Darboux Transformations and Global Explicit Solutions for Nonlocal Davey–Stewartson I Equation

By Zi-Xiang Zhou

For the nonlocal Davey–Stewartson I equation, the Darboux transformation is considered and explicit expressions of the solutions are obtained. Like other nonlocal equations, many solutions of this equation may have singularities. However, by suitable choice of parameters in the solutions of the Lax pair, it is proved that the solutions obtained from seed solutions which are zero and an exponential function of t, respectively, by a Darboux transformation of degree n are global solutions of the nonlocal Davey–Stewartson I equation. The derived solutions are soliton solutions when the seed solution is zero, in the sense that they are bounded and have n peaks, and "dark cross soliton" solutions when the seed solution is an exponential function of t, in the sense that they are bounded and their norms change fast along some crossing straight lines.

1. Introduction

In [1], Ablowitz and Musslimani introduced the nonlocal nonlinear Schrödinger equation and got its explicit solutions by inverse scattering. Quite a lot of work was done after that for this equation and the others [2–14].

In [15], Fokas studied high-dimensional equations and introduced a nonlocal Davey–Stewartson equation (here, we call it nonlocal Davey–Stewartson I equation)

$$\mathrm{i}u_t = u_{xx} + u_{yy} + 2\sigma u^2\bar{u}^* + 2uw_y, \quad w_{xx} - w_{yy} = 2\sigma(u\bar{u}^*)_y, \tag{1}$$

Address for correspondence: Zi-Xiang Zhou, School of Mathematical Sciences, Fudan University, Shanghai 200433, China; e-mail: zxzhou@fudan.edu.cn

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where w satisfies $\bar{w}^* = w$. Here, $\bar{f}(x, y, t) = f(-x, y, t)$ for a function f, $*$ refers to complex conjugation. The solution of (1) is $\mathcal{PT}$-symmetric in the sense that if $(u(x, y, t), w(x, y, t))$ is a solution of (1), then so is $(u^*(-x, y, -t), w^*(-x, y, -t))$. This leads to a conserved density $u\bar{u}^*$, which is invariant under $x \to -x$ together with complex conjugation, and satisfies the conservation law

$$\mathrm{i}(u\bar{u}^*)_t = (u_x\bar{u}^* - u(\bar{u}^*)_x)_x + \left(u_y\bar{u}^* - u\bar{u}^*_y\right)_y. \tag{2}$$

Some global solutions of (1) and related equations have been obtained in [16–19].

As is known, the usual Davey–Stewartson II equation has a Darboux transformation in differential form [20, 21]. However, the usual Davey–Stewartson I equation does not possess a Darboux transformation in differential form. Instead, it has a binary Darboux transformation in integral form [22, 23]. Although the nonlocal Davey–Stewartson I equation looks like the usual Davey–Stewartson I equation, it is essentially more close to the usual Davey–Stewartson II equation. In fact, for analytic solutions, (1) can be derived formally from the standard Davey–Stewartson II equation

$$-\mathrm{i}u_t = -u_{xx} + u_{yy} + 2\sigma u^2u^* + 2uw_y, \quad -w_{xx} - w_{yy} = 2\sigma(uu^*)_y \tag{3}$$

by substituting $x \to \mathrm{i}x$, $t \to -t$ [24].

Therefore, we can construct a Darboux transformation in differential form for the nonlocal Davey–Stewartson I equation (1). Like other nonlocal equations, the derived solutions of this equation may have singularities. Starting from the seed solutions which are zero and an exponential function of t, respectively, we prove that the derived solutions can be globally defined and bounded for all $(x, y, t) \in \mathbf{R}^3$ if the parameters are suitably chosen. Unlike the usual Davey–Stewartson I equation where localized solutions are dromion solutions if the seed solution is zero [25–27], the derived solutions here are soliton solutions in the sense that there are n peaks in each solution $\widetilde{u}$ obtained from a Darboux transformation of degree n. (Because w is only an auxiliary function in (1), we will study mainly the behavior of u.) If the seed solution is an exponential function of t, the norms of the derived solutions $\widetilde{u}$ change a lot along some crossing straight lines. We call them "dark cross soliton" solutions.

In Section 2 of this paper, the Lax pair for the nonlocal Davey–Stewartson I equation is reviewed and its symmetries are considered. Then, the Darboux transformation is constructed and the explicit expressions of the new solutions are derived. In Sections 3 and 4, the soliton solutions and "dark cross soliton" solutions are constructed, respectively. The globalness, boundedness, and the asymptotic behavior of those solutions are discussed.

2. Lax pair and Darboux transformation

Consider the 2×2 linear system

$$
\begin{aligned}
\Phi_x &= U(\partial)\Phi \stackrel{\triangle}{=} J\Phi_y + P\Phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Phi_y + \begin{pmatrix} 0 & -u \\ \sigma\bar{u}^* & 0 \end{pmatrix}\Phi, \\
\Phi_t &= V(\partial)\Phi \stackrel{\triangle}{=} -2\mathrm{i}J\Phi_{yy} - 2\mathrm{i}P\Phi_y + \mathrm{i}Q\Phi \\
&= -2\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Phi_{yy} - 2\mathrm{i}\begin{pmatrix} 0 & -u \\ \sigma\bar{u}^* & 0 \end{pmatrix}\Phi_y \\
&\quad + \mathrm{i}\begin{pmatrix} -\sigma u\bar{u}^* - w_y - w_x & u_x + u_y \\ \sigma(\bar{u}^*)_x - \sigma(\bar{u}^*)_y & \sigma u\bar{u}^* + w_y - w_x \end{pmatrix}\Phi
\end{aligned}
\tag{4}
$$

where u, w are functions of (x, y, t) satisfying $\bar{w}^* = w$, $\partial = \frac{\partial}{\partial y}$, and $U(\partial)$ implies that U is a differential operator with respect to y. The compatibility condition $\Phi_{xt} = \Phi_{tx}$ gives the evolution equation (1).

The coefficients in the Lax pair (4) satisfy

$$
\bar{J}^* = -KJK^{-1}, \quad \bar{P}^* = -KPK^{-1}, \quad \bar{Q}^* = -KQK^{-1}, \tag{5}
$$

where $K = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$. Here, M^* refers to the complex conjugation (without transpose) of a matrix M. (5) can be written equivalently as

$$
\overline{U(\partial)^*} = -KU(\partial)K^{-1}, \quad \overline{V(\partial)^*} = KV(\partial)K^{-1}. \tag{6}
$$

Hence, we have

Lemma 1. *If* $\Phi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ *is a solution of* (4)*, then so is* $K\bar{\Phi}^* = \begin{pmatrix} \sigma\bar{\eta}^* \\ \bar{\xi}^* \end{pmatrix}$.

By Lemma 1, take a solution $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ of (4) and let $H = \begin{pmatrix} \xi & \sigma\bar{\eta}^* \\ \eta & \bar{\xi}^* \end{pmatrix}$, then $G(\partial) = \partial - S$ with $S = H_y H^{-1}$ gives a Darboux transformation [20, 21]. After the action of $G(\partial)$, (u, w) is transformed to $(\widetilde{u}, \widetilde{w})$ by

$$
G(\partial)U(\partial) + G_x(\partial) = \widetilde{U}(\partial)G(\partial), \quad G(\partial)V(\partial) + G_t(\partial) = \widetilde{V}(\partial)G(\partial). \tag{7}
$$

What is more, $G(\partial)$ satisfies $K\bar{G}(\partial)^*K^{-1} = G(\partial)$, which implies that $G(\partial)$ keeps the symmetries in (6) invariant, i.e.,

$$
\overline{\widetilde{U}(\partial)^*} = -K\widetilde{U}(\partial)K^{-1}, \quad \overline{\widetilde{V}(\partial)^*} = K\widetilde{V}(\partial)K^{-1}. \tag{8}
$$

Written explicitly,

$$
S = \frac{1}{\xi\bar{\xi}^* - \sigma\eta\bar{\eta}^*}\begin{pmatrix} \bar{\xi}^*\xi_y - \sigma\eta(\bar{\eta}^*)_y & \sigma\xi(\bar{\eta}^*)_y - \sigma\bar{\eta}^*\xi_y \\ \bar{\xi}^*\eta_y - \eta(\bar{\xi}^*)_y & \xi(\bar{\xi}^*)_y - \sigma\bar{\eta}^*\eta_y. \end{pmatrix}. \tag{9}
$$

This gives the explicit expression of the derived solution

$$\begin{aligned} \widetilde{u} &= u + 2\sigma \frac{\bar{\eta}^* \xi_y - \xi (\bar{\eta}^*)_y}{\bar{\xi}^* \xi - \sigma \bar{\eta}^* \eta}, \\ \widetilde{w} &= w + 2 \frac{(\bar{\xi}^* \xi - \sigma \bar{\eta}^* \eta)_y}{\bar{\xi}^* \xi - \sigma \bar{\eta}^* \eta}. \end{aligned} \tag{10}$$

Clearly, $\widetilde{w}$ satisfies $\bar{\widetilde{w}}^* = \widetilde{w}$.

The Darboux transformation of degree n is given by a 2×2 matrix-valued differential operator (Darboux operator)

$$G(\partial) = \partial^n + G_1 \partial^{n-1} + \cdots + G_n \tag{11}$$

of degree n which is determined by

$$G(\partial) H_j = 0 \quad (j = 1, \dots, n) \tag{12}$$

for n matrix solutions $H_j = \begin{pmatrix} \xi_j & \sigma \bar{\eta}_j^* \\ \eta_j & \bar{\xi}_j^* \end{pmatrix}$ $(j = 1, \dots, n)$ of (4). According to [21], it is a composition of n Darboux operators of degree one. Hence, it still satisfies $K \bar{G}(\partial)^* K^{-1} = G(\partial)$ and keeps the symmetries in (6) invariant.

By comparing the coefficients of ∂^n and ∂^{n-1} in the first equation of (7) and the coefficients of ∂^{n+1} and ∂^n in the second equation of (7), we have

$$\widetilde{P} = P - [J, G_1], \quad \widetilde{Q} = Q + 2[J, G_2] - 2[JG_1 - P, G_1] + 4JG_{1,y} - 2nP_y \tag{13}$$

and

$$\begin{aligned} (G_1)_{11,x} - (G_1)_{11,y} &= -u(G_1)_{21} - 2(G_1)_{12}(G_1)_{21} - \sigma \bar{u}^* (G_1)_{12}, \\ (G_1)_{22,x} + (G_1)_{22,y} &= \sigma \bar{u}^* (G_1)_{12} + 2(G_1)_{12}(G_1)_{21} + u(G_1)_{21}. \end{aligned} \tag{14}$$

(Other two equations concerning the derivatives of $(G_1)_{12}$ and $(G_1)_{21}$ are omitted here because they will not be used.)

With (14) and $(\bar{G}_1)_{12}^* = \sigma (G_1)_{21}$ which is a consequence of $K \bar{G}(\partial)^* K^{-1} = G(\partial)$, we can solve $(\widetilde{u}, \widetilde{w})$ from (13) explicitly. This gives the following lemma.

Lemma 2. *If* $G(\partial)$ *given by* (11) *is a Darboux operator which keeps the symmetries in* (6) *invariant, then* (u, w) *is transformed to*

$$\widetilde{u} = u + 2(G_1)_{12}, \quad \widetilde{w} = w - 2 \operatorname{tr} G_1. \tag{15}$$

$K \bar{G}(\partial)^* K^{-1} = G(\partial)$ also leads to $(\bar{G}_1)_{11}^* = \sigma (G_1)_{22}$, which implies $\bar{\widetilde{w}}^* = \widetilde{w}$.

(12) can be written as

$$\partial^n H_j + G_1 \partial^{n-1} H_j + \cdots + G_n H_j = 0 \quad (j = 1, \dots, n). \tag{16}$$

Hence,

$$
\begin{aligned}
&(G_1 \quad G_2 \quad \cdots \quad G_n)\begin{pmatrix} \partial^{n-1}H_1 & \partial^{n-1}H_2 & \cdots & \partial^{n-1}H_n \\ \partial^{n-2}H_1 & \partial^{n-2}H_2 & \cdots & \partial^{n-2}H_n \\ \vdots & \vdots & & \vdots \\ H_1 & H_2 & \cdots & H_n \end{pmatrix} \\
&\quad = (-\partial^n H_1 \quad -\partial^n H_2 \quad \cdots \quad -\partial^n H_n).
\end{aligned} \tag{17}
$$

Write $H_j = \begin{pmatrix} h_{11}^{(j)} & h_{12}^{(j)} \\ h_{21}^{(j)} & h_{22}^{(j)} \end{pmatrix}$. In order to analyze the behavior of the derived solutions, we reorder the rows and columns to get

$$
\begin{pmatrix} (G_1)_{11} & \cdots & (G_n)_{11} & (G_1)_{12} & \cdots & (G_n)_{12} \\ (G_1)_{21} & \cdots & (G_n)_{21} & (G_1)_{22} & \cdots & (G_n)_{22} \end{pmatrix} W = -R, \tag{18}
$$

where $W = (W_{jk})_{1\le j,k\le 2}$, $R = (R_{jk})_{1\le j,k\le 2}$,

$$
W_{jk} = \begin{pmatrix} \partial^{n-1}h_{jk}^{(1)} & \partial^{n-1}h_{jk}^{(2)} & \cdots & \partial^{n-1}h_{jk}^{(n)} \\ \partial^{n-2}h_{jk}^{(1)} & \partial^{n-2}h_{jk}^{(2)} & \cdots & \partial^{n-2}h_{jk}^{(n)} \\ \vdots & \vdots & & \vdots \\ h_{jk}^{(1)} & h_{jk}^{(2)} & \cdots & h_{jk}^{(n)} \end{pmatrix}, \tag{19}
$$

$$
R_{jk} = \begin{pmatrix} \partial^n h_{jk}^{(1)} & \partial^n h_{jk}^{(2)} & \cdots & \partial^n h_{jk}^{(n)} \end{pmatrix} \quad (j, k = 1, 2). \tag{20}
$$

New solution (15) of Eq. (1) is obtained by solving G_1 from (18).

3. Soliton solutions

3.1. Single soliton solutions

Let $u = 0$, then $\Phi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ satisfies

$$
\Phi_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_y, \quad \Phi_t = -2\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_{yy}. \tag{21}
$$

Take a special solution

$$
\begin{aligned}
\xi &= \mathrm{e}^{\lambda x+\lambda y-2\mathrm{i}\lambda^2 t} + \mathrm{e}^{-\lambda^* x-\lambda^* y-2\mathrm{i}\lambda^{*2} t}, \\
\eta &= a\mathrm{e}^{\lambda x-\lambda y+2\mathrm{i}\lambda^2 t} + b\mathrm{e}^{-\lambda^* x+\lambda^* y+2\mathrm{i}\lambda^{*2} t},
\end{aligned} \tag{22}
$$

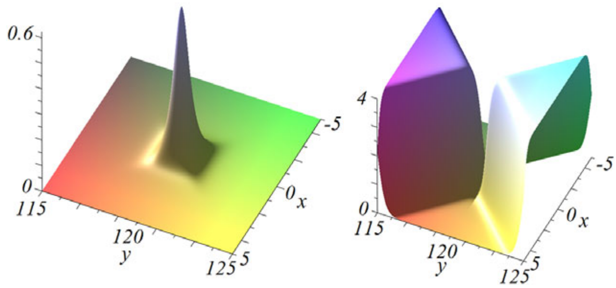

Figure 1. $|\widetilde{u}|$ (left) and $|\widetilde{w}|$ (right) of a 1 soliton solution.

where λ, a, b are complex constants. (10) gives the explicit solution

$$\widetilde{u} = \frac{2\sigma\lambda_R(a^* - b^*)}{D}\mathrm{e}^{2\mathrm{i}\lambda_I y - 4\mathrm{i}\left(\lambda_R^2 - \lambda_I^2\right)t}, \quad \widetilde{w} = \frac{D_y}{D}\mathrm{e}^{-2\mathrm{i}\lambda_I x} \tag{23}$$

of (1) where

$$\begin{aligned} D = {} & (2 - \sigma|a|^2 - \sigma|b|^2)\cosh(2\lambda_R y + 8\lambda_R\lambda_I t) \\ & + \sigma(|a|^2 - |b|^2)\sinh(2\lambda_R y + 8\lambda_R\lambda_I t) \\ & + 2(1 - \sigma\ \mathrm{Re}(ab^*))\cosh(2\lambda_R x) - 2\mathrm{i}\sigma\ \mathrm{Im}(ab^*)\sinh(2\lambda_R x). \end{aligned} \tag{24}$$

Here, $z_R = \mathrm{Re}\ z$ and $z_I = \mathrm{Im}\ z$ for a complex number z.

Note that $(\widetilde{u}, \widetilde{w})$ is global if $|a| < 1$ and $|b| < 1$ because Re $D > 0$ in this case. Moreover, the peak moves in the velocity $(v_x, v_y) = (0, -4\lambda_I)$.

Remark 1. The solution (23) may have singularities when the parameters are not chosen as above, say, when $\sigma = -1$, $a = 1$, $b = -2$, or $\sigma = 1$, $a = 2$, $b = 1/4$.

Figure 1 shows a 1 soliton solution with parameters $\sigma = -1$, $t = 20$, $\lambda = 2 - 1.5\mathrm{i}$, $a = 0.4$, $b = 0.5\mathrm{i}$.

3.2. Multiple soliton solutions

For an $n \times n$ matrix M, define $||M|| = \sup_{x\in\mathbf{C}^n, ||x||=1} ||Mx||$ where $||\cdot||$ is the standard Hermitian norm in $\mathbf{C}^n$. The following facts hold obviously.

(i) $||MN|| \le ||M||\,||N||$.
(ii) Each entry M_{jk} of M satisfies $|M_{jk}| \le ||M||$.
(iii) $|\det M| \le ||M||^n$.
(iv) If $||M|| < 1$, then $||(I + M)^{-1}|| \le (1 - ||M||)^{-1}$.
(v) If $||M|| < 1$, then $|\det(I + M)| \ge (1 - ||M||)^n$.

Now, we construct explicit solutions according to (15). As in (22), take

$$\begin{aligned}\xi_k &= \mathrm{e}^{\lambda_k(x+y)-2\mathrm{i}\lambda_k^2 t} + \mathrm{e}^{-\lambda_k^*(x+y)-2\mathrm{i}\lambda_k^{*2} t},\\ \eta_k &= a_k\mathrm{e}^{\lambda_k(x-y)+2\mathrm{i}\lambda_k^2 t} + b_k\mathrm{e}^{-\lambda_k^*(x-y)+2\mathrm{i}\lambda_k^{*2} t} \quad (k=1,2,\ldots,n)\end{aligned} \tag{25}$$

where $\lambda_1,\ldots,\lambda_n$ are distinct complex constants, then W_{jk}'s in (19) and R_{jk}'s in (20) are

$$\begin{aligned}
&(W_{11})_{jk} = \lambda_k^{n-j} e_{k+} + (-\lambda_k^*)^{n-j} e_{k+}^{*-1},\\
&(W_{12})_{jk} = \sigma a_k^*(-\lambda_k^*)^{n-j} e_{k+}^{*-1} + \sigma b_k^* \lambda_k^{n-j} e_{k+},\\
&(W_{21})_{jk} = a_k(-\lambda_k)^{n-j} e_{k-} + b_k(\lambda_k^*)^{n-j} e_{k-}^{*-1},\\
&(W_{22})_{jk} = (\lambda_k^*)^{n-j} e_{k-}^{*-1} + (-\lambda_k)^{n-j} e_{k-},\\
&(R_{11})_{1k} = \lambda_k^n e_{k+} + (-\lambda_k^*)^n e_{k+}^{*-1},\\
&(R_{12})_{1k} = \sigma a_k^*(-\lambda_k^*)^n e_{k+}^{*-1} + \sigma b_k^* \lambda_k^n e_{k+},\\
&(R_{21})_{1k} = a_k(-\lambda_k)^n e_{k-} + b_k(\lambda_k^*)^n e_{k-}^{*-1},\\
&(R_{22})_{1k} = (\lambda_k^*)^n e_{k-}^{*-1} + (-\lambda_k)^n e_{k-}\\
&(j,k=1,\ldots,n)
\end{aligned} \tag{26}$$

where

$$e_{k\pm} = \mathrm{e}^{\lambda_k(x\pm y)\mp 2\mathrm{i}\lambda_k^2 t}. \tag{27}$$

However, temporarily, we assume e_{k+}, e_{k-} $(k=1,\ldots,n)$ are arbitrary complex numbers instead of those satisfying (27).

Denote $L = \operatorname{diag}((-1)^{n-1}, (-1)^{n-2}, \ldots, -1, 1)$,

$$F = \left(\lambda_k^{n-j}\right)_{1\le j,k\le n}, \quad f = \left(\lambda_1^n,\ldots,\lambda_n^n\right), \tag{28}$$

$$A = \operatorname{diag}(a_1,\ldots,a_n), \quad B = \operatorname{diag}(b_1,\ldots,b_n), \tag{29}$$

$$E_\pm = \operatorname{diag}(e_{1\pm},\ldots,e_{n\pm}). \tag{30}$$

Then,

$$W = \begin{pmatrix} FE_+ + LF^*E_+^{*-1} & \sigma FB^*E_+ + \sigma LF^*A^*E_+^{*-1}\\ LFAE_- + F^*BE_-^{*-1} & LFE_- + F^*E_-^{*-1}\end{pmatrix}, \tag{31}$$

$$R = \begin{pmatrix} fE_+ + (-1)^n f^*E_+^{*-1} & \sigma fB^*E_+ + \sigma(-1)^n f^*A^*E_+^{*-1}\\ (-1)^n fAE_- + f^*BE_-^{*-1} & (-1)^n fE_- + f^*E_-^{*-1}\end{pmatrix}. \tag{32}$$

By using the identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B\Delta^{-1}CA^{-1} & -A^{-1}B\Delta^{-1} \\ -\Delta^{-1}CA^{-1} & \Delta^{-1} \end{pmatrix} \tag{33}$$

for a block matrix where $\Delta = D - CA^{-1}B$, (18) gives

$$((G_1)_{12}, \ldots, (G_n)_{12}) = -(RW^{-1})_{12} = -\left(R_{12} - R_{11}W_{11}^{-1}W_{12}\right)\overset{\circ}{W}^{-1}, \tag{34}$$

where

$$\begin{aligned} \overset{\circ}{W} &= W_{22} - W_{21}W_{11}^{-1}W_{12} \\ &= L\left(FE_- + LF^*E_-^{*-1} - \sigma(FAE_- + LF^*BE_-^{*-1})\cdot\right. \\ &\quad \left.\cdot(FE_+ + LF^*E_+^{*-1})^{-1}(FB^*E_+ + LF^*A^*E_+^{*-1})\right). \end{aligned} \tag{35}$$

LEMMA 3. *Suppose a_j and b_j are nonzero complex constants with $|a_j| < 1$, $|b_j| < 1$ $(j = 1, \ldots, n)$, $\kappa_1, \ldots, \kappa_n$ are nonzero real constants with $|\kappa_j| \neq |\kappa_k| (j, k = 1, \ldots, n;\ j \neq k)$, then there exist positive constants δ, C_1 and C_2, which depend on a_j's, b_j's and κ_j's, such that $|\det W| \geq C_1$ and*

$$|(G_1)_{12}| \leq C_2 \max_{1\leq k\leq n} \frac{|e_{k+}|}{1 + |e_{k+}|^2} \max_{1\leq k\leq n} \frac{|e_{k-}|}{1 + |e_{k-}|^2} \tag{36}$$

hold whenever $|\lambda_j - \mathrm{i}\kappa_j| < \delta$ and $e_{j\pm} \in \mathbf{C} (j = 1, \ldots, n)$.

Proof. Denote $F^{-1}LF^* = I + Z$, then $Z = 0$ if $\lambda_1, \ldots, \lambda_n$ are all purely imaginary. From (31) and (35),

$$\det W = \det(FE_+ + LF^*E_+^{*-1}) \det \overset{\circ}{W}, \tag{37}$$

$$\overset{\circ}{W} = L(FE_- + LF^*E_-^{*-1})(I - \sigma\chi_-\chi_+) \tag{38}$$

where

$$\begin{aligned} \chi_+ &= (FE_+ + LF^*E_+^{*-1})^{-1}(FB^*E_+ + LF^*A^*E_+^{*-1}) \\ &= \Xi_{1+}\Xi_{0+}^{-1} + \Xi_{0+}^{-1}(I + ZE_+^{*-1}\Xi_{0+}^{-1})^{-1}ZE_+^{*-1}(A^* - \Xi_{1+}\Xi_{0+}^{-1}), \\ \chi_- &= (FE_- + LF^*E_-^{*-1})^{-1}(FAE_- + LF^*BE_-^{*-1}) \\ &= \Xi_{1-}\Xi_{0-}^{-1} + \Xi_{0-}^{-1}(I + ZE_-^{*-1}\Xi_{0-}^{-1})^{-1}ZE_-^{*-1}(B - \Xi_{1-}\Xi_{0-}^{-1}), \end{aligned} \tag{39}$$

$$\Xi_{0\pm} = E_\pm + E_\pm^{*-1}, \quad \Xi_{1-} = AE_- + BE_-^{*-1}, \quad \Xi_{1+} = B^*E_+ + A^*E_+^{*-1}. \tag{40}$$

Let $c_0 = \max_{1\le k\le n}\{|a_k|, |b_k|\} < 1$. Suppose $\lambda_1, \ldots, \lambda_n$ are chosen so that $||Z|| < \frac{1-c_0}{2}$, then we have the following estimates.

$$||A|| \le c_0 < 1, \quad ||B|| \le c_0 < 1, \tag{41}$$

$$\left\| E_\pm \Xi_{0\pm}^{-1} \right\| \le 1, \quad \left\| E_\pm^{*-1} \Xi_{0\pm}^{-1} \right\| \le 1, \tag{42}$$

$$||\Xi_{0\pm}|| \ge 2, \quad \left\| \Xi_{0\pm}^{-1} \right\| \le \frac{1}{2}, \quad \left\| \Xi_{1\pm} \Xi_{0\pm}^{-1}| \right\| \le c_0 < 1, \tag{43}$$

$$|\det \Xi_{0\pm}| \ge 2^n, \tag{44}$$

$$\begin{aligned} \left\| E_+^{*-1}(\Xi_{1+}\Xi_{0+}^{-1} - A^*) \right\| &= \max_{1\le k\le n} \frac{|a_k - b_k|\,|e_{k+}|}{1 + |e_{k+}|^2} \le 1, \\ \left\| E_-^{*-1}(\Xi_{1-}\Xi_{0-}^{-1} - B) \right\| &= \max_{1\le k\le n} \frac{|a_k - b_k|\,|e_{k-}|}{1 + |e_{k-}|^2} \le 1, \end{aligned} \tag{45}$$

$$\left\| \left(I + ZE_\pm^{*-1}\Xi_{0\pm}^{-1}\right)^{-1} \right\| \le \left(1 - ||Z||\, \left\| E_\pm^{*-1}\Xi_{0\pm}^{-1} \right\|\right)^{-1} \le (1 - ||Z||)^{-1} \le 2, \tag{46}$$

$$\left|\det\left(I + ZE_\pm^{*-1}\Xi_{0\pm}^{-1}\right)\right| \ge (1 - ||Z||)^{-1} \ge 2^{-n}. \tag{47}$$

Hence, $||\chi_\pm - \Xi_{1\pm}\Xi_{0\pm}^{-1}|| \le ||Z||$ by (39), and

$$||\chi_\pm|| \le c_0 + ||Z|| \le \frac{1 + c_0}{2} < 1. \tag{48}$$

Denote

$$\pi_0 = |\det F|\Big|_{\substack{\lambda_j = \mathrm{i}\kappa_j \\ j=1,\ldots,n}}, \quad \pi_1 = ||F^{-1}||\Big|_{\substack{\lambda_j = \mathrm{i}\kappa_j \\ j=1,\ldots,n}}, \quad \pi_2 = ||f||\Big|_{\substack{\lambda_j = \mathrm{i}\kappa_j \\ j=1,\ldots,n}}. \tag{49}$$

Clearly, π_0, π_1, π_2 are all positive because $\det F|_{\substack{\lambda_j = \mathrm{i}\kappa_j \\ j=1,\ldots,n}}$ is a Vandermonde determinant. By the continuity, there exists $\delta > 0$ such that $\frac{\pi_0}{2} \le |\det F| \le 2\pi_0$, $||F^{-1}|| \le 2\pi_1$, $||f|| \le 2\pi_2$, and $||F^{-1}LF^* - I|| = ||Z|| < \frac{1-c_0}{2}$ whenever $|\lambda_j - \mathrm{i}\kappa_j| < \delta$. By using the estimates (41)–(48), (37) and (38) lead to

$$\begin{aligned} |\det W| = {} & |\det F|^2\, |\det \Xi_{0+}|\, \left|\det\left(I + ZE_+^{*-1}\Xi_{0+}^{-1}\right)\right| \cdot \\ & \cdot |\det \Xi_{0-}|\, \left|\det\left(I + ZE_-^{*-1}\Xi_{0-}^{-1}\right)\right|\, |\det(1 - \sigma\chi_-\chi_+)| \\ \ge {} & \pi_0^2 (1 - ||Z||)^{2n} (1 - ||\chi_+||\, ||\chi_-||)^n \end{aligned}$$

$$\geq \pi_0^2 \left(\frac{1+c_0}{2}\right)^{2n} \left(1 - \left(\frac{1+c_0}{2}\right)^2\right)^n, \tag{50}$$

which is a uniform positive lower bound for any $e_{j\pm} \in \mathbf{C}(j = 1, \ldots, n)$ when $|\lambda_j - \mathrm{i}\kappa_j| < \delta$.

By (31), (32), (34), (35), and (38),

$$\begin{aligned}
((G_1)_{12}, (G_2)_{12}, \ldots, (G_n)_{12}) &= -\left(R_{12} - R_{11} W_{11}^{-1} W_{12}\right) \overset{\circ}{W}^{-1} \\
&= -\sigma f E_+ \Xi_{0+}^{-1} \left(I + Z E_+^{*-1} \Xi_{0+}^{-1}\right)^{-1} (I+Z)(B^* - A^*) E_+^{*-1} \overset{\circ}{W}^{-1} \\
&\quad - \sigma(-1)^n f^* E_+^{*-1} \Xi_{0+}^{-1} \left(I + Z E_+^{*-1} \Xi_{0+}^{-1}\right)^{-1} (A^* - B^*) E_+ \overset{\circ}{W}^{-1} \\
&= -\sigma \left(f - (-1)^n f^* + \left(f E_+ \Xi_{0+}^{-1} + (-1)^n f^* E_+^{*-1} \Xi_{0+}^{-1}\right) \cdot \right. \\
&\quad \cdot \left. \left(I + Z E_+^{*-1} \Xi_{0+}^{-1}\right)^{-1} Z\right) (B^* - A^*) E_+ E_+^{*-1} \Xi_{0+}^{-1} \cdot \\
&\quad \cdot (I - \sigma \chi_- \chi_+)^{-1} \Xi_{0-}^{-1} \left(I + Z E_-^{*-1} \Xi_{0-}^{-1}\right)^{-1} F^{-1} L^{-1}.
\end{aligned} \tag{51}$$

Here, we have used $I + Z = (I + Z E_+^{*-1} \Xi_{0+}^{-1}) + Z E_+ \Xi_{0+}^{-1}$. Hence, by using (41)–(48),

$$\begin{aligned}
|(G_1)_{12}| &\leq 8(||f|| + ||f^*||)||F^{-1}||\, ||(I - \sigma \chi_- \chi_+)^{-1}||\, ||\Xi_{0+}^{-1}||\, ||\Xi_{0-}^{-1}|| \\
&\leq 64 \pi_1 \pi_2 \left(1 - \left(\frac{1+c_0}{2}\right)^2\right)^{-1} \max_{1 \leq k \leq n} \frac{|e_{k+}|}{1 + |e_{k+}|^2} \max_{1 \leq k \leq n} \frac{|e_{k-}|}{1 + |e_{k-}|^2}.
\end{aligned} \tag{52}$$

■

Now, we consider the solutions of the nonlocal Davey–Stewartson I equation. That is, we consider the case where $e_{j\pm}$'s are taken as (27).

THEOREM 1. *Suppose a_j and b_j are nonzero complex constants with $|a_j| < 1$, $|b_j| < 1$ $(j = 1, \ldots, n)$, $\kappa_1, \ldots, \kappa_n$ are nonzero real constants with $|\kappa_j| \neq |\kappa_k| (j, k = 1, \ldots, n;\ j \neq k)$, then there exists a positive constant δ, which depend on a_j's, b_j's and κ_j's, such that the following results hold for the derived solution $\widetilde{u} = 2(G_1)_{12}$ of the nonlocal Davey–Stewartson I equation when* Re $\lambda_j \neq 0$ *and $|\lambda_j - \mathrm{i}\kappa_j| < \delta (j = 1, \ldots, n)$.*

(i) $\widetilde{u}$ is defined globally for $(x, y, t) \in \mathbf{R}^3$.
(ii) For fixed t, $\widetilde{u}$ tends to zero exponentially as $(x, y) \to 0$.
(iii) Let $y = \widetilde{y} + vt$ and keep $(x, \widetilde{y})$ bounded, then $\widetilde{u} \to 0$ as $t \to \infty$ if $v \neq -4\lambda_{kI}$ for all k.

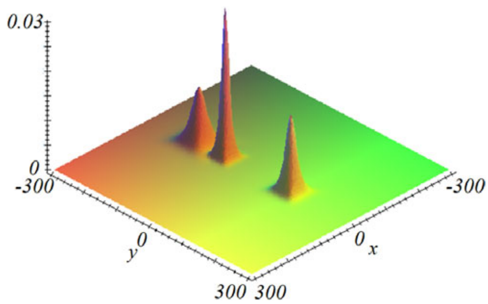

Figure 2. $|\widetilde{u}|$ of a 3 soliton solution.

Proof. We have known that $\widetilde{u}$ is a solution of the nonlocal Davey–Stewartson equation in Section 2.

(i) According to Lemma 3, $|\det W|$ has a uniform positive lower bound. Hence, $\widetilde{u}$ is defined globally.

(ii) When $x \geq 0$ and $y \geq 0$,

$$\begin{aligned} |e_{k+}| &\geq \mathrm{e}^{\lambda_{kR}\sqrt{x^2+y^2}+4\lambda_{kR}\lambda_{kI}t} && \text{if } \lambda_{kR} > 0, \\ |e_{k+}| &\leq \mathrm{e}^{-|\lambda_{kR}|\sqrt{x^2+y^2}+4\lambda_{kR}\lambda_{kI}t} && \text{if } \lambda_{kR} < 0. \end{aligned} \tag{53}$$

Hence, $\max_{1\leq k\leq n} \frac{|e_{k+}|}{1+|e_{k+}|^2}$ tends to zero exponentially when $x \geq 0$, $y \geq 0$ and $(x, y) \to \infty$.

Likewise, we have

$$\begin{aligned} |e_{k-}| &\leq \mathrm{e}^{-\lambda_{kR}\sqrt{x^2+y^2}-4\lambda_{kR}\lambda_{kI}t} && \text{if } x \leq 0,\ y \geq 0,\ \lambda_{kR} > 0, \\ |e_{k-}| &\geq \mathrm{e}^{|\lambda_{kR}|\sqrt{x^2+y^2}-4\lambda_{kR}\lambda_{kI}t} && \text{if } x \leq 0,\ y \geq 0,\ \lambda_{kR} < 0, \\ |e_{k+}| &\leq \mathrm{e}^{-\lambda_{kR}\sqrt{x^2+y^2}+4\lambda_{kR}\lambda_{kI}t} && \text{if } x \leq 0,\ y \leq 0,\ \lambda_{kR} > 0, \\ |e_{k+}| &\geq \mathrm{e}^{|\lambda_{kR}|\sqrt{x^2+y^2}+4\lambda_{kR}\lambda_{kI}t} && \text{if } x \leq 0,\ y \leq 0,\ \lambda_{kR} < 0, \\ |e_{k-}| &\geq \mathrm{e}^{\lambda_{kR}\sqrt{x^2+y^2}-4\lambda_{kR}\lambda_{kI}t} && \text{if } x \geq 0,\ y \leq 0,\ \lambda_{kR} > 0, \\ |e_{k-}| &\leq \mathrm{e}^{-|\lambda_{kR}|\sqrt{x^2+y^2}-4\lambda_{kR}\lambda_{kI}t} && \text{if } x \geq 0,\ y \leq 0,\ \lambda_{kR} < 0. \end{aligned} \tag{54}$$

Lemma 3 implies that $\widetilde{u} \to 0$ exponentially as $(x, y) \to \infty$.

(iii)

$$|e_{k\pm}| = \mathrm{e}^{\lambda_{kR}(x\pm\widetilde{y})\pm\lambda_{kR}(v+4\lambda_{kI})t}. \tag{55}$$

If $v \neq -4\lambda_{kI}$ for all $k = 1, \ldots, n$, then either $e_{k+} \to 0$ or $e_{k+} \to \infty$ for all $k = 1, \ldots, n$ when $t \to \infty$. Lemma 3 implies that $\widetilde{u} \to 0$ when $t \to \infty$. ■

A 3 soliton solution is shown in Fig. 2 where the parameters are $\sigma = -1$, $t = 20$, $\lambda_1 = 0.07 - 1.5\mathrm{i}$, $\lambda_2 = 0.05 + 2\mathrm{i}$, $\lambda_3 = 0.1 + \mathrm{i}$, $a_1 = 0.2$, $a_2 = 0.1\mathrm{i}$,

$a_3 = 0.1$, $b_1 = 0.1\mathrm{i}$, $b_2 = -0.2$, $b_3 = -0.2$. Here, only $|\widetilde{u}|$ is shown because $\widetilde{w}$ is an auxiliary function in (1). The figure of the solution appears similarly if σ is changed to $+1$, although it is not shown here.

4. "Dark cross soliton" solutions

4.1. Single "dark cross soliton" solutions

Now, we take

$$u = \rho \mathrm{e}^{-2\mathrm{i}\sigma|\rho|^2 t}, \quad w = 0 \tag{56}$$

as a solution of (1) where ρ is a complex constant. The Lax pair (4) has a solution

$$\begin{pmatrix} \mathrm{e}^{\alpha(\lambda)x+\beta(\lambda)y+\gamma(\lambda)t} \\ \frac{\lambda}{\rho}\mathrm{e}^{\alpha(\lambda)x+\beta(\lambda)y+(\gamma(\lambda)+2\mathrm{i}\sigma|\rho|^2)t} \end{pmatrix}, \tag{57}$$

where

$$\begin{aligned} &\alpha(\lambda) = \frac{1}{2}\Big(\frac{\sigma|\rho|^2}{\lambda} - \lambda\Big), \quad \beta(\lambda) = \frac{1}{2}\Big(\frac{\sigma|\rho|^2}{\lambda} + \lambda\Big), \\ &\gamma(\lambda) = \mathrm{i}(\alpha(\lambda)^2 - 2\alpha(\lambda)\beta(\lambda) - \beta(\lambda)^2) = \mathrm{i}\lambda^2 - \frac{\mathrm{i}}{2}\left(\frac{\sigma|\rho|^2}{\lambda} + \lambda\right)^2, \end{aligned} \tag{58}$$

λ is a complex constant.

Now, take $\Phi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ where

$$\begin{aligned} \xi &= \mathrm{e}^{\alpha x+\beta y+\gamma t} + \mathrm{e}^{-\alpha^* x-\beta^* y-\gamma^* t}, \\ \eta &= \frac{\lambda}{\rho}\mathrm{e}^{\alpha x+\beta y+(\gamma+2\mathrm{i}\sigma|\rho|^2)t} - \frac{\lambda^*}{\rho}\mathrm{e}^{-\alpha^* x-\beta^* y-(\gamma^*-2\mathrm{i}\sigma|\rho|^2)t}. \end{aligned} \tag{59}$$

Here, $\alpha = \alpha(\lambda)$, $\beta = \beta(\lambda)$, $\gamma = \gamma(\lambda)$. This Φ is a linear combination of the solutions of form (57). Then, (10) gives the new solution

$$\begin{aligned} \widetilde{u} &= \rho \mathrm{e}^{-2\mathrm{i}\sigma|\rho|^2 t} \frac{\frac{\lambda^*}{\lambda}c_1\mathrm{e}^{2\beta_R y+2\gamma_R t} + \frac{\lambda}{\lambda^*}c_1\mathrm{e}^{-2\beta_R y-2\gamma_R t} - c_2\mathrm{e}^{2\alpha_R x} - c_2^*\mathrm{e}^{-2\alpha_R x}}{c_1(\mathrm{e}^{2\beta_R y+2\gamma_R t} + \mathrm{e}^{-2\beta_R y-2\gamma_R t}) + c_2\mathrm{e}^{2\alpha_R x} + c_2^*\mathrm{e}^{-2\alpha_R x}} \\ \widetilde{w} &= \frac{4c_1\beta_R(\mathrm{e}^{2\beta_R y+2\gamma_R t} - \mathrm{e}^{-2\beta_R y-2\gamma_R t})}{c_1(\mathrm{e}^{2\beta_R y+2\gamma_R t} + \mathrm{e}^{-2\beta_R y-2\gamma_R t}) + c_2\mathrm{e}^{2\alpha_R x} + c_2^*\mathrm{e}^{-2\alpha_R x}} \end{aligned} \tag{60}$$

of the nonlocal Davey–Stewartson I equation where

$$c_1 = 1 - \sigma\frac{|\lambda|^2}{|\rho|^2}, \quad c_2 = 1 + \sigma\frac{\lambda^2}{|\rho|^2}. \tag{61}$$

This solution is smooth for all $(x, y, t) \in \mathbf{R}^3$ if $|\lambda| < |\rho|$.

Especially, if λ is real, then $\gamma_R = 0$; thus, we get a standing wave solution $\widetilde{u}$.

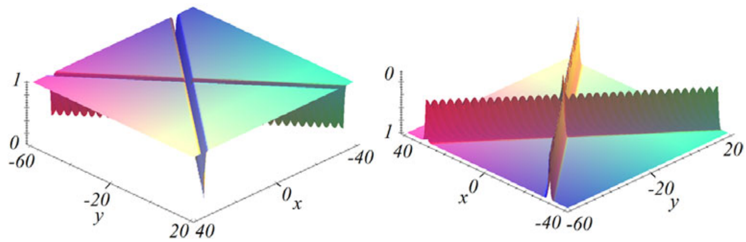

Figure 3. $|\widetilde{u}|$ of a 1 "dark cross soliton" solution.

Figure 3 shows $\widetilde{u}$ of a 1 "dark cross soliton" solution with parameters $\sigma = -1$, $t = 10$, $\rho = 1$, $\lambda = 0.3 + 0.1\mathrm{i}$. The figure on the right describes the same solution but is upside down.

4.2. *Multiple "dark cross soliton" solutions*

Now, we take n solutions

$$\begin{aligned}\xi_k &= \mathrm{e}^{\alpha_k x+\beta_k y+\gamma_k t} + \mathrm{e}^{-\alpha_k^* x-\beta_k^* y-\gamma_k^* t},\\ \eta_k &= \frac{\lambda_k}{\rho}\mathrm{e}^{\alpha_k x+\beta_k y+(\gamma_k+2\mathrm{i}\sigma|\rho|^2)t} - \frac{\lambda_k^*}{\rho}\mathrm{e}^{-\alpha_k^* x-\beta_k^* y-(\gamma_k^*-2\mathrm{i}\sigma|\rho|^2)t}\end{aligned} \tag{62}$$

($k = 1, 2, \ldots, n$) of form (59) with distinct $\lambda_1, \ldots, \lambda_n$ to get multiple "dark cross soliton" solutions. Here,

$$\begin{aligned}\alpha_k &= \frac{1}{2}\left(\frac{\sigma|\rho|^2}{\lambda_k} - \lambda_k\right), \quad \beta_k = \frac{1}{2}\left(\frac{\sigma|\rho|^2}{\lambda_k} + \lambda_k\right),\\ \gamma_k &= \mathrm{i}\left(\alpha_k^2 - 2\alpha_k\beta_k - \beta_k^2\right).\end{aligned} \tag{63}$$

Similar to (31) and (32), W and R in (18) are

$$W = \begin{pmatrix} FE_+ + LF^*E_+^{*-1} & -\sigma\rho^{*-1}\mathrm{e}^{-\mathrm{i}\phi}(LF\Lambda E_- - F^*\Lambda^*E_-^{*-1}) \\ \rho^{-1}\mathrm{e}^{\mathrm{i}\phi}(F\Lambda E_+ - LF^*\Lambda^*E_+^{*-1}) & LFE_- + F^*E_-^{*-1} \end{pmatrix}, \tag{64}$$

$$R = \begin{pmatrix} fE_+ + (-1)^n f^*E_+^{*-1} & -\sigma\rho^{*-1}\mathrm{e}^{-\mathrm{i}\phi}((-1)^n f\Lambda E_- - f^*\Lambda^*E_-^{*-1}) \\ \rho^{-1}\mathrm{e}^{\mathrm{i}\phi}(f\Lambda E_+ - (-1)^n f^*\Lambda^*E_+^{*-1}) & (-1)^n fE_- + f^*E_-^{*-1} \end{pmatrix}, \tag{65}$$

where

$$\begin{aligned} &E_\pm = \mathrm{diag}(e_{k\pm})_{k=1,\ldots,n}, \quad F = \left(\beta_k^{n-j}\right)_{1\le j,k\le n}, \quad f = \left(\beta_1^n, \ldots, \beta_n^n\right),\\ &\Lambda = \mathrm{diag}(\lambda_k)_{k=1,\ldots,n}, \quad \phi = 2\sigma|\rho|^2 t.\end{aligned} \tag{66}$$

Moreover, $e_{k\pm} = \mathrm{e}^{\alpha_k x\pm\beta_k y\pm\gamma_k t}$. However, as in the soliton case, we suppose temporarily that $e_{k\pm}$'s are arbitrary complex numbers.

Hereafter, we look ρ as a parameter of the derived solution.

LEMMA 4. *Suppose $\kappa_1, \dots, \kappa_n$ are distinct nonzero real numbers, then there exist positive constants ρ_0, δ, C_1, and C_2, which depend on κ_j's, such that $|\det W| \geq C_1$ and $|(G_1)_{12}| \leq C_2$ hold whenever $|\rho| > \rho_0$, $|\lambda_j - \mathrm{i}\kappa_j| < \delta$ and $e_{j\pm} \in \mathbf{C}(j = 1, \dots, n)$.*

Proof. Denote $F^{-1}LF^* = I + Z$, then $Z = 0$ if $\lambda_1, \dots, \lambda_n$ are all purely imaginary. Hence, $||Z||$ is small enough if $|\lambda_1 - \mathrm{i}\kappa_1|, \dots, |\lambda_n - \mathrm{i}\kappa_n|$ are all small enough.

Let $c_0 = \max_{1 \leq k \leq n} |\kappa_k|$, $\pi_3 = |\det F||_{\substack{\lambda_k = \mathrm{i}\kappa_k \\ k=1,\dots,n}}$, $\pi_4 = ||F^{-1}LF|||_{\substack{\lambda_k = \mathrm{i}\kappa_k \\ k=1,\dots,n}}$. Then, there exists δ with $0 < \delta < c_0$ such that $||Z|| \leq \frac{1}{2}$, $|\det F| \geq \frac{\pi_3}{2}$, $||F^{-1}LF|| \leq 2\pi_4$ if $|\lambda_k - \mathrm{i}\kappa_k| < \delta$ $(k = 1, \dots, n)$. In this case, $|\lambda_k| < c_0 + \delta < 2c_0$.

From (64),

$$\det W = \det\left(FE_+ + LF^*E_+^{*-1}\right) \det \mathring{W}, \tag{67}$$

where

$$\begin{aligned}\mathring{W} &= LFE_- + F^*E_-^{*-1} + \sigma|\rho|^{-2}\left(F\Lambda E_+ - LF^*\Lambda^*E_+^{*-1}\right) \cdot \\ &\quad \cdot \left(FE_+ + LF^*E_+^{*-1}\right)^{-1} L \left(F\Lambda E_- - LF^*\Lambda^*E_-^{*-1}\right) \\ &= (1 + \sigma|\rho|^{-2}F\chi_+F^{-1}LF\chi_-F^{-1}L^{-1})LF\left(I + ZE_-^{*-1}\Xi_{0-}^{-1}\right)\Xi_{0-},\end{aligned} \tag{68}$$

$$\begin{aligned}\chi_\pm &= F^{-1}\left(F\Lambda E_\pm - LF^*\Lambda^*E_\pm^{*-1}\right)\left(FE_\pm + LF^*E_\pm^{*-1}\right)^{-1}F \\ &= \Xi_{1\pm}\Xi_{0\pm}^{-1} - \left(Z\Lambda^* + \Xi_{1\pm}\Xi_{0\pm}^{-1}Z\right)E_\pm^{*-1}\Xi_{0\pm}^{-1}\left(I + ZE_\pm^{*-1}\Xi_{0\pm}^{-1}\right)^{-1},\end{aligned} \tag{69}$$

$$\Xi_{0\pm} = E_\pm + E_\pm^{*-1}, \quad \Xi_{1\pm} = \Lambda E_\pm - \Lambda^*E_\pm^{*-1}. \tag{70}$$

We have the following estimates:

$$||E_\pm \Xi_{0\pm}^{-1}|| \leq 1, \quad ||E_\pm^{*-1}\Xi_{0\pm}^{-1}|| \leq 1, \quad ||\Xi_{1\pm}\Xi_{0\pm}^{-1}|| \leq 2c_0, \tag{71}$$

$$||\Xi_{0\pm}|| \geq 2, \quad ||\Xi_{0\pm}^{-1}|| \leq \frac{1}{2}, \quad |\det \Xi_{0\pm}| \geq 2^n, \tag{72}$$

$$\left|\left|\left(I + ZE_\pm^{*-1}\Xi_{0\pm}^{-1}\right)^{-1}\right|\right| \leq (1 - ||Z||)^{-1} \leq 2, \tag{73}$$

$$\left|\det\left(I + ZE_\pm^{*-1}\Xi_{0\pm}^{-1}\right)\right| \geq (1 - ||Z||)^n \geq \frac{1}{2^n}. \tag{74}$$

Hence, (69) implies

$$||\chi_\pm|| \leq 2c_0 + 8c_0||Z|| \leq 6c_0, \tag{75}$$

$$||\chi_+F^{-1}LF\chi_-F^{-1}L^{-1}F|| \le ||F^{-1}LF||^2\,||\chi_+||\,||\chi_-|| \le 144c_0^2\pi_4^2. \tag{76}$$

By (67) and (68),

$$|\det W| = |\det F|^2\,|\det \Xi_{0+}|\,|\det \Xi_{0-}|\,\left|\det\left(I + ZE_+^{*-1}\Xi_{0+}^{-1}\right)\right|\cdot$$
$$\cdot\left|\det\left(I + ZE_-^{*-1}\Xi_{0-}^{-1}\right)\right|\,|\det(I+\sigma|\rho|^{-2}\chi_+F^{-1}LF\chi_-F^{-1}L^{-1}F)|$$
$$\ge \frac{\pi_3^2}{4}(1-144c_0^2\pi_4^2|\rho|^{-2}) > 0 \tag{77}$$

if $|\rho| > 12c_0\pi_4$. Therefore, $|\det W|$ has a uniform positive lower bound if $|\rho| > 12c_0\pi_4$ and $|\lambda_k - \mathrm{i}\kappa_k| < \delta (k = 1,\dots,n)$.

By (18), (64), (65), (67), and (68),

$$((G_1)_{12},(G_2)_{12},\dots,(G_n)_{12}) = -(R_{12} - R_{11}W_{11}^{-1}W_{12})\overset{\circ}{W}{}^{-1}$$
$$= \sigma\rho^{*-1}\mathrm{e}^{-\mathrm{i}\phi}(-1)^n\left(f\Lambda E_- - (-1)^n f^*\Lambda^*E_-^{*-1} - (fE_+ + (-1)^n f^*E_+^{*-1})\cdot\right.$$
$$\left.\cdot(-1)^n\left(FE_+ + LF^*E_+^{*-1}\right)^{-1}L(F\Lambda E_- - LF^*\Lambda^*E_-^{*-1})\right)\overset{\circ}{W}{}^{-1}$$
$$= \sigma\rho^{*-1}\mathrm{e}^{-\mathrm{i}\phi}(-1)^n\left(f\Lambda E_-\Xi_{0-}^{-1} - (-1)^n f^*\Lambda^*E_-^{*-1}\Xi_{0-}^{-1} - \right.$$
$$-(-1)^n\left(fE_+\Xi_{0+}^{-1} + (-1)^n f^*E_+^{*-1}\Xi_{0+}^{-1}\right)\left(I + ZE_+^{*-1}\Xi_{0+}^{-1}\right)^{-1}F^{-1}LF\cdot$$
$$\left.\cdot\left(\Xi_{1-}\Xi_{0-}^{-1} - Z\Lambda^*E_-^{*-1}\Xi_{0-}^{-1}\right)\right)\left(I + ZE_-^{*-1}\Xi_{0-}^{-1}\right)^{-1}F^{-1}L^{-1}\cdot$$
$$\cdot\left(1+\sigma|\rho|^{-2}F\chi_+F^{-1}LF\chi_-F^{-1}L^{-1}\right)^{-1}. \tag{78}$$

$(G_1)_{12}$ is bounded when $|\rho| > 12c_0\pi_4$ and $|\lambda_k - \mathrm{i}\kappa_k| < \delta$ $(k = 1,\dots,n)$ because of the estimates (71)–(77). ■

Now, we have the following theorem for the multiple "dark cross soliton" solution.

THEOREM 2. *Suppose $\kappa_1,\dots,\kappa_n$ are distinct nonzero real numbers, then there exist positive constants ρ_0 and δ, which depend on κ_j's, such that the following results hold for the derived solution $\widetilde{u} = u + 2(G_1)_{12}$ of the nonlocal Davey–Stewartson I equation when $|\rho| > \rho_0$ and $|\lambda_j - \mathrm{i}\kappa_j| < \delta$ $(j = 1,\dots,n)$.*

(i) $\widetilde{u}$ is globally defined and bounded for $(x,y,t) \in \mathbf{R}^3$.
(ii) Suppose the real numbers v_x, v_y satisfy $\alpha_{kR}v_x \pm \beta_{kR}v_y \ne 0$ for all $k = 1,\dots,n$ where α_k's and β_k's are given by (63), then $\lim_{s\to+\infty}|\widetilde{u}| = |\rho|$ along the straight line $x = x_0 + v_xs$, $y = y_0 + v_ys$ for arbitrary $x_0, y_0 \in \mathbf{R}$.

Proof. (i) follows directly from Lemma 4. Now, we prove (ii).

Because $|e_{k\pm}| = \mathrm{e}^{(\alpha_{kR}v_x \pm \beta_{kR}v_y)s+(\alpha_{kR}x_0 \pm \beta_{kR}y_0 \pm \gamma_{kR}t)}$ along the straight line $x = x_0 + v_x s$, $y = y_0 + v_y s$, $\alpha_{kR}v_x \pm \beta_{kR}v_y \neq 0$ implies that for each k, $e_{k+} \to 0$ or $e_{k+} \to \infty$, and $e_{k-} \to 0$ or $e_{k-} \to \infty$ as $s \to +\infty$. Let

$$\begin{aligned}
\mu_k &= \begin{cases} \lambda_k & \text{if } \alpha_{kR}v_x + \beta_{kR}v_y > 0, \\ -\lambda_k^* & \text{if } \alpha_{kR}v_x + \beta_{kR}v_y < 0, \end{cases} \\
\nu_k &= \begin{cases} -\lambda_k & \text{if } \alpha_{kR}v_x - \beta_{kR}v_y > 0, \\ \lambda_k^* & \text{if } \alpha_{kR}v_x - \beta_{kR}v_y < 0, \end{cases} \\
a_k &= \beta(\mu_k), \quad b_k = \beta(\nu_k).
\end{aligned} \tag{79}$$

Note that $\alpha(-\lambda) = -\alpha(\lambda)$, $\beta(-\lambda) = -\beta(\lambda)$, $\gamma(-\lambda) = \gamma(\lambda)$, $\alpha(-\lambda^*) = -(\alpha(\lambda))^*$, $\beta(-\lambda^*) = -(\beta(\lambda))^*$, $\gamma(-\lambda^*) = -(\gamma(\lambda))^*$. We have

$$\begin{aligned}
&\xi_k \sim \mathrm{e}^{\alpha(\mu_k)x+\beta(\mu_k)y+\gamma(\mu_k)t}, \quad \eta_k \sim \tfrac{\mu_k}{\rho}\mathrm{e}^{\alpha(\mu_k)x+\beta(\mu_k)y+(\gamma(\mu_k)+2\mathrm{i}\sigma|\rho|^2)t}, \\
&\bar{\xi}_k^* \sim \mathrm{e}^{-\alpha(\nu_k)x+\beta(\nu_k)y-\gamma(\nu_k)t}, \quad \bar{\eta}_k^* \sim \tfrac{\nu_k}{\rho^*}\mathrm{e}^{-\alpha(\nu_k)x+\beta(\nu_k)y+(-\gamma(\nu_k)-2\mathrm{i}\sigma|\rho|^2)t}
\end{aligned} \tag{80}$$

because $s \to +\infty$. Hence,

$$\begin{aligned}
\partial^j \begin{pmatrix} \xi_k & \sigma\bar{\eta}_k^* \\ \eta_k & \bar{\xi}_k^* \end{pmatrix} &\sim \begin{pmatrix} 1 & \\ & \rho^{-1}\mathrm{e}^{\mathrm{i}\phi} \end{pmatrix} \begin{pmatrix} a_k^j & cb_k^j\nu_k \\ a_k^j\mu_k & b_k^j \end{pmatrix} \begin{pmatrix} 1 & \\ & \rho\mathrm{e}^{-\mathrm{i}\phi} \end{pmatrix}, \\
&\cdot \begin{pmatrix} \mathrm{e}^{\alpha(\mu_k)x+\beta(\mu_k)y+\gamma(\mu_k)t} & \\ & \mathrm{e}^{-\alpha(\nu_k)x+\beta(\nu_k)y-\gamma(\nu_k)t} \end{pmatrix}
\end{aligned} \tag{81}$$

where $c = \sigma|\rho|^{-2}$, $\phi = 2\sigma|\rho|^2 t$.

Rewrite (18) as

$$\begin{pmatrix} (G_1)_{11} & \cdots & (G_n)_{11} & \rho^{-1}\mathrm{e}^{\mathrm{i}\phi}(G_1)_{12} & \cdots & \rho^{-1}\mathrm{e}^{\mathrm{i}\phi}(G_n)_{12} \\ (G_1)_{21} & \cdots & (G_n)_{21} & \rho^{-1}\mathrm{e}^{\mathrm{i}\phi}(G_1)_{22} & \cdots & \rho^{-1}\mathrm{e}^{\mathrm{i}\phi}(G_n)_{22} \end{pmatrix} SWS^{-1} = -RS^{-1}, \tag{82}$$

where $S = \begin{pmatrix} I_n & \\ & \rho\mathrm{e}^{-\mathrm{i}\phi}I_n \end{pmatrix}$, I_n is the $n \times n$ identity matrix. Applying Cramer's rule to (82), we have

$$\widetilde{u} = \rho\mathrm{e}^{-\mathrm{i}\phi} + 2(G_1)_{12} = \rho\mathrm{e}^{-\mathrm{i}\phi}\left(1 - 2\frac{\det \widetilde{W}}{\det(SWS^{-1})}\right), \tag{83}$$

where $\widetilde{W}$ is obtained from SWS^{-1} by replacing the $(n+1)$-th row with the first row of RS^{-1}. Hence, (19), (20), and (81) lead to

$$\lim_{s\to+\infty} \widetilde{u} = \rho\mathrm{e}^{-\mathrm{i}\phi}\left(1 - 2\frac{\det W_1}{\det W_0}\right) = \rho\mathrm{e}^{-\mathrm{i}\phi}\frac{\det W_2}{\det W_0}, \tag{84}$$

where

$$W_0 = \begin{pmatrix} a_1^{n-1} & \cdots & a_n^{n-1} & cb_1^{n-1}\nu_1 & \cdots & cb_n^{n-1}\nu_n \\ a_1^{n-2} & \cdots & a_n^{n-2} & cb_1^{n-2}\nu_1 & \cdots & cb_n^{n-2}\nu_n \\ \vdots & & \vdots & \vdots & & \vdots \\ a_1 & \cdots & a_n & cb_1\nu_1 & \cdots & cb_n\nu_n \\ 1 & \cdots & 1 & c\nu_1 & \cdots & c\nu_n \\ a_1^{n-1}\mu_1 & \cdots & a_n^{n-1}\mu_n & b_1^{n-1} & \cdots & b_n^{n-1} \\ a_1^{n-2}\mu_1 & \cdots & a_n^{n-2}\mu_n & b_1^{n-2} & \cdots & b_n^{n-2} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_1\mu_1 & \cdots & a_n\mu_n & b_1 & \cdots & b_n \\ \mu_1 & \cdots & \mu_n & 1 & \cdots & 1 \end{pmatrix}, \tag{85}$$

W_1 is obtained from W_0 by replacing the $(n+1)$-th row with

$$\begin{pmatrix} a_1^n & \cdots & a_n^n & cb_1^n\nu_1 & \cdots & cb_n^n\nu_n \end{pmatrix}, \tag{86}$$

and

$$W_2 = \begin{pmatrix} a_1^{n-1} & \cdots & a_n^{n-1} & cb_1^{n-1}\nu_1 & \cdots & cb_n^{n-1}\nu_n \\ a_1^{n-2} & \cdots & a_n^{n-2} & cb_1^{n-2}\nu_1 & \cdots & cb_n^{n-2}\nu_n \\ \vdots & & \vdots & \vdots & & \vdots \\ a_1 & \cdots & a_n & cb_1\nu_1 & \cdots & cb_n\nu_n \\ 1 & \cdots & 1 & c\nu_1 & \cdots & c\nu_n \\ -c^{-1}a_1^{n-1}\mu_1^{-1} & \cdots & -c^{-1}a_n^{n-1}\mu_n^{-1} & -cb_1^{n-1}\nu_1^2 & \cdots & -cb_n^{n-1}\nu_n^2 \\ a_1^{n-2}\mu_1 & \cdots & a_n^{n-2}\mu_n & b_1^{n-2} & \cdots & b_n^{n-2} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_1\mu_1 & \cdots & a_n\mu_n & b_1 & \cdots & b_n \\ \mu_1 & \cdots & \mu_n & 1 & \cdots & 1 \end{pmatrix}. \tag{87}$$

Denote ROW_k and COL_k to be the k-th row and k-th column of W_2, respectively. The elementary transformations

$$\begin{aligned} &\mathrm{ROW}_{n+k+1} - 2\cdot\mathrm{ROW}_k \to \mathrm{ROW}_{n+k+1} \quad (k=1,\dots,n-1), \\ &\mu_k\cdot\mathrm{COL}_k \to \mathrm{COL}_k \quad (k=1,\dots,n), \\ &\sigma|\rho|^2\nu_k^{-1}\cdot\mathrm{COL}_{n+k} \to \mathrm{COL}_{n+k} \quad (k=1,\dots,n), \\ &-\sigma|\rho|^{-2}\cdot\mathrm{ROW}_{n+k} \to \mathrm{ROW}_{n+k} \quad (k=1,\dots,n), \\ &\mathrm{ROW}_k \leftrightarrow \mathrm{ROW}_{n+k} \quad (k=1,\dots,n) \end{aligned} \tag{88}$$

transform W_2 to W_0. Hence, $\det W_2 = (\prod_{k=1}^n \frac{\nu_k}{\mu_k})\det W_0$. This leads to $\lim_{s\to+\infty}|\widetilde{u}| = |\rho|$ because $\prod_{k=1}^n|\mu_k| = \prod_{k=1}^n|\nu_k| = \prod_{k=1}^n|\lambda_k|$. ■

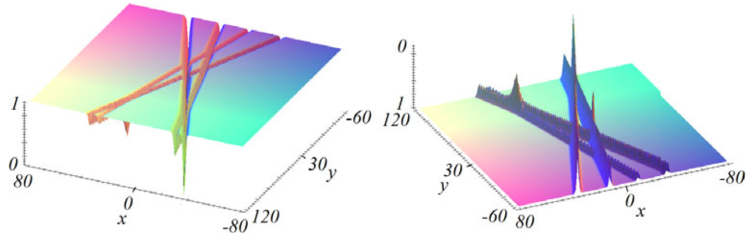

Figure 4. $|\widetilde{u}|$ of a 2 "dark cross soliton" solution.

A 2 "dark cross soliton" solution is shown in Fig. 4 where the parameters are $\sigma = -1$, $t = 10$, $\rho = 1$, $\lambda_1 = 0.8 + 0.1\mathrm{i}$, $\lambda_2 = -0.6 - 0.3\mathrm{i}$. The figure on the right describes the same solution but is upside down.

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Fudan University

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