线性代数

行列式

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目录

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  - 1683年，日本数学家关孝和在其著作《解伏题之法》中也提出了行列式的概念与算法。《解伏题之法》的意思就是“解行列式问题的方法”，书里对行列式的概念和它的展开已经有了清楚的叙述。
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- 在行列式的发展史上，第一个对行列式理论做出连贯的逻辑的阐述，即把行列式理论与线性方程组求解相分离的人，是法国数学家范德蒙。 范德蒙自幼在父亲的指导下学习音乐，但对数学有深厚的兴趣，后来终于成为法兰西科学院院士。他给出了用二阶子式和它们的余子式来展开行列式的法则，就对行列式本身这一点来说，他是这门理论的奠基人。
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2. 行列式的定义
   - 二阶行列式
   - 三阶行列式
   - n阶行列式的定义
3. 行列式的性质
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引例

用消元法求解
$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

消去x_2得
$$(a_{11}a_{22} - a_{12}a_{21})x_1 = b_1a_{22} - b_2a_{12},$$

消去x_1得
$$(a_{11}a_{22} - a_{12}a_{21})x_2 = b_2a_{11} - b_1a_{11}.$$

若 $\boxed{a_{11}a_{22} - a_{12}a_{21} \neq 0}$，则
$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \quad x_2 = \frac{b_2a_{11} - b_1a_{11}}{a_{11}a_{22} - a_{12}a_{21}}.$$

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二阶行列式

由$2^2=4$个数，按下列形式排成2行2列的方形

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

其被定义成一个数

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \equiv D,$$

该数称为由这四个数构成的二阶行列式。

- a_{ij}表示行列式的元素。
  i为行标，表明该元素位于第i行；
  j为列标，表明该元素位于第j列。
- 对角线法则

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

- 类似地，

$$b_1 a_{22} - b_2 a_{12} = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \equiv D_1$$

$$b_2 a_{11} - b_1 a_{21} = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} \equiv D_2$$

则上述方程组的解可表示为

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}.$$

例1

求解二元线性方程组

$$\begin{cases} 3x_1 - 2x_2 = 12, \\ 2x_1 + x_2 = 1. \end{cases}$$

解：因为

$$D = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 7 \neq 0,$$

$$D_1 = \begin{vmatrix} 12 & -2 \\ 1 & 1 \end{vmatrix} = 14,$$

$$D_2 = \begin{vmatrix} 3 & 12 \\ 2 & 1 \end{vmatrix} = -21,$$

因此，

$$x_1 = \frac{D_1}{D} = 2, \quad x_2 = \frac{D_2}{D} = -3.$$

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三阶行列式

由$3^2=9$个数组成的3行3列的三阶行列式，则按如下形式定义一个数

$$D_3=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{array}{l} a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32} \\ -a_{13}a_{22}a_{31}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33} \end{array}$$

- 沙路法

$$
\begin{array}{ccccccc}
 & a_{11} & a_{12} & a_{13} & \color{red}{a_{11}} & \color{red}{a_{12}} & \\
 & a_{21} & a_{22} & a_{23} & \color{red}{a_{21}} & \color{red}{a_{22}} & \\
 & a_{31} & a_{32} & a_{33} & \color{red}{a_{31}} & \color{red}{a_{32}} & \\
- & - & - & & + & + & +
\end{array}
$$

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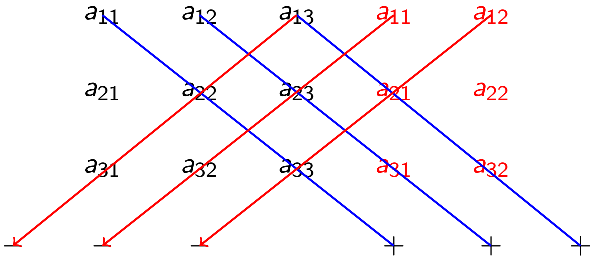
$$\begin{array}{ccccccc}
 & a_{11} & a_{12} & a_{13} & \color{red}{a_{11}} & \color{red}{a_{12}} & \\
 & a_{21} & a_{22} & a_{23} & \color{red}{a_{21}} & \color{red}{a_{22}} & \\
 & a_{31} & a_{32} & a_{33} & \color{red}{a_{31}} & \color{red}{a_{32}} & \\
- & - & - & & + & + & +
\end{array}$$

- 沙路法



例3

计算

$$D_3 = \begin{vmatrix} 1 & 2 & -4 \\ -2 & 2 & 1 \\ -3 & 4 & -2 \end{vmatrix}$$

解：由对角线法则可知，

$$\begin{aligned} D_3 &= 1 \times 2 \times (-2) + 2 \times 1 \times (-3) + (-2) \times 4 \times (-4) \\ &\quad -2 \times (-2) \times (-2) - (-4) \times 2 \times (-3) + 1 \times 1 \times 4 \\ &= -14. \end{aligned}$$

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求方程

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & x \\ 4 & 9 & x^2 \end{vmatrix} = 0$$

解：行列式

$$D = 3x^2 + 18 + 4x - 2x^2 - 12 - 9x = x^2 - 5x + 6$$

由此可知$x = 2$或3。

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如果三元一次方程组

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3, \end{aligned}$$

的系数行列式

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

则用消元法求解可得

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D},$$

其中

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \ D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \ D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}.$$

从二、三阶行列式的展开式中可发现：

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{33} \\ a_{23} & a_{32} \end{vmatrix}}_{M_{11}} - a_{12}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{M_{12}} + a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{M_{13}}$$

这里，M_{11}, M_{12}, M_{13}分别称为a_{11}, a_{12}, a_{13}的余子式，并称

$$A_{11} = (-1)^{1+1}M_{11}, \quad A_{12} = (-1)^{1+2}M_{12}, \quad A_{13} = (-1)^{1+3}M_{13}$$

分别称为a_{11}, a_{12}, a_{13}的代数余子式。这样，D可表示为

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$$=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}$$

$$=a_{11}(a_{22}a_{33}-a_{23}a_{32})-a_{12}(a_{21}a_{33}-a_{23}a_{31})+a_{13}(a_{21}a_{32}-a_{22}a_{31})$$

$$=a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{33} \\ a_{23} & a_{32} \end{vmatrix}}_{M_{11}}-a_{12}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{M_{12}}+a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{M_{13}}$$

这里，M_{11}, M_{12}, M_{13}分别称为a_{11}, a_{12}, a_{13}的余子式，并称

$$A_{11}=(-1)^{1+1}M_{11},\quad A_{12}=(-1)^{1+2}M_{12},\quad A_{13}=(-1)^{1+3}M_{13}$$

分别称为a_{11}, a_{12}, a_{13}的代数余子式。这样，D可表示为

$$D=a_{11}A_{11}+a_{11}A_{13}+a_{13}A_{13}.$$

同样地，

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12},$$

其中

$$A_{11} = (-1)^{1+1}|a_{22}| = a_{22}, \quad A_{11} = (-1)^{1+2}|a_{21}| = -a_{21}.$$

注意这里的$|a_{22}|$，$|a_{21}|$是一阶行列式，而不是绝对值。

我们把一阶行列式$|a|$定义为a。

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注意这里的$|a_{22}|$, $|a_{21}|$是一阶行列式，而不是绝对值。

我们把一阶行列式$|a|$定义为a。

定义

由n^2个数$a_{ij}(i,j=1,2,\cdots,n)$组成的n阶行列式

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{1}$$

是一个数。

定义（续）

- 当$n=1$时，定义$D=|a_{11}|=a_{11}$；
- 当$n\geq 2$时，定义

$$D=a_{11}A_{11}+a_{12}A_{12}+\cdots+a_{1n}A_{1n}, \tag{2}$$

其中

$$A_{1j}=(-1)^{1+j}M_{1j}$$

而M_{1j}是D中划去第1行第j列后，按原顺序排成的$n-1$阶行列式，即

$$M_{1j}=\begin{vmatrix} a_{21} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3,j-1} & a_{3,j+1} & \cdots & a_{3n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} \quad (j=1,2,\cdots,n),$$

并称M_{1j}为a_{1j}的余子式，A_{1j}为a_{1j}的代数余子式.

定义（续）

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并称M_{1j}为a_{1j}的余子式，A_{1j}为a_{1j}的代数余子式.

注

1 在D中，$a_{11}, a_{22}, \cdots, a_{nn}$所在的对角线称为行列式的主对角线，$a_{11}, a_{22}, \cdots, a_{nn}$称为主对角元。

2 行列式D是由n^2个元素构成的n次齐次多项式：

- 二阶行列式的展开式有2!项
- 三阶行列式的展开式有3!项
- n阶行列式的展开式有$n!$项，其中每一项都是不同行不同列的n个元素的乘积，带正号的项与带负号的项各占一半。

例

证明：n阶下三角行列式

$$D_n = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}.$$

证明【数学归纳法】

- 当$n=2$时，结论成立。
- 假设结论对$n-1$阶下三角阵成立，则由定义

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}\cdot(-1)^{1+1}\begin{vmatrix} a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33}\cdots a_{nn}). \quad \square$$

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$$\begin{aligned} \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} &= a_{11}\cdot(-1)^{1+1} \begin{vmatrix} a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33}\cdots a_{nn}). \qquad \square \end{aligned}$$

$$
\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix}
$$

$$
+ \begin{vmatrix} 0 & a_{12} & 0 & \cdots & 0 \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & a_{n3} & \cdots & a_{nn} \end{vmatrix} + \cdots
$$

$$
+ \begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ a_{21} & \cdots & a_{2,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{vmatrix}
$$

同理可证

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

例

计算n阶行列式

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & a_n \\ 0 & 0 & \cdots & a_{n-1} & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_2 & \cdots & * & * \\ a_1 & * & \cdots & * & * \end{vmatrix}$$

解

由行列式定义，

$$D_n = \begin{vmatrix} 0 & 0 & \cdots & 0 & a_n \\ 0 & 0 & \cdots & a_{n-1} & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_2 & \cdots & * & * \\ a_1 & * & \cdots & * & * \end{vmatrix} = (-1)^{1+n} a_n \begin{vmatrix} 0 & 0 & \cdots & a_{n-1} \\ \vdots & \vdots & & \vdots \\ 0 & a_2 & \cdots & * \\ a_1 & * & \cdots & * \end{vmatrix}$$

$$= (-1)^{n-1} a_n D_{n-1}.$$

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解（续）

同理递推，

$$
\begin{aligned}
D_n &= (-1)^{n-1}a_nD_{n-1} = (-1)^{n-1}a_n(-1)^{n-2}a_{n-1}D_{n-2} \\
&\cdots\cdots \\
&= (-1)^{(n-1)+(n-2)+\cdots+2+1}a_na_{n-1}\cdots a_2a_1 \\
&= (-1)^{\frac{n(n-1)}{2}}a_na_{n-1}\cdots a_2a_1.
\end{aligned}
$$

例如，

$$D_2 = -a_1a_2, \quad D_3 = -a_1a_2a_3, \quad D_4 = a_1a_2a_3a_4, \quad D_5 = a_1a_2a_3a_4a_5.$$

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性质 1

互换行列式的行与列，值不变，即

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix} \tag{3}$$

证明【数学归纳法】

将等式两端的行列式分别记为D和D'，对阶数n用归纳法。

- 当$n=2$时，$D=D'$显然成立。
- 假设结论对于阶数小于n的行列式都成立，以下考虑阶数为n的情况。由定义可知，

$$D = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n},$$

$$D' = a_{11}A'_{11} + a_{21}A'_{21} + \cdots + a_{n1}A'_{n1}$$

显然，$A_{11}=A'_{11}$。

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显然，$A_{11} = A'_{11}$。

证明【续】

于是

$$
\begin{aligned}
D' &= a_{11}A_{11} + (-1)^{1+2}a_{21}\begin{vmatrix} a_{12} & a_{32} & \cdots & a_{n2} \\ a_{13} & a_{33} & \cdots & a_{n3} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{3n} & \cdots & a_{nn} \end{vmatrix} \\
&\quad +(-1)^{1+3}a_{31}\begin{vmatrix} a_{12} & a_{22} & a_{42} & \cdots & a_{n2} \\ a_{13} & a_{23} & a_{43} & \cdots & a_{n3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{4n} & \cdots & a_{nn} \end{vmatrix} \\
&\quad + \cdots + (-1)^{1+n}a_{n1}\begin{vmatrix} a_{12} & a_{22} & \cdots & a_{n-1,2} \\ a_{13} & a_{23} & \cdots & a_{n-1,3} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{n-1,n} \end{vmatrix}
\end{aligned}
$$

证明【续】

对$n-1$个行列式按第一行展开，将含a_{12}的项进行合并，可得

$$(-1)^{1+2}a_{21}a_{12}\begin{vmatrix} a_{33} & \cdots & a_{n3} \\ \vdots & & \vdots \\ a_{3n} & \cdots & a_{nn} \end{vmatrix} + (-1)^{1+3}a_{31}a_{12}\begin{vmatrix} a_{23} & a_{43} & \cdots & a_{n3} \\ \vdots & \vdots & & \vdots \\ a_{2n} & a_{4n} & \cdots & a_{nn} \end{vmatrix}$$

$$+\cdots+(-1)^{1+n}a_{n1}a_{12}\begin{vmatrix} a_{23} & \cdots & a_{n-1,3} \\ \vdots & & \vdots \\ a_{2n} & \cdots & a_{n-1,n} \end{vmatrix}$$

证明【续】

$$
\begin{aligned}
= \quad & (-1)^{1+2}a_{12}\left((-1)^{1+1}a_{21}\begin{vmatrix} a_{33} & \cdots & a_{n3} \\ \vdots & & \vdots \\ a_{3n} & \cdots & a_{nn} \end{vmatrix}\right. \\
& +(-1)^{1+2}a_{31}\begin{vmatrix} a_{23} & a_{43} & \cdots & a_{n3} \\ \vdots & \vdots & & \vdots \\ a_{2n} & a_{4n} & \cdots & a_{nn} \end{vmatrix} + \cdots + \\
& \left.(-1)^{1+n-1}a_{n1}\begin{vmatrix} a_{23} & \cdots & a_{n-1,3} \\ \vdots & & \vdots \\ a_{2,n3} & \cdots & a_{n-1,n} \end{vmatrix}\right)
\end{aligned}
$$

证明【续】

$$= (-1)^{1+2}a_{12}\left(\begin{vmatrix} a_{21} & 0 & \cdots & 0 \\ 0 & a_{33} & \cdots & a_{n3} \\ 0 & \vdots & & \vdots \\ 0 & a_{3n} & \cdots & a_{nn} \end{vmatrix}\right.$$

$$+\begin{vmatrix} 0 & a_{31} & 0 & \cdots & 0 \\ a_{23} & 0 & a_{43} & \cdots & a_{n3} \\ \vdots & 0 & \vdots & & \vdots \\ a_{2n} & 0 & a_{4n} & \cdots & a_{nn} \end{vmatrix} + \cdots +$$

$$\left.\begin{vmatrix} 0 & \cdots & 0 & a_{n-1} \\ a_{23} & \cdots & a_{n-1,3} & 0 \\ \vdots & & \vdots & 0 \\ a_{2,n3} & \cdots & a_{n-1,n} & 0 \end{vmatrix}\right)$$

证明【续】

$$= (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{31} & \cdots & a_{n1} \\ a_{23} & a_{33} & \cdots & a_{n3} \\ \vdots & \vdots & & \vdots \\ a_{2n} & a_{3n} & \cdots & a_{nn} \end{vmatrix}$$

$$= (-1)^{1+2}a_{12}M'_{12} = a_{12}A'_{12} = a_{12}A_{12}.$$

同理，含a_{13}的项合并后其值等于$a_{13}A_{13}$，$\cdots$，含a_{1n}的项合并后其值等于$a_{1n}A_{1n}$. 因此，$D = D'$.

有了这个性质，行列式对行成立的性质都适用于列。

证明【续】

$$
\begin{aligned}
&= (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{31} & \cdots & a_{n1} \\ a_{23} & a_{33} & \cdots & a_{n3} \\ \vdots & \vdots & & \vdots \\ a_{2n} & a_{3n} & \cdots & a_{nn} \end{vmatrix} \\
&= (-1)^{1+2} a_{12} M'_{12} = a_{12} A'_{12} = a_{12} A_{12}.
\end{aligned}
$$

同理，含a_{13}的项合并后其值等于$a_{13}A_{13}$，$\cdots$，含a_{1n}的项合并后其值等于$a_{1n}A_{1n}$. 因此，$D = D'$.

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$$
\begin{aligned}
&= (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{31} & \cdots & a_{n1} \\ a_{23} & a_{33} & \cdots & a_{n3} \\ \vdots & \vdots & & \vdots \\ a_{2n} & a_{3n} & \cdots & a_{nn} \end{vmatrix} \\
&= (-1)^{1+2}a_{12}M'_{12} = a_{12}A'_{12} = a_{12}A_{12}.
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同理，含a_{13}的项合并后其值等于$a_{13}A_{13}$，$\cdots$，含a_{1n}的项合并后其值等于$a_{1n}A_{1n}$. 因此，$D = D'$.

有了这个性质，行列式对行成立的性质都适用于列。

性质2

行列式对任一行按下式展开，其值相等，即

$$D = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij}, \quad i = 1, 2, \cdots, n,$$

其中

$$A_{ij} = (-1)^{i+j}M_{ij}$$

而M_{ij}为D中划掉第i行第j列后其余元素按原顺序排成的$n-1$阶行列式，它称为a_{ij}的余子式，A_{ij}称为a_{ij}的代数余子式.

证明[数学归纳法]

- 当$n=2$时，结论显然成立。
- 假设结论对阶数$\leq n-1$的行列式成立，以下考虑阶数为n的情况。

证明【续】

$$
\begin{aligned}
D = {} & (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \\
& + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix}
\end{aligned}
$$

证明【续】

$$+(-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{24} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n4} & \cdots & a_{nn} \end{vmatrix}$$

$$+\cdots+(-1)^{1+n}a_{1n}\begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{i,n-1} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} \end{vmatrix}$$

证明【续】

由归纳假设，按行展开后合并含a_{i1}的项可得

$$(-1)^{(i-1)+1}a_{i1}\left((-1)^{1+2}a_{12}\begin{vmatrix} a_{23} & a_{24} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i-1,3} & a_{i-1,4} & \cdots & a_{i-1,n} \\ a_{i+1,3} & a_{i+1,4} & \cdots & a_{i+1,n} \\ \vdots & \vdots & & \vdots \\ a_{n,3} & a_{n,4} & \cdots & a_{nn} \end{vmatrix}\right.$$

$$+(-1)^{1+3}a_{13}\begin{vmatrix} a_{22} & a_{24} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i-1,2} & a_{i-1,4} & \cdots & a_{i-1,n} \\ a_{i+1,2} & a_{i+1,4} & \cdots & a_{i+1,n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n4} & \cdots & a_{nn} \end{vmatrix}$$

证明【续】

$$
+\cdots+(-1)^{1+n}a_{1n}\begin{vmatrix} a_{22} & \cdots & a_{2,n-1} \\ \vdots & & \vdots \\ a_{i-1,2} & \cdots & a_{i-1,n-1} \\ a_{i+1,2} & \cdots & a_{i+1,n-1} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{n,n-1} \end{vmatrix}\Bigg)
$$

$$
=(-1)^{i+1}a_{i1}\begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1,n} \\ a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1,n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}=(-1)^{i+1}a_{i1}M_{i1}=a_{i1}A_{i1}.
$$

证明【续】

同理可证，含a_{i2}的项合并后其值为$a_{i2}A_{i2}$，$\cdots$，含a_{in}的项合并后其值为$a_{in}A_{in}$.

（线性性质）

1 行列式的某一行（列）中所有的元素都乘以同一个数k，等于用数k乘以此行列式，即

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad (4)$$

（线性性质）

2 若行列式的某一行（列）的元素都是两数之和，如

$$
\begin{vmatrix} a_{11} & \cdots & a_{1j}+b_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j}+b_{2j} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj}+b_{nj} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdots & b_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & b_{2j} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & b_{nj} & \cdots & a_{nn} \end{vmatrix} \tag{5}
$$

一些记号

- $r_i \times k$（$c_i \times k$）：第i行（列）乘以k
- $r_i \div k$（$c_i \div k$）：第i行（列）提取公因子k

例

如果行列式$D=|a_{ij}|_n$的元素$a_{ij}=-a_{ji}(i,j=1,2,\cdots,n)$，就称$D$是反对称行列式（其中$a_{ii}=-a_{ii}\Rightarrow a_{ii}=0, i=1,2,\cdots,n$）.

证明：奇数阶反对称行列式的值为0.

证明

$$D=\begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & 0 \end{vmatrix}$$

例

如果行列式$D=|a_{ij}|_n$的元素$a_{ij}=-a_{ji}(i,j=1,2,\cdots,n)$，就称$D$是反对称行列式（其中$a_{ii}=-a_{ii}\Rightarrow a_{ii}=0, i=1,2,\cdots,n$）.

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证明【续】

$$\xlongequal{\text{性质1}} \begin{vmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ a_{12} & 0 & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & 0 \end{vmatrix}$$

$$\xlongequal[\text{将每行提取公因子}-1]{\text{性质3-1}} (-1)^n \begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & 0 \end{vmatrix} = (-1)^n D.$$

由于n为奇数，故$D = -D$，从而$D = 0$.

证明【续】

$$\begin{vmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ a_{12} & 0 & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & 0 \end{vmatrix} \xlongequal{\text{性质1}}$$

$$\xlongequal[\text{将每行提取公因子}-1]{\text{性质3-1}} (-1)^n \begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & 0 \end{vmatrix} = (-1)^n D.$$

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推论1

若行列式的某行元素全为0，其值为0.

例

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 2 & 5 & 1 \end{vmatrix} = 0.$$

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性质4

若行列式有两行（列）完全相同，其值为0.

证明

不妨设D的第i和j行元素全部相等，即对

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

有$a_{il} = a_{jl} (i \neq j, l = 1, 2, \cdots, n)$.

性质4

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证明

不妨设D的第i和j行元素全部相等，即对

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

有$a_{il} = a_{jl}(i \neq j, l = 1, 2, \cdots, n)$.

证明【续】

对阶数n用数学归纳法。

- 当$n=2$时，结论显然成立。
- 假设结论对阶数为$n-1$的行列式成立，在n阶的情况下，对第$k(k\neq i,j)$行展开，有

$$D=a_{k1}A_{k1}+a_{k2}A_{k2}+\cdots+a_{kn}A_{kn}.$$

注意到余子式$M_{kl}(l=1,2,\cdots,n)$是$n-1$阶行列式，且其中有两行元素相同，故

$$A_{kl}=(-1)^{k+l}M_{kl}=0\quad(l=1,2,\cdots,n),$$

从而$D=0.$

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$$A_{kl}=(-1)^{k+l}M_{kl}=0\quad(l=1,2,\cdots,n),$$

从而$D=0$.

例

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 0.$$

推论2

若行列式中有两行（列）元素成比例，则行列式的值为0.

例

$$\begin{vmatrix} 2 & 0 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 2 & 3 & 4 \end{vmatrix} = 0.$$

例

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性质5

把行列式的某一行（列）的各元素乘以同一个数然后加到另一行（列）对应的元素上去，行列式的值不变。

证明

将数k乘以第j行加到第i行，有

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1}+ka_{j1} & a_{i2}+ka_{j2} & \cdots & a_{in}+ka_{jn} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

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证明【续】

$$
\xlongequal[]{\text{性质3-2}}
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & & \vdots \\
a_{j1} & a_{j2} & \cdots & a_{jn} \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
+
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & & \vdots \\
ka_{j1} & ka_{j2} & \cdots & ka_{jn} \\
\vdots & \vdots & & \vdots \\
a_{j1} & a_{j2} & \cdots & a_{jn} \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
$$

$$
\xlongequal[]{\text{推论2}}
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & & \vdots \\
a_{j1} & a_{j2} & \cdots & a_{jn} \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
$$

证明【续】

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \overset{\text{性质3-2}}{=\!=\!=\!=} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ka_{j1} & ka_{j2} & \cdots & ka_{jn} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\overset{\text{推论2}}{=\!=\!=\!=} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

证明【续】

$$\stackrel{\text{性质3-2}}{=\!=\!=} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ka_{j1} & ka_{j2} & \cdots & ka_{jn} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\stackrel{\text{推论2}}{=\!=\!=} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

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$$\xlongequal{\text{性质3-2}} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ka_{j1} & ka_{j2} & \cdots & ka_{jn} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

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一些记号

- $r_i + r_j \times k$：将第j行乘以k加到第i行
- $c_i + c_j \times k$：将第j列乘以k加到第i列

性质6

互换行列式的两行（列），行列式变号。

证明

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

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$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

证明【续】

$$\xlongequal[r_i+r_j]{\text{性质}5} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1}+a_{j1} & a_{i2}+a_{j2} & \cdots & a_{in}+a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

证明【续】

$$\xlongequal[r_i+r_j]{\text{性质}5} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1}+a_{j1} & a_{i2}+a_{j2} & \cdots & a_{in}+a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

证明【续】

$$\xlongequal[r_i+r_j]{\text{性质}5}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1}+a_{j1} & a_{i2}+a_{j2} & \cdots & a_{in}+a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

证明【续】

$$\xlongequal[r_j - r_i]{\text{性质5}} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1}+a_{j1} & a_{i2}+a_{j2} & \cdots & a_{in}+a_{jn} \\ \vdots & \vdots & & \vdots \\ -a_{i1} & -a_{i2} & \cdots & -a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

证明【续】

$$\xlongequal[r_j-r_i]{\text{性质}5}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1}+a_{j1} & a_{i2}+a_{j2} & \cdots & a_{in}+a_{jn} \\ \vdots & \vdots & & \vdots \\ -a_{i1} & -a_{i2} & \cdots & -a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

证明【续】

$$\xlongequal[r_j - r_i]{\text{性质}5} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1}+a_{j1} & a_{i2}+a_{j2} & \cdots & a_{in}+a_{jn} \\ \vdots & \vdots & & \vdots \\ -a_{i1} & -a_{i2} & \cdots & -a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

证明【续】

$$\xlongequal[r_i+r_j]{\text{性质5}} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ -a_{i1} & -a_{i2} & \cdots & -a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \xlongequal{\text{性质3-1}} -D$$

证明【续】

$$\xlongequal[r_i+r_j]{\text{性质}5} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ -a_{i1} & -a_{i2} & \cdots & -a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \xlongequal{\text{性质}3\text{-}1} -D$$

证明【续】

$$\xlongequal[r_i+r_j]{\text{性质}5} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ -a_{i1} & -a_{i2} & \cdots & -a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \xlongequal{\text{性质}3\text{-}1} -D$$

证明【续】

$$\xlongequal[r_i+r_j]{\text{性质5}} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ -a_{i1} & -a_{i2} & \cdots & -a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \xlongequal{\text{性质3-1}} -D$$

一些记号

- $r_i \leftrightarrow r_j$：互换第i,j行
- $c_i \leftrightarrow c_j$：互换第i,j列

例

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \xlongequal{r_1 \leftrightarrow r_2} -\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \xlongequal{c_1 \leftrightarrow c_2} -\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$$

性质7

行列式某一行的元素乘以另一行对应元素的代数余子式之和等于0，即

$$\sum_{k=1}^{n} a_{ik}A_{jk} = 0 \quad (i \neq j).$$

证明

由性质2，对D的第j行展开得

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{j1}A_{j1} + a_{j2}A_{j2} + \cdots + a_{jn}A_{jn}$$

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行列式某一行的元素乘以另一行对应元素的代数余子式之和等于0，即

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$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{j1}A_{j1} + a_{j2}A_{j2} + \cdots + a_{jn}A_{jn}$$

证明【续】

因此，将D中第j行的元素$a_{j1}, a_{j2}, \cdots, a_{jn}$换成$a_{i1}, a_{i2}, \cdots, a_{in}$后所得的行列式，其展开式就是$\sum_{k=1}^{n} a_{ik}A_{jk}$，即

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \xlongequal{\text{性质4}} 0.$$

结论

- 对行列式D按行展开，有

$$\sum_{k=1} a_{ik}A_{jk} = \delta_{ij}D,$$

其中δ_{ij}为克罗内克（Kronecker）记号，表示为

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

- 对行列式D按列展开，有

$$\sum_{k=1} a_{ki}A_{kj} = \delta_{ij}D,$$

回顾一下行列式的性质

性质1

互换行列式的行与列，值不变

性质2

行列式对任一行按下式展开，其值相等，即

$$D = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij}, \quad i = 1, 2, \cdots, n,$$

其中

$$A_{ij} = (-1)^{i+j} M_{ij}$$

而M_{ij}为D中划掉第i行第j列后其余元素按原顺序排成的$n-1$阶行列式，它称为a_{ij}的余子式，A_{ij}称为a_{ij}的代数余子式.

性质3（线性性质）

1. 行列式的某一行（列）中所有的元素都乘以同一个数k，等于用数k乘以此行列式；
2. 若行列式的某一行（列）的元素都是两数之和，则该行列式可表示为两个行列式的和。

推论1

若行列式的某行元素全为0，其值为0.

性质4

若行列式有两行（列）完全相同，其值为0.

推论2

若行列式中有两行（列）元素成比例，则行列式的值为0.

性质5

把行列式的某一行（列）的各元素乘以同一个数然后加到另一行（列）对应的元素上去，行列式的值不变。

性质6

互换行列式的两行（列），行列式变号。

性质7

行列式某一行的元素乘以另一行对应元素的代数余子式之和等于0，即

$$\sum_{k=1}^{n} a_{ik}A_{jk} = 0 \quad (i \neq j).$$

结论

- 对行列式D按行展开，有

$$\sum_{k=1}^{n} a_{ik}A_{jk} = \delta_{ij}D.$$

- 对行列式D按列展开，有

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例1

计算

$$D=\begin{vmatrix} 3 & 1 & -1 & 2 \\ -5 & 1 & 3 & -4 \\ 2 & 0 & 1 & -1 \\ 1 & -5 & 3 & -3 \end{vmatrix}$$

解：

$$D \xlongequal{c_1 \leftrightarrow c_2} -\begin{vmatrix} 1 & 3 & -1 & 2 \\ 1 & -5 & 3 & -4 \\ 0 & 2 & 1 & -1 \\ -5 & 1 & 3 & -3 \end{vmatrix} \xlongequal[r_4+5r_1]{r_2-r_1} -\begin{vmatrix} 1 & 3 & -1 & 2 \\ 0 & -8 & 4 & -6 \\ 0 & 2 & 1 & -1 \\ 0 & 16 & -2 & 7 \end{vmatrix}$$

$$\xlongequal{r_2 \leftrightarrow r_3} \begin{vmatrix} 1 & 3 & -1 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & -8 & 4 & -6 \\ 0 & 16 & -2 & 7 \end{vmatrix} \xlongequal[r_4-8r_2]{r_3+4r_2} \begin{vmatrix} 1 & 3 & -1 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 8 & -10 \\ 0 & 0 & -10 & 15 \end{vmatrix} = \cdots = 40$$

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计算

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$$D=\begin{vmatrix} 3 & 1 & -1 & 2 \\ -5 & 1 & 3 & -4 \\ 2 & 0 & 1 & -1 \\ 1 & -5 & 3 & -3 \end{vmatrix}$$

解：

$$D \xlongequal{c_1 \leftrightarrow c_2} -\begin{vmatrix} 1 & 3 & -1 & 2 \\ 1 & -5 & 3 & -4 \\ 0 & 2 & 1 & -1 \\ -5 & 1 & 3 & -3 \end{vmatrix} \xlongequal[r_4+5r_1]{r_2-r_1} -\begin{vmatrix} 1 & 3 & -1 & 2 \\ 0 & -8 & 4 & -6 \\ 0 & 2 & 1 & -1 \\ 0 & 16 & -2 & 7 \end{vmatrix}$$

$$\xlongequal{r_2 \leftrightarrow r_3} \begin{vmatrix} 1 & 3 & -1 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & -8 & 4 & -6 \\ 0 & 16 & -2 & 7 \end{vmatrix} \xlongequal[r_4-8r_2]{r_3+4r_2} \begin{vmatrix} 1 & 3 & -1 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 8 & -10 \\ 0 & 0 & -10 & 15 \end{vmatrix} = \cdots = 40$$

例1

计算

$$D=\begin{vmatrix} 3 & 1 & -1 & 2 \\ -5 & 1 & 3 & -4 \\ 2 & 0 & 1 & -1 \\ 1 & -5 & 3 & -3 \end{vmatrix}$$

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例2

计算
$$D=\begin{vmatrix} 3 & 1 & -1 & 2 \\ -5 & 1 & 3 & -4 \\ 2 & 0 & 1 & -1 \\ 1 & -5 & 3 & -3 \end{vmatrix}$$

解：

$$D \xlongequal[c_4+c_3]{c_1-2c_3} \begin{vmatrix} 5 & 1 & -1 & 1 \\ -11 & 1 & 3 & -1 \\ 0 & 0 & 1 & 0 \\ -5 & -5 & 3 & 0 \end{vmatrix}$$

$$=(-1)^{3+3}\begin{vmatrix} 5 & 1 & 1 \\ -11 & 1 & -1 \\ -5 & -5 & 0 \end{vmatrix} \xlongequal{r_2+r_1} \begin{vmatrix} 5 & 1 & 1 \\ -6 & 2 & 0 \\ -5 & -5 & 0 \end{vmatrix}$$

$$=(-1)^{1+3}\begin{vmatrix} -6 & 2 \\ -5 & -5 \end{vmatrix} = 40.$$

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计算

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解：

$$D \overset{c_1-2c_3}{\underset{c_4+c_3}{=\!=}} \begin{vmatrix} 5 & 1 & -1 & 1 \\ -11 & 1 & 3 & -1 \\ 0 & 0 & 1 & 0 \\ -5 & -5 & 3 & 0 \end{vmatrix}$$

$$=(-1)^{3+3}\begin{vmatrix} 5 & 1 & 1 \\ -11 & 1 & -1 \\ -5 & -5 & 0 \end{vmatrix} \overset{r_2+r_1}{=\!=} \begin{vmatrix} 5 & 1 & 1 \\ -6 & 2 & 0 \\ -5 & -5 & 0 \end{vmatrix}$$

$$=(-1)^{1+3}\begin{vmatrix} -6 & 2 \\ -5 & -5 \end{vmatrix}=40.$$

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计算

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$$= (-1)^{1+3} \begin{vmatrix} -6 & 2 \\ -5 & -5 \end{vmatrix} = 40.$$

例3

计算

$$D=\begin{vmatrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ a & 3a+b & 6a+3b+c & 10a+6b+3c+d \end{vmatrix}$$

解：

$$D \xlongequal[r_2-r_1]{\substack{r_4-r_3\\ r_3-r_2}} \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & a & 2a+b & 3a+2b+c \\ 0 & a & 3a+b & 6a+3b+c \end{vmatrix} \xlongequal[r_3-r_2]{r_4-r_3} \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 0 & a & 2a+b \\ 0 & 0 & a & 3a+b \end{vmatrix}$$

$$\xlongequal{r_4-r_3} \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 0 & a & 2a+b \\ 0 & 0 & 0 & a \end{vmatrix} = a^4.$$

例3

计算

$$D=\begin{vmatrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ a & 3a+b & 6a+3b+c & 10a+6b+3c+d \end{vmatrix}$$

解：

$$D \xlongequal[r_2-r_1]{\substack{r_4-r_3\\ r_3-r_2}} \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & a & 2a+b & 3a+2b+c \\ 0 & a & 3a+b & 6a+3b+c \end{vmatrix} \xlongequal[r_3-r_2]{r_4-r_3} \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 0 & a & 2a+b \\ 0 & 0 & a & 3a+b \end{vmatrix}$$

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$$D = \begin{vmatrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ a & 3a+b & 6a+3b+c & 10a+6b+3c+d \end{vmatrix}$$

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解：

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$$\xlongequal{r_4-r_3} \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 0 & a & 2a+b \\ 0 & 0 & 0 & a \end{vmatrix} = a^4.$$

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解：

$$D \xlongequal[r_2-r_1]{\substack{r_4-r_3\\ r_3-r_2}} \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & a & 2a+b & 3a+2b+c \\ 0 & a & 3a+b & 6a+3b+c \end{vmatrix} \xlongequal[r_3-r_2]{r_4-r_3} \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 0 & a & 2a+b \\ 0 & 0 & a & 3a+b \end{vmatrix}$$

$$\xlongequal{r_4-r_3} \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 0 & a & 2a+b \\ 0 & 0 & 0 & a \end{vmatrix} = a^4.$$

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解：

$$D \xlongequal[r_2-r_1]{\substack{r_4-r_3\\ r_3-r_2}} \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & a & 2a+b & 3a+2b+c \\ 0 & a & 3a+b & 6a+3b+c \end{vmatrix} \xlongequal[r_3-r_2]{r_4-r_3} \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 0 & a & 2a+b \\ 0 & 0 & a & 3a+b \end{vmatrix}$$

$$\xlongequal{r_4-r_3} \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 0 & a & 2a+b \\ 0 & 0 & 0 & a \end{vmatrix} = a^4.$$

例4

计算

$$D=\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

解：

$$D_n \xlongequal[i=n,\cdots,2]{r_i-r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i-c_1} \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

例4

计算

$$D=\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

解：

$$D_n \xlongequal[i=n,\cdots,2]{r_i-r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i-c_1} \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

例4

计算

$$D=\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

解：

$$D_n \xlongequal[i=n,\cdots,2]{r_i-r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i-c_1} \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & \end{vmatrix}$$

例4

计算

$$D = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

解：

$$D_n \xlongequal[i=n,\cdots,2]{r_i - r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i - c_1} \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

例4

计算

$$D=\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

解：

$$D_n \xlongequal[i=n,\cdots,2]{r_i-r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i-c_1} \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

(续)

$$
\begin{aligned}
D_n &= \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&\xlongequal[i=2,\cdots,n]{c_i \div n} n^{n-1} \begin{vmatrix} 1 & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&\xlongequal{c_1+c_2+\cdots+c_n} n^{n-1} \begin{vmatrix} 1+\sum_{i=1}^{n-1} \frac{i}{n} & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&= n^{n-1}\left[1+\frac{1}{n}\frac{n(n-1)}{2}\right](-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{n-1} = (-1)^{\frac{(n-1)n}{2}}\frac{n+1}{2}n^{n-1}.
\end{aligned}
$$

(续)

$$D_n = \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i \div n} n^{n-1} \begin{vmatrix} 1 & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$\xlongequal{c_1+c_2+\cdots+c_n} n^{n-1} \begin{vmatrix} 1+\sum_{i=1}^{n-1} \frac{i}{n} & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$= n^{n-1}\left[1+\frac{1}{n}\frac{n(n-1)}{2}\right](-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{n-1} = (-1)^{\frac{(n-1)n}{2}}\frac{n+1}{2}n^{n-1}.$$

(续)

$$
\begin{aligned}
D_n &= \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&\xlongequal[i=2,\cdots,n]{c_i \div n} n^{n-1} \begin{vmatrix} 1 & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&\xlongequal{c_1+c_2+\cdots+c_n} n^{n-1} \begin{vmatrix} 1+\sum_{i=1}^{n-1}\frac{i}{n} & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&= n^{n-1}\left[1+\frac{1}{n}\frac{n(n-1)}{2}\right](-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{n-1} = (-1)^{\frac{(n-1)n}{2}}\frac{n+1}{2}n^{n-1}.
\end{aligned}
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\begin{aligned}
D_n &= \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&\xlongequal[i=2,\cdots,n]{c_i \div n} n^{n-1} \begin{vmatrix} 1 & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&\xlongequal{c_1+c_2+\cdots+c_n} n^{n-1} \begin{vmatrix} 1+\sum_{i=1}^{n-1}\frac{i}{n} & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&= n^{n-1}\left[1+\frac{1}{n}\frac{n(n-1)}{2}\right](-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{n-1} = (-1)^{\frac{(n-1)n}{2}}\frac{n+1}{2}n^{n-1}.
\end{aligned}
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$$D_n = \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i \div n} n^{n-1} \begin{vmatrix} 1 & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

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$$= n^{n-1}\left[1+\frac{1}{n}\frac{n(n-1)}{2}\right](-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{n-1} = (-1)^{\frac{(n-1)n}{2}}\frac{n+1}{2}n^{n-1}.$$

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\begin{aligned}
D_n &= \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&\xlongequal[i=2,\cdots,n]{c_i \div n} n^{n-1} \begin{vmatrix} 1 & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&\xlongequal{c_1+c_2+\cdots+c_n} n^{n-1} \begin{vmatrix} 1+\sum_{i=1}^{n-1}\frac{i}{n} & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&= n^{n-1}\left[1+\frac{1}{n}\frac{n(n-1)}{2}\right](-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{n-1} = (-1)^{\frac{(n-1)n}{2}}\frac{n+1}{2}n^{n-1}.
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$$= n^{n-1}\left[1+\frac{1}{n}\frac{n(n-1)}{2}\right](-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{n-1} = (-1)^{\frac{(n-1)n}{2}}\frac{n+1}{2}n^{n-1}.$$

例5

计算行列式

$$D_{20} = \begin{vmatrix} 1 & 2 & 3 & \cdots & 18 & 19 & 20 \\ 2 & 1 & 2 & \cdots & 17 & 18 & 19 \\ 3 & 2 & 1 & \cdots & 16 & 17 & 18 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 20 & 19 & 18 & \cdots & 3 & 2 & 1 \end{vmatrix}$$

解:

$$D_{20} \xlongequal[i=19,\cdots,1]{c_{i+1}-c_i} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 2 & -1 & 1 & \cdots & 1 & 1 & 1 \\ 3 & -1 & -1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 20 & -1 & -1 & \cdots & -1 & -1 & -1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,20]{r_i+r_1} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 3 & 0 & 2 & \cdots & 2 & 2 & 2 \\ 4 & 0 & 0 & \cdots & 2 & 2 & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 21 & 0 & 0 & \cdots & 0 & 0 & 0 \end{vmatrix} = 21 \times (-1)^{20+1} \times 2^{18} = -21 \times 2^{18}.$$

例5

计算行列式

$$D_{20}=\begin{vmatrix} 1 & 2 & 3 & \cdots & 18 & 19 & 20 \\ 2 & 1 & 2 & \cdots & 17 & 18 & 19 \\ 3 & 2 & 1 & \cdots & 16 & 17 & 18 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 20 & 19 & 18 & \cdots & 3 & 2 & 1 \end{vmatrix}$$

解:

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$$\xlongequal[i=2,\cdots,20]{r_i+r_1} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 3 & 0 & 2 & \cdots & 2 & 2 & 2 \\ 4 & 0 & 0 & \cdots & 2 & 2 & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 21 & 0 & 0 & \cdots & 0 & 0 & 0 \end{vmatrix} = 21\times(-1)^{20+1}\times 2^{18} = -21\times 2^{18}.$$

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例6

计算元素为$a_{ij} = |i - j|$的n阶行列式

解:

$$
\begin{aligned}
D_n &= \begin{vmatrix}
0 & 1 & 2 & \cdots & n-2 & n-1 \\
1 & 0 & 1 & \cdots & n-3 & n-2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
n-2 & n-3 & n-4 & \cdots & 0 & 1 \\
n-1 & n-2 & n-3 & \cdots & 1 & 0
\end{vmatrix} \\
&\xlongequal[i=n-1,\cdots,1]{c_{i+1}-c_i} \begin{vmatrix}
0 & 1 & 1 & \cdots & 1 & 1 \\
1 & -1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
n-2 & -1 & -1 & \cdots & -1 & 1 \\
n-1 & -1 & -1 & \cdots & -1 & -1
\end{vmatrix} \\
&\xlongequal[i=2,\cdots,n]{r_i+r_1} \begin{vmatrix}
0 & 1 & 1 & \cdots & 1 & 1 \\
1 & 0 & 2 & \cdots & 2 & 2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
n-2 & 0 & 0 & \cdots & 0 & 2 \\
n-1 & 0 & 0 & \cdots & 0 & 0
\end{vmatrix} = (-1)^{n-1}(n-1)2^{n-2}.
\end{aligned}
$$

例6

计算元素为$a_{ij}=|i-j|$的n阶行列式

解:

$$D_n = \begin{vmatrix} 0 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 1 & \cdots & n-3 & n-2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & n-3 & n-4 & \cdots & 0 & 1 \\ n-1 & n-2 & n-3 & \cdots & 1 & 0 \end{vmatrix}$$

$$\xlongequal[i=n-1,\cdots,1]{c_{i+1}-c_i} \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & -1 & -1 & \cdots & -1 & 1 \\ n-1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{r_i+r_1} \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 2 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & 0 & 0 & \cdots & 0 & 2 \\ n-1 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix} = (-1)^{n-1}(n-1)2^{n-2}.$$

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计算元素为$a_{ij}=|i-j|$的n阶行列式

解:

$$
\begin{aligned}
D_n &= \begin{vmatrix}
0 & 1 & 2 & \cdots & n-2 & n-1 \\
1 & 0 & 1 & \cdots & n-3 & n-2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
n-2 & n-3 & n-4 & \cdots & 0 & 1 \\
n-1 & n-2 & n-3 & \cdots & 1 & 0
\end{vmatrix} \\
&\xlongequal[i=n-1,\cdots,1]{c_{i+1}-c_i} \begin{vmatrix}
0 & 1 & 1 & \cdots & 1 & 1 \\
1 & -1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
n-2 & -1 & -1 & \cdots & -1 & 1 \\
n-1 & -1 & -1 & \cdots & -1 & -1
\end{vmatrix} \\
&\xlongequal[i=2,\cdots,n]{r_i+r_1} \begin{vmatrix}
0 & 1 & 1 & \cdots & 1 & 1 \\
1 & 0 & 2 & \cdots & 2 & 2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
n-2 & 0 & 0 & \cdots & 0 & 2 \\
n-1 & 0 & 0 & \cdots & 0 & 0
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\end{aligned}
$$

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$$D_n = \begin{vmatrix} 0 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 1 & \cdots & n-3 & n-2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & n-3 & n-4 & \cdots & 0 & 1 \\ n-1 & n-2 & n-3 & \cdots & 1 & 0 \end{vmatrix}$$

$$\xlongequal[i=n-1,\cdots,1]{c_{i+1}-c_i} \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & -1 & -1 & \cdots & -1 & 1 \\ n-1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

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计算元素为$a_{ij} = |i-j|$的n阶行列式

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$$\xlongequal[i=2,\cdots,n]{r_i+r_1} \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 2 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & 0 & 0 & \cdots & 0 & 2 \\ n-1 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix} = (-1)^{n-1}(n-1)2^{n-2}.$$

例6

计算元素为$a_{ij}=|i-j|$的n阶行列式

解:

$$D_n = \begin{vmatrix} 0 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 1 & \cdots & n-3 & n-2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & n-3 & n-4 & \cdots & 0 & 1 \\ n-1 & n-2 & n-3 & \cdots & 1 & 0 \end{vmatrix}$$

$$\xlongequal[i=n-1,\cdots,1]{c_{i+1}-c_i} \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & -1 & -1 & \cdots & -1 & 1 \\ n-1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{r_i+r_1} \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 2 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & 0 & 0 & \cdots & 0 & 2 \\ n-1 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix} = (-1)^{n-1}(n-1)2^{n-2}.$$

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计算元素为$a_{ij}=|i-j|$的n阶行列式

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$$D_n = \begin{vmatrix} 0 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 1 & \cdots & n-3 & n-2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & n-3 & n-4 & \cdots & 0 & 1 \\ n-1 & n-2 & n-3 & \cdots & 1 & 0 \end{vmatrix}$$

$$\overline{\overline{\underset{i=n-1,\cdots,1}{c_{i+1}-c_i}}} \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & -1 & -1 & \cdots & -1 & 1 \\ n-1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

$$\overline{\overline{\underset{i=2,\cdots,n}{r_i+r_1}}} \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 2 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & 0 & 0 & \cdots & 0 & 2 \\ n-1 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix} = (-1)^{n-1}(n-1)2^{n-2}.$$

例7

计算

$$D=\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 0 & \cdots & 0 \\ 3 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & & & \\ n & 0 & 0 & \cdots & n \end{vmatrix}$$

解

$$D \xlongequal[i=2,\cdots,n]{r_1-r_i} \begin{vmatrix} 1-\sum_{i=2}^{n} i & 0 & 0 & \cdots & 0 \\ 2 & 2 & 0 & \cdots & 0 \\ 3 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ n & 0 & 0 & \cdots & n \end{vmatrix}$$

$$= \left(1-\sum_{i=2}^{n} i\right) \cdot 2 \cdot 3 \cdot \cdots \cdot n$$

$$= \left[2-\frac{(n+1)n}{2}\right] n!$$

解

$$D \xlongequal[i=2,\cdots,n]{r_1-r_i} \begin{vmatrix} 1-\sum_{i=2}^{n} i & 0 & 0 & \cdots & 0 \\ 2 & 2 & 0 & \cdots & 0 \\ 3 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ n & 0 & 0 & \cdots & n \end{vmatrix}$$

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解

$$D \xlongequal[i=2,\cdots,n]{r_1-r_i} \begin{vmatrix} 1-\sum_{i=2}^{n} i & 0 & 0 & \cdots & 0 \\ 2 & 2 & 0 & \cdots & 0 \\ 3 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ n & 0 & 0 & \cdots & n \end{vmatrix}$$

$$= \left(1-\sum_{i=2}^{n} i\right)\cdot 2\cdot 3\cdot\cdots\cdot n$$

$$= \left[2-\frac{(n+1)n}{2}\right] n!$$

解

$$D \xlongequal[i=2,\cdots,n]{r_1-r_i} \begin{vmatrix} 1-\sum_{i=2}^n i & 0 & 0 & \cdots & 0 \\ 2 & 2 & 0 & \cdots & 0 \\ 3 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ n & 0 & 0 & \cdots & n \end{vmatrix}$$

$$= \left(1-\sum_{i=2}^n i\right)\cdot 2\cdot 3\cdot\cdots\cdot n$$

$$= \left[2-\frac{(n+1)n}{2}\right] n!$$

解

$$
\begin{aligned}
D &\xlongequal[i=2,\cdots,n]{r_1-r_i} \begin{vmatrix} 1-\sum_{i=2}^{n} i & 0 & 0 & \cdots & 0 \\ 2 & 2 & 0 & \cdots & 0 \\ 3 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ n & 0 & 0 & \cdots & n \end{vmatrix} \\
&= \left(1-\sum_{i=2}^{n} i\right)\cdot 2\cdot 3\cdot\cdots\cdot n \\
&= \left[2-\frac{(n+1)n}{2}\right] n!
\end{aligned}
$$

如何计算“爪形”行列式

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & 0 & 0 & \cdots & a_{nn} \end{vmatrix}$$

其解法固定，即从第二行开始，每行依次乘一个系数然后加到第一行，使得第一行除第一个元素外都为零，从而得到一个下三角行列式。

如何计算“爪形”行列式

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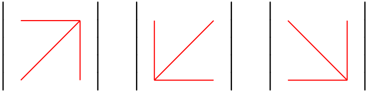
计算行列式（假定$a_i \neq 0$）

$$D_{n+1} = \begin{vmatrix} a_0 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & 0 & \cdots & 0 \\ 1 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & & \\ 1 & 0 & 0 & \cdots & a_n \end{vmatrix} = (a_0 - \sum_{i=1}^{n} \frac{1}{a_i}) a_1 a_2 \cdots a_n.$$

计算行列式（假定$a_i \neq 0$）

$$D_{n+1} = \begin{vmatrix} a_0 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & 0 & \cdots & 0 \\ 1 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & & \\ 1 & 0 & 0 & \cdots & a_n \end{vmatrix} = (a_0 - \sum_{i=1}^{n} \frac{1}{a_i}) a_1 a_2 \cdots a_n.$$

类似的方式还可用于求解如下形式的“爪型行列式”



(a) (b) (c)

例8

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 2 & 1 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & n-1 & \cdots & 0 & 1 \\ n & 0 & \cdots & 0 & 1 \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} n! \left(1 - \sum_{i=2}^{n} \frac{1}{i}\right)$$

例8

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 2 & 1 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & n-1 & \cdots & 0 & 1 \\ n & 0 & \cdots & 0 & 1 \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} n! \left(1 - \sum_{i=2}^{n} \frac{1}{i}\right)$$

例9

计算n阶行列式

$$D_n = \begin{vmatrix} x & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & x \end{vmatrix}$$

方法一

$$D_n \xlongequal{c_1+c_2+\cdots+c_n} \begin{vmatrix} x+(n-1)a & a & \cdots & a \\ x+(n-1)a & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ x+(n-1)a & a & \cdots & x \end{vmatrix}$$

$$\xlongequal{c_1\div[x+(n-1)a]} [x+(n-1)a] \begin{vmatrix} 1 & a & \cdots & a \\ 1 & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ 1 & a & \cdots & x \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{r_i-r_1} [x+(n-1)a] \begin{vmatrix} 1 & a & \cdots & 0 \\ 0 & x-a & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x-a \end{vmatrix}$$

$$= [x+(n-1)a](x-a)^{n-1}$$

方法一

$$D_n \xlongequal{c_1+c_2+\cdots+c_n} \begin{vmatrix} x+(n-1)a & a & \cdots & a \\ x+(n-1)a & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ x+(n-1)a & a & \cdots & x \end{vmatrix}$$

$$\xlongequal{c_1\div[x+(n-1)a]} [x+(n-1)a]\begin{vmatrix} 1 & a & \cdots & a \\ 1 & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ 1 & a & \cdots & x \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{r_i-r_1} [x+(n-1)a]\begin{vmatrix} 1 & a & \cdots & 0 \\ 0 & x-a & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x-a \end{vmatrix}$$

$$= [x+(n-1)a](x-a)^{n-1}$$

方法一

$$D_n \xlongequal{c_1+c_2+\cdots+c_n} \begin{vmatrix} x+(n-1)a & a & \cdots & a \\ x+(n-1)a & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ x+(n-1)a & a & \cdots & x \end{vmatrix}$$

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$$\xlongequal[i=2,\cdots,n]{r_i-r_1} [x+(n-1)a] \begin{vmatrix} 1 & a & \cdots & 0 \\ 0 & x-a & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x-a \end{vmatrix}$$

$$= [x+(n-1)a]\,(x-a)^{n-1}$$

方法一

$$
\begin{aligned}
D_n &\xlongequal{c_1+c_2+\cdots+c_n} \begin{vmatrix} x+(n-1)a & a & \cdots & a \\ x+(n-1)a & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ x+(n-1)a & a & \cdots & x \end{vmatrix} \\
&\xlongequal{c_1\div[x+(n-1)a]} [x+(n-1)a]\begin{vmatrix} 1 & a & \cdots & a \\ 1 & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ 1 & a & \cdots & x \end{vmatrix} \\
&\xlongequal[i=2,\cdots,n]{r_i-r_1} [x+(n-1)a]\begin{vmatrix} 1 & a & \cdots & 0 \\ 0 & x-a & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x-a \end{vmatrix} \\
&= [x+(n-1)a]\,(x-a)^{n-1}
\end{aligned}
$$

方法一

$$D_n \xlongequal{c_1+c_2+\cdots+c_n} \begin{vmatrix} x+(n-1)a & a & \cdots & a \\ x+(n-1)a & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ x+(n-1)a & a & \cdots & x \end{vmatrix}$$

$$\xlongequal{c_1 \div [x+(n-1)a]} [x+(n-1)a] \begin{vmatrix} 1 & a & \cdots & a \\ 1 & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ 1 & a & \cdots & x \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{r_i - r_1} [x+(n-1)a] \begin{vmatrix} 1 & a & \cdots & 0 \\ 0 & x-a & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x-a \end{vmatrix}$$

$$= [x+(n-1)a](x-a)^{n-1}$$

方法一

$$D_n \xlongequal{c_1+c_2+\cdots+c_n} \begin{vmatrix} x+(n-1)a & a & \cdots & a \\ x+(n-1)a & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ x+(n-1)a & a & \cdots & x \end{vmatrix}$$

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$$\xlongequal{c_1 \div [x+(n-1)a]} [x+(n-1)a] \begin{vmatrix} 1 & a & \cdots & a \\ 1 & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ 1 & a & \cdots & x \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{r_i - r_1} [x+(n-1)a] \begin{vmatrix} 1 & a & \cdots & 0 \\ 0 & x-a & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x-a \end{vmatrix}$$

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$$D_n \xlongequal{c_1+c_2+\cdots+c_n} \begin{vmatrix} x+(n-1)a & a & \cdots & a \\ x+(n-1)a & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ x+(n-1)a & a & \cdots & x \end{vmatrix}$$

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$$\xlongequal[i=2,\cdots,n]{r_i - r_1} [x+(n-1)a] \begin{vmatrix} 1 & a & \cdots & 0 \\ 0 & x-a & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x-a \end{vmatrix}$$

$$= [x+(n-1)a]\,(x-a)^{n-1}$$

方法二

$$D_n \xlongequal[i=2,\cdots,n]{r_i-r_1} \begin{vmatrix} x & a & a & \cdots & a \\ a-x & x-a & 0 & \cdots & 0 \\ a-x & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a-x & 0 & 0 & \cdots & x-a \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_1+c_i} \begin{vmatrix} x+(n-1)a & a & a & \cdots & a \\ 0 & x-a & 0 & \cdots & 0 \\ 0 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x-a \end{vmatrix}$$

$$= [x+(n-1)a](x-a)^{n-1}.$$

方法二

$$D_n \xlongequal[i=2,\cdots,n]{r_i - r_1} \begin{vmatrix} x & a & a & \cdots & a \\ a-x & x-a & 0 & \cdots & 0 \\ a-x & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a-x & 0 & 0 & \cdots & x-a \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_1 + c_i} \begin{vmatrix} x+(n-1)a & a & a & \cdots & a \\ 0 & x-a & 0 & \cdots & 0 \\ 0 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x-a \end{vmatrix}$$

$$= [x+(n-1)a](x-a)^{n-1}.$$

方法二

$$D_n \xlongequal[i=2,\cdots,n]{r_i - r_1} \begin{vmatrix} x & a & a & \cdots & a \\ a-x & x-a & 0 & \cdots & 0 \\ a-x & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a-x & 0 & 0 & \cdots & x-a \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_1 + c_i} \begin{vmatrix} x+(n-1)a & a & a & \cdots & a \\ 0 & x-a & 0 & \cdots & 0 \\ 0 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x-a \end{vmatrix}$$

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方法二

$$D_n \xlongequal[i=2,\cdots,n]{r_i - r_1} \begin{vmatrix} x & a & a & \cdots & a \\ a-x & x-a & 0 & \cdots & 0 \\ a-x & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a-x & 0 & 0 & \cdots & x-a \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_1 + c_i} \begin{vmatrix} x+(n-1)a & a & a & \cdots & a \\ 0 & x-a & 0 & \cdots & 0 \\ 0 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x-a \end{vmatrix}$$

$$= [x+(n-1)a](x-a)^{n-1}.$$

方法二

$$D_n \xlongequal[i=2,\cdots,n]{r_i-r_1} \begin{vmatrix} x & a & a & \cdots & a \\ a-x & x-a & 0 & \cdots & 0 \\ a-x & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a-x & 0 & 0 & \cdots & x-a \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_1+c_i} \begin{vmatrix} x+(n-1)a & a & a & \cdots & a \\ 0 & x-a & 0 & \cdots & 0 \\ 0 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x-a \end{vmatrix}$$

$$= [x+(n-1)a](x-a)^{n-1}.$$

方法二

$$
D_n \xlongequal[i=2,\cdots,n]{r_i - r_1} \begin{vmatrix} x & a & a & \cdots & a \\ a-x & x-a & 0 & \cdots & 0 \\ a-x & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a-x & 0 & 0 & \cdots & x-a \end{vmatrix}
$$

$$
\xlongequal[i=2,\cdots,n]{c_1 + c_i} \begin{vmatrix} x+(n-1)a & a & a & \cdots & a \\ 0 & x-a & 0 & \cdots & 0 \\ 0 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x-a \end{vmatrix}
$$

$$
= [x+(n-1)a](x-a)^{n-1}.
$$

方法二

$$D_n \xlongequal[i=2,\cdots,n]{r_i-r_1} \begin{vmatrix} x & a & a & \cdots & a \\ a-x & x-a & 0 & \cdots & 0 \\ a-x & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a-x & 0 & 0 & \cdots & x-a \end{vmatrix}$$

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$$= [x+(n-1)a](x-a)^{n-1}.$$

方法三（升阶法）

$$D_n = \begin{vmatrix} 1 & a & a & \cdots & a \\ 0 & x & a & \cdots & a \\ 0 & a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a & a & \cdots & x \end{vmatrix}_{n+1} \xlongequal[i=2,\cdots,n+1]{r_i - r_1} \begin{vmatrix} 1 & a & a & \cdots & a \\ -1 & x-a & 0 & \cdots & 0 \\ -1 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & x-a \end{vmatrix}_{n+1}$$

- 若$x = a$，则$D_n = 0$。
- 若$x \neq a$，则

$$D_n \xlongequal[j=2,\cdots,n+1]{c_1 + \frac{1}{x-a} c_j} \begin{vmatrix} 1 + \frac{a}{x-a} n & a & a & \cdots & a \\ 0 & x-a & 0 & \cdots & 0 \\ 0 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x-a \end{vmatrix}_{n+1}$$

$$= [x + (n-1)a](x-a)^{n-1}.$$

方法三（升阶法）

$$D_n=\begin{vmatrix}1&a&a&\cdots&a\\0&x&a&\cdots&a\\0&a&x&\cdots&a\\\vdots&\vdots&\vdots&&\vdots\\0&a&a&\cdots&x\end{vmatrix}_{n+1}\xlongequal[i=2,\cdots,n+1]{r_i-r_1}\begin{vmatrix}1&a&a&\cdots&a\\-1&x-a&0&\cdots&0\\-1&0&x-a&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\-1&0&0&\cdots&x-a\end{vmatrix}_{n+1}$$

- 若$x=a$，则$D_n=0$。
- 若$x\neq a$，则

$$D_n\xlongequal[j=2,\cdots,n+1]{c_1+\frac{1}{x-a}c_j}\begin{vmatrix}1+\frac{a}{x-a}n&a&a&\cdots&a\\0&x-a&0&\cdots&0\\0&0&x-a&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&x-a\end{vmatrix}_{n+1}$$

$$=[x+(n-1)a](x-a)^{n-1}.$$

方法三（升阶法）

$$D_n = \begin{vmatrix} 1 & a & a & \cdots & a \\ 0 & x & a & \cdots & a \\ 0 & a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a & a & \cdots & x \end{vmatrix}_{n+1} \xlongequal[i=2,\cdots,n+1]{r_i - r_1} \begin{vmatrix} 1 & a & a & \cdots & a \\ -1 & x-a & 0 & \cdots & 0 \\ -1 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & x-a \end{vmatrix}_{n+1}$$

- 若$x = a$，则$D_n = 0$。
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$$D_n \xlongequal[j=2,\cdots,n+1]{c_1 + \frac{1}{x-a} c_j} \begin{vmatrix} 1+\frac{a}{x-a}n & a & a & \cdots & a \\ 0 & x-a & 0 & \cdots & 0 \\ 0 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x-a \end{vmatrix}_{n+1}$$

$$= [x+(n-1)a](x-a)^{n-1}.$$

方法三（升阶法）

$$D_n = \begin{vmatrix} 1 & a & a & \cdots & a \\ 0 & x & a & \cdots & a \\ 0 & a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a & a & \cdots & x \end{vmatrix}_{n+1} \xlongequal[i=2,\cdots,n+1]{r_i - r_1} \begin{vmatrix} 1 & a & a & \cdots & a \\ -1 & x-a & 0 & \cdots & 0 \\ -1 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & x-a \end{vmatrix}_{n+1}$$

- 若$x = a$，则$D_n = 0$。
- 若$x \neq a$，则

$$D_n \xlongequal[j=2,\cdots,n+1]{c_1 + \frac{1}{x-a} c_j} \begin{vmatrix} 1+\frac{a}{x-a}n & a & a & \cdots & a \\ 0 & x-a & 0 & \cdots & 0 \\ 0 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x-a \end{vmatrix}_{n+1}$$

$$= [x+(n-1)a](x-a)^{n-1}.$$

方法三（升阶法）

$$D_n = \begin{vmatrix} 1 & a & a & \cdots & a \\ 0 & x & a & \cdots & a \\ 0 & a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a & a & \cdots & x \end{vmatrix}_{n+1} \xlongequal[i=2,\cdots,n+1]{r_i - r_1} \begin{vmatrix} 1 & a & a & \cdots & a \\ -1 & x-a & 0 & \cdots & 0 \\ -1 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & x-a \end{vmatrix}_{n+1}$$

- 若 $x = a$，则 $D_n = 0$。
- 若 $x \neq a$，则

$$D_n \xlongequal[j=2,\cdots,n+1]{c_1 + \frac{1}{x-a} c_j} \begin{vmatrix} 1 + \frac{a}{x-a} n & a & a & \cdots & a \\ 0 & x-a & 0 & \cdots & 0 \\ 0 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x-a \end{vmatrix}_{n+1}$$

$$= [x + (n-1)a](x-a)^{n-1}.$$

方法三（升阶法）

$$D_n = \begin{vmatrix} 1 & a & a & \cdots & a \\ 0 & x & a & \cdots & a \\ 0 & a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a & a & \cdots & x \end{vmatrix}_{n+1} \xlongequal[i=2,\cdots,n+1]{r_i - r_1} \begin{vmatrix} 1 & a & a & \cdots & a \\ -1 & x-a & 0 & \cdots & 0 \\ -1 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & x-a \end{vmatrix}_{n+1}$$

- 若$x = a$，则$D_n = 0$。
- 若$x \neq a$，则

$$D_n \xlongequal[j=2,\cdots,n+1]{c_1 + \frac{1}{x-a} c_j} \begin{vmatrix} 1 + \frac{a}{x-a} n & a & a & \cdots & a \\ 0 & x-a & 0 & \cdots & 0 \\ 0 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x-a \end{vmatrix}_{n+1}$$

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$$= [x+(n-1)a](x-a)^{n-1}.$$

方法三（升阶法）

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$$= \quad [x+(n-1)a](x-a)^{n-1}.$$

方法三（升阶法）

$$D_n = \begin{vmatrix} 1 & a & a & \cdots & a \\ 0 & x & a & \cdots & a \\ 0 & a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a & a & \cdots & x \end{vmatrix}_{n+1} \xlongequal[i=2,\cdots,n+1]{r_i - r_1} \begin{vmatrix} 1 & a & a & \cdots & a \\ -1 & x-a & 0 & \cdots & 0 \\ -1 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & x-a \end{vmatrix}_{n+1}$$

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$$= [x + (n-1)a] \, (x-a)^{n-1}.$$

方法四

$$D_n = \begin{vmatrix} x-a & a & \cdots & a \\ 0 & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ 0 & a & \cdots & x \end{vmatrix} + \begin{vmatrix} a & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & x \end{vmatrix}$$

$$= (x-a)D_{n-1} + a(x-a)^{n-1}.$$

于是

$$\begin{cases} D_n = (x-a)D_{n-1} + a(x-a)^{n-1} \\ (x-a)D_{n-1} = (x-a)^2 D_{n-2} + a(x-a)^{n-1} \\ \cdots \\ (x-a)^{n-4}D_4 = (x-a)^{n-3}D_3 + a(x-a)^{n-1} \\ (x-a)^{n-3}D_3 = (x-a)^{n-2}D_2 + a(x-a)^{n-1} \end{cases}$$

因此

$$D_n = (x-a)^{n-2}(x^2-a^2) + (n-2)a(x-a)^{n-1} = [x+(n-1)a](x-a)^{n-1}$$

方法四

$$D_n = \begin{vmatrix} x-a & a & \cdots & a \\ 0 & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ 0 & a & \cdots & x \end{vmatrix} + \begin{vmatrix} a & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & x \end{vmatrix}$$

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方法四

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\begin{aligned}
D_n &= \begin{vmatrix} x-a & a & \cdots & a \\ 0 & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ 0 & a & \cdots & x \end{vmatrix} + \begin{vmatrix} a & a & \cdots & a \\ a & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & x \end{vmatrix} \\
&= (x-a)D_{n-1} + a(x-a)^{n-1}.
\end{aligned}
$$

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$$
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\cdots & \\
(x-a)^{n-4}D_4 &= (x-a)^{n-3}D_3 + a(x-a)^{n-1} \\
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方法四

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方法四

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该行列式经常以不同方式出现，如

- $$\begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 0 \end{vmatrix} = (-1)^{n-1}(n-1)$$

- $$\begin{vmatrix} 1 & a & \cdots & a \\ a & 1 & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & 1 \end{vmatrix} = [1+(n-1)a](1-a)^{n-1}$$

- $$\begin{vmatrix} 1+\lambda & 1 & \cdots & 1 \\ 1 & 1+\lambda & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1+\lambda \end{vmatrix} = (\lambda+n)\lambda^{n-1}$$

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升阶法适用于求形如

$$\begin{vmatrix} x_1 & a & \cdots & a \\ a & x_2 & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & x_n \end{vmatrix}$$

或

$$\begin{vmatrix} x_1 & a_1 & \cdots & a_n \\ a_1 & x_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & x_n \end{vmatrix}$$

的行列式。

$$\begin{vmatrix} x_1 & a & \cdots & a \\ a & x_2 & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & x_n \end{vmatrix} = (1 + \sum_{i=1}^{n} \frac{a}{x_i - a}) \prod_{i=1}^{n} (x_i - a)$$

$$\begin{vmatrix} x_1 & a_1 & \cdots & a_n \\ a_1 & x_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & x_n \end{vmatrix} = (1 + \sum_{i=1}^{n} \frac{a_i}{x_i - a_i}) \prod_{i=1}^{n} (x_i - a_i)$$

常见形式

$$\begin{vmatrix} 1+a & 1 & \cdots & 1 \\ 2 & 2+a & \cdots & 2 \\ \vdots & \vdots & & \vdots \\ n & n & \cdots & n+a \end{vmatrix} = \left[a + \frac{n(n+1)}{2}\right] a^{n-1}$$

或

$$\begin{vmatrix} a_1+b & a_1 & \cdots & a_n \\ a_1 & a_2+b & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n+b \end{vmatrix} = b^{n-1}\left(\sum_{i=1}^{n} a_i + b\right)$$

常见形式

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$$\begin{vmatrix} a_1+b & a_1 & \cdots & a_n \\ a_1 & a_2+b & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n+b \end{vmatrix} = b^{n-1}(\sum_{i=1}^{n} a_i + b)$$

例10

设

$$
\begin{aligned}
D &= \begin{vmatrix} a_{11} & \cdots & a_{1k} & & & \\ \vdots & & \vdots & & & \\ a_{k1} & \cdots & a_{kk} & & & \\ c_{11} & \cdots & c_{1k} & b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{n1} & \cdots & c_{nk} & b_{n1} & \cdots & b_{nn} \end{vmatrix}, \\
D_1 &= det(a_{ij}) = \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix}, \\
D_2 &= det(b_{ij}) = \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{vmatrix}.
\end{aligned}
$$

证明：$D = D_1 D_2$

证明： 对D_1做运算$r_i+\lambda r_j$将它转化成下三角行列式，设为

$$D_1=\begin{vmatrix} p_{11} & & \\ \vdots & \ddots & \\ p_{k1} & \cdots & p_{kk} \end{vmatrix}=p_{11}\cdots p_{kk}.$$

对D_2做运算$c_i+\lambda c_j$将它转化成下三角行列式，设为

$$D_2=\begin{vmatrix} q_{11} & \cdots & q_{1n} \\ & \ddots & \vdots \\ & & q_{nn} \end{vmatrix}=q_{11}\cdots q_{nn}.$$

于是，对D的前k行做运算$r_i+\lambda r_j$，对其后n列做运算$c_i+\lambda c_j$，把D转化为

$$D=\begin{vmatrix} p_{11} & & & & & \\ \vdots & \ddots & & & & \\ p_{k1} & \cdots & p_{kk} & & & \\ c_{11} & \cdots & c_{1k} & q_{11} & & \\ \vdots & & \vdots & \vdots & \ddots & \\ c_{n1} & \cdots & c_{nk} & q_{n1} & \cdots & q_{nn} \end{vmatrix}$$

故$D=p_{11}\cdots p_{kk}q_{11}\cdots q_{nn}=D_1D_2.$

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故$D=p_{11}\cdots p_{kk}q_{11}\cdots q_{nn}=D_1D_2$.

例11

计算$2n$阶行列式

$$D_{2n} = \begin{vmatrix} a & & & & & b \\ & \ddots & & & \iddots & \\ & & a & b & & \\ & & c & d & & \\ & \iddots & & & \ddots & \\ c & & & & & d \end{vmatrix}$$

解： 把D_{2n}中的第$2n$行依次与第$2n-1$行、...、第2行对调（共$2n-2$次相邻对换），在把第$2n$列依次与第$2n-1$列、...、第2列对调，得

$$D_{2n} = \begin{vmatrix} a & b & 0 & & & & & 0 \\ c & d & 0 & & & & & \\ 0 & 0 & a & & & & & b \\ & & & \ddots & & & \iddots & \\ & & & & a & b & & \\ & & & & c & d & & \\ & & & \iddots & & & \ddots & \\ 0 & 0 & c & & & & & d \end{vmatrix}$$

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$$D_{2n}=\begin{vmatrix} a & b & 0 & & & & & 0 \\ c & d & 0 & & & & & \\ 0 & 0 & a & & & & & b \\ & & & \ddots & & & \iddots & \\ & & & & a & b & & \\ & & & & c & d & & \\ & & & \iddots & & & \ddots & \\ 0 & 0 & c & & & & & d \end{vmatrix}$$

故

$$\begin{aligned} D_{2n} &= D_2 D_{2(n-1)} \\ &= (ad-bc)D_{2(n-1)} \\ &= (ad-bc)^2 D_{2(n-2)} \\ &= \cdots \\ &= (ad-bc)^{n-1} D_2 \\ &= (ad-bc)^n. \end{aligned}$$

例12

证明范德蒙德(Vandermonde)行列式

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{n \geq i > j \geq 1} (x_i - x_j).$$

证明：用数学归纳法证明。当$n = 2$时，

$$D_2 = \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1 = \prod_{2 \geq i > j \geq 1} (x_i - x_j),$$

结论成立。

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证明：（续）

现假设结论对$n-1$阶范德蒙德行列式成立，以下证明结论对n阶范德蒙德行列式也成立。

$$D_n \xlongequal[i=n,\cdots,2]{r_i - x_1 r_{i-1}} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\ 0 & x_2(x_2 - x_1) & x_3(x_3 - x_1) & \cdots & x_n(x_n - x_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & x_2^{n-2}(x_2 - x_1) & x_3^{n-2}(x_3 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$

按第1列展开，并把每列的公因子$(x_i - x_1)$提出，就有

$$D_n = (x_2 - x_1)(x_3 - x_1)\cdots(x_n - x_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \end{vmatrix}$$

上式右端的行列式为$n-1$阶范德蒙德行列式，按归纳法假设，它等于所有$(x_i - x_j)$因子的乘积$(n \geq i \geq j \geq 2)$。故

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例13

设a, b, c为互不相同的实数，证明：

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = 0$$

的充要条件是$a+b+c=0$.

解：考察范德蒙德行列式

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & y \\ a^2 & b^2 & c^2 & y^2 \\ a^3 & b^3 & c^3 & y^3 \end{vmatrix} = (a-b)(a-c)(b-c)(a-y)(b-y)(c-y)$$

注意到行列式D可看成是关于y的多项式，比较包含y^2的项：

$$\cdots - \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} y^2 + \cdots = \cdots - (a-b)(a-c)(b-c)(a+b+c)y^2 + \cdots$$

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$$(a-b)(a-c)(b-c)(a+b+c)=\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}=0$$

而a,b,c互不相同，故$a+b+c=0$.

例14

计算三对角行列式

$$D_n = \begin{vmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \\ & & & & c & a \end{vmatrix}$$

解 对D_n按第一行展开

$$D_n = aD_{n-1} + (-1)^{1+2} b \begin{vmatrix} c & b & & & & \\ 0 & a & b & & & \\ 0 & c & a & b & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & c & a & b \\ 0 & 0 & \cdots & 0 & c & a \end{vmatrix} = aD_{n-1} - bcD_{n-2}$$

其中

$$D_1 = a, \quad D_2 = a^2 - bc$$

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$$D_n = aD_{n-1} - bcD_{n-2}$$

改写成

$$D_n - kD_{n-1} = l(D_{n-1} - kD_{n-2})$$

这里

$$k + l = a, \quad kl = bc.$$

令$\Delta_n = D_n - kD_{n-1}$，它满足

$$\begin{cases} \Delta_n = l\Delta_{n-1}, \\ \Delta_2 = D_2 - kD_1 = a^2 - bc - ka = (a-k)a - kl = la - lk = l^2. \end{cases}$$

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$$D_n = aD_{n-1} - bcD_{n-2}$$

改写成

$$D_n - kD_{n-1} = l(D_{n-1} - kD_{n-2})$$

这里

$$k + l = a, \quad kl = bc.$$

令$\Delta_n = D_n - kD_{n-1}$，它满足

$$\begin{cases} \Delta_n = l\Delta_{n-1}, \\ \Delta_2 = D_2 - kD_1 = a^2 - bc - ka = (a-k)a - kl = la - lk = l^2. \end{cases}$$

由此可知

$$\Delta_n = l^{n-2}\Delta_2 = l^2,$$

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考察n元一次方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n. \end{cases} \tag{6}$$

与二、三元线性方程组相类似，它的解可以用n阶行列式表示。

克莱姆法则

如果线性方程组(6)的系数行列式不等于0，即

$$D = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

则方程组(6)存在唯一解

$$x_1 = \frac{D_1}{D}, \ x_2 = \frac{D_2}{D}, \ \cdots, \ x_n = \frac{D_n}{D},$$

其中

$$D_j = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

第j列

证明： 先证存在性：将$x_i = \frac{D_i}{D}$代入第i个方程，则有

$$
\begin{aligned}
& a_{i1}x_1 + \cdots + a_{ii}x_i + \cdots + a_{in}x_n \\
= & \frac{1}{D}(a_{i1}D_1 + \cdots + a_{ii}D_i + \cdots + a_{in}D_n) \\
= & \frac{1}{D}[a_{i1}(b_1A_{11} + \cdots + b_nA_{n1}) + \cdots + a_{ii}(b_1A_{1i} + \cdots + b_nA_{ni}) \\
& + \cdots + a_{in}(b_1A_{1n} + \cdots + b_nA_{nn})] \\
= & \frac{1}{D}[b_1(a_{i1}A_{11} + a_{i2}A_{12} \cdots + a_{in}A_{1n}) + \cdots + b_i(a_{i1}A_{i1} + a_{i2}A_{i2} \cdots + a_{in}A_{in}) \\
& + \cdots + b_n(a_{i1}A_{n1} + a_{i2}A_{n2} \cdots + a_{in}A_{nn})] \\
= & \frac{1}{D}b_iD = b_i.
\end{aligned}
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$$a_{i1}x_1 + \cdots + a_{ii}x_i + \cdots + a_{in}x_n$$

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$$= \frac{1}{D}[a_{i1}(b_1A_{11} + \cdots + b_nA_{n1}) + \cdots + a_{ii}(b_1A_{1i} + \cdots + b_nA_{ni}) + \cdots + a_{in}(b_1A_{1n} + \cdots + b_nA_{nn})]$$

$$= \frac{1}{D}[b_1(a_{i1}A_{11} + a_{i2}A_{12} \cdots + a_{in}A_{1n}) + \cdots + b_i(a_{i1}A_{i1} + a_{i2}A_{i2} \cdots + a_{in}A_{in}) + \cdots + b_n(a_{i1}A_{n1} + a_{i2}A_{n2} \cdots + a_{in}A_{nn})]$$

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证明：　[续]

再证唯一性:设还有一组解$y_i, i=1,2,\cdots,n$，以下证明$y_i = D_i/D$。现构造一个新行列式

$$y_1 D = \begin{vmatrix} a_{11}y_1 & a_{12} & \cdots & a_{1n} \\ a_{21}y_1 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1}y_1 & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

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$$= \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix} = D_1$$

所以$y_1 = D_1/D$。同理可证$y_i = D_i/D, i = 2,\cdots,n$。

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再证唯一性:设还有一组解$y_i, i=1,2,\cdots,n$，以下证明$y_i = D_i/D$。现构造一个新行列式

$$
y_1 D = \begin{vmatrix} a_{11}y_1 & a_{12} & \cdots & a_{1n} \\ a_{21}y_1 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1}y_1 & a_{n2} & \cdots & a_{nn} \end{vmatrix}
$$

$$
\xlongequal{c_1+y_2c_2+\cdots+y_nc_n} \begin{vmatrix} \sum_{k=1}^{n} a_{1k}y_k & a_{12} & \cdots & a_{1n} \\ \sum_{k=1}^{n} a_{2k}y_k & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \sum_{k=1}^{n} a_{2k}y_k & a_{n2} & \cdots & a_{nn} \end{vmatrix}
$$

$$
= \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix} = D_1
$$

所以$y_1 = D_1/D$。同理可证$y_i = D_i/D, i=2,\cdots,n$。

证明： [续]

再证唯一性:设还有一组解$y_i, i = 1, 2, \cdots, n$，以下证明$y_i = D_i/D$。现构造一个新行列式

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$$\xlongequal{c_1 + y_2 c_2 + \cdots + y_n c_n} \begin{vmatrix} \sum_{k=1}^{n} a_{1k}y_k & a_{12} & \cdots & a_{1n} \\ \sum_{k=1}^{n} a_{2k}y_k & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \sum_{k=1}^{n} a_{2k}y_k & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

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\begin{aligned}
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&= \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix} = D_1
\end{aligned}
$$

所以$y_1 = D_1/D$。同理可证$y_i = D_i/D, i = 2, \cdots, n$。

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再证唯一性:设还有一组解$y_i, i = 1, 2, \cdots, n$，以下证明$y_i = D_i/D$。现构造一个新行列式

$$
\begin{aligned}
y_1 D &= \begin{vmatrix} a_{11}y_1 & a_{12} & \cdots & a_{1n} \\ a_{21}y_1 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1}y_1 & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\
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&= \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix} = D_1
\end{aligned}
$$

所以$y_1 = D_1/D$。同理可证$y_i = D_i/D, i = 2, \cdots, n$。

证明： [续]

再证唯一性:设还有一组解$y_i, i=1,2,\cdots,n$，以下证明$y_i = D_i/D$。现构造一个新行列式

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&\xlongequal{c_1+y_2c_2+\cdots+y_nc_n} \begin{vmatrix} \sum_{k=1}^n a_{1k}y_k & a_{12} & \cdots & a_{1n} \\ \sum_{k=1}^n a_{2k}y_k & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \sum_{k=1}^n a_{2k}y_k & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\
&= \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix} = D_1
\end{aligned}
$$

所以$y_1 = D_1/D$。同理可证$y_i = D_i/D, i=2,\cdots,n$。

例

$$\begin{cases} 2x_1 + x_2 - 5x_3 + x_4 = 8, \\ x_1 - 3x_2 - 6x_4 = 9, \\ x_2 - x_3 + 2x_4 = -5, \\ x_1 + 4x_2 - 7x_3 + 6x_4 = 0. \end{cases}$$

解：

$$D = \begin{vmatrix} 2 & 1 & -5 & 1 \\ 1 & -3 & 0 & -6 \\ 0 & 2 & -1 & 2 \\ 1 & 4 & -7 & 6 \end{vmatrix} \xlongequal[r_4-r_2]{r_1-2r_2} \begin{vmatrix} 0 & 7 & -5 & 13 \\ 1 & -3 & 0 & -6 \\ 0 & 2 & -1 & 2 \\ 0 & 7 & -7 & 12 \end{vmatrix}$$

$$= -\begin{vmatrix} 7 & -5 & 13 \\ 2 & -1 & 2 \\ 7 & -7 & 12 \end{vmatrix} \xlongequal[c_3+2c_2]{c_1+2c_2} -\begin{vmatrix} -3 & -5 & 3 \\ 0 & -1 & 0 \\ -7 & -7 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -3 & 3 \\ -7 & -2 \end{vmatrix} = 27.$$

例

$$
\begin{cases}
2x_1 + x_2 - 5x_3 + x_4 = 8, \\
x_1 - 3x_2 - 6x_4 = 9, \\
x_2 - x_3 + 2x_4 = -5, \\
x_1 + 4x_2 - 7x_3 + 6x_4 = 0.
\end{cases}
$$

解：

$$
D = \begin{vmatrix} 2 & 1 & -5 & 1 \\ 1 & -3 & 0 & -6 \\ 0 & 2 & -1 & 2 \\ 1 & 4 & -7 & 6 \end{vmatrix} \xlongequal[r_4 - r_2]{r_1 - 2r_2} \begin{vmatrix} 0 & 7 & -5 & 13 \\ 1 & -3 & 0 & -6 \\ 0 & 2 & -1 & 2 \\ 0 & 7 & -7 & 12 \end{vmatrix}
$$

$$
= -\begin{vmatrix} 7 & -5 & 13 \\ 2 & -1 & 2 \\ 7 & -7 & 12 \end{vmatrix} \xlongequal[c_3 + 2c_2]{c_1 + 2c_2} -\begin{vmatrix} -3 & -5 & 3 \\ 0 & -1 & 0 \\ -7 & -7 & -2 \end{vmatrix}
$$

$$
= \begin{vmatrix} -3 & 3 \\ -7 & -2 \end{vmatrix} = 27.
$$

解：(续)

$$D_1 = \begin{vmatrix} 8 & 1 & -5 & 1 \\ 9 & -3 & 0 & -6 \\ -5 & 2 & -1 & 2 \\ 0 & 4 & -7 & 6 \end{vmatrix} = 81, \quad D_2 = \begin{vmatrix} 2 & 8 & -5 & 1 \\ 1 & 9 & 0 & -6 \\ 0 & -5 & -1 & 2 \\ 1 & 0 & -7 & 6 \end{vmatrix} = -108,$$

$$D_3 = \begin{vmatrix} 2 & 1 & 8 & 1 \\ 1 & -3 & 9 & -6 \\ 0 & 2 & -5 & 2 \\ 1 & 4 & 0 & 6 \end{vmatrix} = -27, \quad D_4 = \begin{vmatrix} 2 & 1 & -5 & 8 \\ 1 & -3 & 0 & 9 \\ 0 & 2 & -1 & -5 \\ 1 & 4 & -7 & 0 \end{vmatrix} = 27$$

于是得

$$x_1 = \frac{D_1}{D} = 3, \quad x_2 = \frac{D_2}{D} = -4, \quad x_3 = \frac{D_3}{D} = -1, \quad x_4 = \frac{D_4}{D} = 1.$$

例

设曲线$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$通过四点$(1,3),(2,4),(3,3),(4,-3)$， 求系数$a_0, a_1, a_2, a_3$。

解：依题意可得线性方程组

$$\begin{cases} a_0 + a_1 + a_2 + a_3 = 3, \\ a_0 + 2a_1 + 4a_2 + 8a_3 = 4, \\ a_0 + 3a_1 + 9a_3 + 27a_3 = 3, \\ a_0 + 4a_1 + 16a_4 + 64a_3 = 3, \end{cases}$$

其系数行列式为

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{vmatrix}$$

是一个范德蒙德行列式，其值为

$$D = 1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 1 = 12$$

例

设曲线$y = a_0 + a_1x + a_2x^2 + a_3x^3$通过四点$(1,3),(2,4),(3,3),(4,-3)$， 求系数$a_0, a_1, a_2, a_3$。

解：依题意可得线性方程组

$$\begin{cases} a_0 + \ \ a_1 + \ \ \ a_2 + \ \ \ a_3 = 3, \\ a_0 + 2a_1 + \ 4a_2 + \ 8a_3 = 4, \\ a_0 + 3a_1 + \ 9a_3 + 27a_3 = 3, \\ a_0 + 4a_1 + 16a_4 + 64a_3 = 3, \end{cases}$$

其系数行列式为

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{vmatrix}$$

是一个范德蒙德行列式，其值为

$$D = 1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 1 = 12$$

解：(续)

$$D_1=\begin{vmatrix} 3 & 1 & 1 & 1\\ 4 & 2 & 4 & 8\\ 3 & 3 & 9 & 27\\ -3 & 4 & 16 & 64 \end{vmatrix}=36,\quad D_2=\begin{vmatrix} 1 & 3 & 1 & 1\\ 1 & 4 & 4 & 8\\ 1 & 3 & 8 & 27\\ 1 & -3 & 16 & 64 \end{vmatrix}=-18,$$

$$D_3=\begin{vmatrix} 1 & 1 & 3 & 1\\ 1 & 2 & 4 & 8\\ 1 & 3 & 3 & 27\\ 1 & 4 & -3 & 64 \end{vmatrix}=24,\quad D_4=\begin{vmatrix} 1 & 1 & 1 & 3\\ 1 & 2 & 4 & 4\\ 1 & 3 & 9 & 3\\ 1 & 4 & 16 & -3 \end{vmatrix}=-6.$$

于是得

$$a_0=\frac{D_1}{D}=3,\quad a_1=\frac{D_2}{D}=-3/2,\quad a_2=\frac{D_3}{D}=2,\quad a_3=\frac{D_4}{D}=-1/2.$$

即曲线方程为

$$y=3-\frac{3}{2}x+2x^2-\frac{1}{2}x^3.$$

定理

如果线性方程组(6)的系数行列式$D \neq 0$，则(6)一定有解，且解是惟一的。

定理

如果线性方程组(6)无解或有两个不同的解，则它的系数行列式必为0。

说明：

- 线性方程组(6)右端的常数项$b_1, b_2, \cdots, b_n$不全为0时，线性方程组(6)叫做**非齐次线性方程组**

  当$b_1, b_2, \cdots, b_n$全为0时，线性方程组(6)叫做**齐次线性方程组**。

齐次线性方程组

对于齐次线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0, \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0. \end{cases} \tag{7}$$

一定有零解，但不一定有非零解。

定理

如果齐次线性方程组(7)的系数行列式$D \neq 0$，则它没有非零解。

定理

如果齐次线性方程组(7)有非零解，则它的系数行列式必为0。

当λ为何值时，齐次线性方程组

$$\begin{cases} (5-\lambda)x + 2y + 2z = 0, \\ 2x + (6-\lambda)y = 0, \\ 2x + (4-\lambda)z = 0. \end{cases}$$

有非零解?

解：由上述定理可知，若所给齐次线性方程组有非零解，则其系数行列式$D=0$。而

$$D = \begin{vmatrix} 5-\lambda & 2 & 2 \\ 2 & 6-\lambda & 0 \\ 2 & 0 & 4-\lambda \end{vmatrix}$$

$$= (5-\lambda)(6-\lambda)(4-\lambda) - 4(4-\lambda) - 4(6-\lambda)$$

$$= (5-\lambda)(2-\lambda)(8-\lambda),$$

故$\lambda=2$、$\lambda=5$或$\lambda=8$。

当λ为何值时，齐次线性方程组

$$\begin{cases} (5-\lambda)x & + & 2y & + & 2z & = & 0, \\ 2x & + & (6-\lambda)y & & & = & 0, \\ 2x & & & + & (4-\lambda)z & = & 0. \end{cases}$$

有非零解?

解： 由上述定理可知，若所给齐次线性方程组有非零解，则其系数行列式$D=0$。而

$$D = \begin{vmatrix} 5-\lambda & 2 & 2 \\ 2 & 6-\lambda & 0 \\ 2 & 0 & 4-\lambda \end{vmatrix}$$

$$= (5-\lambda)(6-\lambda)(4-\lambda) - 4(4-\lambda) - 4(6-\lambda)$$

$$= (5-\lambda)(2-\lambda)(8-\lambda),$$

故$\lambda=2$、$\lambda=5$或$\lambda=8$。

1

$$\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$$

解:

$$原式 = a^2b^2 - (ab)(ab) = 0.$$

1

$$\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$$

解:

$$原式 = a^2b^2 - (ab)(ab) = 0.$$

2

$$\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$$

解:

$$原式 = \cos^2\alpha + \sin^2\alpha = 1.$$

2

$$\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$$

解:

$$原式 = \cos^2\alpha + \sin^2\alpha = 1.$$

3

$$\begin{vmatrix} a+bi & b \\ 2a & a-bi \end{vmatrix}$$

解:

$$原式 = (a+bi)(a-bi) - 2ab = a^2 + b^2 - 2ab = (a-b)^2.$$

3

$$\begin{vmatrix} a+bi & b \\ 2a & a-bi \end{vmatrix}$$

解:

$$\text{原式} = (a+bi)(a-bi) - 2ab = a^2 + b^2 - 2ab = (a-b)^2.$$

4

$$\begin{vmatrix} 3 & 2 & -4 \\ 4 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}$$

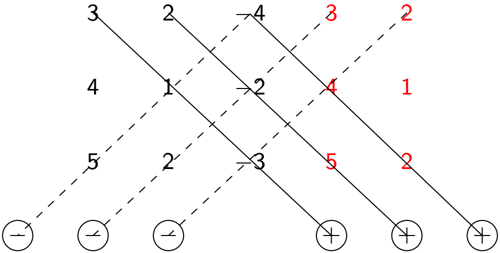
解:

$$\begin{aligned} 原式 &= 3\cdot1\cdot(-3)+2\cdot(-2)\cdot5+(-4)\cdot4\cdot2-(-4)\cdot1\cdot5-3\cdot(-2)\cdot2-2\cdot4\cdot(-3) \\ &= -9-20-32+20+12+24=-5. \end{aligned}$$

4

$$\begin{vmatrix} 3 & 2 & -4 \\ 4 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}$$

解:

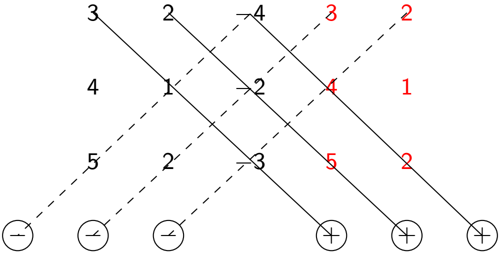


$$\begin{aligned} \text{原式} &= 3\cdot 1\cdot(-3)+2\cdot(-2)\cdot 5+(-4)\cdot 4\cdot 2-(-4)\cdot 1\cdot 5-3\cdot(-2)\cdot 2-2\cdot 4\cdot(-3) \\ &= -9-20-32+20+12+24=-5. \end{aligned}$$

4

$$\begin{vmatrix} 3 & 2 & -4 \\ 4 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}$$

解:



$$\begin{aligned} \text{原式} &= 3\cdot 1\cdot(-3)+2\cdot(-2)\cdot 5+(-4)\cdot 4\cdot 2-(-4)\cdot 1\cdot 5-3\cdot(-2)\cdot 2-2\cdot 4\cdot(-3) \\ &= -9-20-32+20+12+24=-5. \end{aligned}$$

5

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[r_2-r_1]{r_3-r_2} \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} = 0.$$

5

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[r_2-r_1]{r_3-r_2} \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} = 0.$$

6

$$\begin{vmatrix} 2 & 2 & 1 \\ 4 & 1 & -1 \\ 202 & 199 & 101 \end{vmatrix}$$

解:

$$原式 \xlongequal{c_1-c_2} \begin{vmatrix} 0 & 2 & 1 \\ 3 & 1 & -1 \\ 3 & 199 & 101 \end{vmatrix} \xlongequal{r_3-r_2} \begin{vmatrix} 0 & 2 & 1 \\ 3 & 1 & -1 \\ 0 & 198 & 102 \end{vmatrix}$$

$$= (-1)^{2+1} \cdot 3 \cdot \begin{vmatrix} 2 & 1 \\ 198 & 102 \end{vmatrix} = -3 \cdot (2 \cdot 102 - 198) = -18.$$

6

$$\begin{vmatrix} 2 & 2 & 1 \\ 4 & 1 & -1 \\ 202 & 199 & 101 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal{c_1 - c_2} \begin{vmatrix} 0 & 2 & 1 \\ 3 & 1 & -1 \\ 3 & 199 & 101 \end{vmatrix} \xlongequal{r_3 - r_2} \begin{vmatrix} 0 & 2 & 1 \\ 3 & 1 & -1 \\ 0 & 198 & 102 \end{vmatrix}$$

$$= (-1)^{2+1} \cdot 3 \cdot \begin{vmatrix} 2 & 1 \\ 198 & 102 \end{vmatrix} = -3 \cdot (2 \cdot 102 - 198) = -18.$$

7

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{vmatrix}, \quad \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

解：注意到$\omega^3 = 1$，故

$$\omega \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^3 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{vmatrix} = 0,$$

从而

$$原式 = 0.$$

7

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{vmatrix}, \quad \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

解: 注意到$\omega^3 = 1$，故

$$\omega \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^3 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{vmatrix} = 0,$$

从而

$$\text{原式} = 0.$$

8

$$\begin{vmatrix} 1 & x & x \\ x & 2 & x \\ x & x & 3 \end{vmatrix}.$$

解:

$$\text{原式} \xlongequal[r_3 - x r_1]{r_2 - x r_1} \begin{vmatrix} 1 & x & x \\ 0 & 2-x^2 & x-x^2 \\ 0 & x-x^2 & 3-x^2 \end{vmatrix} = \begin{vmatrix} 2-x^2 & x-x^2 \\ x-x^2 & 3-x^2 \end{vmatrix}$$

$$= (2-x^2)(3-x^2) - (x-x^2)^2 = 2x^3 - 6x^2 + 6.$$

8

$$\begin{vmatrix} 1 & x & x \\ x & 2 & x \\ x & x & 3 \end{vmatrix}.$$

解:

$$\text{原式} \xlongequal[r_3 - xr_1]{r_2 - xr_1} \begin{vmatrix} 1 & x & x \\ 0 & 2-x^2 & x-x^2 \\ 0 & x-x^2 & 3-x^2 \end{vmatrix} = \begin{vmatrix} 2-x^2 & x-x^2 \\ x-x^2 & 3-x^2 \end{vmatrix}$$

$$= (2-x^2)(3-x^2) - (x-x^2)^2 = 2x^3 - 6x^2 + 6.$$

9

$$\begin{vmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 3 \\ 0 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix}$$

解:

$$\begin{aligned} \text{原式} &= (-1)^{1+4}\cdot 4\cdot \begin{vmatrix} 0 & 0 & 4 \\ 0 & 4 & 3 \\ 4 & 3 & 2 \end{vmatrix} \\ &= (-1)^{1+4}\cdot 4\cdot (-1)^{1+3}\cdot 4\cdot \begin{vmatrix} 0 & 4 \\ 4 & 3 \end{vmatrix} \\ &= -4\cdot 4\cdot (-16) = 256. \end{aligned}$$

9

$$\begin{vmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 3 \\ 0 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix}$$

解:

$$\begin{aligned} \text{原式} &= (-1)^{1+4} \cdot 4 \cdot \begin{vmatrix} 0 & 0 & 4 \\ 0 & 4 & 3 \\ 4 & 3 & 2 \end{vmatrix} \\ &= (-1)^{1+4} \cdot 4 \cdot (-1)^{1+3} \cdot 4 \cdot \begin{vmatrix} 0 & 4 \\ 4 & 3 \end{vmatrix} \\ &= -4 \cdot 4 \cdot (-16) = 256. \end{aligned}$$

10

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 2 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 8 & \cdots & 0 & 0 & 0 \\ 9 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 10 \end{vmatrix}$$

解:

$$\text{原式} \quad = (-1)^{10+10} \cdot 10 \cdot \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 2 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 8 & \cdots & 0 & 0 \\ 9 & 0 & \cdots & 0 & 0 \end{vmatrix} = 10 \cdot (-1)^{\frac{9\times 8}{2}} 9! = 10!$$

10

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 2 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 8 & \cdots & 0 & 0 & 0 \\ 9 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 10 \end{vmatrix}$$

解:

$$\text{原式} = (-1)^{10+10} \cdot 10 \cdot \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 2 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 8 & \cdots & 0 & 0 \\ 9 & 0 & \cdots & 0 & 0 \end{vmatrix} = 10 \cdot (-1)^{\frac{9\times 8}{2}} 9! = 10!$$

11

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[i=2,3,4]{r_i - r_1} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = (-2)^3 = -8.$$

11

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[i=2,3,4]{r_i - r_1} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = (-2)^3 = -8.$$

12

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[i=4,3,2]{r_i - r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & -3 \\ 1 & 1 & -3 & 1 \\ 1 & -3 & 1 & 1 \end{vmatrix} \xlongequal[i=2,3,4]{c_i - c_1} \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 0 & -4 \\ 1 & 0 & -4 & 0 \\ 1 & -4 & 0 & 0 \end{vmatrix}$$

$$\xlongequal[i=2,3,4]{c_i \div 4} 4^3 \begin{vmatrix} 1 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix} \xlongequal{c_1 + c_2 + c_3 + c_4} 4^3 \begin{vmatrix} 1 + \frac{1+2+3}{4} & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}$$

$$= 4^3 \frac{10}{4} \begin{vmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix} = 160.$$

12

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[i=4,3,2]{r_i - r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & -3 \\ 1 & 1 & -3 & 1 \\ 1 & -3 & 1 & 1 \end{vmatrix} \xlongequal[i=2,3,4]{c_i - c_1} \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 0 & -4 \\ 1 & 0 & -4 & 0 \\ 1 & -4 & 0 & 0 \end{vmatrix}$$

$$\xlongequal[i=2,3,4]{c_i \div 4} 4^3 \begin{vmatrix} 1 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix} \xlongequal{c_1+c_2+c_3+c_4} 4^3 \begin{vmatrix} 1+\frac{1+2+3}{4} & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}$$

$$= 4^3 \frac{10}{4} \begin{vmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix} = 160.$$

12

$$
\begin{vmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3
\end{vmatrix}
$$

解:

$$
\text{原式} \xlongequal[i=4,3,2]{r_i - r_{i-1}}
\begin{vmatrix}
1 & 2 & 3 & 4 \\
1 & 1 & 1 & -3 \\
1 & 1 & -3 & 1 \\
1 & -3 & 1 & 1
\end{vmatrix}
\xlongequal[i=2,3,4]{c_i - c_1}
\begin{vmatrix}
1 & 1 & 2 & 3 \\
1 & 0 & 0 & -4 \\
1 & 0 & -4 & 0 \\
1 & -4 & 0 & 0
\end{vmatrix}
$$

$$
\xlongequal[i=2,3,4]{c_i \div 4} 4^3
\begin{vmatrix}
1 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\
1 & 0 & 0 & -1 \\
1 & 0 & -1 & 0 \\
1 & -1 & 0 & 0
\end{vmatrix}
\xlongequal{c_1 + c_2 + c_3 + c_4} 4^3
\begin{vmatrix}
1 + \frac{1+2+3}{4} & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0
\end{vmatrix}
$$

$$
= 4^3 \frac{10}{4}
\begin{vmatrix}
0 & 0 & -1 \\
0 & -1 & 0 \\
-1 & 0 & 0
\end{vmatrix}
= 160.
$$

12

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$

解:

$$原式 \xlongequal[i=4,3,2]{r_i - r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & -3 \\ 1 & 1 & -3 & 1 \\ 1 & -3 & 1 & 1 \end{vmatrix} \xlongequal[i=2,3,4]{c_i - c_1} \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 0 & -4 \\ 1 & 0 & -4 & 0 \\ 1 & -4 & 0 & 0 \end{vmatrix}$$

$$\xlongequal[i=2,3,4]{c_i \div 4} 4^3 \begin{vmatrix} 1 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix} \xlongequal{c_1+c_2+c_3+c_4} 4^3 \begin{vmatrix} 1+\frac{1+2+3}{4} & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}$$

$$= 4^3 \frac{10}{4} \begin{vmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix} = 160.$$

12

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$

解:

$$原式 \xlongequal[i=4,3,2]{r_i - r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & -3 \\ 1 & 1 & -3 & 1 \\ 1 & -3 & 1 & 1 \end{vmatrix} \xlongequal[i=2,3,4]{c_i - c_1} \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 0 & -4 \\ 1 & 0 & -4 & 0 \\ 1 & -4 & 0 & 0 \end{vmatrix}$$

$$\xlongequal[i=2,3,4]{c_i \div 4} 4^3 \begin{vmatrix} 1 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix} \xlongequal{c_1 + c_2 + c_3 + c_4} 4^3 \begin{vmatrix} 1 + \frac{1+2+3}{4} & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}$$

$$= 4^3 \frac{10}{4} \begin{vmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix} = 160.$$

12

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[i=4,3,2]{r_i - r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & -3 \\ 1 & 1 & -3 & 1 \\ 1 & -3 & 1 & 1 \end{vmatrix} \xlongequal[i=2,3,4]{c_i - c_1} \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 0 & -4 \\ 1 & 0 & -4 & 0 \\ 1 & -4 & 0 & 0 \end{vmatrix}$$

$$\xlongequal[i=2,3,4]{c_i \div 4} 4^3 \begin{vmatrix} 1 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix} \xlongequal{c_1+c_2+c_3+c_4} 4^3 \begin{vmatrix} 1+\frac{1+2+3}{4} & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}$$

$$= 4^3 \frac{10}{4} \begin{vmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix} = 160.$$

13

$$\begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 5 & 0 & 4 & 1 \\ 2 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2} \cdot (-1) \cdot \begin{vmatrix} 5 & 4 & 2 \\ 5 & 4 & 1 \\ 2 & 3 & 2 \end{vmatrix}$$

$$\xlongequal{r_1-r_2} - \begin{vmatrix} 0 & 0 & 1 \\ 5 & 4 & 1 \\ 2 & 3 & 2 \end{vmatrix} = -(-1)^{1+3} \cdot 1 \cdot \begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = -7.$$

13

$$\begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 5 & 0 & 4 & 1 \\ 2 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2}\cdot(-1)\cdot\begin{vmatrix} 5 & 4 & 2 \\ 5 & 4 & 1 \\ 2 & 3 & 2 \end{vmatrix}$$

$$\xlongequal{r_1-r_2} -\begin{vmatrix} 0 & 0 & 1 \\ 5 & 4 & 1 \\ 2 & 3 & 2 \end{vmatrix} = -(-1)^{1+3}\cdot 1\cdot\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = -7.$$

13

$$\begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 5 & 0 & 4 & 1 \\ 2 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2} \cdot (-1) \cdot \begin{vmatrix} 5 & 4 & 2 \\ 5 & 4 & 1 \\ 2 & 3 & 2 \end{vmatrix}$$

$$\xlongequal{r_1-r_2} - \begin{vmatrix} 0 & 0 & 1 \\ 5 & 4 & 1 \\ 2 & 3 & 2 \end{vmatrix} = -(-1)^{1+3} \cdot 1 \cdot \begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = -7.$$

13

$$\begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 5 & 0 & 4 & 1 \\ 2 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2} \cdot (-1) \cdot \begin{vmatrix} 5 & 4 & 2 \\ 5 & 4 & 1 \\ 2 & 3 & 2 \end{vmatrix}$$

$$\xlongequal{r_1-r_2} - \begin{vmatrix} 0 & 0 & 1 \\ 5 & 4 & 1 \\ 2 & 3 & 2 \end{vmatrix} = -(-1)^{1+3} \cdot 1 \cdot \begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = -7.$$

13

$$\begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 5 & 0 & 4 & 1 \\ 2 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2} \cdot (-1) \cdot \begin{vmatrix} 5 & 4 & 2 \\ 5 & 4 & 1 \\ 2 & 3 & 2 \end{vmatrix}$$

$$\xlongequal{r_1-r_2} - \begin{vmatrix} 0 & 0 & 1 \\ 5 & 4 & 1 \\ 2 & 3 & 2 \end{vmatrix} = -(-1)^{1+3} \cdot 1 \cdot \begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = -7.$$

14

$$\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 2 & 5 & 4 & 5 & 3 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal{r_2 \leftrightarrow r_3} -\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 5 & 3 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix} \xlongequal[r_1 - r_3]{\substack{r_2 - r_1 \\ r_4 - r_3}} -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 2 & 5 & 4 & 5 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

$$\xlongequal[r_5 - r_1]{r_3 - 2r_1} -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 & -2 \end{vmatrix} \xlongequal[r_5 + 2r_4]{r_2 \leftrightarrow r_3} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= -12.$$

14

$$\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 2 & 5 & 4 & 5 & 3 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal{r_2 \leftrightarrow r_3} -\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 5 & 3 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix} \xlongequal[r_1 - r_3]{\substack{r_2 - r_1 \\ r_4 - r_3}} -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 2 & 5 & 4 & 5 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

$$\xlongequal[r_5 - r_1]{r_3 - 2r_1} -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 & -2 \end{vmatrix} \xlongequal[r_5 + 2r_4]{r_2 \leftrightarrow r_3} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= -12.$$

14

$$\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 2 & 5 & 4 & 5 & 3 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal{r_2 \leftrightarrow r_3} -\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 5 & 3 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix} \xlongequal[r_1 - r_3]{\substack{r_2 - r_1 \\ r_4 - r_3}} -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 2 & 5 & 4 & 5 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

$$\xlongequal[r_5 - r_1]{r_3 - 2r_1} -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 & -2 \end{vmatrix} \xlongequal[r_5 + 2r_4]{r_2 \leftrightarrow r_3} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= -12.$$

14

$$\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 2 & 5 & 4 & 5 & 3 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal{r_2 \leftrightarrow r_3} - \begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 5 & 3 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix} \xlongequal[r_1 - r_3]{\substack{r_2 - r_1 \\ r_4 - r_3}} - \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 2 & 5 & 4 & 5 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

$$\xlongequal[r_5 - r_1]{r_3 - 2r_1} - \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 & -2 \end{vmatrix} \xlongequal[r_5 + 2r_4]{r_2 \leftrightarrow r_3} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= -12.$$

14

$$\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 2 & 5 & 4 & 5 & 3 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal{r_2 \leftrightarrow r_3} -\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 5 & 3 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix} \xlongequal[r_1 - r_3]{\substack{r_2 - r_1 \\ r_4 - r_3}} -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 2 & 5 & 4 & 5 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

$$\xlongequal[r_5 - r_1]{r_3 - 2r_1} -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 & -2 \end{vmatrix} \xlongequal[r_5 + 2r_4]{r_2 \leftrightarrow r_3} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= -12.$$

14

$$\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 2 & 5 & 4 & 5 & 3 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal{r_2 \leftrightarrow r_3} - \begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 5 & 3 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix} \xlongequal[r_1 - r_3]{\substack{r_2 - r_1 \\ r_4 - r_3}} - \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 2 & 5 & 4 & 5 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

$$\xlongequal[r_5 - r_1]{r_3 - 2r_1} - \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 & -2 \end{vmatrix} \xlongequal[r_5 + 2r_4]{r_2 \leftrightarrow r_3} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= -12.$$

14

$$\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 2 & 5 & 4 & 5 & 3 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal{r_2 \leftrightarrow r_3} -\begin{vmatrix} 3 & 6 & 5 & 6 & 4 \\ 3 & 6 & 3 & 4 & 2 \\ 2 & 5 & 4 & 5 & 3 \\ 2 & 5 & 4 & 6 & 5 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix} \xlongequal[r_1 - r_3]{\substack{r_2 - r_1 \\ r_4 - r_3}} -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 2 & 5 & 4 & 5 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & -1 & -1 \end{vmatrix}$$

$$\xlongequal[r_5 - r_1]{r_3 - 2r_1} -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 & -2 \end{vmatrix} \xlongequal[r_5 + 2r_4]{r_2 \leftrightarrow r_3} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= -12.$$

15

$$\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 5 & 1 \end{vmatrix}$$

解:

$$原式 = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} -1 & 3 \\ 5 & 1 \end{vmatrix} = (-2) \cdot (-16) = 32.$$

15

$$\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 5 & 1 \end{vmatrix}$$

解:

$$\text{原式} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} -1 & 3 \\ 5 & 1 \end{vmatrix} = (-2) \cdot (-16) = 32.$$

16

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 1 & 1 \end{vmatrix}$$

解:

$$\text{原式} \overset{r_3 \leftrightarrow r_5}{=\!=\!=} - \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 0 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= -(-1)^{3+2} \cdot 1 \cdot \begin{vmatrix} 1 & 3 \\ 6 & 8 \end{vmatrix} \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= (-10) \cdot 2 = -20.$$

16

$$\begin{vmatrix}1&2&3&4&5\\6&7&8&9&10\\0&0&0&1&3\\0&0&0&2&4\\0&1&0&1&1\end{vmatrix}$$

解:

$$\text{原式} \xlongequal{r_3\leftrightarrow r_5} -\begin{vmatrix}1&2&3&4&5\\6&7&8&9&10\\0&1&0&1&1\\0&0&0&2&4\\0&0&0&1&3\end{vmatrix} = -\begin{vmatrix}1&2&3\\6&7&8\\0&1&0\end{vmatrix}\cdot\begin{vmatrix}2&4\\1&3\end{vmatrix}$$

$$= -(-1)^{3+2}\cdot 1\cdot\begin{vmatrix}1&3\\6&8\end{vmatrix}\cdot\begin{vmatrix}2&4\\1&3\end{vmatrix}$$

$$= (-10)\cdot 2 = -20.$$

16

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 1 & 1 \end{vmatrix}$$

解:

$$原式 \xlongequal{r_3 \leftrightarrow r_5} -\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 0 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= -(-1)^{3+2} \cdot 1 \cdot \begin{vmatrix} 1 & 3 \\ 6 & 8 \end{vmatrix} \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= (-10) \cdot 2 = -20.$$

16

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 1 & 1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal{r_3 \leftrightarrow r_5} -\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 0 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= -(-1)^{3+2} \cdot 1 \cdot \begin{vmatrix} 1 & 3 \\ 6 & 8 \end{vmatrix} \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= (-10) \cdot 2 = -20.$$

16

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 1 & 1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal{r_3 \leftrightarrow r_5} -\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 0 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= -(-1)^{3+2} \cdot 1 \cdot \begin{vmatrix} 1 & 3 \\ 6 & 8 \end{vmatrix} \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= (-10) \cdot 2 = -20.$$

16

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 1 & 1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal{r_3 \leftrightarrow r_5} -\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 0 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= -(-1)^{3+2} \cdot 1 \cdot \begin{vmatrix} 1 & 3 \\ 6 & 8 \end{vmatrix} \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= (-10) \cdot 2 = -20.$$

17

$$
\begin{vmatrix}
0 & 0 & 1 & -1 & 2 \\
0 & 0 & 3 & 0 & 2 \\
0 & 0 & 2 & 4 & 0 \\
1 & 2 & 4 & 0 & -1 \\
3 & 1 & 2 & 5 & 8
\end{vmatrix}
$$

解:

$$
\text{原式} \xlongequal[c_2 \leftrightarrow c_1]{c_3 \leftrightarrow c_2}
\begin{vmatrix}
1 & 0 & 0 & -1 & 2 \\
3 & 0 & 0 & 0 & 2 \\
2 & 0 & 0 & 4 & 0 \\
4 & 1 & 2 & 0 & -1 \\
2 & 3 & 1 & 5 & 8
\end{vmatrix}
\xlongequal[c_3 \leftrightarrow c_2]{c_4 \leftrightarrow c_3}
\begin{vmatrix}
1 & -1 & 0 & 0 & 2 \\
3 & 0 & 0 & 0 & 2 \\
2 & 4 & 0 & 0 & 0 \\
4 & 0 & 1 & 2 & -1 \\
2 & 5 & 3 & 1 & 8
\end{vmatrix}
$$

$$
\xlongequal[c_4 \leftrightarrow c_3]{c_5 \leftrightarrow c_4}
\begin{vmatrix}
1 & -1 & 2 & 0 & 0 \\
3 & 0 & 2 & 0 & 0 \\
2 & 4 & 0 & 0 & 0 \\
4 & 0 & -1 & 1 & 2 \\
2 & 5 & 8 & 3 & 1
\end{vmatrix}
=
\begin{vmatrix}
1 & -1 & 2 \\
3 & 0 & 2 \\
2 & 4 & 0
\end{vmatrix}
\cdot
\begin{vmatrix}
1 & 2 \\
3 & 1
\end{vmatrix}
$$

$$
\xlongequal{r_2 - r_1}
\begin{vmatrix}
1 & -1 & 2 \\
2 & 1 & 0 \\
2 & 4 & 0
\end{vmatrix}
\begin{vmatrix}
1 & 2 \\
3 & 1
\end{vmatrix}
= 2
\begin{vmatrix}
2 & 1 \\
2 & 4
\end{vmatrix}
\begin{vmatrix}
1 & 2 \\
3 & 1
\end{vmatrix}
= 2 \cdot 6 \cdot (-5) = -60.
$$

17

$$\begin{vmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 4 & 0 \\ 1 & 2 & 4 & 0 & -1 \\ 3 & 1 & 2 & 5 & 8 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[c_2 \leftrightarrow c_1]{c_3 \leftrightarrow c_2} \begin{vmatrix} 1 & 0 & 0 & -1 & 2 \\ 3 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 4 & 0 \\ 4 & 1 & 2 & 0 & -1 \\ 2 & 3 & 1 & 5 & 8 \end{vmatrix} \xlongequal[c_3 \leftrightarrow c_2]{c_4 \leftrightarrow c_3} \begin{vmatrix} 1 & -1 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 2 \\ 2 & 4 & 0 & 0 & 0 \\ 4 & 0 & 1 & 2 & -1 \\ 2 & 5 & 3 & 1 & 8 \end{vmatrix}$$

$$\xlongequal[c_4 \leftrightarrow c_3]{c_5 \leftrightarrow c_4} \begin{vmatrix} 1 & -1 & 2 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 4 & 0 & -1 & 1 & 2 \\ 2 & 5 & 8 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 2 & 4 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$\xlongequal{r_2 - r_1} \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 2 & 4 & 0 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2 \cdot 6 \cdot (-5) = -60.$$

17

$$\begin{vmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 4 & 0 \\ 1 & 2 & 4 & 0 & -1 \\ 3 & 1 & 2 & 5 & 8 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[c_2 \leftrightarrow c_1]{c_3 \leftrightarrow c_2} \begin{vmatrix} 1 & 0 & 0 & -1 & 2 \\ 3 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 4 & 0 \\ 4 & 1 & 2 & 0 & -1 \\ 2 & 3 & 1 & 5 & 8 \end{vmatrix} \xlongequal[c_3 \leftrightarrow c_2]{c_4 \leftrightarrow c_3} \begin{vmatrix} 1 & -1 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 2 \\ 2 & 4 & 0 & 0 & 0 \\ 4 & 0 & 1 & 2 & -1 \\ 2 & 5 & 3 & 1 & 8 \end{vmatrix}$$

$$\xlongequal[c_4 \leftrightarrow c_3]{c_5 \leftrightarrow c_4} \begin{vmatrix} 1 & -1 & 2 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 4 & 0 & -1 & 1 & 2 \\ 2 & 5 & 8 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 2 & 4 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$\xlongequal{r_2 - r_1} \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 2 & 4 & 0 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2 \cdot 6 \cdot (-5) = -60.$$

17

$$\begin{vmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 4 & 0 \\ 1 & 2 & 4 & 0 & -1 \\ 3 & 1 & 2 & 5 & 8 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[c_2 \leftrightarrow c_1]{c_3 \leftrightarrow c_2} \begin{vmatrix} 1 & 0 & 0 & -1 & 2 \\ 3 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 4 & 0 \\ 4 & 1 & 2 & 0 & -1 \\ 2 & 3 & 1 & 5 & 8 \end{vmatrix} \xlongequal[c_3 \leftrightarrow c_2]{c_4 \leftrightarrow c_3} \begin{vmatrix} 1 & -1 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 2 \\ 2 & 4 & 0 & 0 & 0 \\ 4 & 0 & 1 & 2 & -1 \\ 2 & 5 & 3 & 1 & 8 \end{vmatrix}$$

$$\xlongequal[c_4 \leftrightarrow c_3]{c_5 \leftrightarrow c_4} \begin{vmatrix} 1 & -1 & 2 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 4 & 0 & -1 & 1 & 2 \\ 2 & 5 & 8 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 2 & 4 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$\xlongequal{r_2 - r_1} \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 2 & 4 & 0 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2 \cdot 6 \cdot (-5) = -60.$$

17

$$\begin{vmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 4 & 0 \\ 1 & 2 & 4 & 0 & -1 \\ 3 & 1 & 2 & 5 & 8 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[c_2 \leftrightarrow c_1]{c_3 \leftrightarrow c_2} \begin{vmatrix} 1 & 0 & 0 & -1 & 2 \\ 3 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 4 & 0 \\ 4 & 1 & 2 & 0 & -1 \\ 2 & 3 & 1 & 5 & 8 \end{vmatrix} \xlongequal[c_3 \leftrightarrow c_2]{c_4 \leftrightarrow c_3} \begin{vmatrix} 1 & -1 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 2 \\ 2 & 4 & 0 & 0 & 0 \\ 4 & 0 & 1 & 2 & -1 \\ 2 & 5 & 3 & 1 & 8 \end{vmatrix}$$

$$\xlongequal[c_4 \leftrightarrow c_3]{c_5 \leftrightarrow c_4} \begin{vmatrix} 1 & -1 & 2 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 4 & 0 & -1 & 1 & 2 \\ 2 & 5 & 8 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 2 & 4 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$\xlongequal{r_2 - r_1} \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 2 & 4 & 0 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2 \cdot 6 \cdot (-5) = -60.$$

17

$$\begin{vmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 4 & 0 \\ 1 & 2 & 4 & 0 & -1 \\ 3 & 1 & 2 & 5 & 8 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[c_2 \leftrightarrow c_1]{c_3 \leftrightarrow c_2} \begin{vmatrix} 1 & 0 & 0 & -1 & 2 \\ 3 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 4 & 0 \\ 4 & 1 & 2 & 0 & -1 \\ 2 & 3 & 1 & 5 & 8 \end{vmatrix} \xlongequal[c_3 \leftrightarrow c_2]{c_4 \leftrightarrow c_3} \begin{vmatrix} 1 & -1 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 2 \\ 2 & 4 & 0 & 0 & 0 \\ 4 & 0 & 1 & 2 & -1 \\ 2 & 5 & 3 & 1 & 8 \end{vmatrix}$$

$$\xlongequal[c_4 \leftrightarrow c_3]{c_5 \leftrightarrow c_4} \begin{vmatrix} 1 & -1 & 2 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 4 & 0 & -1 & 1 & 2 \\ 2 & 5 & 8 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 2 & 4 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$\xlongequal{r_2 - r_1} \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 2 & 4 & 0 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 2 \cdot 6 \cdot (-5) = -60.$$

17

$$
\begin{vmatrix}
0 & 0 & 1 & -1 & 2 \\
0 & 0 & 3 & 0 & 2 \\
0 & 0 & 2 & 4 & 0 \\
1 & 2 & 4 & 0 & -1 \\
3 & 1 & 2 & 5 & 8
\end{vmatrix}
$$

解:

$$
\text{原式} \xlongequal[c_2\leftrightarrow c_1]{c_3\leftrightarrow c_2}
\begin{vmatrix}
1 & 0 & 0 & -1 & 2 \\
3 & 0 & 0 & 0 & 2 \\
2 & 0 & 0 & 4 & 0 \\
4 & 1 & 2 & 0 & -1 \\
2 & 3 & 1 & 5 & 8
\end{vmatrix}
\xlongequal[c_3\leftrightarrow c_2]{c_4\leftrightarrow c_3}
\begin{vmatrix}
1 & -1 & 0 & 0 & 2 \\
3 & 0 & 0 & 0 & 2 \\
2 & 4 & 0 & 0 & 0 \\
4 & 0 & 1 & 2 & -1 \\
2 & 5 & 3 & 1 & 8
\end{vmatrix}
$$

$$
\xlongequal[c_4\leftrightarrow c_3]{c_5\leftrightarrow c_4}
\begin{vmatrix}
1 & -1 & 2 & 0 & 0 \\
3 & 0 & 2 & 0 & 0 \\
2 & 4 & 0 & 0 & 0 \\
4 & 0 & -1 & 1 & 2 \\
2 & 5 & 8 & 3 & 1
\end{vmatrix}
=
\begin{vmatrix}
1 & -1 & 2 \\
3 & 0 & 2 \\
2 & 4 & 0
\end{vmatrix}
\cdot
\begin{vmatrix}
1 & 2 \\
3 & 1
\end{vmatrix}
$$

$$
\xlongequal{r_2-r_1}
\begin{vmatrix}
1 & -1 & 2 \\
2 & 1 & 0 \\
2 & 4 & 0
\end{vmatrix}
\begin{vmatrix}
1 & 2 \\
3 & 1
\end{vmatrix}
= 2
\begin{vmatrix}
2 & 1 \\
2 & 4
\end{vmatrix}
\begin{vmatrix}
1 & 2 \\
3 & 1
\end{vmatrix}
= 2\cdot 6\cdot(-5) = -60.
$$

18

$$\begin{vmatrix} * & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{vmatrix}, \quad \mathbf{A} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix}, \quad \mathbf{B} = \begin{vmatrix} & & & & -1 \\ & & & -2 & \\ & & -3 & & \\ & -4 & & & \\ -5 & & & & \end{vmatrix}$$

解:

$$\begin{aligned} \text{原式} &= (-1)^{3\times 5} \begin{vmatrix} \mathbf{A} & * \\ \mathbf{0} & \mathbf{B} \end{vmatrix} \\ &= (-1) \cdot |\mathbf{A}| \cdot |\mathbf{B}| \\ &= (-1) \cdot 1 \cdot 2 \cdot 3 \cdot (-1)^{\frac{5\times 4}{2}} (-1)(-2)(-3)(-4)(-5) \\ &= 6 \times 120 = 720 \end{aligned}$$

18

$$\begin{vmatrix} * & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{vmatrix}, \quad \mathbf{A} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix}, \quad \mathbf{B} = \begin{vmatrix} & & & & -1 \\ & & & -2 & \\ & & -3 & & \\ & -4 & & & \\ -5 & & & & \end{vmatrix}$$

解:

$$\begin{aligned} \text{原式} &= (-1)^{3\times5} \begin{vmatrix} \mathbf{A} & * \\ \mathbf{0} & \mathbf{B} \end{vmatrix} \\ &= (-1)\cdot|\mathbf{A}|\cdot|\mathbf{B}| \\ &= (-1)\cdot 1\cdot 2\cdot 3\cdot(-1)^{\frac{5\times4}{2}}(-1)(-2)(-3)(-4)(-5) \\ &= 6\times 120 = 720 \end{aligned}$$

18

$$\begin{vmatrix} * & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{vmatrix}, \quad \mathbf{A} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix}, \quad \mathbf{B} = \begin{vmatrix} & & & & -1 \\ & & & -2 & \\ & & -3 & & \\ & -4 & & & \\ -5 & & & & \end{vmatrix}$$

解:

$$\begin{aligned} \text{原式} &= (-1)^{3\times 5} \begin{vmatrix} \mathbf{A} & * \\ \mathbf{0} & \mathbf{B} \end{vmatrix} \\ &= (-1) \cdot |\mathbf{A}| \cdot |\mathbf{B}| \\ &= (-1) \cdot 1 \cdot 2 \cdot 3 \cdot (-1)^{\frac{5\times 4}{2}} (-1)(-2)(-3)(-4)(-5) \\ &= 6 \times 120 = 720 \end{aligned}$$

18

$$\begin{vmatrix} * & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{vmatrix}, \quad \mathbf{A} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix}, \quad \mathbf{B} = \begin{vmatrix} & & & & -1 \\ & & & -2 & \\ & & -3 & & \\ & -4 & & & \\ -5 & & & & \end{vmatrix}$$

解:

$$\begin{aligned} \text{原式} &= (-1)^{3\times 5} \begin{vmatrix} \mathbf{A} & * \\ \mathbf{0} & \mathbf{B} \end{vmatrix} \\ &= (-1) \cdot |\mathbf{A}| \cdot |\mathbf{B}| \\ &= (-1) \cdot 1 \cdot 2 \cdot 3 \cdot (-1)^{\frac{5\times 4}{2}} (-1)(-2)(-3)(-4)(-5) \\ &= 6 \times 120 = 720 \end{aligned}$$

18

$$\begin{vmatrix} * & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{vmatrix}, \quad \mathbf{A} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix}, \quad \mathbf{B} = \begin{vmatrix} & & & & -1 \\ & & & -2 & \\ & & -3 & & \\ & -4 & & & \\ -5 & & & & \end{vmatrix}$$

解:

$$\begin{aligned} \text{原式} &= (-1)^{3\times 5} \begin{vmatrix} \mathbf{A} & * \\ \mathbf{0} & \mathbf{B} \end{vmatrix} \\ &= (-1) \cdot |\mathbf{A}| \cdot |\mathbf{B}| \\ &= (-1) \cdot 1 \cdot 2 \cdot 3 \cdot (-1)^{\frac{5\times 4}{2}} (-1)(-2)(-3)(-4)(-5) \\ &= 6 \times 120 = 720 \end{aligned}$$

18

$$\begin{vmatrix} * & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{vmatrix}, \quad \mathbf{A} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix}, \quad \mathbf{B} = \begin{vmatrix} & & & & -1 \\ & & & -2 & \\ & & -3 & & \\ & -4 & & & \\ -5 & & & & \end{vmatrix}$$

解:

$$\begin{aligned} \text{原式} &= (-1)^{3\times5} \begin{vmatrix} \mathbf{A} & * \\ \mathbf{0} & \mathbf{B} \end{vmatrix} \\ &= (-1)\cdot|\mathbf{A}|\cdot|\mathbf{B}| \\ &= (-1)\cdot1\cdot2\cdot3\cdot(-1)^{\frac{5\times4}{2}}(-1)(-2)(-3)(-4)(-5) \\ &= 6\times120 = 720 \end{aligned}$$

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证明：

$$\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

证明:

$$\begin{aligned} \text{左边} &= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1x & a_1x+b_1 & c_1 \\ b_2x & a_2x+b_2 & c_2 \\ b_3x & a_3x+b_3 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x+b_1 & c_1 \\ b_2 & a_2x+b_2 & c_2 \\ b_3 & a_3x+b_3 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_1x & c_1 \\ a_2 & a_2x & c_2 \\ a_3 & a_3x & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{右边} \end{aligned}$$

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证明：

$$\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

证明:

$$\begin{aligned} \text{左边} &= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1x & a_1x+b_1 & c_1 \\ b_2x & a_2x+b_2 & c_2 \\ b_3x & a_3x+b_3 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x+b_1 & c_1 \\ b_2 & a_2x+b_2 & c_2 \\ b_3 & a_3x+b_3 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_1x & c_1 \\ a_2 & a_2x & c_2 \\ a_3 & a_3x & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{右边} \end{aligned}$$

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证明：

$$\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

证明:

$$\begin{aligned}
\text{左边} &= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1x & a_1x+b_1 & c_1 \\ b_2x & a_2x+b_2 & c_2 \\ b_3x & a_3x+b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x+b_1 & c_1 \\ b_2 & a_2x+b_2 & c_2 \\ b_3 & a_3x+b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_1x & c_1 \\ a_2 & a_2x & c_2 \\ a_3 & a_3x & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{右边}
\end{aligned}$$

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证明：

$$\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

证明:

$$\begin{aligned}
\text{左边} &= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1x & a_1x+b_1 & c_1 \\ b_2x & a_2x+b_2 & c_2 \\ b_3x & a_3x+b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x+b_1 & c_1 \\ b_2 & a_2x+b_2 & c_2 \\ b_3 & a_3x+b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_1x & c_1 \\ a_2 & a_2x & c_2 \\ a_3 & a_3x & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{右边}
\end{aligned}$$

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证明：

$$\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

证明:

$$\begin{aligned}
\text{左边} &= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1x & a_1x+b_1 & c_1 \\ b_2x & a_2x+b_2 & c_2 \\ b_3x & a_3x+b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x+b_1 & c_1 \\ b_2 & a_2x+b_2 & c_2 \\ b_3 & a_3x+b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_1x & c_1 \\ a_2 & a_2x & c_2 \\ a_3 & a_3x & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{右边}
\end{aligned}$$

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证明：

$$\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

证明:

$$\begin{aligned}
\text{左边} &= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1x & a_1x+b_1 & c_1 \\ b_2x & a_2x+b_2 & c_2 \\ b_3x & a_3x+b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x+b_1 & c_1 \\ b_2 & a_2x+b_2 & c_2 \\ b_3 & a_3x+b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_1x & c_1 \\ a_2 & a_2x & c_2 \\ a_3 & a_3x & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{右边}
\end{aligned}$$

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证明：

$$\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

证明:

$$\begin{aligned}
\text{左边} &= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1x & a_1x+b_1 & c_1 \\ b_2x & a_2x+b_2 & c_2 \\ b_3x & a_3x+b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x+b_1 & c_1 \\ b_2 & a_2x+b_2 & c_2 \\ b_3 & a_3x+b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_1x & c_1 \\ a_2 & a_2x & c_2 \\ a_3 & a_3x & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{右边}
\end{aligned}$$

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证明：

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = x^2y^2.$$

证明:

$$\text{左边} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1+x & 1 & 1 & 1 \\ 0 & 1 & 1-x & 1 & 1 \\ 0 & 1 & 1 & 1+y & 1 \\ 0 & 1 & 1 & 1 & 1-y \end{vmatrix}$$

$$\xlongequal[i=2,3,4,5]{r_i-r_1} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & x & 0 & 0 & 0 \\ -1 & 0 & -x & 0 & 0 \\ -1 & 0 & 0 & y & 0 \\ -1 & 0 & 0 & 0 & -y \end{vmatrix} \xlongequal[\substack{c_1+c_4/y \\ c_1-c_5/y}]{\substack{c_1+c_2/x \\ c_1-c_3/x}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & -y \end{vmatrix}$$

$$= x^2y^2.$$

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证明：

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = x^2y^2.$$

证明:

$$\text{左边} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1+x & 1 & 1 & 1 \\ 0 & 1 & 1-x & 1 & 1 \\ 0 & 1 & 1 & 1+y & 1 \\ 0 & 1 & 1 & 1 & 1-y \end{vmatrix}$$

$$\xlongequal[i=2,3,4,5]{r_i - r_1} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & x & 0 & 0 & 0 \\ -1 & 0 & -x & 0 & 0 \\ -1 & 0 & 0 & y & 0 \\ -1 & 0 & 0 & 0 & -y \end{vmatrix} \xlongequal[\substack{c_1+c_4/y \\ c_1-c_5/y}]{\substack{c_1+c_2/x \\ c_1-c_3/x}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & -y \end{vmatrix}$$

$$= x^2y^2.$$

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证明：

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = x^2y^2.$$

证明:

$$\text{左边} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1+x & 1 & 1 & 1 \\ 0 & 1 & 1-x & 1 & 1 \\ 0 & 1 & 1 & 1+y & 1 \\ 0 & 1 & 1 & 1 & 1-y \end{vmatrix}$$

$$\xlongequal[i=2,3,4,5]{r_i - r_1} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & x & 0 & 0 & 0 \\ -1 & 0 & -x & 0 & 0 \\ -1 & 0 & 0 & y & 0 \\ -1 & 0 & 0 & 0 & -y \end{vmatrix} \xlongequal[\substack{c_1+c_4/y \\ c_1-c_5/y}]{\substack{c_1+c_2/x \\ c_1-c_3/x}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & -y \end{vmatrix}$$

$$= x^2y^2.$$

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证明：

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = x^2y^2.$$

证明:

$$左边 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1+x & 1 & 1 & 1 \\ 0 & 1 & 1-x & 1 & 1 \\ 0 & 1 & 1 & 1+y & 1 \\ 0 & 1 & 1 & 1 & 1-y \end{vmatrix}$$

$$\overline{\overline{\underset{i=2,3,4,5}{r_i - r_1}}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & x & 0 & 0 & 0 \\ -1 & 0 & -x & 0 & 0 \\ -1 & 0 & 0 & y & 0 \\ -1 & 0 & 0 & 0 & -y \end{vmatrix} \overline{\overline{\underset{c_1+c_4/y,\ c_1-c_5/y}{c_1+c_2/x,\ c_1-c_3/x}}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & -y \end{vmatrix}$$

$$= x^2y^2.$$

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证明：

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = x^2y^2.$$

证明:

$$左边 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1+x & 1 & 1 & 1 \\ 0 & 1 & 1-x & 1 & 1 \\ 0 & 1 & 1 & 1+y & 1 \\ 0 & 1 & 1 & 1 & 1-y \end{vmatrix}$$

$$\xlongequal[i=2,3,4,5]{r_i-r_1} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & x & 0 & 0 & 0 \\ -1 & 0 & -x & 0 & 0 \\ -1 & 0 & 0 & y & 0 \\ -1 & 0 & 0 & 0 & -y \end{vmatrix} \xlongequal[\substack{c_1+c_4/y \\ c_1-c_5/y}]{\substack{c_1+c_2/x \\ c_1-c_3/x}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & -y \end{vmatrix}$$

$$= x^2y^2.$$

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证明：

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = x^2y^2.$$

证明:

$$\text{左边} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1+x & 1 & 1 & 1 \\ 0 & 1 & 1-x & 1 & 1 \\ 0 & 1 & 1 & 1+y & 1 \\ 0 & 1 & 1 & 1 & 1-y \end{vmatrix}$$

$$\xlongequal[i=2,3,4,5]{r_i - r_1} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & x & 0 & 0 & 0 \\ -1 & 0 & -x & 0 & 0 \\ -1 & 0 & 0 & y & 0 \\ -1 & 0 & 0 & 0 & -y \end{vmatrix} \xlongequal[\substack{c_1+c_4/y \\ c_1-c_5/y}]{\substack{c_1+c_2/x \\ c_1-c_3/x}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & -y \end{vmatrix}$$

$$= x^2y^2.$$

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证明：

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(a-c)(b-c)(a+b+c).$$

证明：考察范德蒙德行列式

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & y \\ a^2 & b^2 & c^2 & y^2 \\ a^3 & b^3 & c^3 & y^3 \end{vmatrix} = (y-a)(y-b)(y-c)(c-a)(c-b)(b-a)$$

等式两端均为关于y的多项式，比较y^2的系数，可知

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(a-c)(b-c)(a+b+c)$$

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证明：

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(a-c)(b-c)(a+b+c).$$

证明: 考察范德蒙德行列式

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & y \\ a^2 & b^2 & c^2 & y^2 \\ a^3 & b^3 & c^3 & y^3 \end{vmatrix} = (y-a)(y-b)(y-c)(c-a)(c-b)(b-a)$$

等式两端均为关于y的多项式，比较y^2的系数，可知

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(a-c)(b-c)(a+b+c)$$

21

证明：

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(a-c)(b-c)(a+b+c).$$

证明: 考察范德蒙德行列式

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & y \\ a^2 & b^2 & c^2 & y^2 \\ a^3 & b^3 & c^3 & y^3 \end{vmatrix} = (y-a)(y-b)(y-c)(c-a)(c-b)(b-a)$$

等式两端均为关于y的多项式，比较y^2的系数，可知

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(a-c)(b-c)(a+b+c)$$

22

证明：

$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (ab+bc+ca)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

证明：考察范德蒙德行列式

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & y & y^2 & y^3 \end{vmatrix} = (y-a)(y-b)(y-c)(c-a)(c-b)(b-a)$$

等式两端均为关于y的多项式，比较y的系数，可知

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(a-c)(b-c)(ab+bc+ca) = (ab+bc+ca)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

22

证明：

$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (ab+bc+ca)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

证明: 考察范德蒙德行列式

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & y & y^2 & y^3 \end{vmatrix} = (y-a)(y-b)(y-c)(c-a)(c-b)(b-a)$$

等式两端均为关于y的多项式，比较y的系数，可知

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(a-c)(b-c)(ab+bc+ca) = (ab+bc+ca)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

22

证明：

$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (ab+bc+ca)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

证明: 考察范德蒙德行列式

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & y & y^2 & y^3 \end{vmatrix} = (y-a)(y-b)(y-c)(c-a)(c-b)(b-a)$$

等式两端均为关于y的多项式，比较y的系数，可知

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(a-c)(b-c)(ab+bc+ca) = (ab+bc+ca)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

23

计算

$$\begin{vmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{vmatrix}$$

解:

$$\text{左边} \xlongequal{\text{按第4行展开}} (-1)^{4+1} \cdot d \cdot \begin{vmatrix} 0 & 2 & a \\ 0 & b & 0 \\ c & 4 & 5 \end{vmatrix}$$

$$\xlongequal{\text{按第2行展开}} (-d) \cdot (-1)^{2+2} \cdot b \begin{vmatrix} 0 & a \\ c & 5 \end{vmatrix} = abcd.$$

23

计算

$$\begin{vmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{vmatrix}$$

解:

$$\text{左边} \xlongequal{\text{按第4行展开}} (-1)^{4+1} \cdot d \cdot \begin{vmatrix} 0 & 2 & a \\ 0 & b & 0 \\ c & 4 & 5 \end{vmatrix}$$

$$\xlongequal{\text{按第2行展开}} (-d) \cdot (-1)^{2+2} \cdot b \begin{vmatrix} 0 & a \\ c & 5 \end{vmatrix} = abcd.$$

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$$\begin{vmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{vmatrix}$$

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$$\xlongequal{\text{按第2行展开}} (-d) \cdot (-1)^{2+2} \cdot b \begin{vmatrix} 0 & a \\ c & 5 \end{vmatrix} = abcd.$$

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计算

$$\begin{vmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{vmatrix}$$

解:

$$\text{左边} \xlongequal{\text{按第4行展开}} (-1)^{4+1} \cdot d \cdot \begin{vmatrix} 0 & 2 & a \\ 0 & b & 0 \\ c & 4 & 5 \end{vmatrix}$$

$$\xlongequal{\text{按第2行展开}} (-d) \cdot (-1)^{2+2} \cdot b \begin{vmatrix} 0 & a \\ c & 5 \end{vmatrix} = abcd.$$

24

计算
$$\begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix}$$

解:

$$\text{左边} \xlongequal{\text{按第1行展开}} (-1)^{1+1}\cdot a\cdot \begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + (-1)^{1+2}\cdot 1\cdot \begin{vmatrix} -1 & 1 & 0 \\ 0 & c & 1 \\ 0 & -1 & d \end{vmatrix}$$

$$= a\cdot \begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + \begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix}$$

$$\xlongequal{\text{按第1行展开}} a\cdot\left(b\cdot \begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix} + (-1)^{1+2}\cdot 1\cdot \begin{vmatrix} -1 & 1 \\ 0 & d \end{vmatrix}\right) + (cd+1)$$

$$= a(b(cd+1)+d)+(cd+1) = (ab+1)(cd+1)+ad$$

24

计算

$$\begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix}$$

解:

左边 $\xlongequal{\text{按第1行展开}} (-1)^{1+1}\cdot a\cdot \begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + (-1)^{1+2}\cdot 1\cdot \begin{vmatrix} -1 & 1 & 0 \\ 0 & c & 1 \\ 0 & -1 & d \end{vmatrix}$

$$= a\cdot \begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + \begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix}$$

$$\xlongequal{\text{按第1行展开}} a\cdot\left(b\cdot \begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix} + (-1)^{1+2}\cdot 1\cdot \begin{vmatrix} -1 & 1 \\ 0 & d \end{vmatrix}\right) + (cd+1)$$

$$= a(b(cd+1)+d) + (cd+1) = (ab+1)(cd+1) + ad$$

24

计算

$$\begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix}$$

解:

$$\text{左边} \xlongequal{\text{按第1行展开}} (-1)^{1+1} \cdot a \cdot \begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} -1 & 1 & 0 \\ 0 & c & 1 \\ 0 & -1 & d \end{vmatrix}$$

$$= a \cdot \begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + \begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix}$$

$$\xlongequal{\text{按第1行展开}} a \cdot \left(b \cdot \begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix} + (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} -1 & 1 \\ 0 & d \end{vmatrix} \right) + (cd+1)$$

$$= a(b(cd+1)+d) + (cd+1) = (ab+1)(cd+1) + ad$$

24

计算

$$\begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix}$$

解:

$$\text{左边} \xlongequal{\text{按第1行展开}} (-1)^{1+1}\cdot a\cdot\begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + (-1)^{1+2}\cdot 1\cdot\begin{vmatrix} -1 & 1 & 0 \\ 0 & c & 1 \\ 0 & -1 & d \end{vmatrix}$$

$$= a\cdot\begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + \begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix}$$

$$\xlongequal{\text{按第1行展开}} a\cdot\left(b\cdot\begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix} + (-1)^{1+2}\cdot 1\cdot\begin{vmatrix} -1 & 1 \\ 0 & d \end{vmatrix}\right) + (cd+1)$$

$$= a(b(cd+1)+d) + (cd+1) = (ab+1)(cd+1)+ad$$

24

计算

$$\begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix}$$

解:

$$\text{左边} \xlongequal{\text{按第1行展开}} (-1)^{1+1} \cdot a \cdot \begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} -1 & 1 & 0 \\ 0 & c & 1 \\ 0 & -1 & d \end{vmatrix}$$

$$= a \cdot \begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + \begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix}$$

$$\xlongequal{\text{按第1行展开}} a \cdot \left(b \cdot \begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix} + (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} -1 & 1 \\ 0 & d \end{vmatrix} \right) + (cd+1)$$

$$= a(b(cd+1)+d)+(cd+1) = (ab+1)(cd+1)+ad$$

24

计算

$$\begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix}$$

解:

$$\text{左边} \xlongequal{\text{按第1行展开}} (-1)^{1+1} \cdot a \cdot \begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} -1 & 1 & 0 \\ 0 & c & 1 \\ 0 & -1 & d \end{vmatrix}$$

$$= a \cdot \begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + \begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix}$$

$$\xlongequal{\text{按第1行展开}} a \cdot \left(b \cdot \begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix} + (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} -1 & 1 \\ 0 & d \end{vmatrix} \right) + (cd+1)$$

$$= a(b(cd+1)+d) + (cd+1) = (ab+1)(cd+1) + ad$$

25

计算

$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$$

解:

$$\text{左边} \xlongequal[c_2-c_1]{\substack{c_4-c_3\\c_3-c_2}} \begin{vmatrix} a^2 & 2a+1 & 2a+3 & 2a+5 \\ b^2 & 2b+1 & 2b+3 & 2b+5 \\ c^2 & 2c+1 & 2c+3 & 2c+5 \\ d^2 & 2d+1 & 2d+3 & 2d+5 \end{vmatrix}$$

$$\xlongequal[c_3-c_2]{c_4-c_3} \begin{vmatrix} a^2 & 2a+1 & 2 & 2 \\ b^2 & 2b+1 & 2 & 2 \\ c^2 & 2c+1 & 2 & 2 \\ d^2 & 2d+1 & 2 & 2 \end{vmatrix} = 0.$$

25

计算

$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$$

解:

$$\text{左边} \xlongequal[c_2-c_1]{\substack{c_4-c_3\\ c_3-c_2}} \begin{vmatrix} a^2 & 2a+1 & 2a+3 & 2a+5 \\ b^2 & 2b+1 & 2b+3 & 2b+5 \\ c^2 & 2c+1 & 2c+3 & 2c+5 \\ d^2 & 2d+1 & 2d+3 & 2d+5 \end{vmatrix}$$

$$\xlongequal[c_3-c_2]{c_4-c_3} \begin{vmatrix} a^2 & 2a+1 & 2 & 2 \\ b^2 & 2b+1 & 2 & 2 \\ c^2 & 2c+1 & 2 & 2 \\ d^2 & 2d+1 & 2 & 2 \end{vmatrix} = 0.$$

25

计算

$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$$

解:

$$\text{左边} \xlongequal[c_2-c_1]{\substack{c_4-c_3\\ c_3-c_2}} \begin{vmatrix} a^2 & 2a+1 & 2a+3 & 2a+5 \\ b^2 & 2b+1 & 2b+3 & 2b+5 \\ c^2 & 2c+1 & 2c+3 & 2c+5 \\ d^2 & 2d+1 & 2d+3 & 2d+5 \end{vmatrix}$$

$$\xlongequal[c_3-c_2]{c_4-c_3} \begin{vmatrix} a^2 & 2a+1 & 2 & 2 \\ b^2 & 2b+1 & 2 & 2 \\ c^2 & 2c+1 & 2 & 2 \\ d^2 & 2d+1 & 2 & 2 \end{vmatrix} = 0.$$

25

计算

$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$$

解:

$$\text{左边} \xlongequal[c_2-c_1]{\substack{c_4-c_3\\ c_3-c_2}} \begin{vmatrix} a^2 & 2a+1 & 2a+3 & 2a+5 \\ b^2 & 2b+1 & 2b+3 & 2b+5 \\ c^2 & 2c+1 & 2c+3 & 2c+5 \\ d^2 & 2d+1 & 2d+3 & 2d+5 \end{vmatrix}$$

$$\xlongequal[c_3-c_2]{c_4-c_3} \begin{vmatrix} a^2 & 2a+1 & 2 & 2 \\ b^2 & 2b+1 & 2 & 2 \\ c^2 & 2c+1 & 2 & 2 \\ d^2 & 2d+1 & 2 & 2 \end{vmatrix} = 0.$$

26

计算

$$\begin{vmatrix} a & b & c & 1 \\ b & c & a & 1 \\ c & a & b & 1 \\ \frac{b+c}{2} & \frac{c+a}{2} & \frac{a+b}{2} & 1 \end{vmatrix}$$

解：

$$\text{原式} \xlongequal{r_3+r_1+r_2} \begin{vmatrix} a & b & a+b+c & 1 \\ b & c & a+b+c & 1 \\ c & a & a+b+c & 1 \\ \frac{b+c}{2} & \frac{c+a}{2} & a+b+c & 1 \end{vmatrix} = 0.$$

26

计算

$$\begin{vmatrix} a & b & c & 1 \\ b & c & a & 1 \\ c & a & b & 1 \\ \frac{b+c}{2} & \frac{c+a}{2} & \frac{a+b}{2} & 1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal{r_3+r_1+r_2} \begin{vmatrix} a & b & a+b+c & 1 \\ b & c & a+b+c & 1 \\ c & a & a+b+c & 1 \\ \frac{b+c}{2} & \frac{c+a}{2} & a+b+c & 1 \end{vmatrix} = 0.$$

26

计算
$$\begin{vmatrix} a & b & c & 1 \\ b & c & a & 1 \\ c & a & b & 1 \\ \frac{b+c}{2} & \frac{c+a}{2} & \frac{a+b}{2} & 1 \end{vmatrix}$$

解:

$$原式 \xlongequal{r_3+r_1+r_2} \begin{vmatrix} a & b & a+b+c & 1 \\ b & c & a+b+c & 1 \\ c & a & a+b+c & 1 \\ \frac{b+c}{2} & \frac{c+a}{2} & a+b+c & 1 \end{vmatrix} = 0.$$

27

计算

$$\begin{vmatrix} a_1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & b_3 & a_3 & 0 \\ b_4 & 0 & 0 & a_4 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[c_3 \leftrightarrow c_2]{c_4 \leftrightarrow c_3} \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & b_3 & a_3 \\ b_4 & a_4 & 0 & 0 \end{vmatrix}$$

$$\xlongequal[r_3 \leftrightarrow r_2]{r_4 \leftrightarrow r_3} \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ b_4 & a_4 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & b_3 & a_3 \end{vmatrix} = (a_1 a_4 - b_1 b_4)(a_2 a_3 - b_2 b_3).$$

27

计算

$$\begin{vmatrix} a_1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & b_3 & a_3 & 0 \\ b_4 & 0 & 0 & a_4 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[c_3 \leftrightarrow c_2]{c_4 \leftrightarrow c_3} \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & b_3 & a_3 \\ b_4 & a_4 & 0 & 0 \end{vmatrix}$$

$$\xlongequal[r_3 \leftrightarrow r_2]{r_4 \leftrightarrow r_3} \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ b_4 & a_4 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & b_3 & a_3 \end{vmatrix} = (a_1 a_4 - b_1 b_4)(a_2 a_3 - b_2 b_3).$$

27

计算

$$\begin{vmatrix} a_1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & b_3 & a_3 & 0 \\ b_4 & 0 & 0 & a_4 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[c_3 \leftrightarrow c_2]{c_4 \leftrightarrow c_3} \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & b_3 & a_3 \\ b_4 & a_4 & 0 & 0 \end{vmatrix}$$

$$\xlongequal[r_3 \leftrightarrow r_2]{r_4 \leftrightarrow r_3} \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ b_4 & a_4 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & b_3 & a_3 \end{vmatrix} = (a_1 a_4 - b_1 b_4)(a_2 a_3 - b_2 b_3).$$

27

计算

$$\begin{vmatrix} a_1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & b_3 & a_3 & 0 \\ b_4 & 0 & 0 & a_4 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[c_3 \leftrightarrow c_2]{c_4 \leftrightarrow c_3} \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & b_3 & a_3 \\ b_4 & a_4 & 0 & 0 \end{vmatrix}$$

$$\xlongequal[r_3 \leftrightarrow r_2]{r_4 \leftrightarrow r_3} \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ b_4 & a_4 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & b_3 & a_3 \end{vmatrix} = (a_1a_4 - b_1b_4)(a_2a_3 - b_2b_3).$$

27

计算

$$\begin{vmatrix} a_1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & b_3 & a_3 & 0 \\ b_4 & 0 & 0 & a_4 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal[c_3 \leftrightarrow c_2]{c_4 \leftrightarrow c_3} \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & b_3 & a_3 \\ b_4 & a_4 & 0 & 0 \end{vmatrix}$$

$$\xlongequal[r_3 \leftrightarrow r_2]{r_4 \leftrightarrow r_3} \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ b_4 & a_4 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & b_3 & a_3 \end{vmatrix} = (a_1 a_4 - b_1 b_4)(a_2 a_3 - b_2 b_3).$$

28

计算

$$\begin{vmatrix} 1 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 2 & 3 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \cdots & n-1 & 2 \\ 2 & 2 & 2 & \cdots & 2 & n \end{vmatrix}$$

解:

$$原式 = \begin{vmatrix} 1 & 2 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 1 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 2 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 2 & 2 & 3 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 2 & 2 & 2 & \cdots & n-1 & 2 \\ 0 & 2 & 2 & 2 & \cdots & 2 & n \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 & 2 & \cdots & 2 & 2 \\ -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & n-3 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & n-2 \end{vmatrix}$$

28

计算

$$\begin{vmatrix} 1 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 2 & 3 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \cdots & n-1 & 2 \\ 2 & 2 & 2 & \cdots & 2 & n \end{vmatrix}$$

解:

$$\text{原式} = \begin{vmatrix} 1 & 2 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 1 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 2 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 2 & 2 & 3 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 2 & 2 & 2 & \cdots & n-1 & 2 \\ 0 & 2 & 2 & 2 & \cdots & 2 & n \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 & 2 & \cdots & 2 & 2 \\ -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & n-3 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & n-2 \end{vmatrix}$$

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计算

$$\begin{vmatrix} 1 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 2 & 3 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \cdots & n-1 & 2 \\ 2 & 2 & 2 & \cdots & 2 & n \end{vmatrix}$$

解:

$$\text{原式} = \begin{vmatrix} 1 & 2 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 1 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 2 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 2 & 2 & 3 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 2 & 2 & 2 & \cdots & n-1 & 2 \\ 0 & 2 & 2 & 2 & \cdots & 2 & n \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 & 2 & \cdots & 2 & 2 \\ -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & n-3 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & n-2 \end{vmatrix}$$

28

计算

$$\begin{vmatrix} 1 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 2 & 3 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \cdots & n-1 & 2 \\ 2 & 2 & 2 & \cdots & 2 & n \end{vmatrix}$$

解:

$$\text{原式} = \begin{vmatrix} 1 & 2 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 1 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 2 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 2 & 2 & 3 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 2 & 2 & 2 & \cdots & n-1 & 2 \\ 0 & 2 & 2 & 2 & \cdots & 2 & n \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 & 2 & \cdots & 2 & 2 \\ -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & n-3 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & n-2 \end{vmatrix}$$

$$
\text{原式} = \begin{vmatrix}
1 & 2 & 2 & 2 & \cdots & 2 & 2 \\
-1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & 0 & 0 & 0 & \cdots & n-3 & 0 \\
-1 & 0 & 0 & 0 & \cdots & 0 & n-2
\end{vmatrix}
$$

$$
= (-1)^{3+1}(-1) \begin{vmatrix}
2 & 2 & 2 & \cdots & 2 & 2 \\
-1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & n-3 & 0 \\
0 & 0 & 0 & \cdots & 0 & n-2
\end{vmatrix}
$$

$$
= -2(n-2)!
$$

$$\text{原式} = \begin{vmatrix} 1 & 2 & 2 & 2 & \cdots & 2 & 2 \\ -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & n-3 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & n-2 \end{vmatrix}$$

$$= (-1)^{3+1}(-1) \begin{vmatrix} 2 & 2 & 2 & \cdots & 2 & 2 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-3 & 0 \\ 0 & 0 & 0 & \cdots & 0 & n-2 \end{vmatrix}$$

$$= -2(n-2)!$$

$$
原式 = \begin{vmatrix} 1 & 2 & 2 & 2 & \cdots & 2 & 2 \\ -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & n-3 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & n-2 \end{vmatrix}
$$

$$
= (-1)^{3+1}(-1) \begin{vmatrix} 2 & 2 & 2 & \cdots & 2 & 2 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-3 & 0 \\ 0 & 0 & 0 & \cdots & 0 & n-2 \end{vmatrix}
$$

$$
= -2(n-2)!
$$

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计算

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a & a-1 & a-2 & \cdots & a-n \\ a^2 & (a-1)^2 & (a-2)^2 & \cdots & (a-n)^2 \\ \vdots & \vdots & \vdots & & \vdots \\ a^n & (a-1)^n & (a-2)^n & \cdots & (a-n)^n \end{vmatrix}$$

解：该行列式为范德蒙行列式，故

$$\begin{aligned} \text{原式} &= \prod_{n\geq i>j\geq 0}[(a-i)-(a-j)] \\ &= \prod_{n\geq i>j\geq 0}(j-i) \\ &= (-1)^{\frac{n(n+1)}{2}} \prod_{n\geq i>j\geq 0}(i-j) \\ &= (-1)^{\frac{n(n+1)}{2}} \prod_{i=1}^{n} i! \end{aligned}$$

29

计算

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a & a-1 & a-2 & \cdots & a-n \\ a^2 & (a-1)^2 & (a-2)^2 & \cdots & (a-n)^2 \\ \vdots & \vdots & \vdots & & \vdots \\ a^n & (a-1)^n & (a-2)^n & \cdots & (a-n)^n \end{vmatrix}$$

解： 该行列式为范德蒙行列式，故

$$\begin{aligned} \text{原式} &= \prod_{n\ge i>j\ge 0}[(a-i)-(a-j)] \\ &= \prod_{n\ge i>j\ge 0}(j-i) \\ &= (-1)^{\frac{n(n+1)}{2}} \prod_{n\ge i>j\ge 0}(i-j) \\ &= (-1)^{\frac{n(n+1)}{2}} \prod_{i=1}^{n} i! \end{aligned}$$

29

计算

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a & a-1 & a-2 & \cdots & a-n \\ a^2 & (a-1)^2 & (a-2)^2 & \cdots & (a-n)^2 \\ \vdots & \vdots & \vdots & & \vdots \\ a^n & (a-1)^n & (a-2)^n & \cdots & (a-n)^n \end{vmatrix}$$

解: 该行列式为范德蒙行列式，故

$$\begin{aligned} \text{原式} &= \prod_{n\geq i>j\geq 0}[(a-i)-(a-j)] \\ &= \prod_{n\geq i>j\geq 0}(j-i) \\ &= (-1)^{\frac{n(n+1)}{2}} \prod_{n\geq i>j\geq 0}(i-j) \\ &= (-1)^{\frac{n(n+1)}{2}} \prod_{i=1}^{n} i! \end{aligned}$$

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计算

$$\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \cdots & a_1b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \cdots & a_2b_2^{n-1} & b_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \cdots & a_{n+1}b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix}$$

解:

$$\begin{aligned} \text{原式} &= a_1^n a_2^n \cdots a_{n+1}^n \begin{vmatrix} 1 & a_1^{-1}b_1 & (a_1^{-1}b_1)^2 & \cdots & (a_1^{-1}b_1)^{n-1} & (a_1^{-1}b_1)^n \\ 1 & a_2^{-1}b_2 & (a_2^{-1}b_2)^2 & \cdots & (a_2^{-1}b_2)^{n-1} & (a_2^{-1}b_2)^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & a_{n+1}^{-1}b_{n+1} & (a_{n+1}^{-1}b_{n+1})^2 & \cdots & (a_{n+1}^{-1}b_{n+1})^{n-1} & (a_{n+1}^{-1}b_{n+1})^n \end{vmatrix} \\ &= a_1^n a_2^n \cdots a_{n+1}^n \prod_{n+1\geq i>j\geq 1} \left(\frac{b_i}{a_i} - \frac{b_j}{a_j}\right) \\ &= \prod_{n+1\geq i>j\geq 1} (b_i a_j - a_i b_j) \end{aligned}$$

30

计算

$$\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \cdots & a_1 b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \cdots & a_2 b_2^{n-1} & b_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \cdots & a_{n+1} b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix}$$

解:

$$\text{原式} = a_1^n a_2^n \cdots a_{n+1}^n \begin{vmatrix} 1 & a_1^{-1}b_1 & (a_1^{-1}b_1)^2 & \cdots & (a_1^{-1}b_1)^{n-1} & (a_1^{-1}b_1)^n \\ 1 & a_2^{-1}b_2 & (a_2^{-1}b_2)^2 & \cdots & (a_2^{-1}b_2)^{n-1} & (a_2^{-1}b_2)^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & a_{n+1}^{-1}b_{n+1} & (a_{n+1}^{-1}b_{n+1})^2 & \cdots & (a_{n+1}^{-1}b_{n+1})^{n-1} & (a_{n+1}^{-1}b_{n+1})^n \end{vmatrix}$$

$$= a_1^n a_2^n \cdots a_{n+1}^n \prod_{n+1 \geq i > j \geq 1} \left(\frac{b_i}{a_i} - \frac{b_j}{a_j} \right)$$

$$= \prod_{n+1 \geq i > j \geq 1} (b_i a_j - a_i b_j)$$

30

计算

$$\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \cdots & a_1b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \cdots & a_2b_2^{n-1} & b_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \cdots & a_{n+1}b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix}$$

解:

$$\text{原式} = a_1^n a_2^n \cdots a_{n+1}^n \begin{vmatrix} 1 & a_1^{-1}b_1 & (a_1^{-1}b_1)^2 & \cdots & (a_1^{-1}b_1)^{n-1} & (a_1^{-1}b_1)^n \\ 1 & a_2^{-1}b_2 & (a_2^{-1}b_2)^2 & \cdots & (a_2^{-1}b_2)^{n-1} & (a_2^{-1}b_2)^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & a_{n+1}^{-1}b_{n+1} & (a_{n+1}^{-1}b_{n+1})^2 & \cdots & (a_{n+1}^{-1}b_{n+1})^{n-1} & (a_{n+1}^{-1}b_{n+1})^n \end{vmatrix}$$

$$= a_1^n a_2^n \cdots a_{n+1}^n \prod_{n+1\geq i>j\geq 1} \left(\frac{b_i}{a_i} - \frac{b_j}{a_j}\right)$$

$$= \prod_{n+1\geq i>j\geq 1} (b_i a_j - a_i b_j)$$

30

计算

$$\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \cdots & a_1 b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \cdots & a_2 b_2^{n-1} & b_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \cdots & a_{n+1} b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix}$$

解:

$$\text{原式} = a_1^n a_2^n \cdots a_{n+1}^n \begin{vmatrix} 1 & a_1^{-1}b_1 & (a_1^{-1}b_1)^2 & \cdots & (a_1^{-1}b_1)^{n-1} & (a_1^{-1}b_1)^n \\ 1 & a_2^{-1}b_2 & (a_2^{-1}b_2)^2 & \cdots & (a_2^{-1}b_2)^{n-1} & (a_2^{-1}b_2)^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & a_{n+1}^{-1}b_{n+1} & (a_{n+1}^{-1}b_{n+1})^2 & \cdots & (a_{n+1}^{-1}b_{n+1})^{n-1} & (a_{n+1}^{-1}b_{n+1})^n \end{vmatrix}$$

$$= a_1^n a_2^n \cdots a_{n+1}^n \prod_{n+1 \geq i > j \geq 1} \left(\frac{b_i}{a_i} - \frac{b_j}{a_j} \right)$$

$$= \prod_{n+1 \geq i > j \geq 1} (b_i a_j - a_i b_j)$$

30

计算

$$\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \cdots & a_1 b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \cdots & a_2 b_2^{n-1} & b_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \cdots & a_{n+1} b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix}$$

解:

$$\begin{aligned} \text{原式} &= a_1^n a_2^n \cdots a_{n+1}^n \begin{vmatrix} 1 & a_1^{-1}b_1 & (a_1^{-1}b_1)^2 & \cdots & (a_1^{-1}b_1)^{n-1} & (a_1^{-1}b_1)^n \\ 1 & a_2^{-1}b_2 & (a_2^{-1}b_2)^2 & \cdots & (a_2^{-1}b_2)^{n-1} & (a_2^{-1}b_2)^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & a_{n+1}^{-1}b_{n+1} & (a_{n+1}^{-1}b_{n+1})^2 & \cdots & (a_{n+1}^{-1}b_{n+1})^{n-1} & (a_{n+1}^{-1}b_{n+1})^n \end{vmatrix} \\ &= a_1^n a_2^n \cdots a_{n+1}^n \prod_{n+1 \geq i > j \geq 1} \left(\frac{b_i}{a_i} - \frac{b_j}{a_j} \right) \\ &= \prod_{n+1 \geq i > j \geq 1} (b_i a_j - a_i b_j) \end{aligned}$$

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用克拉默法则求

$$\begin{cases} 5x_1 \quad\quad\quad + 4x_3 + 2x_4 = 3, \\ x_1 - x_2 + 2x_3 + x_4 = 1, \\ 4x_1 + x_2 + 2x_3 \quad\quad = 1, \\ x_1 + x_2 + x_3 + x_4 = 0. \end{cases}$$

解:

$$D = \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -7,$$

$$D_1 = \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 0 & 4 & 1 \\ 1 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 3 & 4 & 2 \\ 2 & 4 & 1 \\ 1 & 3 & 2 \end{vmatrix}$$

$$\xlongequal[r_3-2r_2]{r_1-2r_2} - \begin{vmatrix} -1 & -4 & 0 \\ 2 & 4 & 1 \\ -3 & -5 & 0 \end{vmatrix} = -(-1)^{2+3} \cdot \begin{vmatrix} -1 & -4 \\ -3 & -5 \end{vmatrix} = -7,$$

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用克拉默法则求

$$\begin{cases} 5x_1 & & + & 4x_3 & + & 2x_4 & = & 3, \\ x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 1, \\ 4x_1 & + & x_2 & + & 2x_3 & & & = & 1, \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 0. \end{cases}$$

解:

$$D = \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -7,$$

$$D_1 = \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 0 & 4 & 1 \\ 1 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 3 & 4 & 2 \\ 2 & 4 & 1 \\ 1 & 3 & 2 \end{vmatrix}$$

$$\xlongequal[r_3-2r_2]{r_1-2r_2} - \begin{vmatrix} -1 & -4 & 0 \\ 2 & 4 & 1 \\ -3 & -5 & 0 \end{vmatrix} = -(-1)^{2+3} \cdot \begin{vmatrix} -1 & -4 \\ -3 & -5 \end{vmatrix} = -7,$$

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用克拉默法则求

$$\begin{cases} 5x_1 & & + 4x_3 + 2x_4 = 3, \\ x_1 - x_2 + 2x_3 + x_4 = 1, \\ 4x_1 + x_2 + 2x_3 & = 1, \\ x_1 + x_2 + x_3 + x_4 = 0. \end{cases}$$

解:

$$D = \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -7,$$

$$D_1 = \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} \overset{r_3+r_2}{\underset{r_4+r_2}{=\!=\!=}} \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 0 & 4 & 1 \\ 1 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 3 & 4 & 2 \\ 2 & 4 & 1 \\ 1 & 3 & 2 \end{vmatrix}$$

$$\overset{r_1-2r_2}{\underset{r_3-2r_2}{=\!=\!=}} - \begin{vmatrix} -1 & -4 & 0 \\ 2 & 4 & 1 \\ -3 & -5 & 0 \end{vmatrix} = -(-1)^{2+3} \cdot \begin{vmatrix} -1 & -4 \\ -3 & -5 \end{vmatrix} = -7,$$

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用克拉默法则求

$$\begin{cases} 5x_1 + 4x_3 + 2x_4 = 3, \\ x_1 - x_2 + 2x_3 + x_4 = 1, \\ 4x_1 + x_2 + 2x_3 = 1, \\ x_1 + x_2 + x_3 + x_4 = 0. \end{cases}$$

解:

$$D = \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -7,$$

$$D_1 = \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 0 & 4 & 1 \\ 1 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 3 & 4 & 2 \\ 2 & 4 & 1 \\ 1 & 3 & 2 \end{vmatrix}$$

$$\xlongequal[r_3-2r_2]{r_1-2r_2} - \begin{vmatrix} -1 & -4 & 0 \\ 2 & 4 & 1 \\ -3 & -5 & 0 \end{vmatrix} = -(-1)^{2+3} \cdot \begin{vmatrix} -1 & -4 \\ -3 & -5 \end{vmatrix} = -7,$$

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用克拉默法则求

$$\begin{cases} 5x_1 & & & + & 4x_3 & + & 2x_4 & = & 3, \\ x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 1, \\ 4x_1 & + & x_2 & + & 2x_3 & & & = & 1, \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 0. \end{cases}$$

解:

$$D = \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -7,$$

$$D_1 = \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 0 & 4 & 1 \\ 1 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2}(-1)\begin{vmatrix} 3 & 4 & 2 \\ 2 & 4 & 1 \\ 1 & 3 & 2 \end{vmatrix}$$

$$\xlongequal[r_3-2r_2]{r_1-2r_2} -\begin{vmatrix} -1 & -4 & 0 \\ 2 & 4 & 1 \\ -3 & -5 & 0 \end{vmatrix} = -(-1)^{2+3} \cdot \begin{vmatrix} -1 & -4 \\ -3 & -5 \end{vmatrix} = -7,$$

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用克拉默法则求

$$\begin{cases} 5x_1 & & & + & 4x_3 & + & 2x_4 & = & 3, \\ x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 1, \\ 4x_1 & + & x_2 & + & 2x_3 & & & = & 1, \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 0. \end{cases}$$

解:

$$D = \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -7,$$

$$D_1 = \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 0 & 4 & 1 \\ 1 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2}(-1)\begin{vmatrix} 3 & 4 & 2 \\ 2 & 4 & 1 \\ 1 & 3 & 2 \end{vmatrix}$$

$$\xlongequal[r_3-2r_2]{r_1-2r_2} -\begin{vmatrix} -1 & -4 & 0 \\ 2 & 4 & 1 \\ -3 & -5 & 0 \end{vmatrix} = -(-1)^{2+3}\cdot\begin{vmatrix} -1 & -4 \\ -3 & -5 \end{vmatrix} = -7,$$

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用克拉默法则求

$$\begin{cases} 5x_1 & & + 4x_3 + 2x_4 = 3, \\ x_1 - x_2 + 2x_3 + x_4 = 1, \\ 4x_1 + x_2 + 2x_3 = 1, \\ x_1 + x_2 + x_3 + x_4 = 0. \end{cases}$$

解:

$$D = \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -7,$$

$$D_1 = \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 0 & 4 & 1 \\ 1 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 3 & 4 & 2 \\ 2 & 4 & 1 \\ 1 & 3 & 2 \end{vmatrix}$$

$$\xlongequal[r_3-2r_2]{r_1-2r_2} - \begin{vmatrix} -1 & -4 & 0 \\ 2 & 4 & 1 \\ -3 & -5 & 0 \end{vmatrix} = -(-1)^{2+3} \cdot \begin{vmatrix} -1 & -4 \\ -3 & -5 \end{vmatrix} = -7,$$

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用克拉默法则求

$$\begin{cases} 5x_1 \qquad\quad\; + 4x_3 + 2x_4 = 3, \\ \;\; x_1 - x_2 + 2x_3 + \;\; x_4 = 1, \\ 4x_1 + x_2 + 2x_3 \qquad\quad\;\; = 1, \\ \;\; x_1 + x_2 + \;\; x_3 + \;\; x_4 = 0. \end{cases}$$

解:

$$D = \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -7,$$

$$D_1 = \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 0 & 4 & 1 \\ 1 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 3 & 4 & 2 \\ 2 & 4 & 1 \\ 1 & 3 & 2 \end{vmatrix}$$

$$\xlongequal[r_3-2r_2]{r_1-2r_2} - \begin{vmatrix} -1 & -4 & 0 \\ 2 & 4 & 1 \\ -3 & -5 & 0 \end{vmatrix} = -(-1)^{2+3} \cdot \begin{vmatrix} -1 & -4 \\ -3 & -5 \end{vmatrix} = -7,$$

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用克拉默法则求

$$\begin{cases} 5x_1 & & & + & 4x_3 & + & 2x_4 & = & 3, \\ x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 1, \\ 4x_1 & + & x_2 & + & 2x_3 & & & = & 1, \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 0. \end{cases}$$

解:

$$D = \begin{vmatrix} 5 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -7,$$

$$D_1 = \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 0 & 4 & 1 \\ 1 & 0 & 3 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 3 & 4 & 2 \\ 2 & 4 & 1 \\ 1 & 3 & 2 \end{vmatrix}$$

$$\xlongequal[r_3-2r_2]{r_1-2r_2} - \begin{vmatrix} -1 & -4 & 0 \\ 2 & 4 & 1 \\ -3 & -5 & 0 \end{vmatrix} = -(-1)^{2+3} \cdot \begin{vmatrix} -1 & -4 \\ -3 & -5 \end{vmatrix} = -7,$$

$$D_2 = \begin{vmatrix} 5 & 3 & 4 & 2 \\ 1 & 1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix} \xlongequal[c_4-c_1]{c_3-c_1} \begin{vmatrix} 5 & 3 & -1 & -3 \\ 1 & 1 & 1 & 0 \\ 4 & 1 & -2 & -4 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (-1)^{4+1} \cdot \begin{vmatrix} 3 & -1 & -3 \\ 1 & 1 & 0 \\ 1 & -2 & -4 \end{vmatrix}$$

$$\xlongequal{c_1-c_2} - \begin{vmatrix} 4 & -1 & -3 \\ 0 & 1 & 0 \\ 3 & -2 & -4 \end{vmatrix} = -(-1)^{2+2} \cdot \begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix} = 7,$$

$$D_3 = \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 5 & 0 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 5 & 3 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\xlongequal[c_3-2c_2]{c_1-2c_2} - \begin{vmatrix} -1 & 3 & -4 \\ 1 & 2 & -3 \\ 0 & 1 & 0 \end{vmatrix} = -(-1)^{3+2} \cdot \begin{vmatrix} -1 & -4 \\ 1 & -3 \end{vmatrix} = 7$$

$$D_2 = \begin{vmatrix} 5 & 3 & 4 & 2 \\ 1 & 1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix} \xlongequal[c_4-c_1]{c_3-c_1} \begin{vmatrix} 5 & 3 & -1 & -3 \\ 1 & 1 & 1 & 0 \\ 4 & 1 & -2 & -4 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (-1)^{4+1} \cdot \begin{vmatrix} 3 & -1 & -3 \\ 1 & 1 & 0 \\ 1 & -2 & -4 \end{vmatrix}$$

$$\xlongequal{c_1-c_2} - \begin{vmatrix} 4 & -1 & -3 \\ 0 & 1 & 0 \\ 3 & -2 & -4 \end{vmatrix} = -(-1)^{2+2} \cdot \begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix} = 7,$$

$$D_3 = \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 5 & 0 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 5 & 3 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\xlongequal[c_3-2c_2]{c_1-2c_2} - \begin{vmatrix} -1 & 3 & -4 \\ 1 & 2 & -3 \\ 0 & 1 & 0 \end{vmatrix} = -(-1)^{3+2} \cdot \begin{vmatrix} -1 & -4 \\ 1 & -3 \end{vmatrix} = 7$$

$$D_2 = \begin{vmatrix} 5 & 3 & 4 & 2 \\ 1 & 1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix} \xlongequal[c_4-c_1]{c_3-c_1} \begin{vmatrix} 5 & 3 & -1 & -3 \\ 1 & 1 & 1 & 0 \\ 4 & 1 & -2 & -4 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (-1)^{4+1}\cdot \begin{vmatrix} 3 & -1 & -3 \\ 1 & 1 & 0 \\ 1 & -2 & -4 \end{vmatrix}$$

$$\xlongequal{c_1-c_2} -\begin{vmatrix} 4 & -1 & -3 \\ 0 & 1 & 0 \\ 3 & -2 & -4 \end{vmatrix} = -(-1)^{2+2}\cdot \begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix} = 7,$$

$$D_3 = \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 5 & 0 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 5 & 3 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\xlongequal[c_3-2c_2]{c_1-2c_2} -\begin{vmatrix} -1 & 3 & -4 \\ 1 & 2 & -3 \\ 0 & 1 & 0 \end{vmatrix} = -(-1)^{3+2}\cdot \begin{vmatrix} -1 & -4 \\ 1 & -3 \end{vmatrix} = 7$$

$$D_2 = \begin{vmatrix} 5 & 3 & 4 & 2 \\ 1 & 1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix} \xlongequal[c_4-c_1]{c_3-c_1} \begin{vmatrix} 5 & 3 & -1 & -3 \\ 1 & 1 & 1 & 0 \\ 4 & 1 & -2 & -4 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (-1)^{4+1} \cdot \begin{vmatrix} 3 & -1 & -3 \\ 1 & 1 & 0 \\ 1 & -2 & -4 \end{vmatrix}$$

$$\xlongequal{c_1-c_2} - \begin{vmatrix} 4 & -1 & -3 \\ 0 & 1 & 0 \\ 3 & -2 & -4 \end{vmatrix} = -(-1)^{2+2} \cdot \begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix} = 7,$$

$$D_3 = \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 5 & 0 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 5 & 3 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\xlongequal[c_3-2c_2]{c_1-2c_2} - \begin{vmatrix} -1 & 3 & -4 \\ 1 & 2 & -3 \\ 0 & 1 & 0 \end{vmatrix} = -(-1)^{3+2} \cdot \begin{vmatrix} -1 & -4 \\ 1 & -3 \end{vmatrix} = 7$$

$$D_2 = \begin{vmatrix} 5 & 3 & 4 & 2 \\ 1 & 1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix} \xlongequal[c_4-c_1]{c_3-c_1} \begin{vmatrix} 5 & 3 & -1 & -3 \\ 1 & 1 & 1 & 0 \\ 4 & 1 & -2 & -4 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (-1)^{4+1} \cdot \begin{vmatrix} 3 & -1 & -3 \\ 1 & 1 & 0 \\ 1 & -2 & -4 \end{vmatrix}$$

$$\xlongequal{c_1-c_2} - \begin{vmatrix} 4 & -1 & -3 \\ 0 & 1 & 0 \\ 3 & -2 & -4 \end{vmatrix} = -(-1)^{2+2} \cdot \begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix} = 7,$$

$$D_3 = \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 5 & 0 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 5 & 3 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\xlongequal[c_3-2c_2]{c_1-2c_2} - \begin{vmatrix} -1 & 3 & -4 \\ 1 & 2 & -3 \\ 0 & 1 & 0 \end{vmatrix} = -(-1)^{3+2} \cdot \begin{vmatrix} -1 & -4 \\ 1 & -3 \end{vmatrix} = 7$$

$$D_2 = \begin{vmatrix} 5 & 3 & 4 & 2 \\ 1 & 1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix} \xlongequal[c_4-c_1]{c_3-c_1} \begin{vmatrix} 5 & 3 & -1 & -3 \\ 1 & 1 & 1 & 0 \\ 4 & 1 & -2 & -4 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (-1)^{4+1} \cdot \begin{vmatrix} 3 & -1 & -3 \\ 1 & 1 & 0 \\ 1 & -2 & -4 \end{vmatrix}$$

$$\xlongequal{c_1-c_2} - \begin{vmatrix} 4 & -1 & -3 \\ 0 & 1 & 0 \\ 3 & -2 & -4 \end{vmatrix} = -(-1)^{2+2} \cdot \begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix} = 7,$$

$$D_3 = \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 5 & 0 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 5 & 3 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\xlongequal[c_3-2c_2]{c_1-2c_2} - \begin{vmatrix} -1 & 3 & -4 \\ 1 & 2 & -3 \\ 0 & 1 & 0 \end{vmatrix} = -(-1)^{3+2} \cdot \begin{vmatrix} -1 & -4 \\ 1 & -3 \end{vmatrix} = 7$$

$$D_2 = \begin{vmatrix} 5 & 3 & 4 & 2 \\ 1 & 1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix} \xlongequal[c_4-c_1]{c_3-c_1} \begin{vmatrix} 5 & 3 & -1 & -3 \\ 1 & 1 & 1 & 0 \\ 4 & 1 & -2 & -4 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (-1)^{4+1} \cdot \begin{vmatrix} 3 & -1 & -3 \\ 1 & 1 & 0 \\ 1 & -2 & -4 \end{vmatrix}$$

$$\xlongequal{c_1-c_2} - \begin{vmatrix} 4 & -1 & -3 \\ 0 & 1 & 0 \\ 3 & -2 & -4 \end{vmatrix} = -(-1)^{2+2} \cdot \begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix} = 7,$$

$$D_3 = \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 5 & 0 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 5 & 3 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\xlongequal[c_3-2c_2]{c_1-2c_2} - \begin{vmatrix} -1 & 3 & -4 \\ 1 & 2 & -3 \\ 0 & 1 & 0 \end{vmatrix} = -(-1)^{3+2} \cdot \begin{vmatrix} -1 & -4 \\ 1 & -3 \end{vmatrix} = 7$$

$$D_2 = \begin{vmatrix} 5 & 3 & 4 & 2 \\ 1 & 1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix} \xlongequal[c_4-c_1]{c_3-c_1} \begin{vmatrix} 5 & 3 & -1 & -3 \\ 1 & 1 & 1 & 0 \\ 4 & 1 & -2 & -4 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (-1)^{4+1} \cdot \begin{vmatrix} 3 & -1 & -3 \\ 1 & 1 & 0 \\ 1 & -2 & -4 \end{vmatrix}$$

$$\xlongequal{c_1-c_2} - \begin{vmatrix} 4 & -1 & -3 \\ 0 & 1 & 0 \\ 3 & -2 & -4 \end{vmatrix} = -(-1)^{2+2} \cdot \begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix} = 7,$$

$$D_3 = \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 5 & 0 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 5 & 3 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\xlongequal[c_3-2c_2]{c_1-2c_2} - \begin{vmatrix} -1 & 3 & -4 \\ 1 & 2 & -3 \\ 0 & 1 & 0 \end{vmatrix} = -(-1)^{3+2} \cdot \begin{vmatrix} -1 & -4 \\ 1 & -3 \end{vmatrix} = 7$$

$$D_2 = \begin{vmatrix} 5 & 3 & 4 & 2 \\ 1 & 1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix} \xlongequal[c_4-c_1]{c_3-c_1} \begin{vmatrix} 5 & 3 & -1 & -3 \\ 1 & 1 & 1 & 0 \\ 4 & 1 & -2 & -4 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (-1)^{4+1} \cdot \begin{vmatrix} 3 & -1 & -3 \\ 1 & 1 & 0 \\ 1 & -2 & -4 \end{vmatrix}$$

$$\xlongequal{c_1-c_2} - \begin{vmatrix} 4 & -1 & -3 \\ 0 & 1 & 0 \\ 3 & -2 & -4 \end{vmatrix} = -(-1)^{2+2} \cdot \begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix} = 7,$$

$$D_3 = \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 5 & 0 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 5 & 3 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\xlongequal[c_3-2c_2]{c_1-2c_2} - \begin{vmatrix} -1 & 3 & -4 \\ 1 & 2 & -3 \\ 0 & 1 & 0 \end{vmatrix} = -(-1)^{3+2} \cdot \begin{vmatrix} -1 & -4 \\ 1 & -3 \end{vmatrix} = 7$$

$$D_2 = \begin{vmatrix} 5 & 3 & 4 & 2 \\ 1 & 1 & 2 & 1 \\ 4 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix} \xlongequal[c_4-c_1]{c_3-c_1} \begin{vmatrix} 5 & 3 & -1 & -3 \\ 1 & 1 & 1 & 0 \\ 4 & 1 & -2 & -4 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (-1)^{4+1} \cdot \begin{vmatrix} 3 & -1 & -3 \\ 1 & 1 & 0 \\ 1 & -2 & -4 \end{vmatrix}$$

$$\xlongequal{c_1-c_2} - \begin{vmatrix} 4 & -1 & -3 \\ 0 & 1 & 0 \\ 3 & -2 & -4 \end{vmatrix} = -(-1)^{2+2} \cdot \begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix} = 7,$$

$$D_3 = \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 3 & 2 \\ 1 & -1 & 1 & 1 \\ 5 & 0 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{vmatrix} = (-1)^{2+2}(-1) \begin{vmatrix} 5 & 3 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\xlongequal[c_3-2c_2]{c_1-2c_2} - \begin{vmatrix} -1 & 3 & -4 \\ 1 & 2 & -3 \\ 0 & 1 & 0 \end{vmatrix} = -(-1)^{3+2} \cdot \begin{vmatrix} -1 & -4 \\ 1 & -3 \end{vmatrix} = 7$$

$$D_2=\begin{vmatrix}5&3&4&2\\1&1&2&1\\4&1&2&0\\1&0&1&1\end{vmatrix}\xlongequal[c_4-c_1]{c_3-c_1}\begin{vmatrix}5&3&-1&-3\\1&1&1&0\\4&1&-2&-4\\1&0&0&0\end{vmatrix}=(-1)^{4+1}\cdot\begin{vmatrix}3&-1&-3\\1&1&0\\1&-2&-4\end{vmatrix}$$

$$\xlongequal{c_1-c_2}-\begin{vmatrix}4&-1&-3\\0&1&0\\3&-2&-4\end{vmatrix}=-(-1)^{2+2}\cdot\begin{vmatrix}4&-3\\3&-4\end{vmatrix}=7,$$

$$D_3=\begin{vmatrix}5&0&3&2\\1&-1&1&1\\4&1&1&0\\1&1&0&1\end{vmatrix}\xlongequal[r_4+r_2]{r_3+r_2}\begin{vmatrix}5&0&3&2\\1&-1&1&1\\5&0&2&1\\2&0&1&2\end{vmatrix}=(-1)^{2+2}(-1)\begin{vmatrix}5&3&2\\5&2&1\\2&1&2\end{vmatrix}$$

$$\xlongequal[c_3-2c_2]{c_1-2c_2}-\begin{vmatrix}-1&3&-4\\1&2&-3\\0&1&0\end{vmatrix}=-(-1)^{3+2}\cdot\begin{vmatrix}-1&-4\\1&-3\end{vmatrix}=7$$

$$D_4 = \begin{vmatrix} 5 & 0 & 4 & 3 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 4 & 3 \\ 1 & -1 & 2 & 1 \\ 5 & 0 & 4 & 2 \\ 2 & 0 & 3 & 1 \end{vmatrix}$$

$$= - \begin{vmatrix} 5 & 4 & 3 \\ 5 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} \xlongequal{r_1-r_2} - \begin{vmatrix} 0 & 0 & 1 \\ 5 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} = - \begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = -7.$$

由克拉默法则可知，

$$x_1 = \frac{D_1}{D} = 1, \qquad x_2 = \frac{D_2}{D} = -1,$$

$$x_3 = \frac{D_3}{D} = -1, \quad x_4 = \frac{D_4}{D} = 1.$$

$$D_4 = \begin{vmatrix} 5 & 0 & 4 & 3 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 4 & 3 \\ 1 & -1 & 2 & 1 \\ 5 & 0 & 4 & 2 \\ 2 & 0 & 3 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 5 & 4 & 3 \\ 5 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} \xlongequal{r_1-r_2} -\begin{vmatrix} 0 & 0 & 1 \\ 5 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} = -\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = -7.$$

由克拉默法则可知，

$$x_1 = \frac{D_1}{D} = 1, \qquad x_2 = \frac{D_2}{D} = -1,$$

$$x_3 = \frac{D_3}{D} = -1, \quad x_4 = \frac{D_4}{D} = 1.$$

$$D_4 = \begin{vmatrix} 5 & 0 & 4 & 3 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 4 & 3 \\ 1 & -1 & 2 & 1 \\ 5 & 0 & 4 & 2 \\ 2 & 0 & 3 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 5 & 4 & 3 \\ 5 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} \xlongequal{r_1-r_2} -\begin{vmatrix} 0 & 0 & 1 \\ 5 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} = -\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = -7.$$

由克拉默法则可知，

$$x_1 = \frac{D_1}{D} = 1, \qquad x_2 = \frac{D_2}{D} = -1,$$

$$x_3 = \frac{D_3}{D} = -1, \quad x_4 = \frac{D_4}{D} = 1.$$

$$D_4 = \begin{vmatrix} 5 & 0 & 4 & 3 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 4 & 3 \\ 1 & -1 & 2 & 1 \\ 5 & 0 & 4 & 2 \\ 2 & 0 & 3 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 5 & 4 & 3 \\ 5 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} \xlongequal{r_1-r_2} -\begin{vmatrix} 0 & 0 & 1 \\ 5 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} = -\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = -7.$$

由克拉默法则可知，

$$x_1 = \frac{D_1}{D} = 1, \quad x_2 = \frac{D_2}{D} = -1,$$

$$x_3 = \frac{D_3}{D} = -1, \quad x_4 = \frac{D_4}{D} = 1.$$

$$D_4 = \begin{vmatrix} 5 & 0 & 4 & 3 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 4 & 3 \\ 1 & -1 & 2 & 1 \\ 5 & 0 & 4 & 2 \\ 2 & 0 & 3 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 5 & 4 & 3 \\ 5 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} \xlongequal{r_1-r_2} -\begin{vmatrix} 0 & 0 & 1 \\ 5 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} = -\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = -7.$$

由克拉默法则可知，

$$x_1 = \frac{D_1}{D} = 1, \qquad x_2 = \frac{D_2}{D} = -1,$$

$$x_3 = \frac{D_3}{D} = -1, \qquad x_4 = \frac{D_4}{D} = 1.$$

$$D_4=\begin{vmatrix}5&0&4&3\\1&-1&2&1\\4&1&2&1\\1&1&1&0\end{vmatrix}\xlongequal[r_4+r_2]{r_3+r_2}\begin{vmatrix}5&0&4&3\\1&-1&2&1\\5&0&4&2\\2&0&3&1\end{vmatrix}$$

$$=-\begin{vmatrix}5&4&3\\5&4&2\\2&3&1\end{vmatrix}\xlongequal{r_1-r_2}-\begin{vmatrix}0&0&1\\5&4&2\\2&3&1\end{vmatrix}=-\begin{vmatrix}5&4\\2&3\end{vmatrix}=-7.$$

由克拉默法则可知，

$$x_1=\frac{D_1}{D}=1,\qquad x_2=\frac{D_2}{D}=-1,$$

$$x_3=\frac{D_3}{D}=-1,\quad x_4=\frac{D_4}{D}=1.$$

$$D_4 = \begin{vmatrix} 5 & 0 & 4 & 3 \\ 1 & -1 & 2 & 1 \\ 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_4+r_2]{r_3+r_2} \begin{vmatrix} 5 & 0 & 4 & 3 \\ 1 & -1 & 2 & 1 \\ 5 & 0 & 4 & 2 \\ 2 & 0 & 3 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 5 & 4 & 3 \\ 5 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} \xlongequal{r_1-r_2} -\begin{vmatrix} 0 & 0 & 1 \\ 5 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} = -\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = -7.$$

由克拉默法则可知，

$$x_1 = \frac{D_1}{D} = 1, \qquad x_2 = \frac{D_2}{D} = -1,$$

$$x_3 = \frac{D_3}{D} = -1, \quad x_4 = \frac{D_4}{D} = 1.$$

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用克拉默法则求

$$\begin{cases} \quad\quad x_2 + x_3 + x_4 + x_5 = 1, \\ x_1 \quad\quad + x_3 + x_4 + x_5 = 2, \\ x_1 + x_2 \quad\quad + x_4 + x_5 = 3, \\ x_1 + x_2 + x_3 \quad\quad + x_5 = 4, \\ x_1 + x_2 + x_3 + x_4 \quad\quad = 5. \end{cases}$$

解:

$$D = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_1 \div 4]{r_1+r_2+\cdots+r_5} 4 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix},$$

$$\xlongequal[i=2,\cdots,4]{r_i - r_1} 4 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} = 4.$$

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用克拉默法则求

$$\begin{cases} x_2 + x_3 + x_4 + x_5 = 1, \\ x_1 + x_3 + x_4 + x_5 = 2, \\ x_1 + x_2 + x_4 + x_5 = 3, \\ x_1 + x_2 + x_3 + x_5 = 4, \\ x_1 + x_2 + x_3 + x_4 = 5. \end{cases}$$

解:

$$D = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_1 \div 4]{r_1+r_2+\cdots+r_5} 4 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix},$$

$$\xlongequal[i=2,\cdots,4]{r_i - r_1} 4 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} = 4.$$

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用克拉默法则求

$$\begin{cases} x_2 + x_3 + x_4 + x_5 = 1, \\ x_1 + x_3 + x_4 + x_5 = 2, \\ x_1 + x_2 + x_4 + x_5 = 3, \\ x_1 + x_2 + x_3 + x_5 = 4, \\ x_1 + x_2 + x_3 + x_4 = 5. \end{cases}$$

解:

$$D = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_1 \div 4]{r_1+r_2+\cdots+r_5} 4 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix},$$

$$\xlongequal[i=2,\cdots,4]{r_i - r_1} 4 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} = 4.$$

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用克拉默法则求

$$\begin{cases} \quad x_2 + x_3 + x_4 + x_5 = 1, \\ x_1 + x_3 + x_4 + x_5 = 2, \\ x_1 + x_2 + x_4 + x_5 = 3, \\ x_1 + x_2 + x_3 + x_5 = 4, \\ x_1 + x_2 + x_3 + x_4 = 5. \end{cases}$$

解:

$$D = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_1 \div 4]{r_1+r_2+\cdots+r_5} 4 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix},$$

$$\xlongequal[i=2,\cdots,4]{r_i - r_1} 4 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} = 4.$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5-r_1]{\substack{r_3-r_1\\ r_4-r_1}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+2} \cdot \begin{vmatrix} 2 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 11 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} = 11,$$

$$D_2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 1 & 4 & 1 & 0 & 1 \\ 1 & 5 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5-r_2]{\substack{r_3-r_2\\ r_4-r_2}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 7 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} = 7.$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5-r_1]{\substack{r_3-r_1 \\ r_4-r_1}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+2} \cdot \begin{vmatrix} 2 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 11 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} = 11,$$

$$D_2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 1 & 4 & 1 & 0 & 1 \\ 1 & 5 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5-r_2]{\substack{r_3-r_2 \\ r_4-r_2}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 7 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} = 7.$$

$$
D_1 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5 - r_1]{\substack{r_3 - r_1 \\ r_4 - r_1}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & 0 & -1 \end{vmatrix}
$$

$$
= (-1)^{1+2} \cdot \begin{vmatrix} 2 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1 + r_2 + r_3 + r_4} - \begin{vmatrix} 11 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} = 11,
$$

$$
D_2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 1 & 4 & 1 & 0 & 1 \\ 1 & 5 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5 - r_2]{\substack{r_3 - r_2 \\ r_4 - r_2}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 & -1 \end{vmatrix}
$$

$$
= (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1 + r_2 + r_3 + r_4} - \begin{vmatrix} 7 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} = 7.
$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5 - r_1]{\substack{r_3 - r_1 \\ r_4 - r_1}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+2} \cdot \begin{vmatrix} 2 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1 + r_2 + r_3 + r_4} - \begin{vmatrix} 11 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} = 11,$$

$$D_2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 1 & 4 & 1 & 0 & 1 \\ 1 & 5 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5 - r_2]{\substack{r_3 - r_2 \\ r_4 - r_2}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1 + r_2 + r_3 + r_4} - \begin{vmatrix} 7 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} = 7.$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 1 & 0 \end{vmatrix} \overset{\substack{r_3-r_1\\ r_4-r_1}}{\underset{r_5-r_1}{=\!=\!=\!=}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+2} \cdot \begin{vmatrix} 2 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} \overset{r_1+r_2+r_3+r_4}{=\!=\!=\!=\!=} - \begin{vmatrix} 11 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} = 11,$$

$$D_2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 1 & 4 & 1 & 0 & 1 \\ 1 & 5 & 1 & 1 & 0 \end{vmatrix} \overset{\substack{r_3-r_2\\ r_4-r_2}}{\underset{r_5-r_2}{=\!=\!=\!=}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} \overset{r_1+r_2+r_3+r_4}{=\!=\!=\!=\!=} - \begin{vmatrix} 7 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} = 7.$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5-r_1]{\substack{r_3-r_1\\ r_4-r_1}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+2} \cdot \begin{vmatrix} 2 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 11 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} = 11,$$

$$D_2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 1 & 4 & 1 & 0 & 1 \\ 1 & 5 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5-r_2]{\substack{r_3-r_2\\ r_4-r_2}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 7 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} = 7.$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5 - r_1]{\substack{r_3 - r_1 \\ r_4 - r_1}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+2} \cdot \begin{vmatrix} 2 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 11 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} = 11,$$

$$D_2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 1 & 4 & 1 & 0 & 1 \\ 1 & 5 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5 - r_2]{\substack{r_3 - r_2 \\ r_4 - r_2}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 7 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} = 7.$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5-r_1]{\substack{r_3-r_1\\ r_4-r_1}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+2} \cdot \begin{vmatrix} 2 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 11 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} = 11,$$

$$D_2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 1 & 4 & 1 & 0 & 1 \\ 1 & 5 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5-r_2]{\substack{r_3-r_2\\ r_4-r_2}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 7 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} = 7.$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5 - r_1]{\substack{r_3 - r_1 \\ r_4 - r_1}} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+2} \cdot \begin{vmatrix} 2 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 11 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{vmatrix} = 11,$$

$$D_2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 1 & 4 & 1 & 0 & 1 \\ 1 & 5 & 1 & 1 & 0 \end{vmatrix} \xlongequal[r_5 - r_2]{\substack{r_3 - r_2 \\ r_4 - r_2}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 7 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \end{vmatrix} = 7.$$

$$D_3 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 0 & 1 \\ 1 & 1 & 5 & 1 & 0 \end{vmatrix} \xlongequal[r_5 - r_3]{\substack{r_2 - r_3 \\ r_4 - r_3}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+3} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} \xlongequal{r_1 + r_2 + r_3 + r_4} \begin{vmatrix} 0 & 3 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} = 3,$$

$$D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 & 0 \end{vmatrix} \xlongequal[r_5 - r_4]{\substack{r_2 - r_4 \\ r_3 - r_4}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} \xlongequal{r_1 + r_2 + r_3 + r_4} - \begin{vmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1.$$

$$D_3 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 0 & 1 \\ 1 & 1 & 5 & 1 & 0 \end{vmatrix} \xlongequal[r_5 - r_3]{\substack{r_2 - r_3 \\ r_4 - r_3}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+3} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} \xlongequal{r_1 + r_2 + r_3 + r_4} \begin{vmatrix} 0 & 3 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} = 3,$$

$$D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 & 0 \end{vmatrix} \xlongequal[r_5 - r_4]{\substack{r_2 - r_4 \\ r_3 - r_4}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} \xlongequal{r_1 + r_2 + r_3 + r_4} - \begin{vmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1.$$

$$D_3 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 0 & 1 \\ 1 & 1 & 5 & 1 & 0 \end{vmatrix} \xlongequal[r_5 - r_3]{\substack{r_2 - r_3 \\ r_4 - r_3}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+3} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} \xlongequal{r_1 + r_2 + r_3 + r_4} \begin{vmatrix} 0 & 3 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} = 3,$$

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$$= (-1)^{4+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} \xlongequal{r_1 + r_2 + r_3 + r_4} - \begin{vmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1.$$

$$D_3 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 0 & 1 \\ 1 & 1 & 5 & 1 & 0 \end{vmatrix} \xlongequal[r_5 - r_3]{\substack{r_2 - r_3 \\ r_4 - r_3}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+3} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} \begin{vmatrix} 0 & 3 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} = 3,$$

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$$= (-1)^{4+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1.$$

$$D_3 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 0 & 1 \\ 1 & 1 & 5 & 1 & 0 \end{vmatrix} \xlongequal[r_5 - r_3]{\substack{r_2 - r_3 \\ r_4 - r_3}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+3} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} \xlongequal{r_1 + r_2 + r_3 + r_4} \begin{vmatrix} 0 & 3 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} = 3,$$

$$D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 & 0 \end{vmatrix} \xlongequal[r_5 - r_4]{\substack{r_2 - r_4 \\ r_3 - r_4}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} \xlongequal{r_1 + r_2 + r_3 + r_4} - \begin{vmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1.$$

$$D_3 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 0 & 1 \\ 1 & 1 & 5 & 1 & 0 \end{vmatrix} \overset{\substack{r_2-r_3 \\ r_4-r_3}}{\underset{r_5-r_3}{=\!=}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+3} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} \overset{r_1+r_2+r_3+r_4}{=\!=} \begin{vmatrix} 0 & 3 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} = 3,$$

$$D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 & 0 \end{vmatrix} \overset{\substack{r_2-r_4 \\ r_3-r_4}}{\underset{r_5-r_4}{=\!=}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} \overset{r_1+r_2+r_3+r_4}{=\!=} - \begin{vmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1.$$

$$D_3 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 0 & 1 \\ 1 & 1 & 5 & 1 & 0 \end{vmatrix} \xlongequal[r_5 - r_3]{\substack{r_2 - r_3 \\ r_4 - r_3}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+3} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} \xlongequal{r_1 + r_2 + r_3 + r_4} \begin{vmatrix} 0 & 3 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} = 3,$$

$$D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 & 0 \end{vmatrix} \xlongequal[r_5 - r_4]{\substack{r_2 - r_4 \\ r_3 - r_4}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} \xlongequal{r_1 + r_2 + r_3 + r_4} \begin{vmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1.$$

$$D_3 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 0 & 1 \\ 1 & 1 & 5 & 1 & 0 \end{vmatrix} \xlongequal[r_5 - r_3]{\substack{r_2 - r_3 \\ r_4 - r_3}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+3} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} \begin{vmatrix} 0 & 3 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} = 3,$$

$$D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 & 0 \end{vmatrix} \xlongequal[r_5 - r_4]{\substack{r_2 - r_4 \\ r_3 - r_4}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1.$$

$$D_3 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 0 & 1 \\ 1 & 1 & 5 & 1 & 0 \end{vmatrix} \xlongequal[r_5-r_3]{\substack{r_2-r_3 \\ r_4-r_3}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{1+3} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} \begin{vmatrix} 0 & 3 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{vmatrix} = 3,$$

$$D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 & 0 \end{vmatrix} \xlongequal[r_5-r_4]{\substack{r_2-r_4 \\ r_3-r_4}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} - \begin{vmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1.$$

$$D_5 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 3 \\ 1 & 1 & 1 & 0 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{vmatrix} \xlongequal[r_4 - r_5]{\substack{r_2 - r_5 \\ r_3 - r_5}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 5 \end{vmatrix}$$

$$= (-1)^{5+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & -3 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} \begin{vmatrix} 0 & 0 & 0 & -5 \\ -1 & 0 & 0 & -3 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & -1 \end{vmatrix} = -5.$$

由克拉默法则可知，

$$x_1 = \frac{D_1}{D} = \frac{11}{4}, \quad x_2 = \frac{D_2}{D} = \frac{7}{4}, \quad x_3 = \frac{D_3}{D} = \frac{3}{4},$$

$$x_4 = \frac{D_4}{D} = -\frac{1}{4}, \quad x_5 = \frac{D_5}{D} = -\frac{5}{4}.$$

$$D_5 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 3 \\ 1 & 1 & 1 & 0 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{vmatrix} \xlongequal[r_4 - r_5]{\substack{r_2 - r_5 \\ r_3 - r_5}} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 5 \end{vmatrix}$$

$$= (-1)^{5+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & -3 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} \begin{vmatrix} 0 & 0 & 0 & -5 \\ -1 & 0 & 0 & -3 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & -1 \end{vmatrix} = -5.$$

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$$= (-1)^{5+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & -3 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & -1 \end{vmatrix} \xlongequal{r_1+r_2+r_3+r_4} \begin{vmatrix} 0 & 0 & 0 & -5 \\ -1 & 0 & 0 & -3 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & -1 \end{vmatrix} = -5.$$

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$$= (-1)^{5+1} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & -3 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & -1 \end{vmatrix} \xlongequal{r_1 + r_2 + r_3 + r_4} \begin{vmatrix} 0 & 0 & 0 & -5 \\ -1 & 0 & 0 & -3 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & -1 \end{vmatrix} = -5.$$

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齐次线性方程组

$$\begin{cases} x_1 + x_2 + x_3 + ax_4 = 0, \\ x_1 + 2x_2 + x_3 + x_4 = 0, \\ x_1 + x_2 - 3x_3 + x_4 = 0, \\ x_1 + x_2 + ax_3 + bx_4 = 0. \end{cases}$$

有非零解时，a, b必须满足什么条件?

注

齐次线性方程组有非零解的充分必要条件是系数行列式为零。

解:

$$D = \begin{vmatrix} 1 & 1 & 1 & a \\ 1 & 2 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & a & b \end{vmatrix} \xlongequal{r_1 \leftrightarrow r_3} \begin{vmatrix} 1 & 1 & -3 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & a \\ 1 & 1 & a & b \end{vmatrix} \xlongequal[r_4 - r_1]{r_2 - r_1 \\ r_3 - r_1} \begin{vmatrix} 1 & 1 & -3 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 4 & a-1 \\ 0 & 0 & a+3 & b-1 \end{vmatrix} = 0,$$

即 $4(b-1)-(a-1)(a+3)=0$，也就是 $(a-1)^2 = 4b$.

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齐次线性方程组

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即 $4(b-1) - (a-1)(a+3) = 0$，也就是 $(a-1)^2 = 4b$.

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齐次线性方程组

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即 $4(b-1)-(a-1)(a+3)=0$，也就是 $(a-1)^2 = 4b$.

33

齐次线性方程组

$$\begin{cases} x_1 + x_2 + x_3 + ax_4 = 0, \\ x_1 + 2x_2 + x_3 + x_4 = 0, \\ x_1 + x_2 - 3x_3 + x_4 = 0, \\ x_1 + x_2 + ax_3 + bx_4 = 0. \end{cases}$$

有非零解时，a, b必须满足什么条件?

注

齐次线性方程组有非零解的充分必要条件是系数行列式为零。

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34

求平面上过两点(x_1, y_1)和(x_2, y_2)的直线方程（用行列式表示）。

解：直线方程的两点式为

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1},$$

即

$$(y-y_1)(x_2-x_1)=(x-x_1)(y_2-y_1)$$

亦即

$$x(y_1-y_2)+y(x_2-x_1)+x_1y_2-x_2y_1=0.$$

由行列式的按行展开可知，其行列式形式为

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0.$$

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35

求三次多项式$f(x)=a_0+a_1x+a_2x^2+a_3x^3$，使得

$$f(-1)=0,\quad f(1)=4,\quad f(2)=3,\quad f(3)=16.$$

解: 由条件可知，$f(x)$应满足线性方程组

$$\begin{cases} a_0 + (-1)a_1 + (-1)^2a_2 + (-1)^3a_3 = 0, \\ a_0 + a_1 + a_2 + a_3 = 4, \\ a_0 + 2a_1 + (2)^2a_2 + (2)^3a_3 = 3, \\ a_0 + 3a_1 + (3)^2a_2 + (3)^3a_3 = 16. \end{cases}$$

其系数行列式D为范德蒙行列式

$$D=\begin{vmatrix} 1 & -1 & (-1)^2 & (-1)^3 \\ 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \end{vmatrix}=(3+1)(3-1)(3-2)(2+1)(2-1)(1+1)=48.$$

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求三次多项式$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$，使得

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$$D_1 = \begin{vmatrix} 0 & -1 & 1 & -1 \\ 4 & 1 & 1 & 1 \\ 3 & 2 & 4 & 8 \\ 16 & 3 & 9 & 27 \end{vmatrix} \xlongequal[c_4+c_3]{c_2+c_3} \begin{vmatrix} 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & 2 \\ 3 & 6 & 4 & 12 \\ 16 & 12 & 9 & 36 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 2 \\ 3 & 6 & 12 \\ 16 & 12 & 36 \end{vmatrix}$$

$$\xlongequal[c_2-c_3]{c_1-2c_3} \begin{vmatrix} 0 & 0 & 2 \\ -9 & -6 & 12 \\ -8 & -24 & 36 \end{vmatrix} = 48 \times 7,$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 & -1 \\ 1 & 4 & 1 & 1 \\ 1 & 3 & 4 & 8 \\ 1 & 16 & 9 & 27 \end{vmatrix} \xlongequal[c_3+c_4]{c_1+c_4} \begin{vmatrix} 0 & 0 & 0 & -1 \\ 2 & 4 & 2 & 1 \\ 9 & 3 & 12 & 8 \\ 28 & 16 & 36 & 27 \end{vmatrix} = -\begin{vmatrix} 2 & 4 & 2 \\ 9 & 3 & 12 \\ 28 & 16 & 36 \end{vmatrix}$$

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$$D_3 = \begin{vmatrix} 1 & -1 & 0 & -1 \\ 1 & 1 & 4 & 1 \\ 1 & 2 & 3 & 8 \\ 1 & 3 & 16 & 27 \end{vmatrix} \overset{c_2+c_1}{\underset{c_4+c_1}{=\!=\!=}} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 2 \\ 1 & 3 & 3 & 9 \\ 1 & 4 & 16 & 28 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 2 \\ 3 & 3 & 9 \\ 4 & 16 & 28 \end{vmatrix}$$

$$\overset{c_2-2c_1}{\underset{c_2-c_1}{=\!=\!=}} \begin{vmatrix} 2 & 0 & 0 \\ 3 & -3 & 6 \\ 4 & 8 & 24 \end{vmatrix} = 48 \times (-5),$$

$$D_4 = \begin{vmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 16 \end{vmatrix} \overset{c_2+c_1}{\underset{c_3-c_1}{=\!=\!=}} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 4 \\ 1 & 3 & 3 & 3 \\ 1 & 4 & 8 & 16 \end{vmatrix} = - \begin{vmatrix} 2 & 0 & 4 \\ 3 & 3 & 3 \\ 4 & 8 & 16 \end{vmatrix}$$

$$\overset{c_2-2c_1}{\underset{c_3-c_1}{=\!=\!=}} \begin{vmatrix} 2 & 0 & 0 \\ 3 & 3 & -3 \\ 4 & 8 & 4 \end{vmatrix} = 48 \times 2.$$

$$D_3 = \begin{vmatrix} 1 & -1 & 0 & -1 \\ 1 & 1 & 4 & 1 \\ 1 & 2 & 3 & 8 \\ 1 & 3 & 16 & 27 \end{vmatrix} \xlongequal[c_4+c_1]{c_2+c_1} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 2 \\ 1 & 3 & 3 & 9 \\ 1 & 4 & 16 & 28 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 2 \\ 3 & 3 & 9 \\ 4 & 16 & 28 \end{vmatrix}$$

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$$\xlongequal[c_3-c_1]{c_2-2c_1} \begin{vmatrix} 2 & 0 & 0 \\ 3 & 3 & -3 \\ 4 & 8 & 4 \end{vmatrix} = 48 \times 2.$$

$$D_3 = \begin{vmatrix} 1 & -1 & 0 & -1 \\ 1 & 1 & 4 & 1 \\ 1 & 2 & 3 & 8 \\ 1 & 3 & 16 & 27 \end{vmatrix} \xlongequal[c_4+c_1]{c_2+c_1} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 2 \\ 1 & 3 & 3 & 9 \\ 1 & 4 & 16 & 28 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 2 \\ 3 & 3 & 9 \\ 4 & 16 & 28 \end{vmatrix}$$

$$\xlongequal[c_2-c_1]{c_2-2c_1} \begin{vmatrix} 2 & 0 & 0 \\ 3 & -3 & 6 \\ 4 & 8 & 24 \end{vmatrix} = 48 \times (-5),$$

$$D_4 = \begin{vmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 16 \end{vmatrix} \xlongequal[c_3-c_1]{c_2+c_1} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 4 \\ 1 & 3 & 3 & 3 \\ 1 & 4 & 8 & 16 \end{vmatrix} = -\begin{vmatrix} 2 & 0 & 4 \\ 3 & 3 & 3 \\ 4 & 8 & 16 \end{vmatrix}$$

$$\xlongequal[c_3-c_1]{c_2-2c_1} \begin{vmatrix} 2 & 0 & 0 \\ 3 & 3 & -3 \\ 4 & 8 & 4 \end{vmatrix} = 48 \times 2.$$

$$D_3 = \begin{vmatrix} 1 & -1 & 0 & -1 \\ 1 & 1 & 4 & 1 \\ 1 & 2 & 3 & 8 \\ 1 & 3 & 16 & 27 \end{vmatrix} \xlongequal[c_4+c_1]{c_2+c_1} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 2 \\ 1 & 3 & 3 & 9 \\ 1 & 4 & 16 & 28 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 2 \\ 3 & 3 & 9 \\ 4 & 16 & 28 \end{vmatrix}$$

$$\xlongequal[c_2-c_1]{c_2-2c_1} \begin{vmatrix} 2 & 0 & 0 \\ 3 & -3 & 6 \\ 4 & 8 & 24 \end{vmatrix} = 48 \times (-5),$$

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$$\xlongequal[c_3-c_1]{c_2-2c_1} \begin{vmatrix} 2 & 0 & 0 \\ 3 & 3 & -3 \\ 4 & 8 & 4 \end{vmatrix} = 48 \times 2.$$

由克拉默法则可知

$$x_1 = \frac{D_1}{D} = 7,\quad x_2 = \frac{D_2}{D} = 0,\quad x_3 = \frac{D_3}{D} = -5,\quad x_4 = \frac{D_4}{D} = 2.$$

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证明恒等式

$$\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} = \left(1+\sum_{i=1}^{n}\frac{1}{a_i}\right)\prod_{i=1}^{n} a_i$$

证明1:

$$\begin{aligned} \text{左边} &= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} + \begin{vmatrix} a_1 & 1 & \cdots & 1 \\ 0 & 1+a_2 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 1+a_n \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a_n \end{vmatrix} + \begin{vmatrix} a_1 & 1 & \cdots & 1 \\ 0 & 1+a_2 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 1+a_n \end{vmatrix} \end{aligned}$$

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证明恒等式

$$\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} = \left(1+\sum_{i=1}^{n}\frac{1}{a_i}\right)\prod_{i=1}^{n} a_i$$

证明1:

$$\begin{aligned} \text{左边} &= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} + \begin{vmatrix} a_1 & 1 & \cdots & 1 \\ 0 & 1+a_2 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 1+a_n \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a_n \end{vmatrix} + \begin{vmatrix} a_1 & 1 & \cdots & 1 \\ 0 & 1+a_2 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 1+a_n \end{vmatrix} \end{aligned}$$

$$
\begin{aligned}
\text{左边} &= a_2\cdots a_n + a_1\begin{vmatrix} 1+a_2 & 1 & \cdots & 1 \\ 1 & 1+a_3 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} \\
&= a_2\cdots a_n + a_1\left(\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1+a_3 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} + \begin{vmatrix} a_2 & 1 & \cdots & 1 \\ 0 & 1+a_3 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 1+a_n \end{vmatrix}\right) \\
&= a_2\cdots a_n + a_1\left(\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & a_3 & \cdots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a_n \end{vmatrix} + \begin{vmatrix} a_2 & 1 & \cdots & 1 \\ 0 & 1+a_3 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 1+a_n \end{vmatrix}\right) \\
&= a_2\cdots a_n + a_1\left(a_3\cdots a_n + a_2\begin{vmatrix} 1+a_3 & 1 & \cdots & 1 \\ 1 & 1+a_4 & \cdots & 1 \\ \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix}\right) \\
&= \cdots = a_2\cdots a_n + a_1a_3\cdots a_n + \cdots + a_1\cdots a_{n-1} + a_1\cdots a_n.
\end{aligned}
$$

证明2：

$$
\text{左边} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1+a_1 & 1 & \cdots & 1 \\ 0 & 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & 1 & \cdots & 1+a_n \end{vmatrix} \xlongequal[i=2,\cdots,n+1]{r_i - r_1} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & a_1 & 0 & \cdots & 0 \\ -1 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & a_n \end{vmatrix}
$$

$$
\xlongequal{r_1 + \sum_{i=1}^{n} \frac{1}{a_i} r_{i+1}} \begin{vmatrix} 1+\sum_{i=1}^{n}\frac{1}{a_i} & 0 & 0 & \cdots & 0 \\ -1 & a_1 & 0 & \cdots & 0 \\ -1 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & a_n \end{vmatrix} = \left(1+\sum_{i=1}^{n}\frac{1}{a_i}\right)\prod_{i=1}^{n} a_i.
$$

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$$\begin{vmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ 0 & x & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & -1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & x+a_1 \end{vmatrix} = x^n + \sum_{k=1}^{n} a_k x^{n-k}.$$

证明： 记行列式为 D_n，则

$$D_n = xD_{n-1} + (-1)^{n+1} a_n \begin{vmatrix} -1 & 0 & \cdots & 0 & 0 \\ x & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & x & -1 \end{vmatrix} = xD_{n-1} + a_n.$$

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$$\begin{vmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ 0 & x & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & -1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & x+a_1 \end{vmatrix} = x^n + \sum_{k=1}^{n} a_k x^{n-k}.$$

证明： 记行列式为 D_n，则

$$D_n = xD_{n-1} + (-1)^{n+1} a_n \begin{vmatrix} -1 & 0 & \cdots & 0 & 0 \\ x & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & x & -1 \end{vmatrix} = xD_{n-1} + a_n.$$

于是

$$\begin{aligned} D_n &= xD_{n-1} + a_n, \\ D_{n-1} &= xD_{n-2} + a_{n-1}, && \cdots\cdots \times x \\ D_{n-2} &= xD_{n-3} + a_{n-2}, && \cdots\cdots \times x^2 \\ &\cdots \\ D_2 &= xD_1 + a_2. && \cdots\cdots \times x^{n-2} \end{aligned}$$

所以

$$\begin{aligned} D_n &= a_n + a_{n-1}x + \cdots + a_2x^{n-2} + x^{n-1}D_1 \\ &= a_n + a_{n-1}x + \cdots + a_2x^{n-2} + x^{n-1}(x + a_1) = \text{右边} \end{aligned}$$

于是

$$\begin{aligned} D_n &= xD_{n-1} + a_n, \\ D_{n-1} &= xD_{n-2} + a_{n-1}, && \cdots\cdots \times x \\ D_{n-2} &= xD_{n-3} + a_{n-2}, && \cdots\cdots \times x^2 \\ &\cdots \\ D_2 &= xD_1 + a_2. && \cdots\cdots \times x^{n-2} \end{aligned}$$

所以

$$\begin{aligned} D_n &= a_n + a_{n-1}x + \cdots + a_2x^{n-2} + x^{n-1}D_1 \\ &= a_n + a_{n-1}x + \cdots + a_2x^{n-2} + x^{n-1}(x + a_1) = \text{右边} \end{aligned}$$

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证明

$$\begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 & 0 \\ a_2 & x & -1 & \cdots & 0 & 0 \\ a_3 & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & 0 & 0 & \cdots & x & -1 \\ a_n & 0 & 0 & \cdots & 0 & x \end{vmatrix} = \sum_{k=1}^{n} a_k x^{n-k}$$

证明：记行列式为 D_n

$$\begin{aligned} D_n &= (-1)^{n+1} a_n \begin{vmatrix} -1 & 0 & \cdots & 0 & 0 \\ x & -1 & \cdots & 0 & 0 \\ 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & x & -1 \end{vmatrix} + (-1)^{2n} x \begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 \\ a_2 & x & -1 & \cdots & 0 \\ a_3 & 0 & x & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1} & 0 & 0 & \cdots & x \end{vmatrix} \\ &= a_n + x D_{n-1}. \end{aligned}$$

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证明

$$\begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 & 0 \\ a_2 & x & -1 & \cdots & 0 & 0 \\ a_3 & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & 0 & 0 & \cdots & x & -1 \\ a_n & 0 & 0 & \cdots & 0 & x \end{vmatrix} = \sum_{k=1}^{n} a_k x^{n-k}$$

证明： 记行列式为 D_n

$$\begin{aligned} D_n &= (-1)^{n+1} a_n \begin{vmatrix} -1 & 0 & \cdots & 0 & 0 \\ x & -1 & \cdots & 0 & 0 \\ 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & x & -1 \end{vmatrix} + (-1)^{2n} x \begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 \\ a_2 & x & -1 & \cdots & 0 \\ a_3 & 0 & x & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1} & 0 & 0 & \cdots & x \end{vmatrix} \\ &= a_n + x D_{n-1}. \end{aligned}$$

于是

$$
\begin{aligned}
D_n &= xD_{n-1} + a_n, \\
D_{n-1} &= xD_{n-2} + a_{n-1}, \qquad \cdots\cdots \times x \\
D_{n-2} &= xD_{n-3} + a_{n-2}, \qquad \cdots\cdots \times x^2 \\
&\cdots \\
D_2 &= xD_1 + a_2. \qquad\qquad \cdots\cdots \times x^{n-2}
\end{aligned}
$$

所以

$$
\begin{aligned}
D_n &= a_n + a_{n-1}x + \cdots + a_2x^{n-2} + x^{n-1}D_1 \\
&= a_n + a_{n-1}x + \cdots + a_2x^{n-2} + x^{n-1}a_1 = \text{右边}
\end{aligned}
$$

于是

$$\begin{aligned} D_n &= xD_{n-1} + a_n, \\ D_{n-1} &= xD_{n-2} + a_{n-1}, & \cdots\cdots \times x \\ D_{n-2} &= xD_{n-3} + a_{n-2}, & \cdots\cdots \times x^2 \\ &\cdots \\ D_2 &= xD_1 + a_2. & \cdots\cdots \times x^{n-2} \end{aligned}$$

所以

$$\begin{aligned} D_n &= a_n + a_{n-1}x + \cdots + a_2x^{n-2} + x^{n-1}D_1 \\ &= a_n + a_{n-1}x + \cdots + a_2x^{n-2} + x^{n-1}a_1 = \text{右边} \end{aligned}$$

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$$\begin{vmatrix} \cos\theta & 1 & & & \\ 1 & 2\cos\theta & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2\cos\theta & 1 \\ & & & 1 & 2\cos\theta \end{vmatrix} = \cos n\theta$$

证明:

$$\begin{aligned} D_n &= (-1)^{n+(n-1)} \begin{vmatrix} \cos\theta & 1 & & & \\ 1 & 2\cos\theta & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2\cos\theta & 1 \\ & & & 1 & 2\cos\theta \\ & & & & 1 & 1 \end{vmatrix}_{n-1} + 2\cos\theta D_{n-1} \\ &= -D_{n-2} + 2\cos\theta D_{n-1}. \end{aligned}$$

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$$\begin{vmatrix} \cos\theta & 1 & & & \\ 1 & 2\cos\theta & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2\cos\theta & 1 \\ & & & 1 & 2\cos\theta \end{vmatrix} = \cos n\theta$$

证明:

$$\begin{aligned} D_n &= (-1)^{n+(n-1)} \begin{vmatrix} \cos\theta & 1 & & & & \\ 1 & 2\cos\theta & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 1 & 2\cos\theta & 1 & \\ & & & 1 & 2\cos\theta & \\ & & & & 1 & 1 \end{vmatrix}_{n-1} + 2\cos\theta D_{n-1} \\ &= -D_{n-2} + 2\cos\theta D_{n-1}. \end{aligned}$$

用数学归纳法证明。

1^o 当$n=1$时，结论显然成立。

2^o 假设结论对阶数$\leq n-1$的行列式成立，则由上式可知

$$\begin{aligned}
D_n &= -D_{n-2} + 2\cos\theta D_{n-1} \\
&= -\cos(n-2)\theta + 2\cos\theta\cos(n-1)\theta \\
&= -\cos(n-2)\theta + \cos(n-2)\theta\cos n\theta \\
&= \cos n\theta.
\end{aligned}$$

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计算

$$\begin{vmatrix} \frac{1}{3} & -\frac{5}{2} & \frac{2}{5} & \frac{3}{2} \\ 3 & -12 & \frac{21}{5} & 15 \\ \frac{2}{3} & -\frac{9}{2} & \frac{4}{5} & \frac{5}{2} \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \frac{3}{7} \end{vmatrix}$$

解:

$$\text{原式} = \frac{1}{30} \times \frac{3}{5} \times \frac{1}{30} \times \frac{1}{7} \times \begin{vmatrix} 10 & -75 & 12 & 45 \\ 5 & -20 & 7 & 25 \\ 20 & -135 & 24 & 75 \\ -1 & 2 & -1 & 3 \end{vmatrix}$$

$$\xlongequal[r_4+3\times r_1]{\substack{r_2+2\times r_1 \\ r_3-r_1}} \frac{1}{35\times 300} \times \begin{vmatrix} 10 & -55 & 2 & 75 \\ 5 & -10 & 2 & 40 \\ 20 & -95 & 4 & 135 \\ -1 & 0 & 0 & 0 \end{vmatrix}$$

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计算

$$\begin{vmatrix} \frac{1}{3} & -\frac{5}{2} & \frac{2}{5} & \frac{3}{2} \\ 3 & -12 & \frac{21}{5} & 15 \\ \frac{2}{3} & -\frac{9}{2} & \frac{4}{5} & \frac{5}{2} \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \frac{3}{7} \end{vmatrix}$$

解:

$$\text{原式} = \frac{1}{30} \times \frac{3}{5} \times \frac{1}{30} \times \frac{1}{7} \times \begin{vmatrix} 10 & -75 & 12 & 45 \\ 5 & -20 & 7 & 25 \\ 20 & -135 & 24 & 75 \\ -1 & 2 & -1 & 3 \end{vmatrix}$$

$$\xlongequal[r_4+3\times r_1]{\substack{r_2+2\times r_1 \\ r_3-r_1}} \frac{1}{35 \times 300} \times \begin{vmatrix} 10 & -55 & 2 & 75 \\ 5 & -10 & 2 & 40 \\ 20 & -95 & 4 & 135 \\ -1 & 0 & 0 & 0 \end{vmatrix}$$

$$
\begin{aligned}
\text{原式} &= \frac{1}{35\times 300}\times\begin{vmatrix} -55 & 2 & 75 \\ -10 & 2 & 40 \\ -95 & 4 & 135 \end{vmatrix} \\
&\overset{r_2-r_1}{\underset{r_3-2r_1}{=\!=}} \frac{1}{35\times 300}\times\begin{vmatrix} -55 & 2 & 75 \\ 40 & 0 & -35 \\ 15 & 0 & -15 \end{vmatrix} \\
&= \frac{-2}{35\times 300}\times\begin{vmatrix} 40 & -35 \\ 15 & -15 \end{vmatrix} = \frac{-2}{35\times 300}\times 15\times(-45+35) = \frac{1}{35}.
\end{aligned}
$$

41

计算

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & -n \\ 1 & 1 & \cdots & -n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & \cdots & 1 & 1 \\ -n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal{r_1+r_2+\cdots+r_n} \begin{vmatrix} -1 & 1 & \cdots & 1 & -n \\ -1 & 1 & \cdots & -n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & -n & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{r_i-r_1} \begin{vmatrix} -1 & 1 & \cdots & 1 & -n \\ 0 & 0 & \cdots & -n-1 & 1+n \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & -n-1 & \cdots & 0 & 1+n \\ 0 & 0 & \cdots & 0 & 1+n \end{vmatrix}$$

41

计算

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & -n \\ 1 & 1 & \cdots & -n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & \cdots & 1 & 1 \\ -n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

解:

$$\text{原式} \xlongequal{r_1+r_2+\cdots+r_n} \begin{vmatrix} -1 & 1 & \cdots & 1 & -n \\ -1 & 1 & \cdots & -n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & -n & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{r_i-r_1} \begin{vmatrix} -1 & 1 & \cdots & 1 & -n \\ 0 & 0 & \cdots & -n-1 & 1+n \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & -n-1 & \cdots & 0 & 1+n \\ 0 & 0 & \cdots & 0 & 1+n \end{vmatrix}$$

$$
\begin{aligned}
\text{原式} &= -\begin{vmatrix} 0 & \cdots & -n-1 & 1+n \\ \vdots & & \vdots & \vdots \\ -n-1 & \cdots & 0 & 1+n \\ 0 & \cdots & 0 & 1+n \end{vmatrix}_{n-1} \\
&= -(n+1)\begin{vmatrix} 0 & \cdots & -n-1 \\ \vdots & & \vdots \\ -n-1 & \cdots & 0 \end{vmatrix}_{n-2} \\
&= -(n+1)(-1)^{\frac{(n-2)(n-3)}{2}}(-n-1)^{n-2} \\
&= (-1)^{\frac{(n-2)(n-3)}{2}+(n-1)}(n+1)^{n-1} \\
&= (-1)^{\frac{n^2-5n+6+2n-2}{2}}(n+1)^{n-1} \\
&= (-1)^{\frac{n^2-3n+4}{2}}(n+1)^{n-1} \\
&= (-1)^{\frac{n^2+n-4n+4}{2}}(n+1)^{n-1} \\
&= (-1)^{\frac{n^2+n}{2}}(n+1)^{n-1}.
\end{aligned}
$$

42

计算

$$\begin{vmatrix} a_1+\lambda_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & a_2+\lambda_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & a_3+\lambda_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n+\lambda_n \end{vmatrix}$$

解:

$$原式 = \begin{vmatrix} 1 & a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1+\lambda_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2+\lambda_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & a_3+\lambda_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & a_1 & a_2 & a_3 & \cdots & a_n+\lambda_n \end{vmatrix}$$

42

计算

$$\begin{vmatrix} a_1+\lambda_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & a_2+\lambda_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & a_3+\lambda_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n+\lambda_n \end{vmatrix}$$

解:

$$\text{原式} = \begin{vmatrix} 1 & a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1+\lambda_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2+\lambda_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & a_3+\lambda_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & a_1 & a_2 & a_3 & \cdots & a_n+\lambda_n \end{vmatrix}$$

$$
\text{原式} = \begin{vmatrix} 1 & a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1+\lambda_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2+\lambda_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & a_3+\lambda_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & a_1 & a_2 & a_3 & \cdots & a_n+\lambda_n \end{vmatrix}_{n+1}
$$

$$
= \begin{vmatrix} 1 & a_1 & a_2 & a_3 & \cdots & a_n \\ -1 & \lambda_1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & \lambda_2 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & 0 & \cdots & \lambda_n \end{vmatrix}_{n+1}
$$

$$
= \begin{vmatrix} 1+\sum_{i=1}^{n}\frac{a_i}{\lambda_i} & 0 & 0 & 0 & \cdots & 0 \\ -1 & \lambda_1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & \lambda_2 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & 0 & \cdots & \lambda_n \end{vmatrix}_{n+1} = \left(1+\sum_{i=1}^{n}\frac{a_i}{\lambda_i}\right)\prod_{i=1}^{n} a_i.
$$

43

计算

$$D = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

解:

$$D_n \xlongequal[i=n,\cdots,2]{r_i - r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i - c_1} \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

43

计算

$$D=\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

解:

$$D_n \xlongequal[i=n,\cdots,2]{r_i-r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i-c_1} \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

43

计算

$$D=\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

解:

$$D_n \xlongequal[i=n,\cdots,2]{r_i-r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i-c_1} \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

43

计算

$$D=\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

解:

$$D_n \xlongequal[i=n,\cdots,2]{r_i-r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i-c_1} \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

43

计算

$$D=\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

解:

$$D_n \xlongequal[i=n,\cdots,2]{r_i-r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i-c_1} \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

43

计算

$$D=\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$$

解:

$$D_n \xlongequal[i=n,\cdots,2]{r_i-r_{i-1}} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i-c_1} \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$D_n = \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i \div n} n^{n-1} \begin{vmatrix} 1 & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$\xlongequal{c_1+c_2+\cdots+c_n} n^{n-1} \begin{vmatrix} 1+\sum_{i=1}^{n-1}\frac{i}{n} & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$= n^{n-1}\left[1+\frac{1}{n}\frac{n(n-1)}{2}\right](-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{n-1} = (-1)^{\frac{(n-1)n}{2}}\frac{n+1}{2}n^{n-1}.$$

$$D_n = \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i \div n} n^{n-1} \begin{vmatrix} 1 & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$\xlongequal{c_1+c_2+\cdots+c_n} n^{n-1} \begin{vmatrix} 1+\sum_{i=1}^{n-1} \frac{i}{n} & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$= n^{n-1}\left[1+\frac{1}{n}\frac{n(n-1)}{2}\right](-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{n-1} = (-1)^{\frac{(n-1)n}{2}}\frac{n+1}{2}n^{n-1}.$$

$$
D_n = \begin{vmatrix}
1 & 1 & 2 & \cdots & n-2 & n-1 \\
1 & 0 & 0 & \cdots & 0 & -n \\
1 & 0 & 0 & \cdots & -n & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
1 & -n & 0 & \cdots & 0 & 0
\end{vmatrix}
$$

$$
\xlongequal[i=2,\cdots,n]{c_i \div n} n^{n-1} \begin{vmatrix}
1 & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\
1 & 0 & 0 & \cdots & 0 & -1 \\
1 & 0 & 0 & \cdots & -1 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
1 & -1 & 0 & \cdots & 0 & 0
\end{vmatrix}
$$

$$
\xlongequal{c_1+c_2+\cdots+c_n} n^{n-1} \begin{vmatrix}
1+\sum_{i=1}^{n-1} \frac{i}{n} & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\
0 & 0 & 0 & \cdots & 0 & -1 \\
0 & 0 & 0 & \cdots & -1 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & -1 & 0 & \cdots & 0 & 0
\end{vmatrix}
$$

$$
= n^{n-1}\left[1+\frac{1}{n}\frac{n(n-1)}{2}\right](-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{n-1} = (-1)^{\frac{(n-1)n}{2}}\frac{n+1}{2}n^{n-1}.
$$

$$
\begin{aligned}
D_n &= \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&\xlongequal[i=2,\cdots,n]{c_i \div n} n^{n-1} \begin{vmatrix} 1 & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&\xlongequal{c_1+c_2+\cdots+c_n} n^{n-1} \begin{vmatrix} 1+\sum_{i=1}^{n-1}\frac{i}{n} & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&= n^{n-1}\left[1+\frac{1}{n}\frac{n(n-1)}{2}\right](-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{n-1} = (-1)^{\frac{(n-1)n}{2}}\frac{n+1}{2}n^{n-1}.
\end{aligned}
$$

$$
\begin{aligned}
D_n &= \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&\xlongequal[i=2,\cdots,n]{c_i \div n} n^{n-1} \begin{vmatrix} 1 & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&\xlongequal{c_1+c_2+\cdots+c_n} n^{n-1} \begin{vmatrix} 1+\sum_{i-1}^{n-1}\frac{i}{n} & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&= n^{n-1}\left[1+\frac{1}{n}\frac{n(n-1)}{2}\right](-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{n-1} = (-1)^{\frac{(n-1)n}{2}}\frac{n+1}{2}n^{n-1}.
\end{aligned}
$$

$$D_n = \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i \div n} n^{n-1} \begin{vmatrix} 1 & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$\xlongequal{c_1+c_2+\cdots+c_n} n^{n-1} \begin{vmatrix} 1+\sum_{i-1}^{n-1} \frac{i}{n} & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$= n^{n-1}\left[1+\frac{1}{n}\frac{n(n-1)}{2}\right](-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{n-1} = (-1)^{\frac{(n-1)n}{2}}\frac{n+1}{2}n^{n-1}.$$

$$D_n = \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 0 & 0 & \cdots & 0 & -n \\ 1 & 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$\xlongequal[i=2,\cdots,n]{c_i \div n} n^{n-1} \begin{vmatrix} 1 & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$\xlongequal{c_1+c_2+\cdots+c_n} n^{n-1} \begin{vmatrix} 1+\sum_{i-1}^{n-1} \frac{i}{n} & \frac{1}{n} & \frac{2}{n} & \cdots & \frac{n-2}{n} & \frac{n-1}{n} \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$= n^{n-1}\left[1+\frac{1}{n}\frac{n(n-1)}{2}\right](-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{n-1} = (-1)^{\frac{(n-1)n}{2}}\frac{n+1}{2}n^{n-1}.$$

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证明

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \\ x_1^n & x_2^n & x_3^n & \cdots & x_n^n \end{vmatrix} = \sum_{i=1}^{n} x_i \prod_{1\le j<i\le n}(x_i - x_j).$$

证明：考察行列式

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n & y \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 & y^2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_1^{n-2} & x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} & y^{n-2} \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} & y^{n-1} \\ x_1^n & x_2^n & x_3^n & \cdots & x_n^n & y^n \end{vmatrix} = \prod_{i=1}^{n}(y - x_i) \prod_{1\le j<i\le n}(x_i - x_j)$$

等式两端均为关于y的多项式，比较y^{n-1}的系数便得结论。

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证明

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \\ x_1^n & x_2^n & x_3^n & \cdots & x_n^n \end{vmatrix} = \sum_{i=1}^{n} x_i \prod_{1\le j<i\le n} (x_i - x_j).$$

证明： 考察行列式

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n & y \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 & y^2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_1^{n-2} & x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} & y^{n-2} \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} & y^{n-1} \\ x_1^n & x_2^n & x_3^n & \cdots & x_n^n & y^n \end{vmatrix} = \prod_{i=1}^{n}(y - x_i) \prod_{1\le j<i\le n} (x_i - x_j)$$

等式两端均为关于y的多项式，比较y^{n-1}的系数便得结论。

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用数学归纳法证明：

$$\frac{d}{dt}\begin{vmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{vmatrix} = \sum_{j=1}^{n} \begin{vmatrix} a_{11}(t) & \cdots & \frac{d}{dt}a_{1j}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & \cdots & \frac{d}{dt}a_{2j}(t) & \cdots & a_{2n}(t) \\ \vdots & & \vdots & & \vdots \\ a_{n1}(t) & \cdots & \frac{d}{dt}a_{nj}(t) & \cdots & a_{nn}(t) \end{vmatrix}$$

1^o 当 $n=1$ 时，结论显然成立。

2^o 假设结论对阶数 $\leq n-1$ 的行列式成立，考虑阶数为 n 的行列式，对第一列展开得

$$\begin{aligned} D &= a_{11}A_{11}+a_{21}A_{21}+\cdots+a_{n1}A_{n1}, \\ D' &= a'_{11}A_{11}+a'_{21}A_{21}+\cdots+a'_{n1}A_{n1}+ \\ & \quad a_{11}A'_{11}+a_{21}A'_{21}+\cdots+a_{n1}A'_{n1}, \end{aligned}$$

其中

$$a'_{11}(t)A_{11}(t)+a'_{21}(t)A_{21}(t)+\cdots+a'_{n1}(t)A_{n1}(t)=\begin{vmatrix} a'_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a'_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a'_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{vmatrix}$$

$$
a_{11}A'_{11}+a_{21}A'_{21}+\cdots+a_{n1}A'_{n1}=a_{11}\sum_{j=2}^{n}\begin{vmatrix} a_{22}(t) & \cdots & a'_{2j}(t) & \cdots & a_{2n}(t) \\ a_{32}(t) & \cdots & a'_{3j}(t) & \cdots & a_{3n}(t) \\ \vdots & & \vdots & & \vdots \\ a_{n2}(t) & \cdots & a'_{nj}(t) & \cdots & a_{nn}(t) \end{vmatrix}
$$

$$
-a_{21}\sum_{j=2}^{n}\begin{vmatrix} a_{12}(t) & \cdots & a'_{1j}(t) & \cdots & a_{1n}(t) \\ a_{32}(t) & \cdots & a'_{3j}(t) & \cdots & a_{3n}(t) \\ \vdots & & \vdots & & \vdots \\ a_{n2}(t) & \cdots & a'_{nj}(t) & \cdots & a_{nn}(t) \end{vmatrix}+\cdots
$$

$$
+(-1)^{n+1}a_{n1}\sum_{j=2}^{n}\begin{vmatrix} a_{12}(t) & \cdots & a'_{1j}(t) & \cdots & a_{1n}(t) \\ a_{22}(t) & \cdots & a'_{2j}(t) & \cdots & a_{2n}(t) \\ \vdots & & \vdots & & \vdots \\ a_{n-1,2}(t) & \cdots & a'_{n-1,j}(t) & \cdots & a_{n-1,n}(t) \end{vmatrix}
$$

$$
=\sum_{j=2}^{n}\begin{vmatrix} a_{12}(t) & a_{12}(t) & \cdots & a'_{1j}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a'_{2j}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a'_{n,j}(t) & \cdots & a_{nn}(t) \end{vmatrix}
$$

46

设3个点$P_1(x_1, y_1), P_2(x_2, y_2), P_3(x_3, y_3)$不在一条直线上，求过点$P_1, P_2, P_3$的圆的方程。

解：圆的一般方程为

$$a(x^2+y^2)+bx+cy+d=0, \quad a\neq 0$$

因P_1, P_2, P_3在圆上，故

$$\begin{cases} a(x^2+y^2)+bx+cy+d=0, \\ a(x_1^2+y_1^2)+bx_1+cy_1+d=0, \\ a(x_2^2+y_2^2)+bx_2+cy_2+d=0, \\ a(x_3^2+y_3^2)+bx_3+cy_3+d=0, \end{cases}$$

该齐次线性方程组有非零解的充分必要条件是系数行列式为零，即

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

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47

求使3点$(x_1, y_1), (x_2, y_2), (x_3, y_3)$位于一直线上的充分必要条件。

解: 三点位于一直线上的充分必要条件是

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{y_1 - y_3}{x_1 - x_3},$$

即

$$(x_1 - x_3)(y_1 - y_2) = (x_1 - x_2)(y_1 - y_3)$$

亦即

$$x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2) = 0$$

其行列式形式为

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

47

求使3点$(x_1,y_1),(x_2,y_2),(x_3,y_3)$位于一直线上的充分必要条件。

解: 三点位于一直线上的充分必要条件是

$$\frac{y_1-y_2}{x_1-x_2}=\frac{y_1-y_3}{x_1-x_3},$$

即

$$(x_1-x_3)(y_1-y_2)=(x_1-x_2)(y_1-y_3)$$

亦即

$$x_1(y_2-y_3)-x_2(y_1-y_3)+x_3(y_1-y_2)=0$$

其行列式形式为

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0.$$

48

求过3点$(1,1,1)$，$(2,3,-1)$，$(3,-1,-1)$的平面方程。

解：平面方程为

$$ax + by + cz + d = 0,$$

因3点位于平面上，故

$$\begin{cases} ax + by + cz + d = 0, \\ a + b + c + d = 0, \\ 2a + 3b - c + d = 0, \\ 3a - b - c + d = 0 \end{cases}$$

该齐次线性方程组有非零解，故其系数行列式为零，即

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & -1 & 1 \\ 3 & -1 & -1 & 1 \end{vmatrix} = 0.$$

即

$$-8x - 2y - 6z + 16 = 0.$$

亦即

$$4x + y + 3z - 8 = 0.$$

48

求过3点$(1,1,1)$，$(2,3,-1)$，$(3,-1,-1)$的平面方程。

解： 平面方程为

$$ax+by+cz+d=0,$$

因3点位于平面上，故

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$$-8x-2y-6z+16=0.$$

亦即

$$4x+y+3z-8=0.$$

49

求过点$(1,1,1),(1,1,-1),(1,-1,1),(-1,0,0)$的球面方程，并求其中心与半径。

解: 球面的一般方程为

$$a(x^2+y^2+z^2)+bx+cy+dz+e=0.$$

过该四点的球面方程为

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 1^2+1^2+1^2 & 1 & 1 & 1 & 1 \\ 1^2+1^2+(-1)^2 & 1 & 1 & -1 & 1 \\ 1^2+(-1)^2+1^2 & 1 & -1 & 1 & 1 \\ (-1)^2+0^2+0^2 & -1 & 0 & 0 & 1 \end{vmatrix}=0 \Rightarrow \begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 3 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & -1 & 1 \\ 3 & 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 \end{vmatrix}=0$$

按第一行展开可知

$$-8(x^2+y^2+z^2)+8x+16=0,$$

即

$$x^2+y^2+z^2-x-2=0, \quad \Rightarrow (x-\frac{1}{2})^2+y^2+z^2=(\frac{3}{2})^2$$

圆心为$(\frac{1}{2},0,0)$，半径为$\frac{3}{2}$.

49

求过点$(1,1,1),(1,1,-1),(1,-1,1),(-1,0,0)$的球面方程，并求其中心与半径。

解： 球面的一般方程为

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$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 1^2+1^2+1^2 & 1 & 1 & 1 & 1 \\ 1^2+1^2+(-1)^2 & 1 & 1 & -1 & 1 \\ 1^2+(-1)^2+1^2 & 1 & -1 & 1 & 1 \\ (-1)^2+0^2+0^2 & -1 & 0 & 0 & 1 \end{vmatrix}=0 \Rightarrow \begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 3 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & -1 & 1 \\ 3 & 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 \end{vmatrix}=0$$

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按第一行展开可知

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圆心为$(\frac{1}{2},0,0)$，半径为$\frac{3}{2}$.

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已知$a^2 \neq b^2$，证明方程组

$$
\begin{cases}
ax_1 & & & & + & & & & bx_{2n} & = & 1 \\
& ax_2 & & & + & & & bx_{2n-1} & & = & 1 \\
& & \ddots & & & & \iddots & & & & \\
& & & ax_n & + & bx_{n+1} & & & & = & 1 \\
& & & bx_n & + & ax_{n+1} & & & & = & 1 \\
& & \iddots & & & & \ddots & & & & \\
& bx_2 & & & + & & & ax_{2n-1} & & = & 1 \\
bx_1 & & & & + & & & & ax_{2n} & = & 1
\end{cases}
$$

有唯一解，并求解。

解：其系数行列式为 $D_{2n} = \begin{vmatrix} a & & & & & b \\ & \ddots & & & \iddots & \\ & & a & b & & \\ & & b & a & & \\ & \iddots & & & \ddots & \\ b & & & & & a \end{vmatrix}$

50

已知$a^2 \neq b^2$，证明方程组

$$\begin{cases} ax_1 & & & & + & & & & bx_{2n} & = & 1 \\ & ax_2 & & & + & & & bx_{2n-1} & & = & 1 \\ & & \ddots & & & & \iddots & & & & \\ & & & ax_n & + & bx_{n+1} & & & & = & 1 \\ & & & bx_n & + & ax_{n+1} & & & & = & 1 \\ & & \iddots & & & & \ddots & & & & \\ & bx_2 & & & + & & & ax_{2n-1} & & = & 1 \\ bx_1 & & & & + & & & & ax_{2n} & = & 1 \end{cases}$$

有唯一解，并求解。

解：其系数行列式为 $D_{2n} = \begin{vmatrix} a & & & & & b \\ & \ddots & & & \iddots & \\ & & a & b & & \\ & & b & a & & \\ & \iddots & & & \ddots & \\ b & & & & & a \end{vmatrix}$

把D_{2n}中的第$2n$行依次与第$2n-1$行、...、第2行对调（共$2n-2$次相邻对换），再把第$2n$列依次与第$2n-1$列、...、第2列对调，得

$$D_{2n} = \begin{vmatrix} a & b & & & & & & \\ b & a & & & & & & \\ & & a & & & & & b \\ & & & \ddots & & & \iddots & \\ & & & & a & b & & \\ & & & & b & a & & \\ & & & \iddots & & & \ddots & \\ & & b & & & & & a \end{vmatrix}$$

于是

$$\begin{aligned} D_{2n} &= D_2 D_{2(n-1)} \\ &= (a^2-b^2)D_{2(n-1)} = (a^2-b^2)^2 D_{2(n-2)} \\ &= \cdots = (a^2-b^2)^{n-1} D_2 \\ &= (a^2-b^2)^n. \end{aligned}$$

$$
\begin{aligned}
D_{2n}^{(1)} &= \begin{vmatrix}
1 & & & & & b \\
1 & a & & & b & \\
 & & \ddots & \iddots & & \\
1 & & a & b & & \\
1 & & b & a & & \\
 & & \iddots & \ddots & & \\
1 & b & & & a & \\
1 & & & & & a
\end{vmatrix} \\
&= \begin{vmatrix}
a & & & & b & \\
 & \ddots & & \iddots & & \\
 & & a & b & & \\
 & & b & a & & \\
 & \iddots & & \ddots & & \\
b & & & & a & \\
 & & & & & a
\end{vmatrix} + (-1)^{1+2n} b \begin{vmatrix}
1 & a & & & & b \\
 & & \ddots & & \iddots & \\
1 & & & a & b & \\
1 & & & b & a & \\
 & & \iddots & & \ddots & \\
1 & b & & & & a \\
1 & & & & &
\end{vmatrix} \\
&= aD_{2(n-1)} - (-1)^{(2n-1)+1} bD_{2(n-1)} \\
&= (a-b)(a^2-b^2)^{n-1}.
\end{aligned}
$$

同理可证

$$D_{2n}^{(i)} = (a-b)(a^2-b^2)^{n-1}, \quad i = 2, \cdots, 2n.$$

于是

$$x_i = \frac{D_{2n}^{(i)}}{D_{2n}} = \frac{(a-b)(a^2-b^2)^{n-1}}{(a^2-b^2)^n} = \frac{1}{a+b}, \quad i = 1, \cdots, 2n.$$