线性代数

行列式

▷▷

张晓平

武汉大学数学与统计学院

◁◁

2017 年 10 月 31 日

目录

例:

$$\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$$

例:

$$\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$$

解

$$\text{原式} = a^2b^2 - (ab)(ab) = 0.$$

例:

$$\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$$

例:

$$\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$$

解

$$\text{原式} = \cos^2\alpha + \sin^2\alpha = 1.$$

例:

$$\begin{vmatrix} a+bi & b \\ 2a & a-bi \end{vmatrix}$$

例:

$$\begin{vmatrix} a+bi & b \\ 2a & a-bi \end{vmatrix}$$

解

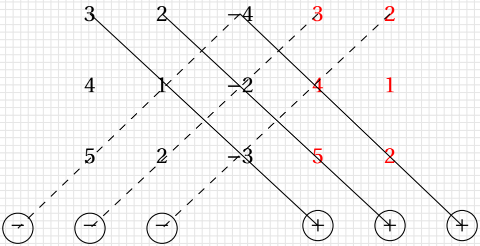
$$\text{原式} = (a+bi)(a-bi) - 2ab = a^2 + b^2 - 2ab = (a-b)^2.$$

例:

$$\begin{vmatrix} 3 & 2 & -4 \\ 4 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}$$

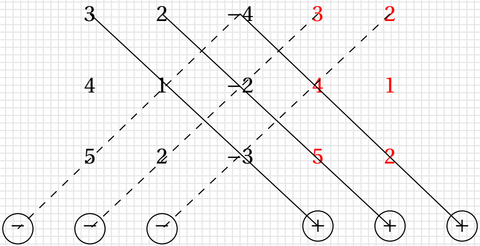
例:

$$\begin{vmatrix} 3 & 2 & -4 \\ 4 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}$$



例:

$$\begin{vmatrix} 3 & 2 & -4 \\ 4 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}$$



解

原式 $= 3\cdot 1\cdot(-3) + 2\cdot(-2)\cdot 5 + (-4)\cdot 4\cdot 2 - (-4)\cdot 1\cdot 5 - 3\cdot(-2)\cdot 2 - 2\cdot 4\cdot(-3)$

$= -9 - 20 - 32 + 20 + 12 + 24 = -5.$

例:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

例:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

解

$$\text{原式} \xlongequal[r_2-r_1]{r_3-r_2} \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} = 0.$$

例:

$$\begin{vmatrix} 2 & 2 & 1 \\ 4 & 1 & -1 \\ 202 & 199 & 101 \end{vmatrix}$$

例:

$$\begin{vmatrix} 2 & 2 & 1 \\ 4 & 1 & -1 \\ 202 & 199 & 101 \end{vmatrix}$$

解

$$\text{原式} \xlongequal{c_1-c_2} \begin{vmatrix} 0 & 2 & 1 \\ 3 & 1 & -1 \\ 3 & 199 & 101 \end{vmatrix} \xlongequal{r_3-r_2} \begin{vmatrix} 0 & 2 & 1 \\ 3 & 1 & -1 \\ 0 & 198 & 102 \end{vmatrix}$$

$$= (-1)^{2+1} \cdot 3 \cdot \begin{vmatrix} 2 & 1 \\ 198 & 102 \end{vmatrix} = -3 \cdot (2 \cdot 102 - 198) = -18.$$

例:

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{vmatrix}, \quad \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

例:

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{vmatrix}, \quad \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

解 注意到 $\omega^3 = 1$，故

$$\omega \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^3 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{vmatrix} = 0,$$

从而

$$\text{原式} = 0.$$

例:

$$\begin{vmatrix} 1 & x & x \\ x & 2 & x \\ x & x & 3 \end{vmatrix}.$$

例:

$$\begin{vmatrix} 1 & x & x \\ x & 2 & x \\ x & x & 3 \end{vmatrix}.$$

解

$$\text{原式} \xlongequal[r_3 - x r_1]{r_2 - x r_1} \begin{vmatrix} 1 & x & x \\ 0 & 2-x^2 & x-x^2 \\ 0 & x-x^2 & 3-x^2 \end{vmatrix} = \begin{vmatrix} 2-x^2 & x-x^2 \\ x-x^2 & 3-x^2 \end{vmatrix}$$

$$= (2-x^2)(3-x^2) - (x-x^2)^2 = 2x^3 - 6x^2 + 6.$$

例:

$$\begin{vmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 3 \\ 0 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix}$$

例:

$$\begin{vmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 3 \\ 0 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix}$$

解

原式 $= (-1)^{1+4} \cdot 4 \cdot \begin{vmatrix} 0 & 0 & 4 \\ 0 & 4 & 3 \\ 4 & 3 & 2 \end{vmatrix}$

$= (-1)^{1+4} \cdot 4 \cdot (-1)^{1+3} \cdot 4 \cdot \begin{vmatrix} 0 & 4 \\ 4 & 3 \end{vmatrix}$

$= -4 \cdot 4 \cdot (-16) = 256.$

例:

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 2 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 8 & \cdots & 0 & 0 & 0 \\ 9 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 10 \end{vmatrix}$$

例:

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 2 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 8 & \cdots & 0 & 0 & 0 \\ 9 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 10 \end{vmatrix}$$

解

原式 $= (-1)^{10+10} \cdot 10 \cdot \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 2 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 8 & \cdots & 0 & 0 \\ 9 & 0 & \cdots & 0 & 0 \end{vmatrix} = 10 \cdot (-1)^{\frac{9\times 8}{2}} 9! = 10!$

例:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$$

例:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$$

解

$$原式 \xlongequal[i=2,3,4]{r_i-r_1} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = (-2)^3 = -8.$$

例:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$

例:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$

解

$$\text{原式} \xlongequal[i=4,3,2]{r_i - r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & -3 \\ 1 & 1 & -3 & 1 \\ 1 & -3 & 1 & 1 \end{vmatrix} \xlongequal[i=2,3,4]{c_i - c_1} \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 0 & -4 \\ 1 & 0 & -4 & 0 \\ 1 & -4 & 0 & 0 \end{vmatrix}$$

$$\xlongequal[i=2,3,4]{c_i \div 4} 4^3 \begin{vmatrix} 1 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix} \xlongequal{c_1 + c_2 + c_3 + c_4} 4^3 \begin{vmatrix} 1 + \frac{1+2+3}{4} & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}$$

$$= 4^3 \frac{10}{4} \begin{vmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix} = 160.$$

例:

$$\begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

例:

$$\begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

解

$$\text{原式} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 5 & 0 & 4 & 1 \\ 2 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2} \cdot (-1) \cdot \begin{vmatrix} 5 & 4 & 2 \\ 5 & 4 & 1 \\ 2 & 3 & 2 \end{vmatrix}$$

$$\xlongequal{r_1-r_2} -\begin{vmatrix} 0 & 0 & 1 \\ 5 & 4 & 1 \\ 2 & 3 & 2 \end{vmatrix} = -(-1)^{1+3} \cdot 1 \cdot \begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = -7.$$

例:

$$\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 2 & 5 & 4 & 5 & 3 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

例:

$$\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 2 & 5 & 4 & 5 & 3 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

解

$$\text{原式} \xlongequal{r_2 \leftrightarrow r_3} -\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 5 & 3 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix} \xlongequal[r_1 - r_3]{\substack{r_2 - r_1 \\ r_4 - r_3}} -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 2 & 5 & 4 & 5 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

$$\xlongequal[r_5 - r_1]{r_3 - 2r_1} -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 & -2 \end{vmatrix} \xlongequal[r_5 + 2r_4]{r_2 \leftrightarrow r_3} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= -12.$$

例:

$$\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 5 & 1 \end{vmatrix}$$

例:

$$\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 5 & 1 \end{vmatrix}$$

解

$$\text{原式} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} -1 & 3 \\ 5 & 1 \end{vmatrix} = (-2) \cdot (-16) = 32.$$

例:

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 1 & 1 \end{vmatrix}$$

例:

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 1 & 1 \end{vmatrix}$$

原式 $\xlongequal{r_3 \leftrightarrow r_5} -\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 0 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$

$$= -(-1)^{3+2} \cdot 1 \cdot \begin{vmatrix} 1 & 3 \\ 6 & 8 \end{vmatrix} \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= (-10) \cdot 2 = -20.$$

例:

$$\begin{vmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 4 & 0 \\ 1 & 2 & 4 & 0 & -1 \\ 3 & 1 & 2 & 5 & 8 \end{vmatrix}$$

例:

$$\begin{vmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 4 & 0 \\ 1 & 2 & 4 & 0 & -1 \\ 3 & 1 & 2 & 5 & 8 \end{vmatrix}$$

解

$$\text{原式} \xlongequal[c_2 \leftrightarrow c_1]{c_3 \leftrightarrow c_2} \begin{vmatrix} 1 & 0 & 0 & -1 & 2 \\ 3 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 4 & 0 \\ 4 & 1 & 2 & 0 & -1 \\ 2 & 3 & 1 & 5 & 8 \end{vmatrix} \xlongequal[c_3 \leftrightarrow c_2]{c_4 \leftrightarrow c_3} \begin{vmatrix} 1 & -1 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 2 \\ 2 & 4 & 0 & 0 & 0 \\ 4 & 0 & 1 & 2 & -1 \\ 2 & 5 & 3 & 1 & 8 \end{vmatrix}$$

$$\xlongequal[c_4 \leftrightarrow c_3]{c_5 \leftrightarrow c_4} \begin{vmatrix} 1 & -1 & 2 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 4 & 0 & -1 & 1 & 2 \\ 2 & 5 & 8 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 2 & 4 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$\xlongequal{r_2 - r_1} \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 2 & 4 & 0 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2 \cdot 6 \cdot (-5) = -60.$$

例:

$$\begin{vmatrix} * & \boldsymbol{A} \\ \boldsymbol{B} & \boldsymbol{0} \end{vmatrix}, \quad \boldsymbol{A} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix}, \quad \boldsymbol{B} = \begin{vmatrix} & & & & -1 \\ & & & -2 & \\ & & -3 & & \\ & -4 & & & \\ -5 & & & & \end{vmatrix}$$

例:

$$\begin{vmatrix} * & \boldsymbol{A} \\ \boldsymbol{B} & \boldsymbol{0} \end{vmatrix}, \quad \boldsymbol{A}=\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix}, \quad \boldsymbol{B}=\begin{vmatrix} & & & & -1 \\ & & & -2 & \\ & & -3 & & \\ & -4 & & & \\ -5 & & & & \end{vmatrix}$$

解

$$\text{原式} = (-1)^{3\times 5}\begin{vmatrix} \boldsymbol{A} & * \\ \boldsymbol{0} & \boldsymbol{B} \end{vmatrix}$$

$$= (-1)\cdot|\boldsymbol{A}|\cdot|\boldsymbol{B}|$$

$$= (-1)\cdot 1\cdot 2\cdot 3\cdot(-1)^{\frac{5\times 4}{2}}(-1)(-2)(-3)(-4)(-5)$$

$$= 6\times 120 = 720$$

例:

证明:

$$\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

例:

证明:

$$\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

证明.

$$\text{左边} = \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1x & a_1x+b_1 & c_1 \\ b_2x & a_2x+b_2 & c_2 \\ b_3x & a_3x+b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x+b_1 & c_1 \\ b_2 & a_2x+b_2 & c_2 \\ b_3 & a_3x+b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_1x & c_1 \\ a_2 & a_2x & c_2 \\ a_3 & a_3x & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{右边}$$

□

例:

证明:

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = x^2 y^2.$$

例:

证明:

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = x^2y^2.$$

证明.

$$\text{左边} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1+x & 1 & 1 & 1 \\ 0 & 1 & 1-x & 1 & 1 \\ 0 & 1 & 1 & 1+y & 1 \\ 0 & 1 & 1 & 1 & 1-y \end{vmatrix}$$

$$\overline{\overline{\underset{i=2,3,4,5}{r_i-r_1}}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & x & 0 & 0 & 0 \\ -1 & 0 & -x & 0 & 0 \\ -1 & 0 & 0 & y & 0 \\ -1 & 0 & 0 & 0 & -y \end{vmatrix} \overline{\overline{\underset{\substack{c_1+c_4/y \\ c_1-c_5/y}}{\overset{\substack{c_1+c_2/x \\ c_1-c_3/x}}{}}}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & -y \end{vmatrix}$$

$$= x^2y^2.$$

□

例:

证明:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(a-c)(b-c)(a+b+c).$$

例:

证明:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(a-c)(b-c)(a+b+c).$$

证明. 考察范德蒙德行列式

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & y \\ a^2 & b^2 & c^2 & y^2 \\ a^3 & b^3 & c^3 & y^3 \end{vmatrix} = (y-a)(y-b)(y-c)(c-a)(c-b)(b-a)$$

等式两端均为关于 y 的多项式，比较 y^2 的系数，可知

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(a-c)(b-c)(a+b+c)$$

□

例:

证明:

$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (ab+bc+ca)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

例:

证明:

$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (ab+bc+ca)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

证明. 考察范德蒙德行列式

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & y & y^2 & y^3 \end{vmatrix} = (y-a)(y-b)(y-c)(c-a)(c-b)(b-a)$$

等式两端均为关于 y 的多项式，比较 y 的系数，可知

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(a-c)(b-c)(ab+bc+ca) = (ab+bc+ca)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

□

例:

计算

$$\begin{vmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{vmatrix}$$

例:

计算

$$\begin{vmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{vmatrix}$$

解

$$\text{左边} \xlongequal{\text{按第 4 行展开}} (-1)^{4+1} \cdot d \cdot \begin{vmatrix} 0 & 2 & a \\ 0 & b & 0 \\ c & 4 & 5 \end{vmatrix}$$

$$\xlongequal{\text{按第 2 行展开}} (-d) \cdot (-1)^{2+2} \cdot b \begin{vmatrix} 0 & a \\ c & 5 \end{vmatrix} = abcd.$$

例:

计算

$$\begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix}$$

例:

计算

$$\begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix}$$

解

$$左边 \xlongequal{按第 1 行展开} (-1)^{1+1}\cdot a\cdot\begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + (-1)^{1+2}\cdot 1\cdot\begin{vmatrix} -1 & 1 & 0 \\ 0 & c & 1 \\ 0 & -1 & d \end{vmatrix}$$

$$= a\cdot\begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + \begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix}$$

$$\xlongequal{按第 1 行展开} a\cdot\left(b\cdot\begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix} + (-1)^{1+2}\cdot 1\cdot\begin{vmatrix} -1 & 1 \\ 0 & d \end{vmatrix}\right) + (cd+1)$$

$$= a(b(cd+1)+d) + (cd+1) = (ab+1)(cd+1) + ad$$

例:

计算

$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$$

例:

计算

$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$$

解

$$\text{左边} \xlongequal[c_2-c_1]{\substack{c_4-c_3 \\ c_3-c_2}} \begin{vmatrix} a^2 & 2a+1 & 2a+3 & 2a+5 \\ b^2 & 2b+1 & 2b+3 & 2b+5 \\ c^2 & 2c+1 & 2c+3 & 2c+5 \\ d^2 & 2d+1 & 2d+3 & 2d+5 \end{vmatrix}$$

$$\xlongequal[c_3-c_2]{c_4-c_3} \begin{vmatrix} a^2 & 2a+1 & 2 & 2 \\ b^2 & 2b+1 & 2 & 2 \\ c^2 & 2c+1 & 2 & 2 \\ d^2 & 2d+1 & 2 & 2 \end{vmatrix} = 0.$$

例:

计算

$$\begin{vmatrix} a & b & c & 1 \\ b & c & a & 1 \\ c & a & b & 1 \\ \frac{b+c}{2} & \frac{c+a}{2} & \frac{a+b}{2} & 1 \end{vmatrix}$$

例:

计算

$$\begin{vmatrix} a & b & c & 1 \\ b & c & a & 1 \\ c & a & b & 1 \\ \frac{b+c}{2} & \frac{c+a}{2} & \frac{a+b}{2} & 1 \end{vmatrix}$$

解

$$原式 \xlongequal{r_3+r_1+r_2} \begin{vmatrix} a & b & a+b+c & 1 \\ b & c & a+b+c & 1 \\ c & a & a+b+c & 1 \\ \frac{b+c}{2} & \frac{c+a}{2} & a+b+c & 1 \end{vmatrix} = 0.$$

例:

计算

$$\begin{vmatrix} a_1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & b_3 & a_3 & 0 \\ b_4 & 0 & 0 & a_4 \end{vmatrix}$$

例:

计算

$$\begin{vmatrix} a_1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & b_3 & a_3 & 0 \\ b_4 & 0 & 0 & a_4 \end{vmatrix}$$

解

$$\text{原式} \xlongequal[c_3\leftrightarrow c_2]{c_4\leftrightarrow c_3} \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & b_3 & a_3 \\ b_4 & a_4 & 0 & 0 \end{vmatrix}$$

$$\xlongequal[r_3\leftrightarrow r_2]{r_4\leftrightarrow r_3} \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ b_4 & a_4 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & b_3 & a_3 \end{vmatrix} = (a_1 a_4 - b_1 b_4)(a_2 a_3 - b_2 b_3).$$

例:

计算

$$\begin{vmatrix} 1 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 2 & 3 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \cdots & n-1 & 2 \\ 2 & 2 & 2 & \cdots & 2 & n \end{vmatrix}$$

解

$$\text{原式}=\begin{vmatrix}1&2&2&2&\cdots&2&2\\0&1&2&2&\cdots&2&2\\0&2&2&2&\cdots&2&2\\0&2&2&3&\cdots&2&2\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&2&2&2&\cdots&n-1&2\\0&2&2&2&\cdots&2&n\end{vmatrix}=\begin{vmatrix}1&2&2&2&\cdots&2&2\\-1&-1&0&0&\cdots&0&0\\-1&0&0&0&\cdots&0&0\\-1&0&0&1&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\-1&0&0&0&\cdots&n-3&0\\-1&0&0&0&\cdots&0&n-2\end{vmatrix}$$

$$=(-1)^{3+1}(-1)\begin{vmatrix}2&2&2&\cdots&2&2\\-1&0&0&\cdots&0&0\\0&0&1&\cdots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\cdots&n-3&0\\0&0&0&\cdots&0&n-2\end{vmatrix}$$

$$=-2(n-2)!$$

例:

计算

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a & a-1 & a-2 & \cdots & a-n \\ a^2 & (a-1)^2 & (a-2)^2 & \cdots & (a-n)^2 \\ \vdots & \vdots & \vdots & & \vdots \\ a^n & (a-1)^n & (a-2)^n & \cdots & (a-n)^n \end{vmatrix}$$

例:

计算

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a & a-1 & a-2 & \cdots & a-n \\ a^2 & (a-1)^2 & (a-2)^2 & \cdots & (a-n)^2 \\ \vdots & \vdots & \vdots & & \vdots \\ a^n & (a-1)^n & (a-2)^n & \cdots & (a-n)^n \end{vmatrix}$$

解　该行列式为范德蒙行列式，故

$$\begin{aligned} \text{原式} \quad &= \prod_{n\ge i>j\ge 0}[(a-i)-(a-j)] \\ &= \prod_{n\ge i>j\ge 0}(j-i) \\ &= (-1)^{\frac{n(n+1)}{2}} \prod_{n\ge i>j\ge 0}(i-j) \\ &= (-1)^{\frac{n(n+1)}{2}} \prod_{i=1}^{n} i! \end{aligned}$$

例:

计算

$$\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \cdots & a_1b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \cdots & a_2b_2^{n-1} & b_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \cdots & a_{n+1}b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix}$$

例:

计算
$$\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \cdots & a_1 b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \cdots & a_2 b_2^{n-1} & b_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \cdots & a_{n+1} b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix}$$

解

原式
$$= a_1^n \cdots a_{n+1}^n \begin{vmatrix} 1 & a_1^{-1}b_1 & (a_1^{-1}b_1)^2 & \cdots & (a_1^{-1}b_1)^{n-1} & (a_1^{-1}b_1)^n \\ 1 & a_2^{-1}b_2 & (a_2^{-1}b_2)^2 & \cdots & (a_2^{-1}b_2)^{n-1} & (a_2^{-1}b_2)^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & a_{n+1}^{-1}b_{n+1} & (a_{n+1}^{-1}b_{n+1})^2 & \cdots & (a_{n+1}^{-1}b_{n+1})^{n-1} & (a_{n+1}^{-1}b_{n+1})^n \end{vmatrix}$$

$$= a_1^n a_2^n \cdots a_{n+1}^n \prod_{n+1 \ge i > j \ge 1} \left(\frac{b_i}{a_i} - \frac{b_j}{a_j} \right)$$

$$= \prod_{n+1 \ge i > j \ge 1} (b_i a_j - a_i b_j)$$

例:

用克拉默法则求

$$\begin{cases} 5x_1 & & + 4x_3 + 2x_4 = 3, \\ x_1 - x_2 + 2x_3 + x_4 = 1, \\ 4x_1 + x_2 + 2x_3 & = 1, \\ x_1 + x_2 + x_3 + x_4 = 0. \end{cases}$$

例:

用克拉默法则求

$$\begin{cases} 5x_1 + 4x_3 + 2x_4 = 3, \\ x_1 - x_2 + 2x_3 + x_4 = 1, \\ 4x_1 + x_2 + 2x_3 = 1, \\ x_1 + x_2 + x_3 + x_4 = 0. \end{cases}$$

解

$$D = \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -7,$$

$$D_1 = \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 0 & 4 & 1 \\ 1 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 3 & 4 & 2 \\ 2 & 4 & 1 \\ 1 & 3 & 2 \end{vmatrix}$$

$$\xlongequal[r_3-2r_2]{r_1-2r_2} - \begin{vmatrix} -1 & -4 & 0 \\ 2 & 4 & 1 \\ -3 & -5 & 0 \end{vmatrix} = -(-1)^{2+3} \cdot \begin{vmatrix} -1 & -4 \\ -3 & -5 \end{vmatrix} = -7,$$

$$D_2 = \begin{vmatrix} 5 & 3 & 4 & 2 \\ 1 & 1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix} \xlongequal[c_4-c_1]{c_3-c_1} \begin{vmatrix} 5 & 3 & -1 & -3 \\ 1 & 1 & 1 & 0 \\ 4 & 1 & -2 & -4 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (-1)^{4+1} \cdot \begin{vmatrix} 3 & -1 & -3 \\ 1 & 1 & 0 \\ 1 & -2 & -4 \end{vmatrix}$$

$$\xlongequal{c_1-c_2} - \begin{vmatrix} 4 & -1 & -3 \\ 0 & 1 & 0 \\ 3 & -2 & -4 \end{vmatrix} = -(-1)^{2+2} \cdot \begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix} = 7,$$

$$D_3 = \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 5 & 0 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 5 & 3 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\xlongequal[c_3-2c_2]{c_1-2c_2} - \begin{vmatrix} -1 & 3 & -4 \\ 1 & 2 & -3 \\ 0 & 1 & 0 \end{vmatrix} = -(-1)^{3+2} \cdot \begin{vmatrix} -1 & -4 \\ 1 & -3 \end{vmatrix} = 7$$

$$D_4 = \begin{vmatrix} 5 & 0 & 4 & 3 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 4 & 3 \\ 1 & -1 & 2 & 1 \\ 5 & 0 & 4 & 2 \\ 2 & 0 & 3 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 5 & 4 & 3 \\ 5 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} \xlongequal{r_1-r_2} -\begin{vmatrix} 0 & 0 & 1 \\ 5 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} = -\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = -7.$$

由克拉默法则可知,

$$x_1 = \frac{D_1}{D} = 1, \qquad x_2 = \frac{D_2}{D} = -1,$$

$$x_3 = \frac{D_3}{D} = -1, \qquad x_4 = \frac{D_4}{D} = 1.$$

例:

用克拉默法则求

$$\begin{cases} \quad\ \ x_2 + x_3 + x_4 + x_5 = 1, \\ x_1 \qquad + x_3 + x_4 + x_5 = 2, \\ x_1 + x_2 \qquad + x_4 + x_5 = 3, \\ x_1 + x_2 + x_3 \qquad + x_5 = 4, \\ x_1 + x_2 + x_3 + x_4 \qquad = 5. \end{cases}$$

解

$$D = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_1 \div 4]{r_1+r_2+\cdots+r_5} 4 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix},$$

$$\xlongequal[i=2,\cdots,4]{r_i - r_1} 4 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} = 4.$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5-r_1]{\substack{r_3-r_1\\ r_4-r_1}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+2} \cdot \begin{vmatrix} 2 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 11 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} = 11,$$

$$D_2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 1 & 4 & 1 & 0 & 1 \\ 1 & 5 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5-r_2]{\substack{r_3-r_2\\ r_4-r_2}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 7 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} = 7.$$

$$D_3 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 0 & 1 \\ 1 & 1 & 5 & 1 & 0 \end{vmatrix} \xlongequal[r_5-r_3]{\substack{r_2-r_3\\ r_4-r_3}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+3} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} \begin{vmatrix} 0 & 3 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} = 3,$$

$$D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 & 0 \end{vmatrix} \xlongequal[r_5-r_4]{\substack{r_2-r_4\\ r_3-r_4}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1.$$

$$D_5=\begin{vmatrix}0&1&1&1&1\\1&0&1&1&2\\1&1&0&1&3\\1&1&1&0&4\\1&1&1&1&5\end{vmatrix}\xlongequal[r_4-r_5]{\substack{r_2-r_5\\r_3-r_5}}\begin{vmatrix}0&1&1&1&1\\0&-1&0&0&-3\\0&0&-1&0&-2\\0&0&0&-1&-1\\1&1&1&1&5\end{vmatrix}$$

$$=(-1)^{5+1}\cdot\begin{vmatrix}1&1&1&1\\-1&0&0&-3\\0&-1&0&-2\\0&0&-1&-1\end{vmatrix}\xlongequal{r_1+r_2+r_3+r_4}\begin{vmatrix}0&0&0&-5\\-1&0&0&-3\\0&-1&0&-2\\0&0&-1&-1\end{vmatrix}=-5.$$

由克拉默法则可知，

$$x_1=\frac{D_1}{D}=\frac{11}{4},\qquad x_2=\frac{D_2}{D}=\frac{7}{4},\qquad x_3=\frac{D_3}{D}=\frac{3}{4},$$

$$x_4=\frac{D_4}{D}=-\frac{1}{4},\qquad x_5=\frac{D_5}{D}=-\frac{5}{4}.$$

例:

齐次线性方程组

$$\begin{cases} x_1 + x_2 + x_3 + ax_4 = 0, \\ x_1 + 2x_2 + x_3 + x_4 = 0, \\ x_1 + x_2 - 3x_3 + x_4 = 0, \\ x_1 + x_2 + ax_3 + bx_4 = 0. \end{cases}$$

有非零解时，a, b 必须满足什么条件？

例:

齐次线性方程组

$$\begin{cases} x_1 + x_2 + x_3 + ax_4 = 0, \\ x_1 + 2x_2 + x_3 + x_4 = 0, \\ x_1 + x_2 - 3x_3 + x_4 = 0, \\ x_1 + x_2 + ax_3 + bx_4 = 0. \end{cases}$$

有非零解时，a, b 必须满足什么条件？

注 1　齐次线性方程组有非零解的充分必要条件是系数行列式为零。

例:

齐次线性方程组

$$\begin{cases} x_1 + x_2 + x_3 + ax_4 = 0, \\ x_1 + 2x_2 + x_3 + x_4 = 0, \\ x_1 + x_2 - 3x_3 + x_4 = 0, \\ x_1 + x_2 + ax_3 + bx_4 = 0. \end{cases}$$

有非零解时，a, b 必须满足什么条件？

注 1　齐次线性方程组有非零解的充分必要条件是系数行列式为零。

解

$$D = \begin{vmatrix} 1 & 1 & 1 & a \\ 1 & 2 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & a & b \end{vmatrix} \overset{r_1 \leftrightarrow r_3}{=\!=\!=} \begin{vmatrix} 1 & 1 & -3 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & a \\ 1 & 1 & a & b \end{vmatrix} \overset{\substack{r_2 - r_1 \\ r_3 - r_1}}{\underset{r_4 - r_1}{=\!=\!=}} \begin{vmatrix} 1 & 1 & -3 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 4 & a-1 \\ 0 & 0 & a+3 & b-1 \end{vmatrix} = 0,$$

即 $4(b-1)-(a-1)(a+3)=0$，也就是 $(a-1)^2 = 4b$.

例:

求平面上过两点 (x_1, y_1) 和 (x_2, y_2) 的直线方程（用行列式表示）。

例:
求平面上过两点 (x_1, y_1) 和 (x_2, y_2) 的直线方程（用行列式表示)。

解 直线方程的两点式为

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1},$$

即

$$(y-y_1)(x_2-x_1) = (x-x_1)(y_2-y_1)$$

亦即

$$x(y_1-y_2) + y(x_2-x_1) + x_1y_2 - x_2y_1 = 0.$$

由行列式的按行展开可知，其行列式形式为

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

例:

求三次多项式 $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$，使得

$$f(-1) = 0, \quad f(1) = 4, \quad f(2) = 3, \quad f(3) = 16.$$

例:

求三次多项式 $f(x)=a_0+a_1x+a_2x^2+a_3x^3$，使得

$$f(-1)=0,\quad f(1)=4,\quad f(2)=3,\quad f(3)=16.$$

解 由条件可知，$f(x)$ 应满足线性方程组

$$\begin{cases} a_0 + (-1)a_1 + (-1)^2a_2 + (-1)^3a_3 = 0, \\ a_0 + a_1 + a_2 + a_3 = 4, \\ a_0 + 2a_1 + (2)^2a_2 + (2)^3a_3 = 3, \\ a_0 + 3a_1 + (3)^2a_2 + (3)^3a_3 = 16. \end{cases}$$

其系数行列式 D 为范德蒙行列式

$$D=\begin{vmatrix} 1 & -1 & (-1)^2 & (-1)^3 \\ 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \end{vmatrix}=(3+1)(3-1)(3-2)(2+1)(2-1)(1+1)=48.$$

$$D_1 = \begin{vmatrix} 0 & -1 & 1 & -1 \\ 4 & 1 & 1 & 1 \\ 3 & 2 & 4 & 8 \\ 16 & 3 & 9 & 27 \end{vmatrix} \xlongequal[c_4+c_3]{c_2+c_3} \begin{vmatrix} 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & 2 \\ 3 & 6 & 4 & 12 \\ 16 & 12 & 9 & 36 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 2 \\ 3 & 6 & 12 \\ 16 & 12 & 36 \end{vmatrix}$$

$$\xlongequal[c_2-c_3]{c_1-2c_3} \begin{vmatrix} 0 & 0 & 2 \\ -9 & -6 & 12 \\ -8 & -24 & 36 \end{vmatrix} = 48 \times 7,$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 & -1 \\ 1 & 4 & 1 & 1 \\ 1 & 3 & 4 & 8 \\ 1 & 16 & 9 & 27 \end{vmatrix} \xlongequal[c_3+c_4]{c_1+c_4} \begin{vmatrix} 0 & 0 & 0 & -1 \\ 2 & 4 & 2 & 1 \\ 9 & 3 & 12 & 8 \\ 28 & 16 & 36 & 27 \end{vmatrix} = -\begin{vmatrix} 2 & 4 & 2 \\ 9 & 3 & 12 \\ 28 & 16 & 36 \end{vmatrix}$$

$$\xlongequal[c_2-c_1]{c_2-2c_1} \begin{vmatrix} 2 & 0 & 0 \\ 9 & -15 & 3 \\ 28 & -40 & 8 \end{vmatrix} = 0.$$

$$D_3 = \begin{vmatrix} 1 & -1 & 0 & -1 \\ 1 & 1 & 4 & 1 \\ 1 & 2 & 3 & 8 \\ 1 & 3 & 16 & 27 \end{vmatrix} \xlongequal[c_4+c_1]{c_2+c_1} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 2 \\ 1 & 3 & 3 & 9 \\ 1 & 4 & 16 & 28 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 2 \\ 3 & 3 & 9 \\ 4 & 16 & 28 \end{vmatrix}$$

$$\xlongequal[c_2-c_1]{c_2-2c_1} \begin{vmatrix} 2 & 0 & 0 \\ 3 & -3 & 6 \\ 4 & 8 & 24 \end{vmatrix} = 48 \times (-5),$$

$$D_4 = \begin{vmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 16 \end{vmatrix} \xlongequal[c_3-c_1]{c_2+c_1} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 4 \\ 1 & 3 & 3 & 3 \\ 1 & 4 & 8 & 16 \end{vmatrix} = -\begin{vmatrix} 2 & 0 & 4 \\ 3 & 3 & 3 \\ 4 & 8 & 16 \end{vmatrix}$$

$$\xlongequal[c_3-c_1]{c_2-2c_1} \begin{vmatrix} 2 & 0 & 0 \\ 3 & 3 & -3 \\ 4 & 8 & 4 \end{vmatrix} = 48 \times 2.$$

由克拉默法则可知

$$a_0 = \frac{D_1}{D} = 7, \quad a_1 = \frac{D_2}{D} = 0, \quad a_2 = \frac{D_3}{D} = -5, \quad a_3 = \frac{D_4}{D} = 2.$$

例:

证明恒等式

$$\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} = \left(1+\sum_{i=1}^{n}\frac{1}{a_i}\right)\prod_{i=1}^{n}a_i$$

例:

证明恒等式

$$\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} = \left(1+\sum_{i=1}^{n}\frac{1}{a_i}\right)\prod_{i=1}^{n}a_i$$

证明 (1):

$$\text{左边} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} + \begin{vmatrix} a_1 & 1 & \cdots & 1 \\ 0 & 1+a_2 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 1+a_n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a_n \end{vmatrix} + \begin{vmatrix} a_1 & 1 & \cdots & 1 \\ 0 & 1+a_2 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 1+a_n \end{vmatrix}$$

$$
\begin{aligned}
&= a_2\cdots a_n + a_1\begin{vmatrix} 1+a_2 & 1 & \cdots & 1 \\ 1 & 1+a_3 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} \\
&= a_2\cdots a_n + a_1\left(\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1+a_3 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} + \begin{vmatrix} a_2 & 1 & \cdots & 1 \\ 0 & 1+a_3 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 1+a_n \end{vmatrix}\right) \\
&= a_2\cdots a_n + a_1\left(\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & a_3 & \cdots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a_n \end{vmatrix} + \begin{vmatrix} a_2 & 1 & \cdots & 1 \\ 0 & 1+a_3 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 1+a_n \end{vmatrix}\right) \\
&= a_2\cdots a_n + a_1\left(a_3\cdots a_n + a_2\begin{vmatrix} 1+a_3 & 1 & \cdots & 1 \\ 1 & 1+a_4 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix}\right) \\
&= \cdots = a_2\cdots a_n + a_1 a_3\cdots a_n + \cdots + a_1\cdots a_{n-1} + a_1\cdots a_n.
\end{aligned}
$$

证明 (2):

$$
\text{左边} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1+a_1 & 1 & \cdots & 1 \\ 0 & 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & 1 & \cdots & 1+a_n \end{vmatrix} \xlongequal[i=2,\cdots,n+1]{r_i - r_1} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & a_1 & 0 & \cdots & 0 \\ -1 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & a_n \end{vmatrix}
$$

$$
\xlongequal{r_1 + \sum_{i=1}^{n} \frac{1}{a_i} r_{i+1}} \begin{vmatrix} 1+\sum_{i=1}^{n} \frac{1}{a_i} & 0 & 0 & \cdots & 0 \\ -1 & a_1 & 0 & \cdots & 0 \\ -1 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & a_n \end{vmatrix} = \left(1 + \sum_{i=1}^{n} \frac{1}{a_i}\right) \prod_{i=1}^{n} a_i .
$$

例:

$$\begin{vmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ 0 & x & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & -1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & x+a_1 \end{vmatrix} = x^n + \sum_{k=1}^{n} a_k x^{n-k}.$$

证明. 记行列式为 D_n，则

$$D_n = xD_{n-1} + (-1)^{n+1} a_n \begin{vmatrix} -1 & 0 & \cdots & 0 & 0 \\ x & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & x & -1 \end{vmatrix} = xD_{n-1} + a_n.$$

于是

$$\begin{aligned} D_n &= xD_{n-1} + a_n, \\ D_{n-1} &= xD_{n-2} + a_{n-1}, & \cdots\cdots \times x \\ D_{n-2} &= xD_{n-3} + a_{n-2}, & \cdots\cdots \times x^2 \\ &\cdots \\ D_2 &= xD_1 + a_2. & \cdots\cdots \times x^{n-2} \end{aligned}$$

所以

$$\begin{aligned} D_n &= a_n + a_{n-1}x + \cdots + a_2 x^{n-2} + x^{n-1} D_1 \\ &= a_n + a_{n-1}x + \cdots + a_2 x^{n-2} + x^{n-1}(x + a_1) = \text{右边} \end{aligned}$$

□

例:

证明

$$\begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 & 0 \\ a_2 & x & -1 & \cdots & 0 & 0 \\ a_3 & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & 0 & 0 & \cdots & x & -1 \\ a_n & 0 & 0 & \cdots & 0 & x \end{vmatrix} = \sum_{k=1}^{n} a_k x^{n-k}$$

证明. 记行列式为 D_n

$$D_n = (-1)^{n+1}a_n \begin{vmatrix} -1 & 0 & \cdots & 0 & 0 \\ x & -1 & \cdots & 0 & 0 \\ 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & x & -1 \end{vmatrix} + (-1)^{2n}x \begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 \\ a_2 & x & -1 & \cdots & 0 \\ a_3 & 0 & x & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1} & 0 & 0 & \cdots & x \end{vmatrix}$$

$$= a_n + xD_{n-1}.$$

于是

$$\begin{aligned} D_n &= xD_{n-1} + a_n, \\ D_{n-1} &= xD_{n-2} + a_{n-1}, & \cdots\cdots \times x \\ D_{n-2} &= xD_{n-3} + a_{n-2}, & \cdots\cdots \times x^2 \\ &\cdots \\ D_2 &= xD_1 + a_2. & \cdots\cdots \times x^{n-2} \end{aligned}$$

所以

$$\begin{aligned} D_n &= a_n + a_{n-1}x + \cdots + a_2x^{n-2} + x^{n-1}D_1 \\ &= a_n + a_{n-1}x + \cdots + a_2x^{n-2} + x^{n-1}a_1 = \text{右边} \end{aligned}$$

□

例:

$$\begin{vmatrix} \cos\theta & 1 & & & \\ 1 & 2\cos\theta & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2\cos\theta & 1 \\ & & & 1 & 2\cos\theta \end{vmatrix} = \cos n\theta$$

证明.

$$
\begin{aligned}
D_n &= (-1)^{n+(n-1)}\begin{vmatrix}
\cos\theta & 1 & & & & \\
1 & 2\cos\theta & 1 & & & \\
 & \ddots & \ddots & \ddots & & \\
 & & 1 & 2\cos\theta & 1 & \\
 & & & 1 & 2\cos\theta & \\
 & & & & 1 & 1
\end{vmatrix}_{n-1} + 2\cos\theta D_{n-1} \\
&= -D_{n-2} + 2\cos\theta D_{n-1}.
\end{aligned}
$$

用数学归纳法证明。

1^o 当 $n=1$ 时，结论显然成立。

2^o 假设结论对阶数 $\leq n-1$ 的行列式成立，则由上式可知

$$
\begin{aligned}
D_n &= -D_{n-2} + 2\cos\theta D_{n-1} \\
&= -\cos(n-2)\theta + 2\cos\theta\cos(n-1)\theta \\
&= -\cos(n-2)\theta + \cos(n-2)\theta\cos n\theta \\
&= \cos n\theta.
\end{aligned}
$$

□

例:

计算

$$\begin{vmatrix} \frac{1}{3} & -\frac{5}{2} & \frac{2}{5} & \frac{3}{2} \\ 3 & -12 & \frac{21}{5} & 15 \\ \frac{2}{3} & -\frac{9}{2} & \frac{4}{5} & \frac{5}{2} \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \frac{3}{7} \end{vmatrix}$$

解

$$\text{原式} = \frac{1}{30}\times\frac{3}{5}\times\frac{1}{30}\times\frac{1}{7}\times\begin{vmatrix} 10 & -75 & 12 & 45 \\ 5 & -20 & 7 & 25 \\ 20 & -135 & 24 & 75 \\ -1 & 2 & -1 & 3 \end{vmatrix}$$

$$\xlongequal[r_4+3\times r_1]{\substack{r_2+2\times r_1 \\ r_3-r_1}} \frac{1}{35\times 300}\times\begin{vmatrix} 10 & -55 & 2 & 75 \\ 5 & -10 & 2 & 40 \\ 20 & -95 & 4 & 135 \\ -1 & 0 & 0 & 0 \end{vmatrix}$$

$$\text{原式} = \frac{1}{\ldots}\begin{vmatrix} -55 & 2 & 75 \\ -10 & 2 & 40 \\ \ldots \end{vmatrix}$$

例:

计算

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & -n \\ 1 & 1 & \cdots & -n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & \cdots & 1 & 1 \\ -n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

解

$$\text{原式} \xlongequal{r_1+r_2+\cdots+r_n} \begin{vmatrix} -1 & 1 & \cdots & 1 & -n \\ -1 & 1 & \cdots & -n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & -n & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{r_i-r_1} \begin{vmatrix} -1 & 1 & \cdots & 1 & -n \\ 0 & 0 & \cdots & -n-1 & 1+n \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & -n-1 & \cdots & 0 & 1+n \\ 0 & 0 & \cdots & 0 & 1+n \end{vmatrix}$$

$$\begin{vmatrix} 0 & \cdots & -n-1 & 1+n \\ \vdots & & \vdots & \vdots \end{vmatrix}$$

例:

计算

$$\begin{vmatrix} a_1+\lambda_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & a_2+\lambda_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & a_3+\lambda_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n+\lambda_n \end{vmatrix}$$

解

$$\text{原式} = \begin{vmatrix} 1 & a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1+\lambda_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2+\lambda_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & a_3+\lambda_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & a_1 & a_2 & a_3 & \cdots & a_n+\lambda_n \end{vmatrix}$$

$$\text{原式} = \begin{vmatrix} 1 & a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1+\lambda_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2+\lambda_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & a_3+\lambda_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & a_1 & a_2 & a_3 & \cdots & a_n+\lambda_n \end{vmatrix}_{n+1}$$

例:

计算

$$D=\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

解

$$D_n \xlongequal[i=n,\cdots,2]{r_i-r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i-c_1} \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \end{vmatrix}$$

例:

证明

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \\ x_1^n & x_2^n & x_3^n & \cdots & x_n^n \end{vmatrix} = \sum_{i=1}^n x_i \prod_{1\le j<i\le n}(x_i - x_j).$$

证明. 考察行列式

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n & y \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 & y^2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_1^{n-2} & x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} & y^{n-2} \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} & y^{n-1} \\ x_1^n & x_2^n & x_3^n & \cdots & x_n^n & y^n \end{vmatrix} = \prod_{i=1}^n (y - x_i) \prod_{1\le j<i\le n}(x_i - x_j)$$

等式两端均为关于 y 的多项式，比较 y^{n-1} 的系数便得结论。

□

例:

用数学归纳法证明:

$$\frac{d}{dt}\begin{vmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{vmatrix} = \sum_{j=1}^{n} \begin{vmatrix} a_{11}(t) & \cdots & \frac{d}{dt}a_{1j}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & \cdots & \frac{d}{dt}a_{2j}(t) & \cdots & a_{2n}(t) \\ \vdots & & \vdots & & \vdots \\ a_{n1}(t) & \cdots & \frac{d}{dt}a_{nj}(t) & \cdots & a_{nn}(t) \end{vmatrix}$$

证明.

1^o 当 $n=1$ 时，结论显然成立。

2^o 假设结论对阶数 $\leq n-1$ 的行列式成立，考虑阶数为 n 的行列式，对第一列展开得

$$\begin{aligned} D &= a_{11}A_{11}+a_{21}A_{21}+\cdots+a_{n1}A_{n1}, \\ D' &= a'_{11}A_{11}+a'_{21}A_{21}+\cdots+a'_{n1}A_{n1}+ \\ & \quad a_{11}A'_{11}+a_{21}A'_{21}+\cdots+a_{n1}A'_{n1}, \end{aligned}$$

其中

$$a'_{11}(t)A_{11}(t)+a'_{21}(t)A_{21}(t)+\cdots+a'_{n1}(t)A_{n1}(t) = \begin{vmatrix} a'_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a'_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \end{vmatrix}$$

例:

设 3 个点 $P_1(x_1,y_1),P_2(x_2,y_2),P_3(x_3,y_3)$ 不在一条直线上，求过点 P_1,P_2,P_3 的圆的方程。

解　圆的一般方程为

$$a(x^2+y^2)+bx+cy+d=0,\quad a\neq 0$$

因 P_1,P_2,P_3 在圆上，故

$$\begin{cases} a(x^2+y^2)+bx+cy+d=0, \\ a(x_1^2+y_1^2)+bx_1+cy_1+d=0, \\ a(x_2^2+y_2^2)+bx_2+cy_2+d=0, \\ a(x_3^2+y_3^2)+bx_3+cy_3+d=0, \end{cases}$$

该齐次线性方程组有非零解的充分必要条件是系数行列式为零，即

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0.$$

例:

求使 3 点 $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ 位于一直线上的充分必要条件。

解　三点位于一直线上的充分必要条件是

$$\frac{y_1-y_2}{x_1-x_2}=\frac{y_1-y_3}{x_1-x_3},$$

即

$$(x_1-x_3)(y_1-y_2)=(x_1-x_2)(y_1-y_3)$$

亦即

$$x_1(y_2-y_3)-x_2(y_1-y_3)+x_3(y_1-y_2)=0$$

其行列式形式为

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0.$$

例:

求过 3 点 $(1,1,1)$，$(2,3,-1)$，$(3,-1,-1)$ 的平面方程。

解　平面方程为

$$ax+by+cz+d=0,$$

因 3 点位于平面上，故

$$\begin{cases} ax+by+cz+d=0, \\ a+b+c+d=0, \\ 2a+3b-c+d=0, \\ 3a-b-c+d=0 \end{cases}$$

该齐次线性方程组有非零解，故其系数行列式为零，即

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & -1 & 1 \\ 3 & -1 & -1 & 1 \end{vmatrix} = 0.$$

即

$$-8x-2y-6z+16=0.$$

亦即

$$4x+y+3z-8=0.$$

例:

求过点 $(1,1,1),(1,1,-1),(1,-1,1),(-1,0,0)$ 的球面方程，并求其中心与半径。

解　球面的一般方程为

$$a(x^2+y^2+z^2)+bx+cy+dz+e=0.$$

过该四点的球面方程为

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 1^2+1^2+1^2 & 1 & 1 & 1 & 1 \\ 1^2+1^2+(-1)^2 & 1 & 1 & -1 & 1 \\ 1^2+(-1)^2+1^2 & 1 & -1 & 1 & 1 \\ (-1)^2+0^2+0^2 & -1 & 0 & 0 & 1 \end{vmatrix}=0 \Rightarrow \begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 3 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & -1 & 1 \\ 3 & 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 \end{vmatrix}=0$$

按第一行展开可知

$$-8(x^2+y^2+z^2)+8x+16=0,$$

即

$$x^2+y^2+z^2-x-2=0, \quad \Rightarrow (x-\frac{1}{2})^2+y^2+z^2=(\frac{3}{2})^2$$

圆心为 $(\frac{1}{2},0,0)$，半径为 $\frac{3}{2}$.

例:

已知 $a^2 \neq b^2$，证明方程组

$$\begin{cases} ax_1 & & & & + & & & & bx_{2n} & = & 1 \\ & ax_2 & & & + & & & bx_{2n-1} & & = & 1 \\ & & \ddots & & & & \iddots & & & & \\ & & & ax_n & + & bx_{n+1} & & & & = & 1 \\ & & & bx_n & + & ax_{n+1} & & & & = & 1 \\ & & \iddots & & & & \ddots & & & & \\ & bx_2 & & & + & & & ax_{2n-1} & & = & 1 \\ bx_1 & & & & + & & & & ax_{2n} & = & 1 \end{cases}$$

有唯一解，并求解。

解 其系数行列式为 $D_{2n} = \begin{vmatrix} a & & & & & b \\ & \ddots & & & \iddots & \\ & & a & b & & \\ & & b & a & & \\ & \iddots & & & \ddots & \\ b & & & & & a \end{vmatrix}$

把 D_{2n} 中的第 $2n$ 行依次与第 $2n-1$ 行、…、第 2 行对调（共 $2n-2$ 次相邻对换），再把第 $2n$ 列依次与第 $2n-1$ 列、…、第 2 列对调，得