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Numerical Computation of High Dimensional Solitons via Darboux Transformation [1]

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Abstract: Darboux transformation gives explicit soliton solutions of nonlinear partial differential equations. Using numerical computation in each step of constructing Darboux transformation, one can get the graphs of the solitons practically. In n dimensions ($n \geq 3$), this method greatly increases the speed and deduces the memory usage of computation comparing to the software for algebraic computation. A technical problem concerning floating overflow is discussed.
Key Words: Darboux transformation, numerical computation

1. Introduction

Darboux transformation is a very powerful method to get explicit solutions of nonlinear partial differential equations. The derived solutions are usually written explicitly in terms of the seed solution and the solutions of the Lax pair. Although they are completely explicit, it is still needed to get some graphs to have better understanding of these solitons. To obtain the graphs, the simplest way is using the software for algebraic computation, such as MAPLE, MATHEMATICA etc. This is quite easy in two (or 1+1) dimensions. However, for the complicated solitons in n dimensions ($n \geq 3$), such as the "3-solitons" of DS equation or "2-solitons" of 3-wave equation under nonlinear constraint, this routine method need tremendous memory and want tremendous speed of the computer. Practically, it is impossible. Therefore, numerical computation should be used in each step of Darboux transformation so that the computation is much faster. Although numerical computation is used here, it differs a lot from the usually numerical way to get solutions, because there is almost no computational error (only the rounding error of the computer). In this paper, we will show a practical algorithm to get the graphs of solutions via Darboux transformation with numerical computation. For two dimensional problem, this method is applicable directly. For three dimensional problem, the nonlinear constraint method [1,2,11] is used so that the differentiation and integration are avoided in the computation.

2. Darboux transformation

There is a general procedure to construct Darboux transformation for the Lax set in n dimensions [4,5,6,10]:

$$\frac{\partial \Phi}{\partial x_i} = U_i(x, \lambda)\Phi, \quad (i = 1, \cdots, n) \tag{1}$$

where $x = (x_1, \cdots, x_n)$, λ is a complex number (spectral parameter), U_i are $N \times N$ matrices which are polynomials of λ.

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The integrability condition of (1) is

$$\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} + [U_i, U_j] = 0, \tag{2}$$

which gives nonlinear partial differential equations.

For simplicity, here we suppose $U_i \in su(N)$, i.e. $U_i^* = -U_i$ for real λ. This guarantees the global existence of the Darboux transformation and the solutions.

Let $\lambda_1, \cdots, \lambda_l$ be l complex numbers such that $\mathrm{Re}\lambda_i \neq 0$, $\lambda_i \neq \lambda_j$ and $\lambda_i \neq \bar{\lambda}_j$ for $i \neq j$. Let H_i be an $N \times s$ matrix solution (rank=min(N, s)) of (1) with $\lambda = \lambda_i$ $(1 \leq s \leq N-1)$. Let

$$\Gamma_{ij} = \frac{H_i^* H_j}{\lambda_j - \bar{\lambda}_i}. \tag{3}$$

Then

$$G(\lambda) = \prod_{j=1}^{l} (\lambda - \bar{\lambda}_j) \Big(1 - \sum_{j,k=1}^{l} \frac{H_j (\Gamma^{-1})_{jk} H_k^*}{\lambda - \bar{\lambda}_k}\Big) \tag{4}$$

is a Darboux matrix of degree l for (1). That is, for any solution Φ of (1), $\Phi' = G\Phi$ satisfies

$$\frac{\partial \Phi'}{\partial x_i} = U_i'(x, \lambda)\Phi', \quad (i = 1, \cdots, n) \tag{5}$$

where U_i' are polynomials of λ[11]. Comparing (5) with (1), U_i' are uniquely determined by

$$U_i' = G U_i G^{-1} + \frac{\partial G}{\partial x_i} G^{-1}. \tag{6}$$

As soon as the solution (U_i) of (2) is known and one can solve (1), the transformation (6) gives the new solution (U_i') of (2). Therefore, from the known solution (U_i) to get new solution (U_i') of (2), it is essential to compute $G(\lambda)$ given by (4).

Remark 1. For some 1+2 dimensional problems, such as DSI equation, there are many works discussing the 2-solitons graphically (see eg. [3,7,9]). It is quite difficult to draw the graphs of more complicated solitons in these ways. Alternately, by the nonlinear constraint method, many 1+2 dimensional problems without spectral parameter can be associated with the system (1) with $n = 3$, which can be dealt with by the above method[2,11,12].

3. Numerical computation of Darboux transformation

(4) is completely explicit. However, usually H_i contain many terms with exponential. It is very easy to get floating overflow in direct computation for large area. To avoid that, we need to modify H_i a bit.

Let $D_i(x)$ be an arbitrary $s \times s$ nondegenerate diagonal matrix $(1 \leq i \leq l)$, $D = \mathrm{diag}(D_1, \cdots, D_l)$. Let $\widetilde{H}_i = H_i D_i^{-1}$, then

$$(\Gamma_{ij}) = D^* (\widetilde{\Gamma}_{ij}) D \tag{7}$$

where

$$\widetilde{\Gamma}_{ij} = \frac{\widetilde{H}_i^* \widetilde{H}_j}{\lambda_j - \bar{\lambda}_i} \tag{8}$$

and

$$G(\lambda) = \prod_{j=1}^{l} (\lambda - \bar{\lambda}_j) \Big(1 - \sum_{j,k=1}^{l} \frac{\widetilde{H}_j (\widetilde{\Gamma}^{-1})_{jk} \widetilde{H}_k^*}{\lambda - \bar{\lambda}_k}\Big). \tag{9}$$

Therefore, the transformation $H_i \to H_i D_i$ does not change the expression of $G(\lambda)$.

Owing to this fact, we can get the graph of solitons through the following procedure by numerical computation:

(i) Take l complex numbers $\lambda_1, \cdots, \lambda_l$ such that $\mathrm{Re}\lambda_i \neq 0$, $\lambda_i \neq \lambda_j$ and $\lambda_i \neq \bar{\lambda}_j$ for $i \neq j$;

(ii) Let $\widetilde{H}_i = (\widetilde{H}^{(i)}_{\alpha\beta})$ be an $N \times s$ matrix solution (rank=min(N, s)) of (1) with $\lambda = \lambda_i$. Let $D_i = \mathrm{diag}(d^{(i)}_1, \cdots, d^{(s)}_l)$ where

$$d^{(i)}_\beta(x) = \max_{1 \leq \alpha \leq N} |\widetilde{H}^{(i)}_{\alpha\beta}(x)|, \quad (\beta = 1, \cdots, s), \tag{10}$$

$H_i = \widetilde{H}_i D_i^{-1}$. Then each entry of H_i has norm ≤ 1;

(iii) Calculate the coefficients of λ in (4).

(iv) Get (U'_i) from (6).

In this algorithm, we need complex matrix computation for a computer language, such as C language.

Remark 2. The crucial points of this algorithm are

1) Use numerical computation instead of software for algebraic computation. This greatly increases the speed and need much less memory.

2) Modify H_i in Step (ii) to avoid floating overflow.

Remark 3. This method can also be applied to very complicated Lax set in high dimensions with spectral parameter, say, some systems derived by [8].

4. An example: "3-soliton" and "4-soliton" solutions of DSI equation

There is a nonlinear constraint for DSI equation. With this nonlinear constraint, We can use Darboux transformation to get the soliton solutions. It is known that the "l-soliton" solution has at most l^2 peaks in its asymptotic solution. To show this fact graphically, computation is not easy. With MAPLE, "2-soliton" solution of DSI equation can be drawn. However, drawing "2-soliton" solution of 3-wave equation is too difficult to do. The computation for "3-soliton" and "4-soliton' solutions of DSI equation is impossible with MAPLE or other software for algebraic computation. Here we use the above mentioned method to draw the following figures for the "3-soliton" and "4-soliton" solutions of DSI equation, which correspond to $n = 3$, $N = 4$, $s = 2$, $l = 3$ or $l = 4$. The computation takes only a few minutes on Pentium. The parameters of the Darboux matrix (see [11]) are

$$\lambda_1 = 1 + 2i, \quad \lambda_2 = 2 - i, \quad \lambda_3 = -2 + 1.5i, \quad \lambda_4 = -3 + 0.5i,$$

$$C^{(1)} = \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}, \quad C^{(2)} = \begin{pmatrix} 1 & 0 \\ 4/3 & 1 \end{pmatrix}, \quad C^{(3)} = \begin{pmatrix} 3/2 & 1 \\ -2 & 2 \end{pmatrix}, \quad C^{(4)} = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix},$$

and the grid is 80×80.

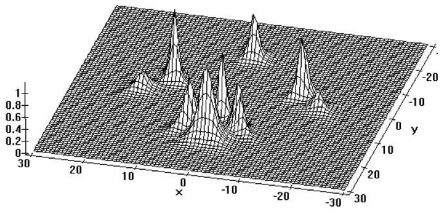

Fig.1: "3-soliton", $t = 2$

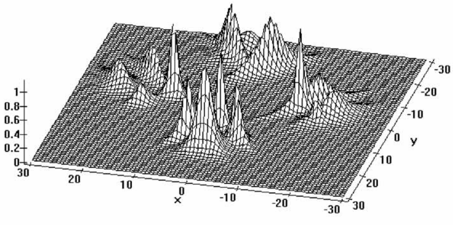

Fig.2: "4-soliton", $t = 2$

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