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Localized solitons of the hyperbolic su(N) AKNS system

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Abstract. Using the nonlinear constraint and Darboux transformation methods, the $(m_1, \ldots, m_N)$ localized solitons of the hyperbolic su(N) AKNS system are constructed. Here 'hyperbolic su(N)' means that the first part of the Lax pair is $\Psi_y = J\Psi_x + U(x, y, t)\Psi$ where J is constant real diagonal and $U^* = -U$. When different solitons move in different velocities, each component U_{ij} of the solution U has at most $m_i m_j$ peaks as $t \to \infty$. This corresponds to the (M, N) solitons for the DSI equation. When all the solitons move in the same velocity, U_{ij} still has at most $m_i m_j$ peaks if the phase differences are large enough.

1. Introduction

Since the discovery of localized solitons for the DSI equation [1], the (M, N) solitons [5], especially the (N, N) solitons, have been discussed in various ways, such as the inverse scattering method [2, 6, 10, 15, 16], the binary Darboux transformation method [14], the Hirota method, etc [11]. On the other hand, the nonlinear constraint method [3] has been developed for $(1 + 2)$-dimensional problems since the work of [12, 4]. Using the Darboux transformation in higher dimensions [7, 9], the solutions can be obtained explicitly. In [17], the nonlocalized solutions of the N-wave equation were derived in this way. In [18], the localized soliton solutions of the hyperbolic su(N) AKNS system were discussed by the nonlinear constraint method which transforms the original problem to a $2N \times 2N$ linear system which separates all the variables. It shows that, if the velocities of the solitons are all different, there are Darboux transformations of rth degree such that the asymptotic solution has at most r^2 peaks. However, the velocities are not arbitrary. For example, the velocities of '1-soliton' should be along the y-axis. In [13], a localized one-soliton solution of the DSIII equation was obtained by the nonlinear constraint method which transforms the original problem to an $(N+1) \times (N+1)$ linear system which separates all of the variables.

In this paper, we show that under the nonlinear constraint, there is a way to construct a Darboux transformation so that the number of peaks and the values of velocities have more freedom. This construction is valid for all the lower ordered equations in the hyperbolic su(N) AKNS system, including the DSI equation, the N-wave equation, etc. For the DSI equation, this corresponds to the (M, N) solitons.

The main conclusions are: (1) there are $m = (m_1, \ldots, m_N)$ solitons $U^{[m]}$ which are localized (tend to 0 as $(x, y) \to \infty$); (2) if different solitons move in different velocities (characterized by an algebraic condition (47) and hold for the DSI equation), then the component $U^{[m]}_{jk}$ of $U^{[m]}$ has at most $m_j m_k$ peaks as $t \to \infty$; (3) if all the solitons move in the same velocity (hold for the N-wave equation), then $U^{[m]}_{jk}$ also has at most $m_j m_k$ peaks as the phase differences are large enough.

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2. The system and its nonlinear constraints

Here we consider the hyperbolic su(N) AKNS system

$$\begin{aligned} \Psi_y &= J\Psi_x + U(x, y, t)\Psi \\ \Psi_t &= \sum_{j=0}^{n} V_j(x, y, t)\partial^{n-j}\Psi \end{aligned} \tag{1}$$

where $\partial = \partial/\partial x$, $J = \mathrm{diag}(J_1, \ldots, J_N)$ is a real constant diagonal $N \times N$ matrix with mutually different entries. $U(x, y, t)$ is off-diagonal with $U^* = -U$. In this case, we call (1) a hyperbolic su(N) AKNS system, since J is real and $U \in \mathrm{su}(N)$.

The integrability conditions of (1) are

$$[J, V_{j+1}^A] = V_{j,y}^A - JV_{j,x}^A - [U, V_j]^A + \sum_{k=0}^{j-1} C_{n-k}^{n-j}(V_k\partial^{j-k}U)^A \tag{2}$$

$$V_{j,y}^D - JV_{j,x}^D = [U, V_j^A]^D - \sum_{k=0}^{j-1} C_{n-k}^{n-j}(V_k\partial^{j-k}U)^D \tag{3}$$

$$U_t = V_{n,y}^A - JV_{n,x}^A - [U, V_n]^A + \sum_{k=0}^{n-1}(V_k\partial^{n-k}U)^A \tag{4}$$

where the superscripts D and A refer to the diagonal and off-diagonal parts of a matrix. As mentioned in [18], (3) and (4) give a system of nonlinear PDEs of U and all the diagonal parts of V_j's, where V_j's are determined by (2) respectively. Usually, only U is physically significant.

There is a connection of (1) with the linear system

$$\begin{aligned} &\Phi_x = \begin{pmatrix} \mathrm{i}\lambda I & \mathrm{i}F \\ \mathrm{i}F^* & 0 \end{pmatrix} \qquad \Phi_y = \begin{pmatrix} \mathrm{i}\lambda J + U & \mathrm{i}JF \\ \mathrm{i}F^*J & 0 \end{pmatrix} \\ &\Phi_t = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \qquad \Phi = \sum_{j=0}^{n} \begin{pmatrix} W_j & X_j \\ -X_j^* & Z_j \end{pmatrix} \lambda^{n-j}\Phi \end{aligned} \tag{5}$$

where F, W_j, X_j, Z_j are $N \times K$, $N \times N$, $N \times K$, $K \times K$ matrices respectively ($K \geqslant 1$) and satisfy $W_j^* = -W_j$, $Z_j^* = -Z_j$. This is a slight generalization of the linear system in [18], where K should be N.

The integrability conditions of (5) include

$$\begin{aligned} F_y &= JF_x + UF \\ \mathrm{i}F_t &= X_{n,x} + \mathrm{i}W_nF - \mathrm{i}FZ_n \end{aligned} \tag{6}$$

$$\begin{aligned} \mathrm{i}X_{j+1} &= X_{j,x} + \mathrm{i}W_jF - \mathrm{i}FZ_j \\ W_{j,x} &= -\mathrm{i}FX_j^* - \mathrm{i}X_jF^* \\ Z_{j,x} &= \mathrm{i}F^*X_j + \mathrm{i}X_j^*F \\ \mathrm{i}[J, W_{j+1}] &= W_{j,y} - [U, W_j] + \mathrm{i}JFX_j^* + \mathrm{i}X_jF^*J \\ Z_{j,y} &= \mathrm{i}F^*JX_j + \mathrm{i}X_j^*JF \end{aligned} \tag{7}$$

$$U_x = [J, FF^*] \tag{8}$$

$$U_t = W_{n,y} - [U, W_n] + \mathrm{i}JFX_n^* + \mathrm{i}X_nF^*J. \tag{9}$$

For $U = 0$, $F = 0$, (7) implies that $W_j(\lambda) = \mathrm{i}\Omega_j(t)$, $X_j = 0$, $Z_j = \mathrm{i}Z_j^0(t)$ where $\Omega_j(t)$'s are real diagonal matrices and $Z_j^0(t)$'s are real matrices.

When $Z_j^0(t) = \zeta_j(t) I_K$ (I_K is the $K \times K$ identity matrix) where $\zeta_j(t)$ is a real function of t, (6) is just the Lax pair (1) for $n = 1, 2, 3$. Equations (7) and (9) give the recursion relations to determine W_j, X_j, Z_j, together with the evolution equations corresponding to (2)–(4), which are the integrability conditions of equation (1). Equation (8) gives a nonlinear constraint between U and F.

This system includes the DSI equation and the N-wave equation as special cases.

In order to consider the asymptotic behaviour of the solution U, here we suppose that Ω_j is independent of t and $\zeta_j = 0$. Denote $\Omega = \sum_{j=0}^{n} \Omega_j \lambda^{n-j}$ and write $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_N)$.

We need the following symbols and simple conclusions.

If M_1, M_2 are $j \times k$ matrices, we write $M_1 \doteq M_2$ if there is a nondegenerate diagonal $k \times k$ matrix A such that $M_2 = M_1 A$.

If L is a diagonal matrix, then $M_1 \doteq M_2$ and $\det M_1 \neq 0$ imply $M_1 L M_1^{-1} = M_2 L M_2^{-1}$.

Let

$$M = \begin{pmatrix} a & -v^*/\bar{a} \\ v & I_K \end{pmatrix} \tag{10}$$

where $v \neq 0$ is an $K \times 1$ vector, $a \neq 0$ is a number. Let

$$\Lambda = \begin{pmatrix} \lambda_0 & \\ & \bar{\lambda}_0 I_K \end{pmatrix}. \tag{11}$$

Then we have

$$M^{-1} = \frac{1}{\Delta} \begin{pmatrix} \bar{a} & v^* \\ -\bar{a} v & \Delta I_K - v v^* \end{pmatrix} \tag{12}$$

$$M \Lambda M^{-1} = \frac{1}{\Delta} \begin{pmatrix} \bar{\lambda}_0 \Delta + (\lambda_0 - \bar{\lambda}_0)|a|^2 & (\lambda_0 - \bar{\lambda}_0) a v^* \\ (\lambda_0 - \bar{\lambda}_0) \bar{a} v & \bar{\lambda}_0 \Delta I_K + (\lambda_0 - \bar{\lambda}_0) v v^* \end{pmatrix} \tag{13}$$

where $\Delta = v^* v + |a|^2$. Moreover,

$$\lim_{a \to \infty} M \Lambda M^{-1} = \begin{pmatrix} \lambda_0 & \\ & \bar{\lambda}_0 I_K \end{pmatrix} \tag{14}$$

$$\lim_{a \to 0} M \Lambda M^{-1} = \begin{pmatrix} \bar{\lambda}_0 & \\ & \bar{\lambda}_0 I_K + (\lambda_0 - \bar{\lambda}_0) \frac{v v^*}{v^* v} \end{pmatrix}. \tag{15}$$

For an $(N + K) \times (N + K)$ matrix M, denote M_{B_N} to be an $N \times N$ matrix containing the first N columns and N rows of M.

3. Darboux transformation and soliton solutions

We now construct the solutions from $U = 0$, $F = 0$. A procedure to construct the Darboux transformation was proposed in [18].

Let λ_α ($\alpha = 1, 2, \ldots, r$) be r complex numbers such that $\lambda_\alpha \neq \lambda_\beta$ for $\alpha \neq \beta$ and $\lambda_\alpha \neq \bar{\lambda}_\beta$ for all α, β. Take

$$\Lambda_\alpha = \mathrm{diag}(\underbrace{\lambda_\alpha, \ldots, \lambda_\alpha}_{N}, \underbrace{\bar{\lambda}_\alpha, \ldots, \bar{\lambda}_\alpha}_{K}) \tag{16}$$

$$H_\alpha = \begin{pmatrix} \exp(Q_\alpha(s)) & -\exp(-Q_\alpha(s)^*) C_\alpha^* \\ C_\alpha & I_K \end{pmatrix} \tag{17}$$

where

$$Q_\alpha(s) = \begin{pmatrix} a_{\alpha 1} s + b_{\alpha 1} & & \\ & \ddots & \\ & & a_{\alpha N} s + b_{\alpha N} \end{pmatrix} \tag{18}$$

$a_{\alpha j}$, $b_{\alpha j}$ are constants,

$$C_\alpha = (0, \ldots, 0, \underset{l_\alpha}{\kappa_\alpha}, 0, \ldots, 0). \tag{19}$$

κ_α is a constant $K \times 1$ nonzero vector which appears at the l_α's column of C_α.

Let

$$\rho_\alpha = \mathrm{Re}\,(a_{\alpha,l_\alpha}) \qquad \phi_\alpha = \mathrm{Im}\,(a_{\alpha,l_\alpha}) \qquad \pi_\alpha = b_{\alpha,l_\alpha}. \tag{20}$$

The Darboux matrices for such $\{\Lambda_\alpha, H_\alpha\}$ can be constructed as follows. Let

$$\begin{aligned}
&G^{(1)}(\lambda) = \lambda - H_1\Lambda_1 H_1^{-1} \qquad H_\alpha^{(1)} = G^{(1)}(\lambda_\alpha)H_\alpha \qquad (\alpha = 2, 3, \ldots, r)\\
&G^{(2)}(\lambda) = \lambda - H_2^{(1)}\Lambda_2 H_2^{(1)-1} \qquad H_\alpha^{(2)} = G^{(2)}(\lambda_\alpha)H_\alpha^{(1)} \qquad (\alpha = 3, 4, \ldots, r)\\
&\cdots\\
&G^{(r)}(\lambda) = \lambda - H_r^{(r-1)}\Lambda_r H_r^{(r-1)-1}
\end{aligned} \tag{21}$$

$$G(\lambda) = G^{(r)}(\lambda)G^{(r-1)}(\lambda)\cdots G^{(1)}(\lambda) \tag{22}$$

then $G(\lambda)$ is a polynomial of λ of degree r. The permutability [8] implies that if $(\Lambda_\alpha, H_\alpha)$ and (Λ_β, H_β) are interchanged, $G(\lambda)$ is invariant.

Let

$$m_j = \#\{\alpha \mid 1 \leqslant \alpha \leqslant r,\ l_\alpha = j\} \qquad m = (m_1, \ldots, m_N) \tag{23}$$

then $m_1 + \cdots + m_N = r$.

Suppose

$$G(\lambda) = \lambda^r - G_1\lambda^{r-1} + \cdots + (-1)^r G_r \tag{24}$$

denote

$$U^{[m]} = \mathrm{i}[J, (G_1)_{B_N}]. \tag{25}$$

For the problem (5) with $U = F = 0$, we have $Q_\alpha = \mathrm{i}\lambda_\alpha(x + Jy) + \mathrm{i}\Omega(\lambda_\alpha)t$, where s can be t or other parameters. The matrix $G(\lambda)$ given by (22) is a Darboux matrix, that is, for any solution Φ of (5), $G\Phi$ satisfies (5) with certain $\widetilde{U}$, $\widetilde{F}$, $\widetilde{W}_j$, $\widetilde{X}_j$, $\widetilde{Z}_j$ replacing U, F, W_j, X_j, Z_j. $U^{[m]}$ is actually the derived solution of (9). Owing to the choice of H_α in (17), $U^{[m]}$ is globally defined. Here we first consider the generalized $Q_\alpha(s)$ in (18) so that the localization, asymptotic properties, etc can be treated uniformly.

Proposition 1.

(1) If there is at most one α $(1 \leqslant \alpha \leqslant r)$ such that $\rho_\alpha = 0$, then $\lim_{s\to\infty} U^{[m]} = 0$.

(2) If $\rho_\alpha = 0$, $\rho_\beta = 0$ $(\alpha \neq \beta)$, and $\rho_\gamma \neq 0$ for all $\gamma \neq \alpha, \beta$, then:

(i) when $l_\alpha = l_\beta$, $\lim_{s\to\infty} U^{[m]} = 0$;

(ii) when $l_\alpha \neq l_\beta$,

$$\lim_{s\to\infty} U_{ab}^{[m]} = 0 \qquad \text{for } (a, b) \neq (l_\alpha, l_\beta) \tag{26}$$

and as $s \to \infty$,

$$U_{l_\alpha,l_\beta}^{[m]} \sim \frac{B_{\alpha\beta}\exp(\mathrm{i}\,\mathrm{Im}\,(\pi_\alpha - \pi_\beta) + \mathrm{i}(\phi_\beta - \phi_\alpha)s)}{A_{\alpha\beta}\cosh(\mathrm{Re}\,(\pi_\alpha + \pi_\beta) - \delta_{\alpha\beta}^{(1)}) + \cosh(\mathrm{Re}\,(\pi_\alpha - \pi_\beta) - \delta_{\alpha\beta}^{(2)})} \tag{27}$$

where $A_{\alpha\beta}$, $\delta_{\alpha\beta}^{(1)}$, $\delta_{\alpha\beta}^{(2)}$, are real constants, $A_{\alpha\beta} > 0$, and $B_{\alpha\beta}$ is a complex constant. Moreover, if $K = 1$, then $B_{\alpha\beta} \neq 0$ if and only if $\kappa_\alpha \neq 0$ and $\kappa_\beta \neq 0$.

Proof. First suppose $\rho_\alpha \neq 0$. By (14) and (15),

$$\lim_{\rho_\alpha s\to\pm\infty} H_\alpha \Lambda_\alpha H_\alpha^{-1} = S_\alpha^{\pm\infty} \tag{28}$$

where

$$\begin{aligned} S_\alpha^{+\infty} &= \begin{pmatrix} \lambda_\alpha I_N & \\ & \bar\lambda_\alpha I_K \end{pmatrix} \\ S_\alpha^{-\infty} &= \begin{pmatrix} \lambda_\alpha I_N + (\bar\lambda_\alpha - \lambda_\alpha) E_{l_\alpha l_\alpha} & \\ & \bar\lambda_\alpha I_K + (\lambda_\alpha - \bar\lambda_\alpha)\frac{\kappa_\alpha \kappa_\alpha^*}{\kappa_\alpha^*\kappa_\alpha} \end{pmatrix} \end{aligned} \tag{29}$$

E_{jk} is an $N \times N$ matrix whose (j,k)th entry is 1 and the remaining entries are zero.

For $\beta \neq \alpha$,

$$(\lambda_\beta - S_\alpha^{\pm\infty}) H_\beta \doteq \begin{pmatrix} \exp(Q_\beta(s)) & -\exp(-Q_\beta(s)^*)\widetilde{C}_\beta^{\pm *} \\ \widetilde{C}_\beta^{\pm} & I_K \end{pmatrix} \tag{30}$$

where

$$\begin{aligned} \widetilde{C}_\beta^{\pm} &= (0,\dots,0,\underset{l_\beta}{\widetilde{\kappa}_\beta^{\pm}},0,\dots,0) \\ \widetilde{\kappa}_\beta^{+} &= \frac{\lambda_\beta - \bar\lambda_\alpha}{\lambda_\beta - \lambda_\alpha}\kappa_\beta \\ \widetilde{\kappa}_\beta^{-} &= \begin{cases} \dfrac{\lambda_\beta - \bar\lambda_\alpha}{\lambda_\beta - \lambda_\alpha}\kappa_\beta - \dfrac{\lambda_\alpha - \bar\lambda_\alpha}{\lambda_\beta - \lambda_\alpha}\dfrac{\kappa_\alpha^*\kappa_\beta}{\kappa_\alpha^*\kappa_\alpha}\kappa_\alpha & \text{if } l_\beta \neq l_\alpha \\ \kappa_\beta - \dfrac{\lambda_\alpha - \bar\lambda_\alpha}{\lambda_\beta - \bar\lambda_\alpha}\dfrac{\kappa_\alpha^*\kappa_\beta}{\kappa_\alpha^*\kappa_\alpha}\kappa_\alpha & \text{if } l_\beta = l_\alpha. \end{cases} \end{aligned} \tag{31}$$

Therefore, if $\rho_\alpha \neq 0$, the action of the limit Darboux matrix $\lambda - S_\alpha^{\pm\infty}$ on H_β $(\beta \neq \alpha)$ does not change the form of H_β, but only changes the constant vector κ_β.

If $K = 1$, then $\kappa_\beta^{\pm *}\kappa_\gamma^{\pm} \neq 0$ implies $\widetilde{\kappa}_\beta^{\pm *}\widetilde{\kappa}_\gamma^{\pm} \neq 0$. When $K > 1$, this does not hold in general.

Now suppose $\rho_\alpha = 0$. Without loss of generality, suppose $l_\alpha = 1$. Then

$$H_\alpha \doteq \begin{pmatrix} \exp(\pi_\alpha) & & & & -\exp(-\bar\pi_\alpha)\kappa_\alpha^* \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ \kappa_\alpha & 0 & \cdots & 0 & I_K \end{pmatrix}. \tag{32}$$

By (13),

$$H_\alpha \Lambda_\alpha H_\alpha^{-1} = \frac{1}{\Delta} \times \begin{pmatrix} \bar\lambda_\alpha \Delta + (\lambda_\alpha - \bar\lambda_\alpha)\exp(\pi_\alpha + \bar\pi_\alpha) & & & & (\lambda_\alpha - \bar\lambda_\alpha)\exp(\pi_\alpha)\kappa_\alpha^* \\ & \lambda_\alpha & & & 0 \\ & & \ddots & & \vdots \\ & & & \lambda_\alpha & 0 \\ (\lambda_\alpha - \bar\lambda_\alpha)\exp(\bar\pi_\alpha)\kappa_\alpha & 0 & \cdots & 0 & \bar\lambda_\alpha \Delta I_K + (\lambda_\alpha - \bar\lambda_\alpha)\kappa_\alpha\kappa_\alpha^* \end{pmatrix} \tag{33}$$

where $\Delta = \exp(\pi_\alpha + \bar\pi_\alpha) + \kappa_\alpha^*\kappa_\alpha$.

Part (1) of the proposition is derived as follows. Owing to the permutability of Darboux transformations, we can suppose $\rho_1 \neq 0, \dots, \rho_{r-1} \neq 0$, $\rho_r = 0$. Then, as $s \to \infty$, $G^{(\alpha)}$

tends to a diagonal matrix for $\alpha \leqslant r-1$. Considering (33), the limit of $(G^{(r)}(\lambda))_{B_N}$ is also diagonal, hence

$$U^{[m]} = \mathrm{i}[J, (G_1)_{B_N}] \to 0. \tag{34}$$

Now we turn to prove part (2) of the proposition. First, suppose $r = 2$. We use an alternative way of constructing Darboux matrices [18]. Let

$$\overset{\circ}{H}_\alpha = \begin{pmatrix} \exp(Q_\alpha(s)) \\ C_\alpha \end{pmatrix} \qquad \Gamma_{\alpha\beta} = \frac{\overset{\circ}{H}{}^*_\alpha \overset{\circ}{H}_\beta}{\lambda_\beta - \bar{\lambda}_\alpha} \tag{35}$$

then

$$G(\lambda) = \prod_{\alpha=1}^{r}(\lambda - \bar{\lambda}_\alpha)\left(1 - \sum_{\alpha,\beta=1}^{r} \frac{\overset{\circ}{H}_\alpha (\Gamma^{-1})_{\alpha\beta} \overset{\circ}{H}{}^*_\beta}{\lambda - \bar{\lambda}_\beta}\right) \tag{36}$$

and

$$U^{[m]} = \mathrm{i}\left[J, \sum_{\alpha,\beta=1}^{r} \left(\overset{\circ}{H}_\alpha (\Gamma^{-1})_{\alpha\beta} \overset{\circ}{H}{}^*_\beta\right)_{B_N}\right]. \tag{37}$$

- Case (i): $l_\alpha = l_\beta$.

$$\begin{aligned}
\overset{\circ}{H}_\alpha &\doteq \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& \exp(\pi_\alpha) &&& \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \\ 0 & \cdots & 0 & \underset{l_\alpha}{\kappa_\alpha} & 0 & \cdots & 0 \end{pmatrix} \\
\overset{\circ}{H}_\beta &\doteq \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& \exp(\pi_\beta) &&& \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \\ 0 & \cdots & 0 & \underset{l_\beta}{\kappa_\beta} & 0 & \cdots & 0 \end{pmatrix}
\end{aligned} \tag{38}$$

where $l_\beta = l_\alpha$, then $\Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}$ where Γ_{jk}'s are $N \times N$ diagonal matrices. Therefore, $\Gamma^{-1} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ where Σ_{jk}'s are also $N \times N$ diagonal matrices. This implies that $U^{[m]} = 0$.

- Case (ii): $l_\alpha \neq l_\beta$. Suppose $\overset{\circ}{H}_\alpha$, $\overset{\circ}{H}_\beta$ are given by (38) with $l_\beta \neq l_\alpha$. Denote

$$\theta_{\alpha\beta} = \frac{\kappa^*_\alpha \kappa_\beta}{\sqrt{\kappa^*_\alpha \kappa_\alpha \kappa^*_\beta \kappa_\beta}} \tag{39}$$

$$g_{\alpha\beta} = 1 - \frac{4 \mathrm{Im}\, \lambda_\alpha \mathrm{Im}\, \lambda_\beta}{|\lambda_\beta - \bar{\lambda}_\alpha|^2} |\theta_{\alpha\beta}|^2 > 0 \tag{40}$$

then, by direct computation, we have

$$\begin{aligned}
&\lim_{s\to\infty} U^{[m]}_{l_\alpha,l_\beta} \exp(\mathrm{i}(\phi_\beta-\phi_\alpha)s) = \frac{2\mathrm{i}(J_{l_\beta}-J_{l_\alpha})\mathrm{Im}\,\lambda_\alpha \mathrm{Im}\,\lambda_\beta\theta_{\alpha\beta}}{\bar{\lambda}_\beta-\lambda_\alpha}\frac{\exp(\mathrm{i}\,\mathrm{Im}\,(\pi_\alpha-\pi_\beta))}{D}\\
&D = \sqrt{g_{\alpha\beta}}\cosh(\mathrm{Re}\,(\pi_\alpha+\pi_\beta)-\delta_1)+\cosh(\mathrm{Re}\,(\pi_\alpha-\pi_\beta)-\delta_2)\\
&\delta_1 = \frac{1}{2}\ln g_{\alpha\beta}+\frac{1}{2}\ln(\kappa_\alpha^*\kappa_\alpha\kappa_\beta^*\kappa_\beta)+2\ln\left|\frac{\lambda_\beta-\lambda_\alpha}{\lambda_\beta-\bar{\lambda}_\alpha}\right|\\
&\delta_2 = \frac{1}{2}\ln\frac{\kappa_\alpha^*\kappa_\alpha}{\kappa_\beta^*\kappa_\beta}
\end{aligned}\tag{41}$$

and $U_{\mu\nu}\to 0$ if $(\mu,\nu)\neq(l_\alpha,l_\beta)$.

When $r>2$, we still use the permutability of Darboux transformations and suppose $\rho_1\neq 0,\ldots,\rho_{r-2}\neq 0$, $\rho_{r-1}=\rho_r=0$. Then, after the action of $G^{(r-2)}(\lambda)\cdots G^{(1)}(\lambda)$, the derived $H^{(r-2)}_{r-1}$, $H^{(r-2)}_r$ has the same asymptotic form as H_{r-1}, H_r, provided that the constant vectors κ_{r-1}, κ_r are changed to $\kappa^{(r-2)}_{r-1}$, $\kappa^{(r-2)}_r$. Therefore, as in the case $r=2$, the limit of U_{l_{r-1},l_r} has the desired form, and the limits of the other components of U are all zero. The proposition is proved. $\square$

4. Localization of the solutions

Now we consider (5), for this system,

$$Q_\alpha = \mathrm{i}\lambda_\alpha(x+Jy)+\mathrm{i}\omega(\lambda_\alpha)t. \tag{42}$$

We consider the limit of the solution as $(x,y)\to\infty$ along a straight line $x=\xi+v_xs$, $y=\eta+v_ys$ $(v_x^2+v_y^2>0)$, then

$$Q_\alpha = \mathrm{i}\lambda_\alpha(\xi+J\eta)+\mathrm{i}\omega(\lambda_\alpha)t+\mathrm{i}\lambda_\alpha(v_x+Jv_y)s. \tag{43}$$

Now

$$\rho_\alpha = \mathrm{Re}\,(\mathrm{i}\lambda_\alpha(v_x+J_{l_\alpha}v_y)) = -\mathrm{Im}\,\lambda_\alpha(v_x+J_{l_\alpha}v_y). \tag{44}$$

If there is at most one $\rho_\alpha=0$, then proposition 1 implies that $U^{[m]}\to 0$ as $s\to\infty$. If $\rho_\alpha=0$, $\rho_\beta=0$ $(\alpha\neq\beta)$, then $l_\alpha=l_\beta$ since $J_{l_\alpha}\neq J_{l_\beta}$ if $l_\alpha\neq l_\beta$. Hence, proposition 1 also implies $U^{[m]}\to 0$ as $s\to\infty$. Therefore, we have the following.

Theorem 1. *$U^{[m]}\to 0$ as $(x,y)\to\infty$ in any direction.*

5. Asymptotic behaviour of the solutions as $t\to\infty$

Now we use a frame (ξ,η) which moves in a fixed velocity (v_x,v_y), that is, let $x=\xi+v_xt$, $y=\eta+v_yt$, then

$$Q_\alpha = \mathrm{i}\lambda_\alpha(\xi+J\eta)+(\mathrm{i}\lambda_\alpha(v_x+Jv_y)+\mathrm{i}\omega(\lambda_\alpha))t \tag{45}$$

$$\rho_\alpha = -\mathrm{Im}\,\lambda_\alpha(v_x+J_{l_\alpha}v_y)-\mathrm{Im}\,(\omega_{l_\alpha}(\lambda_\alpha)). \tag{46}$$

Suppose that for mutually different α,β,γ,

$$\det\begin{pmatrix}1 & J_{l_\alpha} & \sigma_\alpha\\ 1 & J_{l_\beta} & \sigma_\beta\\ 1 & J_{l_\gamma} & \sigma_\gamma\end{pmatrix}\neq 0 \tag{47}$$

where

$$\sigma_\alpha = \mathrm{Im}\,(\omega_{l_\alpha}(\lambda_\alpha))/\mathrm{Im}\,(\lambda_\alpha). \tag{48}$$

Then there are at most two $\rho_\alpha = 0$ $(\alpha = 1, \ldots, r)$. By proposition 1, $U^{[m]}_{l_\alpha, l_\beta} \not\to 0$ only if $\rho_\alpha = 0$, $\rho_\beta = 0$. This leads to

$$v_x = \frac{J_{l_\alpha}\sigma_\beta - J_{l_\beta}\sigma_\alpha}{J_{l_\beta} - J_{l_\alpha}} \qquad v_y = \frac{\sigma_\alpha - \sigma_\beta}{J_{l_\beta} - J_{l_\alpha}}. \tag{49}$$

For $U^{[m]}_{jk} \not\to 0$, α, β can take m_j and m_k values respectively, hence there are at most $m_j m_k$ velocities (v_x, v_y) such that $U^{[m]}_{jk} \not\to 0$. Therefore, we have the following.

Theorem 2. *Suppose (47) is satisfied. Then as $t \to \infty$, the asymptotic solution of $U^{[m]}_{jk}$ has at most $m_j m_k$ peaks whose velocities are given by (49) ($l_\alpha = j$, $l_\beta = k$). If a peak has velocity (v_x, v_y), then, in the coordinate $\xi = x - v_x t$, $\eta = y - v_y t$, $\lim_{t\to\infty} U_{ab} = 0$ for all $(a, b) \neq (j, k)$, and as $t \to \infty$*

$$\begin{aligned} U^{[m]}_{jk} &\sim \frac{B_{\alpha\beta}\exp(\mathrm{i}\,\mathrm{Re}\,(\lambda_\alpha - \lambda_\beta)\xi + \mathrm{i}(\lambda_\alpha J_j - \lambda_\beta J_k)\eta + \mathrm{i}(\phi_\alpha - \phi_\beta)t)}{D} \\ D &= A_{\alpha\beta}\cosh(\mathrm{Im}\,(\lambda_\alpha + \lambda_\beta)\xi + \mathrm{Im}\,(\lambda_\alpha J_j + \lambda_\beta J_k)\eta + \delta^{(1)}_{\alpha\beta}) \\ &\quad + \cosh(\mathrm{Im}\,(\lambda_\alpha - \lambda_\beta)\xi + \mathrm{Im}\,(\lambda_\alpha J_j - \lambda_\beta J_k)\eta + \delta^{(2)}_{\alpha\beta}) \end{aligned} \tag{50}$$

where $A_{\alpha\beta}$, $\delta^{(1)}_{\alpha\beta}$, $\delta^{(2)}_{\alpha\beta}$ are real constants, $A_{\alpha\beta} > 0$, and $B_{\alpha\beta}$ is a complex constant,

$$\phi_\gamma = \mathrm{Re}\,\lambda_\gamma(v_x + J_{l_\gamma} v_y) + \mathrm{Re}\,(\omega_{l_\gamma}(\lambda_\gamma)) \qquad (\gamma = \alpha, \beta). \tag{51}$$

Remark. Condition (47) implies that the velocities of the solitons are all different. This is true for the DSI equation. However, for the N-wave equation, all the solitons move in the same velocity. We will discuss this problem in the next section.

Example. DSI equation.

Let $n = 2$, $N = 2$,

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad U = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} \qquad \omega = -2\mathrm{i}J\lambda^2 \tag{52}$$

then we have

$$\begin{aligned} F_y &= JF_x + UF \\ F_t &= 2\mathrm{i}JF_{xx} + 2\mathrm{i}UF_x + \mathrm{i}\begin{pmatrix} |u|^2 + 2q_1 & u_x + u_y \\ -\bar{u}_x + \bar{u}_y & -|u|^2 - 2q_2 \end{pmatrix} F \end{aligned} \tag{53}$$

$$\begin{aligned} -\mathrm{i}u_t &= u_{xx} + u_{yy} + 2|u|^2 u + 2(q_1 + q_2)u \\ q_{1,x} - q_{1,y} &= q_{2,x} + q_{2,y} = -(|u|^2)_x \end{aligned} \tag{54}$$

$$(FF^*)^D = \tfrac{1}{2}\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \qquad [J, FF^*] = U_x \tag{55}$$

(54) is the DSI equation.

If we construct the solution $U^{[m]}$ as above, then theorem 1 implies that $U^{[m]} \to 0$ as $(x, y) \to \infty$ in any directions. If $\mathrm{Re}\,\lambda_\alpha \neq \mathrm{Re}\,\lambda_\beta$ for $\alpha \neq \beta$ and $l_\alpha = l_\beta$, then, theorem 2 implies that as $t \to \infty$, u has at most $m_1 m_2$ peaks $(m_1 + m_2 = r)$. From (48), $\sigma_\alpha = -4J_{l_\alpha}\mathrm{Re}\,\lambda_\alpha$, hence (49) implies that each peak has the velocity $v_x = 2\mathrm{Re}\,(\lambda_\alpha - \lambda_\beta)$,

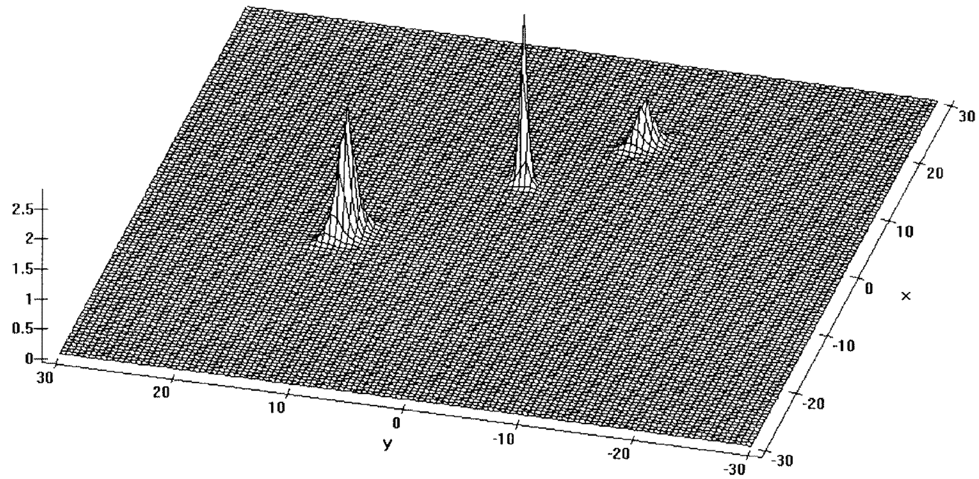

Figure 1. $u^{[1,3]}$ of the DSI equation.

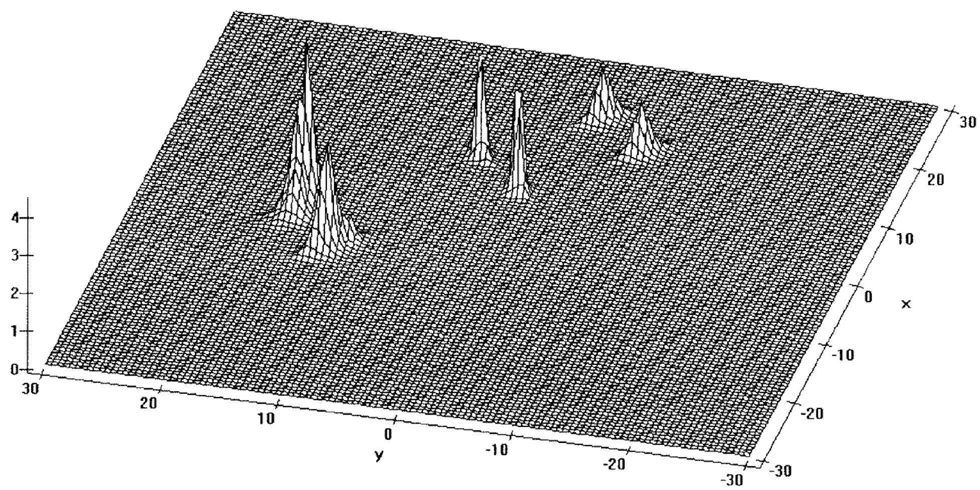

Figure 2. $u^{[2,3]}$ of the DSI equation.

$v_y = 2\mathrm{Re}\,(\lambda_\alpha + \lambda_\beta)$ $(l_\alpha = 1,\ l_\beta = 2)$. This is the (m_1, m_2) solitons [5]. When $K = 1$, these peaks do not vanish if and only if κ_α's are all nonzero.

Figures 1–3 show the solitons $u^{[1,3]}$, $u^{[2,3]}$ and $u^{[3,3]}$ respectively. The parameters are $K = 1$, $t = 2$, $\lambda_1 = 1 - 2\mathrm{i}$, $\lambda_2 = -3 - \mathrm{i}$, $\lambda_3 = 2 + \mathrm{i}$, $\lambda_4 = -1 + 3\mathrm{i}$, $\lambda_5 = 2 + 1.5\mathrm{i}$, $\lambda_6 = -0.5 - 1.5\mathrm{i}$, $C_1 = (1, 0)$, $C_2 = (0, 1)$, $C_3 = (0, 2)$, $C_4 = (0, -2)$, $C_5 = (2, 0)$, $C_6 = (-2, 0)$. Some techniques on drawing such figures were explained in [19].

6. Asymptotic solutions as the phases differences tend to infinity

For the equations whose solitons move in the same speed, like the N-wave equation, the peaks do not separate as $t \to \infty$. However, we can still see some peaks in the figures

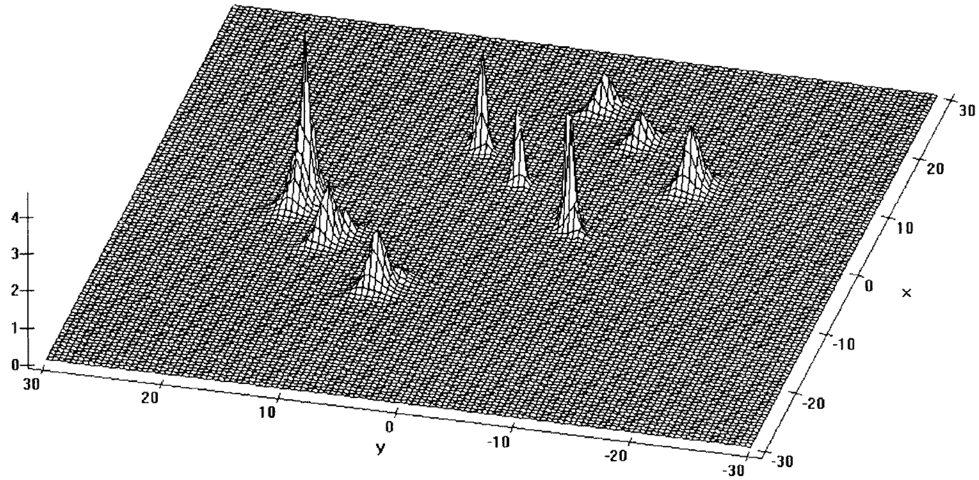

Figure 3. $u^{[3,3]}$ of the DSI equation.

[20]. In [20], we obtained the asymptotic behaviour of the r^2 solitons. Here we will get the corresponding asymptotic properties of more general solitons obtained in this paper. We have the following.

Theorem 3. *Let p_α $(\alpha = 1, \ldots, r)$ be constant real numbers satisfying*

$$\det\begin{pmatrix} 1 & J_{l_\alpha} & p_\alpha/\mathrm{Im}\,\lambda_\alpha \\ 1 & J_{l_\beta} & p_\beta/\mathrm{Im}\,\lambda_\beta \\ 1 & J_{l_\gamma} & p_\gamma/\mathrm{Im}\,\lambda_\gamma \end{pmatrix} \neq 0 \tag{56}$$

for mutually different α, β, γ. Let d_α be complex constant $K \times 1$ vectors, $\kappa_\alpha = d_\alpha \exp(p_\alpha \tau)$ and construct the Darboux transformations as above. Let $x = \xi + v_x\tau$, $y = \eta + v_y\tau$, then, for any j, k with $1 \leqslant j, k \leqslant N$, $j \neq k$, $\lim_{\tau\to\infty} U_{jk}^{[m]} \neq 0$ only if (v_x, v_y) takes specific $m_j m_k$ values.

Proof. As in section 4, here

$$Q_\alpha = \mathrm{i}\lambda_\alpha(\xi + J\eta) + \mathrm{i}\omega(\lambda_\alpha)t + \mathrm{i}\lambda_\alpha(v_x + Jv_y)\tau \tag{57}$$

hence

$$\overset{\circ}{H}_\alpha \doteq \begin{pmatrix} \exp(\widetilde{Q}_\alpha(\tau)) \\ D_\alpha \end{pmatrix} \tag{58}$$

where

$$D_\alpha = (0, \ldots, 0, \underset{l_\alpha}{d_\alpha}, 0, \ldots, 0) \tag{59}$$

$$\widetilde{Q}_\alpha(\tau) = \mathrm{i}\lambda_\alpha(\xi + J\eta) + \mathrm{i}\omega(\lambda_\alpha)t + (\mathrm{i}\lambda_\alpha(v_x + Jv_y) - p_\alpha)\tau \tag{60}$$

hence the real part of the coefficient of τ in $\widetilde{Q}_\alpha(\tau)$ is

$$\widetilde{\rho}_\alpha = -\mathrm{Im}\,\lambda_\alpha(v_x + Jv_y) - p_\alpha. \tag{61}$$

Condition (56) implies that there are at most two $\widetilde{\rho}_\alpha$'s such that $\widetilde{\rho}_\alpha = 0$. According to proposition 1, as $\tau \to \infty$, $U_{jk}^{[m]} \not\to 0$ only if there exist $\widetilde{\rho}_\alpha = 0$, $\widetilde{\rho}_\beta = 0$, $\alpha \neq \beta$, $l_\alpha = j$, $l_\beta = k$. Therefore, the theorem is verified. □

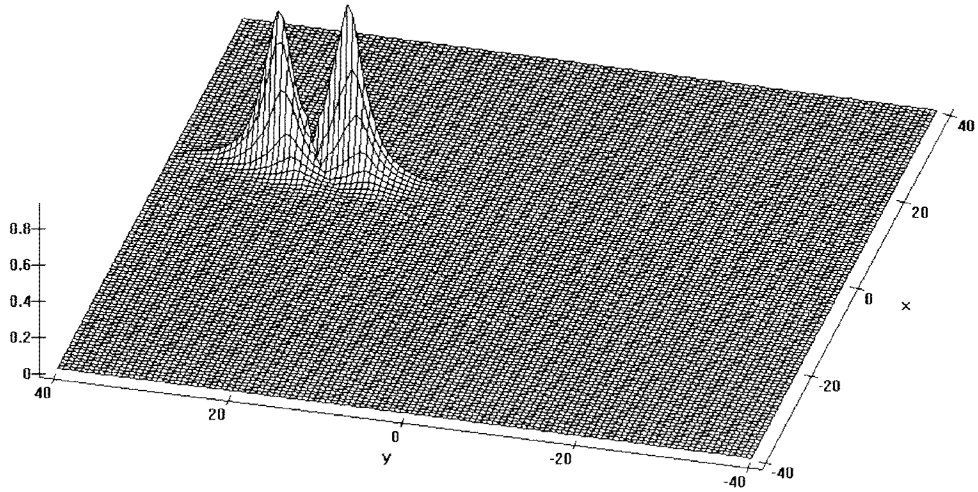

Figure 4. $U^{[0,1,2]}$ of the 3-wave equation.

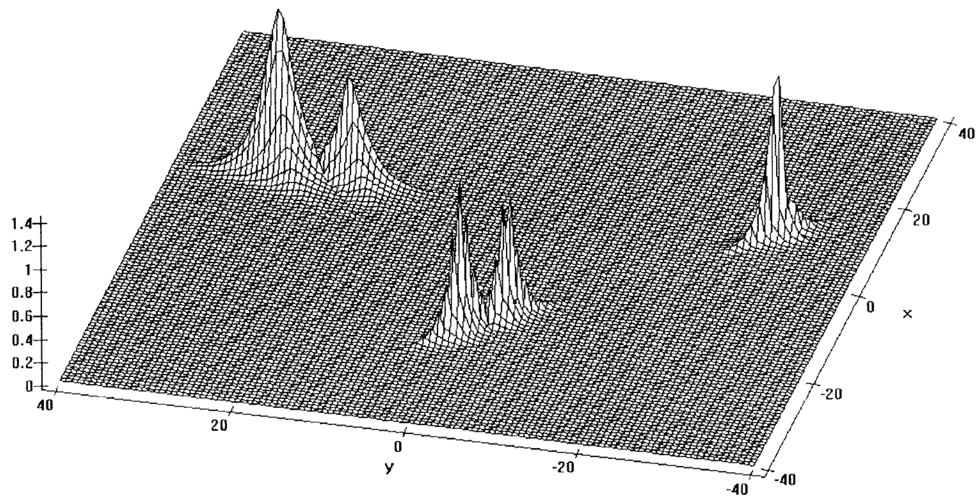

Figure 5. $U^{[1,1,2]}$ of the 3-wave equation.

When condition (47) holds, this theorem is useless, because the evolution will always separate the peaks. However, when (47) does not hold, especially when it is never satisfied, this theorem reveals a fact of the separation of the peaks.

Example. N-wave equation.

Let $n = 1$, $\omega = L\lambda$ where $L = \text{diag}(L_1, \ldots, L_N)$ is a constant real diagonal matrix such that $L_j \neq L_k$ for $j \neq k$. Then, the integrability conditions (6)–(9) imply

$$F_y = JF_x + UF \qquad F_t = LF_x + VF \tag{62}$$

$$[J, V] = [L, U] \qquad U_t - V_y + [U.V] + JV_x - LU_x = 0 \tag{63}$$

$$U_x = [J, FF^*] \tag{64}$$

(63) is just the N-wave equation.

Suppose $U^{[m]}$ is constructed as above, then theorem 1 implies that $U^{[m]} \to 0$ as $(x, y) \to \infty$ in any directions. Theorem 2 cannot be applied here. The reason is: condition (47) only holds if $l_\alpha \neq l_\beta$ for $\alpha \neq \beta$. Hence for any j, $m_j = 0$ or 1. This implies that (47) does not hold generally unless $m_j = 0$ or 1 for all $1 \leqslant j \leqslant N$. Therefore, we apply theorem 3 to the previous problem. Theorem 3 implies that if we choose $\{p_\alpha\}$ such that (56) is satisfied, then, for each (j, k), $\lim_{\tau\to\infty} U_{jk}^{[m]}$ has at most $m_j m_k$ peaks. When $K = 1$, these peaks do not vanish if and only if κ_α's are all nonzero.

Remark. Here $\tau \to \infty$ means that the phase differences of different peaks tend to infinity. Therefore, the peaks are separated by enlarging the phase differences.

Here are the figures describing the solutions $U^{[0,1,2]}$ and $U^{[1,1,2]}$ of the 3-wave equation. The vertical axis is $(|u_{12}|^2 + |u_{13}|^2 + |u_{23}|^2)^{1/4}$ so that all the components are shown in one figure. The parameters are

$$J = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \qquad L = \begin{pmatrix} 2 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

$K = 1$, $t = 10$, $\lambda_1 = 1 - 2\mathrm{i}$, $\lambda_2 = -3 - \mathrm{i}$, $\lambda_3 = 2 + \mathrm{i}$, $\lambda_4 = -1 + 3\mathrm{i}$, $C_1 = (0, 1, 0)$, $C_2 = (0, 0, 1)$, $C_3 = (0, 0, 4096)$, $C_4 = (1, 0, 0)$. Note that for $U^{[0,1,2]}$, only U_{23} has two peaks, and for $U^{[1,1,2]}$, U_{12}, U_{13}, U_{23} have one, two, two peaks respectively.

Acknowledgments

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