Nonlinearity 14 (2001) 701–717 www.iop.org/Journals/no PII: S0951-7715(01)18448-3

Finite-dimensional integrable systems associated with the Davey–Stewartson I equation

Zixiang Zhou[1], **Wen-Xiu Ma**[2] **and Ruguang Zhou**[2,3]

[1] Institute of Mathematics, Fudan University, Shanghai 200433, People's Republic of China
[2] Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong, People's Republic of China
[3] Department of Mathematics, Xuzhou Normal University, Xuzhou 221009, People's Republic of China

E-mail: zxzhou@guomai.sh.cn, mawx@cityu.edu.hk and rgzhou@public.xz.js.cn

Received 11 October 2000, in final form 16 March 2001
Recommended by P Deift

Abstract
For the Davey–Stewartson I equation, which is an integrable equation in 1 + 2 dimensions, we have already found its Lax pair in (1 + 1)-dimensional form by nonlinear constraints. This paper deals with the second nonlinearization of this (1 + 1)-dimensional system to obtain three (1 + 0)-dimensional Hamiltonian systems with a constraint of Neumann type. The full set of involutive conserved integrals is obtained and their functional independence is proved. Therefore, the Hamiltonian systems are completely integrable in the Liouville sense. A periodic solution of the Davey–Stewartson I equation is obtained by solving these classical Hamiltonian systems as an example.

PACS numbers: 0545Y, 0230J

1. Introduction

The Davey–Stewartson I (DSI) equation is a famous (1 + 2)-dimensional integrable equation describing the motion of a water wave [6]. It has been discussed using various methods. Soliton solutions were obtained by the inverse scattering method [7, 13], Bäcklund transformation [1], binary Darboux transformation [12], nonlinearization method [17, 18], etc. Almost-periodic solutions were also obtained [11].

In [17, 18], this (1 + 2)-dimensional problem was nonlinearized to an essentially (1 + 1)-dimensional linear system (1) where all the differentials are separated. This system is very useful for obtaining localized soliton solutions using the Darboux transformation in 1 + 1 dimensions.

On the other hand, many (1 + 1)-dimensional integrable systems can be nonlinearized to (1 + 0)-dimensional (or so-called finite-dimensional) integrable systems [2, 9, 15, 16], and the

 © 2001 IOP Publishing Ltd and LMS Publishing Ltd Printed in the UK

idea of nonlinearization was proposed by Cao [2]. Some (1 + 1)-dimensional problems were completely solved and the periodic or quasi-periodic solutions were obtained.

In [3], the KP equation, which is (1 + 2) dimensional, was nonlinearized not only to (1 + 1)-dimensional [4, 8], but also to (1 + 0)-dimensional integrable Hamiltonian systems.

In the present paper, we show that the (1 + 1)-dimensional system obtained by nonlinearizing the Lax pair of the DSI equation can also be nonlinearized to three (1 + 0)-dimensional Hamiltonian systems. We find a full set of involutive conserved integrals and prove their functional independence. Therefore, these systems are completely integrable in the Liouville sense. As an example, when the number of eigenvalues is two, we solve the systems directly to obtain a periodic solution of the DSI equation.

It is well known that the DSI equation has a Lax pair in 1 + 2 dimensions. In [17, 18], a new integrable system was presented, which is essentially (1 + 1) dimensional, since all the differentials are separated. This system can be written explicitly as

$$\Phi_y = V\Phi = \begin{pmatrix} \mathrm{i}\lambda & u & \mathrm{i}f \\ -\bar{u} & -\mathrm{i}\lambda & -\mathrm{i}g \\ \mathrm{i}\bar{f} & -\mathrm{i}\bar{g} & 0 \end{pmatrix}\Phi \qquad \Phi_x = U\Phi = \begin{pmatrix} \mathrm{i}\lambda & 0 & \mathrm{i}f \\ 0 & \mathrm{i}\lambda & \mathrm{i}g \\ \mathrm{i}\bar{f} & \mathrm{i}\bar{g} & 0 \end{pmatrix}\Phi$$
$$\Phi_t = W\Phi = \begin{pmatrix} -2\mathrm{i}\lambda^2 + \mathrm{i}|u|^2 + \mathrm{i}v_1 & -2u\lambda + \mathrm{i}u_y & -2\mathrm{i}f\lambda - 2f_y \\ 2\bar{u}\lambda + \mathrm{i}\bar{u}_y & 2\mathrm{i}\lambda^2 - \mathrm{i}|u|^2 - \mathrm{i}v_2 & 2\mathrm{i}g\lambda - 2g_y \\ -2\mathrm{i}\bar{f}\lambda + 2\bar{f}_y & 2\mathrm{i}\bar{g}\lambda + 2\bar{g}_y & -2\mathrm{i}(|f|^2 - |g|^2) \end{pmatrix}\Phi. \tag{1}$$

Here u, f and g are complex functions, v_1 and v_2 are real functions.

Its integrability conditions $\Phi_{xy} = \Phi_{yx}$, $\Phi_{yt} = \Phi_{ty}$ and $\Phi_{xt} = \Phi_{tx}$ consist of the following three parts.

(1) DSI equation

$$\begin{aligned} -\mathrm{i}u_t &= u_{xx} + u_{yy} + 2|u|^2u + 2(v_1 + v_2)u \\ v_{1,x} - v_{1,y} &= v_{2,x} + v_{2,y} = -(|u|^2)_x. \end{aligned} \tag{2}$$

(2) Standard Lax pair of the DSI equation

$$F_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F_x + \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} F$$
$$F_t = 2\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F_{xx} + 2\mathrm{i}\begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} F_x + \mathrm{i}\begin{pmatrix} |u|^2 + 2v_1 & u_x + u_y \\ -\bar{u}_x + \bar{u}_y & -|u|^2 - 2v_2 \end{pmatrix} F \tag{3}$$

where

$$F = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{4}$$

(3) Nonlinear constraint

$$FF^* = \frac{1}{2}\begin{pmatrix} v_1 & u_x \\ \bar{u}_x & v_2 \end{pmatrix}. \tag{5}$$

Therefore, the nonlinear equations we will consider consist of the DSI equation, its standard Lax pair in 1 + 2 dimensions and the nonlinear constraint. As soon as this problem is solved, we obtain the solution of the DSI equation, although only part of the solutions can be obtained by this method. In [18], localized (M, N) soliton solutions were obtained from the Darboux transformation for (1).

As for many other (1 + 1)-dimensional problems, here we want to find the nonlinear constraint for (1) and perform nonlinearization again to obtain (1+0)-dimensional Hamiltonian systems.

2. Nonlinearization and Hamiltonians

In order to obtain a nonlinear constraint which is compatible with all the x, y and t equations in (1), we must consider the y equation first.

Note that if Φ is a solution of (1) for real λ, then

$$\bar{\Phi}_y = -V^T\bar{\Phi} \qquad \bar{\Phi}_x = -U^T\bar{\Phi} \qquad \bar{\Phi}_t = -W^T\bar{\Phi}. \tag{6}$$

Suppose $\Phi = (\phi_1, \phi_2, \phi_3)^T$, then we may choose $(\mathrm{i}\bar{\phi}_1, \mathrm{i}\bar{\phi}_2, \mathrm{i}\bar{\phi}_3)^T$ to be the corresponding conjugate coordinates.

The first element of the Lenard sequence corresponding to $w = (u, -\bar{u}, \mathrm{i}f, \mathrm{i}\bar{f}, -\mathrm{i}g, -\mathrm{i}\bar{g})$ is [14]

$$G_0 = (0, 0, \mathrm{i}\bar{f}, \mathrm{i}f, \mathrm{i}\bar{g}, \mathrm{i}g). \tag{7}$$

By computing the variation of λ [9], we have

$$\delta\lambda/\delta\omega = (\mathrm{i}\bar{\phi}_1\phi_2, \mathrm{i}\bar{\phi}_2\phi_1, \mathrm{i}\bar{\phi}_1\phi_3, \mathrm{i}\bar{\phi}_3\phi_1, \mathrm{i}\bar{\phi}_2\phi_3, \mathrm{i}\bar{\phi}_3\phi_2). \tag{8}$$

Now take N distinct eigenvalues $\lambda_j \in \mathbb{R}$ $(N \geqslant 2)$. Suppose the corresponding solution of the Lax pair for $\lambda = \lambda_j$ is $(\phi_{1j}, \phi_{2j}, \phi_{3j})^T$. Then define

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \qquad \Phi_j = (\phi_{j1}, \ldots, \phi_{jN})^T \qquad (j = 1, 2, 3). \tag{9}$$

Each Φ_j is a column vector, and Λ is a constant real diagonal matrix.

We choose $(\phi_{1j}, \phi_{2j}, \phi_{3j}, \mathrm{i}\bar{\phi}_{1j}, \mathrm{i}\bar{\phi}_{2j}, \mathrm{i}\bar{\phi}_{3j})$ as the coordinates instead of the real ones $\mathrm{Re}(\phi_{jk})$ and $\mathrm{Im}(\phi_{jk})$ $(1 \leqslant j \leqslant 3, 1 \leqslant k \leqslant N)$. The standard symplectic form of $\mathbb{R}^{6N}$ is given by

$$\omega = 2 \sum_{\substack{1\leqslant j\leqslant N\\ 1\leqslant\alpha\leqslant 3}} \mathrm{d}\,\mathrm{Im}(\phi_{j\alpha}) \wedge \mathrm{d}\,\mathrm{Re}(\phi_{j\alpha}) = \sum_{\substack{1\leqslant j\leqslant N\\ 1\leqslant\alpha\leqslant 3}} \mathrm{i}\,\mathrm{d}\bar{\phi}_{j\alpha} \wedge \mathrm{d}\phi_{j\alpha}. \tag{10}$$

Denote $\langle V_1, V_2\rangle = V_1^* V_2$ for two column vectors V_1 and V_2.

Define the nonlinear constraint

$$G_0 = \sum_j \delta\lambda_j/\delta\omega \tag{11}$$

which is

$$\langle\Phi_1, \Phi_2\rangle = 0 \qquad \langle\Phi_3, \Phi_1\rangle = f \qquad \langle\Phi_3, \Phi_2\rangle = g. \tag{12}$$

By differentiating these constraints and using (1), we have

$$\begin{aligned}
&\langle\Phi_1, \Phi_1\rangle_x = \langle\Phi_2, \Phi_2\rangle_x = \langle\Phi_3, \Phi_3\rangle_x = 0\\
&\langle\Phi_1, \Phi_1\rangle_y = \langle\Phi_2, \Phi_2\rangle_y = \langle\Phi_3, \Phi_3\rangle_y = 0\\
&\langle\Phi_1, \Phi_1\rangle_t = \langle\Phi_2, \Phi_2\rangle_t = \langle\Phi_3, \Phi_3\rangle_t = 0
\end{aligned} \tag{13}$$

$$
\begin{aligned}
f_x &= \mathrm{i}\langle\Phi_3, \Lambda\Phi_1\rangle + \mathrm{i}f(\langle\Phi_3, \Phi_3\rangle - \langle\Phi_1, \Phi_1\rangle) \\
g_x &= \mathrm{i}\langle\Phi_3, \Lambda\Phi_2\rangle + \mathrm{i}g(\langle\Phi_3, \Phi_3\rangle - \langle\Phi_2, \Phi_2\rangle) \\
f_y &= \mathrm{i}\langle\Phi_3, \Lambda\Phi_1\rangle + ug + \mathrm{i}f(\langle\Phi_3, \Phi_3\rangle - \langle\Phi_1, \Phi_1\rangle) \\
g_y &= -\mathrm{i}\langle\Phi_3, \Lambda\Phi_2\rangle - \bar{u}f - \mathrm{i}g(\langle\Phi_3, \Phi_3\rangle - \langle\Phi_2, \Phi_2\rangle)
\end{aligned} \tag{14}
$$

$$
u = \frac{2\mathrm{i}\langle\Phi_2, \Lambda\Phi_1\rangle + 2\mathrm{i}\langle\Phi_2, \Phi_3\rangle\langle\Phi_3, \Phi_1\rangle}{\langle\Phi_1, \Phi_1\rangle - \langle\Phi_2, \Phi_2\rangle} \tag{15}
$$

$$
u_x = 2f\bar{g} \tag{16}
$$

$$
\begin{aligned}
u_y = q \equiv \frac{2\mathrm{i}}{\langle\Phi_1, \Phi_1\rangle - \langle\Phi_2, \Phi_2\rangle}\big[&2\mathrm{i}\langle\Phi_2, \Lambda^2\Phi_1\rangle + 2\mathrm{i}f\langle\Phi_2, \Lambda\Phi_3\rangle + 2\mathrm{i}\bar{g}\langle\Phi_3, \Lambda\Phi_1\rangle \\
&+\mathrm{i}f\bar{g}(2\langle\Phi_3, \Phi_3\rangle - \langle\Phi_1, \Phi_1\rangle - \langle\Phi_2, \Phi_2\rangle) \\
&+u(|g|^2 - |f|^2) + u(\langle\Phi_2, \Lambda\Phi_2\rangle - \langle\Phi_1, \Lambda\Phi_1\rangle)\big]
\end{aligned} \tag{17}
$$

$$
v_1 = 2|f|^2 \qquad v_2 = 2|g|^2. \tag{18}
$$

Therefore, the Lax pair (1) becomes

$$
\Phi_{1,x} = \mathrm{i}\Lambda\Phi_1 + \mathrm{i}f\Phi_3 \qquad \Phi_{2,x} = \mathrm{i}\Lambda\Phi_2 + \mathrm{i}g\Phi_3 \qquad \Phi_{3,x} = \mathrm{i}\bar{f}\Phi_1 + \mathrm{i}\bar{g}\Phi_2 \tag{19}
$$

$$
\begin{aligned}
&\Phi_{1,y} = \mathrm{i}\Lambda\Phi_1 + u\Phi_2 + \mathrm{i}f\Phi_3 \qquad \Phi_{2,y} = -\bar{u}\Phi_1 - \mathrm{i}\Lambda\Phi_2 - \mathrm{i}g\Phi_3 \\
&\Phi_{3,y} = \mathrm{i}\bar{f}\Phi_1 - \mathrm{i}\bar{g}\Phi_2
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
\Phi_{1,t} &= (-2\mathrm{i}\Lambda^2 + \mathrm{i}|u|^2 + \mathrm{i}v_1)\Phi_1 + (-2u\Lambda + \mathrm{i}u_y)\Phi_2 + (-2\mathrm{i}f\Lambda - 2f_y)\Phi_3 \\
\Phi_{2,t} &= (2\bar{u}\Lambda + \mathrm{i}\bar{u}_y)\Phi_1 + (2\mathrm{i}\Lambda^2 - \mathrm{i}|u|^2 - \mathrm{i}v_2)\Phi_2 + (2\mathrm{i}g\Lambda - 2g_y)\Phi_3 \\
\Phi_{3,t} &= (-2\mathrm{i}\bar{f}\Lambda + 2\bar{f}_y)\Phi_1 + (2\mathrm{i}\bar{g}\Lambda + 2\bar{g}_y)\Phi_2 - 2\mathrm{i}(|f|^2 - |g|^2)\Phi_3
\end{aligned} \tag{21}
$$

where $u, u_y, v_1, v_2, f, g, f_y, g_y$ are given by (12)–(18), respectively.

Corresponding to the symplectic form (10), the Poisson bracket of two functions ξ and η of $\{\phi_{jk}, \mathrm{i}\bar{\phi}_{jk}\}$ is given by

$$
\begin{aligned}
\{\xi, \eta\} &= \frac{1}{2}\sum_{j,k}\left(\frac{\partial\xi}{\partial\,\mathrm{Re}(\phi_{jk})}\frac{\partial\eta}{\partial\,\mathrm{Im}(\phi_{jk})} - \frac{\partial\xi}{\partial\,\mathrm{Im}(\phi_{jk})}\frac{\partial\eta}{\partial\,\mathrm{Re}(\phi_{jk})}\right) \\
&= \frac{1}{\mathrm{i}}\sum_{j,k}\left(\frac{\partial\xi}{\partial\phi_{jk}}\frac{\partial\eta}{\partial\bar{\phi}_{jk}} - \frac{\partial\xi}{\partial\bar{\phi}_{jk}}\frac{\partial\eta}{\partial\phi_{jk}}\right).
\end{aligned} \tag{22}
$$

Hereafter, we always look at ϕ_{jk} and $\bar{\phi}_{jk}$ as independent variables in differentiations.

Theorem 1. *The Hamiltonians for (19)–(21) are given by*

$$
H^x = -\langle\Phi_1, \Lambda\Phi_1\rangle - \langle\Phi_2, \Lambda\Phi_2\rangle - |\langle\Phi_1, \Phi_3\rangle|^2 - |\langle\Phi_2, \Phi_3\rangle|^2 \tag{23}
$$

$$
H^y = -\langle\Phi_1, \Lambda\Phi_1\rangle + \langle\Phi_2, \Lambda\Phi_2\rangle - |\langle\Phi_1, \Phi_3\rangle|^2 + |\langle\Phi_2, \Phi_3\rangle|^2 + \mathrm{i}u\langle\Phi_1, \Phi_2\rangle - \mathrm{i}\bar{u}\langle\Phi_2, \Phi_1\rangle \tag{24}
$$

$$
\begin{aligned}
H^t = 2\langle\Phi_1, \Lambda^2\Phi_1\rangle - 2\langle\Phi_2, \Lambda^2\Phi_2\rangle &+ 4\,\mathrm{Re}(\langle\Phi_1, \Phi_3\rangle\langle\Phi_3, \Lambda\Phi_1\rangle) \\
&-4\,\mathrm{Re}(\langle\Phi_2, \Phi_3\rangle\langle\Phi_3, \Lambda\Phi_2\rangle) + 2(\langle\Phi_3, \Phi_3\rangle - \langle\Phi_1, \Phi_1\rangle)\langle\Phi_1, \Phi_3\rangle\langle\Phi_3, \Phi_1\rangle \\
&-2(\langle\Phi_3, \Phi_3\rangle - \langle\Phi_2, \Phi_2\rangle)\langle\Phi_2, \Phi_3\rangle\langle\Phi_3, \Phi_2\rangle \\
&+\frac{4}{\langle\Phi_1, \Phi_1\rangle - \langle\Phi_2, \Phi_2\rangle}\,|\langle\Phi_2, \Lambda\Phi_1\rangle + \langle\Phi_2, \Phi_3\rangle\langle\Phi_3, \Phi_1\rangle|^2 - 2\,\mathrm{Re}(q\langle\Phi_1, \Phi_2\rangle)
\end{aligned} \tag{25}
$$

respectively, which satisfy

$$\{H^x, \langle\Phi_1, \Phi_2\rangle\} = \{H^y, \langle\Phi_1, \Phi_2\rangle\} = \{H^t, \langle\Phi_1, \Phi_2\rangle\} = 0. \tag{26}$$

Here u is given by (15) and q is given by (17).

The proof is obtained by direct computation.

Therefore, we obtain Hamiltonian systems (23)–(25) with Neumann-type constraint, $\langle\Phi_1, \Phi_2\rangle = 0$. Any solution of the Hamiltonian equations

$$\begin{array}{lll} \mathrm{i}\phi_{jk,x} = \dfrac{\partial H^x}{\partial\bar{\phi}_{jk}} & \mathrm{i}\phi_{jk,y} = \dfrac{\partial H^y}{\partial\bar{\phi}_{jk}} & \mathrm{i}\phi_{jk,t} = \dfrac{\partial H^t}{\partial\bar{\phi}_{jk}} \\ -\mathrm{i}\bar{\phi}_{jk,x} = \dfrac{\partial H^x}{\partial\phi_{jk}} & -\mathrm{i}\bar{\phi}_{jk,y} = \dfrac{\partial H^y}{\partial\phi_{jk}} & -\mathrm{i}\bar{\phi}_{jk,t} = \dfrac{\partial H^t}{\partial\phi_{jk}} \end{array} \tag{27}$$

gives a solution of the DSI equation (2), where u, v_1, v_2 are given by (15) and (18), respectively.

3. Integrability

Now we consider the integrability of the Hamiltonian systems given by theorem 1 on the submanifold

$$S = \{(\Phi_1, \Phi_2, \Phi_3, \mathrm{i}\bar{\Phi}_1, \mathrm{i}\bar{\Phi}_2, \mathrm{i}\bar{\Phi}_3) \in \mathbb{R}^{6N} \mid \langle\Phi_1, \Phi_2\rangle = 0, \langle\Phi_1, \Phi_1\rangle \neq \langle\Phi_2, \Phi_2\rangle\}. \tag{28}$$

Here we still use $3N$ complex numbers and their complex conjugates $(\Phi_1, \Phi_2, \Phi_3, \mathrm{i}\bar{\Phi}_1, \mathrm{i}\bar{\Phi}_2, \mathrm{i}\bar{\Phi}_3)$ to represent a point in $\mathbb{R}^{6N}$. Clearly, S has two connected components characterized by $\langle\Phi_1, \Phi_1\rangle > \langle\Phi_2, \Phi_2\rangle$ and $\langle\Phi_2, \Phi_2\rangle > \langle\Phi_1, \Phi_1\rangle$, respectively. Since $\langle\Phi_1, \Phi_1\rangle \neq \langle\Phi_2, \Phi_2\rangle$, $0 \notin S$. S is a $(6N-2)$-dimensional real analytic manifold, on which the coordinates can be given by $\phi_{31}, \ldots, \phi_{3N}$ (with their complex conjugates) and $2N-1$ of $\phi_{11}, \ldots, \phi_{1N}, \phi_{21}, \ldots, \phi_{2N}$ (with their complex conjugates) whenever the remaining one is non-zero.

Define

$$\begin{aligned} \gamma_1 &= \mathrm{Re}\langle\Phi_1, \Phi_2\rangle = \frac{1}{2}(\langle\Phi_1, \Phi_2\rangle + \langle\Phi_2, \Phi_1\rangle) \\ \gamma_2 &= \mathrm{Im}\langle\Phi_1, \Phi_2\rangle = \frac{1}{2\mathrm{i}}(\langle\Phi_1, \Phi_2\rangle - \langle\Phi_2, \Phi_1\rangle) \end{aligned} \tag{29}$$

then S is defined by two real-valued functions $\gamma_1 = \gamma_2 = 0$. Since

$$\{\gamma_1, \gamma_2\} = \tfrac{1}{2}(\langle\Phi_1, \Phi_1\rangle - \langle\Phi_2, \Phi_2\rangle) \tag{30}$$

is never zero on S, the symplectic form (10) on $\mathbb{R}^{6N}$ naturally induces a (non-degenerate) symplectic form on S. The corresponding Poisson bracket of two functions ξ, η on S is still given by (22) if they satisfy $\{\xi, \gamma_j\} = 0$, $\{\eta, \gamma_j\} = 0$ $(j = 1, 2)$.

Hereafter, the Poisson bracket $\{\ \}$ always denotes the standard Poisson bracket (22) on $\mathbb{R}^{6N}$.

Let

$$L(\lambda) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} \bar{\phi}_{1j}\phi_{1j} & \bar{\phi}_{2j}\phi_{1j} & \bar{\phi}_{3j}\phi_{1j} \\ \bar{\phi}_{1j}\phi_{2j} & \bar{\phi}_{2j}\phi_{2j} & \bar{\phi}_{3j}\phi_{2j} \\ \bar{\phi}_{1j}\phi_{3j} & \bar{\phi}_{2j}\phi_{3j} & \bar{\phi}_{3j}\phi_{3j} \end{pmatrix}. \tag{31}$$

Then we have

Lemma 1. $L(\lambda)$ *satisfies the Lax equations*

$$L_y = [V, L] \qquad L_x = [U, L] \qquad L_t = [W, L] \tag{32}$$

if and only if the constraints (12) hold.

Proof. Let $J = \mathrm{diag}(1, -1, 0)$, $L_0 = \mathrm{diag}(1, 1, 0)$, $\phi_j = (\phi_{1j}, \phi_{2j}, \phi_{3j})^T$, then

$$L(\lambda) = L_0 + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \phi_j \phi_j^*. \tag{33}$$

Since $V(\lambda_j)^* = -V(\lambda_j)$,

$$\begin{aligned} L_y(\lambda) &= \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} [V(\lambda_j), \phi_j \phi_j^*] \\ &= \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} [V(\lambda), \phi_j \phi_j^*] - \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} [V(\lambda) - V(\lambda_j), \phi_j \phi_j^*] \\ &= [V(\lambda), L(\lambda) - L_0] - \mathrm{i}\left[J, \sum_{j=1}^{N} \phi_j \phi_j^*\right]. \end{aligned} \tag{34}$$

Hence $L_y(\lambda) = [V(\lambda), L(\lambda)]$ if and only if

$$[L_0, V(\lambda)] = \mathrm{i}\left[J, \sum_{j=1}^{N} \phi_j \phi_j^*\right]. \tag{35}$$

Written in the components, this is exactly the constraints (12). This proves that the first equation of (32) is equivalent to (12). When (12) holds, the other two equations of (32) are obtained similarly to the first one. The lemma is proved. □

By lemma 1, $\mathrm{tr}\, L^k$ $(k \geqslant 1)$ are all conserved. Expand $\mathrm{tr}\, L^k$ as a Laurent series

$$\mathrm{tr}\, L^k = 2 + \sum_{j=0}^{\infty} \frac{\widetilde{\mathcal{E}}_j^{(k)}}{\lambda^{j+1}} \tag{36}$$

which is convergent absolutely and uniformly as $|\lambda| > \max_{1 \leqslant j \leqslant N} |\lambda_j|$, then all $\{\widetilde{\mathcal{E}}_j^{(k)}\}$ are conserved.

Moreover, we can show that any two of $\{\widetilde{\mathcal{E}}_j^{(k)}\}$ commute with each other. This follows from the following more general lemma.

Lemma 2. *Suppose* $\mathbb{R}^{2nr} = \{(q_{11}, \ldots, q_{1n}, q_{21}, \ldots, q_{2n}, \ldots, q_{r1}, \ldots, q_{rn}, p_{11}, \ldots, p_{1n}, \ldots, p_{r1}, \ldots, p_{rn})\}$ *is equipped with the standard symplectic form*

$$\omega = \sum_{\substack{1 \leqslant j \leqslant r \\ 1 \leqslant \alpha \leqslant n}} \mathrm{d}p_{j\alpha} \wedge \mathrm{d}q_{j\alpha}. \tag{37}$$

Denote $q_j = (q_{j1}, \ldots, q_{jn})^T$, $p_j = (p_{j1}, \ldots, p_{jn})^T$. *Let* $\lambda_1, \ldots, \lambda_n$ *be* n *distinct real numbers and* A *be an* $r \times r$ *constant matrix,*

$$M(\lambda) = A + \sum_{\alpha=1}^{n} \frac{1}{\lambda - \lambda_\alpha} \begin{pmatrix} p_{1\alpha}q_{1\alpha} & p_{2\alpha}q_{1\alpha} & \cdots & p_{r\alpha}q_{1\alpha} \\ p_{1\alpha}q_{2\alpha} & p_{2\alpha}q_{2\alpha} & \cdots & p_{r\alpha}q_{2\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1\alpha}q_{r\alpha} & p_{2\alpha}q_{r\alpha} & \cdots & p_{r\alpha}q_{r\alpha} \end{pmatrix}. \tag{38}$$

Then for any two complex numbers λ, μ *and any positive integers* k, l, a, b *with* $1 \leqslant a, b \leqslant r$,

$$\begin{aligned} &\{\operatorname{tr} M^k(\lambda), \operatorname{tr} M^l(\mu)\} = 0 \\ &\{\operatorname{tr} M^k(\lambda), \langle p_b, q_a\rangle\} = -k[A, M^{k-1}(\lambda)]_{ab}. \end{aligned} \tag{39}$$

Here $\langle p_b, q_a\rangle = \sum_{\alpha=1}^{n} p_{b\alpha}q_{a\alpha}$.

Proof. The jth row of

$$\frac{\partial M(\lambda)}{\partial q_{j\alpha}}$$

is

$$\frac{1}{\lambda - \lambda_\alpha}(p_{1\alpha}, \ldots, p_{r\alpha})$$

and the other rows are zero. Similarly, the jth column of

$$\frac{\partial M(\lambda)}{\partial p_{j\alpha}}$$

is

$$\frac{1}{\lambda - \lambda_\alpha}(q_{1\alpha}, \ldots, q_{r\alpha})^T$$

and the other columns are zero. Hence

$$\begin{aligned} \frac{1}{kl}\left\{\operatorname{tr} M^k(\lambda), \operatorname{tr} M^l(\mu)\right\} &= \frac{1}{kl}\sum_{j,\alpha}\left(\frac{\partial}{\partial q_{j\alpha}}\big(\operatorname{tr} M^k(\lambda)\big)\frac{\partial}{\partial p_{j\alpha}}\big(\operatorname{tr} M^l(\mu)\big)\right. \\ &\quad \left. -\frac{\partial}{\partial q_{j\alpha}}\big(\operatorname{tr} M^l(\mu)\big)\frac{\partial}{\partial p_{j\alpha}}\big(\operatorname{tr} M^k(\lambda)\big)\right) \\ &= \sum_{j,\alpha}\left(\operatorname{tr}\left(M^{k-1}(\lambda)\frac{\partial M(\lambda)}{\partial q_{j\alpha}}\right)\operatorname{tr}\left(M^{l-1}(\mu)\frac{\partial M(\mu)}{\partial p_{j\alpha}}\right)\right. \\ &\quad \left. -\operatorname{tr}\left(M^{l-1}(\mu)\frac{\partial M(\mu)}{\partial q_{j\alpha}}\right)\operatorname{tr}\left(M^{k-1}(\lambda)\frac{\partial M(\lambda)}{\partial p_{j\alpha}}\right)\right) \end{aligned}$$

$$= \sum_{a,b,j,\alpha} \left(\frac{1}{\lambda - \lambda_\alpha} (M^{k-1}(\lambda))_{aj} p_{a\alpha} \cdot \frac{1}{\mu - \lambda_\alpha} (M^{l-1}(\mu))_{jb} q_{b\alpha} \right.$$

$$\left. - \frac{1}{\mu - \lambda_\alpha} (M^{l-1}(\mu))_{aj} p_{a\alpha} \cdot \frac{1}{\lambda - \lambda_\alpha} (M^{k-1}(\lambda))_{jb} q_{b\alpha} \right)$$

$$= \sum_{a,b,\alpha} \frac{1}{\lambda - \lambda_\alpha} \frac{1}{\mu - \lambda_\alpha} p_{a\alpha} q_{b\alpha} [M^{k-1}(\lambda), M^{l-1}(\mu)]_{ab}$$

$$= \sum_{a,b,\alpha} \frac{1}{\mu - \lambda} \left(\frac{1}{\lambda - \lambda_\alpha} - \frac{1}{\mu - \lambda_\alpha} \right) p_{a\alpha} q_{b\alpha} [M^{k-1}(\lambda), M^{l-1}(\mu)]_{ab}$$

$$= \frac{1}{\mu - \lambda} \operatorname{tr} \big((M(\lambda) - M(\mu)) [M^{k-1}(\lambda), M^{l-1}(\mu)] \big) = 0. \tag{40}$$

This proves the first part. The second part is proved as follows:

$$\frac{1}{k} \left\{ \operatorname{tr} M^k(\lambda), \langle p_b, q_a \rangle \right\} = \frac{1}{k} \sum_\alpha \left(\frac{\partial}{\partial q_{b\alpha}} \big(\operatorname{tr} M^k(\lambda) \big) q_{a\alpha} - \frac{\partial}{\partial p_{a\alpha}} \big(\operatorname{tr} M^k(\lambda) \big) p_{b\alpha} \right)$$

$$= \sum_{l,\alpha} \left(\frac{1}{\lambda - \lambda_\alpha} (M^{k-1}(\lambda))_{lb} p_{l\alpha} q_{a\alpha} - \frac{1}{\lambda - \lambda_\alpha} (M^{k-1}(\lambda))_{al} q_{l\alpha} p_{b\alpha} \right)$$

$$= \Big((M(\lambda) - A) M^{k-1}(\lambda) - M^{k-1}(\lambda)(M(\lambda) - A) \Big)_{ab}$$

$$= -[A, M^{k-1}(\lambda)]_{ab}. \tag{41}$$

The lemma is proved. □

From this lemma, we know that the set $\{\widetilde{\mathcal{E}}_j^{(k)}\}$ is in involution.

Remark 1. For the three-wave equation, [14] wrote down such $\{\widetilde{\mathcal{E}}_j^{(k)}\}$. It was proved that they were in involution and $3N$ of them were independent. However, for the three-wave equation, the first term of L is $\operatorname{diag}(\beta_1, \beta_2, \beta_3)$ ($\beta_1, \beta_2, \beta_3$ are distinct) rather than $\operatorname{diag}(1, 1, 0)$ here. In the case $\beta_1 = 1$, $\beta_2 = 1$, $\beta_3 = 0$, those $\{\widetilde{\mathcal{E}}_j^{(k)}\}$ are obviously still in involution. Hence we can also use the result of [14] to find the involution. However, we prove it more directly and easily here. On the other hand, with the constraint $\langle \Phi_1, \Phi_2 \rangle = 0$, the independence of $\{\widetilde{\mathcal{E}}_j^{(k)}\}$ changes completely. Hence we should prove the independence in this constrained case.

For real λ, suppose the eigenvalues of the Hermitian matrix $L(\lambda)$ are ν_1, ν_2 and ν_3, then $\operatorname{tr} L^k = \sum_{j=1}^3 \nu_j^k$, while

$$\det(\mu - L(\lambda)) = \prod_{j=1}^3 (\mu - \nu_j)$$

$$= \mu^3 - (\nu_1 + \nu_2 + \nu_3)\mu^2 + (\nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1)\mu - \nu_1\nu_2\nu_3 \tag{42}$$

for any μ. Hence $\operatorname{tr} L^k$ can be expressed by the coefficients of μ in $\det(\mu - L(\lambda))$.

Suppose $L(\lambda) = (L_{jk})_{1 \leqslant j,k \leqslant 3}$, then

$$\nu_1 + \nu_2 + \nu_3 = \operatorname{tr} L \qquad \nu_1\nu_2\nu_3 = \det L$$
$$\nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1 = \sum_{1 \leqslant j < k \leqslant 3} \begin{vmatrix} L_{jj} & L_{jk} \\ L_{kj} & L_{kk} \end{vmatrix}. \tag{43}$$

Define

$$\operatorname{tr} L = \sum_{k=-1}^{\infty} \frac{\mathcal{E}_k^{(0)}}{\lambda^{k+1}} \qquad \det L = \sum_{k=-1}^{\infty} \frac{\mathcal{E}_k^{(2)}}{\lambda^{k+1}}$$
$$\sum_{1\leqslant j<k\leqslant 3} \begin{vmatrix} L_{jj} & L_{jk} \\ L_{kj} & L_{kk} \end{vmatrix} = \sum_{k=-1}^{\infty} \frac{\mathcal{E}_k^{(1)}}{\lambda^{k+1}}. \tag{44}$$

These $\{\mathcal{E}_k^{(j)}\}$ differ from the $\{F_k^{(j)}\}$ in [14] with a multiple i.

To simplify the expressions of $\mathcal{E}_k^{(j)}$s, we define the following simpler but equivalent conserved integrals $E_k^{(j)}$s, which are non-degenerate linear combinations of $\mathcal{E}_k^{(j)}$s. This also makes the expressions (50)–(52) of the Hamiltonians and the proof of theorem 4 simpler. For $m \geqslant 0$, define

$$E_m^{(1)} = 2\mathcal{E}_m^{(0)} - \mathcal{E}_m^{(1)} = \langle\Phi_1, \Lambda^m\Phi_1\rangle + \langle\Phi_2, \Lambda^m\Phi_2\rangle - \sum_{1\leqslant i<j\leqslant 3}\sum_{l=1}^{m} \begin{vmatrix} \langle\Phi_i, \Lambda^{l-1}\Phi_i\rangle & \langle\Phi_j, \Lambda^{m-l}\Phi_i\rangle \\ \langle\Phi_i, \Lambda^{l-1}\Phi_j\rangle & \langle\Phi_j, \Lambda^{m-l}\Phi_j\rangle \end{vmatrix} \tag{45}$$

$$E_m^{(2)} = -\mathcal{E}_{m+1}^{(2)} + \mathcal{E}_{m+1}^{(1)} - \mathcal{E}_{m+1}^{(0)} = \sum_{l=0}^{m} \begin{vmatrix} \langle\Phi_1, \Lambda^{l}\Phi_1\rangle & \langle\Phi_2, \Lambda^{m-l}\Phi_1\rangle \\ \langle\Phi_1, \Lambda^{l}\Phi_2\rangle & \langle\Phi_2, \Lambda^{m-l}\Phi_2\rangle \end{vmatrix} - \sum_{\substack{i+j+k=m-1\\ i,j,k\geqslant 0}} \begin{vmatrix} \langle\Phi_1, \Lambda^{i}\Phi_1\rangle & \langle\Phi_2, \Lambda^{j}\Phi_1\rangle & \langle\Phi_3, \Lambda^{k}\Phi_1\rangle \\ \langle\Phi_1, \Lambda^{i}\Phi_2\rangle & \langle\Phi_2, \Lambda^{j}\Phi_2\rangle & \langle\Phi_3, \Lambda^{k}\Phi_2\rangle \\ \langle\Phi_1, \Lambda^{i}\Phi_3\rangle & \langle\Phi_2, \Lambda^{j}\Phi_3\rangle & \langle\Phi_3, \Lambda^{k}\Phi_3\rangle \end{vmatrix} \tag{46}$$

$$E_m^{(3)} = \mathcal{E}_m^{(1)} - \mathcal{E}_m^{(0)} = \langle\Phi_3, \Lambda^m\Phi_3\rangle + \sum_{1\leqslant i<j\leqslant 3}\sum_{l=1}^{m} \begin{vmatrix} \langle\Phi_i, \Lambda^{l-1}\Phi_i\rangle & \langle\Phi_j, \Lambda^{m-l}\Phi_i\rangle \\ \langle\Phi_i, \Lambda^{l-1}\Phi_j\rangle & \langle\Phi_j, \Lambda^{m-l}\Phi_j\rangle \end{vmatrix}. \tag{47}$$

The above sums are zero if the upper bound is smaller than the lower bound. According to lemma 2, we have

Theorem 2. $\{E_m^{(j)}, E_n^{(k)}\} = 0$ *and* $\{E_m^{(j)}, \langle\Phi_1, \Phi_2\rangle\} = 0$ *for any* $j, k = 1, 2, 3$ *and* $m, n \geqslant 0$. *Therefore,* $\{E_m^{(j)}\}$ *are in involution on S.*

Define

$$\Omega_1 = \langle\Phi_1, \Phi_1\rangle \qquad \Omega_2 = \langle\Phi_2, \Phi_2\rangle \qquad \Omega_3 = \langle\Phi_3, \Phi_3\rangle. \tag{48}$$

By (45)–(47),

$$E_0^{(1)} = \Omega_1 + \Omega_2 \qquad E_0^{(2)} = \Omega_1\Omega_2 - |\langle\Phi_1, \Phi_2\rangle|^2 \qquad E_0^{(3)} = \Omega_3. \tag{49}$$

Hence $\Omega_1, \Omega_2, \Omega_3$ are expressed by $E_k^{(j)}$s and $|\langle\Phi_1, \Phi_2\rangle|^2$.

The Hamiltonians in theorem 1 can be expressed as

$$H^x = -E_1^{(1)} - E_0^{(1)}E_0^{(3)} - E_0^{(2)} \tag{50}$$

$$H^y = \frac{1}{\Omega_1 - \Omega_2}\left(-E_0^{(1)}E_1^{(1)} + 2E_1^{(2)} - E_0^{(1)}E_0^{(2)} - E_0^{(1)2}E_0^{(3)} + 2E_0^{(2)}E_0^{(3)}\right) \tag{51}$$

$$H^t = \frac{1}{\Omega_1 - \Omega_2}\big(2E_0^{(1)}E_2^{(1)} - 4E_2^{(2)} + 2(E_0^{(1)2} - 2E_0^{(2)})(E_1^{(1)} + E_1^{(3)})$$
$$+(E_0^{(1)2} - 4E_0^{(2)})H^x + (E_0^{(1)} - 2E_0^{(3)})(\Omega_1 - \Omega_2)H^y + (H^x)^2 - (H^y)^2$$
$$-(u\langle\Phi_1, \Phi_2\rangle - \bar{u}\langle\Phi_2, \Phi_1\rangle)^2 + 4(|\langle\Phi_1, \Phi_3\rangle|^2 + |\langle\Phi_2, \Phi_3\rangle|^2)|\langle\Phi_1, \Phi_2\rangle|^2\big). \tag{52}$$

Each Hamiltonian is a function of $E_k^{(j)}$s and $|\langle\Phi_1, \Phi_2\rangle|^2$, $\langle\Phi_1, \Phi_2\rangle^2$, $\langle\Phi_2, \Phi_1\rangle^2$, which is smooth near S.

Let γ_1 and γ_2 be two real functions in (29) defining S. For any two functions H_1 and H_2 on $\mathbb{R}^{6N}$ with $\{H_j, \gamma_k\} = 0$ $(j, k = 1, 2)$, define $H_1 \doteq H_2$ if $H_1 = H_2$ and $\nabla H_1 = \nabla H_2$ on S. In this case, H_1 and H_2 have the same Hamiltonian vector field which is tangent to S. Hence they give the same Hamiltonian systems on S. Moreover, for any function F on $\mathbb{R}^{6N}$ with $\{F, \gamma_k\} = 0$ $(k = 1, 2)$, $\{H_1, F\} = \{H_2, F\}$ holds.

From (49),

$$\Omega_1 - \Omega_2 = \sigma^{-1}\sqrt{E_0^{(1)2} - 4(E_0^{(2)} + |\langle\Phi_1, \Phi_2\rangle|^2)} \tag{53}$$

where $\sigma = \pm 1$ depends on the connected component of S. Let

$$\begin{aligned} \Delta &= \sqrt{E_0^{(1)2} - 4E_0^{(2)}} \\ K^y &= (\Omega_1 - \Omega_2)H^y = -E_0^{(1)}E_1^{(1)} + 2E_1^{(2)} - E_0^{(1)}E_0^{(2)} - E_0^{(1)2}E_0^{(3)} + 2E_0^{(2)}E_0^{(3)}. \end{aligned} \tag{54}$$

For γ_1 and γ_2, only the quadratic terms appear in the expressions of H^y and H^t. Hence we have

$$H^y \doteq \frac{\sigma}{\Delta}K^y \tag{55}$$

$$H^t \doteq \frac{\sigma}{\Delta}\bigg(2E_0^{(1)}E_2^{(1)} - 4E_2^{(2)} + 2(E_0^{(1)2} - 2E_0^{(2)})(E_1^{(1)} + E_1^{(3)})$$
$$+(E_0^{(1)2} - 4E_0^{(2)})H^x + (E_0^{(1)} - 2E_0^{(3)})K^y + (H^x)^2 - \frac{1}{\Delta^2}(K^y)^2\bigg). \tag{56}$$

By (50), H^x depends on $E_k^{(j)}$s only. On the other hand, the right-hand sides of (55) and (56) also depend on $E_k^{(j)}$s only. Therefore, we obtain the following result from theorems 1 and 2.

Theorem 3. *Three Hamiltonians H^x, H^y, H^t defined by theorem 1 commute with each other:*

$$\{H^x, H^y\} = \{H^x, H^t\} = \{H^y, H^t\} = 0 \tag{57}$$

and they satisfy

$$\{H^x, \langle\Phi_1, \Phi_2\rangle\} = \{H^y, \langle\Phi_1, \Phi_2\rangle\} = \{H^t, \langle\Phi_1, \Phi_2\rangle\} = 0. \tag{58}$$

Moreover, each $E_m^{(j)}$ is conserved under the Hamiltonian flows given by H^x, H^y, H^t, respectively.

Next we shall prove the integrability of these Hamiltonian systems. That is

Theorem 4. $3N - 1$ *real-valued functions* $\{E_m^{(j)}\}$ $(j = 1, 3;\ 0 \leqslant m \leqslant N - 1)$ *and* $\{E_m^{(2)}\}$ $(0 \leqslant m \leqslant N - 2)$ *are functionally independent in a dense open subset of S.*

Proof. Let

$$
\begin{aligned}
\widetilde{S}_1 &= \{(\Phi_1, \Phi_2, \Phi_3, \mathrm{i}\bar{\Phi}_1, \mathrm{i}\bar{\Phi}_2, \mathrm{i}\bar{\Phi}_3) \in S \mid \phi_{1,N} \neq 0, \bar{\phi}_{1,N} \neq 0, \langle \Phi_1, \Phi_1 \rangle > \langle \Phi_2, \Phi_2 \rangle\} \\
\widetilde{S}_2 &= \{(\Phi_1, \Phi_2, \Phi_3, \mathrm{i}\bar{\Phi}_1, \mathrm{i}\bar{\Phi}_2, \mathrm{i}\bar{\Phi}_3) \in S \mid \phi_{1,N} \neq 0, \bar{\phi}_{1,N} \neq 0, \langle \Phi_1, \Phi_1 \rangle < \langle \Phi_2, \Phi_2 \rangle\} \\
\widetilde{S} &= \widetilde{S}_1 \cup \widetilde{S}_2
\end{aligned}
\tag{59}
$$

then $\widetilde{S}$ is a dense open subset of S. Similar to S, $\widetilde{S}$ has also two connected components, which are $\widetilde{S}_1$ and $\widetilde{S}_2$. In $\widetilde{S}$, we can solve $\phi_{2,N}, \bar{\phi}_{2,N}$ from the constraint $\langle \Phi_1, \Phi_2 \rangle = 0$ as

$$
\phi_{2,N} = -\sum_{j=1}^{N-1} \frac{\bar{\phi}_{1j}}{\bar{\phi}_{1N}} \phi_{2j} \qquad \bar{\phi}_{2,N} = -\sum_{j=1}^{N-1} \frac{\phi_{1j}}{\phi_{1N}} \bar{\phi}_{2j}. \tag{60}
$$

Hence $\widetilde{S}_1$ has global coordinates

$$
\Theta = \{\phi_{1j}, \mathrm{i}\bar{\phi}_{1j}\ (1 \leqslant j \leqslant N);\ \phi_{2j}, \mathrm{i}\bar{\phi}_{2j}\ (1 \leqslant j \leqslant N-1);\ \phi_{3j}, \mathrm{i}\bar{\phi}_{3j}\ (1 \leqslant j \leqslant N)\}. \tag{61}
$$

Let $P_0 \in \widetilde{S}_1$ be given by $\Phi_1 = (1, 1, \ldots, 1)^T$, $\Phi_2 = \epsilon(1, 1, \ldots, 1, -N+1)^T$, $\Phi_3 = \epsilon(1, 1, \ldots, 1)^T$, where ϵ is a small real constant. Here Φ_3 is chosen to be parallel with Φ_1 so that the following computation will be simplified. Since ϕ_{2N} and $\bar{\phi}_{2N}$ are functions of the variables in Θ, we have, at P_0,

$$
\begin{aligned}
&\frac{\partial E_m^{(1)}}{\partial \bar{\phi}_{1j}} = \lambda_j^m + \mathrm{O}(\epsilon) \qquad \frac{\partial E_m^{(1)}}{\partial \bar{\phi}_{2j}} = \mathrm{O}(\epsilon) \qquad \frac{\partial E_m^{(1)}}{\partial \bar{\phi}_{3j}} = \mathrm{O}(\epsilon) \\
&\frac{\partial E_m^{(2)}}{\partial \bar{\phi}_{1j}} = \mathrm{O}(\epsilon^2) \qquad \frac{\partial E_m^{(2)}}{\partial \bar{\phi}_{3j}} = \mathrm{O}(\epsilon^3) \\
&\frac{\partial E_m^{(2)}}{\partial \bar{\phi}_{2j}} = \sum_{l=0}^{m} \begin{vmatrix} \langle \Phi_1, \Lambda^l \Phi_1 \rangle & \lambda_j^{m-l} \phi_{1j} \\ \langle \Phi_1, \Lambda^l \Phi_2 \rangle & \lambda_j^{m-l} \phi_{2j} \end{vmatrix} - \frac{\phi_{1j}}{\phi_{1N}} \sum_{l=0}^{m} \begin{vmatrix} \langle \Phi_1, \Lambda^l \Phi_1 \rangle & \lambda_N^{m-l} \phi_{1N} \\ \langle \Phi_1, \Lambda^l \Phi_2 \rangle & \lambda_N^{m-l} \phi_{2N} \end{vmatrix} + \mathrm{O}(\epsilon^3) \\
&\qquad = \epsilon \sum_{l=0}^{m} \begin{vmatrix} \lambda_1^l + \cdots + \lambda_N^l & \lambda_j^{m-l} - \lambda_N^{m-l} \\ \lambda_1^l + \cdots + \lambda_{N-1}^l - (N-1)\lambda_N^l & \lambda_j^{m-l} + (N-1)\lambda_N^{m-l} \end{vmatrix} + \mathrm{O}(\epsilon^3) \\
&\qquad = N\epsilon \sum_{l=0}^{m} \lambda_N^{m-l} (\lambda_1^l + \cdots + \lambda_{N-1}^l + \lambda_j^l) + \mathrm{O}(\epsilon^3) \\
&\frac{\partial E_m^{(3)}}{\partial \bar{\phi}_{1j}} = \mathrm{O}(\epsilon^2) \qquad \frac{\partial E_m^{(3)}}{\partial \bar{\phi}_{2j}} = \mathrm{O}(\epsilon) \qquad \frac{\partial E_m^{(3)}}{\partial \bar{\phi}_{3j}} = \epsilon \lambda_j^m + \mathrm{O}(\epsilon^3).
\end{aligned}
\tag{62}
$$

Here the subscript j is taken from 1 to N for ϕ_{1j}, ϕ_{3j}, and from 1 to $N-1$ for ϕ_{2j}. It can be checked that

$$
\det \left(\sum_{l=0}^{m} \lambda_N^{m-l} (\lambda_1^l + \cdots + \lambda_{N-1}^l + \lambda_j^l) \right)_{\substack{0 \leqslant m \leqslant N-2 \\ 1 \leqslant j \leqslant N-1}} = N \prod_{1 \leqslant j < k \leqslant N-1} (\lambda_j - \lambda_k). \tag{63}
$$

Hence the Jacobian determinant

$$
\begin{aligned}
J &\equiv \frac{\partial(E_0^{(1)}, \ldots, E_{N-1}^{(1)}, E_0^{(2)}, \ldots, E_{N-2}^{(2)}, E_0^{(3)}, \ldots, E_{N-1}^{(3)})}{\partial(\bar{\phi}_{11}, \ldots, \bar{\phi}_{1,N}, \bar{\phi}_{21}, \ldots, \bar{\phi}_{2,N-1}, \bar{\phi}_{31}, \ldots, \bar{\phi}_{3,N})} \\
&= N^N \epsilon^{2N-1} \left(\prod_{1 \leqslant j < k \leqslant N} (\lambda_j - \lambda_k) \right)^2 \prod_{1 \leqslant j < k \leqslant N-1} (\lambda_j - \lambda_k) + \mathrm{O}(\epsilon^{2N}).
\end{aligned}
\tag{64}
$$

When ϵ is small enough and $\epsilon \neq 0$, J is non-zero near the point P_0. Since J is a rational function of

$$\{\phi_{1j}, \mathrm{i}\bar{\phi}_{1j}\ (1 \leqslant j \leqslant N);\ \phi_{2j}, \mathrm{i}\bar{\phi}_{2j}\ (1 \leqslant j \leqslant N-1);\ \phi_{3j}, \mathrm{i}\bar{\phi}_{3j}\ (1 \leqslant j \leqslant N)\} \tag{65}$$

$J = 0$ identically if J is zero in an open subset of $\widetilde{S}_1$. Therefore, J is non-zero in a dense open subset of $\widetilde{S}_1$. Similarly, J is non-zero in a dense open subset of $\widetilde{S}_2$. Since all $E_m^{(j)}$ are real-valued functions, the Jacobian matrix of $(E_0^{(1)}, \ldots, E_{N-1}^{(1)}, E_0^{(2)}, \ldots, E_{N-2}^{(2)}, E_0^{(3)}, \ldots, E_{N-1}^{(3)})$ with respect to the real coordinates

$$\begin{aligned} &\mathrm{Re}(\phi_{11}), \ldots, \mathrm{Re}(\phi_{1,N}), \mathrm{Re}(\phi_{21}), \ldots, \mathrm{Re}(\phi_{2,N-1}), \mathrm{Re}(\phi_{31}), \ldots, \mathrm{Re}(\phi_{3,N}) \\ &\mathrm{Im}(\phi_{11}), \ldots, \mathrm{Im}(\phi_{1,N}), \mathrm{Im}(\phi_{21}), \ldots, \mathrm{Im}(\phi_{2,N-1}), \mathrm{Im}(\phi_{31}), \ldots, \mathrm{Im}(\phi_{3,N}) \end{aligned} \tag{66}$$

is of full rank $3N - 1$. The theorem is proved. □

Theorem 5. *The Hamiltonian systems given by theorem 1 are completely integrable on S in the Liouville sense.*

Proof. We have proved: (1) $\{E_m^{(j)}\}$ $(j = 1, 2, 3; m = 0, 1, 2, \ldots)$ are in involution on S (theorem 2). (2) $\{E_m^{(j)}\}$ $(j = 1, 3;\ 0 \leqslant m \leqslant N-1)$ and $\{E_m^{(2)}\}$ $(0 \leqslant m \leqslant N-2)$ are functionally independent in a dense open subset of S (theorem 4). It remains to prove that the Hamiltonian vector fields of all $\{E_m^{(j)}\}$ are complete. This follows from the compactness of each level set, which is a closed subset of the compact set

$$\{(\Phi_1, \Phi_2, \Phi_3, \mathrm{i}\bar{\Phi}_1, \mathrm{i}\bar{\Phi}_2, \mathrm{i}\bar{\Phi}_3) \in S \mid \langle \Phi_j, \Phi_j \rangle = \Omega_{j0}, j = 1, 2, 3\} \tag{67}$$

where Ω_{j0} $(j = 1, 2, 3)$ are constants. Therefore, the Hamiltonian systems given by theorem 1 are completely integrable [5]. □

4. Example: an explicit solution for $N = 2$

Now suppose $N = 2$,

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \qquad \Phi_j = \begin{pmatrix} \phi_j \\ \psi_j \end{pmatrix}.$$

Let R_1, R_2, R_3, G and K be defined by

$$\begin{aligned} R_j^2 &= |\phi_j|^2 + |\psi_j|^2 \quad (j = 1, 2, 3) \\ G &= |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 - R_2^2 \\ K &= (\langle \Phi_1, \Lambda\Phi_1 \rangle + \langle \Phi_1, \Phi_3 \rangle \langle \Phi_3, \Phi_1 \rangle)/R_1^2 \end{aligned} \tag{68}$$

then from the above list of conserved integrals, we know that R_1, R_2, R_3, G and K are all constants. Moreover,

$$\langle \Phi_2, \Lambda\Phi_2 \rangle + \langle \Phi_2, \Phi_3 \rangle \langle \Phi_3, \Phi_2 \rangle = R_2^2(R_3^2 + \lambda + \mu - K). \tag{69}$$

Let

$$\phi_j = R_j \cos\theta_j \exp(\mathrm{i}\alpha_j) \qquad \psi_j = R_j \sin\theta_j \exp(\mathrm{i}\beta_j) \tag{70}$$

then the constraint $\langle \Phi_2, \Phi_1 \rangle = 0$ leads to

$$\theta_2 = \sigma\theta_1 + \pi/2 + l\pi \qquad \exp(\mathrm{i}(\beta_2 - \beta_1 - \alpha_2 + \alpha_1)) = \sigma \tag{71}$$

where l is an integer and $\sigma = \pm 1$. Note that (70) is invariant under the transformation $\theta_1 \to -\theta_1$, $\beta_1 \to \beta_1 + \pi$. Hence we can always choose $\sigma = 1$.

Let $\delta = \beta_1 - \beta_3 - \alpha_1 + \alpha_3$ and $\rho = \cos^2\theta_1$. Substituting (70) into the second equation of (68), we obtain

$$R_3^2 \cos^2\theta_3 = G - (R_1^2 - R_2^2)\rho. \tag{72}$$

The third equation of (68) gives

$$\begin{aligned}\lambda\cos^2\theta_1 + \mu\sin^2\theta_1 + R_3^2(\cos^2\theta_1\cos^2\theta_3 + \sin^2\theta_1\sin^2\theta_3)& \\ +2R_3^2\cos\theta_1\sin\theta_1\cos\theta_3\sin\theta_3\cos\delta = K&\end{aligned} \tag{73}$$

from which δ can be solved as a function of ρ.

From equation (19), we have

$$\begin{aligned}
\theta_{1,x} &= R_3^2\cos\theta_3\sin\theta_3\sin\delta \\
\theta_{3,x} &= (R_2^2 - R_1^2)\cos\theta_1\sin\theta_1\sin\delta \\
\alpha_{1,x} &= \lambda + R_3^2\cos^2\theta_3 + R_3^2\cos\theta_3\sin\theta_3\tan\theta_1\cos\delta \\
\alpha_{2,x} &= \lambda + R_3^2\cos^2\theta_3 - R_3^2\cos\theta_3\sin\theta_3\cot\theta_1\cos\delta \\
\alpha_{3,x} &= R_1^2\cos^2\theta_1 + R_2^2\sin^2\theta_1 + (R_1^2 - R_2^2)\cos\theta_1\sin\theta_1\tan\theta_3\cos\delta \\
\beta_{1,x} &= \mu + R_3^2\sin^2\theta_3 + R_3^2\cos\theta_3\sin\theta_3\cot\theta_1\cos\delta \\
\beta_{2,x} &= \mu + R_3^2\sin^2\theta_3 - R_3^2\cos\theta_3\sin\theta_3\tan\theta_1\cos\delta \\
\beta_{3,x} &= R_1^2\sin^2\theta_1 + R_2^2\cos^2\theta_1 + (R_1^2 - R_2^2)\cos\theta_1\sin\theta_1\cot\theta_3\cos\delta.
\end{aligned} \tag{74}$$

The first equation of the above system leads to

$$\rho_x = -2R_3^2\cos\theta_1\sin\theta_1\cos\theta_3\sin\theta_3\sin\delta. \tag{75}$$

Solving $\cos\delta$ from (73), we obtain the equation of ρ:

$$\rho_x = -\sqrt{P(\rho)} \tag{76}$$

where

$$\begin{aligned}P(\rho) = 4b(\lambda - \mu)\rho^3 &+ \big(4(\mu - \lambda)G + 4b(\mu - K) - (\lambda - \mu + b - a)^2\big)\rho^2 \\ &+\big(4(K - \mu)G + 2(\lambda - \mu + b - a)(K + G - \mu - a)\big)\rho \\ &-(K + G - \mu - a)^2\end{aligned} \tag{77}$$

which is a cubic polynomial,

$$a = R_3^2 \qquad b = R_1^2 - R_2^2. \tag{78}$$

Suppose $b > 0$, $\lambda > \mu$ and P has three different real roots $\rho_1 < \rho_2 < \rho_3$. Moreover, suppose

$$\begin{aligned}&K + G - \mu - a \neq 0 \qquad K - G - \lambda + b \neq 0 \\ &\max\left(0, \frac{G - a}{b}\right) < \min\left(1, \frac{G}{b}\right).\end{aligned} \tag{79}$$

Then the solution ρ can be expressed by elliptic functions of x. Let $\rho = \rho_1 + (\rho_2 - \rho_1)\omega^2$, then

$$\omega_x = \pm p\sqrt{(1 - \omega^2)(1 - k^2\omega^2)} \tag{80}$$

where

$$k = \sqrt{(\rho_2 - \rho_1)/(\rho_3 - \rho_1)} \qquad p = \sqrt{b(\lambda - \mu)(\rho_3 - \rho_1)}. \tag{81}$$

Hence $\omega = \pm\,\mathrm{sn}(p(x - \widetilde{x}_0))$,

$$\rho = \rho_1 + (\rho_2 - \rho_1)\,\mathrm{sn}^2(p(x - \widetilde{x}_0)) \tag{82}$$

where $\widetilde{x}_0$ is independent of x, but may depend on y and t. ρ is a periodic function of x.

Remark 2. Since ρ_j is a root of P, (77) leads to

$$4\rho_j(1-\rho_j)(G - b\rho_j)(a - G + b\rho_j) = \big(K - \lambda\rho_j - \mu(1-\rho_j) - \rho_j(G - b\rho_j) - (1-\rho_j)(a - G + b\rho_j)\big)^2 \geqslant 0. \tag{83}$$

(This is actually equivalent to (73).) Hence, $\max(0, \frac{G-a}{b}) \leqslant \rho_1 < \rho_2 \leqslant \min(1, \frac{G}{b})$ holds if there is a solution locally, since $0 \leqslant \cos^2\theta_1 \leqslant 1$ and $0 \leqslant \cos^2\theta_3 \leqslant 1$ should be satisfied. This also guarantees that the solution is global because $\rho_1 \leqslant \rho \leqslant \rho_2$. Moreover, under the assumption $K + G - \mu - a \neq 0$ and $K - G - \lambda + b \neq 0$, $P(0) \neq 0$, $P(1) \neq 0$. Hence $0 < \rho_1 < \rho_2 < 1$ and $0 < \rho < 1$.

Remark 3. Using the formulae

$$1 - k^2\,\mathrm{sn}^2(\xi) = \mathrm{dn}^2(\xi) = \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\ln\Theta(\xi) + \widetilde{C} \tag{84}$$

where $\widetilde{C}$ is a certain constant, the previous solution ρ can be expressed as a Θ function.

In order to compute the y and t equations, we first write down the expressions for u, f and g. They are

$$\begin{aligned}
u &= \frac{2\mathrm{i}R_1R_2}{R_1^2 - R_2^2}\exp(\mathrm{i}(\alpha_1 - \alpha_2))\big((K - \lambda - R_3^2\cos^2\theta_3)\cot\theta_1 - R_3^2\cos\theta_3\sin\theta_3\exp(\mathrm{i}\delta)\big) \\
&= \frac{2\mathrm{i}R_1R_2}{R_1^2 - R_2^2}\exp(\mathrm{i}(\alpha_1 - \alpha_2))\big((\mu + R_3^2\sin^2\theta_3 - K)\tan\theta_1 \\
&\quad + R_3^2\cos\theta_3\sin\theta_3\exp(-\mathrm{i}\delta)\big)
\end{aligned} \tag{85}$$

$$\begin{aligned}
|u|^2 &= \frac{4R_1^2R_2^2}{(R_1^2 - R_2^2)^2}\big(K(R_3^2 + \lambda + \mu - K) - \lambda\mu - \lambda R_3^2\sin^2\theta_3 - \mu R_3^2\cos^2\theta_3\big) \\
f &= R_1R_3\exp(\mathrm{i}(\alpha_1 - \alpha_3))\big(\cos\theta_3\cos\theta_1 + \sin\theta_3\sin\theta_1\exp(\mathrm{i}\delta)\big) \\
g &= R_2R_3\exp(\mathrm{i}(\alpha_2 - \alpha_3))\big(-\cos\theta_3\sin\theta_1 + \sin\theta_3\cos\theta_1\exp(\mathrm{i}\delta)\big).
\end{aligned}$$

With the help of MAPLE, equations (20) and (21) are reduced to the following simple equations:

$$\begin{aligned}
&\theta_{1,y} = \gamma_1\theta_{1,x} \qquad \theta_{3,y} = \gamma_1\theta_{3,x} \qquad \theta_{1,t} = \gamma_2\theta_{1,x} \qquad \theta_{3,t} = \gamma_2\theta_{3,x} \\
&\alpha_{1,y} = \gamma_1\alpha_{1,x} - \frac{2R_2^2K}{R_1^2 - R_2^2} \qquad \alpha_{2,y} = \gamma_1\alpha_{2,x} - \frac{2R_1^2(R_3^2 + \lambda + \mu - K)}{R_1^2 - R_2^2} \\
&\alpha_{3,y} = \gamma_1\alpha_{3,x} - \frac{2R_2^2R_2^2}{R_1^2 - R_2^2} \\
&\alpha_{1,t} = \gamma_2\alpha_{1,x} + C_{12} \qquad \alpha_{2,t} = \gamma_2\alpha_{2,x} - C_{21} \qquad \alpha_{3,t} = \gamma_2\alpha_{3,x} + C_3 \\
&(\beta_j - \alpha_j)_y = \gamma_1(\beta_j - \alpha_j)_x \qquad (\beta_j - \alpha_j)_t = \gamma_2(\beta_j - \alpha_j)_x + C_0 \quad (j = 1, 2, 3)
\end{aligned} \tag{86}$$

where the constants γ_1, γ_2, C_0, C_{12}, C_{21} and C_3 are given by

$$\begin{aligned}\gamma_1 &= \frac{R_1^2 + R_2^2}{R_1^2 - R_2^2}\\ \gamma_2 &= \frac{2(R_1^4 + R_2^4 - (R_1^2 + R_2^2)(R_3^2 + \lambda + \mu))}{R_1^2 - R_2^2} - \frac{4R_1^2R_2^2(R_3^2 + \lambda + \mu - 2K)}{(R_1^2 - R_2^2)^2}\end{aligned} \tag{87}$$

$$C_0 = \frac{2(\lambda - \mu)(R_1^4 + R_2^4)}{R_1^2 - R_2^2}$$

$$\begin{aligned}C_{ij} = \frac{2}{(R_1^2 - R_2^2)^2}&\Bigg((R_j^4 + 2R_i^2R_j^2 - R_i^4)\Bigg((\lambda - \mu)\Bigg(G - \frac{R_i^2 - R_j^2}{2}\Bigg)\\ &+(R_i^2 - R_j^2)\Bigg(\frac{\lambda + \mu}{2} - K_i\Bigg) - \lambda\mu - \lambda R_3^2\Bigg) + 4R_1^2R_2^2K(R_3^2 + \lambda + \mu - K)\\ &-\lambda(R_i^2 - R_j^2)(R_1^4 + R_2^4) - 2R_j^4K_i^2\Bigg) \qquad (i, j) = (1, 2) \text{ or } (2, 1)\end{aligned} \tag{88}$$

$$K_1 = K \qquad K_2 = R_3^2 + \lambda + \mu - K$$

$$\begin{aligned}C_3 = \frac{2}{(R_1^2 - R_2^2)^2}&\big(R_2^2(R_2^4 + 3R_1^4 - R_1^2R_2^2)(\lambda + R_3^2) + R_1^2(R_1^4 - R_2^4 + R_1^2R_2^2)\mu\\ &-2(R_1^2 + R_2^2)(R_1^4 + R_2^4)K - 2R_1^2R_2^2(R_1^4 - R_2^4)\big).\end{aligned}$$

Hence

$$\rho = \rho_1 + (\rho_2 - \rho_1)\,\mathrm{sn}^2(p(x + \gamma_1 y + \gamma_2 t - x_0)) \tag{89}$$

where x_0 is an arbitrary constant, p is given by (81) and the parameter of the function sn is k given by (81).

The solutions of the DSI equation are

$$\begin{aligned}u = \pm\frac{\mathrm{i}R_1R_2}{R_1^2 - R_2^2}\,&\frac{1}{\rho(\xi)\sqrt{1 - \rho(\xi)}}\\ &\times\Big((2K - \lambda - \mu - a + b)\rho(\xi) + (\mu + a - K - G) - \mathrm{i}\sqrt{P(\rho(\xi))}\Big)\\ &\times\exp\left(\mathrm{i}\int Q(\rho(\xi))\,\mathrm{d}\xi + \mathrm{i}\alpha(x - \gamma_1 y + \gamma_2 t) + \mathrm{i}(C_{12} + C_{21})t\right)\end{aligned} \tag{90}$$

and

$$\begin{aligned}v_1 &= 2R_1^2((\mu - \lambda)\rho(\xi) + K - \mu)\\ v_2 &= 2R_2^2((\lambda - \mu)\rho(\xi) + K - \mu + a)\end{aligned} \tag{91}$$

where

$$\begin{aligned}\xi &= x + \gamma_1 y + \gamma_2 t - x_0\\ \alpha &= \frac{R_2^2K - R_1^2(R_3^2 + \lambda + \mu - K)}{R_1^2 + R_2^2}\\ Q(\rho) &= \frac{2b\rho^2 + (\mu - b - 2G)\rho + (K + G - \mu - a)}{2\rho(1 - \rho)}\\ \rho(\xi) &= \rho_1 + (\rho_2 - \rho_1)\,\mathrm{sn}^2(p\xi)\end{aligned} \tag{92}$$

and the parameter k of the function sn is given by (81).

u has no singularity when (79) holds because in this case $0 < \rho < 1$.

Suppose the minimal positive period of the function sn with parameter k is $T(k)$ and

$$A = \frac{p}{T(k)} \int_0^{T(k)/p} Q(\rho(\xi))\, \mathrm{d}\xi. \tag{93}$$

Then we have the following properties of the solutions:

(1) u is a double periodic function on the (x, y)-plane. The period for $x + \gamma_1 y$ is $T(k)/p$, while the period for $(A + \alpha)x + (A - \alpha)\gamma_1 y$ is 2π.
(2) u is periodic with respect to t if and only if

$$\frac{2\pi p}{(C_{12} + C_{21} + A\gamma_2 + \alpha\gamma_2)T(k)}$$

is a rational number.
(3) $|u|^2$, v_1 and v_2 are periodic functions of $x + \gamma_1 y + \gamma_2 t$ only, and they extend constantly in a transversal direction on the (x, y)-plane.
(4) The phase of u depends not only on the linear functions of x, y and t, but also on an sn function of $x + \gamma_1 y + \gamma_2 t$. This can be obtained from (90) and

$$(\arg u)_x = \mathrm{Re}\, \frac{1}{\mathrm{i}} \frac{u_x}{u} = \mathrm{Re}\, \frac{2 f \bar{g}}{\mathrm{i} u} \neq \text{constant} \tag{94}$$

by using (85) and a tedious computation.

It is still interesting to solve more general periodic solutions using this method.

Acknowledgments

This work was supported by the Chinese National Research Project 'Nonlinear Science', the City University of Hong Kong (grant no 7001041), the Research Grants Council of Hong Kong (grant nos 9040395, 9040466), the Doctoral Program Foundation and the Key Project for Young Teachers of the Ministry of Education of China, the National Natural Science Foundation of China (Project 19801031) and the Special Grant of Excellent PhD Thesis. The authors thank the referee for helpful suggestions. The first author (ZXZ) is also grateful to the Department of Mathematics of the City University of Hong Kong for their hospitality.

References

[1] Boiti M, Konopelchenko B G and Pempinelli F 1985 Bäcklund transformations via Gauge transformations in 1 + 2 dimensions *Inverse Problems* **1** 33
[2] Cao C W 1990 Nonlinearization of the Lax system for AKNS hierarchy *Sci. China* A **33** 528
[3] Cao C W, Wu Y T and Geng X G 1999 Relation between the Kadometsev–Petviashvili equation and the confocal involutive system *J. Math. Phys.* **40** 3948
[4] Cheng Y and Li Y S 1991 The constraint of the KP equation and its special solutions *Phys. Lett.* A **157** 22
[5] Cushman R H and Bates L M 1997 *Global Aspects of Classical Integrable Systems* (Boston, MA: Birkhäuser)
[6] Davey A and Stewartson K 1974 On three-dimensional packets of surface waves *Proc. R. Soc.* A **338** 101
[7] Fokas A S and Santini P M 1989 Coherent structures in multidimensions *Phys. Rev. Lett.* **63** 1329
[8] Konopelchenko B, Sidorenko J and Strampp W 1991 (1 + 1)-dimensional integrable systems as symmetry constraints of (2 + 1) dimensional systems *Phys. Lett.* A **157** 17
[9] Ma W X, Fuchsteiner B and Oevel W 1996 A 3 × 3 matrix spectral problem for AKNS hierarchy and its binary nonlinearization *Physica* A **233** 331
[10] Ma W X, Ding Q, Zhang W G and Lu B Q 1996 Binary nonlinearization of Lax pairs of Kaup–Newell soliton hierarchy *Nuovo Cimento* B **111** 1135

[11] Malanyuk T M 1994 Finite-gap solutions of the Davey–Stewartson equations *J. Nonlinear Sci.* **4** 1
[12] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[13] Pempinelli F 1995 New features of soliton dynamics in 2 + 1 dimensions *Acta Appl. Math.* **39** 445
[14] Wu Y T and Geng X G 1999 A finite-dimensional integrable system associated with the three-wave interaction equations *J. Math. Phys.* **40** 3409
[15] Zeng Y B and Lin R L 1998 Families of dynamical r-matrices and Jacobi inverse problems for nonlinear evolution equations *J. Math. Phys.* **39** 5964
[16] Zhou R G 1997 The finite-band solution of the Jaulent–Miodek equation *J. Math. Phys.* **38** 2535
[17] Zhou Z X 1996 Soliton solutions for some equations in 1 + 2 dimensional hyperbolic $su(N)$ AKNS system *Inverse Problems* **12** 89
[18] Zhou Z X 1998 Localized solitons of hyperbolic $su(N)$ AKNS system *Inverse Problems* **14** 1371