MAT1856/APM466 Assignment 1

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February, 2024

Fundamental Questions - 25 points

1.

   (a) Governments issue bonds instead of printing more money to (1) maintain control over inflation, (2) uphold confidence in the currency, and (3) influence interest rates.

   (b) One hypothetical example is the scenario that central bank will take on a series of interest rate cuts in the future to simulate economy growth. Then they may anticipate lower long-term interest rate, which leads to flatten long-term part of a yield curve.

   (c) Quantitative easing is a monetary policy strategy used by a central bank, purchases securities from the open market to reduce interest rates and increase the money supply. At the beginning of the COVID-19, the Fed purchased massive amount of debt securities such as treasury securities and government-guaranteed mortgage-backed securities

2. Selected 10 bonds:"CAN 2.25 3/1/2024", "CAN 1.50 9/1/2024", "CAN 1.25 3/1/2025", "CAN 0.50 9/1/2025", "CAN 0.25 3/1/2026", " CAN 1.00 9/1/2026", "CAN 1.25 3/1/2027", "CAN 2.75 9/1/2027", "CAN 3.50 3/1/2028", "CAN 3.25 9/1/2028".

   Since we need to choose 10 bonds to construct a yield a "0-5 year" yield, then we should select a series of bonds, and the bonds are separated by six months. The most suitable bonds provided in the links are listed above, as they are the most evenly distributed bonds in maturity date, and none of them is out of 5 years range.

3. The eigenvalues and eigenvectors associated with the covariance matrix tell us the characteristics about the yield curve. Eigenvalues explain the amount of variance of the data along each eigenvector direction. Each eigenvector points in a direction that captures the maximum variance in the data. The eigenvectors associated with larger eigenvalues explain the most significant patterns of variation among the processes.

Empirical Questions - 75 points

4.

   Assumption: 1 year = 360 days, 1 month = 30 days, Face Value = 100
   Variables: **C** = coupon payment, which is coupon/2; **PV** = present value; **N** = number of days until maturity date; **n** = number of coupon payments remaining.

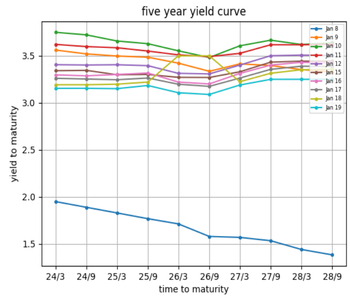
   (a) *S*tep 1: Calculate present value. Present value is the dirty price. We have the equation: Dirty Price = Accrued Interest + Clean Price, where clean prices are the prices we collected as close price.
   For the accrued interest, we have Accrued Interest = $\dfrac{180 - (30 - \text{days}) - 30}{360}$
   where "days" stands for the date we collected close prices for each bond.

Step 2: Calculate YTM: A bond's present value (PV) is calculated by discounting the future cash flows back to the present time. Also, the rates are compounded semi-annually. The formula is:

$$PV = \sum_{i=1}^{n} \frac{C}{(1+\frac{y}{2})^i} + \frac{C+100}{(1+\frac{y}{2})^N}$$

As we know the present value PV, coupon payment C, and the number of periods. As for the methodology, we first set $y = couponerate$, then constantly increase or decrease y until the PV equilibrium reached.

Step 3: Plot Yield Curve. After deriving every yield corresponding to each bond at each date, we have a matrix aobut yield, where row stands for certain date, and column stands for one bond. Then, each row is a yield curve and we plot a 5-year yield curve superimposed on-top of each other.



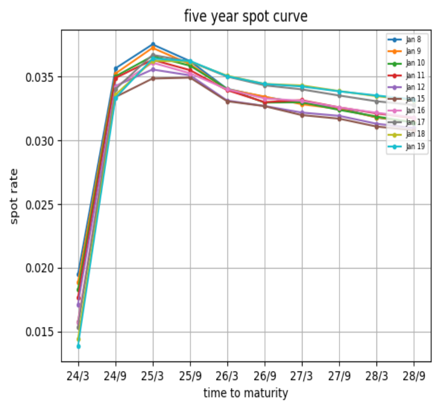
(b) To calculate the spot rate, we need to use the formula:

$$PV = \sum_{i=1}^{n-1} \frac{C}{(1+\frac{S_i}{2})^i} + \frac{C+100}{(1+\frac{S_n}{2})^N}$$

where S_i stands for the spot rate at time i, S_n is the spot rate of the bond at maturity date. We used bootstrapping algorithm to solve S_n, as we know every previous S_i.
Pseudo code and labeled graph are shown below:

Algorithm 1 Bootstrapping Spot Rates

Initialize *spot_rates* list with 10 entries, each initialized to 0
for i **in** $range(10)$ **do**
  if $i = 0$ **then**
    $spot\_rates[0] = (\frac{100+c}{PV})^{(\frac{1}{t})} - 1 \times 2$
  else
    $discounted\ cash\ flow = 0$
    for j **in** $range(1, i+1)$ **do**
      $discounted\ cash\ flow = discounted\ cash\ flow + \frac{c}{(1+\frac{spot\ rates[j-1]}{2})^t}$
      $t = t + 1$
    end for
    $residual = PV - discounted\ cash\ flow$
    $spot\ rate = (\frac{100+c}{residual})^{(\frac{1}{t})} - 1 \times 2$
    $spot\ rates[i] = spot\ rate$
  end if
end for
return *spot rates*



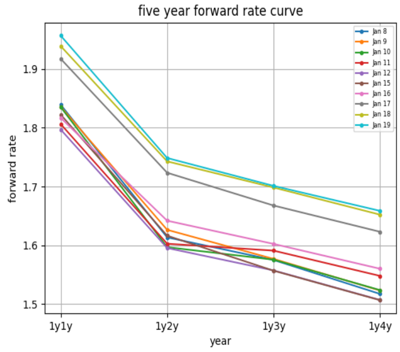
(c) To calculate the forward rate, we need to use the formula (notice that it is semi-annual compounding):

$$F_{t,t+n} = (\frac{(1+S_{t+n})^{2(t+n)}}{((1+S_t)^{2t}})^{\frac{1}{2n}} - 1$$

where S_{t+n} is the spot rate for period ending in $t+n$, S_t is the spot rate for period ending in t, and $F_{t,t+n}$ is the forward rate from time t to $t+n$. Using spot rates calculated by (b), we can then have the forward rates. Pseudo code and labeled graph are shown below:

Algorithm 1 Calculate Forward Rates

```
function FORWARD RATE(dateindex)
    Initialize an empty list forward list
    s = super_spot[dateindex][1]
    t = 1
    for i = 1 to 4 do
        t = t + 2
        sp = super spot[dateindex][t]
        numerator = (1 + sp/2)^(2×(1+i))
        denominator = (1 + s/2)^2
        forward rate = (numerator/denominator)^(1/(2×i)) − 1
        Append forward rate × 100 to forward list
    end for
    return forward list
end function
```



5. Covariance matrix(5×5) for the time series of daily log-returns of yield:

$$\begin{pmatrix} 1.83371221e-04 & 7.11936474e-05 & 8.98720759e-05 & 1.54674736e-04 & 1.80957465e-04 \\ 7.11936474e-05 & 1.33398757e-04 & 1.37369092e-04 & 1.53249087e-04 & 1.26092469e-04 \\ 8.98720759e-05 & 1.37369092e-04 & 1.82361818e-04 & 1.86991956e-04 & 1.65671713e-04 \\ 1.54674736e-04 & 1.53249087e-04 & 1.86991956e-04 & 2.48878913e-04 & 2.36327503e-04 \\ 1.80957465e-04 & 1.26092469e-04 & 1.65671713e-04 & 2.36327503e-04 & 2.54937423e-04 \end{pmatrix}$$

Covariance matrix(4×4) for the time series of forward rates:

$$\begin{pmatrix} 0.00039606 & 0.000288 & 0.00021245 & 0.00018989 \\ 0.000288 & 0.00032707 & 0.00025771 & 0.00025957 \\ 0.00021245 & 0.00025771 & 0.00033208 & 0.00036019 \\ 0.00018989 & 0.00025957 & 0.00036019 & 0.00040076 \end{pmatrix}$$

6. **log-returns of yield:** eigenvalues and their associated eigenvectors are listed as below.
$8.29183743e-04$, $[0.37341037\ 0.68279811\ 0.55591557\ 0.26606085\ \ -0.12050285]^t$
$1.25532115e-04$, $[0.33567483\ \ -0.46071562\ 0.61504445\ \ -0.41290197\ 0.3553812]^t$
$2.57953961e-05$, $[0.41564398\ \ -0.4827102\ \ -0.05908566\ 0.76823531\ \ -0.02354422]^t$
$1.65518373e-05$, $[0.53926118\ \ -0.0933093\ \ -0.21533297\ \ -0.38868817\ \ -0.70925593]^t$
$5.88504130e-06$, $[0.53322094\ 0.28251033\ \ -0.51265855\ \ -0.13213476\ 0.59630945]^t$

forward rates: eigenvalues and their associated eigenvectors are listed as below.
$1.15013885e-03$, $[-0.46495627 - 0.72920047 - 0.492867680.09572765]^t$
$2.54431722e-04$, $[-0.49025075 - 0.249810880.834889290.01444287]^t$
$4.98688842e-05$, $[-0.509699650.34753699 - 0.18206413 - 0.76568725]^t$
$1.52983954e-06$, $[-0.532612560.53392812 - 0.163993790.63588573]^t$

The first (in terms of size) eigenvalue indicates the amount of variance captured by its associated eigenvector. The associated eigenvector represents the direction in which the data varies the most. It indicates the main direction along which the data points are most spread out.

References and GitHub Link to Code

GitHub Link to Code:
`https://github.com/zxuanHuang/mathematical-finance-Yield-Curves.git`