Due: Tues. 1/15/2018, in class

1. Fill in the steps that were skipped over in class during the derivation of the fine and hyperfine structure of the hydrogen atom within the framework of non-relativistic quantum mechanics. I don't mind if you closely follow the derivations given in Sakurai. You may take the radial integrals stated in class and given in App. B of Sakurai as given. Be sure to clearly define the basis states that you are using and why.

   (a) (7 Pt) Derive the first-order energy shift for hydrogenic states due to the Darwin term
   $$\hat{H}'_D = \frac{\pi \hbar^2 e^2}{2m^2c^2}\,\delta(\mathbf{r})$$

   (b) (8 Pt) Derive the first-order energy shift for hydrogenic states due to the relativistic correction to the kinetic energy
   $$\hat{H}'_{kin} = -\frac{1}{8}\frac{\hat{\mathbf{p}}^4}{m^3c^2} = -\frac{\hat{T}^2}{2mc^2}$$
   where $\hat{T} = \hat{\mathbf{p}}^2/(2m)$ is the kinetic energy operator.

   (c) (6 Pt) Taking the first-order shift due to the spin-orbit interaction as given
   $$\Delta E_{s-o} = \begin{cases} |E_n^{(0)}|\,\frac{\alpha^2}{n}\left(\frac{1}{l+1/2} - \frac{1}{j+1/2}\right), & l \neq 0 \\ 0\,, & l = 0 \end{cases} \tag{1}$$
   obtain the complete expression for the first-order fine-structure correction to the energy levels of hydrogen.

   (d) (18 Pt) As a perturbation on top of the fine structure, derive the first-order energy shift for hydrogenic states due to the hyperfine interaction
   $$\hat{H}'_{HF} = \frac{g_N e^2}{2m_p mc^2}\frac{1}{r^3}\,\mathbf{I}\cdot\mathbf{L} - \frac{8\pi}{3}\boldsymbol{\mu}_\mathrm{e}\cdot\boldsymbol{\mu}_\mathrm{N}\,\delta(\mathbf{r}) \quad (l=0)$$
   $$\hat{H}'_{HF} = \frac{g_N e^2}{2m_p mc^2}\frac{1}{r^3}\,\mathbf{I}\cdot\mathbf{L} + \frac{1}{r^3}\left[\boldsymbol{\mu}_\mathrm{e}\cdot\boldsymbol{\mu}_\mathrm{N} - 3(\boldsymbol{\mu}_\mathrm{e}\cdot\hat{\mathbf{r}})(\boldsymbol{\mu}_\mathrm{N}\cdot\hat{\mathbf{r}})\right] \quad (l\neq 0)$$
   where $\hat{\mathbf{r}} = \mathbf{r}/r$ and the magnetic moments of the proton and electron are $\boldsymbol{\mu}_\mathrm{N} = \frac{g_N\mu_\mathrm{nuc}}{\hbar}\,\mathbf{I}$ and $\boldsymbol{\mu}_\mathrm{e} = (e/mc)\mathbf{S}$.

2. Consider the hydrogen problem in its simplest form by ignoring fine-structure corrections, etc. Suppose we prepare the atom in an initial state given by
   $$\Psi(t=0) = R_{10}(r)\,Y_{3,0}(\theta,\varphi)$$
   where $R_{nl}(r)$ is the hydrogen radial wavefunction, and $Y_{l,m}$ the spherical harmonic function.

(a) (7 Pt) Is this a stationary state?

(b) (7 Pt) Does this state possess definite angular momentum?

(c) (12 Pt) Calculate the expectation value of the energy in this state.

(d) (7 Pt) How (qualitatively) would this state evolve in time?

3. For this problem, consider only the non-relativistic Hamiltonian of the hydrogen atom, i.e., neglect fine structure and any higher order corrections. Assume that the proton is a uniformly charged sphere of radius r_0 which is much less than the Bohr radius. This changes the Coulomb potential felt by the electron in hydrogen atom to (as an exercise, you should try to derive this from Gauss's law)

$$V(r) = \begin{cases} \frac{e^2}{2r_0^3}\left(r^2 - 3r_0^2\right), & r \leq r_0 \\ -\frac{e^2}{r}, & r > r_0 \end{cases}$$

(a) (12 Pt) Use first-order perturbation theory to find the shift of the ground-state (1S state) energy due to this finite-size effect, and calculate the fractional change in the ground-state energy using $r_0 = 0.8\,\text{fm}$. (1 fm $= 10^{-15}$m)

(b) (8 Pt) Do you expect the effect to be larger or smaller for the 2S state? For the 2P state? Why?

4. A deuterium atom is a hydrogen atom with the nucleus consisting of a deuteron (1 proton and 1 neutron, $m_d \approx 2m_p$, total nuclear spin $I = 1$, g-factor $g_d = 0.86$, where the magnetic moment is $\hat{\boldsymbol{\mu}}_d = g_d \mu_{nuc} \hat{\mathbf{I}}/\hbar$). The ground state of the atom is still $1S_{1/2}$. When hyperfine structure is taken into account, this level splits into two energy levels labelled by F where $\hat{\mathbf{F}} = \hat{\mathbf{I}} + \hat{\mathbf{J}}$ is the total angular momentum including nuclear spin.

(a) (8 Pt) Calculate the hyperfine splitting (energy difference between the two levels) in the ground state. You may use the result of problem 1(d) from above without re-derivation, just changing constants as appropriate. (Look carefully at your derivation and the definition of the magnetic moment and g-factor. Make sure you don't change too many constants.) Give the splitting in terms of a frequency. Divide energy by h, not $\hbar$!