Mathematical Foundation of Finite Element Methods

Chapter 2: 2D/3D Finite Element Spaces

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Outline

Outline

Triangular mesh: uniform partition

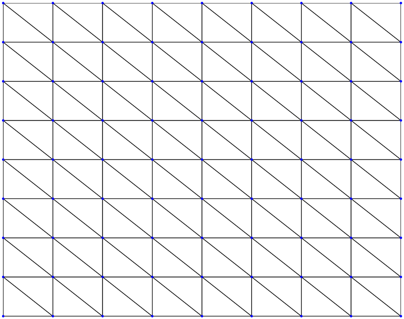
- Consider $\Omega = [left, right] \times [bottom, top]$.
- First, we form a uniform rectangular partition of Ω into N_1 elements in $x-axis$ and N_2 elements in $y-axis$ with mesh size
$$h = [h_1, h_2] = [\frac{right - left}{N_1}, \frac{top - bottom}{N_2}].$$

Triangular mesh: global indices

- For example, when $N_1 = N_2 = 8$, we have

Triangular mesh: global indices

- Then we divide each rectangular element into two triangular elements by connecting the left-top corner and the right-lower corner of the rectangular element.
- For example, when $N_1 = N_2 = 8$, we have

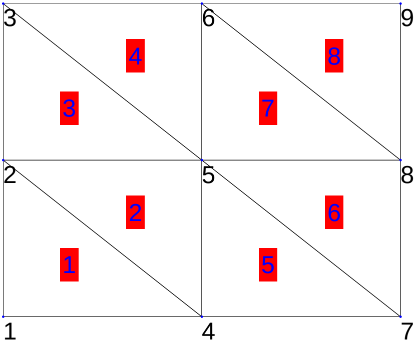


Triangular mesh: global indices

- This would give an uniform triangular partition.
- There are $N = 2N_1N_2$ elements and $N_m = (N_1 + 1)(N_2 + 1)$ mesh nodes.

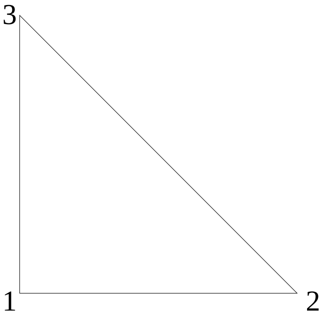
Triangular mesh: global indices

- Define your global indices for all the mesh elements E_n $(n = 1, \cdots, N)$ and mesh nodes Z_k $(k = 1, \cdots, N_m)$.
- For example, when $N_1 = N_2 = 2$, we have



Triangular mesh: local node index

- Let N_l denote the number of local mesh nodes in a mesh element. Define your index for the local mesh nodes in a mesh element.



Triangular mesh: information matrices

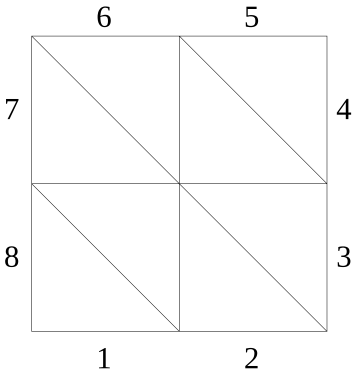
- Define matrix P to be an information matrix consisting of the coordinates of all mesh nodes.
- Define matrix T to be an information matrix consisting of the global node indices of the mesh nodes of all the mesh elements.
- We can use the j^{th} column of the matrix P to store the coordinates of the j^{th} mesh node and the n^{th} column of the matrix T to store the global node indices of the mesh nodes of the n^{th} mesh element. For example, when $N_1 = N_2 = 2$, we have

$$P = \begin{pmatrix} 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0.5 & 1 \end{pmatrix},$$

$$T = \begin{pmatrix} 1 & 2 & 2 & 3 & 4 & 5 & 5 & 6 \\ 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 \\ 2 & 5 & 3 & 6 & 5 & 8 & 6 & 9 \end{pmatrix}.$$

Triangular mesh: boundary edge index

- Define your index for the boundary edges.
- For example, when $N_1 = N_2 = 2$, we have



Triangular mesh: boundary edge information matrix

- Matrix *boundaryedges*:
- $boundaryedges(1, k)$ is the type of the k^{th} boundary edge e_k: Dirichlet (-1), Neumann (-2), Robin (-3)......
- $boundaryedges(2, k)$ is the index of the element which contains the k^{th} boundary edge e_k.
- Each boundary edge has two end nodes. We index them as the first and the second counterclock wise along the boundary.
- $boundaryedges(3, k)$ is the global node index of the first end node of the k^{th} boundary boundary edge e_k.
- $boundaryedges(4, k)$ is the global node index of the second end node of the k^{th} boundary boundary edge e_k.
- Set $nbe = size(boundaryedges, 2)$ to be the number of boundary edges;

Triangular mesh: boundary edge information matrix

- For the mesh with $N_1 = N_2 = 2$ and all Dirichlet boundary condition, we have:

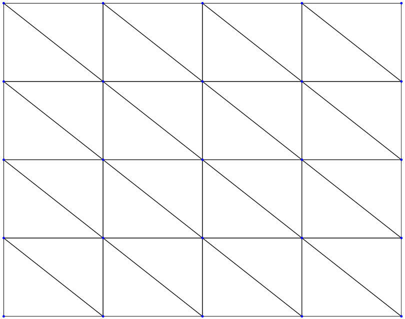
$$boundaryedges = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 5 & 6 & 8 & 8 & 4 & 3 & 1 \\ 1 & 4 & 7 & 8 & 9 & 6 & 3 & 2 \\ 4 & 7 & 8 & 9 & 6 & 3 & 2 & 1 \end{pmatrix}.$$

Triangular mesh

- What are the information matrices

$$P, \ T, \ boundaryedges$$

for the following mesh?



Triangular mesh

- What are the information matrices

$$P, \ T, \ boundaryedges$$

for a general uniform triangular mesh with the mesh size

$$h = [h_1, h_2] = [\frac{right - left}{N_1}, \frac{top - bottom}{N_2}]$$

in the domain

$$\Omega = [left, right] \times [bottom, top]?$$

Rectangular mesh: uniform partition

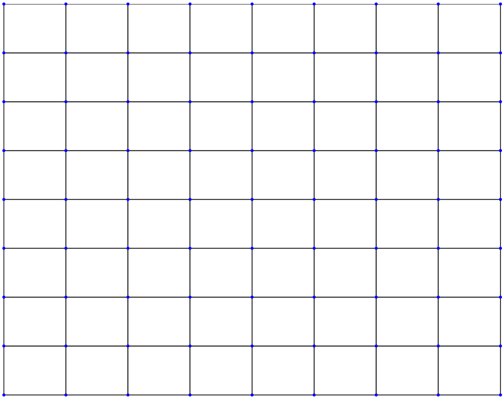
- Consider $\Omega = [left, right] \times [bottom, top]$.
- Consider a uniform rectangular partition of Ω into N_1 elements in $x - axis$ and N_2 elements in $y - axis$ with mesh size

$$h = [h_1, h_2] = [\frac{right - left}{N_1}, \frac{top - bottom}{N_2}].$$

- There are $N = N_1 N_2$ elements and $N_m = (N_1 + 1)(N_2 + 1)$ mesh nodes.

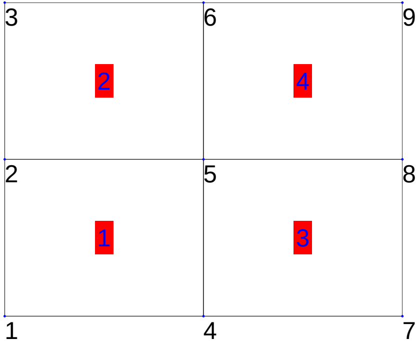
Rectangular mesh: uniform partition

- For example, when $N_1 = N_2 = 8$, we have



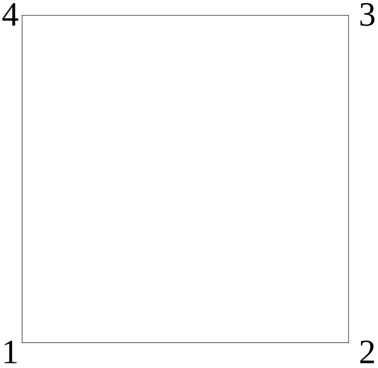
Rectangular mesh: global indices

- Define your global indices for all the mesh elements E_n $(n = 1, \cdots, N)$ and mesh nodes Z_k $(k = 1, \cdots, N_m)$.
- For example, when $N_1 = N_2 = 2$, we have



Rectangular mesh: local node index

- Let N_l denote the number of local mesh nodes in a mesh element. Define your index for the local mesh nodes in a mesh element.



Rectangular mesh: information matrices

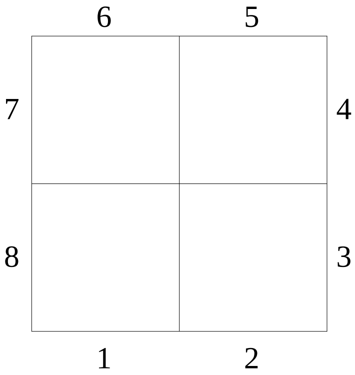
- Define matrix P to be an information matrix consisting of the coordinates of all mesh nodes.
- Define matrix T to be an information matrix consisting of the global node indices of the mesh nodes of all the mesh elements.
- For example, when $N_1 = N_2 = 2$, we have

$$P = \begin{pmatrix} 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0.5 & 1 \end{pmatrix},$$

$$T = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 4 & 5 & 7 & 8 \\ 5 & 6 & 8 & 9 \\ 2 & 3 & 5 & 6 \end{pmatrix}.$$

Rectangular mesh: boundary edge index

- Define your index for the boundary edges.
- For example, when $N_1 = N_2 = 2$, we have



Rectangular mesh: boundary edge information matrix

- Matrix *boundaryedges*:
- $boundaryedges(1,k)$ is the type of the k^{th} boundary edge e_k: Dirichlet (-1), Neumann (-2), Robin (-3)......
- $boundaryedges(2,k)$ is the index of the element which contains the k^{th} boundary edge e_k.
- Each boundary edge has two end nodes. We index them as the first and the second counterclock wise along the boundary.
- $boundaryedges(3,k)$ is the global node index of the first end node of the k^{th} boundary boundary edge e_k.
- $boundaryedges(4,k)$ is the global node index of the second end node of the k^{th} boundary boundary edge e_k.
- Set $nbe = size(boundaryedges, 2)$ to be the number of boundary edges;

Rectangular mesh: boundary edge information matrix

- For example, when $N_1 = N_2 = 2$ and all the boundary are Dirichlet type, we have:

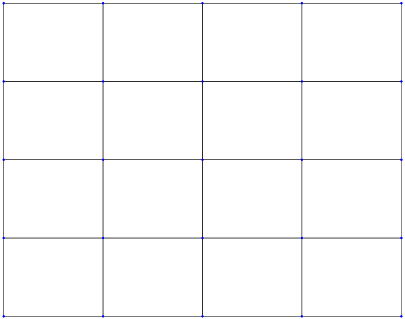
$$boundaryedges = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 3 & 3 & 4 & 4 & 2 & 2 & 1 \\ 1 & 4 & 7 & 8 & 9 & 6 & 3 & 2 \\ 4 & 7 & 8 & 9 & 6 & 3 & 2 & 1 \end{pmatrix}.$$

Rectangular mesh

- What are the information matrices

$$P, \; T, \; boundaryedges$$

for the following mesh?



Rectangular mesh

- What are the information matrices

$$P, \ T, \ boundaryedges$$

for a general uniform rectangular mesh with the mesh size

$$h = [h_1, h_2] = [\frac{right - left}{N_1}, \frac{top - bottom}{N_2}]$$

in the domain

$$\Omega = [left, right] \times [bottom, top]?$$

Outline

2D linear finite element: reference basis functions

- The "reference→ local → global" framework will be used to construct the finite element spaces.
- We only consider the nodal basis functions (Lagrange type) in this course.
- We first consider the reference 2D linear basis functions on the reference triangular element $\hat{E} = \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3$ where $\hat{A}_1 = (0,0)$, $\hat{A}_2 = (1,0)$, and $\hat{A}_3 = (0,1)$.
- Define three reference 2D linear basis functions

$$\hat{\psi}_j(\hat{x},\hat{y}) \quad = \quad a_j\hat{x} + b_j\hat{y} + c_j, \;\; j = 1,2,3,$$

such that

$$\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, 2, 3$.

2D linear finite element: reference basis functions

- Then it's easy to obtain

$$
\begin{aligned}
\hat{\psi}_1(\hat{A}_1) = 1 \quad &\Rightarrow \quad c_1 = 1, \\
\hat{\psi}_1(\hat{A}_2) = 0 \quad &\Rightarrow \quad a_1 + c_1 = 0, \\
\hat{\psi}_1(\hat{A}_3) = 0 \quad &\Rightarrow \quad b_1 + c_1 = 0, \\
\hat{\psi}_2(\hat{A}_1) = 0 \quad &\Rightarrow \quad c_2 = 0, \\
\hat{\psi}_2(\hat{A}_2) = 1 \quad &\Rightarrow \quad a_2 + c_2 = 1, \\
\hat{\psi}_2(\hat{A}_3) = 0 \quad &\Rightarrow \quad b_2 + c_2 = 0, \\
\hat{\psi}_3(\hat{A}_1) = 0 \quad &\Rightarrow \quad c_3 = 0, \\
\hat{\psi}_3(\hat{A}_2) = 0 \quad &\Rightarrow \quad a_3 + c_3 = 0, \\
\hat{\psi}_3(\hat{A}_3) = 1 \quad &\Rightarrow \quad b_3 + c_3 = 1.
\end{aligned}
$$

2D linear finite element: reference basis functions

- Hence

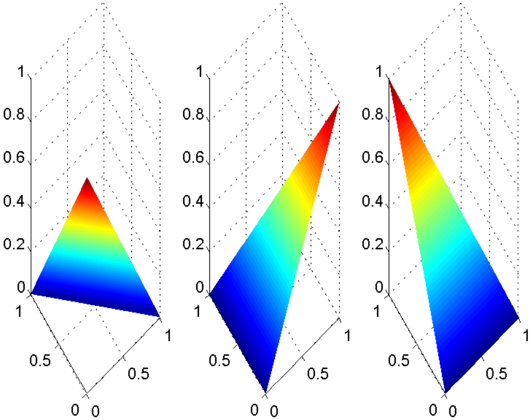
$$
\begin{aligned}
a_1 &= -1, b_1 = -1, c_1 = 1, \\
a_2 &= 1, b_2 = 0, c_2 = 0, \\
a_3 &= 0, b_3 = 1, c_3 = 0.
\end{aligned}
$$

- Then the three reference 2D linear basis functions are

$$
\begin{aligned}
\hat{\psi}_1(\hat{x}, \hat{y}) &= -\hat{x} - \hat{y} + 1, \\
\hat{\psi}_2(\hat{x}, \hat{y}) &= \hat{x}, \\
\hat{\psi}_3(\hat{x}, \hat{y}) &= \hat{y}.
\end{aligned}
$$

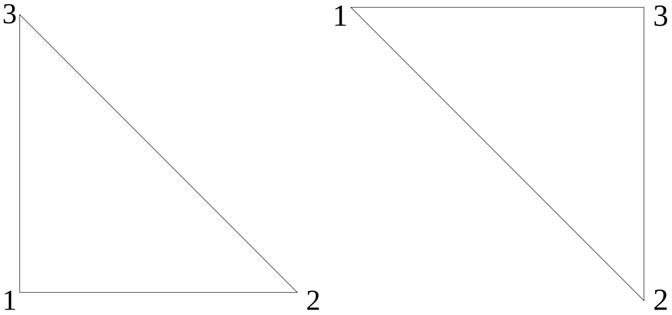
2D linear finite element: reference basis functions

- Plots of the three linear basis functions on the reference triangle:



2D linear finite element: local node index

- Let N_{lb} denote the number of local finite element nodes (local finite element basis functions) in a mesh element. Here $N_{lb} = 3$. Define your index for the local finite element nodes in a mesh element.

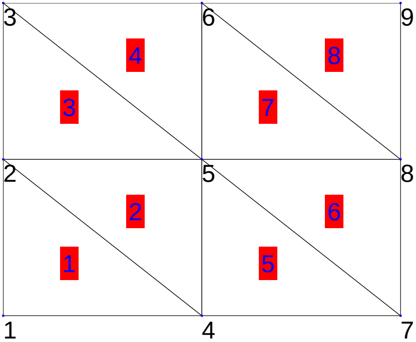


2D linear finite element: information matrices

- The mesh information matrices P and T are for the mesh nodes.
- We also need similar finite element information matrices P_b and T_b for the finite elements nodes, which are the nodes corresponding to the finite element basis functions.
- Note: For the nodal finite element basis functions, the correspondence between the finite elements nodes and the finite element basis functions is one-to-one in a straightforward way. But it could be more complicated for other types of finite element basis functions in the future.
- Let N_b denote the total number of the finite element basis functions (= the number of unknowns = the total number of the finite element nodes). Here $N_b = N_m = (N_1 + 1)(N_2 + 1)$.

2D linear finite element: information matrices

- Define your global indices for all the mesh elements E_n $(n = 1, \cdots, N)$ and finite element nodes X_j $(j = 1, \cdots, N_b)$ (or the finite element basis functions).
- For example, when $N_1 = N_2 = 2$, we have



2D linear finite element: information matrices

- Define matrix P_b to be an information matrix consisting of the coordinates of all finite element nodes.
- Define matrix T_b to be an information matrix consisting of the global node indices of the finite element nodes of all the mesh elements.

2D linear finite element: information matrices

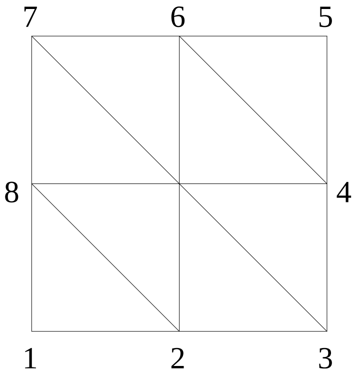
- For the 2D linear finite elements, P_b and T_b are the same as the P and T of the triangular mesh since the nodes of the 2D linear finite element basis functions are the same as those of the mesh. For example, when $N_1 = N_2 = 2$, we have

$$P_b = P = \begin{pmatrix} 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0.5 & 1 \end{pmatrix},$$

$$T_b = T = \begin{pmatrix} 1 & 2 & 2 & 3 & 4 & 5 & 5 & 6 \\ 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 \\ 2 & 5 & 3 & 6 & 5 & 8 & 6 & 9 \end{pmatrix}.$$

2D linear finite element: boundary node index

- Define your index for the boundary finite element nodes.
- For example, when $N_1 = N_2 = 2$, we have,



2D linear finite element: boundary node information matrix

- Matrix *boundarynodes*:
- $boundarynodes(1, k)$ is the type of the k^{th} boundary finite element node: Dirichlet (-1), Neumann (-2), Robin (-3)......
- The intersection nodes of Dirichlet boundary condition and other boundary conditions usually need to be treated as Dirichlet boundary nodes.
- $boundarynodes(2, k)$ is the global node index of the k^{th} boundary boundary finite element node.
- Set $nbn = size(boundarynodes, 2)$ to be the number of boundary finite element nodes;
- For the above example with all Dirichlet boundary condition, we have:

$$boundarynodes = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 4 & 7 & 8 & 9 & 6 & 3 & 2 \end{pmatrix}.$$

2D linear finite element: affine mapping

- Now we can use the affine mapping between an arbitrary triangle $E = \triangle A_1 A_2 A_3$ and the reference triangle $\hat{E} = \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3$ to construct the local basis functions from the reference ones.
- Assume

$$A_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad i = 1, 2, 3.$$

- Consider the affine mapping

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} A_2 - A_1, A_3 - A_1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + A_1 \\ &= \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \end{aligned}$$

2D linear finite element: affine mapping

- The affine mapping actually maps

$$\hat{A}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = A_1,$$

$$\hat{A}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = A_2,$$

$$\hat{A}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = A_3.$$

- Hence the affine mapping maps $\triangle \hat{A}_1 \hat{A}_2 \hat{A}_3$ to $\triangle A_1 A_2 A_3$.
- Also,

$$\hat{x} = \frac{(y_3 - y_1)(x - x_1) - (x_3 - x_1)(y - y_1)}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)},$$

$$\hat{y} = \frac{(y_2 - y_1)(x - x_1) - (x_2 - x_1)(y - y_1)}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}.$$

2D linear finite element: affine mapping

- Define the Jacobi matrix:

$$J = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}.$$

- Then

$$|J| = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1),$$

and

$$\begin{aligned} \hat{x} &= \frac{(y_3 - y_1)(x - x_1) - (x_3 - x_1)(y - y_1)}{|J|}, \\ \hat{y} &= \frac{-(y_2 - y_1)(x - x_1) + (x_2 - x_1)(y - y_1)}{|J|}. \end{aligned}$$

2D linear finite element: affine mapping

- For a given function $\hat{\psi}(\hat{x}, \hat{y})$ where $(\hat{x}, \hat{y}) \in \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3$, we can define the corresponding function for $(x, y) \in \triangle A_1 A_2 A_3$ as follows:

$$\psi(x, y) = \hat{\psi}(\hat{x}, \hat{y}),$$

where

$$\begin{aligned} \hat{x} &= \frac{(y_3 - y_1)(x - x_1) - (x_3 - x_1)(y - y_1)}{|J|}, \\ \hat{y} &= \frac{-(y_2 - y_1)(x - x_1) + (x_2 - x_1)(y - y_1)}{|J|}. \end{aligned}$$

2D linear finite element: affine mapping

- Then by chain rule, we get

$$
\begin{aligned}
\frac{\partial \psi}{\partial x} &= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial x} \\
&= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{y_3 - y_1}{|J|} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{y_1 - y_2}{|J|}, \\
\frac{\partial \psi}{\partial y} &= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial y} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial y} \\
&= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{x_1 - x_3}{|J|} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{x_2 - x_1}{|J|}.
\end{aligned}
$$

2D linear finite element: local basis functions

- Consider the n^{th} element $E_n = \triangle A_{n1} A_{n2} A_{n3}$ where

$$A_{ni} = \begin{pmatrix} x_{ni} \\ y_{ni} \end{pmatrix} \quad (i = 1, 2, 3).$$

- The three local 2D linear basis functions are

$$\psi_{ni}(x, y) = \hat{\psi}_i(\hat{x}, \hat{y}), \ i = 1, 2, 3,$$

where

$$\begin{aligned}
\hat{x} &= \frac{(y_{n3} - y_{n1})(x - x_{n1}) - (x_{n3} - x_{n1})(y - y_{n1})}{|J_n|}, \\
\hat{y} &= \frac{-(y_{n2} - y_{n1})(x - x_{n1}) + (x_{n2} - x_{n1})(y - y_{n1})}{|J_n|}, \\
|J_n| &= (x_{n2} - x_{n1})(y_{n3} - y_{n1}) - (x_{n3} - x_{n1})(y_{n2} - y_{n1}).
\end{aligned}$$

2D linear finite element: local basis functions

- And for $i = 1, 2, 3$,

$$
\begin{aligned}
\frac{\partial \psi_{ni}}{\partial x} &= \frac{\partial \hat{\psi}_i}{\partial \hat{x}} \frac{y_{n3} - y_{n1}}{|J_n|} + \frac{\partial \hat{\psi}_i}{\partial \hat{y}} \frac{y_{n1} - y_{n2}}{|J_n|}, \\
\frac{\partial \psi_{ni}}{\partial y} &= \frac{\partial \hat{\psi}_i}{\partial \hat{x}} \frac{x_{n1} - x_{n3}}{|J_n|} + \frac{\partial \hat{\psi}_i}{\partial \hat{y}} \frac{x_{n2} - x_{n1}}{|J_n|}.
\end{aligned}
$$

- The reference and local basis functions defined in this section are what you need to input into the code in order to use the "reference $\rightarrow$ local" framework to define the local basis functions.

2D linear finite element: local basis functions

- In more details, we have

$$
\begin{aligned}
\psi_{n1}(x,y) &= \hat{\psi}_1(\hat{x},\hat{y}) = -\hat{x}-\hat{y}+1 \\
&= -\frac{(y_{n3}-y_{n1})(x-x_{n1})-(x_{n3}-x_{n1})(y-y_{n1})}{|J_n|} \\
&\quad -\frac{-(y_{n2}-y_{n1})(x-x_{n1})+(x_{n2}-x_{n1})(y-y_{n1})}{|J_n|}+1, \\
\psi_{n2}(x,y) &= \hat{\psi}_2(\hat{x},\hat{y}) = \hat{x} \\
&= \frac{(y_{n3}-y_{n1})(x-x_{n1})-(x_{n3}-x_{n1})(y-y_{n1})}{|J_n|}, \\
\psi_{n3}(x,y) &= \hat{\psi}_3(\hat{x},\hat{y}) = \hat{y} \\
&= \frac{-(y_{n2}-y_{n1})(x-x_{n1})+(x_{n2}-x_{n1})(y-y_{n1})}{|J_n|}.
\end{aligned}
$$

2D linear finite element: local basis functions

- And

$$
\begin{aligned}
\frac{\partial \psi_{n1}}{\partial x} &= -\frac{y_{n3}-y_{n1}}{|J_n|}+\frac{y_{n2}-y_{n1}}{|J_n|}=\frac{y_{n2}-y_{n3}}{|J_n|},\\
\frac{\partial \psi_{n2}}{\partial x} &= \frac{y_{n3}-y_{n1}}{|J_n|},\\
\frac{\partial \psi_{n3}}{\partial x} &= -\frac{y_{n2}-y_{n1}}{|J_n|},\\
\frac{\partial \psi_{n1}}{\partial y} &= \frac{x_{n3}-x_{n1}}{|J_n|}-\frac{x_{n2}-x_{n1}}{|J_n|}=\frac{x_{n3}-x_{n2}}{|J_n|},\\
\frac{\partial \psi_{n2}}{\partial y} &= -\frac{x_{n3}-x_{n1}}{|J_n|},\\
\frac{\partial \psi_{n3}}{\partial y} &= \frac{x_{n2}-x_{n1}}{|J_n|}.
\end{aligned}
$$

- You can also directly input these local basis functions and their derivatives into your code.

2D linear finite element: local basis functions

- In another way, the local basis functions can be also directly formed on the n^{th} element $E_n = \triangle A_{n1}A_{n2}A_{n3}$ as follows:

$$\psi_{nj}(x,y) = a_{nj}x + b_{nj}y + c_{nj}, \; j = 1,2,3,$$

such that

$$\psi_{nj}(A_{ni}) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, 2, 3$.

- Obtain the local basis functions in the above way and compare them with the ψ_{n1}, ψ_{n2}, and ψ_{n3} obtained before.
- They are the same!

2D linear finite element: global basis functions

"local $\to$ global" framework:

- Define the local finite element space

$$S_h(E_n) = span\{\psi_{n1}, \psi_{n2}, \psi_{n3}\}.$$

- At each finite element node X_j ($j = 1, \cdots, N_b$), define the corresponding global linear basis function ϕ_j such that $\phi_j|_{E_n} \in S_h(E_n)$ and

$$\phi_j(X_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, \cdots, N_b$.

- Then define the global finite element space to be

$$U_h = span\{\phi_j\}_{j=1}^{N_b}.$$

2D linear finite element: global basis functions

- Hence

$$\phi_j|_{E_n} = \begin{cases} \psi_{n1}, & \text{if } j = T_b(1,n), \\ \psi_{n2}, & \text{if } j = T_b(2,n), \\ \psi_{n3}, & \text{if } j = T_b(3,n), \\ 0, & \text{otherwise.} \end{cases}$$

for $j = 1, \cdots, N_b$ and $n = 1, \cdots, N$.

2D quadratic finite element: reference basis functions

- We first consider the reference 2D quadratic basis functions on the reference triangular element $\hat{E} = \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3$ where $\hat{A}_1 = (0,0)$, $\hat{A}_2 = (1,0)$, and $\hat{A}_3 = (0,1)$. Define $\hat{A}_4 = (0.5,0)$, $\hat{A}_5 = (0.5,0.5)$, and $\hat{A}_6 = (0,0.5)$.
- Define six reference 2D linear basis functions

$$\hat{\psi}_j(\hat{x},\hat{y}) \quad = \quad a_j\hat{x}^2 + b_j\hat{y}^2 + c_j\hat{x}\hat{y} + d_j\hat{y} + e_j\hat{x} + f_j, \; j = 1, \cdots, 6,$$

such that

$$\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, \cdots, 6$.

2D quadratic finite element: reference basis functions

- For $\hat{\psi}_1$, it's easy to obtain

$$\begin{aligned}
\hat{\psi}_1(\hat{A}_1) = 1 &\quad\Rightarrow\quad f_1 = 1,\\
\hat{\psi}_1(\hat{A}_2) = 0 &\quad\Rightarrow\quad a_1 + e_1 + f_1 = 0,\\
\hat{\psi}_1(\hat{A}_3) = 0 &\quad\Rightarrow\quad b_1 + d_1 + f_1 = 0,\\
\hat{\psi}_1(\hat{A}_4) = 0 &\quad\Rightarrow\quad 0.25a_1 + 0.5e_1 + f_1 = 0,\\
\hat{\psi}_1(\hat{A}_5) = 0 &\quad\Rightarrow\quad 0.25a_1 + 0.25b_1 + 0.25c_1 + 0.5d_1 + 0.5e_1 + f_1 = 0,\\
\hat{\psi}_1(\hat{A}_6) = 0 &\quad\Rightarrow\quad 0.25b_1 + 0.5d_1 + f_1 = 0.
\end{aligned}$$

- Hence

$$a_1 = 2, b_1 = 2, c_1 = 4, d_1 = -3, e_1 = -3, f_1 = 1.$$

- Then

$$\hat{\psi}_1(\hat{x}, \hat{y}) = 2\hat{x}^2 + 2\hat{y}^2 + 4\hat{x}\hat{y} - 3\hat{y} - 3\hat{x} + 1.$$

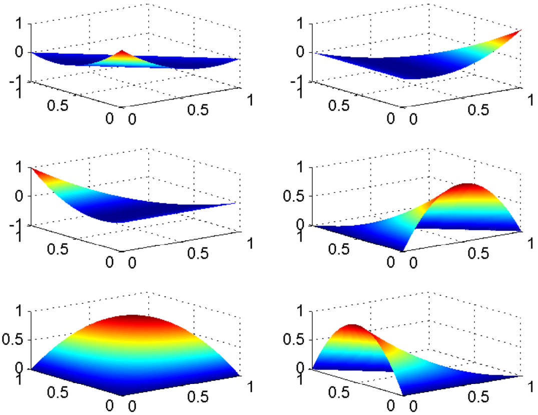
2D quadratic finite element: reference basis functions

- Similarly, we can obtain all the six reference 2D quadratic basis functions

$$
\begin{aligned}
\hat{\psi}_1(\hat{x},\hat{y}) &= 2\hat{x}^2+2\hat{y}^2+4\hat{x}\hat{y}-3\hat{y}-3\hat{x}+1,\\
\hat{\psi}_2(\hat{x},\hat{y}) &= 2\hat{x}^2-\hat{x},\\
\hat{\psi}_3(\hat{x},\hat{y}) &= 2\hat{y}^2-\hat{y},\\
\hat{\psi}_4(\hat{x},\hat{y}) &= -4\hat{x}^2-4\hat{x}\hat{y}+4\hat{x},\\
\hat{\psi}_5(\hat{x},\hat{y}) &= 4\hat{x}\hat{y},\\
\hat{\psi}_6(\hat{x},\hat{y}) &= -4\hat{y}^2-4\hat{x}\hat{y}+4\hat{y}.
\end{aligned}
$$

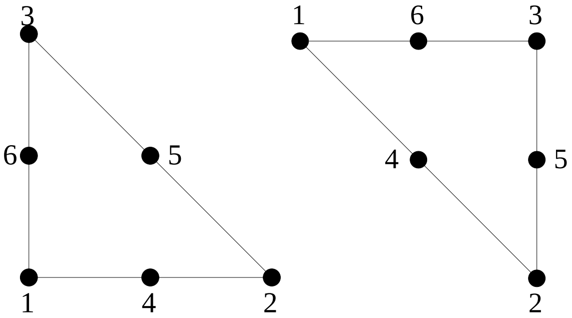
2D quadratic finite element: reference basis functions

- Plots of the six quadratic basis functions on the reference triangle:



2D quadratic finite element: local node index

- Define your index for the local finite element nodes in a mesh element with $N_{lb} = 6$.

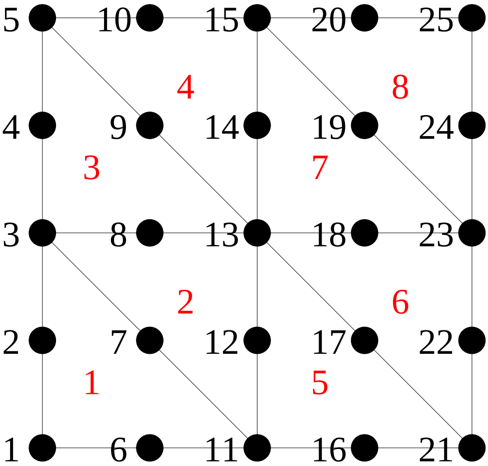


2D quadratic finite element: information matrices

- Define your global indices for all the mesh elements E_n $(n = 1, \cdots, N)$ and finite element nodes X_j $(j = 1, \cdots, N_b)$ (or the finite element basis functions) with $N_b = (2N_1 + 1)(2N_2 + 1) \neq N_m$.

2D quadratic finite element: information matrices

- For example, when $N_1 = N_2 = 2$, we have



2D quadratic finite element: information matrices

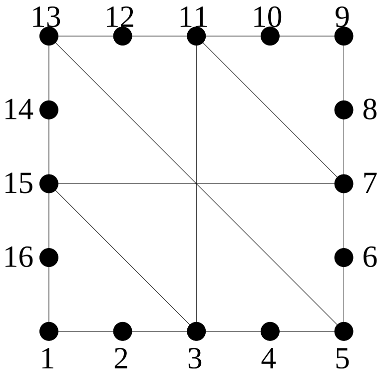
- The P_b and T_b for 2D quadratic finite element are different from the P and T for the triangular mesh. For the above example we have

$$P_b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & \cdots & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{pmatrix},$$

$$T_b = \begin{pmatrix} 1 & 3 & 3 & 5 & 11 & 13 & 13 & 15 \\ 11 & 11 & 13 & 13 & 21 & 21 & 23 & 23 \\ 3 & 13 & 5 & 15 & 13 & 23 & 15 & 25 \\ 6 & 7 & 8 & 9 & 16 & 17 & 18 & 19 \\ 7 & 12 & 9 & 14 & 17 & 22 & 19 & 24 \\ 2 & 8 & 4 & 10 & 12 & 18 & 14 & 20 \end{pmatrix}.$$

2D quadratic finite element: boundary node index

- Define your index for the boundary finite element nodes.
- For example, when $N_1 = N_2 = 2$, we have,



2D quadratic finite element: boundary node information matrix

- Matrix *boundarynodes*:
- For example, when $N_1 = N_2 = 2$ and all the boundary is Dirichlet type, we have:

$$boundarynodes = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & \cdots & -1 & \cdots & -1 & \cdots & -1 \\ 1 & 6 & 11 & 16 & 21 & \cdots & 25 & \cdots & 5 & \cdots & 2 \end{pmatrix}.$$

2D quadratic finite element: affine mapping

- The affine mapping we use here is exactly the same as the previous one!
- Recall: for a given function $\hat{\psi}(\hat{x}, \hat{y})$ where $(\hat{x}, \hat{y}) \in \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3$, we can define the corresponding function for $(x, y) \in \triangle A_1 A_2 A_3$ as follows:

$$\psi(x, y) = \hat{\psi}(\hat{x}, \hat{y}),$$

where

$$\begin{aligned} \hat{x} &= \frac{(y_3 - y_1)(x - x_1) - (x_3 - x_1)(y - y_1)}{|J|}, \\ \hat{y} &= \frac{-(y_2 - y_1)(x - x_1) + (x_2 - x_1)(y - y_1)}{|J|}. \end{aligned}$$

2D quadratic finite element: affine mapping

- Recall: by chain rule, we get

$$
\begin{aligned}
\frac{\partial \psi}{\partial x} &= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial x} \\
&= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{y_3 - y_1}{|J|} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{y_1 - y_2}{|J|}, \\
\frac{\partial \psi}{\partial y} &= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial y} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial y} \\
&= \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{x_1 - x_3}{|J|} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{x_2 - x_1}{|J|}.
\end{aligned}
$$

2D quadratic finite element: affine mapping

- By chain rule again, we get

$$\begin{aligned}
\frac{\partial^2 \psi}{\partial x^2} &= \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2}\frac{\partial \hat{x}}{\partial x}\frac{y_3 - y_1}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{x}\partial \hat{y}}\frac{\partial \hat{x}}{\partial x}\frac{y_1 - y_2}{|J|} \\
&\quad + \frac{\partial^2 \hat{\psi}}{\partial \hat{x}\partial \hat{y}}\frac{\partial \hat{y}}{\partial x}\frac{y_3 - y_1}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2}\frac{\partial \hat{y}}{\partial x}\frac{y_1 - y_2}{|J|} \\
&= \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2}\frac{(y_3 - y_1)^2}{|J|^2} + 2\frac{\partial^2 \hat{\psi}}{\partial \hat{x}\partial \hat{y}}\frac{(y_3 - y_1)(y_1 - y_2)}{|J|^2} \\
&\quad + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2}\frac{(y_1 - y_2)^2}{|J|^2}.
\end{aligned}$$

2D quadratic finite element: affine mapping

- And

$$
\begin{aligned}
\frac{\partial^2 \psi}{\partial y^2} &= \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2}\frac{\partial \hat{x}}{\partial y}\frac{x_1 - x_3}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{x}\partial \hat{y}}\frac{\partial \hat{x}}{\partial y}\frac{x_2 - x_1}{|J|} \\
&+ \frac{\partial^2 \hat{\psi}}{\partial \hat{x}\partial \hat{y}}\frac{\partial \hat{y}}{\partial y}\frac{x_1 - x_3}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2}\frac{\partial \hat{y}}{\partial y}\frac{x_2 - x_1}{|J|} \\
&= \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2}\frac{(x_1 - x_3)^2}{|J|^2} + 2\frac{\partial^2 \hat{\psi}}{\partial \hat{x}\partial \hat{y}}\frac{(x_1 - x_3)(x_2 - x_1)}{|J|^2} \\
&+ \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2}\frac{(x_2 - x_1)^2}{|J|^2}.
\end{aligned}
$$

2D quadratic finite element: affine mapping

- And

$$
\begin{aligned}
\frac{\partial^2 \psi}{\partial x \partial y} &= \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{\partial \hat{x}}{\partial y} \frac{y_3 - y_1}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{x}}{\partial y} \frac{y_1 - y_2}{|J|} \\
&+ \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{y}}{\partial y} \frac{y_3 - y_1}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} \frac{\partial \hat{y}}{\partial y} \frac{y_1 - y_2}{|J|} \\
&= \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{(x_1 - x_3)(y_3 - y_1)}{|J|^2} + \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{(x_1 - x_3)(y_1 - y_2)}{|J|^2} \\
&+ \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{(x_2 - x_1)(y_3 - y_1)}{|J|^2} + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} \frac{(x_2 - x_1)(y_1 - y_2)}{|J|^2}.
\end{aligned}
$$

2D quadratic finite element: local basis functions

- Consider the n^{th} element $E_n = \triangle A_{n1} A_{n2} A_{n3}$ where

$$A_{ni} = \begin{pmatrix} x_{ni} \\ y_{ni} \end{pmatrix}, \quad i = 1, 2, 3.$$

- Define

$$A_{n4} = \frac{A_{n1} + A_{n2}}{2}, \quad A_{n5} = \frac{A_{n2} + A_{n3}}{2}, \quad A_{n6} = \frac{A_{n3} + A_{n1}}{2}.$$

2D quadratic finite element: local basis functions

- The six local 2D linear basis functions are

$$\psi_{ni}(x,y) \quad = \quad \hat{\psi}_i(\hat{x},\hat{y}), \ i = 1, \cdots, 6,$$

where

$$\begin{aligned}
\hat{x} &= \frac{(y_{n3} - y_{n1})(x - x_{n1}) - (x_{n3} - x_{n1})(y - y_{n1})}{|J_n|}, \\
\hat{y} &= \frac{-(y_{n2} - y_{n1})(x - x_{n1}) + (x_{n2} - x_{n1})(y - y_{n1})}{|J_n|}, \\
|J_n| &= (x_{n2} - x_{n1})(y_{n3} - y_{n1}) - (x_{n3} - x_{n1})(y_{n2} - y_{n1}).
\end{aligned}$$

2D quadratic finite element: local basis functions

- And for $i = 1, \cdots, 6$,

$$
\begin{aligned}
\frac{\partial \psi_{ni}}{\partial x} &= \frac{\partial \hat{\psi}_i}{\partial \hat{x}} \frac{y_{n3} - y_{n1}}{|J_n|} + \frac{\partial \hat{\psi}_i}{\partial \hat{y}} \frac{y_{n1} - y_{n2}}{|J_n|}, \\
\frac{\partial \psi_{ni}}{\partial y} &= \frac{\partial \hat{\psi}_i}{\partial \hat{x}} \frac{x_{n1} - x_{n3}}{|J_n|} + \frac{\partial \hat{\psi}_i}{\partial \hat{y}} \frac{x_{n2} - x_{n1}}{|J_n|}, \\
\frac{\partial^2 \psi_{ni}}{\partial x^2} &= \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x}^2} \frac{(y_3 - y_1)^2}{|J|^2} + 2\frac{\partial^2 \hat{\psi}_i}{\partial \hat{x} \partial \hat{y}} \frac{(y_3 - y_1)(y_1 - y_2)}{|J|^2} + \frac{\partial^2 \hat{\psi}_i}{\partial \hat{y}^2} \frac{(y_1 - y_2)^2}{|J|^2}, \\
\frac{\partial^2 \psi_{ni}}{\partial y^2} &= \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x}^2} \frac{(x_1 - x_3)^2}{|J|^2} + 2\frac{\partial^2 \hat{\psi}_i}{\partial \hat{x} \partial \hat{y}} \frac{(x_1 - x_3)(x_2 - x_1)}{|J|^2} + \frac{\partial^2 \hat{\psi}_i}{\partial \hat{y}^2} \frac{(x_2 - x_1)^2}{|J|^2}, \\
\frac{\partial^2 \psi_{ni}}{\partial x \partial y} &= \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x}^2} \frac{(x_1 - x_3)(y_3 - y_1)}{|J|^2} + \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x} \partial \hat{y}} \frac{(x_1 - x_3)(y_1 - y_2)}{|J|^2} \\
&\quad + \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x} \partial \hat{y}} \frac{(x_2 - x_1)(y_3 - y_1)}{|J|^2} + \frac{\partial^2 \hat{\psi}_i}{\partial \hat{y}^2} \frac{(x_2 - x_1)(y_1 - y_2)}{|J|^2}.
\end{aligned}
$$

2D quadratic finite element: local basis functions

- In another way, the local basis functions can be also directly formed on the n^{th} element $E_n = \triangle A_{n1}A_{n2}A_{n3}$ with edge middle points A_{n4}, A_{n5}, and A_{n6}: Define

$$\begin{aligned}\psi_{nj}(x,y) &= a_{nj}x^2 + b_{nj}y^2 + c_{nj}xy + d_{nj}y + e_{nj}x + f_{nj},\\ & j = 1, \cdots, 6,\end{aligned}$$

such that

$$\psi_{nj}(A_{ni}) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, \cdots, 6$.

2D quadratic finite element: global basis functions

"local $\rightarrow$ global" framework:

- Define the local finite element space

$$S_h(E_n) = span\{\psi_{n1}, \cdots, \psi_{n6}\}.$$

- At each finite element node X_j ($j = 1, \cdots, N_b$), define the corresponding global linear basis function ϕ_j such that $\phi_j|_{E_n} \in S_h(E_n)$ and

$$\phi_j(X_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

  for $i, j = 1, \cdots, N_b$.

- Then define the global finite element space to be

$$U_h = span\{\phi_j\}_{j=1}^{N_b}.$$

2D quadratic finite element: global basis functions

- Hence

$$
\phi_j|_{E_n} = \begin{cases} \psi_{n1}, & \text{if } j = T_b(1,n), \\ \psi_{n2}, & \text{if } j = T_b(2,n), \\ \psi_{n3}, & \text{if } j = T_b(3,n), \\ \psi_{n4}, & \text{if } j = T_b(4,n), \\ \psi_{n5}, & \text{if } j = T_b(5,n), \\ \psi_{n6}, & \text{if } j = T_b(6,n), \\ 0, & \text{otherwise.} \end{cases}
$$

for $j = 1, \cdots, N_b$ and $n = 1, \cdots, N$.

Outline

Bilinear finite element: reference basis functions

- If we consider the reference bilinear basis functions on the reference rectangular element $\hat{E} = \square \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4$ where $\hat{A}_1 = (0,0)$, $\hat{A}_2 = (1,0)$, , $\hat{A}_3 = (1,1)$, and $\hat{A}_4 = (0,1)$, then the formation of these basis functions is very similar that of the reference 2D linear basis functions.
- Also, the affine mapping between $\hat{E} = \square \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4$ and $e = \square A_1 A_2 A_3 A_4$ is very similar to the one we use for the triangular mesh. The only change is to use $\hat{A}_4$ and A_4 to replace $\hat{A}_3$ and A_3 respectively. Think about why!
- Hence the formation of the local and global bilinear basis functions is also very similar to that of the local and global 2D linear basis functions.
- Derive the reference, local and global bilinear basis functions in the above way by yourself.

Bilinear finite element: reference basis functions

- In this section, we consider the reference bilinear basis functions on another reference rectangular element $\hat{E} = \square \hat{A}_1\hat{A}_2\hat{A}_3\hat{A}_4$ where $\hat{A}_1 = (-1,-1)$, $\hat{A}_2 = (1,-1)$, , $\hat{A}_3 = (1,1)$, and $\hat{A}_4 = (-1,1)$. We will also take a look at a different affine mapping.
- Define four reference bilinear basis functions

$$\hat{\psi}_j(\hat{x},\hat{y}) \quad = \quad a_j + b_j\hat{x} + c_j\hat{y} + d_j\hat{x}\hat{y}, \; j = 1,2,3,4$$

such that

$$\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i,j = 1,2,3,4$.

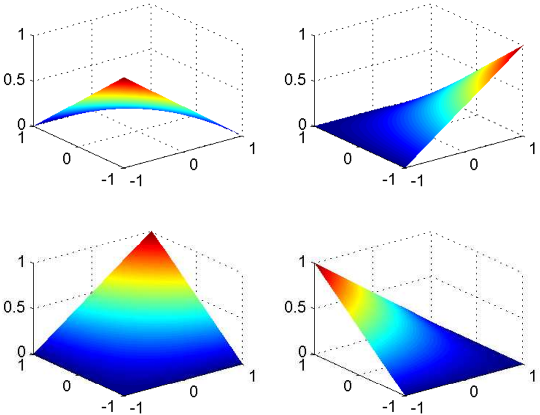
Bilinear finite element: reference basis functions

- Then the four reference bilinear basis functions are

$$
\begin{aligned}
\hat{\psi}_1(\hat{x}, \hat{y}) &= \frac{1 - \hat{x} - \hat{y} + \hat{x}\hat{y}}{4}, \\
\hat{\psi}_2(\hat{x}, \hat{y}) &= \frac{1 + \hat{x} - \hat{y} - \hat{x}\hat{y}}{4}, \\
\hat{\psi}_3(\hat{x}, \hat{y}) &= \frac{1 + \hat{x} + \hat{y} + \hat{x}\hat{y}}{4}, \\
\hat{\psi}_4(\hat{x}, \hat{y}) &= \frac{1 - \hat{x} + \hat{y} - \hat{x}\hat{y}}{4}.
\end{aligned}
$$

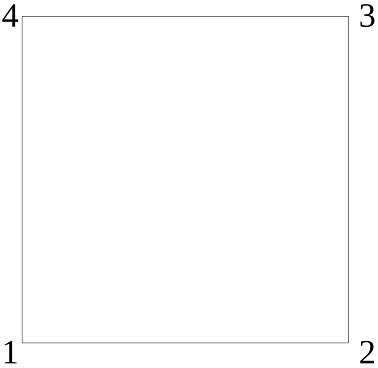
Bilinear finite element: reference basis functions

- Plots of the four bilinear basis functions on the reference triangle:



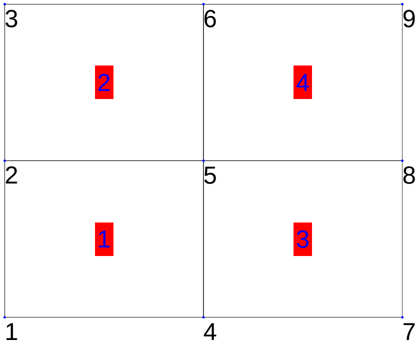
Bilinear finite element: local node index

- Define your index for the local finite element nodes in a mesh element with $N_{lb} = 4$.



Bilinear finite element: information matrices

- Define your global indices for all the mesh elements E_n $(n = 1, \cdots, N)$ and finite element nodes X_j $(j = 1, \cdots, N_b)$ (or the finite element basis functions) with $N_b = N_m = (N_1 + 1)(N_2 + 1)$.
- For example, when $N_1 = N_2 = 2$, we have



Bilinear finite element: information matrices

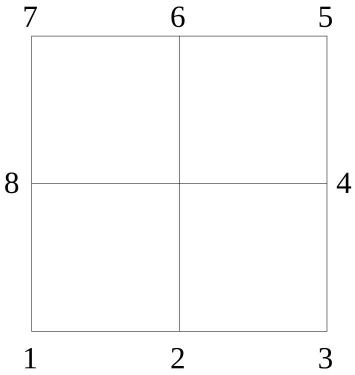
- For the bilinear finite elements, P_b and T_b are the same as the P and T of the rectangular mesh since the nodes of the bilinear finite element basis functions are the same as those of the mesh. For example, when $N_1 = N_2 = 2$, we have

$$P_b = P = \begin{pmatrix} 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0.5 & 1 \end{pmatrix},$$

$$T_b = T = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 4 & 5 & 7 & 8 \\ 5 & 6 & 8 & 9 \\ 2 & 3 & 5 & 6 \end{pmatrix}.$$

Bilinear finite element: boundary node index

- Define your index for the boundary finite element nodes.
- For example, when $N_1 = N_2 = 2$, we have



Bilinear finite element: boundary node information matrix

- Matrix *boundarynodes*:
- For example, when $N_1 = N_2 = 2$ and all the boundary is Dirichlet type, we have:

$$boundarynodes = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 4 & 7 & 8 & 9 & 6 & 3 & 2 \end{pmatrix}.$$

Bilinear finite element: affine mapping

- Now we can use the affine mapping between an arbitrary rectangle $E = \square A_1A_2A_3A_4$ and the reference triangle $\hat{E} = \square \hat{A}_1\hat{A}_2\hat{A}_3\hat{A}_4$ to construct the local basis functions from the reference ones.
- Assume A_1, A_2, A_3, and A_4 are the left-lower, right-lower, right-upper, and left-upper vertices respectively.
- Assume

$$A_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \ (i = 1, 2, 3, 4), \ h_1 = x_2 - x_1, \ h_2 = y_4 - y_1.$$

- Consider the affine mapping

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}h_1 & 0 \\ 0 & \frac{1}{2}h_2 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} x_1 + \frac{1}{2}h_1 \\ y_1 + \frac{1}{2}h_2 \end{pmatrix}.$$

Bilinear finite element: affine mapping

- The affine mapping actually maps

$$\hat{A}_i \quad \rightarrow \quad A_i, \;\; i = 1, 2, 3, 4.$$

- Hence the affine mapping maps $\square \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4$ to $\square A_1 A_2 A_3 A_4$.
- Also,

$$\begin{aligned} \hat{x} &= \frac{2x - 2x_1 - h_1}{h_1}, \\ \hat{y} &= \frac{2y - 2y_1 - h_2}{h_2}. \end{aligned}$$

Bilinear finite element: affine mapping

- For a given function $\hat{\psi}(\hat{x}, \hat{y})$ where $(\hat{x}, \hat{y}) \in \square\hat{A}_1\hat{A}_2\hat{A}_3\hat{A}_4$, we can define the corresponding function for $(x, y) \in \square A_1A_2A_3A_4$ as follows:

$$\psi(x, y) = \hat{\psi}(\hat{x}, \hat{y}),$$

where

$$\hat{x} = \frac{2x - 2x_1 - h_1}{h_1},$$
$$\hat{y} = \frac{2y - 2y_1 - h_2}{h_2}.$$

Bilinear finite element: affine mapping

- Then by chain rule, we get

$$
\begin{aligned}
\frac{\partial \psi}{\partial x} &= \frac{\partial \hat{\psi}}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{\psi}}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial x} \\
&= \frac{\partial \hat{\psi}}{\partial \hat{x}}\frac{2}{h_1}, \\
\frac{\partial \psi}{\partial y} &= \frac{\partial \hat{\psi}}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial y} + \frac{\partial \hat{\psi}}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial y} \\
&= \frac{\partial \hat{\psi}}{\partial \hat{y}}\frac{2}{h_2}, \\
\frac{\partial^2 \psi}{\partial x \partial y} &= \frac{2}{h_1}\frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2}\frac{\partial \hat{x}}{\partial y} + \frac{2}{h_1}\frac{\partial^2 \hat{\psi}}{\partial \hat{x}\partial \hat{y}}\frac{\partial \hat{y}}{\partial y} \\
&= \frac{4}{h_1 h_2}\frac{\partial^2 \hat{\psi}}{\partial \hat{x}\partial \hat{y}}.
\end{aligned}
$$

Bilinear finite element: local basis functions

- Consider the n^{th} element $E_n = \triangle A_{n1}A_{n2}A_{n3}A_{n4}$ where

$$A_{ni} = \begin{pmatrix} x_{ni} \\ y_{ni} \end{pmatrix}.$$

Recall that the mesh size $h = (h_1, h_2)$.

- The four local bilinear basis functions are

$$\psi_{ni}(x, y) = \hat{\psi}_i(\hat{x}, \hat{y}), \ i = 1, 2, 3, 4$$

where

$$\hat{x} = \frac{2x - 2x_{n1} - h_1}{h_1},$$
$$\hat{y} = \frac{2y - 2y_{n1} - h_2}{h_2}.$$

Bilinear finite element: local basis functions

- And for $i = 1, 2, 3, 4$,

$$\begin{aligned}
\frac{\partial \psi_{ni}}{\partial x} &= \frac{\partial \hat{\psi}_i}{\partial \hat{x}} + \frac{\partial \hat{\psi}_i}{\partial \hat{y}}, \\
\frac{\partial \psi_{ni}}{\partial y} &= \frac{\partial \hat{\psi}_i}{\partial \hat{x}} + \frac{\partial \hat{\psi}_i}{\partial \hat{y}}, \\
\frac{\partial^2 \psi_{ni}}{\partial x \partial y} &= \frac{4}{h_1 h_2} \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x} \partial \hat{y}}.
\end{aligned}$$

- The reference and local functions defined in this section are what you will need to input into the code!

Bilinear finite element: local basis functions

- In another way, the local basis functions can be also directly formed on the n^{th} element $E_n = \triangle A_{n1}A_{n2}A_{n3}A_{n4}$ as follows:

$$\psi_{nj}(x,y) = a_{nj} + b_{nj}x + c_{nj}y + d_{nj}, \ j = 1,2,3,4,$$

such that

$$\psi_{nj}(A_{ni}) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i,j = 1,2,3,4$.

Bilinear finite element: global basis functions

"local $\rightarrow$ global" framework:

- Define the local finite element space

$$S_h(E_n) = span\{\psi_{n1}, \psi_{n2}, \psi_{n3}, \psi_{n4}\}.$$

- At each finite element node X_j ($j = 1, \cdots, N_b$), define the corresponding global linear basis function ϕ_j such that $\phi_j|_{E_n} \in S_h(E_n)$ and

$$\phi_j(X_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

  for $i, j = 1, \cdots, N_b$.

- Then define the global finite element space to be

$$U_h = span\{\phi_j\}_{j=1}^{N_b}.$$

Bilinear finite element: global basis functions

- Hence

$$\phi_j|_{E_n} = \begin{cases} \psi_{n1}, & \text{if } j = T_b(1,n), \\ \psi_{n2}, & \text{if } j = T_b(2,n), \\ \psi_{n3}, & \text{if } j = T_b(3,n), \\ \psi_{n4}, & \text{if } j = T_b(4,n), \\ 0, & \text{otherwise.} \end{cases}$$

for $j = 1, \cdots, N_b$ and $n = 1, \cdots, N$.

Biquadratic finite element: reference basis functions

- We consider the reference biquadratic basis functions on the reference rectangular element $\hat{E} = \square\hat{A}_1\hat{A}_2\hat{A}_3\hat{A}_4$ where $\hat{A}_1 = (-1,-1)$, $\hat{A}_2 = (1,-1)$, , $\hat{A}_3 = (1,1)$, and $\hat{A}_4 = (-1,1)$. Define $\hat{A}_5 = (0,-1)$, $\hat{A}_6 = (1,0)$, , $\hat{A}_7 = (0,1)$, $\hat{A}_8 = (-1,0)$, and $\hat{A}_9 = (0,0)$.
- Define nine reference biquadratic basis functions

$$\begin{aligned}\hat{\psi}_j(\hat{x},\hat{y}) \quad = \quad & a_j + b_j\hat{x} + c_j\hat{y} + d_j\hat{x}\hat{y} + e_j\hat{x}^2 + f_j\hat{y}^2 \\ & + g_j\hat{x}^2\hat{y} + h_j\hat{x}\hat{y}^2 + k_j\hat{x}^2\hat{y}^2, \ j = 1,\cdots,9\end{aligned}$$

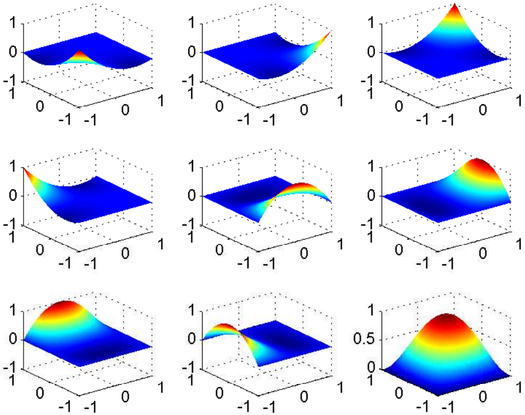
such that

$$\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i,j = 1,\cdots,9$.

Biquadratic finite element: reference basis functions

- Plots of the nine biquadratic basis functions on the reference triangle:



Outline

3D linear finite element: reference basis functions

- We consider the reference 3D linear basis functions on the reference tetrahedron element $E = \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4$ where $\hat{A}_1 = (0, 0, 0)$, $\hat{A}_2 = (1, 0, 0)$, , $\hat{A}_3 = (0, 1, 0)$, and $\hat{A}_4 = (0, 0, 1)$.
- Define four reference 3D linear basis functions

$$\hat{\psi}_j(\hat{x}, \hat{y}, \hat{z}) = a_j\hat{x} + b_j\hat{y} + c_j\hat{z} + d_j, \ j = 1, 2, 3, 4$$

such that

$$\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, 2, 3, 4$.

3D linear finite element: reference basis functions

- Then it's easy to obtain

$$
\begin{aligned}
\hat{\psi}_1(\hat{A}_1) = 1 \quad &\Rightarrow \quad d_1 = 1,\\
\hat{\psi}_1(\hat{A}_2) = 0 \quad &\Rightarrow \quad a_1 + d_1 = 0,\\
\hat{\psi}_1(\hat{A}_3) = 0 \quad &\Rightarrow \quad b_1 + d_1 = 0,\\
\hat{\psi}_1(\hat{A}_4) = 0 \quad &\Rightarrow \quad c_1 + d_1 = 0,\\
\hat{\psi}_2(\hat{A}_1) = 0 \quad &\Rightarrow \quad d_2 = 0,\\
\hat{\psi}_2(\hat{A}_2) = 1 \quad &\Rightarrow \quad a_2 + d_2 = 1,\\
\hat{\psi}_2(\hat{A}_3) = 0 \quad &\Rightarrow \quad b_2 + d_2 = 0,\\
\hat{\psi}_2(\hat{A}_4) = 0 \quad &\Rightarrow \quad c_2 + d_2 = 0,
\end{aligned}
$$

3D linear finite element: reference basis functions

- and

$$
\begin{aligned}
\hat{\psi}_3(\hat{A}_1) = 0 \quad &\Rightarrow \quad d_3 = 0,\\
\hat{\psi}_3(\hat{A}_2) = 0 \quad &\Rightarrow \quad a_3 + d_3 = 0,\\
\hat{\psi}_3(\hat{A}_3) = 0 \quad &\Rightarrow \quad b_3 + d_3 = 1,\\
\hat{\psi}_3(\hat{A}_4) = 1 \quad &\Rightarrow \quad c_3 + d_3 = 0,\\
\hat{\psi}_4(\hat{A}_1) = 0 \quad &\Rightarrow \quad d_4 = 0,\\
\hat{\psi}_4(\hat{A}_2) = 0 \quad &\Rightarrow \quad a_4 + d_4 = 0,\\
\hat{\psi}_4(\hat{A}_3) = 0 \quad &\Rightarrow \quad b_4 + d_4 = 0,\\
\hat{\psi}_4(\hat{A}_4) = 1 \quad &\Rightarrow \quad c_4 + d_4 = 1.
\end{aligned}
$$

3D linear finite element: reference basis functions

- Hence

$$\begin{aligned}
&a_1 = -1, b_1 = -1, c_1 = -1, d_1 = 1,\\
&a_2 = 1, b_2 = 0, c_2 = 0, d_2 = 0,\\
&a_3 = 0, b_3 = 1, c_3 = 0, d_3 = 0,\\
&a_4 = 0, b_4 = 0, c_4 = 1, d_4 = 0.
\end{aligned}$$

- Then the four reference 3D linear basis functions are

$$\begin{aligned}
\hat{\psi}_1(\hat{x}, \hat{y}, \hat{z}) &= -\hat{x} - \hat{y} - \hat{z} + 1,\\
\hat{\psi}_2(\hat{x}, \hat{y}, \hat{z}) &= \hat{x},\\
\hat{\psi}_3(\hat{x}, \hat{y}, \hat{z}) &= \hat{y},\\
\hat{\psi}_4(\hat{x}, \hat{y}, \hat{z}) &= \hat{z}.
\end{aligned}$$

Trilinear finite element: reference basis functions

- We consider the reference trilinear basis functions on the reference cube element $E = \triangle \hat{A}_1\hat{A}_2\hat{A}_3\hat{A}_4\hat{A}_5\hat{A}_6\hat{A}_7\hat{A}_8$ where $\hat{A}_1 = (0,0,0)$, $\hat{A}_2 = (1,0,0)$, , $\hat{A}_3 = (1,1,0)$, $\hat{A}_4 = (0,1,0)$, $\hat{A}_5 = (0,0,1)$, $\hat{A}_6 = (1,0,1)$, $\hat{A}_7 = (1,1,1)$, and $\hat{A}_8 = (0,1,1)$.
- Define eight reference 3D trilinear basis functions

$$\begin{aligned}\hat{\psi}_j(\hat{x},\hat{y},\hat{z}) &= a_j + b_j\hat{x} + c_j\hat{y} + d_j\hat{z} + e_j\hat{x}\hat{y} + f_j\hat{x}\hat{z} \\ &\quad + g_j\hat{y}\hat{z} + h_j\hat{x}\hat{y}\hat{z},\ \ j = 1,\cdots,8\end{aligned}$$

such that

$$\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, \cdots, 8$.

Outline

More topics for finite elements

- Higher degree finite elements......
- Mixed finite elements: Raviart-Thomas elements, Taylor-Hood elements, Mini elements......
- Hermitian types of finite elements
- Nonconforming finite elements
- Another way to construct the basis functions: use the product of 1D basis functions to form the corresponding basis functions on rectangle or cube elements.

Approximation capability of the finite element spaces

- Question: Given a function u and a finite element space $U_h = span\{\phi_j\}_{j=1}^{N_b}$ with finite element nodes X_j $(j = 1, \cdots, N_b)$, how small is $\inf\limits_{w \in U_h} \|u - w\|$?
- Finite element interpolation

$$u_I = \sum_{j=1}^{N_b} u(X_j)\phi_j.$$

- Since $u_I \in U_h$, then

$$\inf_{w \in U_h} \|u - w\| \leq \|u - u_I\|.$$

- The finite element interpolation error $\|u - u_I\|$ is a traditional tool to evaluate the approximation capability of a finite element space. Here the norm $\|\cdot\|$ needs to be chosen properly according to the interpolated basis function u. For example, if $u \in H^1(\Omega)$, then $\|\cdot\|$ can be chosen as the L^2 norm $\|\cdot\|_0$ or H^1 norm $\|\cdot\|_1$.