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Collocation method for one dimensional nonlocal diffusion equations

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Abstract

In this paper, the collocation method for solving one dimensional steady state and time dependent nonlocal diffusion equations is analyzed. The difficulty of applying collocation method to nonlocal diffusion equations comes from the singularity of the kernel. If $s < \frac{1}{2}$ the kernel is weakly singular, however, if $s \geq \frac{1}{2}$ the kernel is not integrable in Riemann sense. So that the Hadamard finite part integral is introduced to overcome this difficulty. For analysis and performance, a "balance" term is added to discretize the nonlocal operator. Numerical results validate the theorems.

KEYWORDS

collocation method, convergence rate, Hadamard finite part integral, nonlocal diffusion equation, stability

1 | INTRODUCTION

Nonlocal diffusion equations are widely used in science and engineering [1–6], finite element methods for solving both steady state and time-dependent problems were studied in References [7–13], finite difference and high order methods can be found in References [14–17], collocation methods for 1-D and 2-D steady state nonlocal diffusion equations with the kernel $s < \frac{1}{2}$ were considered in References [18, 19], a reproducing kernel collocation method for multi-dimensional problem is analyzed in Reference [20]. However our method is a different approach, the time-dependent problem, stability and convergence analysis still need to be done. In this paper, the collocation schemes employ the Hadamard finite part integral to deal with the case $s \geq \frac{1}{2}$, and use uniform mesh, continuous piece-wise linear basis functions. This choice is natural for local problem but not so for nonlocal problem due to the singularity of the kernel. Our method is to use the simple basis functions without further processing for the kernel integration, just integrate the discrete form in Riemann or Hadamard finite part sense. Clearly, the convergence rate is low compared with previously mentioned methods. However, this method can be used in complex geometry for 2D or 3D problem with appropriate Hadamard finite part integral or small s. But here we focus on the 1D problem.

Let Ω be a finite bar in $\mathbb{R}$. Without loss of generality, we take $\Omega = (0, 1)$. A nonlocal operator $\mathcal{L}_\delta$ is defined as, for any function $u = u(x) : \Omega \to \mathbb{R}$,

$$\mathcal{L}_\delta u(x) = \int_{B_\delta(x)} (u(y) - u(x))\gamma_\delta(|x-y|)dy = \int_0^\delta [u(x-z) + u(x+z) - 2u(x)]\gamma_\delta(z)dz$$

where $x \in \Omega$ and $B_\delta(x) = \{y \in \mathbb{R} : |y - x| \leq \delta\}$ denoting a neighborhood centered at x of radius δ, which is the horizon parameter, and $\gamma_\delta(|x-y|) = 0$ if $y \notin B_\delta(x)$. The following nonlocal constrained value problem on a nonzero measure volume is our main subject of interest here:

$$\begin{cases} -\mathcal{L}_\delta u + cu = f_\delta \ \text{ on } \ \Omega, \\ \qquad\qquad u = g, \ \text{ on } \ \Omega_\mathcal{I} \end{cases} \tag{1}$$

$$\begin{cases} u_t - \mathcal{L}_\delta u = f_\delta \quad \text{on} \ \ \Omega \times (0, T], \\ \qquad\quad u = g, \quad \text{on} \ \ \Omega_\mathcal{I} \times [0, T] \\ \quad u(x, 0) = u_0, \ \text{ on } \ \Omega \end{cases} \tag{2}$$

with the kernel

$$\gamma_\delta(z) = \frac{2-2s}{\delta^{2-2s}} \frac{1}{|z|^{1+2s}}, \tag{3}$$

where $0 \leq s < 1$, $c \geq 0$, $\Omega_\mathcal{I} = [-\delta, 0] \cup [1, 1+\delta]$.

Then, we introduce the definitions of the Hadamard finite part integrals:

$$\int_a^b \frac{u(y)}{(y-a)^{2s}}dy \quad \text{or} \quad \int_a^b \frac{u(y)}{(b-y)^{2s}}dy \tag{4}$$

If $s \in \left(0, \frac{1}{2}\right)$, they are just weakly singular integrals. However, when $s \in \left[\frac{1}{2}, 1\right)$, such kind of integrals are not integrable in classic Riemann sense, and they should be understood in the Hadamard finite part sense. There are several definitions, equivalent mathematically, for this finite-part integral in the literature [21]. Here we adopt the following definitions:

$$\int_a^b \frac{u(y)}{(y-a)^{2s}}dy = \begin{cases} \lim\limits_{\varepsilon\to 0}\left(\int_{a+\varepsilon}^b \frac{u(y)}{y-a}dy + u(a)\ln\varepsilon\right), & s = \frac{1}{2}, \\ \lim\limits_{\varepsilon\to 0}\left(\int_{a+\varepsilon}^b \frac{u(y)}{(y-a)^{2s}}dy + u(a)\frac{\varepsilon^{1-2s}}{1-2s}\right), & s \in \left(\frac{1}{2}, 1\right). \end{cases} \tag{5}$$

Similarly,

$$\int_a^b \frac{u(y)}{(b-y)^{2s}}dy = \begin{cases} \lim\limits_{\varepsilon\to 0}\left(\int_a^{b-\varepsilon} \frac{u(y)}{b-y}dy + u(b)\ln\varepsilon\right), & s = \frac{1}{2}, \\ \lim\limits_{\varepsilon\to 0}\left(\int_a^{b-\varepsilon} \frac{u(y)}{(b-y)^{2s}}dy + u(b)\frac{\varepsilon^{1-2s}}{1-2s}\right), & s \in \left(\frac{1}{2}, 1\right). \end{cases} \tag{6}$$

This paper is arranged as follows. In Section 1, the background and the definition of one dimensional Hadamard finite part integrals are given. In Section 2, the nonlocal operator is discretized by collocation method, the matrix of the discrete system is computed explicitly, and the analysis of truncation error is done. In Section 3, collocation method for the steady state problem is analyzed. In Section 4, forward, backward Euler and Crank–Nicolson methods are proposed to solve time-dependent problem. The stability and convergence theorems are obtained. In Section 5, the numerical results illustrate the theorems.

2 | DISCRETIZATION OF THE NONLOCAL OPERATOR

Let $0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1$ be a uniform partition on $[0, 1]$ with the mesh size $h = 1/(n+1)$. Denote by

$$I_h u(x) = \sum_{j=0}^{n+1} u(x_j)\phi_j(x) \tag{7}$$

the piece-wise linear interpolation of $u(x)$, where $\phi_j(x)$ is a basis function of piece-wise linear interpolation, satisfying $\phi_j(x_i) = \delta_{ij}$. Substituting (7) into the nonlocal operator (1) with the kernel (3), applying Hadamard finite part integral when necessary, yields the quadrature:

$$\begin{aligned}\mathcal{L}_\delta^{\mathcal{H}} I_h u(x_i) &= \frac{2-2s}{\delta^{2-2s}} \int_0^\delta \frac{\sum_{j=0}^{n+1} u(x_j)(\phi_j(x_i - z) + \phi_j(x_i + z)) - 2u(x_i)}{z^{1+2s}}\, dz \\ &:= \frac{2-2s}{\delta^{2-2s}} \sum_{j=0}^{n+1} \omega_{ij} u(x_j).\end{aligned} \tag{8}$$

In the following, we will study how to obtain the explicit expression of the coefficient ω_{ij}. Before the detailed process, we first define a quantity

$$\zeta_i := \min\left(i, \frac{\delta}{h}\right), \quad i = 1, \cdots, n. \tag{9}$$

Throughout the paper, we denote $\int_a^b f = 0$ if $a \geq b$. Moreover, let $r = \left\lfloor \frac{\delta}{h} \right\rfloor$.

2.1 | Explicit expression of ω_{ij}

It is easy to see that $\omega_{i,i-m} = \omega_{i,i+m}$ for $m \geq 1$.

If $m = 0$,

$$\begin{aligned}\omega_{ii} &= \int_0^{\min(\delta,h)} \frac{\phi_i(x_i - z) + \phi_i(x_i + z) - 2}{z^{1+2s}}\, dz - 2\int_{\min(\delta,h)}^{\delta} \frac{1}{z^{1+2s}}\, dz \\ &= -2h^{-1} \int_0^{\min(\delta,h)} \frac{1}{z^{2s}}\, dz - 2\int_{\min(\delta,h)}^{\delta} \frac{1}{z^{1+2s}}\, dz \\ &= \omega_{i,i}^{(1)} + \omega_{i,i}^{(2)}\end{aligned} \tag{10}$$

$$\omega_{i,i}^{(1)} = \begin{cases} -2\zeta_1, & s = 0, \\ -2h^{-1}\ln(\zeta_1 h), & s = \frac{1}{2}, \\ -2\frac{h^{-2s}\zeta_1^{1-2s}}{1-2s}, & s \in (0,1)\backslash\left\{\frac{1}{2}\right\}, \end{cases} \tag{11}$$

$$\omega_{i,i}^{(2)} = \begin{cases} -2(\ln\delta - \ln(\zeta_1 h)), & s = 0, \\ -2\left(\frac{1}{\zeta_1 h} - \frac{1}{\delta}\right), & s = \frac{1}{2}, \\ -2\left(\frac{(\zeta_1 h)^{-2s}}{2s} - \frac{\delta^{-2s}}{2s}\right), & s \in (0,1)\backslash\left\{\frac{1}{2}\right\}, \end{cases} \tag{12}$$

for $\delta > h$, and otherwise vanishes, and we have used the definition (5) if $s \geq \frac{1}{2}$.

If $m = 1$,

$$\omega_{i,i+1} = \int_0^{\min(\delta,h)} \frac{\phi_{i+1}(x_i+z)}{z^{1+2s}}\,dz + \int_h^{\min(\delta,2h)} \frac{\phi_{i+1}(x_i+z)}{z^{1+2s}}\,dz = \omega_{i,i+1}^{(1)} + \omega_{i,i+1}^{(2)} \tag{13}$$

where $\omega_{i,i+1}^{(1)} = -\frac{1}{2}\omega_{i,i}^{(1)}$, and

$$\omega_{i,i+1}^{(2)} = \begin{cases} 1-\zeta_2+2\ln\zeta_2, & s=0, \\ h^{-1}(-\ln\zeta_2+2-2\zeta_2^{-1}), & s=\frac{1}{2}, \\ -h^{-2s}\left(\frac{\zeta_2^{-2s}-1}{s}+\frac{\zeta_2^{1-2s}-1}{1-2s}\right), & s\in(0,1)\setminus\left\{\frac{1}{2}\right\} \end{cases} \tag{14}$$

for $\delta > h$, and otherwise vanishes.

If $m > 1$, then

$$\omega_{i,i+m} = \int_{(m-1)h}^{\min(\delta,mh)} \frac{\phi_{i+m}(x_i+z)}{z^{1+2s}}\,dz + \int_{mh}^{\min(\delta,(m+1)h)} \frac{\phi_{i+m}(x_i+z)}{z^{1+2s}}\,dz = \omega_{i,i+m}^{(1)} + \omega_{i,i+m}^{(2)} \tag{15}$$

where

$$\omega_{i,i+m}^{(1)} = \begin{cases} \zeta_m-(m-1)-(m-1)\ln\frac{\zeta_m}{m-1}, & s=0, \\ h^{-1}\ln\frac{\zeta_m}{m-1}+(m-1)\frac{\zeta_m^{-1}-(m-1)^{-1}}{h}, & s=\frac{1}{2}, \\ h^{-2s}\left(\frac{\zeta_m^{1-2s}-(m-1)^{1-2s}}{1-2s}+(m-1)\frac{\zeta_m^{-2s}-(m-1)^{-2s}}{2s}\right), & \text{elsewhere} \end{cases}$$

for $\delta > (m-1)h$ or otherwise vanishes, and

$$\omega_{i,i+m}^{(2)} = \begin{cases} m-\zeta_{m+1}+(m+1)\ln\frac{\zeta_{m+1}}{m}, & s=0, \\ -h^{-1}\ln\frac{\zeta_{m+1}}{m}-(m+1)\frac{\zeta_{m+1}^{-1}-m^{-1}}{h}, & s=\frac{1}{2}, \\ -h^{-2s}\left(\frac{\zeta_{m+1}^{1-2s}-m^{1-2s}}{1-2s}+(m+1)\frac{\zeta_{m+1}^{-2s}-m^{-2s}}{2s}\right), & s\in(0,1)\setminus\left\{\frac{1}{2}\right\} \end{cases}$$

for $\delta > mh$ or otherwise vanishes. It is easy to see that $\omega_{i,i+m} = 0$ if $m > r+1$.

2.2 | Truncation error analysis

Theorem 1 *Let $\mathcal{L}_\delta^{\mathcal{H}} I_h u$ be defined in (8), then for $u \in C^2(\Omega\cup\Omega_I)$ we have*

$$|\mathcal{E}(x_i;u)| = |\mathcal{L}_\delta u(x_i) - \mathcal{L}_\delta^{\mathcal{H}} I_h u(x_i)| \le \begin{cases} \frac{2-2s}{\delta^{2-2s}}Ch^2|\ln h|, & s=0, \\ \frac{2-2s}{\delta^{2-2s}}Ch|\ln h|, & s=\frac{1}{2}, \\ \frac{2-2s}{\delta^{2-2s}}Ch^{2-2s}, & s\in(0,1)\setminus\left\{\frac{1}{2}\right\}, \end{cases} \tag{16}$$

where $i = 1,\cdots,n$, $\delta > h$, and C depends on u'', does not depend on h.

Proof. Define the error function $\mathcal{E}(x_i;u) = \mathcal{L}_\delta u(x_i) - \mathcal{L}_\delta^{\mathcal{H}} I_h u(x_i)$, $e(y) = u(y) - I_h u(y)$, and $\omega_{i,i}^{(1)}$ as in (11). Then we have

$$\frac{\delta^{2-2s}}{2-2s}\mathcal{E}(x_i;u) = \int_{x_i-\delta}^{x_{i-1}} \frac{e(y)}{(x_i-y)^{1+2s}}\,dy + \int_{x_{i+1}}^{x_i+\delta} \frac{e(y)}{(y-x_i)^{1+2s}}\,dy + \int_{x_{i-1}}^{x_{i+1}} \frac{u(y)-I_h u(y)}{|y-x_i|^{1+2s}}\,dy.$$

By using the definitions (5) and (6) we have

$$\int_{x_{i-1}}^{x_{i+1}} \frac{I_h u(y)}{|y-x_i|^{1+2s}}\,dy = [2u(x_i)-u(x_{i-1})-u(x_{i+1})]\frac{1}{2}\omega_{i,i}^{(1)} + u(x_i)\int_{x_{i-1}}^{x_{i+1}} \frac{1}{|y-x_i|^{1+2s}}\,dy$$

Hence,

$$\begin{aligned}\frac{\delta^{2-2s}}{2-2s}\mathcal{E}(x_i;u) &= \int_{x_i-\delta}^{x_{i-1}} \frac{e(y)}{(x_i-y)^{1+2s}}\,dy + \int_{x_{i+1}}^{x_i+\delta} \frac{e(y)}{(y-x_i)^{1+2s}}\,dy \\ &\quad + \int_{x_{i-1}}^{x_{i+1}} \frac{u(y)-u(x_i)-u'(x_i)(y-x_i)}{|y-x_i|^{1+2s}}\,dy - [2u(x_i)-u(x_{i-1})-u(x_{i+1})]\frac{1}{2}\omega_{i,i}^{(1)} \\ &\coloneqq \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3-[2u(x_i)-u(x_{i-1})-u(x_{i+1})]\frac{1}{2}\omega_{i,i}^{(1)}\end{aligned}$$

where we have used the fact

$$\int_{x_{i-1}}^{x_{i+1}} \frac{y-x_i}{|y-x_i|^{1+2s}}\,dy = 0.$$

By noting that $u \in C^2(\Omega\cup\Omega_I)$ and using the property of Lagrange interpolation, we obtain

$$|e(y)| = |u(y)-I_h u(y)| \le \max_{0\le i\le n}\max_{y\in[x_i,x_{i+1}]}\left|\frac{1}{2}(y-x_i)(y-x_{i+1})u''(\xi_y)\right| \le Ch^2$$

and by Taylor expansion for $\mathcal{I}_3$, bounds for $\mathcal{I}_i$ are given as follows

$$|\mathcal{I}_i| \le \begin{cases} Ch^2|\ln h|, & s=0 \\ Ch^{2-2s}, & s\in(0,1), \end{cases} \quad i=1,2,3.$$

also with (11), (16) follows immediately and the proof is completed. ■

Definition 1

$$\widetilde{\mathcal{L}}_\delta^{\mathcal{H}} I_h u(x_i) = \mathcal{L}_\delta^{\mathcal{H}} I_h u(x_i) - \frac{2-2s}{\delta^{2-2s}}[2u(x_i)-u(x_{i-1})-u(x_{i+1})]\frac{\omega_{i,i}^{(1)}}{2}\chi_{\left[s\ge\frac{1}{2}\right]} \tag{17}$$

Remark 1 From the proof of Theorem 1, we can prove that

$$\widetilde{\mathcal{E}}(x_i;u) \coloneqq \mathcal{L}_\delta u(x_i) - \widetilde{\mathcal{L}}_\delta^{\mathcal{H}} I_h u(x_i), \quad i=1,\cdots,n$$

also possesses the accuracy as (16) in Theorem 1 when $s \ne 1/2$, but has higher order rate h compared with $h|\ln h|$ if $s = 1/2$.

$$|\widetilde{\mathcal{E}}(x_i;u)| \le \begin{cases} \frac{2-2s}{\delta^{2-2s}}Ch^2|\ln h|, & s=0 \\ \frac{2-2s}{\delta^{2-2s}}Ch^{2-2s}, & s\in(0,1), \end{cases} \quad i=1,2,3.$$

And this modified operator $\widetilde{\mathcal{L}}_\delta^{\mathcal{H}}$ with the "balance" term in (17) can guarantee that the stiffness matrix is a M-matrix for $s \ge 1/2$.

From now on, we assume $\delta > h$, we will use $\widetilde{\mathcal{L}}_\delta^{\mathcal{H}} u_h(x_i)$ for solving the model problem (1), where $u_h(x) = \sum_{i=0}^{n+1} u_i\phi_i(x),\ x\in\Omega$ is the numerical solution.

3 | STEADY STATE PROBLEM

Consider the following problem

$$\begin{cases} -\widetilde{\mathcal{L}}_\delta^{\mathcal{H}} u_h(x_i) + cu_i = f_\delta(x_i), & x_i \in \Omega, \\ u_h(x) = g(x), & x \in \Omega_{\mathcal{I}}, \end{cases} \tag{18}$$

From (8), (17), and (18), we get

$$-\frac{2-2s}{\delta^{2-2s}}\left(\sum_{j=1}^{n}\omega_{ij}u_j - \frac{\omega_{i,i}^{(1)}}{2}(2u_i - u_{i-1} - u_{i+1})\chi_{\left[s\geq\frac{1}{2}\right]}\right) + cu_i = f_i, \tag{19}$$

where $1 \leq i \leq n$, and

$$f_i = f_\delta(x_i) + \int_{ih}^{\max(\delta,ih)} \frac{g(x_i - z)}{z^{1+2s}}\,dz + \int_{(n+1-i)h}^{\max(\delta,(n+1-i)h)} \frac{g(x_i + z)}{z^{1+2s}}\,dz.$$

Then we have the matrix form of (19) which is

$$(W + cI)\mathbf{U} = \mathbf{f}, \tag{20}$$

where $\mathbf{U} = (u_1, u_2, \cdots, u_n)^T,\ \mathbf{f} = (f_1, f_2, \cdots, f_n)^T$ and

$$W = (W_{i,j}) \in \mathbb{R}^{n\times n}$$

with

$$\begin{aligned} W_{i,i} &= -\frac{2-2s}{\delta^{2-2s}}\left(\omega_{i,i} - \omega_{i,i}^{(1)}\chi_{\left[s\geq\frac{1}{2}\right]}\right), \\ W_{i,i+1} &= -\frac{2-2s}{\delta^{2-2s}}\left(\omega_{i,i+1} + \frac{\omega_{i,i}^{(1)}\chi_{\left[s\geq\frac{1}{2}\right]}}{2}\right), \\ W_{i,j} &= -\frac{2-2s}{\delta^{2-2s}}\omega_{i,j}, \quad |j-i| > 1. \end{aligned} \tag{21}$$

Lemma 1 *Let W be defined by (21), then W is a M-matrix.*

Proof. As assumed $\delta > h$, from (10)–(14) and (21), we can see that

$$W_{i,i} > 0, \quad W_{i,j} \leq 0, \quad |j-i| > 0,$$

and

$$W_{i,i} \geq \sum_{j\neq i} |W_{i,j}|.$$

The proof is completed. ■

Before stating a lemma in the proof of our main result, we now introduce an important conclusion given in Varga [22]. For simplicity, we define

$$\mathcal{A} := M + cI = (a_{i,j}) \in \mathbb{R}^{n\times n}.$$

$\mathcal{A}$ is a strictly diagonally dominant matrix if $c > 0$ and

$$\|\mathcal{A}^{-1}\|_\infty \leq \max_{1\leq i\leq n} \frac{1}{|a_{ii}| - \sum_{j\neq i}|a_{ij}|}, \quad 1 \leq i \leq n. \tag{22}$$

Lemma 2 *Let $\mathcal{A}^{-1} = (b_{ij})$ be the inverse of $\mathcal{A}$ with $c > 0$, then*

$$\|\mathcal{A}^{-1}\|_\infty \le \frac{1}{c}. \tag{23}$$

Proof. From Lemma 1, for $\delta < 1/2$, $i = r+1, \cdots, n-r$,

$$|a_{ii}| - \sum_{j\neq i} |a_{ij}| = c.$$

By (22), we get (23). The proof is complete. ■

Remark 2 If $\delta > 1/2$, it is easy to see, from the explicit entries, W is strictly diagonally dominant but denser.

Theorem 2 *Assume that $u(x)$, the solution of* Equation (1), *belongs to $C^2(\Omega \cup \Omega_I)$. Then, for the solution of linear system* (20), *there holds for sufficiently small h*

$$\|u - u_h\|_\infty \le \begin{cases} \frac{2-2s}{\delta^{2-2s}} C h^2 |\ln h|, & s = 0, \\ \frac{2-2s}{\delta^{2-2s}} C h^{2-2s}, & s \in (0,1), \end{cases}$$

where u_h comes from (20), $c \ge 0$, *C does not depend on h.*

Proof. From (1) and (18), we have

$$-\widetilde{\mathcal{L}}_\delta^{\mathcal{H}} (u_h - I_h u)(x_i) + c(u_i - u(x_i)) = \widetilde{\mathcal{L}}_\delta^{\mathcal{H}} I_h u(x_i) - \mathcal{L}_\delta u(x_i), \quad i = 1, \cdots, n,$$

which is

$$\mathbf{U} - \mathbf{u} = \mathcal{A}^{-1}\mathbf{E}$$

where $\mathbf{u} = (u(x_1), \cdots, u(x_n))^T$ be the exact solution vector, and

$$\mathbf{E} = (\widetilde{\mathcal{E}}(x_1; u), \cdots, \widetilde{\mathcal{E}}(x_n; u))^T$$

where $\widetilde{\mathcal{E}}(x_i; u)$ is defined in Remark 1.

Then, for $c > 0$, by Theorem 1 and Lemma 2, we have

$$\|\mathbf{U} - \mathbf{u}\|_\infty \le \|\mathcal{A}^{-1}\|_\infty \|\mathbf{E}\|_\infty \le \|\mathbf{E}\|_\infty / c.$$

If $c = 0$, for simplicity, we assume $\delta = mh$, the proof is similar if $\delta \neq mh$, we extend the uniform mesh onto $[-\delta, 0]$ and $[1, 1+\delta]$, $I_h u$ is also extended, $u_h = I_h u$ on Ω_I, then for $i = 0, 1, \cdots, n+1$

$$\widetilde{\mathcal{L}}_\delta^{\mathcal{H}} u_h(x_i) = -\sum_{j=-m}^{m} W_{i,i+j} u_{i+j}$$

where $W_{0,k} = W_{1,1+k}$, $W_{n+1,n+1-k} = W_{1,1+k}$ for $k = 0, 1, \cdots, m$. If $i+j < 0$ or $i+j > n+1$, $W_{i,i+j} = W_{i,i-j}$, $u_{i+j} = u((i+j)h)$. Since

$$-W_{i,i} < 0, -W_{i,j} \ge 0, \quad |j - i| > 0,$$

and

$$W_{i,i} \ge \sum_{j \neq i} - W_{i,j}$$

the discrete Maximum Principle is valid, which means, if

$$\widetilde{\mathcal{L}}_\delta^{\mathcal{H}} u_h(x_i) \ge 0$$

we have

$$\max_{1\le i\le n} u_i \le \max_{j\le 0 \text{ or } j\ge n+1} u_j.$$

Apply the operator $\widetilde{\mathcal{L}}_\delta^{\mathcal{H}}$ to $q(x)=x^2$, for $i=0,\cdots,n+1$, we have

$$\begin{aligned}
\widetilde{\mathcal{L}}_\delta^{\mathcal{H}} I_h q(x_i) &= -\sum_{j=-m}^{m} W_{i,i+j}(x_i+jh)^2 = -\sum_{j=-m}^{m} W_{i,i+j}(jh)^2 \\
&= -2\sum_{j=1}^{m} W_{i,i+j}(jh)^2 \ge \frac{4-4s}{\delta^{2-2s}}\int_{x_{i+1}}^{x_i+\delta} \frac{\sum_{j=1}^{m}(jh)^2\phi_{i+j}(y)}{(y-x_i)^{1+2s}}\,dy \\
&\ge \frac{4-4s}{\delta^{2-2s}}\int_{x_{i+1}}^{x_i+\delta} \frac{(y-x_i)^2}{(y-x_i)^{1+2s}}\,dy = 2-2\frac{h^{2-2s}}{\delta^{2-2s}}.
\end{aligned}$$

So that with h small enough, $1 \le \widetilde{\mathcal{L}}_\delta^{\mathcal{H}} I_h q(x_i) \le 4$, the upper bound 4 is obvious. Then follow the proof in Smith [23] for the finite difference approximation to Poisson's equation, the conclusion can be obtained. ■

4 | TIME DEPENDENT PROBLEM

Firstly, we have the semi-discrete form of problem (2) as:

$$\begin{cases}
\frac{\partial u_h(x_i,t)}{\partial t} - \widetilde{\mathcal{L}}_\delta^{\mathcal{H}} u_h(x_i,t) = f_\delta(x_i,t), & \text{on } \Omega\times(0,T], \\
u_h(x,t) = g(x,t), & \text{on } \Omega_{\mathcal{I}}\times[0,T], \\
u_h(x,0) = u_0(x), & \text{on } \Omega,
\end{cases} \tag{24}$$

where $u_h(x,t) = \sum_{i=0}^{n+1} u_i(t)\phi_i(x),\ (x,t)\in\Omega\times(0,T]$. Then we define

$$f_i(t) = f_\delta(x_i,t) + \int_{ih}^{\max(\delta,ih)} \frac{g(x_i-z,t)}{z^{1+2s}}\,dz + \int_{(n+1-i)h}^{\max(\delta,(n+1-i)h)} \frac{g(x_i+z,t)}{z^{1+2s}}\,dz.$$

Then we have the matrix form of (24) which is

$$\frac{\partial}{\partial t}\mathbf{U}(t) + W\mathbf{U}(t) = \mathbf{f}(t), \tag{25}$$

where the matrix W is defined in (21) and

$$\mathbf{U} = (u_1(t),\cdots,u_n(t))^T,\ \ \mathbf{f} = (f_1(t),\cdots,f_n(t))^T.$$

4.1 | Stability of collocation method

In this section we will consider the stability of fully discrete collocation schemes from Equation (24) for Equation (2).

4.1.1 | Backward Euler method

Let $\Delta t = \frac{T}{M}$, we can get the backward Euler scheme for Equation (25) as:

$$\frac{\mathbf{U}^{m+1}-\mathbf{U}^m}{\Delta t} = -W\mathbf{U}^{m+1} + \mathbf{f}^{m+1},\ \ m=0,1,\cdots,M-1, \tag{26}$$

where W is defined in (21) and

$$\mathbf{U}^m = (u_1^m, \cdots, u_n^m)^T, \ \ \mathbf{f}^m = (f_1(t_m), \cdots, f_n(t_m))^T.$$

Theorem 3 *The backward Euler scheme* (26) *is unconditionally stable.*

Proof. Equation (26) can be written as

$$(I + \Delta t W)\mathbf{U}^{m+1} = \mathbf{U}^m + \Delta t \mathbf{f}^{m+1},$$

let $Q = I + \Delta t W$, we have

$$\mathbf{U}^{m+1} = Q^{-1}\mathbf{U}^m + \Delta t Q^{-1}\mathbf{f}^{m+1}.$$

Then we will consider the eigenvalue of matrix Q. By the Gerschgorin's theorem, if λ is an eigenvalue of Q, then

$$|\lambda - (1 + w_0\Delta t)| \leq w_0\Delta t,$$

where $w_0 = W_{i,i}, \ i = 1, \cdots, n$, so that we have

$$1 \leq \lambda \leq 1 + 2w_0\Delta t,$$

and $\frac{1}{\lambda} < 1$, is the eigenvalue of Q^{-1}. And we know that W is a symmetric matrix, so that Q and Q^{-1} are both symmetric. We define $\overline{Q} = Q^{-1}$, then we have

$$||\overline{Q}||_2 = \rho(\overline{Q}) \leq 1,$$

and then,

$$\begin{aligned}
||\mathbf{U}^{m+1}||_2 &= ||\overline{Q}\mathbf{U}^m + \Delta t\overline{Q}\mathbf{f}^{m+1}||_2, \\
&\leq ||\overline{Q}||_2||\mathbf{U}^m||_2 + \Delta t||\overline{Q}||_2||\mathbf{f}^{m+1}||_2, \\
&\leq ||\mathbf{U}^m||_2 + \Delta t||\mathbf{f}^{m+1}||_2, \\
&\leq ||\mathbf{U}^0||_2 + \sum_{k=1}^{m+1}\Delta t||\mathbf{f}^k||_2,
\end{aligned}$$

Let

$$||\mathbf{U}||_{2,h} = \left(\sum_{i=1}^{N} h(u_i)^2\right)^{1/2},$$

then we have

$$||\mathbf{U}^{m+1}||_{2,h} \leq ||\mathbf{U}^0||_{2,h} + \sum_{k=1}^{m+1}\Delta t||\mathbf{f}^k||_{2,h}.$$

So that the backward Euler method is unconditionally stable. ∎

4.1.2 | Crank–Nicolson method

Let $\Delta t = \frac{T}{M}$, we can get the Crank–Nicolson scheme for Equation (2) as:

$$\frac{\mathbf{U}^{m+1} - \mathbf{U}^m}{\Delta t} = -W\frac{\mathbf{U}^{m+1} + \mathbf{U}^m}{2} + \frac{\mathbf{f}^{m+1} + \mathbf{f}^m}{2}, \ \ m = 0, 1, \cdots, M-1, \tag{27}$$

where $\mathbf{U}^m, \mathbf{f}^m$ and W are defined as in (26), (21).

Theorem 4 *The Crank–Nicolson Scheme (27) is unconditionally stable.*

Proof. Equation (27) can be written as

$$\left(I + \frac{\Delta t}{2} W\right) \mathbf{U}^{m+1} = \left(I - \frac{\Delta t}{2} W\right) \mathbf{U}^{m} + \frac{\Delta t}{2}(\mathbf{f}^{m+1} + \mathbf{f}^{m}).$$

we define $B = I + \frac{\Delta t}{2} W$, we have

$$B\mathbf{U}^{m+1} = (2I - B)\mathbf{U}^{m} + \frac{\Delta t}{2}(\mathbf{f}^{m+1} + \mathbf{f}^{m}),$$

which when multiplied by B^{-1} yields

$$\mathbf{U}^{m+1} = (2B^{-1} - I)\mathbf{U}^{m} + \frac{\Delta t}{2} B^{-1}(\mathbf{f}^{m+1} + \mathbf{f}^{m}),$$

let $\overline{Q} = 2B^{-1} - I$, if λ is an eigenvalue of B, then $\mu = \frac{2}{\lambda} - 1$ is an eigenvalue of $\overline{Q}$. In order that the eigenvalues of $\overline{Q}$ be less than or equal to 1 in magnitude, we must have

$$\left|\frac{2}{\lambda} - 1\right| \leq 1.$$

If λ is positive, this is the same as $\lambda \geq 1$.

By the Gerschgorin's theorem, if λ is an eigenvalue of B, then

$$\left|\lambda - \left(1 + \frac{w_0 \Delta t}{2}\right)\right| \leq \frac{w_0 \Delta t}{2},$$

so that we have

$$1 \leq \lambda \leq 1 + w_0 \Delta t.$$

Thus we see that λ is always greater than or equal to 1. This then implies that the eigenvalues of $\overline{Q}$ are always less than 1 in magnitude.

And we know that W is a symmetric matrix, so that B and B^{-1} are both symmetric. Then $\overline{Q}$ is symmetric and we have

$$\|\overline{Q}\|_2 = \rho(\overline{Q}) \leq 1,$$

and then,

$$\begin{aligned}
\|\mathbf{U}^{m+1}\|_2 &= \|\overline{Q}\mathbf{U}^{m} + \frac{\Delta t}{2} B^{-1}(\mathbf{f}^{m+1} + \mathbf{f}^{m})\|_2, \\
&\leq \|\overline{Q}\|_2 \|\mathbf{U}^{m}\|_2 + \frac{\Delta t}{2} \|B^{-1}\|_2 \|\mathbf{f}^{m+1} + \mathbf{f}^{m}\|_2, \\
&\leq \|\mathbf{U}^{m}\|_2 + \frac{\Delta t}{2} \|\mathbf{f}^{m+1} + \mathbf{f}^{m}\|_2, \\
&\leq \|\mathbf{U}^{0}\|_2 + \sum_{k=1}^{m+1} \frac{\Delta t}{2} \|\mathbf{f}^{k} + \mathbf{f}^{k-1}\|_2,
\end{aligned}$$

then we have

$$\|\mathbf{U}^{m+1}\|_{2,h} \leq \|\mathbf{U}^{0}\|_{2,h} + \sum_{k=1}^{m+1} \frac{\Delta t}{2} \|\mathbf{f}^{k} + \mathbf{f}^{k-1}\|_{2,h}.$$

So that the Crank–Nicolson method is unconditionally stable. ∎

4.1.3 | Forward Euler method

Let $\Delta t = \frac{T}{M}$, we can get the forward Euler method for Equation (2) as:

$$\frac{\mathbf{U}^{m+1} - \mathbf{U}^m}{\Delta t} = -W\mathbf{U}^m + \mathbf{f}^m, \quad m = 0, 1, \cdots, M-1, \tag{28}$$

where $\mathbf{U}^m, \mathbf{f}^m$ and W are defined as in Equations (26) and (21).

Theorem 5 *The forward Euler scheme* (28) *is conditionally stable provided*

$$\Delta t \leq \frac{1}{w_0}$$

with $\delta > h$ *and*

$$w_0 = \begin{cases} \frac{2-2s}{\delta^{2-2s}}(2 + 2\ln(\delta/h)), & s = 0, \\ \frac{2-2s}{\delta^{2-2s}} \frac{2\delta - 2h}{h\delta}, & s = \frac{1}{2}, \\ \frac{2-2s}{\delta^{2-2s}} \frac{h^{-2s} - \delta^{-2s}}{2s}, & s \in (0,1) \backslash \left\{ \frac{1}{2} \right\}. \end{cases} \tag{29}$$

Proof. Equation (28) can be written as

$$\mathbf{U}^{m+1} = (I - \Delta t W)\mathbf{U}^m + \Delta t \mathbf{f}^m,$$

Let $Q = I - \Delta t W$, we have

$$\mathbf{U}^{m+1} = Q\mathbf{U}^m + \Delta t \mathbf{f}^m.$$

Then we will consider the eigenvalues of matrix Q. Define w_0 as the diagonal entry of W, by the Gerschgorin's theorem, if λ is an eigenvalue of Q, then

$$\begin{aligned} |\lambda - (1 - w_0\Delta t)| &\leq w_0\Delta t, \\ |\lambda| - |1 - w_0\Delta t| &\leq w_0\Delta t, \\ |\lambda| &\leq |1 - w_0\Delta t| + w_0\Delta t. \end{aligned}$$

If $0 \leq w_0\Delta t \leq 1$, then we have $|\lambda| \leq 1$, and Q is a symmetric matrix,

$$\|Q\|_2 = \rho(Q) \leq 1,$$

so that

$$\begin{aligned} \|\mathbf{U}^{m+1}\|_2 &= \|Q\mathbf{U}^m + \Delta t \mathbf{f}^m\|_2, \\ &\leq \|Q\|_2\|\mathbf{U}^m\|_2 + \Delta t\|\mathbf{f}^m\|_2, \\ &\leq \|\mathbf{U}^m\|_2 + \Delta t\|\mathbf{f}^m\|_2, \\ &\leq \|\mathbf{U}^0\|_2 + \sum_{k=0}^{m} \Delta t\|\mathbf{f}^k\|_2, \end{aligned}$$

which is

$$\|\mathbf{U}^{m+1}\|_{2,h} \leq \|\mathbf{U}^0\|_{2,h} + \sum_{k=0}^{m} \Delta t\|\mathbf{f}^k\|_{2,h}.$$

so we get the sufficient stability condition for forward Euler method as

$$\Delta t \leq \frac{1}{w_0}.$$

■

4.2 | Convergence of time-dependent problem

In this section we will prove the convergence theorems for time dependent problem.

4.2.1 | Convergence of Euler methods

Theorem 6 *If the solution of* (2) $u \in C^{2,2}(\Omega \cup \Omega_I \times [0,T])$, *and* Δt *satisfies the stability condition for Euler* schemes (26) and (28), *then we have the convergence result*

$$\max_{1\le j\le M}\|\mathbf{u}^j-\mathbf{U}^j\|_{2,h}\le\begin{cases}\frac{2-2s}{\delta^{2-2s}}C(\Delta t+h^2|\ln h|), & s=0,\\ \frac{2-2s}{\delta^{2-2s}}C(\Delta t+h^{2-2s}), & s\in(0,1).\end{cases}$$

where $\mathbf{u}^j=(u(x_1,t_j),\cdots,u(x_n,t_j))^T$, *and* $\mathbf{U}^j$ *is the numerical solution of Euler schemes.*

Proof. Firstly, we consider backward Euler method. Let $u\in C^{2,2}(\Omega\cup\Omega_I\times[0,T])$, from Equation (2), we have

$$\frac{u(x_i,t_{m+1})-u(x_i,t_m)}{\Delta t}-\widetilde{\mathcal{L}}_\delta^{\mathcal{H}} I_h u(x_i,t_{m+1})=f_i(t_{m+1})+\widetilde{\mathcal{E}}_i^{m+1}+\eta_i^{m+1} \tag{30}$$

where $f_i(t_{m+1})$ as in Equation (25), $I_h u(x,t_{m+1})=\sum_{j=0}^{n+1}u(x_j,t_{m+1})\phi_j(x),\ x\in\Omega$, and

$$\begin{aligned}\widetilde{\mathcal{E}}_i^{m+1}&=\mathcal{L}_\delta u(x_i,t_{m+1})-\widetilde{\mathcal{L}}_\delta^{\mathcal{H}} I_h u(x_i,t_{m+1}),\\ \eta_i^{m+1}&=-\frac{\partial u}{\partial t}(x_i,t_{m+1})+\frac{u(x_i,t_{m+1})-u(x_i,t_m)}{\Delta t}.\end{aligned}$$

We define

$$\begin{aligned}\mathbf{u}^{m+1}&=(u(x_1,t_{m+1}),\cdots,u(x_n,t_{m+1}))^T,\\ \mathbf{E}^{m+1}&=(\widetilde{\mathcal{E}}_1^{m+1},\cdots,\widetilde{\mathcal{E}}_n^{m+1})^T,\\ \mathbf{n}^{m+1}&=(\eta_1^{m+1},\cdots,\eta_n^{m+1})^T.\end{aligned}$$

Then the matrix form of Equation (30) is

$$\frac{\mathbf{u}^{m+1}-\mathbf{u}^m}{\Delta t}=-W\mathbf{u}^{m+1}+\mathbf{f}^{m+1}+\mathbf{E}^{m+1}+\mathbf{n}^{m+1}. \tag{31}$$

Let Equations (31)–(26), we have

$$\frac{\mathbf{e}^{m+1}-\mathbf{e}^m}{\Delta t}=-W\mathbf{e}^{m+1}+\mathbf{E}^{m+1}+\mathbf{n}^{m+1},$$

where $\mathbf{e}^{m+1}=\mathbf{u}^{m+1}-\mathbf{U}^{m+1}$. From the stability theorem and Remark 1, we have

$$\|\mathbf{e}^{m+1}\|_{2,h}\le\|\mathbf{e}^0\|_{2,h}+\sum_{k=1}^{m+1}\Delta t(\|\mathbf{E}^k\|_{2,h}+\|\mathbf{n}^k\|_{2,h}),$$

which is

$$\max_{1\le j\le M}\|\mathbf{e}^j\|_{2,h}\le\begin{cases}\frac{2-2s}{\delta^{2-2s}}C(\Delta t+h^2|\ln h|), & s=0,\\ \frac{2-2s}{\delta^{2-2s}}C(\Delta t+h^{2-2s}), & s\in(0,1).\end{cases}$$

Similarly, we can get the same result for forward Euler method under the stability condition. ∎

4.2.2 | Convergence of Crank–Nicolson method

Theorem 7 *If the solution of Equation (2) $u \in C^{2,3}(\Omega \cup \Omega_I \times [0,T])$, then we have the convergence result*

$$\max_{1\le j\le M} \|\mathbf{u}^j - \mathbf{U}^j\|_{2,h} \le \begin{cases} \frac{2-2s}{\delta^{2-2s}} C(\Delta t^2 + h^2|\ln h|), & s=0, \\ \frac{2-2s}{\delta^{2-2s}} C(\Delta t^2 + h^{2-2s}), & s\in(0,1). \end{cases}$$

where $\mathbf{u}^j = (u(x_1,t_j),\cdots,u(x_n,t_j))^T$, and $\mathbf{U}^j$ is the numerical solution of Equation (27).

Proof. For Crank–Nicolson method. Let $u \in C^{2,3}(\Omega \cup \Omega_I \times [0,T])$, we have

$$\frac{\partial u}{\partial t}(x_i,t_{m+1}) - \mathcal{L}_\delta u(x_i,t_{m+1}) = f_i(t_{m+1}), \tag{32}$$

$$\frac{\partial u}{\partial t}(x_i,t_m) - \mathcal{L}_\delta u(x_i,t_m) = f_i(t_m). \tag{33}$$

We define

$$\begin{aligned} \widetilde{\mathcal{E}}_i^{m+1} &= \mathcal{L}_\delta u(x_i,t_{m+1}) - \widetilde{\mathcal{L}}_\delta^{\mathcal{H}} I_h u(x_i,t_{m+1}), \\ \widetilde{\eta}_i^{m+1} &= -\frac{1}{2}\left(\frac{\partial u}{\partial t}(x_i,t_{m+1}) + \frac{\partial u}{\partial t}(x_i,t_m)\right) + \frac{u(x_i,t_{m+1})-u(x_i,t_m)}{\Delta t}. \end{aligned}$$

So add (32)–(33), we get

$$\begin{aligned} &\frac{u(x_i,t_{m+1})-u(x_i,t_m)}{\Delta t} - \frac{1}{2}(\widetilde{\mathcal{L}}_\delta^{\mathcal{H}} I_h u(x_i,t_{m+1}) + \widetilde{\mathcal{L}}_\delta^{\mathcal{H}} I_h u(x_i,t_m)) \\ &= \frac{1}{2}(f_i(t_{m+1}) + f_i(t_m)) + \frac{\widetilde{\mathcal{E}}_i^{m+1} + \widetilde{\mathcal{E}}_i^{m}}{2} + \widetilde{\eta}_i^{m+1}. \end{aligned} \tag{34}$$

First, by Taylor expansion, we have

$$\begin{aligned} u(x_i,t_m) &= u(x_i,t_{m+1}) - \Delta t u_t(x_i,t_{m+1}) + \frac{\Delta t^2}{2}u_{tt}(x_i,t_{m+1}) - \frac{\Delta t^3}{6}u_{ttt}(x_i,\xi_1), \\ &= u(x_i,t_{m+1}) - \Delta t u_t(x_i,t_{m+1}) + \frac{\Delta t^2}{2}u_{tt}(x_i,t_m) + \frac{\Delta t^3}{2}u_{tt}(x_i,\xi_2) - \frac{\Delta t^3}{6}u_{ttt}(x_i,\xi_1), \end{aligned}$$

and

$$u(x_i,t_{m+1}) = u(x_i,t_m) + \Delta t u_t(x_i,t_m) + \frac{\Delta t^2}{2}u_{tt}(x_i,t_m) + \frac{\Delta t^3}{6}u_{ttt}(x_i,\xi_3),$$

so that

$$\widetilde{\eta}_i^{m+1} = -\frac{\Delta t^2}{4}u_{tt}(x_i,\xi_2) + \frac{\Delta t^2}{12}u_{ttt}(x_i,\xi_1) + \frac{\Delta t^2}{12}u_{ttt}(x_i,\xi_3).$$

We define $\mathbf{N}^{m+1} = (\widetilde{\eta}_1^{m+1},\cdots,\widetilde{\eta}_n^{m+1})^T$. Then we have the matrix form of Equation (34)

$$\frac{\mathbf{u}^{m+1}-\mathbf{u}^m}{\Delta t} = -W\frac{\mathbf{u}^{m+1}+\mathbf{u}^m}{2} + \frac{\mathbf{f}^{m+1}+\mathbf{f}^m}{2} + \frac{\mathbf{E}^{m+1}+\mathbf{E}^m}{2} + \mathbf{N}^{m+1}. \tag{35}$$

Let (35)–(27), we have

$$\frac{\mathbf{e}^{m+1}-\mathbf{e}^m}{\Delta t} = -W\frac{\mathbf{e}^{m+1}+\mathbf{e}^m}{2} + \frac{\mathbf{E}^{m+1}+\mathbf{E}^m}{2} + \mathbf{N}^{m+1}. \tag{36}$$

From the stability property we have

$$\|\mathbf{e}^{m+1}\|_{2,h} \le \|\mathbf{e}^0\|_{2,h} + \sum_{k=1}^{m+1}\Delta t\left(\frac{\|\mathbf{E}^k\|_{2,h} + \|\mathbf{E}^{k-1}\|_{2,h}}{2} + \|\mathbf{N}^k\|_{2,h}\right),$$

so that

$$\max_{1\le j\le M}\|\mathbf{e}^j\|_{2,h} \le \begin{cases} \frac{2-2s}{\delta^{2-2s}}C(\Delta t^2 + h^2|\ln h|), & s=0,\\ \frac{2-2s}{\delta^{2-2s}}C(\Delta t^2 + h^{2-2s}), & s\in(0,1). \end{cases} \qquad \blacksquare$$

5 | NUMERICAL EXPERIMENT

In this section we will consider two examples with exact solutions the first one is a steady state problem the second one is time-dependent. Both examples illustrate our analysis results. We define the L^∞ error as $\|e\|_\infty$, the truncation error as $e_{\text{truncation}}$, and the L^∞ norm for the inverse of stiffness matrix as $\|\mathcal{A}^{-1}\|_\infty$.

Example 1 Let $u(x) = x^2(1-x^2)$, $c=0$, then

$$-\mathcal{L}_\delta u = \frac{2-2s}{\delta^{2-2s}}\int_{x-\delta}^{x+\delta}\frac{u(x)-u(y)}{|x-y|^{1+2s}}dy = (12x^2-2) + \frac{2s-2}{s-2}\delta^2$$

Example 2 Consider

$$\begin{cases} u_t - \mathcal{L}_\delta u = \left((12x^2-2) + \frac{2s-2}{s-2}\delta^2\right)\sin t + x^2(1-x^2) \quad \text{on } \Omega\times(0,1]\\ u = x^2(1-x^2)\sin t, \quad \text{on } \Omega_{\mathcal{I}}\times[0,1]\\ u(x,0)=0, \quad \text{on } \Omega \end{cases} \tag{37}$$

with exact solution $u(x,t) = x^2(1-x^2)\sin t$.

In Tables 1–5, δ is fixed as 0.1, s ranges from 0.1 to 0.9. The convergence rates match well with the theorem, and the rate for $\|e\|_\infty$ is slightly higher when s is close to 1. This is also found in time dependent problems see Tables 7, 9, and 11. Also, the L^∞ norm for the inverse of stiffness matrix tends to be stable as h decreases for given s, and $\|\mathcal{A}^{-1}\|_\infty$ decreases as s increases.

Tables 6 and 7 show the numerical results for forward Euler method solving Example 2, we can see that as s increases, to get a stable scheme, Δt should be smaller, this agrees with the Theorem 5.

In Tables 8–11, two unconditionally stable schemes, 8–9 for backward Euler, 10–11 for Crank–Nicolson, are used to solve Example 2, the errors are consistent with the convergence rates obtained in Theorem 6 and 7.

TABLE 1 Numerical results of Example 1 ($\delta = 0.1, s = 0.1$)

h	$\|e\|_\infty$	Rate	$e_{\text{truncation}}$	Rate	$\|\mathcal{A}^{-1}\|_\infty$
1/8	1.593196e-01		1.153957e-01		5.047659e+00
1/16	9.502453e-02	0.746	4.211551e-02	1.454	9.051536e+00
1/32	4.231699e-02	1.167	1.385465e-02	1.604	1.241474e+01
1/64	1.547239e-02	1.452	4.489262e-03	1.626	1.422312e+01
1/128	5.085781e-03	1.605	1.417351e-03	1.663	1.499308e+01
1/256	1.588387e-03	1.679	4.398864e-04	1.688	1.529535e+01
1/512	4.842902e-04	1.714	1.347824e-04	1.706	1.541417e+01
1/1024	1.456743e-04	1.733	4.088709e-05	1.721	1.546268e+01
1/2048	4.342223e-05	1.746	1.230024e-05	1.733	1.548344e+01
1/4096	1.285839e-05	1.756	3.675163e-06	1.743	1.549272e+01

TABLE 2 Numerical results of Example 1 ($\delta = 0.1, s = 0.4$)

h	$\|e\|_\infty$	Rate	$e_{\text{truncation}}$	Rate	$\|\mathcal{A}^{-1}\|_\infty$
1/8	2.135781e-01		2.468391e+00		3.169786e-01
1/16	1.863756e-01	0.197	1.287209e+00	0.939	6.143939e-01
1/32	1.428161e-01	0.384	6.106285e-01	1.076	1.060865e+00
1/64	9.306979e-02	0.618	2.775233e-01	1.138	1.566544e+00
1/128	5.189687e-02	0.843	1.234706e-01	1.168	1.986530e+00
1/256	2.581967e-02	1.007	5.438204e-02	1.183	2.255554e+00
1/512	1.201548e-02	1.104	2.391942e-02	1.185	2.401109e+00
1/1024	5.400836e-03	1.154	1.048225e-02	1.190	2.473039e+00
1/2048	2.386882e-03	1.178	4.584957e-03	1.193	2.507105e+00
1/4096	1.046428e-03	1.190	2.002777e-03	1.195	2.522957e+00

TABLE 3 Numerical results of Example 1 ($\delta = 0.1, s = 0.5$)

h	$\|e\|_\infty$	Rate	$e_{\text{truncation}}$	Rate	$\|\mathcal{A}^{-1}\|_\infty$
1/8	1.120111e+00		9.207689e-01		4.481420e+00
1/16	3.768083e-01	1.572	5.219356e-01	0.819	3.170576e+00
1/32	1.041444e-01	1.855	2.758462e-01	0.920	1.827800e+00
1/64	4.233145e-02	1.299	1.410940e-01	0.967	1.530115e+00
1/128	1.932604e-02	1.131	7.125268e-02	0.986	1.418917e+00
1/256	9.253058e-03	1.063	3.578355e-02	0.994	1.369462e+00
1/512	4.527239e-03	1.031	1.792537e-02	0.997	1.345758e+00
1/1024	2.238820e-03	1.016	8.969673e-03	0.999	1.333999e+00
1/2048	1.113131e-03	1.008	4.486251e-03	1.000	1.328069e+00
1/4096	5.549646e-04	1.004	2.243397e-03	1.000	1.325055e+00

TABLE 4 Numerical results of Example 1 ($\delta = 0.1, s = 0.6$)

h	$\|e\|_\infty$	Rate	$e_{\text{truncation}}$	Rate	$\|\mathcal{A}^{-1}\|_\infty$
1/8	1.362766e+00		1.737165e+00		2.891150e+00
1/16	5.026394e-01	1.439	1.142448e+00	0.605	1.914829e+00
1/32	1.533591e-01	1.713	6.988938e-01	0.709	1.043245e+00
1/64	6.948869e-02	1.142	4.131606e-01	0.758	8.370253e-01
1/128	3.560824e-02	0.965	2.406352e-01	0.780	7.528877e-01
1/256	1.925267e-02	0.887	1.391594e-01	0.790	7.115317e-01
1/512	1.069229e-02	0.848	8.019510e-02	0.795	6.894626e-01
1/1024	6.025212e-03	0.827	4.613623e-02	0.798	6.772092e-01
1/2048	3.423004e-03	0.816	2.651998e-02	0.799	6.702679e-01
1/4096	1.953681e-03	0.809	1.523790e-02	0.799	6.662955e-01

TABLE 5 Numerical results of Example 1 ($\delta = 0.1, s = 0.9$)

h	$\lVert e \rVert_\infty$	Rate	$e_{\text{truncation}}$	Rate	$\lVert \mathcal{A}^{-1} \rVert_\infty$
1/8	5.008943e+00		2.384427e+01		7.755381e-01
1/16	2.410037e+00	1.055	2.445495e+01	−0.036	4.199832e-01
1/32	9.472193e-01	1.347	2.300362e+01	0.088	1.889865e-01
1/64	5.592037e-01	0.760	2.079200e+01	0.146	1.284872e-01
1/128	3.792972e-01	0.560	1.843938e+01	0.173	1.003342e-01
1/256	2.762895e-01	0.457	1.620117e+01	0.187	8.413289e-02
1/512	2.103532e-01	0.393	1.416895e+01	0.193	7.370509e-02
1/1024	1.650180e-01	0.350	1.236315e+01	0.197	6.650512e-02
1/2048	1.322528e-01	0.319	1.077511e+01	0.198	6.128640e-02
1/4096	1.076886e-01	0.296	9.385661e+00	0.199	5.736671e-02

TABLE 6 Forward Euler scheme of Example 2 ($\delta = 0.1, s = 0.0$)

h	Δt	$\lVert e \rVert_\infty$	Rate	$\lVert e \rVert_{2,h}$	Rate
1/4	1/16	3.184×10^{-2}		1.688×10^{-2}	
1/8	1/64	1.774×10^{-2}	0.84	9.047×10^{-3}	0.90
1/16	1/256	6.292×10^{-3}	1.50	3.134×10^{-3}	1.53
1/32	1/1024	1.986×10^{-3}	1.66	9.925×10^{-4}	1.66
1/64	1/4096	6.049×10^{-4}	1.72	3.004×10^{-4}	1.72

TABLE 7 Forward Euler scheme of Example 2 ($\delta = 0.1, s = 0.8$)

h	Δt	$\lVert e \rVert_\infty$	Rate	$\lVert e \rVert_{2,h}$	Rate
1/4	1/16	2.690×10^{5}		1.904×10^{5}	
1/8	1/64	$9.212e \times 10^{5}$	−1.78	5.766×10^{5}	−1.60
1/16	1/256	6.675×10^{-1}	20.40	3.996×10^{-1}	20.46
1/32	1/1024	2.920×10^{-1}	1.19	1.823×10^{-1}	1.13
1/64	1/4096	1.683×10^{-1}	0.79	1.061×10^{-1}	0.78

TABLE 8 Backward Euler scheme of Example 2 ($\delta = 0.1, s = 0.0$)

h	Δt	$\lVert e \rVert_\infty$	Rate	$\lVert e \rVert_{2,h}$	Rate
1/16	1/256	6.455×10^{-2}		4.128×10^{-2}	
1/32	1/1024	2.515×10^{-2}	1.36	1.606×10^{-2}	1.36
1/64	1/4096	8.226×10^{-3}	1.61	5.270×10^{-3}	1.61
1/128	1/16384	2.460×10^{-3}	1.74	1.579×10^{-3}	1.74

TABLE 9 Backward Euler scheme of Example 2 ($\delta = 0.1, s = 0.8$)

h	Δt	$\lVert e \rVert_\infty$	Rate	$\lVert e \rVert_{2,h}$	Rate
1/16	1/256	6.687×10^{-1}		4.005×10^{-1}	
1/32	1/1024	2.922×10^{-1}	1.19	1.825×10^{-1}	1.13
1/64	1/4096	1.684×10^{-1}	0.79	1.061×10^{-1}	0.78
1/128	1/16384	1.077×10^{-1}	0.65	6.810×10^{-2}	0.64

TABLE 10 C-N scheme of Example 2 ($\delta = 0.1, s = 0.0$)

h	Δt	$\|\|e\|\|_\infty$	Rate	$\|\|e\|\|_{2,h}$	Rate
1/16	1/16	6.453×10^{-2}		4.127×10^{-2}	
1/32	1/32	2.514×10^{-2}	1.36	1.605×10^{-2}	1.36
1/64	1/64	8.225×10^{-3}	1.61	5.269×10^{-3}	1.61
1/128	1/128	2.459×10^{-3}	1.74	1.579×10^{-3}	1.74
1/256	1/256	7.030×10^{-4}	1.81	4.520×10^{-4}	1.80

TABLE 11 C-N scheme of Example 2 ($\delta = 0.1, s = 0.8$)

h	Δt	$\|\|e\|\|_\infty$	Rate	$\|\|e\|\|_{2,h}$	Rate
1/16	1/16	6.684×10^{-1}		4.003×10^{-1}	
1/32	1/32	2.922×10^{-1}	1.19	1.825×10^{-1}	1.13
1/64	1/64	1.684×10^{-1}	0.79	1.061×10^{-1}	0.78
1/128	1/128	1.077×10^{-1}	0.65	6.810×10^{-2}	0.64
1/256	1/256	7.280×10^{-2}	0.56	4.611×10^{-2}	0.56

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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