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Finite Dimensional Integrable Hamiltonian Systems Associated with DSI Equation by Bargmann Constraints

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The Davey-Stewartson I equation is a typical integrable equation in 2+1 dimensions. Its Lax system being essentially in 1+1 dimensional form has been found through nonlinearization from 2+1 dimensions to 1+1 dimensions. In the present paper, this essentially 1+1 dimensional Lax system is further nonlinearized into 1+0 dimensional Hamiltonian systems by taking the Bargmann constraints. It is shown that the resulting 1+0 dimensional Hamiltonian systems are completely integrable in Liouville sense by finding a full set of integrals of motion and proving their functional independence.

KEYWORDS: DSI equation, finite dimensional Hamiltonian system

§1. Introduction

The Davey-Stewartson I (DSI) equation is a famous 2+1 dimensional integrable equation which describes the motion of water wave.[1)] This equation has localized soliton solutions and has been studied in various ways, such as inverse scattering,[2,3)] binary Darboux transformation,[4)] nonlinearization to 1+1 dimensional problems[5,6)] etc.

For 1+1 dimensional integrable systems, the nonlinearization procedure, both mono-nonlinearization[7)] and binary nonlinearization,[8)] reduces them to finite dimensional (1+0 dimensional) integrable Hamiltonian systems.[9–12)] Therefore, it transforms a partial differential equation to a systems of ordinary differential equations. This greatly simplifies the procedure of getting solutions, at least numerical solutions. Some important explicit solutions, especially periodic or quasi-periodic solutions have been obtained in this way.

The nonlinear constraint method has also been applied to some 2+1 dimensional equations like the KP, MKP, N-wave equations etc.[13–15)] For the DSI equation, we have already found its new Lax system (2.1) by nonlinearization in which all the derivatives are separated. Each pair of equations in this system is 1+1 dimensional. Hence the derived Lax system is looked as essentially 1+1 dimensional because we can use 1+1 dimensional method to solve it. It is possible to nonlinearize this essentially 1+1 dimensional system again to get finite dimensional Hamiltonian systems.

In the present paper, we show that there are Bargmann constraints which reduce the DSI equation to finite dimensional Hamiltonian systems. These Hamiltonian systems have a full set of integrals of motions and these integrals of motion are functionally independent in a dense open subset of the phase space. Therefore, these Hamiltonian systems are completely integrable in Liouville sense.

§2. Nonlinearization

We consider the following Lax system[5)]

$$\Phi_x = W^x\Phi = \begin{pmatrix} \mathrm{i}\lambda & 0 & \mathrm{i}f \\ 0 & \mathrm{i}\lambda & \mathrm{i}g \\ \mathrm{i}\bar{f} & \mathrm{i}\bar{g} & 0 \end{pmatrix}\Phi$$

$$\Phi_y = W^y\Phi = \begin{pmatrix} \mathrm{i}\lambda & u & \mathrm{i}f \\ -\bar{u} & -\mathrm{i}\lambda & -\mathrm{i}g \\ \mathrm{i}\bar{f} & -\mathrm{i}\bar{g} & 0 \end{pmatrix}\Phi$$

$$\Phi_t = W^t\Phi = \begin{pmatrix} -2\mathrm{i}\lambda^2+\mathrm{i}|u|^2+\mathrm{i}v_1 & -2u\lambda+\mathrm{i}u_y & -2\mathrm{i}f\lambda-2f_y \\ 2\bar{u}\lambda+\mathrm{i}\bar{u}_y & 2\mathrm{i}\lambda^2-\mathrm{i}|u|^2-\mathrm{i}v_2 & 2\mathrm{i}g\lambda-2g_y \\ -2\mathrm{i}\bar{f}\lambda+2\bar{f}_y & 2\mathrm{i}\bar{g}\lambda+2\bar{g}_y & -2\mathrm{i}(|f|^2-|g|^2) \end{pmatrix}\Phi. \tag{2.1}$$

Here u, f and g are complex functions, v_1 and v_2 are real functions.

Its integrability conditions $\Phi_{xy}=\Phi_{yx}$, $\Phi_{xt}=\Phi_{tx}$ and $\Phi_{yt}=\Phi_{ty}$ consist of the following three parts.

(1) DSI equation

$$\begin{aligned} &-\mathrm{i}u_t = u_{xx}+u_{yy}+2|u|^2u+2(v_1+v_2)u \\ &v_{1,x}-v_{1,y} = v_{2,x}+v_{2,y} = -(|u|^2)_x. \end{aligned} \tag{2.2}$$

(2) Standard Lax pair of the DSI equation

$$F_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}F_x + \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}F$$

$$F_t = 2\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}F_{xx} + 2\mathrm{i}\begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}F_x$$

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$$+\mathrm{i}\begin{pmatrix}|u|^2+2v_1 & u_x+u_y\\ -\bar{u}_x+\bar{u}_y & -|u|^2-2v_2\end{pmatrix}F, \tag{2.3}$$

where $F=(f,g)^T$.

(3) Nonlinear constraint

$$FF^*=\frac{1}{2}\begin{pmatrix}v_1 & u_x\\ \bar{u}_x & v_2\end{pmatrix}. \tag{2.4}$$

Hence any solution of (2.2)–(2.4) gives a solution of the DSI equation.

Notice that if Φ is a vector solution of (2.1) for real λ, then $\Psi=\mathrm{i}\bar{\Phi}$ is a solution of the adjoint equations

$$\Psi_x=-(W^x)^T\Psi \qquad \Psi_y=-(W^y)^T\Psi$$
$$\Psi_t=-(W^t)^T\Psi, \tag{2.5}$$

where each entry of $\bar{\Phi}$ is the complex conjugation of the corresponding entry of Φ.

In order to obtain the finite dimensional Hamiltonian systems, we first nonlinearize the y-equation of (2.1) in the following way.

Consider the pair $\Phi_y=W^y\Phi$ and $\Phi_t=W^t\Phi$. Let $w=(u,-\bar{u},\mathrm{i}f,\mathrm{i}\bar{f},-\mathrm{i}g,-\mathrm{i}\bar{g})$ containing all the variables in W^y.[10] Then the recursion relations of this AKNS system can be expressed in Lenard form

$$JG_l=KG_{l-1} \qquad (l=1,2,\cdots), \tag{2.6}$$

where (J,K) is the Lenard pair (J is a non-degenerate constant matrix) which was given by[10] (here $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3$ in[10] are $\alpha_1=\mathrm{i}$, $\alpha_2=-\mathrm{i}$, $\alpha_3=0$, $\beta_1=-2\mathrm{i}$, $\beta_2=2\mathrm{i}$, $\beta_3=0$) and $\{G_0,G_1,G_2,\cdots\}$ is the Lenard sequence. The first element of this Lenard sequence is given by[10] as

$$G_0=-2(-\bar{u},u,\mathrm{i}\bar{f},\mathrm{i}f,-\mathrm{i}\bar{g},-\mathrm{i}g), \tag{2.7}$$

which is a kernel of K.

On the other hand, the variation of spectral parameter can be computed by the general formula[17]

$$\begin{aligned}\frac{\delta\lambda}{\delta w}&=C_0\mathrm{tr}\left(\Phi\Psi^T\frac{\partial W^y}{\partial w}\right)\\ &=C_0(\mathrm{i}\bar{\phi}_1\phi_2,\mathrm{i}\bar{\phi}_2\phi_1,\mathrm{i}\bar{\phi}_1\phi_3,\mathrm{i}\bar{\phi}_3\phi_1,\mathrm{i}\bar{\phi}_2\phi_3,\mathrm{i}\bar{\phi}_3\phi_2),\end{aligned} \tag{2.8}$$

where C_0 is a constant.

Now let $\lambda_1,\cdots,\lambda_N$ be N distinct non-zero real numbers, $(\phi_{1\alpha},\phi_{2\alpha},\phi_{3\alpha})^T$ be the corresponding solution of the Lax system (2.1) for $\lambda=\lambda_\alpha$. Let $\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_N)$, $\Phi_j=(\phi_{j1},\cdots,\phi_{jN})^T$.

By the general formulation of nonlinearization, we impose the nonlinear constraint

$$G_0=-2\sum_{j=1}^N\frac{\delta\lambda_j}{\delta w}, \tag{2.9}$$

which gives the relations

$$\langle\Phi_2,\Phi_1\rangle=-\mathrm{i}u \quad \langle\Phi_3,\Phi_1\rangle=f \quad \langle\Phi_3,\Phi_2\rangle=-g, \tag{2.10}$$

where $\langle V_1,V_2\rangle=V_1^*V_2$ for any two vectors V_1 and V_2. These are the Bargmann constraints between (u,f,g) and (Φ_1,Φ_2,Φ_3).

Remark 1 *If we consider the pair $\Phi_y=W^y\Phi$ and $\Phi_x=W^x\Phi$, G_0 is different. In that case, we can not obtain Bargmann constraints, but only Neumann constraints.*[16]

Let

$$L(\lambda)=\begin{pmatrix}1 & & \\ & -1 & \\ & & 0\end{pmatrix}+\sum_{\alpha=1}^N\frac{1}{\lambda-\lambda_\alpha}\begin{pmatrix}\bar{\phi}_{1\alpha}\phi_{1\alpha} & \bar{\phi}_{2\alpha}\phi_{1\alpha} & \bar{\phi}_{3\alpha}\phi_{1\alpha}\\ \bar{\phi}_{1\alpha}\phi_{2\alpha} & \bar{\phi}_{2\alpha}\phi_{2\alpha} & \bar{\phi}_{3\alpha}\phi_{2\alpha}\\ \bar{\phi}_{1\alpha}\phi_{3\alpha} & \bar{\phi}_{2\alpha}\phi_{3\alpha} & \bar{\phi}_{3\alpha}\phi_{3\alpha}\end{pmatrix}. \tag{2.11}$$

Lemma 1 *The Lax equations*

$$L_x=[W^x,L] \quad L_y=[W^y,L] \quad L_t=[W^t,L], \tag{2.12}$$

hold if and only if (2.10) *holds.*

Proof. Let $F_\alpha=(\phi_{1\alpha},\phi_{2\alpha},\phi_{3\alpha})^T$, then

$$L(\lambda)=C+\sum_{\alpha=1}^N\frac{1}{\lambda-\lambda_\alpha}F_\alpha F_\alpha^*. \tag{2.13}$$

Since $F_{\alpha,y}=W^y(\lambda_\alpha)F_\alpha$ and $(W^y(\lambda_\alpha))^*=-W^y(\lambda_\alpha)$, we have

$$\begin{aligned}L_y(\lambda)&=\sum_\alpha\frac{1}{\lambda-\lambda_\alpha}[W^y(\lambda_\alpha),F_\alpha F_\alpha^*]\\ &=[W^y(\lambda),L(\lambda)-C]\\ &\quad-\sum_\alpha\frac{1}{\lambda-\lambda_\alpha}[W^y(\lambda)-W^y(\lambda_\alpha),F_\alpha F_\alpha^*]\\ &=[W^y(\lambda),L(\lambda)-C]-\mathrm{i}\left[C,\sum_\alpha F_\alpha F_\alpha^*\right].\end{aligned} \tag{2.14}$$

Hence $L_y(\lambda)=[W^y(\lambda),L(\lambda)]$ if and only if

$$\mathrm{i}\left[C,\sum_\alpha F_\alpha F_\alpha^*\right]=[C,W^y(\lambda)]. \tag{2.15}$$

Written in the components, it becomes (2.10). If (2.10) holds, the other two equations in (2.12) are obtained similarly. The lemma is proved.

From the above constraints, we have

$$\begin{aligned}f_x&=\mathrm{i}\langle\Phi_3,\Lambda\Phi_1\rangle+\mathrm{i}f(\langle\Phi_3,\Phi_3\rangle-\langle\Phi_1,\Phi_1\rangle)-ug\\ g_x&=-\mathrm{i}\langle\Phi_3,\Lambda\Phi_2\rangle-\mathrm{i}g(\langle\Phi_3,\Phi_3\rangle-\langle\Phi_2,\Phi_2\rangle)-\bar{u}f\\ f_y&=\mathrm{i}\langle\Phi_3,\Lambda\Phi_1\rangle+\mathrm{i}f(\langle\Phi_3,\Phi_3\rangle-\langle\Phi_1,\Phi_1\rangle)\\ g_y&=\mathrm{i}\langle\Phi_3,\Lambda\Phi_2\rangle+\mathrm{i}g(\langle\Phi_3,\Phi_3\rangle-\langle\Phi_2,\Phi_2\rangle)\\ u_x&=2f\bar{g}\\ u_y&=-2\langle\Phi_2,\Lambda\Phi_1\rangle+\langle\Phi_2,\Phi_1\rangle(\langle\Phi_1,\Phi_1\rangle-\langle\Phi_2,\Phi_2\rangle)\\ v_1&=2\langle\Phi_1,\Phi_3\rangle\langle\Phi_3,\Phi_1\rangle \quad v_2=2\langle\Phi_2,\Phi_3\rangle\langle\Phi_3,\Phi_2\rangle.\end{aligned} \tag{2.16}$$

With the constraints (2.10), the system (2.1) is changed to a system of ordinary differential equations

$$\Phi_{1,x} = \mathrm{i}\Lambda\Phi_1 + \mathrm{i}f\Phi_3 \quad \Phi_{2,x} = \mathrm{i}\Lambda\Phi_2 + \mathrm{i}g\Phi_3$$
$$\Phi_{3,x} = \mathrm{i}\bar{f}\Phi_1 + \mathrm{i}\bar{g}\Phi_2$$
$$\Phi_{1,y} = \mathrm{i}\Lambda\Phi_1 + u\Phi_2 + \mathrm{i}f\Phi_3$$
$$\Phi_{2,y} = -\bar{u}\Phi_1 - \mathrm{i}\Lambda\Phi_2 - \mathrm{i}g\Phi_3$$
$$\Phi_{3,y} = \mathrm{i}\bar{f}\Phi_1 - \mathrm{i}\bar{g}\Phi_2$$
$$\begin{aligned}\Phi_{1,t} &= (-2\mathrm{i}\Lambda^2 + \mathrm{i}|u|^2 + \mathrm{i}v_1)\Phi_1 + (-2u\Lambda + \mathrm{i}u_y)\Phi_2 \\ &\quad + (-2\mathrm{i}f\Lambda - 2f_y)\Phi_3 \\ \Phi_{2,t} &= (2\bar{u}\Lambda + \mathrm{i}\bar{u}_y)\Phi_1 + (2\mathrm{i}\Lambda^2 - \mathrm{i}|u|^2 - \mathrm{i}v_2)\Phi_2 \\ &\quad + (2\mathrm{i}g\Lambda - 2g_y)\Phi_3 \\ \Phi_{3,t} &= (-2\mathrm{i}\bar{f}\Lambda + 2\bar{f}_y)\Phi_1 + (2\mathrm{i}\bar{g}\Lambda + 2\bar{g}_y)\Phi_2 \\ &\quad - 2\mathrm{i}(|f|^2 - |g|^2)\Phi_3, \end{aligned} \tag{2.17}$$

where u, u_y, v_1, v_2, f, g, f_y, g_y are given by (2.10) and (2.16).

$\mathrm{Re}\,\phi_{j\alpha}$ and $\mathrm{Im}\,\phi_{j\alpha}$ ($j = 1, 2, 3$; $\alpha = 1, \cdots, N$) form a system of coordinates of the phase space $\boldsymbol{R}^{6N}$. For simplicity, we use the complex coordinates $\phi_{j\alpha}$ and $\bar{\phi}_{j\alpha}$ instead of $\mathrm{Re}\,\phi_{j\alpha}$ and $\mathrm{Im}\,\phi_{j\alpha}$. $\boldsymbol{R}^{6N}$ has the standard symplectic form

$$\begin{aligned}\omega &= 2\sum_{j=1}^{3}\sum_{\alpha=1}^{N} \mathrm{d}\,\mathrm{Im}(\phi_{j\alpha}) \wedge \mathrm{d}\,\mathrm{Re}(\bar{\phi}_{j\alpha}) \\ &= \mathrm{i}\sum_{j=1}^{3}\sum_{\alpha=1}^{N} \mathrm{d}\bar{\phi}_{j\alpha} \wedge \mathrm{d}\phi_{j\alpha}. \end{aligned} \tag{2.18}$$

The corresponding Poisson bracket of two functions ξ and η is

$$\{\xi, \eta\} = -\mathrm{i}\sum_{j=1}^{3}\sum_{\alpha=1}^{N}\left(\frac{\partial\xi}{\partial\phi_{j\alpha}}\frac{\partial\eta}{\partial\bar{\phi}_{j\alpha}} - \frac{\partial\xi}{\partial\bar{\phi}_{j\alpha}}\frac{\partial\eta}{\partial\phi_{j\alpha}}\right). \tag{2.19}$$

By direct computation, we have

Lemma 2 (2.17) *is equivalent to three Hamiltonian equations*

$$\begin{aligned}\mathrm{i}\phi_{j\alpha,x} &= \frac{\partial H^x}{\partial\bar{\phi}_{j\alpha}} \quad -\mathrm{i}\bar{\phi}_{j\alpha,x} = \frac{\partial H^x}{\partial\phi_{j\alpha}} \\ \mathrm{i}\phi_{j\alpha,y} &= \frac{\partial H^y}{\partial\bar{\phi}_{j\alpha}} \quad -\mathrm{i}\bar{\phi}_{j\alpha,y} = \frac{\partial H^y}{\partial\phi_{j\alpha}} \\ \mathrm{i}\phi_{j\alpha,t} &= \frac{\partial H^t}{\partial\bar{\phi}_{j\alpha}} \quad -\mathrm{i}\bar{\phi}_{j\alpha,t} = \frac{\partial H^t}{\partial\phi_{j\alpha}}, \end{aligned} \tag{2.20}$$

where

$$\begin{aligned}H^x &= -\langle\Phi_1, \Lambda\Phi_1\rangle - \langle\Phi_2, \Lambda\Phi_2\rangle - |\langle\Phi_1, \Phi_3\rangle|^2 \\ &\quad + |\langle\Phi_2, \Phi_3\rangle|^2 \\ H^y &= -\langle\Phi_1, \Lambda\Phi_1\rangle + \langle\Phi_2, \Lambda\Phi_2\rangle - |\langle\Phi_1, \Phi_3\rangle|^2 \\ &\quad - |\langle\Phi_2, \Phi_3\rangle|^2 - |\langle\Phi_1, \Phi_2\rangle|^2 \\ H^t &= 2\langle\Phi_1, \Lambda^2\Phi_1\rangle - 2\langle\Phi_2, \Lambda^2\Phi_2\rangle \\ &\quad + 4\mathrm{Re}(\langle\Phi_1, \Phi_3\rangle\langle\Phi_3, \Lambda\Phi_1\rangle) \\ &\quad + 4\mathrm{Re}(\langle\Phi_2, \Lambda\Phi_3\rangle\langle\Phi_3, \Lambda\Phi_2\rangle) \\ &\quad + 4\mathrm{Re}(\langle\Phi_1, \Phi_2\rangle\langle\Phi_2, \Lambda\Phi_1\rangle) \\ &\quad + 2(\langle\Phi_3, \Phi_3\rangle - \langle\Phi_1, \Phi_1\rangle)|\langle\Phi_1, \Phi_3\rangle|^2 \\ &\quad - 2(\langle\Phi_3, \Phi_3\rangle - \langle\Phi_2, \Phi_2\rangle)|\langle\Phi_2, \Phi_3\rangle|^2 \\ &\quad - (\langle\Phi_1, \Phi_1\rangle - \langle\Phi_2, \Phi_2\rangle)|\langle\Phi_1, \Phi_2\rangle|^2. \end{aligned} \tag{2.21}$$

§3. Integrability

Lemma 3 *For any two complex numbers λ, μ and any positive integers k, l,*

$$\{\mathrm{tr}L^k(\lambda), \mathrm{tr}L^l(\mu)\} = 0. \tag{3.1}$$

Proof. Denote $e_j = (0, \cdots, 0, \frac{1}{j}, 0, \cdots, 0)^T$, then

$$\frac{\partial L(\lambda)}{\partial\bar{\phi}_{j\alpha}} = \frac{1}{\lambda - \lambda_\alpha}F_\alpha e_j^T \quad \frac{\partial L(\lambda)}{\partial\phi_{j\alpha}} = \frac{1}{\lambda - \lambda_\alpha}e_j F_\alpha^*. \tag{3.2}$$

Since

$$\begin{aligned}\mathrm{tr}\left(\frac{\partial L^k(\lambda)}{\partial\phi_{j\alpha}}\right) &= \mathrm{tr}\sum_{r=1}^{k} L^{r-1}(\lambda)\frac{\partial L(\lambda)}{\partial\phi_{j\alpha}}L^{k-r}(\lambda) \\ &= k\mathrm{tr}\left(L^{k-1}(\lambda)\frac{\partial L(\lambda)}{\partial\phi_{j\alpha}}\right), \end{aligned} \tag{3.3}$$

$$\mathrm{tr}\left(\frac{\partial L^k(\lambda)}{\partial\bar{\phi}_{j\alpha}}\right) = k\mathrm{tr}\left(L^{k-1}(\lambda)\frac{\partial L(\lambda)}{\partial\bar{\phi}_{j\alpha}}\right),$$

we have

$$\begin{aligned}&\frac{\mathrm{i}}{kl}\{\mathrm{tr}L^k(\lambda), \mathrm{tr}L^l(\mu)\} \\ &= \sum_{j,\alpha}\left(\mathrm{tr}(L^{k-1}(\lambda)\frac{\partial L(\lambda)}{\partial\phi_{j\alpha}})\mathrm{tr}(L^{l-1}(\mu)\frac{\partial L(\mu)}{\partial\bar{\phi}_{j\alpha}})\right. \\ &\quad \left. - \mathrm{tr}(L^{l-1}(\mu)\frac{\partial L(\mu)}{\partial\phi_{j\alpha}})\mathrm{tr}(L^{k-1}(\lambda)\frac{\partial L(\lambda)}{\partial\bar{\phi}_{j\alpha}})\right) \\ &= \sum_{a,b,j,\alpha}\frac{1}{\lambda - \lambda_\alpha}(L^{k-1}(\lambda))_{aj}\bar{\phi}_{a\alpha}\frac{1}{\mu - \lambda_\alpha}(L^{l-1}(\mu))_{jb}\phi_{b\alpha} \\ &\quad - \sum_{a,b,j,\alpha}\frac{1}{\mu - \lambda_\alpha}(L^{l-1}(\mu))_{aj}\bar{\phi}_{a\alpha}\frac{1}{\lambda - \lambda_\alpha}(L^{k-1}(\lambda))_{jb}\phi_{b\alpha} \\ &= \sum_{a,b,\alpha}\frac{1}{\lambda - \lambda_\alpha}\frac{1}{\mu - \lambda_\alpha}\bar{\phi}_{a\alpha}\phi_{b\alpha}[L^{k-1}(\lambda), L^{l-1}(\mu)]_{ab} \\ &= \sum_{a,b,\alpha}\frac{1}{\mu - \lambda}\left(\frac{1}{\lambda - \lambda_\alpha} - \frac{1}{\mu - \lambda_\alpha}\right)\bar{\phi}_{a\alpha}\phi_{b\alpha} \\ &\quad \cdot [L^{k-1}(\lambda), L^{l-1}(\mu)]_{ab} \\ &= \frac{1}{\mu - \lambda}\mathrm{tr}\Big((L(\lambda) - L(\mu))[L^{k-1}(\lambda), L^{l-1}(\mu)]\Big) = 0. \end{aligned} \tag{3.4}$$

The lemma is proved.

Using this lemma, we can construct involutive integrals of motion from $\mathrm{tr}L^k$ ($k = 1, 2, \cdots$).

For any complex number ξ, let

$$\det(\xi - L(\lambda)) = \xi^3 - p_1(\lambda)\xi^2 + p_2(\lambda)\xi - p_3(\lambda) \quad (3.5)$$

then $p_k(\lambda)$ is the sum of all the determinants of the principal submatrices of $L(\lambda)$ of order k. Suppose the eigenvalues of $L(\lambda)$ are $\nu_1(\lambda), \nu_2(\lambda), \nu_3(\lambda)$, then

$$\mathrm{tr}L^k(\lambda) = \sum_{j=1}^{3} \nu_j^k(\lambda)$$

$$p_k(\lambda) = \sum_{1\le j_1<\cdots<j_k\le 3} \nu_{j_1}(\lambda)\cdots\nu_{j_k}(\lambda). \quad (3.6)$$

Since $p_k(\lambda)$ can be expressed as a polynomial of $\mathrm{tr}L^l(\lambda)$ ($l = 1, 2, \cdots$), $\{p_j(\lambda), p_k(\mu)\} = 0$ for any two complex numbers λ and μ and two integers $j, k \ge 0$.

Expand $p_k(\lambda)$ as a Laurent series of λ

$$p_k(\lambda) = \sum_{m=-1}^{\infty} E_m^{(k)} \lambda^{-m-1}, \quad (3.7)$$

which is convergent when $|\lambda| > \max_{1\le\alpha\le N} |\lambda_\alpha|$, then

$$\begin{aligned}
E_m^{(1)} &= \langle\Phi_1, \Lambda^m\Phi_1\rangle + \langle\Phi_2, \Lambda^m\Phi_2\rangle + \langle\Phi_3, \Lambda^m\Phi_3\rangle \\
E_m^{(2)} &= -\langle\Phi_1, \Lambda^m\Phi_1\rangle + \langle\Phi_2, \Lambda^m\Phi_2\rangle \\
&\quad + \sum_{1\le i<j\le 3}\sum_{l=1}^{m} \begin{vmatrix} \langle\Phi_i, \Lambda^{l-1}\Phi_i\rangle & \langle\Phi_j, \Lambda^{m-l}\Phi_i\rangle \\ \langle\Phi_i, \Lambda^{l-1}\Phi_j\rangle & \langle\Phi_j, \Lambda^{m-l}\Phi_j\rangle \end{vmatrix} \\
E_m^{(3)} &= -\langle\Phi_3, \Lambda^m\Phi_3\rangle \\
&\quad - \sum_{l=1}^{m} \begin{vmatrix} \langle\Phi_1, \Lambda^{l-1}\Phi_1\rangle & \langle\Phi_3, \Lambda^{m-l}\Phi_1\rangle \\ \langle\Phi_1, \Lambda^{l-1}\Phi_3\rangle & \langle\Phi_3, \Lambda^{m-l}\Phi_3\rangle \end{vmatrix} \\
&\quad + \sum_{l=1}^{m} \begin{vmatrix} \langle\Phi_2, \Lambda^{l-1}\Phi_2\rangle & \langle\Phi_3, \Lambda^{m-l}\Phi_2\rangle \\ \langle\Phi_2, \Lambda^{l-1}\Phi_3\rangle & \langle\Phi_3, \Lambda^{m-l}\Phi_3\rangle \end{vmatrix} \\
&\quad + \sum_{\substack{i+j+k=m-2\\ i,j,k\ge 0}} \begin{vmatrix} \langle\Phi_1, \Lambda^i\Phi_1\rangle & \langle\Phi_2, \Lambda^j\Phi_1\rangle & \langle\Phi_3, \Lambda^k\Phi_1\rangle \\ \langle\Phi_1, \Lambda^i\Phi_2\rangle & \langle\Phi_2, \Lambda^j\Phi_2\rangle & \langle\Phi_3, \Lambda^k\Phi_2\rangle \\ \langle\Phi_1, \Lambda^i\Phi_3\rangle & \langle\Phi_2, \Lambda^j\Phi_3\rangle & \langle\Phi_3, \Lambda^k\Phi_3\rangle \end{vmatrix}.
\end{aligned} \quad (3.8)$$

The sums are zero if the lower bound is greater than the upper bound. These $E_m^{(k)}$'s are in involution.

Let

$$\begin{aligned}
\Omega_1 &= \langle\Phi_1, \Phi_1\rangle = (E_0^{(1)} - E_0^{(2)} + E_0^{(3)})/2 \\
\Omega_2 &= \langle\Phi_2, \Phi_2\rangle = (E_0^{(1)} + E_0^{(2)} + E_0^{(3)})/2 \\
\Omega_3 &= \langle\Phi_3, \Phi_3\rangle = -E_0^{(3)},
\end{aligned} \quad (3.9)$$

then

$$\begin{aligned}
H^x &= -E_1^{(1)} - E_1^{(3)} - (\Omega_1 - \Omega_2)\Omega_3 \\
H^y &= E_1^{(2)} - \Omega_1\Omega_2 - \Omega_1\Omega_3 - \Omega_2\Omega_3 \\
H^t &= -2E_2^{(2)} + 2(\Omega_1 + \Omega_2)E_1^{(1)} \\
&\quad + (\Omega_1 + \Omega_2 - 2\Omega_3)H^x + (\Omega_1 - \Omega_2)H^y,
\end{aligned} \quad (3.10)$$

which have all been expressed as polynomials of $E_m^{(k)}$'s.

Hence the following lemma holds.

Lemma 4

$$\{H^x, H^y\} = \{H^x, H^t\} = \{H^y, H^t\} = 0, \quad (3.11)$$

$$\{H^x, E_m^{(k)}\} = \{H^y, E_m^{(k)}\} = \{H^t, E_m^{(k)}\} = 0, \quad (3.12)$$

and

$$\{E_m^{(k)}, E_p^{(l)}\} = 0, \quad (3.13)$$

for all $k, l = 1, 2, 3$; $m, p = 0, 1, 2, \cdots$.

Now we consider the independence of $E_m^{(k)}$'s.

Lemma 5 $E_m^{(k)}$ $(1 \le k \le 3; 0 \le m \le N-1)$ *are functionally independent in a dense open subset of* $\boldsymbol{R}^{6N}$.

Proof Let P_0 in $\boldsymbol{R}^{6N}$ be given by $\phi_{j\alpha} = \epsilon$ ($j = 1, 2, 3$; $\alpha = 1, 2, \cdots, N$) where ϵ is a small real constant. Then, at P_0,

$$\frac{\partial E_m^{(k)}}{\partial\bar\phi_{j\alpha}} = \varepsilon\lambda_\alpha^m b_{kj} + O(\varepsilon^3), \quad (3.14)$$

where

$$(b_{kj}) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (3.15)$$

The Jacobian determinant

$$\begin{aligned}
J &\equiv \frac{\partial(E_0^{(1)}, \cdots, E_{N-1}^{(1)}, E_0^{(2)}, \cdots, E_{N-1}^{(2)}, E_0^{(3)}, \cdots, E_{N-1}^{(3)})}{\partial(\bar\phi_{11}, \cdots, \bar\phi_{1N}, \bar\phi_{21}, \cdots, \bar\phi_{2N}, \bar\phi_{31}, \cdots, \bar\phi_{3N})} \\
&= -2\epsilon^{3N}\left(\prod_{1\le\alpha<\beta\le N}(\lambda_\beta - \lambda_\alpha)\right)^3 + O(\epsilon^{3N+2}).
\end{aligned} \quad (3.16)$$

$J \ne 0$ at P_0 if $\epsilon \ne 0$ is small enough. Since J is a real analytic function of

$$(\phi_{11}, \cdots, \phi_{3N}, \bar\phi_{11}, \cdots, \bar\phi_{3N}),$$

$J \ne 0$ in a dense open subset of $\boldsymbol{R}^{6N}$. Therefore, the Jacobian determinant of

$$(E_0^{(1)}, \cdots, E_{N-1}^{(1)}, E_0^{(2)}, \cdots, E_{N-1}^{(2)}, E_0^{(3)}, \cdots, E_{N-1}^{(3)}),$$

to $6N$ real coordinates

$$(\mathrm{Re}(\phi_{11}), \cdots, \mathrm{Re}(\phi_{3N}), \mathrm{Im}(\phi_{11}), \cdots, \mathrm{Im}(\phi_{3N})),$$

is of full rank $3N$. The lemma is proved.

In summary, from Lemma 2–5, we have

Theorem 1 The Hamiltonian systems given by the Hamiltonians (2.21) *are involutive and completely integrable in Liouville sense. The integrals of motion are* $E_m^{(k)}$ $(k = 1, 2, 3;\ m = 0, \cdots, N-1)$ *given by* (3.8), *which are in involution and functionally independent in a dense open subset of the phase space* $\boldsymbol{R}^{6N}$. *Moreover, each solution of these Hamiltonian systems gives a solution of the DSI equation.*

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