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In Memory of Gu Chaohao

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Contents

Separation of Variables for the Lax Pair of the Bogomolny Equation in 2+1 Dimensional Anti-de Sitter Space-time

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A Lax system separating all the variables is reduced from the Lax pair of the Bogomolny equation for the Yang–Mills–Higgs field in 2+1 dimensional anti-de Sitter space-time. It also contains the Bogomolny equation in 2+1 dimensional Minkowski space-time (Ward equation) under further reduction or limit. The Darboux transformation can be constructed in the usual way to get explicit solutions of the Bogomolny equation.

1. Introduction

The Bogomolny equation for the Yang–Mills–Higgs field in the 2+1 dimensional anti-de Sitter space-time and Minkowski space-time are known to be integrable.[8,15] They have been studied and solved explicitly by several ways and various kinds of solutions have been obtained[5,8–10,14,15,17,18]

On the other hand, a lot of integrable systems in 2+1 dimensions without spectral parameters can be reduced by separating variables in the Lax pair. A typical way is the nonlinear constraint method,[1,2,11] in which a 2+1 dimensional integrable system can be reduced to several 1+1 dimensional systems.[7] Under this kind of constraint, we can obtain explicit solutions like the dromions[19] and the quasi-periodic solutions[1,20] of the 2+1 dimensional integrable system,

In this paper, we will separate the variables for the Lax pair (with a spectral parameter) of the Bogomolny equation in 2+1 dimensional anti-de

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Sitter space-time. The derived system is written in 1+1 dimensional form. The Darboux transformation can be applied to this derived system.

By various limit or reduction process, the results are also valid for the Bogomolny equation in 2+1 dimensional Minkowski space-time[3,5,8] (i.e. Ward equation, or the 2+1 dimensional principal chiral field with Wess–Zumino–Witten term[15]) and the standard 1+1 dimensional principal chiral field.[6,12,13,16]

2. Bogomolny Equation in 2+1 Dimensional Anti-de Sitter Space-time

The 2+1 dimensional anti-de Sitter space-time (AdS) of constant curvature $-1/\rho^2$ $(\rho > 0)$ is the hyperboloid

$$U^2 + V^2 - X^2 - Y^2 = \rho^2 \tag{1}$$

in $\mathbf{R}^{2,2}$ with the metric

$$\mathrm{d}s^2 = -\mathrm{d}U^2 - \mathrm{d}V^2 + \mathrm{d}X^2 + \mathrm{d}Y^2. \tag{2}$$

By defining

$$r = \frac{\rho}{U+X} - \rho + 1, \quad y = \frac{Y}{U+X}, \quad t = -\frac{V}{U+X}, \tag{3}$$

a part of the AdS with $U+X > 0$ is represented by the Poincaré coordinates (r, y, t) with $r > -\rho + 1$ and the metric is

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\rho^2}{(r+\rho-1)^2}(-\mathrm{d}t^2 + \mathrm{d}r^2 + \mathrm{d}y^2) \\ &= \frac{\rho^2}{(x_2+\rho-1)^2}(\mathrm{d}x_2^2 - 4\,\mathrm{d}x_1\,\mathrm{d}x_3) \end{aligned} \tag{4}$$

where $x_1 = (y+t)/2$, $x_2 = r$, $x_3 = (t-y)/2$.

For simplicity, denote

$$\gamma = \frac{x_2 + \rho - 1}{\rho}. \tag{5}$$

Here are some basic notions for the Yang–Mills–Higgs field.

Let G be an $N \times N$ matrix Lie group, $\mathcal{G}$ be its Lie algebra. The Yang–Mills–Higgs field in AdS is given by the Yang–Mills potentials $\{A_\mu\}$ and the Higgs field Φ with $A_\mu \in \mathcal{G}$, $\Phi \in \mathcal{G}$. Denote $\partial_\mu = \partial/\partial x_\mu$ where $\mu = 1, 2, 3$. The covariant derivative of an N-dimensional vector ψ on which $\mathcal{G}$ acts is $D_\mu\psi = \partial_\mu\psi + A_\mu\psi$. The strength of the Yang–Mills field is defined by

$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, and the covariant derivative of the Higgs field Φ is defined by $D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi]$.

For $g(x) \in G$, the transformation

$$A'_\mu = gA_\mu g^{-1} - (\partial_\mu g)g^{-1}, \quad \Phi' = g\Phi g^{-1} \tag{6}$$

is called a gauge transformation. Under the gauge transformation, the strength is changed to

$$F'_{\mu\nu} = gF_{\mu\nu}g^{-1}. \tag{7}$$

The Bogomolny equation with the metric (4) is[18]

$$D_1\Phi = \gamma F_{12}, \quad D_2\Phi = \frac{\gamma}{2}F_{13}, \quad D_3\Phi = \gamma F_{23}. \tag{8}$$

By (6) and (7), a gauge transformation transforms a solution of (8) to another solution of it.

Lemma 1. *For any Yang–Mills–Higgs field (A_μ, Φ) satisfying the Bogomolny equation (8) on AdS, there exists a gauge transformation such that $A'_1 = 0$, $A'_2 = \gamma^{-1}\Phi'$ where γ is defined by (5).*

Proof: After the gauge transformation given by $g \in G$,

$$\begin{aligned} A'_1 &= gA_1g^{-1} - (\partial_1 g)g^{-1}, \\ A'_2 - \gamma^{-1}\Phi' &= g(A_2 - \gamma^{-1}\Phi)g^{-1} - (\partial_2 g)g^{-1}. \end{aligned} \tag{9}$$

If we want $A'_1 = 0$, $A'_2 = \gamma^{-1}\Phi'$, then g should satisfy

$$\partial_1 g = gA_1, \quad \partial_2 g = g(A_2 - \gamma^{-1}\Phi). \tag{10}$$

The integrability condition of (10) is

$$\partial_2 A_1 - \partial_1(A_2 - \gamma^{-1}\Phi) - [A_1, A_2 - \gamma^{-1}\Phi] = 0, \tag{11}$$

which is the first equation of (8). Hence g always exists. Moreover, since $A_1 \in \mathcal{G}$ and $A_2 - \gamma^{-1}\Phi \in \mathcal{G}$, we can want $g \in G$. The lemma is proved.

Thus, we shall always choose the gauge so that $A_1 = 0$, $A_2 = \gamma^{-1}\Phi$ hold identically.

It was known that (8) has a Lax pair[18]

$$\begin{aligned} &((x_2 + \rho - 1)D_2 + \rho\Phi - (\rho\lambda - 2x_1)D_1)\eta = 0, \\ &\left(D_3 - \frac{\rho\lambda - 2x_1}{x_2 + \rho - 1}D_2 + \frac{\rho(\rho\lambda - 2x_1)}{(x_2 + \rho - 1)^2}\Phi\right)\eta = 0 \end{aligned} \tag{12}$$

which was presented by Ref. 15. This means that (8) is the integrability condition of (12).

Considering the gauge fixed in Lemma 1, (12) can be written as

$$\begin{aligned} \partial_2\eta &= \sigma(\lambda)\partial_1\eta - P\eta, \\ \partial_3\eta &= \sigma(\lambda)\partial_2\eta - Q\eta \end{aligned} \tag{13}$$

where

$$\sigma(\lambda) = \frac{\rho\lambda - 2x_1}{x_2 + \rho - 1}, \tag{14}$$

$$P = 2\gamma^{-1}\Phi = 2A_2, \quad Q = A_3. \tag{15}$$

There are two special cases. When $\rho \to +\infty$, the space-time tends to the Minkowski space-time, and $\sigma(\lambda) \to \lambda$. When $\rho = 1$, the space-time becomes the AdS with curvature -1, and $\sigma = \dfrac{\lambda - 2x_1}{x_2}$

3. Separation of Variables

Now we introduce a new Lax system

$$\partial_j\Psi = \sum_{\alpha=1}^{r} \frac{\sigma_\alpha^{j-1}B_\alpha}{\lambda - \lambda_\alpha}\Psi \quad (j = 1, 2, 3). \tag{16}$$

Here $\lambda_1, \cdots, \lambda_r$ are distinct complex numbers, $\sigma_\alpha = \sigma(\lambda_\alpha)$, $B_1, \cdots, B_r$ are $N \times N$ matrix functions to be determined.

The integrability conditions of (16) are

$$\begin{aligned} &\partial_l(\sigma_\alpha^{k-1}B_\alpha) - \partial_k(\sigma_\alpha^{l-1}B_\alpha) \\ &+ \sum_{\beta\neq\alpha} \frac{\sigma_\alpha^{k-1}\sigma_\beta^{l-1} - \sigma_\alpha^{l-1}\sigma_\beta^{k-1}}{\lambda_\alpha - \lambda_\beta}[B_\alpha, B_\beta] = 0 \end{aligned} \tag{17}$$

$(1 \le k < l \le 3)$. In fact, (17) is equivalent to two equations within it with $k = 1, l = 2$ and $k = 2, l = 3$ respectively:

$$\begin{aligned} \partial_2 B_\alpha &= \partial_1(\sigma_\alpha B_\alpha) + \Big[B_\alpha, \sum_{\beta=1}^{r}\gamma^{-1}B_\beta\Big], \\ \partial_3(\sigma_\alpha B_\alpha) &= \partial_2(\sigma_\alpha^2 B_\alpha) + \Big[\sigma_\alpha B_\alpha, \sum_{\beta=1}^{r}\gamma^{-1}\sigma_\beta B_\beta\Big]. \end{aligned} \tag{18}$$

B_α's can be constructed from the solutions of the original Lax pair (13) and its conjugate, as in the nonlinear constraint method for other equations.[2]

Suppose that F_α is an $N \times p$ solution ($1 \le p \le N$) of (13) with $\lambda = \lambda_\alpha$, that is, F_α satisfies

$$\begin{aligned} \partial_2 F_\alpha &= \sigma_\alpha \partial_1 F_\alpha - P F_\alpha, \\ \sigma_\alpha \partial_3 F_\alpha &= \sigma_\alpha^2 \partial_2 F_\alpha - \sigma_\alpha Q F_\alpha. \end{aligned} \tag{19}$$

Suppose also that G_α is an $N \times p$ solution of

$$\begin{aligned} \partial_2 G_\alpha &= \partial_1(\sigma_\alpha G_\alpha) + P^T G_\alpha, \\ \sigma_\alpha \partial_3 G_\alpha &= \partial_2(\sigma_\alpha^2 G_\alpha) + \sigma_\alpha Q^T G_\alpha, \end{aligned} \tag{20}$$

which is conjugate to (19).

Let $B_\alpha = \kappa_\alpha F_\alpha G_\alpha^T$ where κ_α is a complex constant, then B_α satisfies

$$\begin{aligned} \partial_2 B_\alpha &= \partial_1(\sigma_\alpha B_\alpha) + [B_\alpha, P], \\ \partial_3(\sigma_\alpha B_\alpha) &= \partial_2(\sigma_\alpha^2 B_\alpha) + [\sigma_\alpha B_\alpha, Q]. \end{aligned} \tag{21}$$

Comparing (21) with (18), we know that when

$$P = \gamma^{-1} \sum_{\beta=1}^{r} B_\beta, \quad Q = \gamma^{-1} \sum_{\beta=1}^{r} \sigma_\beta B_\beta, \tag{22}$$

(18) is consistent with (21).

Neglecting the special expressions of B_α's, we have the following general theorem.

Theorem 1. *Suppose $\lambda_1, \cdots, \lambda_r$ are distinct complex constants, and $B_1, \cdots, B_r$ are $N \times N$ matrices satisfying the integrability conditions (18) of (16), then*

$$\begin{aligned} A_1 = 0, \quad A_2 &= \frac{1}{2}\gamma^{-1} \sum_{\beta=1}^{r} B_\beta, \\ A_3 = \gamma^{-1} \sum_{\beta=1}^{r} \sigma_\beta B_\beta, \quad \Phi &= \frac{1}{2} \sum_{\beta=1}^{r} B_\beta \end{aligned} \tag{23}$$

give a solution of the Bogomolny equation (8).

Proof: Since the gauge has already been fixed so that $A_1 = 0$, $A_2 = \gamma^{-1}\Phi$, we have

$$D_1 \Phi = \gamma \partial_1 A_2 = \gamma F_{12}, \tag{24}$$

which is the first equation of (8). Now we verify the other two equations of (8). Summing up (18) on α, we have

$$2D_2\Phi = 2\partial_2 \Phi = \sum_{\alpha=1}^{r} \partial_2 B_\alpha = \gamma \partial_1 A_3 = \gamma F_{13}, \tag{25}$$

$$
\begin{aligned}
2D_3\Phi &= 2\partial_3\Phi + 2[A_3,\Phi] = \partial_3\Big(\sum_{\alpha=1}^r B_\alpha\Big) - \Big[\sum_{\alpha=1}^r B_\alpha, \sum_{\beta=1}^r \gamma^{-1}\sigma_\beta B_\beta\Big] \\
&= \partial_2\Big(\sum_{\alpha=1}^r \sigma_\alpha B_\alpha\Big) + \sum_{\alpha=1}^r (\partial_2\sigma_\alpha) B_\alpha = \partial_2(\gamma A_3) - \rho^{-1}A_3 \\
&= \gamma\partial_2 A_3 = \gamma F_{23} + D_3\Phi.
\end{aligned} \tag{26}
$$

So $D_3\Phi = \gamma F_{23}$. The theorem is proved.

The solutions of (18) can be obtained by standard ways in 1+1 dimensions. In the next section, we will use the Darboux transformation method to solve them.

Remark 1. (16) can be generalized to $j = 1, \cdots, n$ with any positive integer n. For $n = 2$, $r = 2$, $\lambda_1 = 1$, $\lambda_2 = -1$, and $\rho \to +\infty$, the Lax system is

$$
\begin{aligned}
\partial_1\Psi &= \Big(\frac{B_1}{\lambda - 1} + \frac{B_2}{\lambda + 1}\Big)\Psi, \\
\partial_2\Psi &= \Big(\frac{B_1}{\lambda - 1} - \frac{B_2}{\lambda + 1}\Big)\Psi.
\end{aligned} \tag{27}
$$

This is the well-known Lax pair of the 1+1 dimensional principal chiral field (harmonic map from $\mathbf{R}^{1,1}$ to the Lie group G). A lot of explicit solutions of it have been known already.[6,12,13,16]

4. Darboux Transformation

We use the general procedure of constructing Darboux transformation for (16).[4]

Let $\mu_1, \cdots, \mu_N$ be non-zero complex numbers such that $\mu_1, \cdots, \mu_N$, $\lambda_1, \cdots, \lambda_r$ are distinct, h_j be a solution of (16) with $\lambda = \mu_j$ $(j = 1, \cdots, N)$. Let

$$
\begin{aligned}
&\Lambda = \operatorname{diag}(\mu_1, \cdots, \mu_N), \quad H = (h_1, \cdots, h_N), \\
&S = H\Lambda H^{-1}.
\end{aligned} \tag{28}
$$

Then, for any solution Ψ of (16), $\widetilde{\Psi} = (\lambda I - S)\Psi$ satisfies

$$
\partial_j\widetilde{\Psi} = \sum_{\alpha=1}^r \frac{\sigma_\alpha^{j-1}\widetilde{B}_\alpha}{\lambda - \lambda_\alpha}\widetilde{\Psi} \tag{29}
$$

where

$$
\widetilde{B}_\alpha = (\lambda_\alpha I - S)B_\alpha(\lambda_\alpha I - S)^{-1} \tag{30}
$$

and S satisfies

$$\partial_j S = \sum_{\alpha=1}^{r} \sigma_\alpha^{j-1}(B_\alpha - \widetilde{B}_\alpha). \tag{31}$$

(31) can also be written as

$$\sum_{\alpha=1}^{r} \sigma_\alpha^{j-1}\widetilde{B}_\alpha = \sum_{\alpha=1}^{r} \sigma_\alpha^{j-1} B_\alpha - \partial_j S. \tag{32}$$

From (23), this gives a Darboux transformation as stated in the following theorem.

Theorem 2. *Suppose B_α's are solutions of (18), S is constructed as above, (A_1, A_2, A_3, Φ) are given by (23), then*

$$\begin{aligned} &\widetilde{A}_1 = 0, \quad \widetilde{A}_2 = A_2 - \frac{\rho}{2(x_2+\rho-1)}\partial_1 S, \\ &\widetilde{A}_3 = A_3 - \frac{\rho}{x_2+\rho-1}\partial_2 S, \quad \widetilde{\Phi} = \Phi - \frac{1}{2}\partial_1 S \end{aligned} \tag{33}$$

also give a solution of the Bogomolny equation (17).

This is the Darboux transformation of degree one. An infinite series of solutions can be obtained by successive actions of Darboux transformations of degree one.

Example. To use the Darboux transformation, we need a seed solution of (18). Suppose the seed solution is $B_\alpha = u_\alpha(x_1, x_2, x_3)C_\alpha$ $(\alpha = 1, \cdots, r)$ where u_α's are functions, C_α's are constant $N \times N$ matrices such that $[C_\alpha, C_\beta] = 0$ for $\alpha, \beta = 1, \cdots, r$. By (18), u_α's satisfy

$$\begin{aligned} &\partial_2 u_\alpha = \partial_1(\sigma_\alpha u_\alpha), \\ &\partial_3(\sigma_\alpha u_\alpha) = \partial_2(\sigma_\alpha^2 u_\alpha). \end{aligned} \tag{34}$$

The general solution is

$$u_\alpha = \Big(\frac{x_2+\rho-1}{\rho\lambda_\alpha - 2x_1}\Big)^2 f_\alpha(\theta_\alpha(x_1, x_2, x_3)) \tag{35}$$

where

$$\theta_\alpha(x_1, x_2, x_3) = \frac{(x_2+\rho-1)^2}{2(\rho\lambda_\alpha - 2x_1)} + x_3 - \frac{\rho-2}{2\lambda_\alpha}, \tag{36}$$

f_α's are arbitrary meromorphic functions.

Remark 2. In (36), the constant $-\dfrac{\rho-2}{2\lambda_\alpha}$ is added so that θ_α converges as $\rho \to +\infty$. Hence, in the Minkowski space-time, $u_a = \lambda_\alpha^{-2} f_\alpha(x_3 + \lambda_\alpha^{-1}x_2 + \lambda_\alpha^{-2}x_1)$.

(35) gives the seed solution of (18) for the Darboux transformation, and the corresponding

$$\begin{aligned} &A_1 = 0, \quad A_2 = \frac{\rho}{2(x_2+\rho-1)} \sum_{\alpha=1}^{r} u_\alpha C_\alpha, \\ &A_3 = \sum_{\alpha=1}^{r} \frac{\rho(\rho\lambda_\alpha - 2x_1)}{(x_2+\rho-1)^2} u_\alpha C_\alpha, \quad \Phi = \frac{1}{2}\sum_{\alpha=1}^{r} u_\alpha C_\alpha. \end{aligned} \tag{37}$$

The fundamental solution of (16) is

$$\Psi = \exp\left(\sum_{\alpha=1}^{r} \frac{F_\alpha(\theta_\alpha) C_\alpha}{\lambda - \lambda_\alpha}\right)\Psi_0 \tag{38}$$

where Ψ_0 is an $N \times N$ constant matrix, $F'_\alpha = f_\alpha$.

With the seed solution (37), we can get new explicit solutions according to Theorem 2.

Especially, take $G = SU(2)$, $C_\alpha = \mathrm{diag}(\mathrm{i}c_\alpha, -\mathrm{i}c_\alpha)$ where c_α's are constant real numbers. Then, as usual construction for $SU(2)$ solutions, we should take $\mu_2 = \bar{\mu}_1$ and $h_2^* h_1 = 0$. Hence

$$H = \begin{pmatrix} \alpha_1 g & -\bar{\alpha}_2 \bar{g}^{-1} \\ \alpha_2 g^{-1} & \bar{\alpha}_1 \bar{g} \end{pmatrix} \tag{39}$$

where

$$g = \exp\left(\mathrm{i}\sum_{\alpha=1}^{r} \frac{F_\alpha(\theta_\alpha) c_\alpha}{\mu_1 - \lambda_\alpha}\right), \tag{40}$$

α_1 and α_2 are complex constants.

$$\begin{aligned} S &= H\Lambda H^{-1} \\ &= \frac{\mu_1 + \bar{\mu}_1}{2} I + \frac{\mu_1 - \bar{\mu}_1}{2} \frac{1}{|\alpha_1|^2|g|^2 + |\alpha_2|^2|g|^{-2}} \cdot \\ &\cdot \begin{pmatrix} |\alpha_1|^2|g|^2 - |\alpha_2|^2|g|^{-2} & 2\alpha_1\bar{\alpha}_2 g\bar{g}^{-1} \\ 2\bar{\alpha}_1\alpha_2 \bar{g} g^{-1} & -|\alpha_1|^2|g|^2 + |\alpha_2|^2|g|^{-2} \end{pmatrix}, \end{aligned} \tag{41}$$

and

$$\begin{aligned} &\widetilde{A}_1 = 0, \\ &\widetilde{A}_2 = \frac{\rho}{2(x_2+\rho-1)} \sum_{\alpha=1}^{r} u_\alpha C_\alpha - \frac{\rho}{2(x_2+\rho-1)} \partial_1 S, \\ &\widetilde{A}_3 = \sum_{\alpha=1}^{r} \frac{\rho(\rho\lambda_\beta - 2x_1)}{(x_2+\rho-1)^2} u_\alpha C_\alpha - \frac{\rho}{x_2+\rho-1} \partial_2 S, \\ &\widetilde{\Phi} = \frac{1}{2}\sum_{\alpha=1}^{r} u_\alpha C_\alpha - \frac{1}{2}\partial_1 S. \end{aligned} \tag{42}$$

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