# Modification of Gilbert's Model With 1D Ising Model

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In 2007, David Vokoun et al. derived a formula for the force of interaction between magnets. The formula is called the Gilbert's Model. According to the Gilbert's Model, the force between two ferromagnets is given by a constant factor proportional to the saturation magnetization of each magnet multiplied by a function of the separation distance and geometry of the magnets. However, it is known that the magnetization will not stay constant due to the disruption of the magnetic domain. Hence, We were able to show that the assumed constant is better described as a function of hyperbolic tangent of the separation distance due to the effects of magnetic field interactions that affects the magnet's domain alignment and the magnetizations of each magnet, and we demonstrate that the inclusion of a simple toy 1D Ising model acting as a perturbation on the background magnetizations better predicts the magnetic coupling of cylindrical magnets over small distances.

Keywords: Magnets, Magnetic force

## I. INTRODUCTION

In this experiment, we will analyze magnets and its magnetic force. We decided to focus on ferromagnetic interactions. Ferromagnetism is a mechanism involving permanent magnets and also certain metals. More formally, ferromagnetism is the physical phenomenon in which electrically uncharged materials strongly attract others[1]. One example of naturally occurring ferromagnetic materials is iron.

Magnets can create a magnetic field. The phenomenon associated with magnetic field is called magnetism. Magnetism is associated with movement of charges, either in the form of electron or electric current. These electric charges will create a force, resulting from the magnetic fields interacting with each other when they are exposed towards each other. The domains of the magnets defines how strong the magnets are at attraction and repulsion on other magnetic objects. Not only that, the domains of magnets also determine its polarity. From the behavior expressed by the magnets, we can figure out which way the field is pointing. Magnetic field with domains opposing each other will repel, on the other hand, magnetic field with domains in the same direction will attract each other[3].

These magnetic field interaction will exert a force. This force is very hard to calculate, however, in 2007, David Vokoun et al. derived a formula to measure the force between magnets. The formula is called the Gilbert's Model, and is given by:

$$F(x) \simeq \frac{\pi\mu_0}{4} M^2 R^4 \left[\frac{1}{x^2} + \frac{1}{(x+2L)^2} - \frac{2}{(x+L)^2}\right] \quad (1)$$

Where $\mu_0$ is the permeability of space, given by $4\pi \times 10^-7$ Tm/A, M is the magnetization of the magnets, $x$ is the separation distance of the magnets in meters, L is the length of the magnet, and R is the radius of the magnets [2].

In the equation given by the Gilbert's Model, we can see that the equation is divided into two parts, a part we decided to call g(x), which is dependent on the separation distance, shown at Equation2. There is also another part that is supposed to be constant assuming if we used the same magnets. As shown in Equation3, this part does not depend on the separation distance between the magnets, but only depend on the magnetization and the radius of the magnets.

$$g(x) = \left[\frac{1}{x^2} + \frac{1}{(x+2L)^2} - \frac{2}{(x+L)^2}\right] \quad (2)$$

$$Magnetizationconstant = \frac{\pi\mu_0}{4} M^2 R^4 \quad (3)$$

The important part of this equation is that the equation has some limitations. It is applied only to the part where $x >> R$, or when the ratio $x/R >> 1$. However, it was never mentioned how small R needs to be compared to x. Since there is no set value, we decided to test the theory.

Another important part of this paper is the theoretical support of our data. We used the 1D Ising Model in order to explain the behavior of magnets. The 1D Ising model is a model explaining the spins in a domain and how external field affects the direction of the spins. As we all know, magnets have their own domain, and the domain can be expressed as a countless number of spins[5]. The spins can have two types of arrangements, spin up and spin down. These spins will interact with each other (mainly neighboring spins) as well as an external force which will affect the energy of each spin.

Neighboring spins will want to align with each other, given that spins that align with each other allows the

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system to have less energy. On the other hand, if the spins are opposed to each other, it will result in higher energy in the system. External magnetic field also affects the spins greatly. The spins will want to follow the direction of the external magnetic field to reduce energy. This will result in the magnetic spin turning, modifying the domain of the magnet. As mentioned before, the domain of the magnets defines the strength of the magnet, thus when a magnets is exposed to an external magnetic field, it will change the strength of the magnet, which reduces the magnetization. This piece of information is very important in our derivation shown in later steps.

There are two important functions of the 1D Ising model, the Hamiltonian and also the partition function. The Hamiltonian of the 1D Ising Model allows us to figure out the value of the total energy in a system. The Hamiltonian is given by Equation4, where $-J\sum_s s_i s_j$ is the spin-spin coupling and the $-h\sum_i s_i$ gives the effect of the external magnetic field[4].

$$\mathcal{H} = -J\sum_s s_i s_j - h\sum_i s_i \tag{4}$$

On the other hand, the partition function, shown by Equation5 allows us to calculate the sum over all possible configurations for the system to have a certain energy. In the partition function, c is the maximum number of configuration, and $\beta$ is given as $1/KT$, where K is Boltzmann's constant and T is the absolute temperature.

$$\mathcal{Z} = \sum_c e^{-\beta\mathcal{H}} \tag{5}$$

In this paper, we will discuss how the "magnetization constant" part actually behaves in a condition where the separation distance is close to the radius of the magnets. We will also use the theory mentioned to support our modification

## II. EXPERIMENT

### A. Background

When magnets repel each other, they will exert a certain amount of force towards each other. In this experiment, we will try to calculate the repulsion force using the mass exerted by the magnets when the magnets are aligned and repelling each other. We will use Newton's second law to convert the mass into force in order to find the relationship between the separation distance and force.

### B. Materials and Apparatus

#### 1. Materials

1. Magnets (cylindrical, diameter 11.8mm, length 11.2mm)
2. Masking tape

#### 2. Apparatus

1. Vernier caliper
2. Weighing scale
3. Stand
4. Ruler
5. Clamps
6. Spirit level

### C. Methodology

We taped a magnet on a wooden block, and another on a center of the ruler; we ensure that both magnets facing each other are like poles, therefore ensuring the force exerted will be repelling force instead of attractive force. Taping the magnet on the wooden block prevents the magnetic field of the magnet to affect the reading on the scale; which might result in fluctuation of readings. We positioned the wooden block that has the magnet on the scale, and tare the scale to zero reading. We aligned the stand to that they are straight and level, then positioned the ruler with the stand and the clamps, such that the magnet underneath of the ruler is closely aligned to the magnet on the scale. We then start on the highest position possible on the stand. After making sure the ruler is levelled using a spirit level, we measure the separation distance between magnets using the upper jaws of the electronic vernier caliper. At the same time, we also read the readings on the scale. Since the reading on the scale is the mass exerted by the repulsion force, we multiply it by the acceleration of gravity, $9.81m/s^2$, in order to get the force exerted by one magnet to the other at different separation distance. We continue getting more data points by adjusting the clamps that is holding the ruler down by a small distance, and repeat the data recording process. After the whole process is done and the weighing scale cannot read the mass exerted anymore or when the separation distance is very small, we stopped collecting the data. The data is then analyzed and plotted. By calculating g(x) using the separation distance and the length of each magnet, we can find the "magnetization constant" value of the magnet at that certain point. We also used R program to analyze our data further. We repeated the experiment with different cylindrical magnet of different radius.

## III. DATA AND ANALYSIS

### A. First experiment

To analyze our data, we use Microsoft Excel to help us with computing all the values we have. First, we convert the values we got to the standard units, then we calculated the force of the repelling force of the magnets. From the separation distance and the dimensions of the magnet, we can find the value of g(x) for every data point we have. After calculating g(x) and also the force, we can divide the force by g(x) in order to find the value of the "magnetization constant" part.

In this experiment, we are able to show that the "magnetization constant" value is not constant as the original model had predicted it to be. The data we collected explains how the separation distance of the magnets affect the magnetization of the magnets, thus affecting the force exerted by the magnets. However, the graph does show that it is approaching a constant at a y-value asymptote.

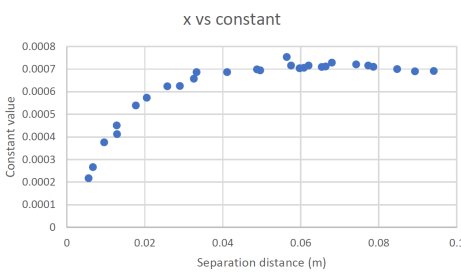


FIG. 1. Constant Values for Different Distances

As shown in figure 1. The constant value is definitely not constant. This graph seems similar to a hyperbolic tangent function. To analyze our data further, we tried a hyperbolic tangent plot using R to try and see if our data fits the function. We get a graph as shown in 2. Where the constant value is a function given by $(3.277 \times 10^-4)tanh(34.01x)$

The hyperbolic tangent is shown as the red curve, and the data is shown as the circles on the graph. Since the original model has the boundary of $x >> R$, we noticed that the limitations must be the point where the graph approaches an asymptote. In this case, the asymptote must be the condition where both magnets are at their most saturated state.

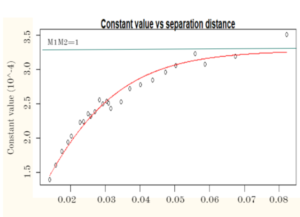


FIG. 2. Hyperbolic tangent function with our data

After getting the result, we added the hyperbolic tangent function to the original Gilbert's Model. This results in the equation shown in Equation6

$$F(x) = \frac{\pi\mu_0}{4} M^2 R^4 (3.277 \times 10^-4) tanh(34.01x) \tag{6}$$

$$\left[\frac{1}{x^2} + \frac{1}{(x+2L)^2} - \frac{2}{(x+L)^2}\right]$$

After looking at our data, we looked at the 1D Ising Model in order to try to understand and explain our findings and find theoretical support. We derived the partition function of the 1D Ising model which results in

$$< m >= -\frac{\partial}{\partial\beta} ln(\mathcal{Z}) \tag{7}$$

From Equation7, we can see that $< m >$ is the perturbation to the magnetization. The perturbation can be further solved as shown in Equation8, where N is the number of spins and h is the external magnetic field of the opposing magnet.

$$< m >= -Nhtanh(\beta h) \tag{8}$$

From the 1D Ising Model, we were able to understand the reason the magnetization has a hyperbolic tangent nature as it allows us the understand the spin in the magnets. When there is an external magnetic field acting on a magnet, the spin of the magnet tend to have the tendency to change into the direction of the external field. From that, we can show that there is a perturbation to the magnetization, which is a hyperbolic tangent function with respect to the external magnetic field. This supports our modification of the Gilbert's Model, showing that there is a function of hyperbolic tangent affecting the magnetization.

We can then further say that the perturbation affects the magnets such that:

$$M_2 = M_2saturated + < m > \tag{9}$$

We then checked and see if our modification improves the model. We reused the same values for the distance and also the force exerted, adding the hyperbolic tangent modification. We graphed the modified data, which results in 3

Comparing this to the graph before modification, this results in a more linear graph. It also has a smaller slope and a larger $R^2$ value.

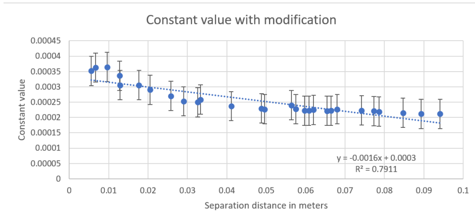


FIG. 3. Constant Values after modification

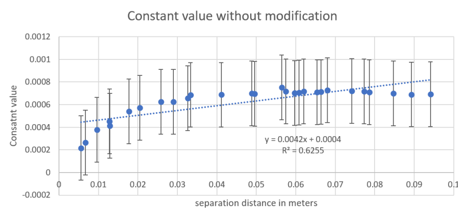


FIG. 4. Constant Value before modification

We also further solved the magnetization by using the formula, we can see the large difference in the $R^2$ value before and after the modification, even though the slope is similarly large.

## IV. CONCLUSIONS

In conclusion, we showed that the Gilbert's Model is not very accurate when the separation distance x is very similar to R, the radius of the magnet. We provided a better modification of this model supported and derived by the 1D Ising Model. The modified Gilbert's Model does seem to show a great result and reflects the initial theory where the magnetization is a constant.

## V. FUTURE WORK

We wish to conduct the experiment with a more reliable equipment and set up. There were instances where we were not able to determine the alignment of the two magnets as the repulsion force would slightly shift the magnet that is attached on the wooden block. Furthermore, the magnet that is attached underneath the ruler may not be parallel to the surface of the magnet on top of the wooden block. We also would like to get more data on the magnetization constant of different radius of

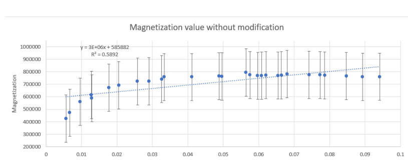


FIG. 5. Magnetization before modification

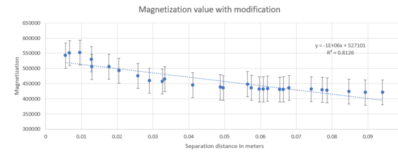


FIG. 6. Magnetization after modification

cylindrical magnet and understand the relations of a and b from the improvised Gilbert's model based on different types of cylindrical magnet. We wish to also investigate whether the height of the cylindrical magnets has any effect with the magnetization constant. Lastly, we found our last modification, but we haven't been able to find a good explanation for the linear modification, we would like to investigate it further if given the chance.

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### Appendix A: Solving for the hyperbolic tangent

We will use the partition function and the Hamiltonian function to solve for the hyperbolic tangent

$$\mathcal{H} = -J\sum_{s} s_i s_j - h\sum_{i} s_i \tag{A1}$$

EquationA1 is the Hamiltonian for the 1D Ising Model. In this Ising Model, there are two distinct parts, the spin-spin coupling, which is given by EquationA2, and the external magnetic factor, which is given by EquationA3.

$$-J\sum_{s} s_i s_j \tag{A2}$$

The spin-spin coupling considers the energy exerted between domains. In this equation, $S_i$ indicates the spin we are currently looking at, and the $S_j$ indicates the spin beside $S_i$, which we can refer to as a neighboring spin. On this experiment, however, we will assume that g(x) of the Gilbert's Model is taking care of this complicated portion of the model, this means we can assume $J = 0$.

$$-h\sum_{i} s_i \tag{A3}$$

On the other hand, the external magnetic factor is the one we will be focusing in this paper. The external magnetic factor takes account the external magnetic field exerted by the other magnet to the magnet we are looking at. The external magnetic field will make the internal magnetic domain want to align with it, thus reducing the magnetic strength of the magnet. Therefore, we can rewrite the Hamiltonian as:

$$\mathcal{H} = -h\sum_i s_i \tag{A4}$$

As mentioned in the introduction, the partition function is expressed as:

$$\mathcal{Z} = \sum_c e^{-\beta\mathcal{H}} \tag{A5}$$

After rewriting the Hamiltonian function, we can substitute the Hamiltonian to the partition function, getting the equation:

$$\mathcal{Z} = \sum_c e^{-\beta h \sum_i S_i} \tag{A6}$$

We can write the partition function as a product

$$\mathcal{Z} = \prod_{i=1}^{N} e^{\beta h} + e^{-\beta h} \tag{A7}$$

Where the positive exponent expresses spin up, and the negative exponent expresses spin down. This function looks really similar to the hyperbolic cosine function, but we need to manipulate its expression, hence it needs the function to be divided by 2. So we multiplied it by 2/2, which is equivalent to 1. Therefore we can derive the expression further, resulting in:

$$\mathcal{Z} = \prod_{i=1}^{N} 2\frac{e^{\beta h} + e^{-\beta h}}{2} \tag{A8}$$

Now we can actually rewrite that as a hyperbolic cosine function. And also, we can write the product as an exponent, therefore achieving the expression:

$$\mathcal{Z} = 2^N(cosh(\beta h))^N \tag{A9}$$

We then can take the partial derivative of the partition function with respect to $\beta$ (expressed in EquationA10. Also multiplying the partial differential of the partition function with the negative reciprocal of the partition function. This allows us to get EquationA11

$$\frac{\partial\mathcal{Z}}{\partial\beta} = -\sum_s (h\sum_i s_i)e^{-\beta h \sum_i s_i} \tag{A10}$$

$$-\frac{1}{\mathcal{Z}}\frac{\partial\mathcal{Z}}{\partial\beta} = \frac{\sum_s (h\sum_i s_i)e^{-\beta h \sum_i s_i}}{\sum_s e^{-\beta h \sum_i s_i}} \tag{A11}$$

The equation EquationA11 is can be further simplified as EquationA12. $h < s >$ can also be called the perturbation of the magnetization, which we will express as $< m >$.

$$-\frac{1}{\mathcal{Z}}\frac{\partial\mathcal{Z}}{\partial\beta} = h < s > \tag{A12}$$

Therefore we can simply write the perturbation to the magnetization as:

$$< m > = -\frac{\partial}{\partial\beta}ln\mathcal{Z} \tag{A13}$$

We can further solve the $ln\mathcal{Z}$ to

$$ln\mathcal{Z} = Nln2 + Nlncosh(\beta h) \tag{A14}$$

$$\frac{\partial}{\partial\beta}ln\mathcal{Z} = \frac{N}{cosh(\beta h)} \times h \times sinh(\beta h) \tag{A15}$$

Therefore, simplifying the fractions:

$$\frac{\partial}{\partial\beta}ln\mathcal{Z} = Nhtanh(\beta h) \tag{A16}$$

Hence we can confirm that it does result in a hyperbolic tangent function, supporting our experimental results with a theoretical background.

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