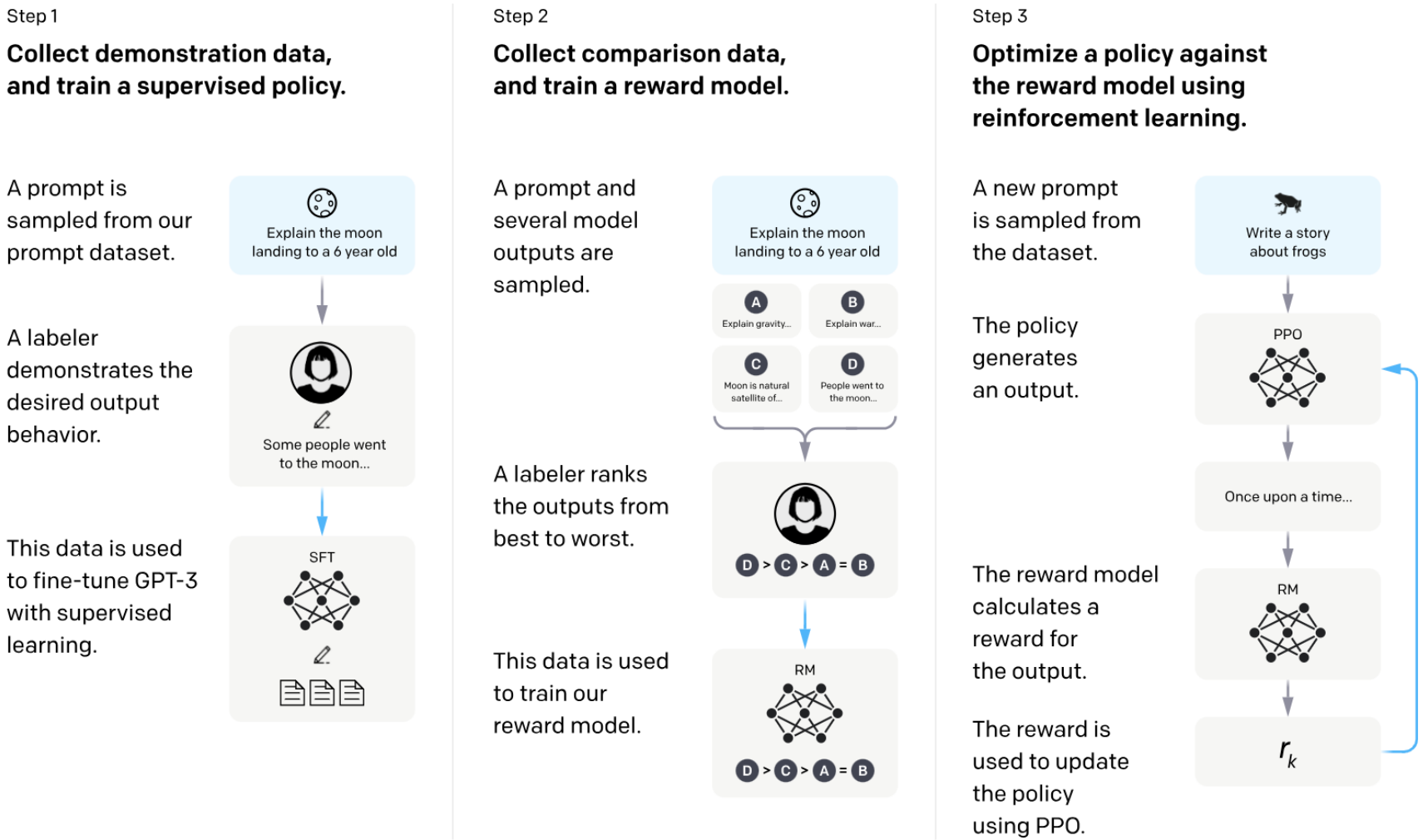


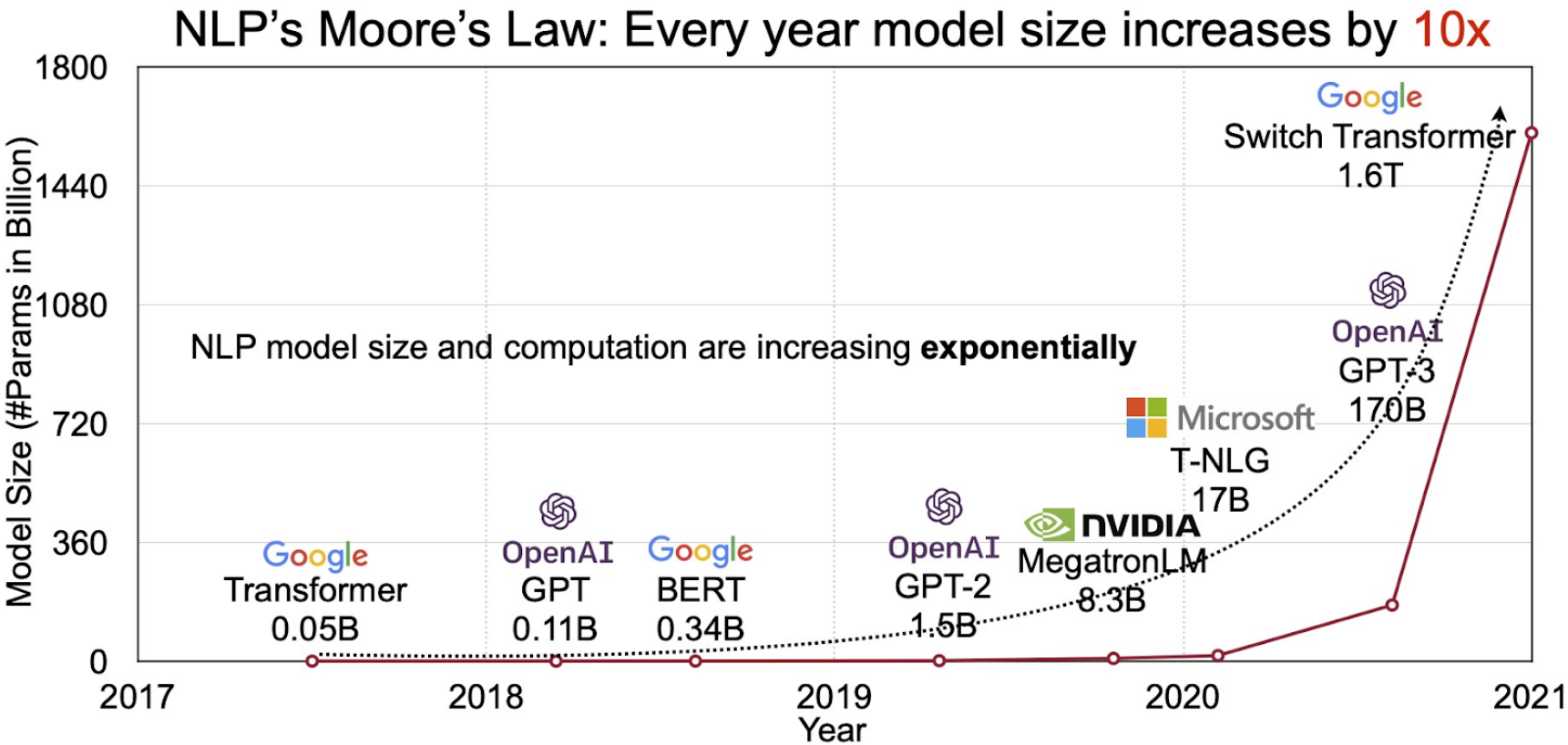
Figure 3.3: On TriviaQA GPT3's performance grows smoothly with model size, suggesting that language models continue to absorb knowledge as their capacity increases. One-shot and few-shot performance make significant gains over zero-shot behavior, matching and exceeding the performance of the SOTA fine-tuned open-domain model, RAG [LPP+20]

GPTv3.5 [20]

- From GPT to "ChatGPT" through ***alignment***
  - Supervised (multi-task) instruction tuning
  - Reinforcement learning from human feedback (RLHF)



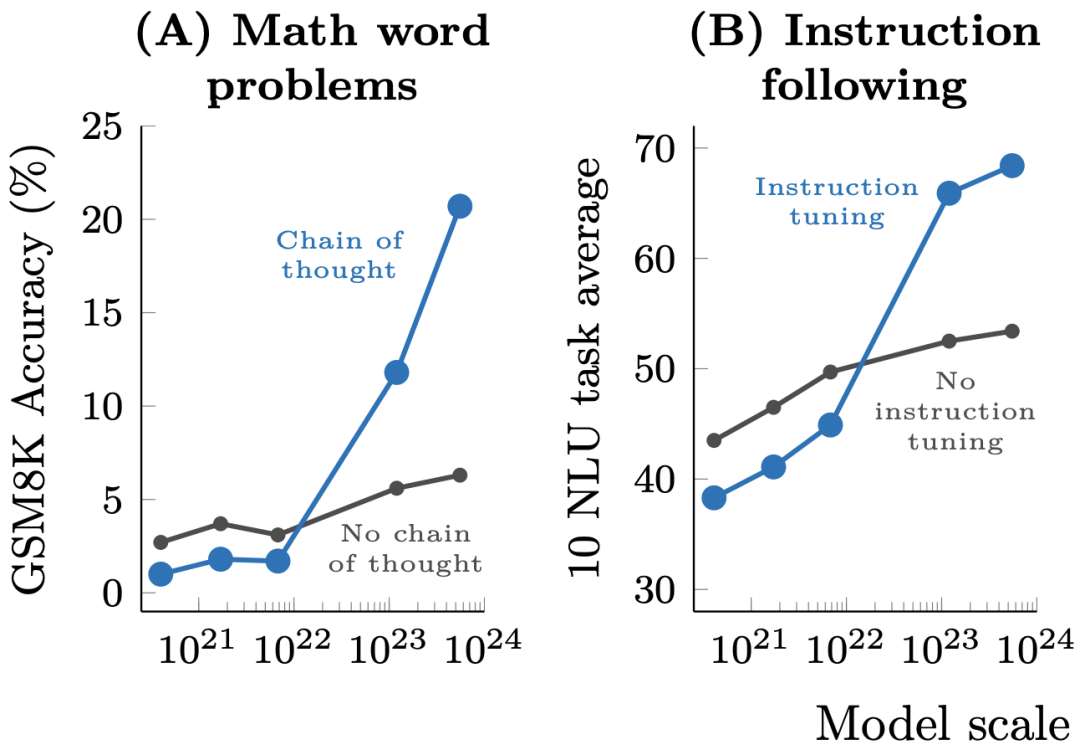
Sizes of Large Language Models (LLMs)



- Training costs [29]:
  - 110M params: \$2.5k–\$50k
  - 340M params: \$10k–\$200k
  - 1.5B param: \$80k–\$1.6m

Size Does Matter!

- Emerging abilities of LLMs [35]



- A balance: 70B parameters + 1.4T training tokens [6]

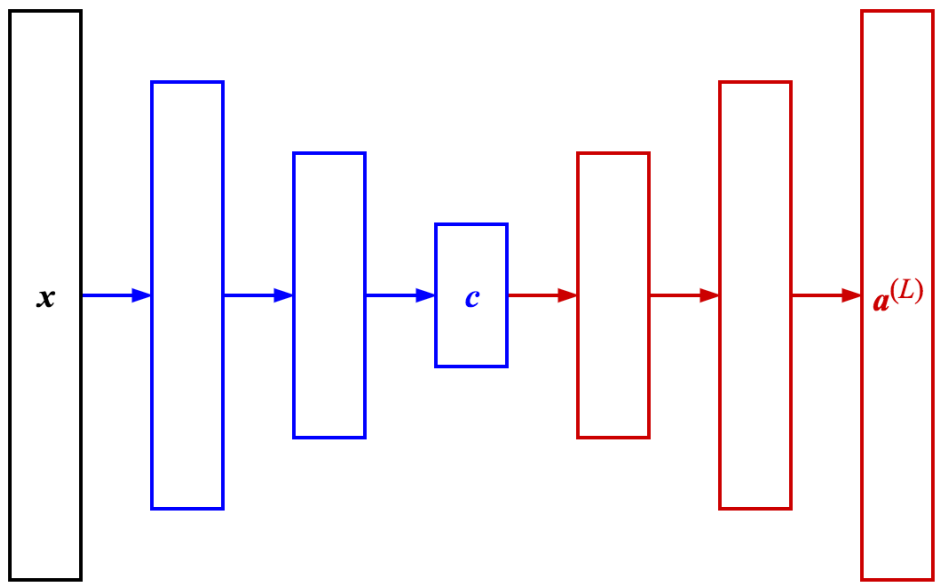
GPT Variants

- Low-rank adaptation (LoRA) [7]: to efficiently fine-tune LLMs
- WebGPT [19]: GPT that can search the web
- Retrieval-augmented generation (RAG) [13]: GPT to query external knowledge (vector) base
- GPTs that can reflect on their answers [1, 18]

Outline

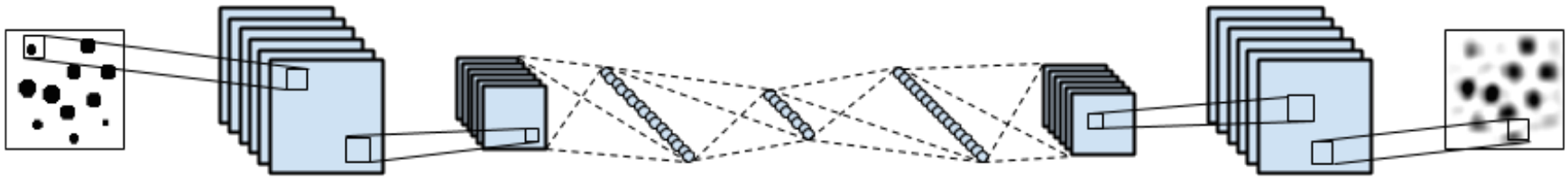
Autoencoders (AE)

- ***Encoder***: to learn a low dimensional representation $\mathbf{c}$ (called ***code***) of input $\mathbf{x}$
- ***Decoder***: to reconstruct $\mathbf{x}$ from $\mathbf{c}$
- Objective: $\arg\max_\Theta \log \mathrm{P}(\mathbb{X} \mid \Theta) = \arg\max_\Theta \sum_i \log \mathrm{P}(\boldsymbol{x}^{(i)} \mid \Theta)$
- Assuming that $\mathbf{x} \sim \mathcal{N}(\mu, \cdot)$, we have linear output units $\boldsymbol{a}^{(L)} = \boldsymbol{z}^{(L)} = \hat{\mu}$
  - $\log \mathrm{P}(\boldsymbol{x}^{(i)} \mid \Theta) \propto -\|\boldsymbol{x}^{(i)} - \boldsymbol{a}^{(i,L)}\|^2$
  - $\arg\max_\Theta \sum_i \log \mathrm{P}(\boldsymbol{x}^{(i)} \mid \Theta) = \arg\min_\Theta \sum_i \|\boldsymbol{x}^{(i)} - \boldsymbol{a}^{(i,L)}\|^2$ (minimizing reconstruct error)



Convolutional Autoencoders

- Convolution + deconvolution layers:



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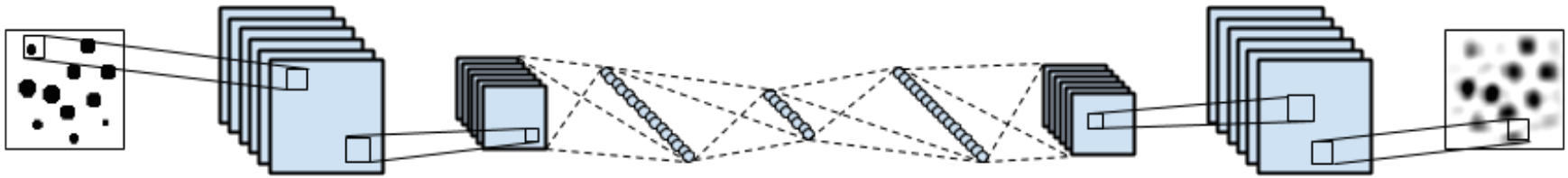

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Convolutional Autoencoders

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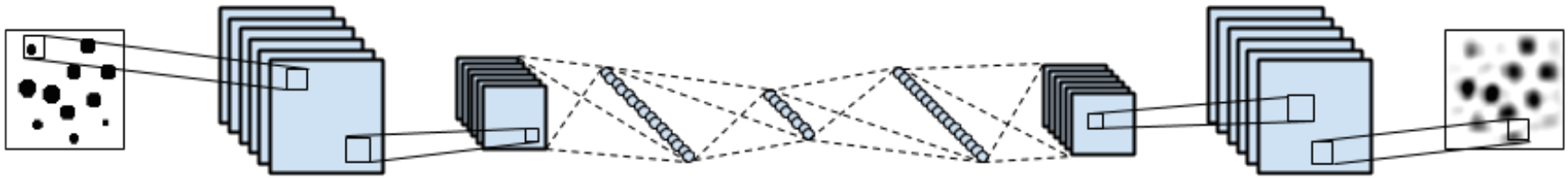

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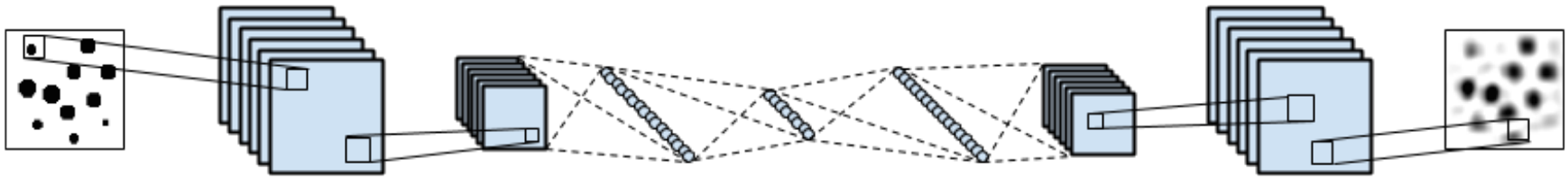

- Decoder is a simplified DeconvNet [36] trained from scratch:
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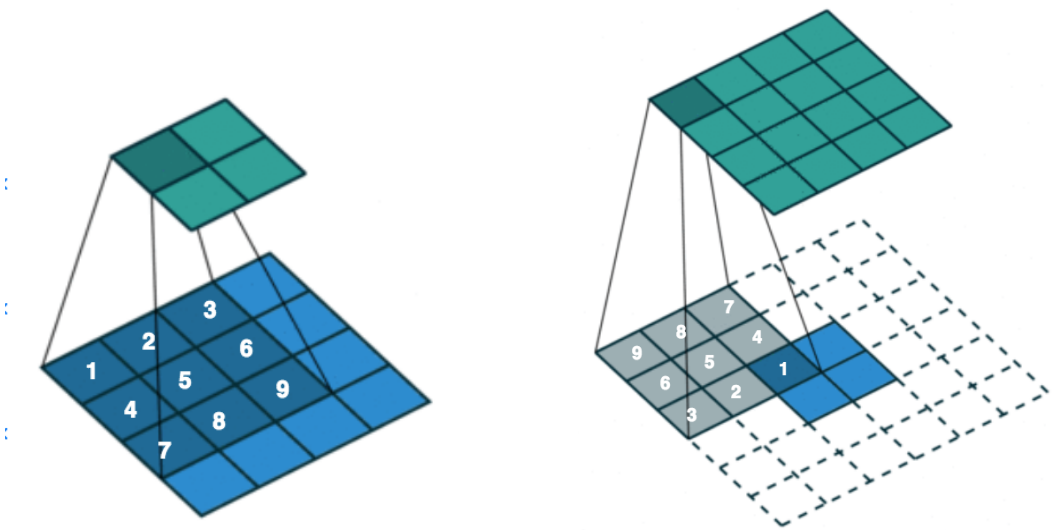
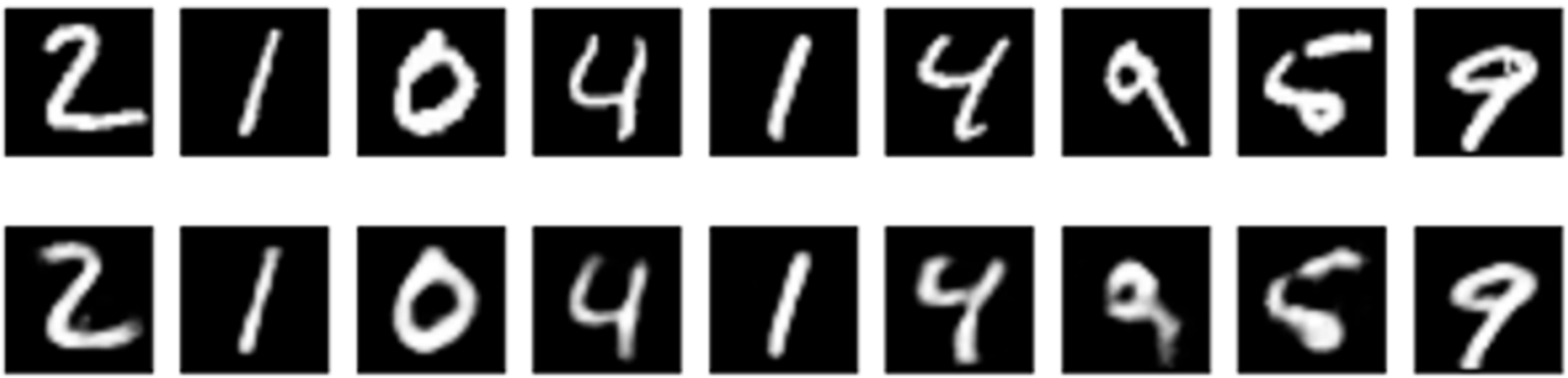


Codes & Reconstructed $\mathbf{x}$

- A 32-bit code can roughly represents a 32×32 (1024 dimensional) MNIST image



Outline

Manifolds I

- In many applications, data concentrate around one or more low-dimensional ***manifolds***

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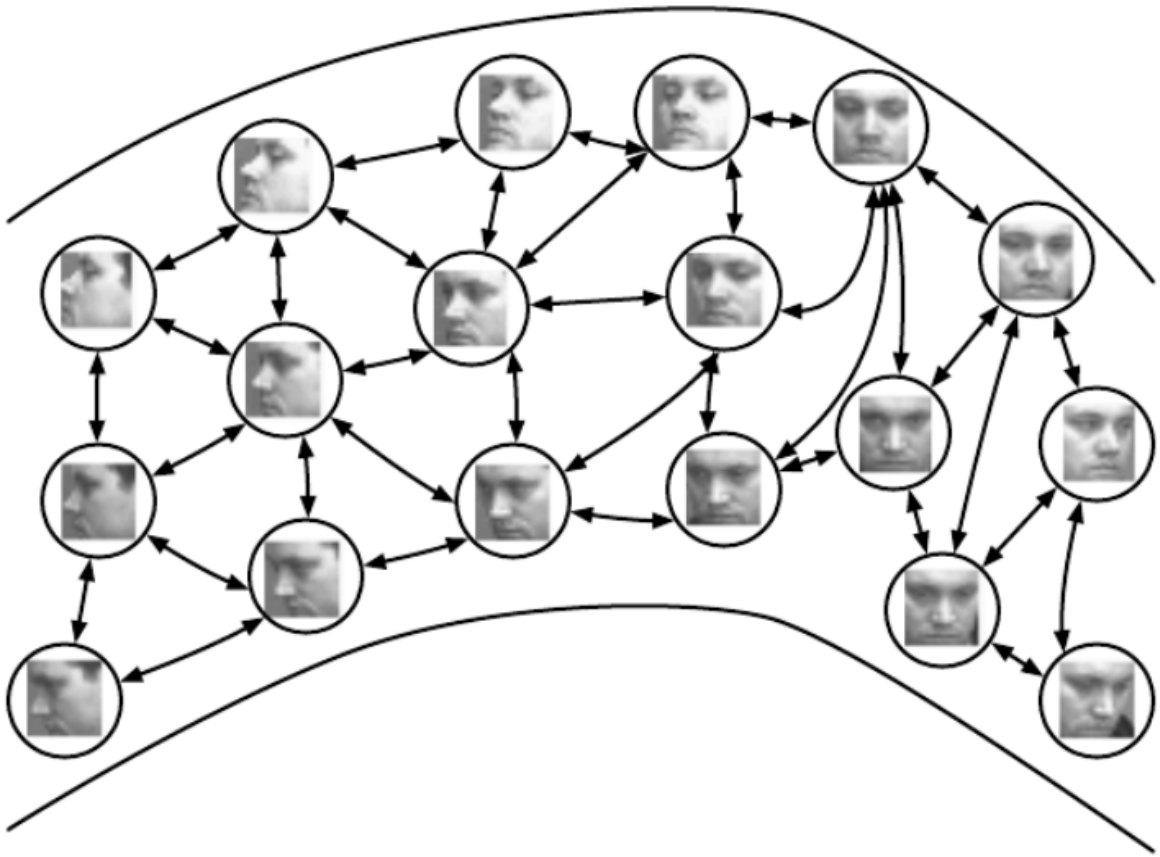
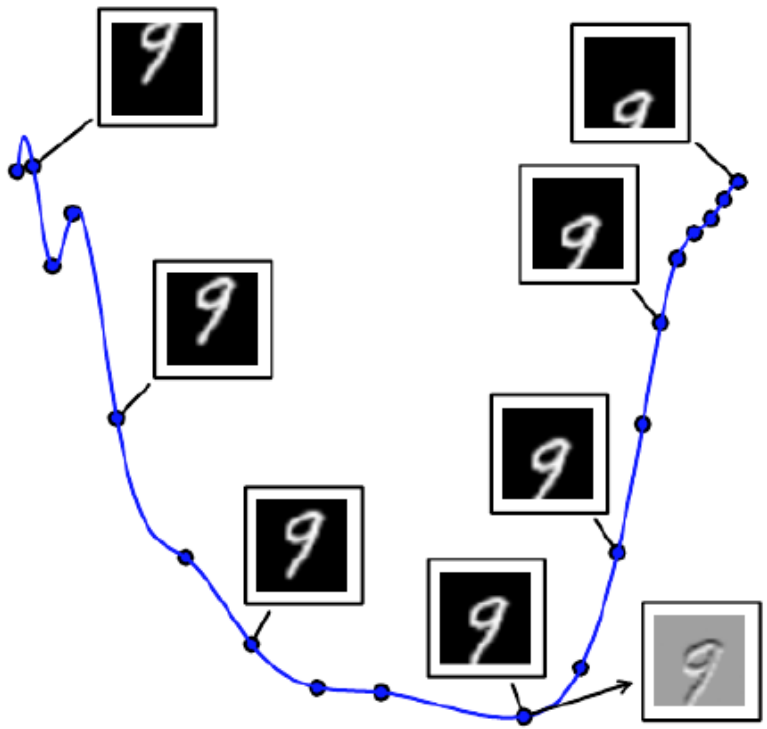
- A manifold is a topological space that are ***linear locally***



Manifolds II

- For each point $\boldsymbol{x}$ on a manifold, we have its ***tangent space*** spanned by ***tangent vectors***
  - Local directions specify how one can change $\boldsymbol{x}$ infinitesimally while staying on the manifold



Learning Manifolds I

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- Contractive autoencoder [24]: regularizes the code $\mathbf{c}$ such that it is invariant to local changes of $\mathbf{x}$:

$$\Omega(\mathbf{c}) = \sum_i \left\| \frac{\partial \boldsymbol{c}^{(i)}}{\partial \boldsymbol{x}^{(i)}} \right\|_F^2$$

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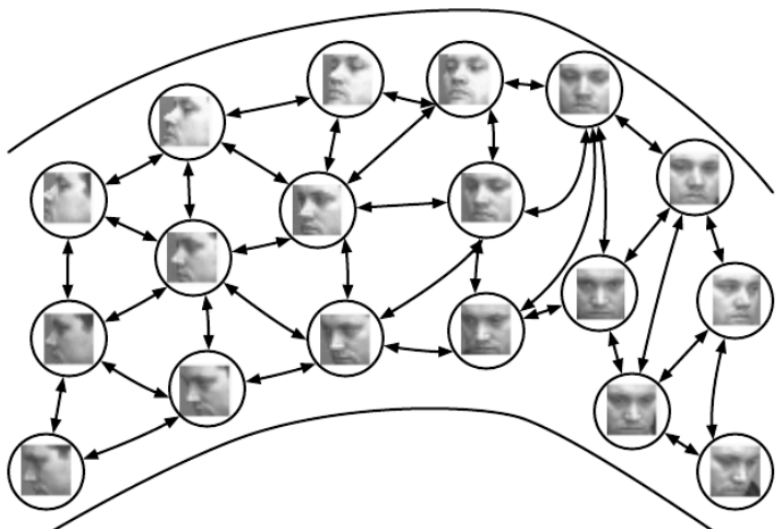
- Hence, $\mathbf{c}$ represents only the variations needed to reconstruct $\mathbf{x}$
  - I.e., $\mathbf{c}$ changes most along tangent vectors



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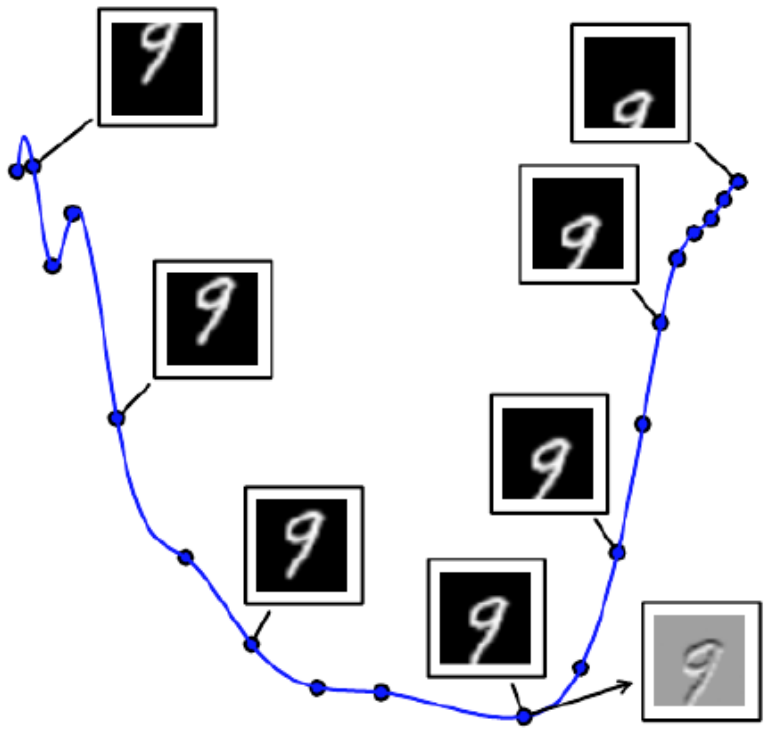
  - Encoder: to encode $\mathbf{x}$ ***with*** random noises
  - Decoder: to reconstruct $\mathbf{x}$ ***without*** noises

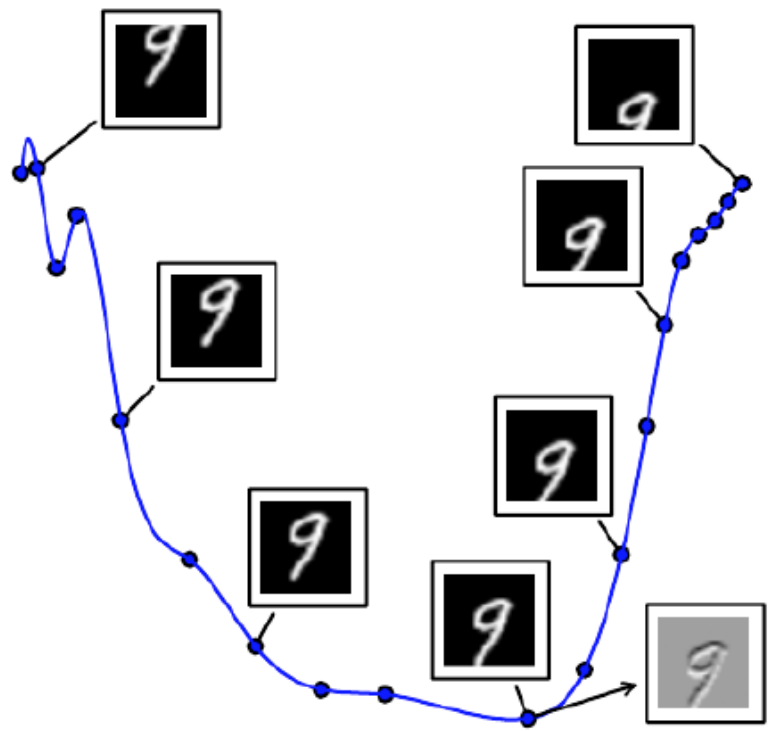
Getting Tangent Vectors I

- The code $\mathbf{c}$ represents a coordinate on a low dimensional manifold
  - E.g., the blue line
- How to get the tangent vectors of a given $\boldsymbol{c}$?



Getting Tangent Vectors II

- Recall: directions in the input space that ***changes c most*** should be tangent vectors



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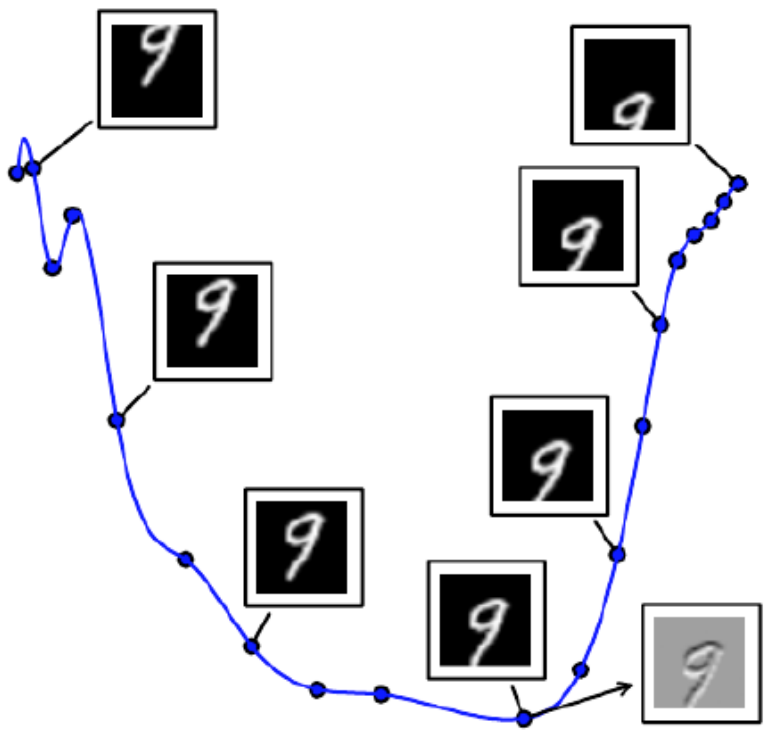
- Recall: directions in the input space that ***changes c most*** should be tangent vectors

- Given a point $\boldsymbol{x}$, let $\boldsymbol{c}$ be the code of $\boldsymbol{x}$ and $\boldsymbol{J}(\boldsymbol{x}) = \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{x}}$ be the Jacobian matrix of $\boldsymbol{c}$ at $\boldsymbol{x}$



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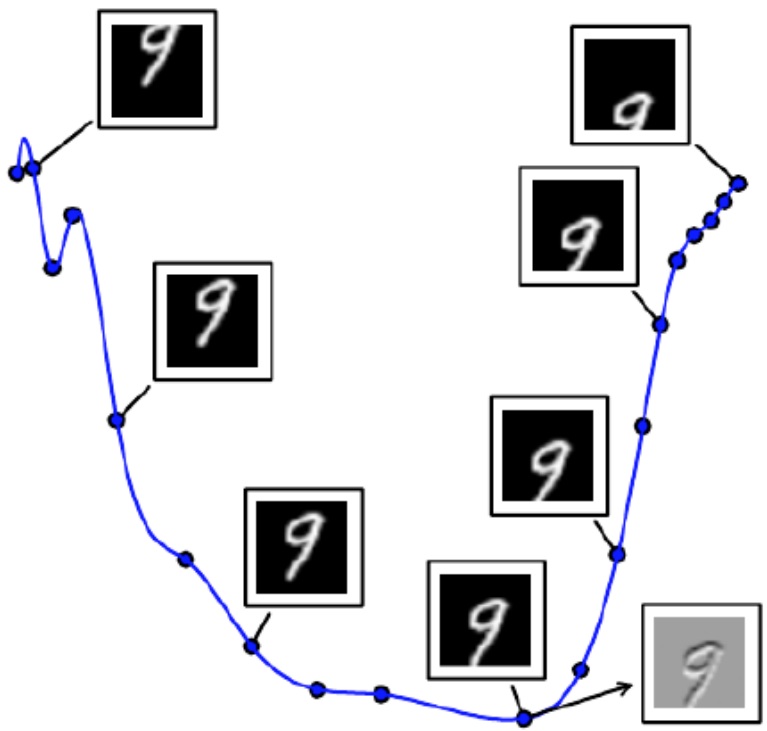

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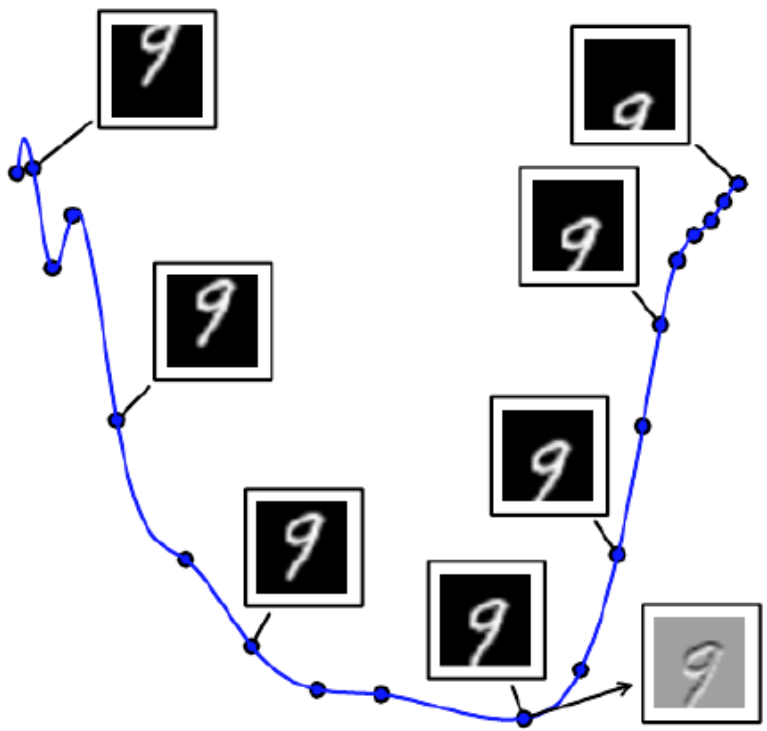
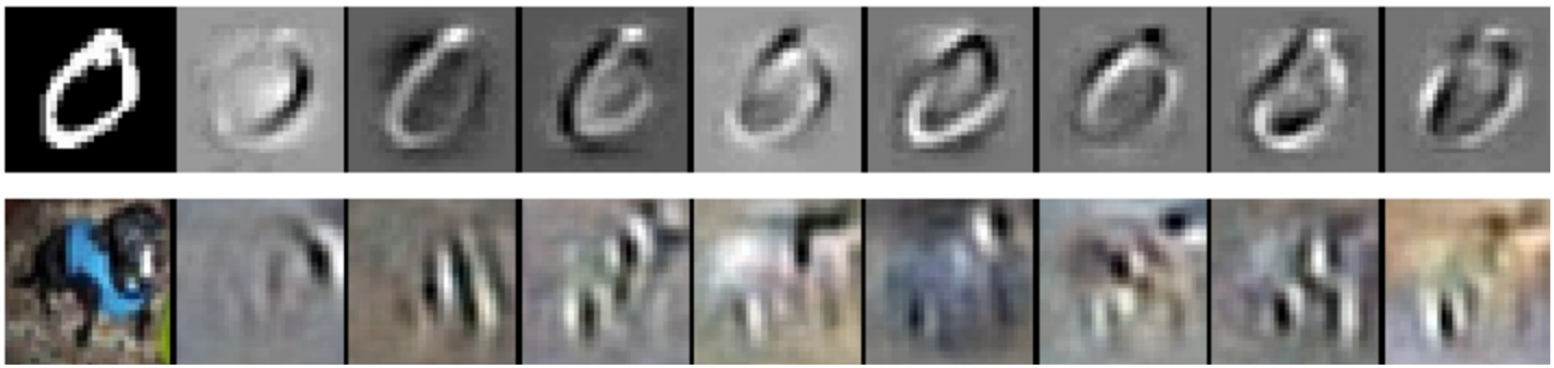
1. Decompose $\boldsymbol{J}(\boldsymbol{x})$ using SVD such that $\boldsymbol{J}(\boldsymbol{x}) = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^\top$
2. Let tangent vectors be ***rows of V corresponding to the largest singular values in D***



Getting Tangent Vectors III

- In practice, $\boldsymbol{J}(\boldsymbol{x})$ usually has few large singular values
- Tangent vectors found by contractive/denoising autoencoders:



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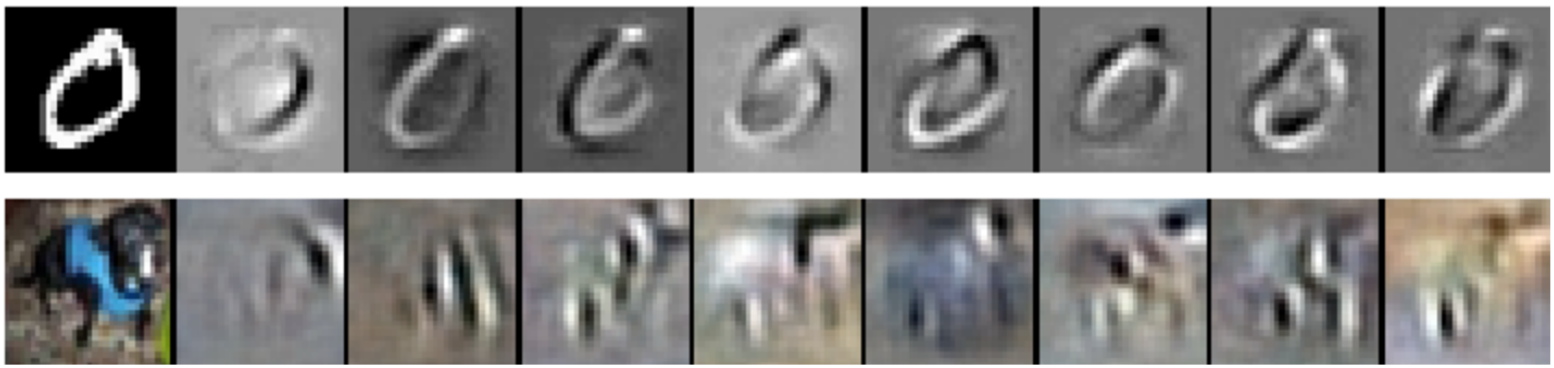
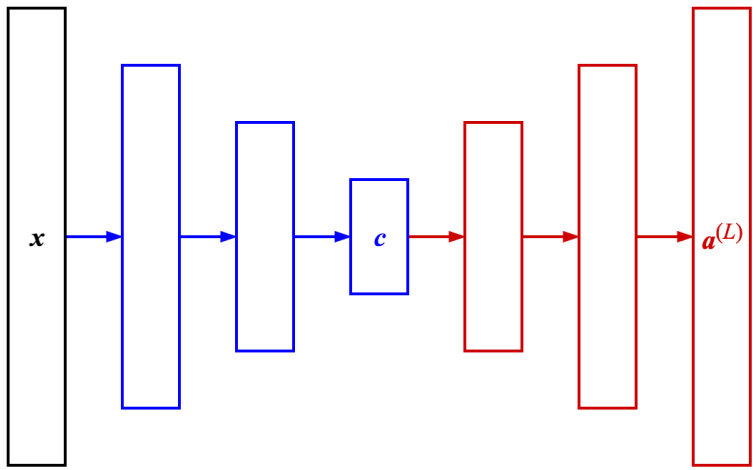


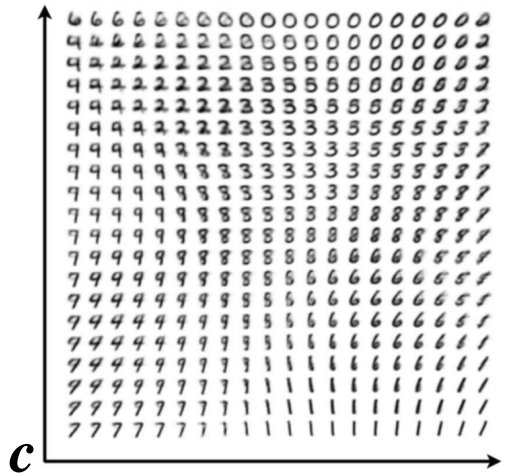
- Can be used by Tangent Prop [30]:
- Let $\{\boldsymbol{v}^{(i,j)}\}_j$ be tangent vectors of each example $\boldsymbol{x}^{(i)}$
- Trains an NN classifier f with cost penalty: $\Omega[f] = \sum_{i,j} \nabla_{\boldsymbol{x}} f(\boldsymbol{x}^{(i)})^\top \boldsymbol{v}^{(i,j)}$
  - Points in the same manifold share the same label

Outline

Problems of Autoencoder I

- Ideally, the decoder of an autoencoder can be used to generate images even with ***synthetic codes***





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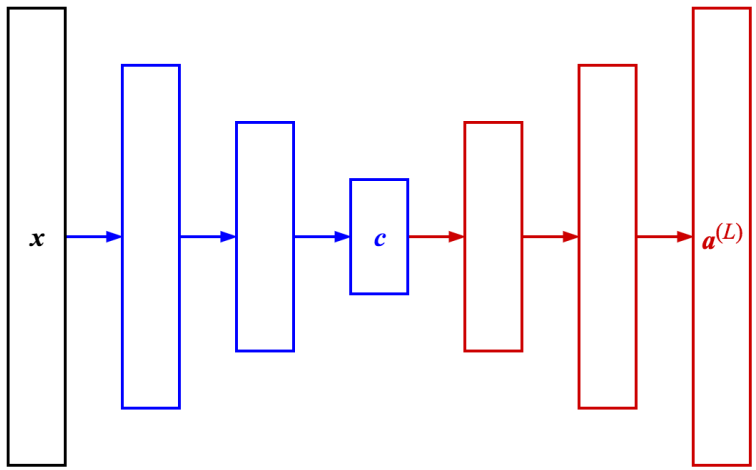
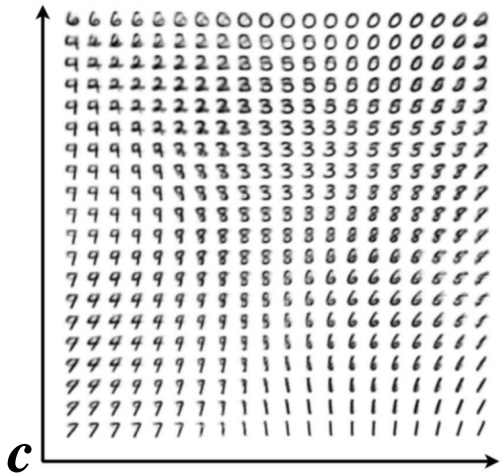




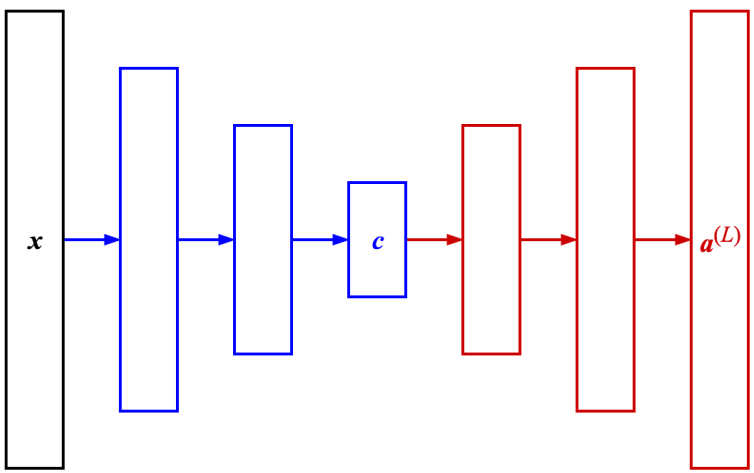
- In reality, the learnt $\mathbf{c}$ in code space has many "holes" that fail to map to images
  - More complex image patterns
  - Training data are never enough

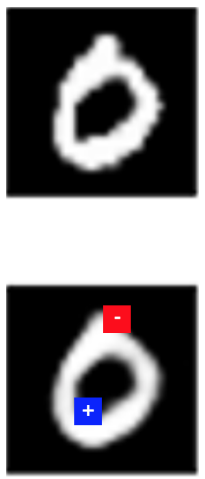
Problems of Autoencoder II

- Blurry images: the objective

$$\arg\max_{\Theta} \sum_i \log \mathrm{P}(\boldsymbol{x}^{(i)} \mid \Theta) = \arg\min_{\Theta} \sum_i \|\boldsymbol{x}^{(i)} - \boldsymbol{a}^{(i,L)}\|^2$$

does not panelize Gaussian pixel noises in $\boldsymbol{a}^{(i,L)}$





Variational Autoencoders (VAE) [8]

- Encoder $f(\cdot;\Theta_f)$: maps each sample of $\mathbf{x}$ (i.e., $\boldsymbol{x}^{(i)} \in \mathbb{X}$) to an ***axis-aligned normal distribution*** $\mathcal{N}(\mu, \sigma)$
  - Each code dimension is independent with each other
  - $f(\boldsymbol{x}) = (\mu, \sigma)$
- Decoder $g(\cdot;\Theta_g)$: same as that of AE
- How to minimize the objective $\arg\max_{\Theta_f,\Theta_g} \log \mathrm{P}(\mathbb{X} \mid \Theta_f, \Theta_g))$?

VAE Objective

- Objective:
  $\arg\max_{\Theta_f,\Theta_g} \log \mathrm{P}(\mathbb{X}\,|\,\Theta_f,\Theta_g) = \arg\max_{\Theta_f,\Theta_g} \sum_i \log \mathrm{P}(\boldsymbol{x}^{(i)}\,|\,\Theta_f,\Theta_g)$
- Considering $\log \mathrm{P}(\boldsymbol{x})$ for any sample $\boldsymbol{x}$, we have

$$
\begin{aligned}
\log \mathrm{P}(\boldsymbol{x}) &= \textstyle\int_{\boldsymbol{c}} \mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}) \log \mathrm{P}(\boldsymbol{x})\, d\boldsymbol{c} \;\; // \text{ Q can be any distribution} \\
&= \textstyle\int_{\boldsymbol{c}} \mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}) \log \left(\frac{\mathrm{P}(\boldsymbol{c},\boldsymbol{x})}{\mathrm{P}(\boldsymbol{c}|\boldsymbol{x})} \right) d\boldsymbol{c} = \int_{\boldsymbol{c}} \mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}) \log \left(\frac{\mathrm{P}(\boldsymbol{c},\boldsymbol{x})}{\mathrm{Q}(\boldsymbol{c}|\boldsymbol{x})} \frac{\mathrm{Q}(\boldsymbol{c}|\boldsymbol{x})}{\mathrm{P}(\boldsymbol{c}|\boldsymbol{x})} \right) d\boldsymbol{c} \\
&= \textstyle\int_{\boldsymbol{c}} \mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}) \log \left(\frac{\mathrm{P}(\boldsymbol{c},\boldsymbol{x})}{\mathrm{Q}(\boldsymbol{c}|\boldsymbol{x})} \right) d\boldsymbol{c} + \int_{\boldsymbol{c}} \mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}) \log \left(\frac{\mathrm{Q}(\boldsymbol{c}|\boldsymbol{x})}{\mathrm{P}(\boldsymbol{c}|\boldsymbol{x})} \right) d\boldsymbol{c} \\
&= \textstyle\int_{\boldsymbol{c}} \mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}) \log \left(\frac{\mathrm{P}(\boldsymbol{c},\boldsymbol{x})}{\mathrm{Q}(\boldsymbol{c}|\boldsymbol{x})} \right) d\boldsymbol{c} + \mathrm{D}_{\mathrm{KL}} \left(\mathrm{Q}(\mathbf{c}|\boldsymbol{x}) \| \mathrm{P}(\mathbf{c}|\boldsymbol{x}) \right) \\
&\geq \textstyle\int_{\boldsymbol{c}} \mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}) \log \left(\frac{\mathrm{P}(\boldsymbol{c},\boldsymbol{x})}{\mathrm{Q}(\boldsymbol{c}|\boldsymbol{x})} \right) d\boldsymbol{c} \;\; // \text{ lower bound} \\
&= \textstyle\int_{\boldsymbol{c}} \mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}) \log \left(\frac{\mathrm{P}(\boldsymbol{x}|\boldsymbol{c})\mathrm{P}(\boldsymbol{c})}{\mathrm{Q}(\boldsymbol{c}|\boldsymbol{x})} \right) d\boldsymbol{c}
\end{aligned}
$$

- VAE lets $\mathrm{Q}(\cdot\,|\boldsymbol{x}) = \mathcal{N}(f(\boldsymbol{x};\Theta_f))$ and $\mathrm{P}(\cdot\,|\boldsymbol{c}) = \mathcal{N}(g(\boldsymbol{c};\Theta_g))$
  - So $\mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}) = \mathrm{Q}(\boldsymbol{c}|\boldsymbol{x},\Theta_f)$ and $\mathrm{P}(\boldsymbol{x}|\boldsymbol{c}) = \mathrm{P}(\boldsymbol{x}|\boldsymbol{c},\Theta_g)$
- New objective: finds Θ_f and Θ_g that maximize the lower bound
  - Not necessarily maximize $\log \mathrm{P}(\boldsymbol{x}|\,\Theta_f,\Theta_g)$

Maximizing Lower Bound

$$
\begin{aligned}
&\int_{\boldsymbol{c}} \mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}) \log \left(\frac{\mathrm{P}(\boldsymbol{x}|\boldsymbol{c})\mathrm{P}(\boldsymbol{c})}{\mathrm{Q}(\boldsymbol{c}|\boldsymbol{x})} \right) d\boldsymbol{c} \\
&= \int_{\boldsymbol{c}} \mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}, \Theta_f) \log \left(\frac{\mathrm{P}(\boldsymbol{x}|\boldsymbol{c}, \Theta_g)\mathrm{P}(\boldsymbol{c})}{\mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}, \Theta_f)} \right) d\boldsymbol{c} \\
&= \int_{\boldsymbol{c}} \mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}, \Theta_f) \log \left(\frac{\mathrm{P}(\boldsymbol{c})}{Q(\boldsymbol{c}|\boldsymbol{x}, \Theta_f)} \right) d\boldsymbol{c} + \int_{\boldsymbol{c}} \mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}, \Theta_f) \log \mathrm{P}(\boldsymbol{x}|\boldsymbol{c}, \Theta_g) d\boldsymbol{c} \\
&= -\mathrm{D}_{\mathrm{KL}} \left(\mathrm{Q}(\mathbf{c}|\boldsymbol{x}, \Theta_f) \| \mathrm{P}(\mathbf{c}) \right) + \mathrm{E}_{(\mathbf{c}|\boldsymbol{x}, \Theta_f) \sim \mathrm{Q}} \left[\log \mathrm{P}(\boldsymbol{x}|\mathbf{c}, \Theta_g) \right]
\end{aligned}
$$

- For $\mathrm{P}(\mathbf{c}) = \mathcal{N}(\mathbf{0}, \boldsymbol{I})$:
- To minimize the first term, the encoder $f(\boldsymbol{x}; \Theta_f) = (\mu, \sigma)$ has a loss term [8]

$$\exp(\sigma) - (\mathbf{1} + \sigma) + \|\mu\|^2$$

- To maximize the second term, the decoder
  1. Samples $\boldsymbol{c}$ from $\mathcal{N}(\mu, \sigma)$ to get $g(\boldsymbol{c}; \Theta_g)$, the mean of output $\mathcal{N}$
  2. Minimizes a loss term $\|\boldsymbol{x} - \boldsymbol{a}^{(L)}\|^2$ as in AE

Outline

Problems of VAE

- Only maximize a lower bound of the likelihood $\mathrm{P}(\mathbb{X}|\,\Theta_f, \Theta_g)$
  - Θ_f and Θ_g are encoder and decoder weights, respectively
- Still blurry images
  - The decoder's loss is the same with that of AE

Flow-based Models

- Idea: let the decoder $g(\cdot;\Theta_g)$ be a ***deterministic invertible*** function
  - Given $\mathbf{c} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I})$, we have $\mathbf{x} = g(\mathbf{c};\Theta_g)$ of complex distribution
  - Conversely, given $\mathbf{x}$, we have $\mathbf{c} = g^{-1}(\mathbf{x};\Theta_g)$
- The likelihood can be maximize directly:

$$
\begin{aligned}
&\arg\max_g \log \mathrm{P}_g(\mathbb{X}) \\
&= \arg\max_g \textstyle\sum_i \log \mathrm{P}_{\mathbf{x}}(g(\boldsymbol{c}^{(i)})) \\
&= \arg\max_g \textstyle\sum_i \log \left[\mathrm{P}_{\mathbf{c}}(g^{-1}(\boldsymbol{x}^{(i)}))\left|\det\left(\boldsymbol{J}(g^{-1})(\boldsymbol{x}^{(i)})\right)\right|\right] \\
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$$

- First term: finds g^{-1} (Θ_g) that maps all $\boldsymbol{x}^{(i)}$ to 0 (mean of $\mathcal{N}$)
- Second term: prevents g^{-1} (Θ_g) from mapping all $\boldsymbol{x}^{(i)}$ to only 0
  - As $\log|\det(\boldsymbol{O})| = -\inf$

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- Second term: prevents g^{-1} (Θ_g) from mapping all $\boldsymbol{x}^{(i)}$ to only 0
  - As $\log|\det(\boldsymbol{O})| = -\inf$
- But how to ensure the followings during training?
  - g is invertible
  - $\det\left(\boldsymbol{J}(g^{-1})(\cdot)\right)$ can be easily computed

Ensuring Invertibility

- Glow [9]: make g^{-1} an 1×1 convolution layer
  - $\mathbf{x}_{i,j,:} = \begin{bmatrix} \mathrm{x}_{i,j,1} \\ \mathrm{x}_{i,j,2} \\ \mathrm{x}_{i,j,3} \end{bmatrix}, \quad g^{-1} = \boldsymbol{W}_{3\times 3},$

$$g^{-1}(\mathbf{x}_{i,j,:}) = \boldsymbol{W}_{3\times 3} \begin{bmatrix} \mathrm{x}_{i,j,1} \\ \mathrm{x}_{i,j,2} \\ \mathrm{x}_{i,j,3} \end{bmatrix} = \begin{bmatrix} \mathrm{c}_{i,j,1} \\ \mathrm{c}_{i,j,2} \\ \mathrm{c}_{i,j,3} \end{bmatrix}$$

- At training time, initialize $\boldsymbol{W}_{3\times 3}$ as an invertible matrix
  - $g^{-1} = \boldsymbol{W}_{3\times 3}$ is likely to be invertible after SGD updates
  - Determinant is easy to compute: $\det\left(\boldsymbol{J}(g^{-1})(\boldsymbol{x}^{(i)})\right) = \det(\boldsymbol{W}_{3\times 3})^{W\times H}$
- At inference time, use $g = \boldsymbol{W}_{3\times 3}^{-1}$ to generate images

Ensuring Invertibility

- Glow [9]: make g^{-1} an 1×1 convolution layer
  - $\mathbf{x}_{i,j,:} = \begin{bmatrix} \mathrm{x}_{i,j,1} \\ \mathrm{x}_{i,j,2} \\ \mathrm{x}_{i,j,3} \end{bmatrix}, \quad g^{-1} = \boldsymbol{W}_{3\times 3},$

$$g^{-1}(\mathbf{x}_{i,j,:}) = \boldsymbol{W}_{3\times 3} \begin{bmatrix} \mathrm{x}_{i,j,1} \\ \mathrm{x}_{i,j,2} \\ \mathrm{x}_{i,j,3} \end{bmatrix} = \begin{bmatrix} \mathrm{c}_{i,j,1} \\ \mathrm{c}_{i,j,2} \\ \mathrm{c}_{i,j,3} \end{bmatrix}$$

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- At inference time, use $g = \boldsymbol{W}_{3\times 3}^{-1}$ to generate images
- Problem: g has limited expressiveness

Step-wise Generation

- Cascade multiple invertible g to have a more complex one:

$$\mathbf{x} = g^{(T)}(\cdots g^{(2)}(g^{(1)}(\mathbf{c})))$$

Step-wise Generation

- Cascade multiple invertible g to have a more complex one:

$$\mathbf{x} = g^{(T)}(\cdots g^{(2)}(g^{(1)}(\mathbf{c})))$$

- Result: sharp images

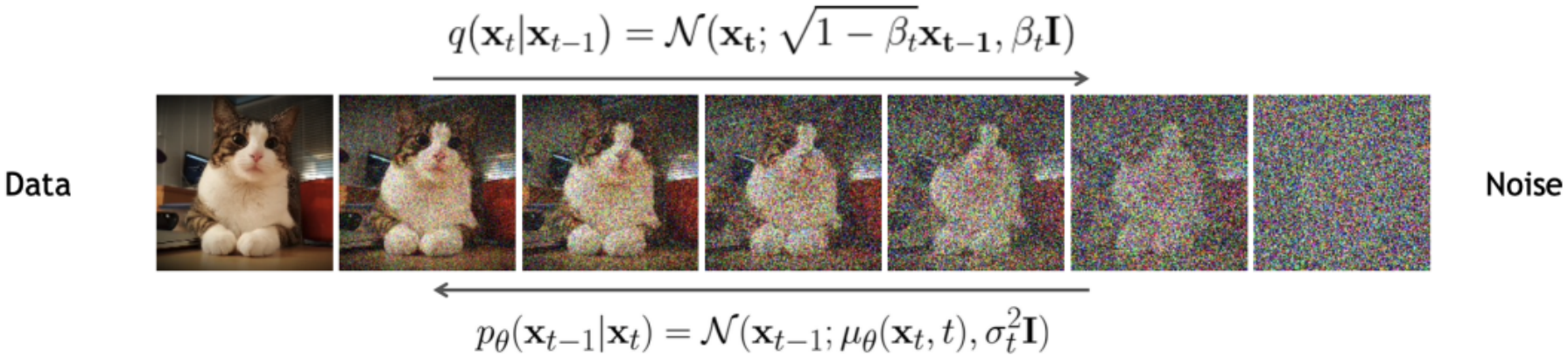
Outline

Problems of Flow-based Models

- Limited model expressiveness
- Slow training and inference
  - #steps can be large

Denoising Diffusion Probabilistic Models (DDPM) [5]

- Borrow some good ideas from previous works
  - Probabilistic formulation of VAE that models encoder in the objective
  - Step-wise encoding/decoding in generative flows
- But, unlike flows, the encoding steps
  - Are ***predefined***; no parameter to learn
  - Can be simplified to ***one encoding step***



Encoding: multi-steps = single-step

- Predefined encoding functions $f^{(t)}(\cdot;\beta^{(t)})$, $t=1,\cdots,T$:
  - $\beta^{(t)}$, $\forall t$, are hyperparameters; no learning needed
- Let each sample $\boldsymbol{x}=\boldsymbol{x}^{(0)}$
- $\mathbf{x}^{(1)}=f^{(1)}(\boldsymbol{x}^{(0)};\beta^{(1)})=\sqrt{1-\beta^{(1)}}\boldsymbol{x}^{(0)}+\sqrt{\beta^{(1)}}\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}\sim\mathcal{N}(\mathbf{0},\boldsymbol{I})$
  - So, $(\mathbf{x}^{(1)}|\mathbf{x}^{(0)}=\boldsymbol{x}^{(0)})\sim\mathcal{N}(\sqrt{1-\beta^{(1)}}\boldsymbol{x}^{(0)},\beta^{(1)}\boldsymbol{I})$

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  - So, $(\mathbf{x}^{(1)}|\mathbf{x}^{(0)}=\boldsymbol{x}^{(0)})\sim\mathcal{N}(\sqrt{1-\beta^{(1)}}\boldsymbol{x}^{(0)},\beta^{(1)}\boldsymbol{I})$
- $$\begin{aligned}\mathbf{x}^{(2)} &= f^{(2)}(\boldsymbol{x}^{(1)};\beta^{(t)})=\sqrt{1-\beta^{(2)}}\boldsymbol{x}^{(1)}+\sqrt{\beta^{(2)}}\varepsilon \\ &= \sqrt{1-\beta^{(2)}}\sqrt{1-\beta^{(1)}}\boldsymbol{x}^{(0)}+\sqrt{1-(1-\beta^{(2)})(1-\beta^{(1)})}\varepsilon \\ &= \sqrt{\alpha^{(2)}\alpha^{(1)}}\boldsymbol{x}^{(0)}+\sqrt{1-\alpha^{(2)}\alpha^{(1)}}\varepsilon\end{aligned}$$
  - $\sqrt{1-\beta^{(1)}}\sqrt{\beta^{(2)}}\varepsilon+\sqrt{\beta^{(2)}}\varepsilon\sim\mathcal{N}(\mathbf{0},((1-\beta^{(1)})\beta^{(2)}+\beta^{(2)})\boldsymbol{I})$
  - Let $\alpha^{(t)}=1-\beta^{(t)}$ (derived hyperparameter)
  - Only one ε is added

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  - Let $\alpha^{(t)}=1-\beta^{(t)}$ (derived hyperparameter)
  - Only one ε is added
- $\mathbf{x}^{(t)}=\sqrt{\bar{\alpha}^{(t)}}\boldsymbol{x}^{(0)}+\sqrt{1-\bar{\alpha}^{(t)}}\varepsilon$
  - Let $\bar{\alpha}^{(t)}=\alpha^{(t)}\alpha^{(t-1)}\cdots\alpha^{(1)}$ (derived hyperparameter)

Objective

- As VAE, DDPM maximizes a lower bound of $\mathrm{P}(\mathbb{X})$
- VAE: for each $\boldsymbol{x}$, where $\mathrm{P}(\boldsymbol{x}) = \int_{\boldsymbol{c}} \mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}) \log \mathrm{P}(\boldsymbol{x})\, d\boldsymbol{c}$, maximize:

$$\int_{\boldsymbol{c}} \mathrm{Q}(\boldsymbol{c}|\boldsymbol{x}) \log \left(\frac{\mathrm{P}(\boldsymbol{x}, \boldsymbol{c})}{\mathrm{Q}(\boldsymbol{c}|\boldsymbol{x})} \right) d\boldsymbol{c} = \mathrm{E}_{\mathbf{c}|\boldsymbol{x} \sim \mathrm{Q}} \left[\log \left(\frac{\mathrm{P}(\boldsymbol{x}, \mathbf{c})}{\mathrm{Q}(\mathbf{c}|\boldsymbol{x})} \right) \right] \leq \mathrm{P}(\boldsymbol{x})$$

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- DDPM: for each $\boldsymbol{x}$ where $\mathrm{P}(\boldsymbol{x}^{(0)}) = \int_{\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(T)}} \mathrm{Q}(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(T)}|\boldsymbol{x}^{(0)}) \log \mathrm{P}(\boldsymbol{x}^{(0)})\, d\boldsymbol{x}^{(1)} \cdots d\boldsymbol{x}^{(T)}$, maximize:

$$\mathrm{E}_{(\mathbf{x}^{(1)},\cdots,\mathbf{x}^{(T)}|\boldsymbol{x}^{(0)})\sim\mathrm{Q}} \left[\log \left(\frac{\mathrm{P}(\boldsymbol{x}^{(0)},\mathbf{x}^{(1)},\cdots,\mathbf{x}^{(T)})}{\mathrm{Q}(\mathbf{x}^{(1)},\cdots,\mathbf{x}^{(T)}|\boldsymbol{x}^{(0)})} \right) \right],$$

  which can be simplified to [15]:

$$-\mathrm{D}_{\mathrm{KL}}\left(\mathrm{Q}(\mathbf{x}^{(T)}|\boldsymbol{x}^{(0)}) \| \mathrm{P}(\mathbf{x}^{(T)})\right) + \mathrm{E}_{\mathbf{x}^{(1)}|\boldsymbol{x}^{(0)}\sim\mathrm{Q}}\left[\log \mathrm{P}(\boldsymbol{x}^{(0)}|\mathbf{x}^{(1)})\right] + \\ -\sum_{t=2}^{T} \mathrm{E}_{\mathbf{x}^{(t)}|\boldsymbol{x}^{(0)}\sim\mathrm{Q}}\left[\mathrm{D}_{\mathrm{KL}}\left(\mathrm{Q}(\mathbf{x}^{(t-1)}|\mathbf{x}^{(t)},\boldsymbol{x}^{(0)}) \| \mathrm{P}(\mathbf{x}^{(t-1)}|\mathbf{x}^{(t)})\right)\right]$$

  - First term controlled by encoding process (predefined)
  - Second & third term controlled by denoising process (learnable)

Denoising I

- For simplicity, we focus on maximizing the third term:

$$-\sum_{t=2}^{T} \mathrm{E}_{\mathbf{x}^{(t)}|\boldsymbol{x}^{(0)}\sim \mathrm{Q}} \left[\mathrm{D}_{\mathrm{KL}} \left(\mathrm{Q}(\mathbf{x}^{(t-1)}|\mathbf{x}^{(t)},\boldsymbol{x}^{(0)}) \| \mathrm{P}(\mathbf{x}^{(t-1)}|\mathbf{x}^{(t)}) \right) \right]$$

- Goal: for each observed $\boldsymbol{x}^{(t)}$, minimize

$$\mathrm{D}_{\mathrm{KL}} \left(\mathrm{Q}(\mathbf{x}^{(t-1)}|\boldsymbol{x}^{(t)},\boldsymbol{x}^{(0)}) \| \mathrm{P}(\mathbf{x}^{(t-1)}|\boldsymbol{x}^{(t)}) \right)$$

- Note that

$$\begin{aligned} \mathrm{Q}(\mathbf{x}^{(t-1)}|\boldsymbol{x}^{(t)},\boldsymbol{x}^{(0)}) &= \frac{\mathrm{Q}(\mathbf{x}^{(t-1)},\boldsymbol{x}^{(t)},\boldsymbol{x}^{(0)})}{\mathrm{Q}(\boldsymbol{x}^{(t)},\boldsymbol{x}^{(0)})} \\ &= \frac{\mathrm{Q}(\boldsymbol{x}^{(t)}|\mathbf{x}^{(t-1)})\mathrm{Q}(\mathbf{x}^{(t-1)}|\boldsymbol{x}^{(0)})\mathrm{Q}(\boldsymbol{x}^{(0)})}{\mathrm{Q}(\boldsymbol{x}^{(t)}|\boldsymbol{x}^{(0)})\mathrm{Q}(\boldsymbol{x}^{(0)})} \\ &= \frac{\mathrm{Q}(\boldsymbol{x}^{(t)}|\mathbf{x}^{(t-1)})\mathrm{Q}(\mathbf{x}^{(t-1)}|\boldsymbol{x}^{(0)})}{\mathrm{Q}(\boldsymbol{x}^{(t)}|\boldsymbol{x}^{(0)})} \end{aligned}$$

- Since $\mathrm{Q}(\boldsymbol{x}^{(t)}|\mathbf{x}^{(t-1)})$ & $\mathrm{Q}(\mathbf{x}^{(t-1)}|\boldsymbol{x}^{(0)})$ are Gaussian, we have [15]:
$\mathrm{Q}(\mathbf{x}^{(t-1)}|\boldsymbol{x}^{(t)},\boldsymbol{x}^{(0)}) = \mathcal{N}\left(\frac{\sqrt{\alpha^{(t)}}(1-\alpha^{(t-1)})\boldsymbol{x}^{(t)} + \sqrt{\bar{\alpha}^{(t-1)}}\beta^{(t)}\boldsymbol{x}^{(0)}}{1-\bar{\alpha}^{(t)}}, \frac{1-\bar{\alpha}^{(t-1)}}{1-\bar{\alpha}^{(t)}}\beta^{(t)}\boldsymbol{I} \right)$

Denoising II

- Goal: for each observed $\boldsymbol{x}^{(t)}$, minimize

$$\mathrm{D}_{\mathrm{KL}}\left(\mathrm{Q}(\mathbf{x}^{(t-1)}|\boldsymbol{x}^{(t)},\boldsymbol{x}^{(0)})||\mathrm{P}(\mathbf{x}^{(t-1)}|\boldsymbol{x}^{(t)})\right),$$

  where $\mathrm{Q}(\cdots)=\mathcal{N}\left(\frac{\sqrt{\alpha^{(t)}}(1-\alpha^{(t-1)})\boldsymbol{x}^{(t)}+\sqrt{\bar{\alpha}^{(t-1)}}\beta^{(t)}\boldsymbol{x}^{(0)}}{1-\bar{\alpha}^{(t)}},\cdots\right)$ is ***fixed***

- DDPM finds Θ that move the mean of $\mathrm{P}(\mathbf{x}^{(t-1)}|\boldsymbol{x}^{(t)},\Theta)$ (also Gaussian) toward $\mathrm{Q}(\cdots)$'s mean:

$$\frac{\sqrt{\alpha^{(t)}}(1-\alpha^{(t-1)})\boldsymbol{x}^{(t)}+\sqrt{\bar{\alpha}^{(t-1)}}\beta^{(t)}\boldsymbol{x}^{(0)}}{1-\bar{\alpha}^{(t)}}$$

Noise Predictor

- Θ moves $\mathrm{P}(\mathbf{x}^{(t-1)}|\boldsymbol{x}^{(t)},\Theta)$'s mean toward

$$\begin{aligned}
&\frac{\sqrt{\alpha^{(t)}}(1-\alpha^{(t-1)})\boldsymbol{x}^{(t)}+\sqrt{\bar{\alpha}^{(t-1)}}\beta^{(t)}\boldsymbol{x}^{(0)}}{1-\bar{\alpha}^{(t)}} \\
&= \frac{\sqrt{\alpha^{(t)}}(1-\alpha^{(t-1)})\boldsymbol{x}^{(t)}+\sqrt{\bar{\alpha}^{(t-1)}}\beta^{(t)}\frac{\boldsymbol{x}^{(t)}-\sqrt{1-\bar{\alpha}^{(t)}}\boldsymbol{\varepsilon}}{\sqrt{\bar{\alpha}^{(t)}}}}{1-\bar{\alpha}^{(t)}} \\
&= \frac{1}{\sqrt{\alpha^{(t)}}}\left(\boldsymbol{x}^{(t)}-\frac{1-\alpha^{(t)}}{\sqrt{1-\bar{\alpha}^{(t)}}}\boldsymbol{\varepsilon}\right)
\end{aligned}$$

- What's its corresponding network?

Noise Predictor

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- By definition: let Θ parametrize a network outputting $\boldsymbol{x}^{(t-1)}$ given $\boldsymbol{x}^{(t)}$
  - But the input $\boldsymbol{x}^{(t)}$ also resides in the output target

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&= \frac{1}{\sqrt{\alpha^{(t)}}}\left(\boldsymbol{x}^{(t)}-\frac{1-\alpha^{(t)}}{\sqrt{1-\bar{\alpha}^{(t)}}}\boldsymbol{\varepsilon}\right)
\end{aligned}$$

- What's its corresponding network?
- By definition: let Θ parametrize a network outputting $\boldsymbol{x}^{(t-1)}$ given $\boldsymbol{x}^{(t)}$
  - But the input $\boldsymbol{x}^{(t)}$ also resides in the output target
- DDPM: let Θ parametrize a ***noise predictor*** outputting $\boldsymbol{\varepsilon}$ given $\boldsymbol{x}^{(t)}$
  - Objective: $\arg\min_{\Theta} \|\boldsymbol{\varepsilon}-e(\boldsymbol{x}^{(t)},t;\Theta)\|^2$
  - Of U-Net [26] architecture
  - Shared between all t

Training & Inference Algorithms

- One-step encoding during training time
- Muti-step inference (sampling), with each intermediate decoding step t, $t > 1$, comes with extra noise $\sigma^{(t)}\boldsymbol{z}$
  - Similar to output token sampling in GPT
  - Improves performance empirically

Algorithm 1 Training

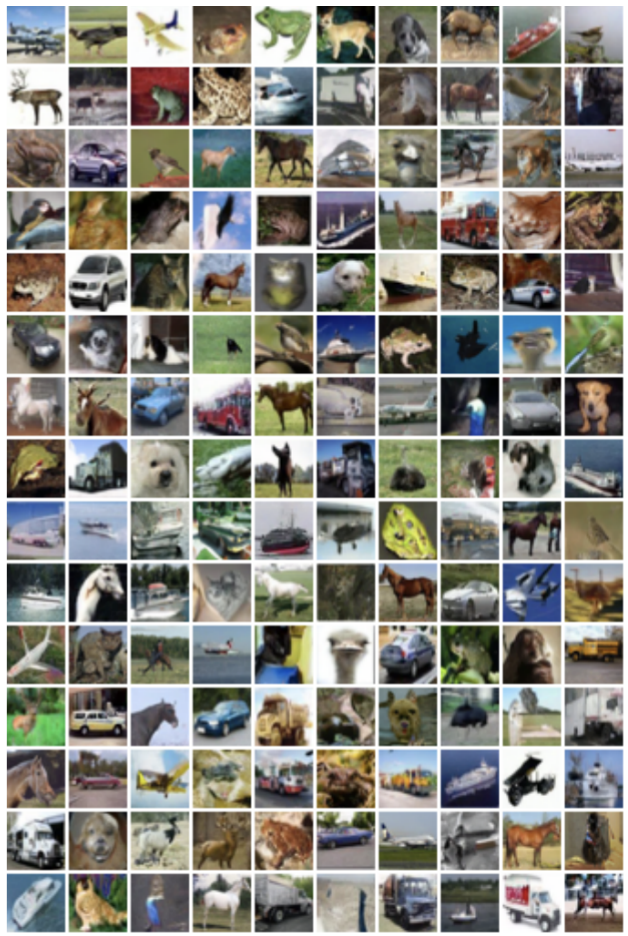
1: **repeat**
2: $\mathbf{x}_0 \sim q(\mathbf{x}_0)$
3: $t \sim \text{Uniform}(\{1, \dots, T\})$
4: $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$
5: Take gradient descent step on
$$\nabla_\theta \left\| \boldsymbol{\epsilon} - \boldsymbol{\epsilon}_\theta(\sqrt{\bar{\alpha}_t}\mathbf{x}_0 + \sqrt{1-\bar{\alpha}_t}\boldsymbol{\epsilon}, t) \right\|^2$$
6: **until** converged

Algorithm 2 Sampling

1: $\mathbf{x}_T \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$
2: **for** $t = T, \dots, 1$ **do**
3: $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ if $t > 1$, else $\mathbf{z} = \mathbf{0}$
4: $\mathbf{x}_{t-1} = \frac{1}{\sqrt{\alpha_t}} \left(\mathbf{x}_t - \frac{1-\alpha_t}{\sqrt{1-\bar{\alpha}_t}} \boldsymbol{\epsilon}_\theta(\mathbf{x}_t, t) \right) + \sigma_t \mathbf{z}$
5: **end for**
6: **return** $\mathbf{x}_0$

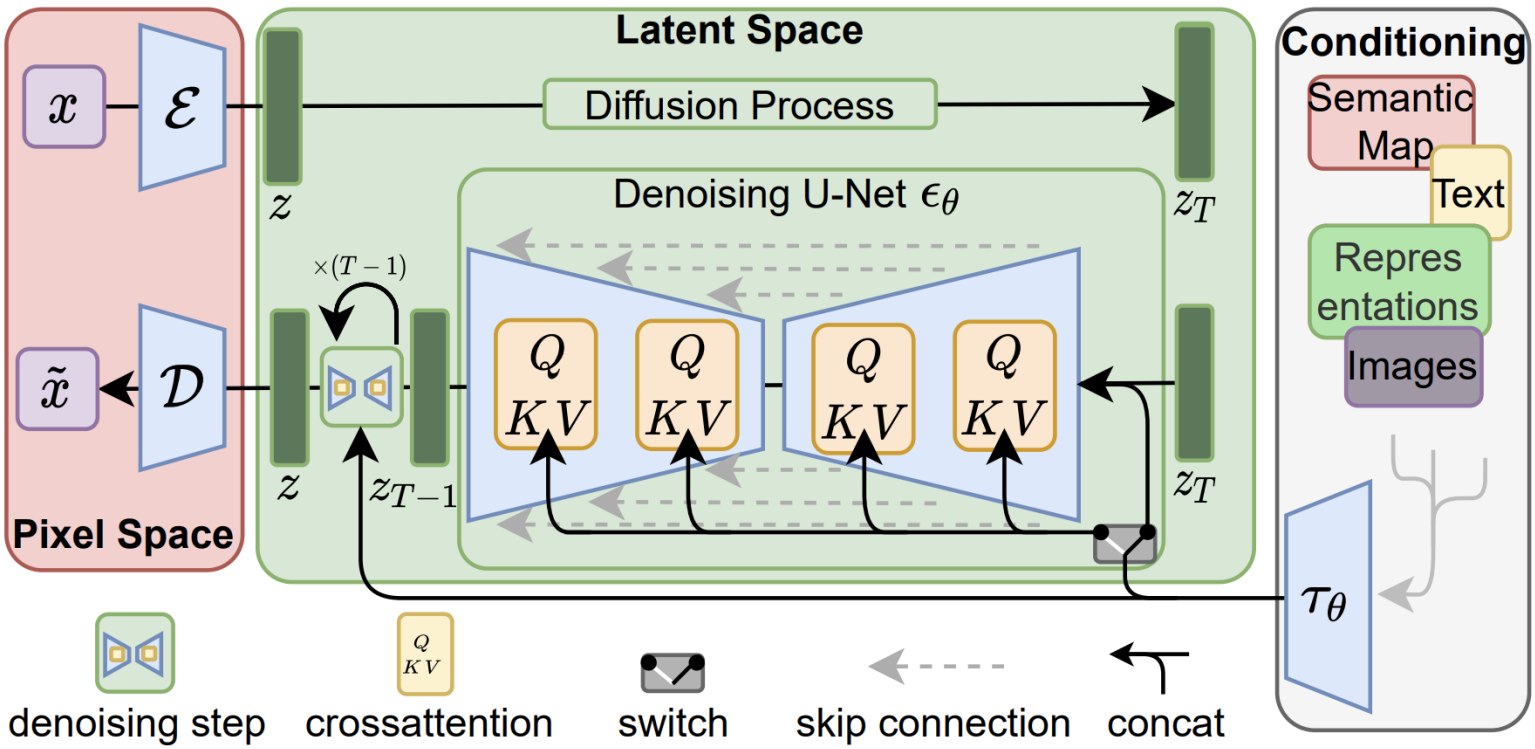
Results

- Sharp and coherent



Conditioning & Scaling

- Stable Diffusion [25]: separately train 3 networks
  - Text/condition embedding: only needs text data
  - Diffusion: needs paired (text-image) data but works at ***latent*** (low dimensional) space
  - Image encoder/decoder: only needs image data
- Text embedding as extra input for denoising net
- Other models like DALL-E [23] also use similar strategies



Results

Text-to-Image Synthesis on LAION. 1.45B Model.

'A street sign that reads "Latent Diffusion" '	*'A zombie in the style of Picasso'*	*'An image of an animal half mouse half octopus'*	*'An illustration of a slightly conscious neural network'*	*'A painting of a squirrel eating a burger'*	*'A watercolor painting of a chair that looks like an octopus'*	*'A shirt with the inscription: "I love generative models!" '*

- Even the smallest text-image training set (LAION [28, 27]) has >400M samples!
- MS COCO [14] "only" has 328K images

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