Upper bounds on the number of eigenvalues of stationary Schrödinger operators

Dai Shi[a)]

School of Mathematical Sciences, Fudan University, 220 Handan Road, Shanghai 200433, China and ICME, Stanford University, Stanford, California 94305, USA

(Received 10 May 2010; accepted 15 June 2010; published online 30 August 2010)

In this article we provide several upper bounds on the number of eigenvalues, respectively, for Schrödinger operators of one-dimensional case, central potential case, and the case with point-interactions. Compared with Bargman's and Calogero's results, the new bounds can give finite estimates when the potential function has a heavy tail. Besides, a generalization of Calogero's bound is given, which can be applied to bell shaped potential functions rather than merely monotonous ones. With respect to Schrödinger operators containing Dirac functions, some bounds are also established. © *2010 American Institute of Physics.* [doi:10.1063/1.3461876]

I. INTRODUCTION

A lot of work has been done to estimate the number of eigenvalues of Schrödinger operators. A one-dimensional stationary Schrödinger equation can be presented as the following spectrum problem:

$$H\psi := \left(-\frac{d^2}{dx^2} - u\right)\psi = \lambda\psi, \tag{1.1}$$

where $u(x)$ is piecewise continuous and satisfies

$$\lim_{x\to\infty} x^2 u(x) = 0. \tag{1.2}$$

In (1.1), we call H the Schrödinger operator, λ the binding energy, and $\psi(x)$ the bound state.

For a Schrödinger operator H, we have the following properties (see pp. 105–110 of Ref. 1 and pp. 87 and 88 of Ref. 2).

Proposition A:

(1) *H is a self-adjoint operator with respect to the Hilbert space L^2 and the standard inner product.*

(2) *H is uniformly bounded below, i.e., there exists a constant $C>0$, such that $\forall \psi$, we have $\langle\psi, H\psi\rangle \geq -C\langle\psi,\psi\rangle$.*

(3) *The continuous spectrum of H is $[0,+\infty)$, and the eigenvalues of H form a finite subset of $[-C,0)$.*

(4) *For each bound state of H, the degeneracy is 1.*

Moreover, the following theorem provides an excellent starting point to estimate the number of eigenvalues of the Schrödinger operator H (see pp. 110 and 111 of Ref. 1 and 12). This is often referred to as the Sturm oscillation theorem. For simplicity, for any function ψ we define $N(\psi)$ to be the number of zeros of ψ.

[a)]Electronic mail: shidai@stanford.edu.

Theorem B: *In addition, if the potential function $u(x)$ satisfies*

$$\int_{-\infty}^{+\infty} |x||u(x)|dx < +\infty, \tag{1.3}$$

then the following two statements hold.

(1) If $\psi_1(x)$, $\psi_2(x)$ are the eigenfunctions corresponding to eigenvalues $\lambda_1 < \lambda_2 < 0$ of H, then $N(\psi_2(x)) \geq N(\psi_1(x)) + 1$.
(2) If $\psi_1(x)$ is the eigenfunction corresponding to the eigenvalue $\lambda_1 < 0$, whereas $\psi_2(x)$ is the solution of (1.1) with $\lambda_2 = 0$, then the same result holds as in (1).

By Theorem B, if $\psi(x)$ is the solution of (1.1) with $\lambda = 0$, then the number of zeros of $\psi(x)$ is greater than or equal to the number of eigenvalues of H. Thus, we can reduce estimating the number of eigenvalues of H to estimating $N(\psi(x))$.

Now we generalize the problem to the three-dimensional case, where the spectrum problem becomes

$$H\psi := (-\triangle - u)\psi = \lambda\psi, \tag{1.4}$$

where $u(x,y,z) \in C_1(\mathbb{R}^3)$ and satisfies

$$\lim_{r\to\infty} r^2 u(x,y,z) = 0. \tag{1.5}$$

In this case it becomes too complex for us to analyze the spectrum. But things get better in the central potential case. That is, $u(x,y,z)$ is the function of only the radius $r = \sqrt{x^2+y^2+z^2}$.

Now separation of variables can be applied. If we denote $\psi(x,y,z) = r^{-1} f(r) Y(\theta,\phi)$, we can get a spectrum problem with respect to r,

$$\left(-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{r^2} - u(r)\right) f(r) = \lambda f(r). \tag{1.6}$$

Here ℓ is called the angular momentum quantum number. $f(r)$ satisfies $f(0)=0$ and $f(r)\to 0$ as $r \to 0$, which represents the bound state. For the central potential case, some necessary and sufficient conditions for the existence of bound states are obtained in Refs. 3 and 4.

For the notational consistency of the one-dimensional case, we denote $f(r)$ as $\psi(r)$. Let $\psi_\ell(r;\lambda)$ be the solution of (1.6) satisfying $\psi_\ell(0;\lambda)=0$ and $\lim_{r\to 0+} r^{-\ell-1}\psi_\ell(r;\lambda) = 1$. It can be shown that such $\psi_\ell(r;\lambda)$ uniquely exists. Denote $N(\psi_\ell(r;\lambda))$ as the number of zeros of $\psi_\ell(r;\lambda)$ on $\mathbb{R}^+$. We have the following theorem in Ref. 2, pp. 90–93.

Theorem C: *If $u(r) \in C_0^\infty$, then the following statements hold.*

(1) If $\lambda < 0$, then $N(\psi_\ell(r;\lambda)) < \infty$.
(2) $N(\psi_\ell(r;\lambda))$ is monotonously increasing with respect to λ, while monotonously decreasing with respect to ℓ.
(3) When $\lambda < 0$, $N(\psi_\ell(r;\lambda))$ equals the number of eigenvalues less than λ. In particular, $N(\psi_\ell(r;\lambda))$ equals the overall number of eigenvalues.

With Theorem C, the estimation of (1.6) can be reduced to calculating $N(\psi_\ell)$ which is the number of zeros on $\mathbb{R}^+$ of the solution of

$$\psi_\ell''(r) = \left(-u(r) + \frac{\ell(\ell+1)}{r^2}\right)\psi_\ell(r) \tag{1.7}$$

with the initial condition $\psi_\ell(0)=0$.

Several upper bounds have been obtained in terms of the central potential case in Refs. 5–8. We list some previous estimates below.

Theorem D:

(1) Bargmann's estimate (see Ref. 9),

$$N(\psi_\ell) \le \frac{1}{2\ell+1}\int_0^{+\infty} r|u(r)|dr.$$

(2) Calogero's estimate (see Ref. 5). If $u(r)$ is monotonously decreasing,

$$N(\psi_\ell) \le \frac{2}{\pi}\int_0^{+\infty}\sqrt{|u(r)|}dr.$$

(3) Glaser–Grosse–Martin–Thirring (GGMT) estimate (see Ref. 11). $\forall p \ge 1$,

$$N(\psi_\ell) \le \frac{c_p}{(2\ell+1)^{2p-1}}\int_0^{+\infty} r^{2p-1}|u(r)|^p dr, \quad \textit{where}\ \ c_p = \frac{(p-1)^{p-1}\Gamma(2p)}{p^p\Gamma(p)^2}.$$

These three estimates can be translated to the one-dimensional case with little change. In Sec. II, we focus on the one-dimensional case and point out pros and cons of various estimates based on examples. In Sec. III, we turn to analyzing the central potential case.

In Sec. IV, we introduce Dirac's function $\delta(x)$ into the potential function $u(x)$ and give a few estimates. Some upper bounds of negative eigenvalues of a Schrödinger operator with only point-interactions are provided in Refs. 10 and 6. But in this paper our potential function is a mixture of δ functions and continuous functions. Below we list several main results of this article.

For the one-dimensional case, we denote $M_0 = \sup u(x)$, $M_2 = \sup x^2 u(x)$ and G_1 as the real number satisfying $\forall |x| > G_1$, $x^2 u(x) < 1/4$. Suppose $\psi(x)$ satisfies $\psi'' = -u\psi$, then the theorem below holds.

Theorem 2.6:

(1) If $M_1 > 1/4$ and $\sqrt{4M_2-1} < G_1\sqrt{M_0}$,

$$N(\psi) \le \frac{2\sqrt{4M_2-1}}{\pi}(1+\ln(G_1\sqrt{M_0}) - \ln\sqrt{4M_2-1}) + 5. \tag{1.8}$$

(2) If $M_2 > 1/4$ and $\sqrt{4M_2-1} \ge G_1\sqrt{M_0}$, $N(\psi) \le 2G_1\sqrt{M_0}/\pi + 3$.
(3) If $M_2 \le 1/4$, $N(\psi) \le 1$.

Moreover, we generalize Calogero's results.

Theorem 2.7:

$$N(\psi) \le 3 + \frac{2}{\pi}\int_{-\infty}^{+\infty}\sqrt{u(x)}dx + \frac{1}{\pi}\int_0^{+\infty}\frac{\max\{u'(x),0\}}{2u(x)}dx + \frac{1}{\pi}\int_{-\infty}^{0}\frac{\max\{-u'(x),0\}}{2u(x)}dx.$$

For the central potential case, we still define M_0, M_2 the same as before, and G_ℓ as the real number satisfying $\forall |x| > G_\ell$, $x^2u(x) < (\ell+1/2)^2$. Then the theorem below holds.

Theorem 3.3: *We have the following estimates on $N(\psi_\ell)$ which is the number of zeros of the solution satisfying (1.7).*

(1) If $\ell \ge \ell_0$, $N(\psi_\ell) = 0$.
(2) If $\ell < \ell_0$, we split the problem to the following subcases.

(2a) If $\sqrt{4M_2-(2\ell+1)^2} < G_\ell\sqrt{M_0}$,

$$N(\psi_\ell) < \frac{\sqrt{4M_2-(2\ell+1)^2}}{\pi}(1+\ln(G_1\sqrt{M_0}) - \ln\sqrt{4M_2-(2\ell+1)^2}) + 2.$$

(2b) If $\sqrt{4M_2-(2\ell+1)^2} \geq G_\ell\sqrt{M_0}$, $N(\psi_\ell) < G_\ell\sqrt{M_0}/\pi+1$.

For the case with generalized potential functions, the equation becomes $\psi''(x) = -u(x)\psi(x) - \Sigma_{i=1}^{N} c_i\delta(x-\gamma_i)\psi(x)$. We also have estimates on the number of zeros denoted by $N(\psi)$.

Theorem 4.5:

$$N(\psi) \leq \frac{2\sqrt{M_0}G}{\pi} + \frac{2}{\pi}\sum_{i=1}^{N}\arctan\frac{c_i}{2\sqrt{M_0}} + 3.$$

II. ESTIMATES FOR ONE-DIMENSIONAL CASE

In this section, we will estimate the number of eigenvalues of (1.1). From the discussion in Sec. I, we need only to estimate the number of zeros of the solution of

$$\psi''(x) = -u(x)\psi(x). \tag{2.1}$$

We have the following lemma, whose proof we omit here. This is often referred to as the Sturm's comparison theorem (see Ref. 12).

Lemma 2.1: Suppose $\psi_{(x)}$ *and* $\psi_2(x)$ *satisfy*

$$\psi_1''(x) = -u_1(x)\psi_1(x), \tag{2.2}$$

$$\psi_2''(x) = -u_2(x)\psi_2(x), \tag{2.3}$$

and $u_1(x) \leq u_2(x)$. *If we denote* α, β *as the two successive zeros of* $\psi_1(x)$ *and set* $\psi_2(x)$ *to have the same value and the same first order derivative as* $\psi_1(x)$ *at* $x=\alpha$, *then* $\exists\gamma\in(\alpha,\beta]$, *such that* $\psi_2(\gamma)=0$.

Lemma 2.1 provides some inspiration to the problem. If we want to estimate the number of zeros of $\psi_1(x)$ satisfying (2.2), we just need to construct the function $\psi_2(x)$ satisfying (2.3), where $u_2(x)$ should be carefully chosen such that $u_2(x) \geq u_1(x)$ and $u_2(x)$ is simple enough for us to give an upper bound of the number of zeros of $\psi_2(x)$. Thus, by Lemma 2.1, the same bound holds for $\psi_1(x)$. This technique is widely used in the proofs of the following a few theorems.

For simplicity in stating our results, we define $M_0 = \sup_{x\in\mathbb{R}} u(x)$. Then the following lemma holds.

Lemma 2.2: In Eq. (2.1), suppose $\psi(x)$ *has two successive zeros* $\alpha<\beta$ *in* $[a,b]$, *then* $\beta-\alpha \geq \pi/\sqrt{M_0}$. *We further deduce that the number of zeros of* $\psi(x)$ *in* $[a,b]$ *is at most* $\sqrt{M_0}(b-a)/\pi+1$.

Proof: Define $\psi_1(x)$ to satisfy $\psi_1''(x) = -M_0\psi_1(x)$ with initial conditions $\psi_1(\alpha)=0$ and $\psi_1'(\alpha) = \psi'(\alpha)$. Then through solving the equation, $\psi_1(x) = \psi'(\alpha)\sin(\sqrt{M_0}(x-\alpha))/\sqrt{M_0}$. The distance between α and the next zero of $\psi_1(x)$ is exactly $\pi/\sqrt{M_0}$. From Lemma 2.1, we have $\beta-\alpha \geq \pi/\sqrt{M_0}$. ■

According to asymptotic behavior (1.5) of $u(x)$, there exist $G_1>0$, such that $\forall|x| > G_1, x^2u(x) \leq 1/4$. We have the following lemma.

Lemma 2.3: If $\psi(x)$ *satisfies Eq. (2.1), then* $\psi(x)$ *has at most one zero in* $[G_1,+\infty)$. *Also, it has at most one zero in* $(-\infty,-G_1]$.

Proof: We prove the first part of the lemma by contradiction. Suppose there exist $\beta > \alpha \geq G_1$, such that $\psi(\alpha)=\psi(\beta)=0$, then we define $\psi_1(x)$ to be the solution of $x^2\psi_1''(x) = -\psi_1(x)/4$ with initial conditions $\psi_1(\alpha)=0$ and $\psi_1'(\alpha)=\psi'(\alpha)$. By solving the equation, we have $\psi_1(x)=\sqrt{x}(C_1+C_2\log x)$, where C_1, C_2 are two constants satisfying

$$C_1 + C_2 \log \alpha = 0, \tag{2.4}$$

$$\frac{d}{dx}(\sqrt{x}(C_1 + C_2 \log x))\big|_{x=\alpha} = \psi'(\alpha). \tag{2.5}$$

Note that $C_2 \neq 0$ [otherwise $C_1=0$ followed by (2.4), which means $\psi_1(x)\equiv 0$], hence $y=C_1+C_2\log x$ is monotonously increasing with respect to x. Thus, $\forall x>\alpha, \psi_1(x)\neq 0$. But from Lemma 2.1, $\exists \gamma\in(\alpha,\beta]$, such that $\psi_1(\gamma)=0$. That is a contradiction. The same proof holds for $x\in(-\infty,-G_1]$. ■

If we use Lemma 2.2 and Lemma 2.3, to estimate the number of zeros in, respectively, $[-G_1,G_1]$ and $(-\infty,-G_1]\cup[G_1,+\infty)$, then we can get the first upper bound.

Theorem 2.1: *The number of zeros of the solution of (2.1), denoted by n, satisfies* $n\leq 2G_1\sqrt{M_0}/\pi+3$.

In order to give a stronger estimate, we establish the following lemma.

Lemma 2.4: In Eq. (2.1), if $\psi(x)$ *has two successive zeros* $0<\alpha<\beta$ *in* $[a,b]$ $(a>0)$*, and* $M_2=\max_{x\in[a,b]} x^2u(x)>1/4$*, then* $\beta\geq\alpha\exp(\pi/\sqrt{4M_2-1})$*. From this we deduce that the number of zeros of* $\psi(x)$ *in* $[a,b]$ *satisfies the estimate* $n\leq\sqrt{4M_2-1}\log(b/a)/\pi+1$.

Proof: Define $\psi_1(x)$ to be the solution of $x^2\psi_1''(x)=-M_2\psi_1(x)$ with the initial conditions $\psi_1(\alpha)=0$ and $\psi_1'(\alpha)=\psi'(\alpha)$. By solving the equation, we have $\psi_1(x)=\sqrt{Ax}\sin(\sqrt{4M_2-1}\log x+\theta)$, where A and θ are two constants. According to the initial condition, we have $\sin(\sqrt{4M_2-1}\log\alpha+\theta)=0$. Thus, the next zero of $\psi_1(x)$, denoted by γ, is $\gamma=\alpha\exp(\pi/\sqrt{4M_2-1})$. By Lemma 2.1, $\beta\geq\gamma=\alpha\exp(\pi/\sqrt{4M_2-1})$.

Now we consider the number of zeros of $\psi(x)$ in the interval $[a,b]$. Suppose it has n zeros $\alpha_1,\alpha_2,\ldots,\alpha_n$. Then according to what we have proven,

$$\alpha_n \geq \alpha_{n-1}\exp\frac{\pi}{\sqrt{4M_2-1}} \geq \alpha_{n-2}\exp\frac{2\pi}{\sqrt{4M_2-1}} \geq \ldots \geq \alpha_1\exp\frac{\pi(n-1)}{\sqrt{4M_2-1}} \geq a\exp\frac{\pi(n-1)}{\sqrt{4M_2-1}}. \tag{2.6}$$

Since in (2.6), $\alpha_n\leq b$, we get $n\leq\sqrt{4M_2-1}\log(b/a)/\pi+1$, which ends the proof. ■

Remark 2.1: If $M_2<1/4$*, then by Lemma 2.3,* $\psi(x)$ *has at most one zero in* $[a,b]$.

Remark 2.2: In Lemma 2.4, if the condition becomes $\alpha<\beta<0$ *and* $b<0$*, then the same result also holds.*

Next we consider the number of zeros of $\psi(x)$ satisfying (2.1). Suppose $\psi(x)$ has n_1 zeros in $[-G_1,-A]\cup[A,G_1]$ and n_2 zeros in $[-A,A]$, where A is an undetermined constant. If we change the definition of M_2 into $M_2=\sup_{x\in\mathbb{R}} x^2u(x)$ and assume $M_2>1/4$, then from Lemma 2.2 and Lemma 2.4, we have $n_1\leq 2\sqrt{4M_2-1}\log(G_1/A)/\pi+2$ and $n_2\leq 2\sqrt{M_0}A/\pi+1$. Adding them together, we get

$$n \leq \frac{2}{\pi}\sqrt{4M_2-1}\log\frac{G_1}{A} + \frac{2\sqrt{M_0}A}{\pi} + 3 := f(A).$$

Now we want to minimize $f(A)$. Set $f'(A)=-2\sqrt{4M_2-1}/(\pi A)+2\sqrt{M_0}/\pi=0$, we get $A=A_0 := \sqrt{4M_2-1}/\sqrt{M_0}$.

When $A_0<G_1$, or $\sqrt{4M_2-1}<G_1\sqrt{M_0}$, $f(A)$ reaches its minimum at A_0,

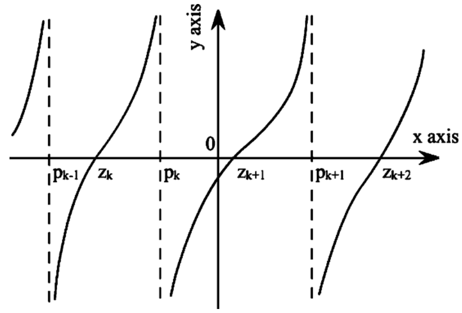

FIG. 1. Graph of $\phi(x)$.

$$f_{\min} = \frac{2\sqrt{4M_2 - 1}}{\pi}(1 + \log(G_1\sqrt{M_0}) - \log\sqrt{4M_2 - 1}) + 3.$$

When $\sqrt{4M_2-1} \geq G_1\sqrt{M_0}$, $f(A)$ reaches its minimum at G_1, in which case $f_{\min}=2G_1\sqrt{M_0}/\pi+3$. But we can use Lemma 2.2 to give a more accurate bound $2G_1\sqrt{M_0}/\pi+1$. When $M_2 \leq 1/4$, by Lemma 2.3, $\psi(x)$ has at most one zero in $\mathbb{R}$.

Combining these results with Lemma 2.3, we summarize our estimate into the following theorem.

Theorem 2.2: *The number of zeros of $\psi(x)$, denoted by $N(\psi)$, has the following estimate.*

(1) If $M_1 > 1/4$ and $\sqrt{4M_2-1} < G_1\sqrt{M_0}$,

$$N(\psi) \leq \frac{2\sqrt{4M_2 - 1}}{\pi}(1 + \log(G_1\sqrt{M_0}) - \log\sqrt{4M_2 - 1}) + 5. \tag{2.7}$$

(2) If $M_2 > 1/4$ and $\sqrt{4M_2-1} \geq G_1\sqrt{M_0}$, $N(\psi) \leq 2G_1\sqrt{M_0}/\pi+3$.
(3) If $M_2 \leq 1/4$, $N(\psi) \leq 1$.

Since the estimate in Theorem 2.2 is obtained by finding the minimum of $f(A)$ while the estimate in Theorem 2.1 is nothing but $f(G_1)$, we deduce that Theorem 2.2 is stronger than the previous one. The drawback is that it involves another parameter M_2.

Next we generalize Calogero's estimate. We assume the potential function $u(x) > 0$, $\forall x \in \mathbb{R}$, which is quite common in reality. We denote $\psi(x)$ as the solution of Eq. (2.1). Define $\phi(x) = \psi(x)/\psi'(x)$ when $\psi'(x) \neq 0$. According to the definition, $\phi(x)=0$ if and only if $\psi(x)=0$. Moreover, as x approaches to the zero of $\psi'(x)$, $\phi(x)$ approaches to infinity.

Since $\psi(x)=\phi(x)\psi'(x)$, we differentiate it at both sides and get

$$\psi'(x) = \phi'(x)\psi'(x) + \phi(x)\psi''(x) = \phi'(x)\psi'(x) - u(x)\phi(x)\psi(x) = \phi'(x)\psi'(x) - u(x)\phi^2(x)\psi'(x).$$

At the both sides the $\psi'(x)$ term cancels out and we get $\phi'(x)=1+u(x)\phi^2(x) > 0$. Thus, $\phi(x)$ is monotonous increasing in each interval where it is continuous. We can depict a graph of $\phi(x)$, as shown in Fig. 1.

Let z_i and p_i, respectively, be the zeros and the poles of $\phi(x)$. Thus, we have $\psi(z_i)=\psi'(p_i)=0$. What we want to estimate is the number of z_i's, which we denote by $n=N(\psi)$. Suppose the ordering of $\{z_i\}$ and $\{p_i\}$ is

$$\infty = p_0 < z_1 < p_1 < z_2 < p_2 < \dots < p_{n-1} < z_n < p_n = +\infty.$$

We further define k such that $p_k \leq 0 < p_{k+1}$. Let $\Phi(x) = \phi(x)\sqrt{u(x)}$. By differentiating $\Phi(x)$ we have

$$\begin{aligned}\Phi'(x) &= \phi'(x)\sqrt{u(x)} + \frac{\phi(x)u'(x)}{2\sqrt{u(x)}}\\ &= (1+\sqrt{u(x)}\phi^2(x))\sqrt{u(x)} + \frac{\phi(x)u'(x)}{2\sqrt{u(x)}}\\ &= (1+\Phi^2(x))\sqrt{u(x)} + \frac{\Phi(x)u'(x)}{2u(x)}.\end{aligned}$$

When $x \in [z_i, p_i]$, $\phi(x) \geq 0$. Thus,

$$\Phi'(x) \leq (1+\Phi^2(x))\sqrt{u(x)} + \frac{(1+\Phi^2(x))\max\{u'(x),0\}}{4u(x)},$$

$$\frac{\Phi'(x)}{1+\Phi^2(x)} \leq \sqrt{u(x)} + \frac{\max\{u'(x),0\}}{4u(x)}.$$

We integrate both sides from z_i to p_i and get

$$\frac{\pi}{2} = \int_{z_i}^{p_i} \frac{\Phi'(x)}{1+\Phi^2(x)}dx \leq \int_{z_i}^{p_i} \left(\sqrt{u(x)} + \frac{\max\{u'(x),0\}}{4u(x)}\right)dx.$$

Summing them together for all $i \geq k+2$ gives

$$n-k-2 \leq \frac{2}{\pi}\int_0^{+\infty} \left(\sqrt{u(x)} + \frac{\max\{u'(x),0\}}{4u(x)}\right)dx.$$

Following exactly the same procedure, we also have

$$k-1 \leq \frac{2}{\pi}\int_{-\infty}^{0} \left(\sqrt{u(x)} + \frac{\max\{u'(x),0\}}{4u(x)}\right)dx.$$

Summing the two inequalities together proves the following theorem.

Theorem 2.3: *The number of zeros, say n, of the solution of Eq. (2.1) has the estimate*

$$n \leq 3 + \frac{2}{\pi}\int_{-\infty}^{+\infty}\sqrt{u(x)}dx + \frac{1}{\pi}\int_0^{+\infty}\frac{\max\{u'(x),0\}}{2u(x)}dx + \frac{1}{\pi}\int_{-\infty}^{0}\frac{\max\{-u'(x),0\}}{2u(x)}dx.$$

Corollary 2.1: If $u(x)$ is monotonously decreasing when $x>0$ and monotonously increasing when $x<0$, we have the following estimate:

$$n \leq 3 + \frac{2}{\pi}\int_{-\infty}^{+\infty}\sqrt{u(x)}dx.$$

This is similar to Calogero's estimate.

Proof: If $u(x)$ is monotonously decreasing, $u'(x) \leq 0$ and hence $\max\{u'(x),0\}=0$. Similarly, $\max\{-u'(x),0\}=0$ when $u(x)$ is monotonously increasing. Substituting them in Theorem 2.3 completes the proof. ∎

Remark 2.3: In Theorem 2.3, $u'(x)/u(x) = (\log u(x))'$ has an extremely heavy tail such that we cannot guarantee the finiteness of the integral for almost all the common functions. Nevertheless, if there exists $X>0$ such that $u(x)$ is monotonously decreasing in $[X,+\infty)$ and monotonously increasing in $(-\infty, X]$, we can avoid the infinite integral and give a finite estimate. This is a generalization of Calogero's estimate.

We give a specific example to compare the various estimates. Consider the hyperbolic potential $u(x)=-k(k+1)/\cosh^2(x)$ where $k \in \mathbb{N}$. Existed results show that the corresponding Schrödinger operator H has k eigenvalues. According to Bargmann's estimate, the number of eigenvalues, say n, has an upper bound $O(k^2)$ and GGMT estimate provides an upper bound $O(k^{2p})$ in which $p \geq 1$.

Meanwhile, $M_2=\sup x^2u(x)=O(k^2)$, $M_0=\sup u(x)=O(k^2)$, and $G_1=O(\log k)$. Thus, in the formula given by Theorem 2.2, n has an upper bound $O(k \log k)$. Moreover, Corollary 2.1 gives an upper bound $O(k)$. The above results show that in this situation Corollary 2.1 provides the most accurate estimate, Theorem 2.2 stays the second, while Bargmann's and GGMT estimates are quite far away from the exact result.

Up until now, all the results are based on assumptions (1.5) and (1.3), in which (1.3) guarantees the decay of the Jost function corresponding to $\lambda=0$ and (1.5) guarantees the existence of G_1. However, we need only

$$\limsup_{|x|\to\infty} u(x) < 1/4 \tag{2.8}$$

to make sure that G_1 exists. At the end of this section, we consider the case that $u(x)$ only satisfies (2.8).

In this case, although (1.3) no longer holds, the first part of Theorem B keeps true. For the second part, we can prove the following weaker result.

Theorem 2.4: *Denote $\psi_1(x)$ to be the eigenfunction of (1.1) corresponding to the eigenvalue $\lambda_1<0$ and $\psi_2(x)$ to be the solution of (1.1) corresponding to $\lambda_2=0$. Then $N(\psi_2(x)) \geq N(\psi_1(x))$.*

The proof is exactly the same as in Lemma 2.1. Thus we omit it here.

Comparing between Theorem 2.4 and the second part of Theorem B, we deduce that we can get the estimate under condition (2.8) through adding the estimate in Theorem 2.2 by 1. Note that the formula in Theorem 2.2 is finite even if $u(x)$ only satisfies (2.8). We can get a finite estimate while the existed results in Theorem D are infinite. That means the Theorem 2.2 provides a better estimate than Theorem D at least when $u(x) \sim O(1/x^2)$ at infinity.

III. ESTIMATES FOR CENTRAL POTENTIAL CASE

The central potential case is quite similar to the one-dimensional case. According to Theorem C, we only need to estimate $N(\psi_\ell(r;\lambda))$ which is the number of zeros of $\psi_\ell(x)$ in $(0,+\infty)$. Many results in Sec. II can be translated.

First, we need the following lemma to "eliminate" the singularity of $x=0$.

Lemma 3.1: Suppose $\psi(x)$ is a nontrivial solution of (1.7) satisfying $\psi(0)=0$. Then $\psi(x)$ has only finite zeros in a neighborhood of $x=0$.

Proof: Since $\lim_{r\to 0+}(-u(r)+\ell(\ell+1)/r^2)=+\infty$, there exists $\delta>0$, $\forall r \in (0,\delta)$, $-u(r)+\ell(\ell+1)/r^2>0$. Now we prove $\psi(x)$ has no zeros in $(0,\delta)$. Suppose the contrary is true, i.e., there exists $\alpha \in (0,\delta)$ such that $\psi(\alpha)=0$. Then by Rolle's theorem, $A \coloneqq \{\beta \in (0,\alpha) : \psi'(\beta)=0\} \neq \emptyset$. Let $\beta_0=\max A$, then we have $\psi(\beta_0)\neq 0$ or otherwise $\psi(r)\equiv 0$. Assume, without losing generality, that $\psi(\beta_0)>0$, then $\psi''(\beta_0)>0$. Hence there exists $\gamma>\beta_0$, such that $\psi'(r)>\psi'(\beta_0)=0$, for all $r \in (\beta_0,\gamma)$. That means $\psi(r)$ is monotonously increasing in the interval. Moreover, $\psi(\alpha)=0$, thus there exists $\beta_1 \in (\beta_0,\alpha)$, such that $\psi(\beta_1)=\psi(\beta_0)$. Again, by Rolle's theorem, there further exists $\beta_2 \in (\beta_0,\beta_1)$ satisfying $\psi'(\beta_2)=0$. This contradicts with $\beta_0=\max A$. ∎

Remark 3.1: Lemma 3.1 shows that in the central potential case, all the zeros of $\psi(r)$ are isolated as in the one-dimensional case.

Lemma 2.1 can be translated to the following lemma.

Lemma 3.2: If $\psi_1(x)$ and $\psi_2(x)$ satisfy the equations

$$\psi_1''(r) = (-u_1(r) + \ell(\ell+1)/r^2)\psi_1(r),$$

$$\psi_2''(r) = (-u_2(r) + \ell(\ell+1)/r^2)\psi_2(r),$$

where $u_1(r)\le u_2(r)$. Denote α, β to be the two successive zeros of $\psi_1(r)$, then $\exists \gamma \in (\alpha,\beta]$, such that $\psi_2(\gamma)=0$.

We observe that if we regard $u(r)-\ell(\ell+1)/r^2$ as the new potential function, Lemma 2.2, Lemma 2.3, and Lemma 2.4 hold too in the central potential case. Now we estimate the number of zeros of $\psi_\ell(x)$ which is the solution to (1.7). The technique is exactly the same as what we have used in Sec. II. We define $M_0=\max_{r\ge 0} u(r)$, $M_2=\max_{r\ge 0} r^2u(r)$. We further define $\ell_0=\min\{\ell:\ell(\ell+1)\ge M_2-1/4\}$ or, equivalently, $\ell_0=\lfloor\sqrt{M_2-1/2}\rfloor+1$, where $\lfloor\cdot\rfloor$ is the floor function. Then $\forall \ell\ge\ell_0$, $\max\{r^2(u(r)-\ell(\ell+1)/r^2)\}=M_2-\ell(\ell+1)\le 1/4$.

Define $\psi_1(r)$ to satisfy $\psi_1(0)=0$ and $\psi_1''(r)=-\psi_1(r)/(4r^2)$. We have $\psi_1(r)=k\sqrt{r}$, where k is a constant. Since $\psi_1(r)$ has no zeros in $\mathbb{R}^2$, we deduce that, by Lemma 3.2, the same is true for $\psi_\ell(r)$. That is, $n_\ell=0$ for $\ell\ge\ell_0$.

Now we consider the case when $\ell<\ell_0$. Since $\limsup r^2u(r)<1/4$, there exists G_ℓ, $\forall r>G_\ell$, $r^2u(r)<1/4+\ell(\ell+1)$. Thus, according to Lemma 2.3, $\psi(r)$ has at most one zeros in $[G_\ell,+\infty)$.

On the other hand, there exists H_ℓ, $\forall r\in(0,H_\ell)$, $r^2u(r)<\ell(\ell+1)$. According the proof in Lemma 3.1, $\psi(r)$ has no zeros in $(0,H_\ell)$.

Next we estimate the number of zeros in $(0,G_\ell]$. Actually we just need to consider the interval $[H_\ell,G_\ell]$. Similar to what we have done in Sec. II, we split the interval into two parts $[H_\ell,A_\ell]$ and $[A_\ell,G_\ell]$. Define $n_{\ell 1}$ and $n_{\ell 2}$ to be the numbers of zeros of $\psi_\ell(r)$ in, respectively, $[H_\ell,A_\ell]$ and $[A_\ell,G_\ell]$. By Lemma 2.3, $n_{1\ell}<\sqrt{M_0}(A_\ell-H_\ell)/\pi+1<\sqrt{M_0}A_\ell/\pi+1$. By Lemma 2.4, $n_{2\ell}<\sqrt{4M_2-(2\ell+1)^2}\log(G_\ell/A_\ell)/\pi+1$. Thus,

$$n_\ell := n_{\ell 1}+n_{\ell 2} < \frac{1}{\pi}\sqrt{4M_2-(2\ell+1)^2}\log\frac{G_\ell}{A_\ell}+\frac{\sqrt{M_0}A_\ell}{\pi}+2 := f(A_\ell).$$

Now we want to choose A_ℓ to minimize $f(A_\ell)$. Through solving the equation $f'(A_\ell)=0$, we get $A_\ell=A_{\ell 0}:=\sqrt{4M_2-(2\ell+1)^2}/\sqrt{M_0}$.

If $A_{\ell 0}<G_\ell$, or $\sqrt{4M_2-(2\ell+1)^2}<G_\ell\sqrt{M_0}$, $f(A_\ell)$ reaches its minimum at $A_{\ell 0}$. Otherwise $f(A_\ell)$ reaches its minimum at G_ℓ. Combining the fact that $\psi_\ell(r)$ has at most one zero in $[G_\ell,+\infty)$, we have proved the following estimate on $N(\psi_\ell)$.

Theorem 3.1:

(1) If $\ell\ge\ell_0$, $N(\psi_\ell)=0$.
(2) If $\ell<\ell_0$, we split the problem to the following two subcases.

(2a) If $\sqrt{4M_2-(2\ell+1)^2}<G_\ell\sqrt{M_0}$,

$$N(\psi_\ell) < \frac{\sqrt{4M_2-(2\ell+1)^2}}{\pi}(1+\ln(G_1\sqrt{M_0})-\ln\sqrt{4M_2-(2\ell+1)^2})+2.$$

(2b) If $\sqrt{4M_2-(2\ell+1)^2}\ge G_\ell\sqrt{M_0}$, $N(\psi_\ell)<G_\ell\sqrt{M_0}/\pi+1$.

IV. ESTIMATE FOR POTENTIALS WITH DIRAC FUNCTIONS

In this section, we will consider the spectrum problem,

$$-\psi''(x)-u(x)\psi(x)-\sum_{i=1}^{N}c_i\delta(x-\gamma_i)\psi(x)=\lambda\psi(x), \tag{4.1}$$

where $c_i>0$ and γ_i are constants, $u(x)\in C^1$ and satisfies (1.5) and (1.2). $\delta(x)$ is the Dirac function. For this case, it can be shown that Theorem B also holds. Once again, the problem is reduced to estimating the number of zeros of the solution of

$$\psi''(x)=-u(x)\psi(x)-\sum_{i=1}^{N}c_i\delta(x-\gamma_i)\psi(x). \tag{4.2}$$

We observe that Lemma 2.1 can be translated as follows. Again, we omit the proof.

Lemma 4.1: If $\psi_1(x)$ and $\psi_2(x)$ satisfy

$$\psi_1''(x)=-u_1(x)\psi_1(x)-\sum_{i=1}^{N}c_i\delta(x-\gamma_i)\psi_1(x),$$

$$\psi_2''(x)=-u_2(x)\psi_2(x)-\sum_{i=1}^{N}c_i\delta(x-\gamma_i)\psi_2(x),$$

where $u_1(x)\le u_2(x)$. Thus, if α, β are the two successive zeros of $\psi_1(x)$, and we further suppose $\psi_2(x)$ has the same value and the first order derivative as $\psi_1(x)$ at $x=\alpha$, then $\exists\gamma\in(\alpha,\beta]$, such that $\psi_2(\gamma)=0$.

By Lemma 4.1, we can use the comparing technique as we used in the previous two chapters. First we give a naive estimate. Let $M_0=\max_{x\in\mathbb{R}}|u(x)|$. By (1.2), there exists $G>0$, such that $\forall 1\le i\le N, |\gamma_i|<G$ and $\forall x>G$, $x^2u(x)<1/4$. By Lemma 2.3, $\psi(x)$ has at most one zeros in either $(-\infty,-G]$ or $[G,+\infty)$. Now we estimate the number of zeros in $[-G,G]$.

Suppose $\{\gamma_i\}_{i=1}^{N}$ divide $[-G,G]$ into $N+1$ segments $[-G,G]=\cap_{i=1}^{N+1}[\gamma_{i-1},\gamma_i]$, where $\gamma_0=-G$ and $\gamma_{N+1}=G$. In each of the interval $[\gamma_{i-1},\gamma_i]$, $\psi(x)$ satisfies a Schrödinger equation without δ potentials. Then $\psi(x)$ has at most $\sqrt{M_0}(\gamma_i-\gamma_{i-1})/\pi+1$ zeros by Lemma 2.2. Summing them together for all $i=1,\ldots,N$ gives the upper bound of the number of zeros in $[-G,G]$, which is $2\sqrt{M_0}G/\pi+N+1$. Thus, we have the following theorem. Again we denote $N(\psi)$ as the number of zeros of $\psi(x)$ satisfying (4.2).

Theorem 4.1:

$$n\le\frac{2\sqrt{M_0}G}{\pi}+N+3. \tag{4.3}$$

Next we try to get a more accurate estimate of $N(\psi)$. Suppose $\psi(x)$ satisfies

$$\psi''(x)=-M_0\psi(x)-\sum_{i=1}^{N}c_i\delta(x-\gamma_i)\psi(x) \tag{4.4}$$

with the initial condition $\psi(-G)=0$. Then by Lemma 4.1 we can get an upper bound of $N(\psi)$ through calculating the number of zeros of $\psi(x)$ in $[-G,G]$. We can choose $\psi(x)$ to be $\sin(\sqrt{M_0}x+\sqrt{M_0}G)$ in $[\gamma_0,\gamma_1]$. We further suppose that $\psi(x)=A\sin(\sqrt{M_0}x+\sqrt{M_0}G+\theta_1)$ in $[\gamma_1,\gamma_2]$. Denote $C_1=c_1/\sqrt{M_0}$, $\alpha=\sqrt{M_0}\gamma_1+\sqrt{M_0}G$. From $\psi(\gamma_1-)=\psi(\gamma_1+)$ and $\psi'(\gamma_1+)-\psi'(\gamma_1-)=-c_1\psi(\gamma_1)$, we get $A\sin(\alpha+\theta_1)=\sin\alpha$ and $A\cos(\alpha+\theta_1)=\cos\alpha-C_1\sin\alpha$. Division between the two equations gives

$$\tan(\alpha+\theta_1) = \frac{\tan\alpha}{-C_1\tan\alpha+1}. \tag{4.5}$$

Solving for θ_1 from (4.5), we get

$$\tan\theta_1 = \frac{C_1\tan^2\alpha}{\tan^2\alpha - C_1\tan\alpha+1}. \tag{4.6}$$

Thus, we can choose

$$\theta_1 = \begin{cases} \pi+\arctan\left(\dfrac{C_1\tan^2\alpha}{\tan^2\alpha-C_1\tan\alpha+1}\right) & \text{if } C_1^2>4 \text{ and } \tan^2\alpha - C_1\tan\alpha+1<0 \\ \arctan\left(\dfrac{C_1\tan^2\alpha}{\tan^2\alpha-C_1\tan\alpha+1}\right) & \text{otherwise.} \end{cases}$$

We have the following two lemmas.

Lemma 4.2: Let k be the integer such that $k\pi\le\alpha<(k+1)\pi$. *Then we have* $k\pi\le\alpha+\theta_1<(k+1)\pi$.

Proof: We split the problem into two cases.

Case 1: $C_1^2-4\le 0$. In this case, the quadratic function $\tan^2\alpha - C_1\tan\alpha+1$ with respect to $\tan\alpha$ has discriminant ≤ 0. Thus, $\tan^2\alpha - C_1\tan\alpha+1\ge 0$. According to (4.6), $\theta_1\in(0,\pi/2]$.

If $\alpha\in[k\pi,k\pi+\pi/2)$, then $\theta_1+\alpha\in(k\pi,k\pi+\pi)$. The statement is true.

If $\alpha\in[k\pi+\pi/2,k\pi+\pi)$, then $\tan\alpha<0$ and $\theta_1+\alpha>k\pi$. Moreover, $\alpha+\theta_1\in[k\pi+\pi/2,k\pi+3\pi/2)$. According to (4.5), $\tan(\alpha+\theta_1)<0$. Thus, $\alpha+\theta_1\in[k\pi+\pi/2,k\pi+\pi)$. The statement is true.

Case 2: $C_1^2-4>0$. If $\tan^2\alpha-C_1\tan\alpha+1\ge 0$, then by (4.6), $\tan\theta_1>0$, $\theta_1\in(0,\pi/2]$. By exactly the same proof as in Case 1, the statement is true.

If $\tan^2\alpha-C_1\tan\alpha+1<0$, then $\theta_1\in(\pi/2,\pi)$ according to the formula to choose θ_1. In addition, $C_1\tan\alpha>\tan^2\alpha+1>0$ gives $\alpha\in(k\pi,k\pi+\pi/2)$. Thus, $\theta_1+\alpha\in(k\pi+\pi/2,k\pi+3\pi/2)$. But by (4.5), we have $\tan(\alpha+\theta_1)<0$. Hence $\theta_1+\alpha\in(k\pi+\pi/2,k\pi+\pi)$. The statement is true.

In summary, the proof is complete. ■

Lemma 4.3: We have the following estimate of θ_1:

$$\theta_1\le 2\arctan\frac{C_1}{2}.$$

Proof: We split the problem into three cases.

Case 1: $C_1^2\le 4$. In this case

$$\tan\theta_1 = \frac{C_1}{(\tan\alpha)^{-2}-C_1(\tan\alpha)^{-1}+1} = \frac{C_1}{((\tan\alpha)^{-1}-C_1/2)^2+1-C_1^2/4}>0. \tag{4.7}$$

The monotony range of the inverse proportion function gives $\tan\theta_1\le C_1/(1-C_1^2/4)$. Hence $\theta_1\le\arctan(C_1/(1-C_1^2/4))=2\arctan(C_1/2)$.

Case 2: $\tan^2\alpha-C_1\tan\alpha+1\ge 0$ and $C_1^2>4$. In this case, $\theta_1<\pi/2$, $2\arctan(C_1/2)>\pi/2$. The statement trivially holds.

Case 3: $\tan^2\alpha-C_1\tan\alpha+1<0$ and $C_1^2>4$. In this case, (4.7) and the monotony range of the inverse proportion function gives $\tan\theta_1\le C_1/(1-C_1^2/4)$. Hence $\theta_1\le\pi+\arctan(C_1/(1-C_1^2/4))=2\arctan(C_1/2)$.

In summary, the proof is complete. ■

We can regard $\psi(x)$ as a sine wave with respect to x. As x increases, so does the phase. We observe that x is the zero of $\psi(x)$ if and only if the corresponding phase is multiples of π. Due to the δ function, a sudden change by θ_1 in the phase occurs when $x=\gamma_1$. But by Lemma 4.2, the changed phase still stays in $[k\pi,(k+1)\pi)$.

Similarly, we can construct the recursive series

$$\alpha_1 = \sqrt{M_0}(\gamma_1 + G), \quad \alpha_{i+1} = \alpha_i + \sqrt{M_0}(\gamma_{i+1} - \gamma_i) + \theta_i, \quad 1 \le i \le N.$$

Moreover, for $q \le i \le N$,

$$\theta_i = \begin{cases} \pi + \arctan\left(\dfrac{C_i \tan^2 \alpha_i}{\tan^2 \alpha_i - C_i \tan \alpha_i + 1}\right) & \text{if } C_i^2 > 4 \text{ and } \tan^2 \alpha_i - C_i \tan \alpha_i + 1 < 0 \\ \arctan\left(\dfrac{C_i \tan^2 \alpha_i}{\tan^2 \alpha_i - C_i \tan \alpha_i + 1}\right) & \text{otherwise.} \end{cases}$$

Here α_i is the phase when x reaches γ_i and θ_i is the sudden change of the phase at that point. Hence when $x=G$ the phase is $2\sqrt{M_0}G+\sum_{i=1}^N \theta_i$. Note that the initial phase is 0 when $x=-G$. Thus, the formula above is also the total change of the phase. Recall that $\psi(x)$ is the solution of (4.4). If we denote n as the number of zeros of $\psi(x)$ in $[-G,G]$, then $n=[2\sqrt{M_0}G/\pi+\sum_{i=1}^N \theta_i/\pi]+1$, where $[\cdot]$ is the floor function.

Combined with Lemma 4.3 and Lemma 4.1, we have the following theorem.

Theorem 4.2: *The number of zeros of the solution of (4.2) satisfies the following estimate:*

$$N(\psi) \le \frac{2\sqrt{M_0}G}{\pi} + \frac{2}{\pi}\sum_{i=1}^{N} \arctan\frac{c_i}{2\sqrt{M_0}} + 3. \tag{4.8}$$

Remark 4.1: Since $2\arctan(C_i/2) < \pi$, *we have*

$$\frac{2}{\pi}\sum_{i=1}^{N} \arctan\frac{c_i}{2\sqrt{M_0}} < N.$$

Hence the estimate in (4.8) is better than (4.3).

ACKNOWLEDGMENTS

I would like to thank Professor Zixiang Zhou for providing precious advices. The work is partially sponsored by National NSFC of China (Grant No. J0730103) and National Basic Research Program of China (Grant No. 2007CB814800).

[1] W. Eckhaus and A. Van Harten, *The Inverse Scattering Transformation and the Theory of Solitons: An Introduction* (Elsevier, New York, 1981).
[2] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators* (Academic, New York, 1978).
[3] F. Brau, "Sufficient conditions for the existence of bound states in a central potential," J. Phys. A **37**, 6687 (2004).
[4] F. Brau, "Necessary and sufficient conditions for the existence of bound states in a central potential," J. Phys. A **36**, 9907 (2003).
[5] F. Calogero, "Upper and lower bounds for the number of bound states in a given central potential," Commun. Math. Phys. **1**, 80 (1965).
[6] O. Ogurisu, "On the number of negative eigenvalues of a Schrödinger operator with poing interactions," Lett. Math. Phys. **85**, 129 (2008).
[7] F. Brau and F. Calogero, "Upper and lower limits for the number of S-wave bound states in an attractive potential," J. Math. Phys. **44**, 1554 (2003).
[8] K. Chadan, R. Kobayashi, A. Martin, and J. Stubbe, "Generalization of the Calogero-Cohn bound on the number of bound states," J. Math. Phys. **37**, 1106 (1996).
[9] V. Bargmann, "On the number of bound states in a central field of force," Proc. Natl. Acad. Sci. U.S.A. **38**, 961 (1952).
[10] S. Albeverio and L. Nizhnik, "On the number of negative eigenvalues of a one-dimensional Schrödinger operator with

poing interactions," Lett. Math. Phys. **65**, 27 (2003).
[11] V. Glaser, H. Grosse, A. Martin, and W. Thirring, *Studies in Mathematical Physics* (Princeton University Press, Princeton, 1976), pp. 169–194.
[12] W. Amrein, A. Hinz, and D. Pearson, *Sturm-Liouville Theory: Past and Present* (Birkhäuser, Basel, 2005), pp. 29–43.