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Binary Darboux transformations for Manakov triad (II) ☆

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Abstract

A general formula for the closed one-form in a binary Darboux transformation for a Manakov triad of arbitrary order is obtained. The one-form given by this formula coincides with those for many special PDEs. This is a generalization of results obtained before [Phys. Lett. A 195 (1994) 339].

1. Introduction

A Manakov triad consists of three differential operators (L, N, W) satisfying

$$L_t = WL - LN, \tag{1.1}$$

where

$$L = \sum_{i+j\le n} L_{ij}\partial_x^i\partial_y^j, \qquad N = \sum N_{ij}\partial_x^i\partial_y^j, \qquad W = \sum W_{ij}\partial_x^i\partial_y^j, \tag{1.2}$$

and the coefficients L_{ij}, N_{ij}, $W_{ij} \in C^\infty(\mathbb{R}^3)\otimes\mathscr{M}_r$. Here $\mathscr{M}_r$ denotes the set of all $r\times r$ matrices.

As soon as (1.1) holds, the linear system

$$L\Psi = 0, \qquad \Psi_t = N\Psi \tag{1.3}$$

is integrable. It is clear that the Manakov triad is a generalization of the Lax pair (in that case $W = N$), which proved to be very important in solving nonlinear PDEs [1–4].

Darboux transformation (DT) methods in differential form are very useful to obtain explicit solutions for quite many nonlinear PDEs with Lax pairs. But it seems that the binary Darboux transformation (BDT), the Darboux transformation in integral form first given by Matveev and Salle, is more powerful in considering the nonlinear PDEs with Manakov triads. The combinatorial use of BT and BDT for the Davey–Stewartson (DS) equation leads to many interesting soliton solutions [2,3,5]. The BDT also works for the Kadomtsev–Petviashvili

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(KP) equation, the Novikov–Veselov (NV) equation [3] and so on. All these BDTs are based on the integral of a closed one-form which depends on the solutions of (1.3) and the solutions of its dual system. For a general 1 + 1 dimensional Lax pair, the formula is already known. This formula is similar to that in Example 1. It is applied to the system in which only the first derivatives of y and t are concerned. On the other hand, for the system whose (x, y) parts are of second-order, the BDT is given by Ref. [4] with only recursive relations. Here we shall give a general explicit formula for the closed one-form related with a Manakov triad. This one-form is the same as those in many special cases like the KP, DS, NV equations etc. and it is valid for all the systems in (1.3). Using this one-form, we propose a general algorithm to obtain new solutions of the nonlinear PDE (1.1).

It is known that due to (1.1), not only the system (1.3), but also its dual system

$$L^*\Phi^* = 0, \qquad \Phi_t^* = -W^*\Phi^* \tag{1.4}$$

is integrable. Here L^*, W^* are the formal adjoint operators of L and W respectively, i.e.

$$L^*f = \sum_{i+j\leqslant n} (-1)^{i+j}\partial_x^i\partial_y^j\left(L_{ij}^{\mathrm{T}}f\right), \tag{1.5}$$

so is M^*, and L_{ij}^{T} is the transpose of the matrix L_{ij}.

From (1.1), we have

$$L_t^* = (-N)^* L^* - L^*(-W^*), \tag{1.6}$$

hence, there is a duality

$$(L, N, W, \Psi, \Phi) \Leftrightarrow (L^*, -W^*, -N^*, \Phi^*, \Psi^*), \tag{1.7}$$

that is, any result holding for (L, N, W, Ψ, Φ) also holds for $(L^*, -W^*, -N^*, \Phi^*, \Psi^*)$.

2. The closed one-form related with the Manakov triad

To express the one-form, we need the notation of formal pseudo-differential operators in multi-variables. Let $\mathscr{D}_{xy}$ be an associate algebra containing all the formal expansions

$$\sum_{i=-\infty}^{M}\sum_{j=-\infty}^{N} A_{ij}\partial_x^i\partial_y^j, \qquad A_{ij}\in \mathrm{C}^\infty(\mathbb{R}^2)\otimes\mathscr{M}_r, \tag{2.1}$$

where the generators ∂_x, ∂_y, ∂_x^{-1}, ∂_y^{-1} should satisfy the relations

$$\begin{gathered}
\partial_x\circ\partial_x^{-1} = \partial_x^{-1}\circ\partial_x = \partial_y\circ\partial_y^{-1} = \partial_y^{-1}\circ\partial_y = 1,\\
\partial_x\circ\partial_y = \partial_y\circ\partial_x, \qquad \partial_x\circ\partial_y^{-1} = \partial_y^{-1}\circ\partial_x, \qquad \partial_x^{-1}\circ\partial_y = \partial_y\circ\partial_x^{-1}, \qquad \partial_x^{-1}\circ\partial_y^{-1} = \partial_y^{-1}\circ\partial_x^{-1},\\
\partial_x\circ f = f\partial_x + f^{(1,0)}, \qquad \partial_y\circ f = f\partial_y + f^{(0,1)},\\
\partial_x^{-1}\circ f = \sum_{i=0}^{\infty}(-1)^i f^{(i,0)}\partial_x^{-i-1}, \qquad \partial_y^{-1}\circ f = \sum_{j=0}^{\infty}(-1)^j f^{(0,j)}\partial_y^{-j-1},
\end{gathered} \tag{2.2}$$

where $f\in \mathrm{C}^\infty(\mathbb{R}^2)\times\mathscr{M}_r$, $\partial_x^{-i} = (\partial_x^{-1})^i$, $\partial_y^{-j} = (\partial_y^{-1})^j$ for $i, j\geqslant 0$, and $f^{(i,j)} = \partial_x^i\partial_y^j f$.

If $A = \sum_{i=-\infty}^{M}\sum_{j=-\infty}^{N} A_{ij}\partial_x^i\partial_y^j \in\mathscr{D}_{xy}$, define

$$\operatorname*{Res}_{\partial_x} A = A_{-1,0}, \qquad \operatorname*{Res}_{\partial_y} A = A_{0,-1}, \qquad \operatorname*{Res}_{\partial_x\partial_y} A = A_{-1,-1}.$$

Let

$$\omega_x^{(1)}(\Phi,\Psi)=\operatorname*{Res}_{\partial_y}\left(\partial_y^{-1}\circ\Phi\circ L\circ\Psi\circ\partial_y^{-1}\right),\qquad \omega_x^{(2)}(\Phi,\Psi)=-\operatorname*{Res}_{\partial_y}\left(\partial_y^{-1}\circ\Psi^*\circ L^*\circ\Phi^*\circ\partial_y^{-1}\right)^*,$$

$$\omega_y^{(1)}(\Phi,\Psi)=-\operatorname*{Res}_{\partial_x}\left(\partial_x^{-1}\circ\Phi\circ L\circ\Psi\circ\partial_x^{-1}\right),\qquad \omega_y^{(2)}(\Phi,\Psi)=\operatorname*{Res}_{\partial_x}\left(\partial_x^{-1}\circ\Psi^*\circ L^*\circ\Phi^*\circ\partial_x^{-1}\right)^*,$$

$$\omega_x(\Phi,\Psi)=\omega_x^{(1)}(\Phi,\Psi)+\omega_x^{(2)}(\Phi,\Psi),\qquad \omega_y(\Phi,\Psi)=\omega_y^{(1)}(\Phi,\Psi)+\omega_y^{(2)}(\Phi,\Psi),\tag{2.3}$$

then we have

Lemma. Let $\Psi(x,y)$ be a solution of $L\Psi=0$, and $\Phi(x,y)$ be a solution of $L^*\Phi^*=0$, then

$$\omega_{xy}(\Phi,\Psi)=\omega_x(\Phi,\Psi)\,\mathrm{d}x+\omega_y(\Phi,\Psi)\,\mathrm{d}y\tag{2.4}$$

is a closed one-form.

Proof. To write $\omega_y^{(1)}(\Phi,\Psi)$ in terms of the coefficients, we have

$$\begin{aligned}
\omega_y^{(1)}(\Phi,\Psi)&=-\sum_{i+j\leqslant n}\operatorname*{Res}_{\partial_x}\left(\partial_x^{-1}\circ\Phi L_{ij}\partial_x^i\partial_y^j\circ\Psi\partial_x^{-1}\right)\\
&=-\sum_{i+j\leqslant n}\sum_{\alpha\leqslant i}C_i^\alpha\operatorname*{Res}_{\partial_x}\left(\partial_x^{-1}\circ\Phi L_{ij}\Psi^{(i-\alpha,j)}\partial_x^{\alpha-1}\right)\\
&=-\sum_{i+j\leqslant n}\sum_{\alpha\leqslant i}\sum_{k=0}^{\infty}(-1)^kC_i^\alpha\operatorname*{Res}_{\partial_x}\left(\left(\Phi L_{ij}\Psi^{(i-\alpha,j)}\right)^{(k,0)}\partial_x^{\alpha-k-2}\right)\\
&=-\sum_{i+j\leqslant n}\sum_{\alpha=1}^{i}(-1)^{\alpha-1}C_i^\alpha\left(\Phi L_{ij}\Psi^{(i-\alpha,j)}\right)^{(\alpha-1,0)}\\
&=-\sum_{i+j\leqslant n,i\geqslant 1}\sum_{\alpha=1}^{i}\sum_{\sigma=0}^{\alpha-1}(-1)^{\alpha-1}C_i^\alpha C_{\alpha-1}^\sigma\left(\Phi L_{ij}\right)^{(\sigma,0)}\Psi^{(i-1-\sigma,j)}\\
&=-\sum_{i+j\leqslant n,i\geqslant 1}\sum_{\sigma=0}^{i-1}(-1)^\sigma\left(\Phi L_{ij}\right)^{(\sigma,0)}\Psi^{(i-1-\sigma,j)},
\end{aligned}\tag{2.5}$$

$$\partial_x\omega_y^{(1)}(\Phi,\Psi)=\sum_{i+j\leqslant n}(-1)^i\left(\Phi L_{ij}\right)^{(i,0)}\Psi^{(0,j)}\tag{2.6}$$

by $L\Psi=0$ and (A.2). Similarly,

$$\omega_y^{(2)}(\Phi,\Psi)=\sum_{i+j\leqslant n,i\geqslant 1}(-1)^{i+j}\sum_{\sigma=0}^{i-1}(-1)^\sigma\left(\Phi L_{ij}\right)^{(i-1-\sigma,j)}\Psi^{(\sigma,0)},\tag{2.7}$$

$$\partial_x\omega_y^{(2)}(\Phi,\Psi)=-\sum_{i+j\leqslant n}(-1)^j\left(\Phi L_{ij}\right)^{(0,j)}\Psi^{(i,0)}\tag{2.8}$$

by $L^*\Phi^*=0$. Therefore,

$$\partial_x\omega_y(\Phi,\Psi)=\sum_{i+j\leqslant n}\left[(-1)^i\left(\Phi L_{ij}\right)^{(i,0)}\Psi^{(0,j)}-(-1)^j\left(\Phi L_{ij}\right)^{(0,j)}\Psi^{(i,0)}\right].\tag{2.9}$$

This expression is antisymmetric with respect to x and y. Hence, comparing ω_x and ω_y, we obtain $\mathrm{d}\omega_{xy}(\Phi,\Psi)=0$. The Lemma is proved.

Remark 1. Generally,

$$\omega_y^{(2)} - \omega_y^{(1)} = \sum_{\substack{i+j\leqslant n\\ i\geqslant 1}} \sum_{\sigma=0}^{i-1} \left[(-1)^{\sigma} (\Phi L_{ij})^{(\sigma,0)} \Psi^{(i-1-\sigma,j)} + (-1)^{i+j+\sigma} (\Phi L_{ij})^{(i-1-\sigma,j)} \Psi^{(\sigma,0)} \right]$$

$$= \frac{\partial}{\partial y} \sum_{i+j\leqslant n, i,j\geqslant 1} \sum_{\sigma}^{i-1} \sum_{\tau}^{j-1} (-1)^{\sigma+\tau} (\Phi L_{ij})^{(\sigma,\tau)} \Psi^{(i-1-\sigma,j-1-\tau)}.$$

If L is of first order, like in the DS equation, or L is of second order whose $\partial_x \partial_y$ coefficient vanishes, like in the KP equation, the identities $\omega_x^{(1)} = \omega_x^{(2)}$, $\omega_y^{(1)} = \omega_y^{(2)}$ always hold. In those cases, we can simply define $w_x = w_x^{(1)}$, $w_y = w_y^{(1)}$.

Now consider the problem with t part.

We use the following symbols. Suppose

$$N = \sum_{i,j} N_{ij} \partial_x^i \partial_y^j$$

is any differential operator, Ψ is a matrix function, then

$$N\Psi = \sum_{i,j} N_{ij} \Psi^{(i,j)}, \qquad N \circ \Psi = \sum_{i,j,\alpha,\beta} C_i^{\alpha} C_j^{\beta} N_{ij} \Psi^{(i-\alpha,j-\beta)} \partial_x^{\alpha} \partial_y^{\beta},$$

$$N \circ_x \Psi = \sum_{i,j,\beta} C_j^{\beta} N_{ij} \Psi^{(i,j-\beta)} \partial_y^{\beta}, \qquad N \circ_y \Psi = \sum_{i,j,\alpha} C_i^{\alpha} N_{ij} \Psi^{(i-\alpha,j)} \partial_x^{\alpha}. \tag{2.10}$$

Theorem. Suppose (1.1) holds. Let $\Psi(x, y, t)$ be a solution of (1.3) and $\Phi(x, y, t)$ be a solution of (1.4), then

$$\omega(\Phi, \Psi) = \omega_x(\Phi, \Psi)\, dx + \omega_y(\Phi, \Psi)\, dy + \omega_t(\Phi, \Psi)\, dt$$

$$\equiv \left(\operatorname*{Res}_{\partial_y} \left(\partial_y^{-1} \circ \Phi \circ L \circ \Psi \circ \partial_y^{-1} \right) - \operatorname*{Res}_{\partial_y} \left(\partial_y^{-1} \circ \Psi^* \circ L^* \circ \Phi^* \circ \partial_y^{-1} \right)^* \right) dx$$

$$+ \left(-\operatorname*{Res}_{\partial_x} \left(\partial_x^{-1} \circ \Phi \circ L \circ \Psi \circ \partial_x^{-1} \right) + \operatorname*{Res}_{\partial_x} \left(\partial_x^{-1} \circ \Psi^* \circ L^* \circ \Phi^* \circ \partial_x^{-1} \right)^* \right) dy$$

$$+ \left(\operatorname*{Res}_{\partial_x\partial_y} \left(\partial_x^{-1} \partial_y^{-1} \circ \Phi \circ L \circ (N \circ_y \Psi - N \circ_x \Psi) \circ \partial_x^{-1} \partial_y^{-1} \right) \right.$$

$$\left. + \operatorname*{Res}_{\partial_x\partial_y} \left(\partial_x^{-1} \partial_y^{-1} \circ \Psi^* \circ L^* \circ \left(W^* \circ_y \Phi^* - W^* \circ_x \Phi^* \right) \circ \partial_x^{-1} \partial_y^{-1} \right)^* \right) dt \tag{2.11}$$

is a closed one-form.

Sketch of proof. By the Lemma, we only need to show that $\partial_x \omega_t = \partial_t \omega_x$, $\partial_y \omega_t = \partial_t \omega_y$. Owing to the antisymmetry between x and y in (2.11), it is enough to prove the second equality. For simplicity, denote $Z = N\Psi$, $S = \Phi \circ L$, $R = S \circ N = \Phi \circ L \circ N$. Moreover, we will often omit the range of summation. All the indices in the summation take all possible values.

Now we compute $\partial_t \omega_y(\Phi, \Psi)$. Using (1.3) and (1.4), $\partial_t \omega_y(\Phi, \Psi) = \Theta_N + \Theta_W$, where

$$\Theta_N = -\operatorname*{Res}_{\partial_x} \left(\partial_x^{-1} \circ \Phi \circ L \circ (N\Psi - N \circ \Psi) \circ \partial_x^{-1} \right)$$

$$+ \operatorname*{Res}_{\partial_x} \left(\partial_x^{-1} \circ \left[(N\Psi)^* - \Psi^* \circ N^* \right] \circ L^* \circ \Phi^* \circ \partial_x^{-1} \right)^*, \tag{2.12}$$

$$\Theta_W = -\operatorname*{Res}_{\partial_x}\left(\partial_x^{-1} \circ \Psi^* \circ L^* \circ (W^* \Phi^* - W^* \circ \Phi^*) \circ \partial_x^{-1}\right)^* + \operatorname*{Res}_{\partial_x}\left(\partial_x^{-1} \circ \left[(W^* \Phi^*)^* - \Phi \circ W\right] \circ L \circ \Psi \circ \partial_x^{-1}\right). \tag{2.13}$$

Here we only write down the calculation for Θ_N because the calculation for Θ_W is similar. Written in terms of Z, S and R,

$$\Theta_N = -\operatorname*{Res}_{\partial_x}\left(\partial_x^{-1} \circ (S \circ Z - R \circ \Psi) \circ \partial_x^{-1}\right) + \operatorname*{Res}_{\partial_x}\left(\partial_x^{-1} \circ (Z^* \circ S^* - \Psi^* \circ R^*) \circ \partial_x^{-1}\right)^*.$$

By a tedious calculation, we have

$$\begin{aligned}\Theta_N &= \frac{\partial}{\partial y}\Bigg(\sum_{i,j,\alpha,\beta,p,q,\gamma,\delta} (-1)^{\beta+\gamma+i+p+1} C_{p-\gamma-1}^{\alpha-i} C_{j-\beta-1}^{\delta-q} S_{ij}^{(\alpha,\beta)} N_{pq}^{(i+p-\alpha-\gamma-1,\, j+q-\beta-\delta-1)} \Psi^{(\gamma,\delta)} \\ &\quad - \sum_{i,j,\alpha,\beta,p,q,\gamma,\delta} (-1)^{\alpha+\delta+j+q+1} C_{i-\alpha-1}^{\gamma-p} C_{q-\delta-1}^{\beta-j} S_{ij}^{(\alpha,\beta)} N_{pq}^{(i+p-\alpha-\gamma-1,\, j+q-\beta-\delta-1)} \Psi^{(\gamma,\delta)}\Bigg) \\ &= \frac{\partial}{\partial y} \operatorname*{Res}_{\partial_x \partial_y}\left(\partial_x^{-1}\partial_y^{-1} \circ \Phi \circ L \circ \left(N \circ_y \Psi - N \circ_x \Psi\right) \circ \partial_x^{-1}\partial_y^{-1}\right)\end{aligned}$$

by equalities (A.1), (A.2) and $L^* \Phi^* = 0$. Using the duality, a similar equality holds for the terms with W. Therefore, $\partial_t \omega_y = \partial_y \omega_t$. The theorem is proved.

From the Theorem, the indefinite integral

$$\Omega(\Phi, \Psi) = \int \omega(\Phi, \Psi) \tag{2.14}$$

is independent of the path of integration.

3. Binary Darboux transformation

For equations such as the KP, DS, NV ones etc., the one-form given here is just the one used for constructing the BDT before [3,5]. Here there are two more known examples.

Example 1. The Lax pair

$$\Psi_y = M\Psi \equiv \sum_{i=0}^{m} M_i \partial_x^i \Psi, \qquad \Psi_t = N\Psi \equiv \sum_{j=0}^{n} N_i \partial_x^j \Psi \tag{3.1}$$

has the integrability condition

$$M_t - N_y + [M, N] = 0, \tag{3.2}$$

which gives the nonlinear PDEs such as the KP, DS equations etc.

For this system, the closed one-form is

$$\omega(\Phi, \Psi) = \Phi\Psi \,\mathrm{d}x + \operatorname*{Res}_{\partial_x}\left(\partial_x^{-1} \circ \Phi M \circ \Psi \partial_x^{-1}\right) \mathrm{d}y + \operatorname*{Res}_{\partial_x}\left(\partial_x^{-1} \circ \Phi N \circ \Psi \partial_x^{-1}\right) \mathrm{d}t, \tag{3.3}$$

which is derived from the Theorem and Remark 1.

Take $\tilde{\Psi} = \Psi - \pi\Omega^{-1}(\lambda, \pi)\Omega(\lambda, \Psi)$, then we can prove that there exists differential operators $\tilde{M}$, $\tilde{N}$ with respect to x such that

$$\tilde{\Psi}_y = \tilde{M}\tilde{G}, \qquad \tilde{\Psi}_t = \tilde{N}\tilde{G}. \tag{3.4}$$

This gives the new solution $(\tilde{M}, \tilde{N})$ of (3.2). The proof is similar to that in Ref. [4].

Example 2. The second order system

$$L_{11}\Psi_{xx} + L_{12}\Psi_{xy} + L_{22}\Psi_{yy} + L_{10}\Psi_x + L_{01}\Psi_y + L_{00}\Psi = 0, \qquad \Psi_t = \sum_{j=0}^{n} N_j \partial^j \Psi \tag{3.5}$$

was discussed in Ref. [4]. The closed one-form given here is equal to that in Ref. [4] up to a complete differential. The BDT is the same. Therefore, the new solution of (3.5) can be obtained from a know solution.

Now return to the general case. For (1.1), (1.3), we have the following procedure to construct the BDT.

Proposition. Suppose L_{0n} is nondegenerate. Then generically, there exist differential operators $P_s(\partial_x, \partial_y)$ of sth order with $s \geqslant s(n)$, and a differential operator $\tilde{L}(\partial_x, \partial_y)$ of nth order, whose coefficients are $r \times r$ matrices, such that for any solution Ψ of (1.3),

$$\tilde{\Psi} = P_s(\partial_x, \partial_y)\Psi - \Omega(\lambda, \Psi) \tag{3.6}$$

satisfies $\tilde{L}\tilde{\Psi} = 0$, $\tilde{\Psi}_t = \tilde{N}\tilde{\Psi}$. Here λ is a solution of (1.4),

$$\begin{aligned} s(n) &= 0, && \text{if } n = 1, 2, \\ &= 2n - 3, && \text{if } n \geqslant 3. \end{aligned} \tag{3.7}$$

Proof. The equation $L\Psi = 0$ has $(n+1) + n + \ldots + 1 = \frac{1}{2}(n+1)(n+2)$ coefficients (each $r \times r$ matrix is counted as 1). Considering the homogeneity, the number of independent coefficients is

$$N_c = \tfrac{1}{2}(n+1)(n+2) - 1 = \tfrac{1}{2}n(n+3). \tag{3.8}$$

For the undetermined

$$P_s = \sum_{i+j \leqslant s} P_{ij}^s \partial_x^i \partial_y^j,$$

with $s \geqslant n-2$ (this assumption comes from the fact that the first derivative of Ω will create $(n-1$th derivatives of Φ and Ψ), there are $\frac{1}{2}(s+1)(s+2)$ coefficients. Now take $\frac{1}{2}(s+1)(s+2)$ $r \times r$ matrix solutions $\pi_1, \ldots, \pi_{(s+1)(s+2)/2}$ of (1.3), and let

$$\tilde{\pi}_i = P_s(\partial_x, \partial_y)\pi_i - \Omega(\lambda, \pi_i) = 0 \qquad (i = 1, \ldots, \tfrac{1}{2}(s+1)(s+2)), \tag{3.9}$$

then generically, when the coefficient matrix for $\{P_{ij}^s \mid i+j \leqslant s\}$ is nondegenerate, the $\{P_{ij}^s\}$ is determined by (3.9) uniquely. Now consider $\tilde{L}\tilde{\Psi} = 0$. This is of the form

$$\sum_{i+j \leqslant n+s} Q_{ij}\partial_x^i \partial_y^j \Psi + R\Omega(\lambda, \Psi) = 0.$$

However, all the terms of $\partial_x^i \partial_y^j \Psi$ with $j \geqslant n$ can be replaced by the terms $\partial_x^i \partial_y^j \Psi$ with $j < n$ by $L\Psi = 0$. Hence, $\tilde{L}\tilde{\Psi} = 0$ is actually equivalent to

$$\sum_{i+j \leqslant n+s,\, j \leqslant n-1} \tilde{Q}_{ij}\partial_x^i \partial_y^j \Psi + R\Omega(\lambda, \Psi) = 0. \tag{3.10}$$

The number of independent coefficients is

$$\tilde{N}_c = (s+2) + \ldots + (n+s+1) + 1 = \tfrac{1}{2}n^2 + ns + \tfrac{3}{2}n + 1. \tag{3.11}$$

Generically, we can determine the coefficients $\tilde{L}_{ij}$ by letting the coefficients of N_c terms on the left-hand side of (3.10) by zero. Hence only $\tilde{N}_c - N_c$ terms are left in (3.10). That is,

$$\tilde{L}\tilde{\Psi} = \sum_{i,j\in J} R_{ij}\partial_x^i\partial_y^j\Psi + R\Omega(\lambda, \Psi) = 0,$$

where $\#J \leqslant \tilde{N}_c - N_c - 1$.

Now let $\Psi = \pi_\alpha$ $(\alpha = 1, \dots, \frac{1}{2}(s+1)(s+2))$, then

$$\sum_{i,j\in J} R_{ij}\partial_x^i\partial_y^j\pi_\alpha + R\Omega(\lambda, \pi_\alpha) = 0, \qquad \alpha = 1, \dots, \tfrac{1}{2}(s+1)(s+2) \tag{3.12}$$

If

$$\tilde{N}_c - N_c \leqslant \tfrac{1}{2}(s+1)(s+2), \tag{3.13}$$

then when the coefficient matrix is nondegenerate, all R_{ij} and R should be zero. Therefore, $\tilde{L}_{ij}$ exists when the coefficient matrices are nondegenerate. Condition (3.13) implies $s \geqslant 2n-3$ for $n \geqslant 3$, and $s \geqslant 0$ for $n = 1, 2$.

For the t-part, suppose N is of order p, then $\tilde{\Psi}_t - \tilde{N}\tilde{\Psi}$ is of the form

$$\sum_{i+j\leqslant h+s} Q'_{ij}\partial_x^i\partial_y^j\Psi + R'\Omega(\lambda, \Psi) = 0, \tag{3.14}$$

when $s \geqslant n-2$. Using $L\Psi = 0$, (3.14) is equivalent to an equation of the form

$$\sum_{i+j\leqslant h+s,\, j\leqslant n-1} \tilde{Q}'_{ij}\partial_x^i\partial_y^j\Psi + R'\Omega(\lambda, \Psi) = 0. \tag{3.15}$$

The number of independent coefficients is

$$\tilde{N}'_c = (s+h-n-2) + \dots + (s+h+1) + 1, \tag{3.16}$$

while the number of independent coefficients in $\tilde{\Psi}_t = \tilde{N}\tilde{\Psi}$ is

$$N'_c = \tfrac{1}{2}(h+1)(h+2). \tag{3.17}$$

Hence, when

$$\tilde{N}'_c - N'_c \leqslant \tfrac{1}{2}(s+1)(s+2), \tag{3.18}$$

the identity $\tilde{\Psi}_t = \tilde{N}\tilde{\Psi}$ holds generically. (3.18) implies

$$s^2 - (2n-3)s + h^2 - (2n-3)h + n^2 - 3n + 2 \geqslant 0. \tag{3.19}$$

Whenever $s \geqslant s(n)$, (3.15) holds for any h. Hence the proposition is true.

Remark 2. Using the above proposition, we can obtain the new solutions of (1.1) when there are no restrictions on the components of L, N and W. For concrete problems, this gives an algorithm to obtain new solutions, although the restrictions on L, N and W should be considered.

Remark 3. Here we have some conditions concerning the nondegenerateness of some matrices. Therefore, the general result is local.

Remark 4. We only give the transformation for L and Ψ in the Proposition. However, in the general case, the transformation for Φ is still unknown. Therefore, we can apply the BDT only once from the known solution. On the other hand, in many special cases, the transformations of Φ are known and an infinite sequence of explicit solutions can be obtained.

Appendix

For nonpositive integer M, S and integer T,

$$\sum_{k=\max(0,-T)}^{M} \frac{(-1)^k (k+S)!}{k!(M-k)!(k+T)!} = (-1)^M C_{S-T}^{M} \frac{S!}{(T+M)!}, \quad S \geqslant M+T,$$
$$= 0, \quad T \leqslant S < M+T,$$
$$= C_{M+T-S-1}^{M} \frac{S!}{(T+M)!}, \quad 0 \leqslant S < T. \tag{A.1}$$

If A, B are positive integers, $A \leqslant B$, then

$$\sum_{\alpha=A}^{B} \frac{(-1)^\alpha}{\alpha(\alpha-A)!(B-\alpha)!} = \frac{(-1)^A (A-1)!}{B!}. \tag{A.2}$$

These two equalities can be proved by recursion and standard calculus.

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