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A universal way to determine Hirota's bilinear equation of KdV type

Yi-Chao Ye[a] and Zi-Xiang Zhou[b]
School of Mathematical Sciences, Fudan University, Shanghai 200433, China

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A universal way, which is independent of computer programming, is presented to determine whether a given nonlinear partial differential equation can be transformed to a bilinear equation of KdV type via a logarithmic transformation. When it can be transformed, all the parameters are determined so that the bilinear equation is written explicitly. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4818836]

I. INTRODUCTION

In 1971, R. Hirota introduced a direct method to construct multi-soliton solutions of integrable nonlinear partial differential equation (PDE).[8–10] The equation is transformed to Hirota's bilinear form, and the solutions are constructed by a formal perturbation expansion. A lot of work has been done for various equations after that (cf. Ref. 11 and the references therein). In late 1980s, Hietarinta designed a program to search for integrable bilinear equations of Korteweg-de Vries (KdV) type,[4] modified Korteweg-de Vries (mKdV) type,[5] sine-Gordon (SG) type,[6] and nonlinear Schrödinger (NLS) type.[7] Apart from the KdV equation, those of KdV type include Sawada-Kotera equation,[19] the Boussinesq equation,[10] the shallow water waves equation,[12] the Ito equation,[13] the bidirectional Sawada-Kotera equation,[1] the Kadomtsev-Petviashvili (KP) equation,[18] the $(2+1)$-dimensional breaking soliton equation,[17] the $(2+1)$-dimensional Sawada-Kotera equation,[14] the $(3+1)$-dimensional KdV equation,[16] etc. To transform a nonlinear equation to bilinear form of KdV type, it is essential to determine the vectors ρ, ω, and the complex number a in (15). This can be realized equation by equation, as in the above cited works. In Ref. 15, a quite general method to determine the parameter ρ was given. On the other hand, various computer algorithms were presented for generating bilinear form.[2,3,20,22,23] Especially, a computer algorithm to determine ρ, ω, and a for bilinear form of KdV type is presented in Ref. 21.

In the present paper, a universal way is presented to determine whether a given nonlinear partial differential equation can be transformed to a bilinear equation of KdV type via a logarithmic transformation. When it can be transformed, all the parameters are determined so that the bilinear equation is written explicitly. Some classical equations whose bilinear form is of KdV type are discussed as examples to show the validity of the method.

Comparing with the known results, this method is universal to all the equations of KdV type and do not need any help of computer programming. Although the derivation of conclusions is quite complicated, the result (Steps 1–4 in Sec. VI) can be used easily to any specific equation. What is more, with those steps, a simple computer program can be written, although it is no longer necessary.

The paper is organized as follows. In Sec. II, we give some basic notations. In Sec. III, the bilinear equation of KdV type is introduced. Some properties of differential operators and Hirota derivatives are discussed in Sec. IV, while the properties of bilinear equation of KdV type are discussed in Sec. V. In Sec. VI, the main results are presented, which give a universal way to determine all the necessary parameters for transforming the equation to a bilinear equation of KdV type. The Sawada-Kotera equation is used as an example in that section. In Sec. VII, some more equations are discussed as examples.

[a] 11210180007@fudan.edu.cn
[b] zxzhou@fudan.edu.cn

II. BASIC NOTATIONS

Throughout this paper, we will consider a nonlinear PDE in m variables.

Let $\mathbf{N}$ be the set of non-negative integers,

$$\Pi = \overbrace{\mathbf{N} \times \cdots \times \mathbf{N}}^{m}, \quad \Pi^* = \Pi \backslash \{(0, \cdots, 0)\}. \tag{1}$$

For $\alpha = (\alpha_1, \cdots, \alpha_m) \in \Pi$, denote

$$|\alpha| = \alpha_1 + \cdots + \alpha_m, \quad \alpha! = \alpha_1! \cdots \alpha_m!. \tag{2}$$

We call α even if $|\alpha|$ is even.

For any two elements $\alpha = (\alpha_1, \cdots, \alpha_m)$ and $\beta = (\beta_1, \cdots, \beta_m)$ in Π, define $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for all $i = 1, \cdots, m$. Denote $v^\alpha = v_1^{\alpha_1} \cdots v_m^{\alpha_m}$ for a vector $v = (v_1, \cdots, v_m)$, and

$$\binom{\alpha}{\beta} = \prod_{i=1}^{m} \binom{\alpha_i}{\beta_i} \tag{3}$$

for combinatory coefficient.

For a positive integer k, let

$$\mathfrak{D}_k = \overbrace{\Pi \times \cdots \times \Pi}^{k} / \sim, \tag{4}$$

where two elements $(\alpha^{(1)}, \cdots, \alpha^{(k)})$ and $(\beta^{(1)}, \cdots, \beta^{(k)})$ in $\overbrace{\Pi \times \cdots \times \Pi}^{k}$ are equivalent if and only if $(\alpha^{(1)}, \cdots, \alpha^{(k)})$ is a permutation of $(\beta^{(1)}, \cdots, \beta^{(k)})$. Let $[\alpha^{(1)}, \cdots, \alpha^{(k)}] \in \mathfrak{D}_k$ be the equivalence class containing $(\alpha^{(1)}, \cdots, \alpha^{(k)})$. Similarly, $\mathfrak{D}_k^* = \overbrace{\Pi^* \times \cdots \times \Pi^*}^{k} / \sim$ where $\sim$ is the same as above. Let $\mathfrak{D} = \bigcup_{k=1}^{\infty} \mathfrak{D}_k$ and $\mathfrak{D}^* = \bigcup_{k=1}^{\infty} \mathfrak{D}_k^*$.

For $I = [\alpha^{(1)}, \cdots, \alpha^{(k)}] \in \mathfrak{D}_k$ with $\alpha^{(i)} = (\alpha_1^{(i)}, \cdots, \alpha_m^{(i)})$ and $\alpha_j^{(i)} \in \mathbf{N}$ $(i = 1, \cdots, k; j = 1, \cdots, m)$, define

$$\deg(I) = k, \quad |I| = \sum_{j=1}^{k} |\alpha^{(j)}|, \quad I! = \alpha^{(1)}! \cdots \alpha^{(k)}!, \tag{5}$$

$$s_j(I) = \alpha_j^{(1)} + \cdots + \alpha_j^{(k)} \ (j = 1, \cdots, m), \quad s(I) = (s_1(I), \cdots, s_m(I)). \tag{6}$$

I is called even if all $\alpha^{(j)}$ $(j = 1, \cdots, k)$ are even.

Let $x = (x_1, \cdots, x_m)$ be the coordinates of $\mathbf{R}^m$. Denote $\partial_j = \dfrac{\partial}{\partial x_j}$, $\partial = (\partial_1, \cdots, \partial_m)$. For $\alpha = (\alpha_1, \cdots, \alpha_m)$, denote $\partial^\alpha u = \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m} u$. Moreover, for $I = [\alpha^{(1)}, \cdots, \alpha^{(k)}] \in \mathfrak{D}_k$, define the nonlinear differential operator ∂^I as $\partial^I u = (\partial^{\alpha^{(1)}} u) \cdots (\partial^{\alpha^{(k)}} u)$. A nonlinear partial differential equation considered in this paper is of form

$$W[u] \equiv \sum_{I \in \mathfrak{D}} w_I \partial^I u = 0, \tag{7}$$

where $w_I \in \mathbf{R}$ $(I \in \mathfrak{D})$ are constants. Hereafter, we always suppose that there are only finite number of w_I such that $w_I \neq 0$.

For a polynomial

$$P(\tau) = \sum_{\alpha \in \Pi} p_\alpha \tau^\alpha \tag{8}$$

of variables $\tau = (\tau_1, \tau_2, \cdots, \tau_m)$ where p_α's are constants, define

$$P(\partial) f = \sum_{\alpha \in \Pi} p_\alpha \partial^\alpha f \tag{9}$$

for a function f.

Example. $u^2u_{xx}^2u_{xt}$ can be written as $\partial^I u$ with $I = [(0, 0), (0, 0), (2, 0), (2, 0), (1, 1)]$. $\deg(I) = 5$ is the degree of $u^2u_{xx}^2u_{xt}$ as a polynomial of u, u_{xx}, u_{xt}. $|I| = 6$ is the total order of the derivatives with respect to both x and t, and $s(I) = (5, 1)$ gives the total order of derivatives with respect to x and t respectively.

III. BILINEAR EQUATION OF KdV TYPE

For $\alpha = (\alpha_1, \cdots, \alpha_m) \in \Pi$, the Hirota derivative for two functions f and g is defined by

$$D^\alpha f \cdot g = (\partial_1 - \partial_1')^{\alpha_1} \cdots (\partial_m - \partial_m')^{\alpha_m} f(x)g(x')\big|_{x'=x}, \tag{10}$$

where $x, x' \in \mathbf{R}^m$. Then, for a polynomial P in the form (8), define

$$P(D)f \cdot f = \sum_{\alpha\in\Pi} p_\alpha D^\alpha f \cdot f. \tag{11}$$

When $|\alpha|$ is odd, exchange of x' and x in the Hirota derivative (10) implies that $D^\alpha f \cdot f = 0$. Hence we may always assume that each monomial in P is of even degree.

$D^\alpha f \cdot f$ is a differential polynomial of f, which can be expressed explicitly by expanding

$$\cosh\Big(\sum_{j=1}^m \varepsilon_j D_j\Big) f \cdot f = \exp\Big(2\cosh\Big(\sum_{j=1}^m \varepsilon_j \partial_j\Big)\ln f\Big), \tag{12}$$

where $\varepsilon_1, \cdots, \varepsilon_m$ are arbitrary constants.[11]

For $\omega \in \Pi$, the bilinear equation

$$\partial^\omega\Big(\frac{P(D)f \cdot f}{f^2}\Big) = 0 \tag{13}$$

is of KdV type. Apart from the KdV equation, there are a lot of bilinear equations of KdV type, as mentioned in the Introduction.

We want to determine whether a given equation

$$W[u] \equiv \sum_{I\in\mathfrak{D}} w_I \partial^I u = 0 \tag{14}$$

can be written in the bilinear form of KdV type under a logarithm transformation $u = 2a\partial^\rho \ln f$, i.e., whether we have the identity

$$\partial^\omega\Big(\frac{P(D)f \cdot f}{f^2}\Big) = \sum_{I\in\mathfrak{D}} w_I \partial^I u\Big|_{u=2a\partial^\rho \ln f}. \tag{15}$$

In order that $\ln f$ does not appear explicitly in the equation, we always suppose that $w_I = 0$ if $\rho = 0$ and $I \notin \mathfrak{D}^*$.

What we need to do is to determine $\rho, \omega \in \Pi$, $a \in \mathbf{R}$, and the polynomial P.

IV. PROPERTIES OF DIFFERENTIAL OPERATORS AND HIROTA DERIVATIVE

Hereafter, we will denote

$$S_k\Big(\sum_{j\ge 1}\sum_{I\in\mathfrak{D}^*} \frac{c_{jI}\partial^I f}{f^j}\Big) = \sum_{I\in\mathfrak{D}^*} c_{kI}, \tag{16}$$

where c_{jI}'s are constants. Moreover, denote $o\left(\frac{1}{f^k}\right)$ to be a Laurent polynomial of form

$$\sum_{j\ge k+1}\sum_{I\in\mathfrak{D}^*} \frac{c_{jI}\partial^I f}{f^j}. \tag{17}$$

Lemma 1. (1)

$$\partial^\lambda\left(\frac{(\partial^\alpha f)(\partial^\beta f)}{f^2}\right)=\frac{1}{f^2}\sum_{\mu\in\Pi,\,\mu\le\lambda}\binom{\lambda}{\mu}(\partial^{\lambda-\mu}\partial^\alpha f)(\partial^\mu\partial^\beta f)+o\left(\frac{1}{f^2}\right), \tag{18}$$

$$S_2\left(\partial^\lambda\left(\frac{(\partial^\alpha f)(\partial^\beta f)}{f^2}\right)\right)=2^{|\lambda|}. \tag{19}$$

(2)

$$\partial^\lambda\left(\frac{\partial^\alpha f}{f}\right)=\frac{1}{f}\partial^\lambda\partial^\alpha f-\frac{1}{f^2}\sum_{\mu\in\Pi^*,\,\mu\le\lambda}\binom{\lambda}{\mu}(\partial^{\lambda-\mu}\partial^\alpha f)(\partial^\mu f)+o\left(\frac{1}{f^2}\right), \tag{20}$$

$$S_2\left(\partial^\lambda\left(\frac{\partial^\alpha f}{f}\right)\right)=1-2^{|\lambda|}. \tag{21}$$

Proof. (1) For any $j, l\in\mathbf{N}$, with $1\le j\le m$ and $l>0$,

$$\partial_j^l\left(\frac{1}{f^2}\right)=o\left(\frac{1}{f^2}\right). \tag{22}$$

Hence, for $\lambda=(\lambda_1,\cdots,\lambda_m)$,

$$\begin{aligned}\partial^\lambda\left(\frac{(\partial^\alpha f)(\partial^\beta f)}{f^2}\right)&=\frac{\partial^\lambda((\partial^\alpha f)(\partial^\beta f))}{f^2}+o\left(\frac{1}{f^2}\right)\\&=\frac{1}{f^2}\sum_{\mu_1=0}^{\lambda_1}\cdots\sum_{\mu_m=0}^{\lambda_m}\binom{\lambda_1}{\mu_1}\cdots\binom{\lambda_m}{\mu_m}\cdot\\&\quad\cdot(\partial_1^{\lambda_1-\mu_1}\cdots\partial_m^{\lambda_m-\mu_m}\partial^\alpha f)(\partial_1^{\mu_1}\cdots\partial_m^{\mu_m}\partial^\beta f)+o\left(\frac{1}{f^2}\right)\\&=\frac{1}{f^2}\sum_{\mu\in\Pi,\,\mu\le\lambda}\binom{\lambda}{\mu}(\partial^{\lambda-\mu}\partial^\alpha f)(\partial^\mu\partial^\beta f)+o\left(\frac{1}{f^2}\right).\end{aligned} \tag{23}$$

Therefore,

$$S_2\left(\partial^\lambda\left(\frac{(\partial^\alpha f)(\partial^\beta f)}{f^2}\right)\right)=\sum_{\mu\in\Pi,\,\mu\le\lambda}\binom{\lambda}{\mu}=2^{|\lambda|}. \tag{24}$$

(2) If $l>0$, then

$$\partial_j^l\left(\frac{1}{f}\right)=\partial_j^{l-1}\left(-\frac{\partial_j f}{f^2}\right)=-\frac{\partial_j^l f}{f^2}+o\left(\frac{1}{f^2}\right). \tag{25}$$

Hence,

$$\partial^\lambda\left(\frac{1}{f}\right)=-\frac{\partial^\lambda f}{f^2}+o\left(\frac{1}{f^2}\right). \tag{26}$$

Let $\lambda = (\lambda_1, \cdots, \lambda_m)$, then

$$\begin{aligned}
\partial^\lambda\left(\frac{\partial^\alpha f}{f}\right) &= \textstyle\sum_{\mu_1=0}^{\lambda_1}\cdots\sum_{\mu_m=0}^{\lambda_m}\binom{\lambda_1}{\mu_1}\cdots\binom{\lambda_m}{\mu_m}\cdot \\
&\quad \cdot(\partial_1^{\lambda_1-\mu_1}\cdots\partial_m^{\lambda_m-\mu_m}\partial^\alpha f)\partial_1^{\mu_1}\cdots\partial_m^{\mu_m}\left(\frac{1}{f}\right) \\
&= \frac{\partial^\lambda\partial^\alpha f}{f} - \sum_{\substack{0\le\mu_1\le\lambda_1,\cdots,0\le\mu_m\le\lambda_m\\(\mu_1,\cdots,\mu_m)\neq 0}}\binom{\lambda_1}{\mu_1}\cdots\binom{\lambda_m}{\mu_m}\cdot \\
&\quad \cdot\frac{(\partial_1^{\lambda_1-\mu_1}\cdots\partial_m^{\lambda_m-\mu_m}\partial^\alpha f)(\partial_1^{\mu_1}\cdots\partial_m^{\mu_m} f)}{f^2} + o\left(\frac{1}{f^2}\right) \\
&= \frac{\partial^\lambda\partial^\alpha f}{f} - \sum_{\mu\in\Pi^*,\,\mu\le\lambda}\binom{\lambda}{\mu}\frac{(\partial^{\lambda-\mu}\partial^\alpha f)(\partial^\mu f)}{f^2} + o\left(\frac{1}{f^2}\right).
\end{aligned} \tag{27}$$

Hence,

$$S_1\left(\partial^\lambda\left(\tfrac{\partial^\alpha f}{f}\right)\right) = 1, \quad S_2\left(\partial^\lambda\left(\tfrac{\partial^\alpha f}{f}\right)\right) = 1 - 2^{|\lambda|}. \tag{28}$$

The lemma is proved.

Lemma 2. Let $u = 2a\partial^\rho \ln f$, then the following results hold.
(1) If $I = [\lambda] \in \mathfrak{D}_1$ with $|\lambda| + |\rho| > 0$, then

$$\begin{aligned}
\partial^\lambda u &= \frac{2a\partial^{\lambda+\rho} f}{f} - \frac{a}{f^2}\sum_{\substack{\mu\in\Pi,\,\mu\le\lambda+\rho\\ \mu\neq 0,\lambda+\rho}}\binom{\lambda+\rho}{\mu}(\partial^{\lambda+\rho-\mu} f)(\partial^\mu f) + o\left(\frac{1}{f^2}\right), \\
S_2(\partial^\lambda u) &= 2a(1 - 2^{|\lambda|+|\rho|-1}).
\end{aligned} \tag{29}$$

(2) If $I \in \mathfrak{D}_2$ with $|I| + |\rho| > 0$, then

$$S_2(\partial^I u) = 4a^2. \tag{30}$$

Proof. (1) Suppose $\rho = (\rho_1, \cdots, \rho_m)$ and $\lambda = (\lambda_1, \cdots, \lambda_m)$. Since $|\lambda| + |\rho| > 0$, we may assume, without loss of generality, that $\lambda_1 + \rho_1 > 0$, and denote $\beta = (\lambda_1 + \rho_1 - 1, \lambda_2 + \rho_2, \cdots, \lambda_m + \rho_m)$. By Lemma 1,

$$\begin{aligned}
\partial^\lambda u &= 2a\partial^\lambda\partial^\rho \ln f = 2a\partial^\beta\left(\frac{\partial_1 f}{f}\right) \\
&= \frac{2a}{f}\partial^\beta\partial_1 f - \frac{2a}{f^2}\sum_{\mu\in\Pi^*,\,\mu\le\beta}\binom{\beta}{\mu}(\partial^{\beta-\mu}\partial_1 f)(\partial^\mu f) + o\left(\frac{1}{f^2}\right) \\
&= \frac{2a}{f}\partial^{\lambda+\rho} f - \frac{2a}{f^2}\sum_{\substack{\mu\in\Pi,\,\mu\le\beta\\ \mu\neq 0}}\binom{\beta}{\mu}(\partial^{\lambda+\rho-\mu} f)(\partial^\mu f) + o\left(\frac{1}{f^2}\right).
\end{aligned} \tag{31}$$

Rewrite μ in the summation as $\lambda + \rho - \mu$, we have

$$\partial^\lambda u = \frac{2a}{f}\partial^{\lambda+\rho} f - \frac{2a}{f^2}\sum_{\substack{\mu\in\Pi,\,\mu\le\lambda+\rho\\ \mu\neq\lambda+\rho\\ \mu_1\ge 1}}\binom{\beta}{\lambda+\rho-\mu}(\partial^{\lambda+\rho-\mu} f)(\partial^\mu f) + o\left(\frac{1}{f^2}\right). \tag{32}$$

Hence,

$$\begin{aligned}
\partial^\lambda u &= \frac{2a}{f}\partial^{\lambda+\rho} f - \frac{a}{f^2}\Bigg[\sum_{\mu\in\Pi,\,\mu\leq\beta}\binom{\beta_2}{\mu_2}\cdots\binom{\beta_m}{\mu_m}\left(\binom{\beta_1}{\mu_1}+\binom{\beta_1}{\beta_1+1-\mu_1}\right)\cdot \\
&\quad \cdot(\partial^{\lambda+\rho-\mu} f)(\partial^\mu f) - 2(\partial^{\lambda+\rho} f) f\Bigg] + o\left(\frac{1}{f^2}\right) \\
&= \frac{2a}{f}\partial^{\lambda+\rho} f - \frac{a}{f^2}\Bigg[\sum_{\mu\in\Pi,\,\mu\leq\beta}\binom{\beta_2}{\mu_2}\cdots\binom{\beta_m}{\mu_m}\binom{\beta_1+1}{\mu_1}(\partial^{\lambda+\rho-\mu} f)(\partial^\mu f) \\
&\quad -2(\partial^{\lambda+\rho} f) f\Bigg] + o\left(\frac{1}{f^2}\right) \\
&= \frac{2a}{f}\partial^{\lambda+\rho} f - \frac{a}{f^2}\sum_{\substack{\mu\in\Pi,\,\mu\leq\lambda+\rho\\ \mu\neq 0,\lambda+\rho}}\binom{\lambda+\rho}{\mu}(\partial^{\lambda+\rho-\mu} f)(\partial^\mu f) + o\left(\frac{1}{f^2}\right).
\end{aligned} \tag{33}$$

Here we have supposed $\binom{n}{k}=0$ if $k<0$ or $k>n$. Then

$$\begin{aligned}
S_2(\partial^\lambda u) &= -a\sum_{\substack{\mu\in\Pi,\,\mu\leq\lambda+\rho\\ \mu\neq 0,\lambda+\rho}}\binom{\lambda+\rho}{\mu} \\
&= -a(2^{|\lambda|+|\rho|}-2) = 2a(1-2^{|\lambda|+|\rho|-1}).
\end{aligned} \tag{34}$$

(2) Suppose $I=[\mu,\nu]\in\mathfrak{D}_2$, then by (29),

$$\partial^\mu u = \frac{2a\partial^\mu\partial^\rho f}{f} + o\left(\frac{1}{f}\right), \quad \partial^\nu u = \frac{2a\partial^\nu\partial^\rho f}{f} + o\left(\frac{1}{f}\right). \tag{35}$$

Hence,

$$S_2(\partial^I u) = S_2((\partial^\mu u)(\partial^\nu u)) = (2a)^2 S_2\left(\frac{(\partial^{\rho+\mu} f)(\partial^{\rho+\nu} f)}{f^2}\right) = 4a^2. \tag{36}$$

The lemma is proved.

Lemma 3. Suppose $\omega,\lambda\in\Pi$ where $|\lambda|>0$ is even, then

$$\frac{D^\lambda f\cdot f}{f^2} = \frac{2\partial^\lambda f}{f} + \sum_{\substack{\mu\in\Pi,\,\mu\leq\lambda\\ \mu\neq 0,\lambda}}\frac{(-1)^{|\mu|}\binom{\lambda}{\mu}(\partial^{\lambda-\mu} f)(\partial^\mu f)}{f^2}, \tag{37}$$

$$\begin{aligned}
S_1\left(\partial^\omega\Big(\frac{D^\lambda f\cdot f}{f^2}\Big)\right) &= 2, \\
S_2\left(\partial^\omega\left(\frac{D^\lambda f\cdot f}{f^2}\right)\right) &= 2(1-2^{|\omega|+1}).
\end{aligned} \tag{38}$$

Proof. Denote $\partial_j' = \frac{\partial}{\partial x_j'}$. For $\lambda=(\lambda_1,\cdots,\lambda_m)$,

$$\begin{aligned}
D^\lambda f\cdot f &= \sum_{\mu_1=0}^{\lambda_1}\cdots\sum_{\mu_m=0}^{\lambda_m}\binom{\lambda_1}{\mu_1}\cdots\binom{\lambda_m}{\mu_m}\cdot \\
&\quad \cdot\partial_1^{\lambda_1-\mu_1}\cdots\partial_m^{\lambda_m-\mu_m}(-\partial_1')^{\mu_1}\cdots(-\partial_m')^{\mu_m} f(x)f(x')|_{x'=x} \\
&= \sum_{\mu_1=0}^{\lambda_1}\cdots\sum_{\mu_m=0}^{\lambda_m}(-1)^{\mu_1+\cdots+\mu_m}\binom{\lambda_1}{\mu_1}\cdots\binom{\lambda_m}{\mu_m}\cdot \\
&\quad \cdot(\partial_1^{\lambda_1-\mu_1}\cdots\partial_m^{\lambda_m-\mu_m} f(x))(\partial_1^{\mu_1}\cdots\partial_m^{\mu_m} f(x)) \\
&= \sum_{\mu\in\Pi,\,\mu\leq\lambda}(-1)^{|\mu|}\binom{\lambda}{\mu}(\partial^{\lambda-\mu} f(x))(\partial^\mu f(x)).
\end{aligned} \tag{39}$$

Hence (37) holds.

Clearly, $S_1\left(\partial^\omega\Big(\frac{D^\lambda f\cdot f}{f^2}\Big)\right)=2$. On the other hand, by Lemma 1,

$$\begin{aligned}
S_2\left(\partial^\omega\left(\frac{D^\lambda f\cdot f}{f^2}\right)\right) &= 2S_2\left(\partial^\omega\left(\frac{\partial^\lambda f}{f}\right)\right)+\\
&\quad+\sum_{\substack{\mu\in\Pi,\,\mu\le\lambda\\ \mu\ne 0,\lambda}}(-1)^{|\mu|}\binom{\lambda}{\mu}S_2\left(\partial^\omega\left(\frac{(\partial^{\lambda-\mu}f)(\partial^\mu f)}{f^2}\right)\right)\\
&= 2(1-2^{|\omega|})+\sum_{\substack{\mu\in\Pi,\,\mu\le\lambda\\ \mu\ne 0,\lambda}}(-1)^{|\mu|}\binom{\lambda}{\mu}2^{|\omega|}\\
&= 2(1-2^{|\omega|})+\left(\prod_{j=1}^{m}(1-1)^{\lambda_j}-2\right)2^{|\omega|}\\
&= 2(1-2^{|\omega|+1}).
\end{aligned}\tag{40}$$

The lemma is proved.

V. PROPERTIES OF THE BILINEAR EQUATION OF KdV TYPE

Consider the linear term

$$W_1[u]\equiv\sum_{[\mu]\in\mathfrak{D}_1}w_{[\mu]}\partial^\mu u\tag{41}$$

in $W[u]$.

Lemma 4. Suppose (15) holds where $P(\tau)=\sum_{\beta\in\Pi}p_\beta\partial^\beta\tau$ for $\tau\in\mathbf{R}^m$. Then for $\beta\in\Pi$,

$$p_\beta=aw_{[\beta+\omega-\rho]},\tag{42}$$

or equivalently,

$$a\sum_{[\mu]\in\mathfrak{D}_1}w_{[\mu]}\partial^{\mu+\rho}f=\partial^\omega P(\partial)f.\tag{43}$$

Proof. Lemma 2 implies that

$$\sum_{I\in\mathfrak{D}}w_I\partial^I u=2a\sum_{[\mu]\in\mathfrak{D}_1}\frac{w_{[\mu]}\partial^\mu\partial^\rho f}{f}+o\left(\frac{1}{f}\right).\tag{44}$$

On the other hand, according to Lemma 3,

$$\frac{P(D)f\cdot f}{f^2}=\frac{2P(\partial)f}{f}+o\left(\frac{1}{f}\right),\tag{45}$$

hence,

$$\begin{aligned}
\partial^\omega\left(\frac{P(D)f\cdot f}{f^2}\right) &= \partial^\omega\left(\frac{2P(\partial)f}{f}\right)+o\left(\frac{1}{f}\right)\\
&= \frac{2\partial^\omega P(\partial)f}{f}+o\left(\frac{1}{f}\right).
\end{aligned}\tag{46}$$

Considering the identity (15), Eqs. (44) and (46) lead to (43). (42) follows by comparing the coefficients. This proves the lemma.

For given $\rho,\sigma\in\Pi$, and $k\in\mathbf{N}$, define

$$\mathfrak{D}_k^{\rho,\sigma}=\{I\in\mathfrak{D}_k\mid s(I)+\deg(I)\rho=\sigma\},\quad \mathfrak{D}^{\rho,\sigma}=\bigcup_{k=1}^{\infty}\mathfrak{D}_k^{\rho,\sigma}.\tag{47}$$

Remark 1. Clearly, $\mathfrak{D}^{\rho,\sigma} \neq \emptyset$ *only if* $\rho \leq \sigma$*. When* $\rho \leq \sigma$ *holds,* $\mathfrak{D}_1^{\rho,\sigma} = \{[\alpha] \in \mathfrak{D}_1 \,|\, \alpha + \rho = \sigma\}$ *contains exactly one element* $[\sigma - \rho]$.

Theorem 1. *If*

$$\partial^{\omega}\left(\frac{D^{\lambda} f \cdot f}{f^2}\right) = \sum_{I \in \mathfrak{D}} c_I \partial^I u\big|_{u=2a\partial^{\rho}\ln f} \tag{48}$$

($c_I = 0$ *if* $\rho = 0$ *and* $I \notin \mathfrak{D}^*$*) where* $|\lambda| > 0$ *is even, then* $I \in \mathfrak{D}^{\rho,\omega+\lambda}$ *if* $c_I \neq 0$.

Therefore, if

$$W[u] = \sum_{\lambda \in \Pi^*} p_{\lambda} \partial^{\omega}\left(\frac{D^{\lambda} f \cdot f}{f^2}\right) \tag{49}$$

with substitution $u = 2a\partial^{\rho}\ln f$*, then*

$$W[u] = \sum_{\lambda \in \Pi^*} W_{\lambda}[u], \tag{50}$$

where

$$W_{\lambda}[u] = \sum_{I \in \mathfrak{D}^{\rho,\omega+\lambda}} w_I \partial^I u = p_{\lambda}\partial^{\omega}\left(\frac{D^{\lambda} f \cdot f}{f^2}\right). \tag{51}$$

Proof. Let $\lambda = (\lambda_1, \cdots, \lambda_m)$, $\omega = (\omega_1, \cdots, \omega_m)$, then by Leibniz rule,

$$\partial^{\omega}\left(\frac{D^{\lambda} f \cdot f}{f^2}\right) = \sum_{k \geq 1}\sum_{I \in \mathfrak{D}^*} d_{kI}\frac{\partial^I f}{f^k}, \tag{52}$$

where d_{kI}'s are constants.

In (48), write $I = [\alpha^{(1)}, \cdots, \alpha^{(k)}] \in \mathfrak{D}_k$ where $\alpha^{(i)} = (\alpha_1^{(i)}, \cdots, \alpha_m^{(i)}) \in \Pi$. Then when $c_I \neq 0$, the total order of derivatives with respect to x_i in both sides of (48) is

$$\lambda_i + \omega_i = \sum_{j=1}^{k}(\alpha_i^{(j)} + \rho_i) = s_i(I) + \deg(I)\rho_i. \tag{53}$$

This implies that $I \in \mathfrak{D}_k^{\rho,\omega+\lambda}$. The theorem is proved.

Remark 2. Actually, Theorem 1 gives a classification of all the terms $w_I\partial^I u$ *in* $W[u]$ *according to the bilinear form corresponding to* $W[u]$*. Two terms* $w_I\partial^I u$ *and* $w_J\partial^J u$ *are in the same class if* I *and* J *are in the same* $\mathfrak{D}^{\rho,\omega+\lambda}$*. In this case, they come from the same term* $D^{\lambda}f \cdot f$ *in the bilinear form. By Remark 1, there is exactly one linear term in each class.*

However, to obtain this classification, we do not need to derive the bilinear form in advance. Both we need are ρ *and* ω*, which will be determined later. In fact, the explicit expression of the bilinear form will depend on this classification.*

Lemma 5. Suppose $v = 2a\ln f$*,* $\lambda \in \Pi^*$ *with* $|\lambda|$ *even, then* $\dfrac{D^{\lambda} f \cdot f}{f^2}$ *can be written as*

$$\frac{D^{\lambda} f \cdot f}{f^2} = \sum_{J \in \mathfrak{D}^*} b_J \partial^J v, \tag{54}$$

where $b_J \geq 0$ *(*$J \in \mathfrak{D}^*$*) are constants. Moreover,* $b_J > 0$ *if and only if* $J \in \mathfrak{D}_k^{0,\lambda} \cap \mathfrak{D}^*$ *is even with* $1 \leq k \leq |\lambda|/2$.

Proof. Since $f = \exp\left(\frac{v}{2a}\right)$, $\frac{\partial^{\lambda} f}{f}$ is a polynomial of $\{\partial^{\alpha} v \,|\, \alpha \in \Pi^*\}$. According to Lemma 3, $\frac{D^{\lambda} f \cdot f}{f^2}$ is of form $\frac{2\partial^{\lambda} f}{f} + \frac{Q}{f^2}$, where Q is a quadratic polynomial of $\{\partial^{\alpha} f | \alpha \in \Pi^*\}$. Write $Q =$

$\sum_{\alpha,\beta\in\Pi^*} d_{\alpha\beta}(\partial^\alpha f)(\partial^\beta f)$, then

$$\frac{D^\lambda f\cdot f}{f^2}=2\frac{\partial^\lambda f}{f}+\sum_{\alpha,\beta\in\Pi^*} d_{\alpha\beta}\frac{\partial^\alpha f}{f}\frac{\partial^\beta f}{f} \tag{55}$$

is a polynomial of $\{\partial^\alpha v\,|\,\alpha\in\Pi^*\}$.

If $b_J\neq 0$, then $s(J)=\lambda$ by comparing the order of derivatives on both sides of (54).

Conversely, suppose $s(J)=\lambda$. The left-hand side of (12) is

$$\begin{aligned}&\sum_{l=0}^{\infty}\frac{1}{(2l)!}\Big(\sum_{j=1}^{m}\varepsilon_j D_j\Big)^{2l} f\cdot f\\&=\sum_{l=0}^{\infty}\frac{1}{(2l)!}\sum_{\lambda_1+\cdots+\lambda_m=2l}\frac{(2l)!}{\lambda_1!\cdots\lambda_m!}\varepsilon_1^{\lambda_1}\cdots\varepsilon_m^{\lambda_m}D_1^{\lambda_1}\cdots D_m^{\lambda_m} f\cdot f\\&=\sum_{l=0}^{\infty}\sum_{\lambda\in\Pi,|\lambda|=2l}\frac{1}{\lambda!}\varepsilon^\lambda D^\lambda f\cdot f.\end{aligned} \tag{56}$$

Suppose $J=[\lambda^{(1)},\cdots,\lambda^{(k)}]\in\mathfrak{D}$. Let B_J be the number of different permutations (including identity) of $(\lambda^{(1)},\cdots,\lambda^{(k)})$. The right-hand side of (12) is

$$\begin{aligned}&\exp\left(2\cosh\Big(\sum_{j=1}^{m}\varepsilon_j\partial_j\Big)\ln f\right)\\&=\exp\left(2\ln f+\sum_{i=1}^{\infty}\frac{2}{(2i)!}\Big(\sum_{j=1}^{m}\varepsilon_j\partial_j\Big)^{2i}\ln f\right)\\&=f^2\sum_{l=0}^{\infty}\frac{1}{l!}\left(\sum_{i=1}^{\infty}\frac{2}{(2i)!}\Big(\sum_{j=1}^{m}\varepsilon_j\partial_j\Big)^{2i}\ln f\right)^l\\&=f^2\sum_{l=0}^{\infty}\frac{1}{l!}\left(\sum_{i=1}^{\infty}\sum_{\lambda\in\Pi,|\lambda|=2i}\frac{2}{\lambda!}\varepsilon^\lambda\partial^\lambda\ln f\right)^l\\&=f^2\sum_{l=0}^{\infty}\frac{1}{l!}\left(\sum_{\lambda\in\Pi^*,\ |\lambda|\text{ is even}}\frac{2}{\lambda!}\varepsilon^\lambda\partial^\lambda\ln f\right)^l\\&=f^2\sum_{J\in\mathfrak{D}^*,\ J\text{ is even}}\frac{1}{\deg(J)!}\frac{2^{\deg(J)}B_J}{J!}\varepsilon^{s(J)}\partial^J\ln f.\end{aligned} \tag{57}$$

Comparing the coefficients of ε^λ, we have

$$b_J=\frac{B_J s(J)!}{a^{\deg(J)}\deg(J)!J!}>0, \tag{58}$$

whenever $J\in\mathfrak{D}^*$ satisfies $s(J)=\lambda$ and J is even. This implies that $b_J>0$ if and only if $J\in\mathfrak{D}_k^{0,\lambda}\cap\mathfrak{D}^*$ is even.

If $J=(\beta^{(1)},\cdots,\beta^{(k)})\in\mathfrak{D}_k^{0,\lambda}\cap\mathfrak{D}^*$ is even, then $|J|=\sum_{j=1}^{k}|\beta^{(k)}|\geq 2k$, hence $1\leq k\leq|\lambda|/2$ when $\mathfrak{D}_k^{0,\lambda}\neq\emptyset$. On the other hand, suppose k satisfies $1\leq k\leq|\lambda|/2$, then there is a partition $\lambda=\lambda^{(1)}+\cdots+\lambda^{(k)}$ such that $\lambda^{(j)}\in\Pi$ $(j=1,\cdots,k)$, $|\lambda^{(j)}|=2$ for $j=1,\cdots,k-1$ and $|\lambda^{(k)}|=|\lambda|-2k+2\geq 2$. Hence, $[\lambda^{(1)},\cdots,\lambda^{(k)}]\in\mathfrak{D}_k^{0,\lambda}$, which implies that $\mathfrak{D}_k^{0,\lambda}\neq\emptyset$. Therefore, $b_J>0$ if and only if $J\in\mathfrak{D}_k^{0,\lambda}\cap\mathfrak{D}^*$ is even with $1\leq k\leq|\lambda|/2$. The lemma is proved.

Theorem 2. *Suppose*

$$\partial^{\omega}\left(\frac{D^{\lambda} f\cdot f}{f^2}\right) = \sum_{I\in\mathfrak{D}^{\rho,\omega+\lambda}} c_I \partial^I u\big|_{u=2a\partial^{\rho}\ln f} \tag{59}$$

($c_I = 0$ if $\rho = 0$ and $I \notin \mathfrak{D}^*$), then $c_I \geq 0$ for all $I \in \mathfrak{D}^{\rho,\omega+\lambda}$. Moreover, there exists $I \in \mathfrak{D}_k^{\rho,\omega+\lambda}$ such that $c_I > 0$ if and only if $1 \leq k \leq |\lambda|/2$.

Proof. (59) can be rewritten as

$$\partial^{\omega}\left(\frac{D^{\lambda} f\cdot f}{f^2}\right) = \sum_{I\in\mathfrak{D}^{\rho,\omega+\lambda}} c_I \partial^I (\partial^{\rho} v), \tag{60}$$

where $v = 2a\ln f$ ($c_I = 0$ if $\rho = 0$ and $I \notin \mathfrak{D}^*$). On the other hand, according to Lemma 5, we can write

$$\frac{D^{\lambda} f\cdot f}{f^2} = \sum_{J\in\mathfrak{D}^{0,\lambda}\cap\mathfrak{D}^*} b_J \partial^J v, \tag{61}$$

where $b_J > 0$ if and only if $J \in \mathfrak{D}_k^{0,\lambda}\cap\mathfrak{D}^*$ is even with $1 \leq k \leq |\lambda|/2$. Hence,

$$\sum_{I\in\mathfrak{D}^{\rho,\omega+\lambda}} c_I \partial^I (\partial^{\rho} v) = \partial^{\omega}\Big(\sum_{J\in\mathfrak{D}^{0,\lambda}\cap\mathfrak{D}^*} b_J \partial^J v\Big). \tag{62}$$

If $I = [\alpha^{(1)},\cdots,\alpha^{(k)}] \in \mathfrak{D}_k^{\rho,\lambda+\omega}$ and $c_I \neq 0$, then by Leibniz rule, there exists $J = [\beta^{(1)},\cdots,\beta^{(k)}] \in \mathfrak{D}_k^{0,\lambda}\cap\mathfrak{D}^*$ with $b_J \neq 0$ and a partition $\omega = \omega^{(1)} + \cdots + \omega^{(k)}$ ($\omega^{(j)} \in \Pi$, $j = 1,\cdots,k$) such that $\alpha^{(j)} + \rho = \beta^{(j)} + \omega^{(j)}$. $b_J \neq 0$ implies $1 \leq k \leq |\lambda|/2$.

Conversely, for k satisfying $1 \leq k \leq |\lambda|/2$, there exists $J = [\beta^{(1)},\cdots,\beta^{(k)}] \in \mathfrak{D}_k^{0,\lambda}\cap\mathfrak{D}^*$ such that $b_J > 0$. Let $K = [\beta^{(1)},\cdots,\beta^{(k-1)},\beta^{(k)} + \omega]$, then by Leibniz rule, there exists $I = [\alpha^{(1)},\cdots,\alpha^{(k)}] \in \mathfrak{D}_k^{\rho,\omega+\lambda}$ with $c_I \neq 0$ such that $K = [\alpha^{(1)} + \rho,\cdots,\alpha^{(k)} + \rho]$. The theorem is proved.

Remark 3. The right-hand side of (59) is linear with respect to u if $|\lambda| \leq 2$, and is nonlinear with respect to u if $|\lambda| > 2$.

From the above theorem, we obtain immediately the following.

Corollary 1. Suppose

$$\partial^{\omega}\left(\frac{D^{\lambda} f\cdot f}{f^2}\right) = \sum_{I\in\mathfrak{D}^{\rho,\omega+\lambda}} c_I \partial^I u\big|_{u=2a\partial^{\rho}\ln f} \tag{63}$$

($c_I = 0$ if $\rho = 0$ and $I \notin \mathfrak{D}^*$) where $|\lambda| > 0$ is even, then there exists $I = [\alpha^{(1)},\cdots,\alpha^{(k)}] \in \mathfrak{D}$ with $k = |\lambda|/2$ such that $|\alpha^{(j)}| + |\rho| = 2$ for all $j = 1,\cdots,k$ and $c_I \neq 0$. Therefore, $|\rho| \leq 2$ always holds.

VI. DETERMINATION OF PARAMETERS IN THE BILINEAR FORM

For given Eq. (14), suppose the identity (15) holds. We will determine ρ, $\omega \in \Pi$, $a \in \mathbf{R}$, and the polynomial P.

To illustrate the method, the Sawada-Kotera equation

$$u_t + 45u^2u_x - 15u_xu_{xx} - 15uu_{xxx} + u_{xxxxx} = 0 \tag{64}$$

will be used as an example.

A. Determination of θ

For $I, J \in \mathfrak{D}_k$, we write $I \prec J$ or $J \succ I$ if (1) $|I| < |J|$, or (2) $|I| = |J|$ and there exists j ($1 \leq j \leq m$) such that $s_1(I) = s_1(J),\cdots,s_{j-1}(I) = s_{j-1}(J)$, $s_j(I) < s_j(J)$, or (3) $s(I) = s(J)$. For a given nonlinear PDE (15), I is called maximal in $\mathfrak{D}_k$ if $I \in \mathfrak{D}_k$, $w_I \neq 0$, and for all $J \in \mathfrak{D}_k$, $I \succ J$ holds.

Obviously there exists unique maximal element $[\theta]$ in $\mathfrak{D}_1$. Since $s([\theta]) + \deg([\theta])\rho = \theta + \rho$, $[\theta]$ must be in $\mathfrak{D}_1^{\rho,\theta+\rho}$ for any given $\rho \in \Pi$.

Example. For Eq. (64), let

$$\Sigma = \{[(0,1)], [(0,0),(0,0),(1,0)], [(1,0),(2,0)], [(0,0),(3,0)], [(5,0)]\} \subset \mathfrak{D}, \tag{65}$$

then $w_I \neq 0$ if and only if $I \in \Sigma$. We have the form

	$I \in \mathfrak{D}$	$\lvert I\rvert$	$\deg(I)$
u_t	$[(0,1)]$	1	1
u^2u_x	$[(0,0),(0,0),(1,0)]$	1	3
u_xu_{xx}	$[(1,0),(2,0)]$	3	2
uu_{xxx}	$[(0,0),(3,0)]$	3	2
u_{xxxxx}	$[(5,0)]$	5	1

(66)

Since $\Sigma \cap \mathfrak{D}_1 = \{[(0,1)], [(5,0)]\}$, the unique maximal element in $\mathfrak{D}_1$ is $[\theta] = [(5,0)]$.

B. Determination of ρ

When the equation is nonlinear, Theorem 2 implies that there exists $J \in \mathfrak{D}_2^{\rho,\theta+\rho}$ such that $w_J \neq 0$. However, J may not be unique.

Theorem 3. *Let θ be the maximal element in $\mathfrak{D}_1$. If J is a maximal element in $\mathfrak{D}_2$, then $J \in \mathfrak{D}_2^{\rho,\theta+\rho}$. Moreover, $\rho = \theta - s(J)$ (independent of the choice of J).*

Proof. We have known that there exists $K \in \mathfrak{D}_2^{\rho,\theta+\rho}$ such that $w_K \neq 0$. Suppose $J \in \mathfrak{D}_2$ is maximal and $J \in \mathfrak{D}_2^{\rho,\nu+\rho}$, then

$$s(J) + 2\rho = \nu + \rho, \quad s(K) + 2\rho = \theta + \rho, \tag{67}$$

which lead to

$$s(K) - s(J) = \theta - \nu, \quad |K| - |J| = |\theta| - |\nu|. \tag{68}$$

$K \prec J$ in $\mathfrak{D}_2$ implies $\theta \prec \nu$ in $\mathfrak{D}_1$. θ is maximal in $\mathfrak{D}_1$ implies $\nu = \theta$. Hence any maximal element in $\mathfrak{D}_2$ are in $\mathfrak{D}_2^{\rho,\theta+\rho}$. (67) gives the expression of ρ. The lemma is proved.

We call $\mathfrak{D}^{\rho,\theta+\rho}$ the top class of Eq. (15).

Example. For Eq. (64), the form (66) implies that the maximal elements in $\mathfrak{D}_2$ are $I_1 = [(1,0),(2,0)]$ and $I_2 = [(0,0),(3,0)]$. Since $s(I_1) = s(I_2) = (3,0)$, $\rho = \theta - s(I_1) = (5,0) - (3,0) = (2,0)$. Then we have

	$I \in \mathfrak{D}$	$\lvert I\rvert$	$\deg(I)$		$s(I) + \deg(I)\rho$
u_t	$[(0,1)]$	1	1	$\theta = (5,0)$	$(2,1)$
u^2u_x	$[(0,0),(0,0),(1,0)]$	1	3	$\rho = (2,0)$	$(7,0)$
u_xu_{xx}	$[(1,0),(2,0)]$	3	2		$(7,0)$
uu_{xxx}	$[(0,0),(3,0)]$	3	2		$(7,0)$
u_{xxxxx}	$[(5,0)]$	5	1		$(7,0)$

(69)

Hence,

$$\begin{aligned} &\mathfrak{D}^{(2,0),(2,1)} \cap \Sigma = \{[(0,1)]\}, \\ &\mathfrak{D}^{(2,0),(7,0)} \cap \Sigma = \{[(0,0),(0,0),(1,0)], [(1,0),(2,0)], [(0,0),(3,0)], [(5,0)]\}. \end{aligned} \tag{70}$$

The top class is $\mathfrak{D}^{(2,0),(7,0)}$.

C. Determination of ω

For a non-empty set $S\subset\Pi$, denote $d(S) = (\beta_1, \cdots, \beta_m)$, where

$$\beta_j = \min_{(\alpha_1,\cdots,\alpha_k)\in S} \alpha_j. \tag{71}$$

Theorem 4. *Suppose* $\partial^\omega\left(\frac{D^\lambda f\cdot f}{f^2}\right) = \sum_{I\in\mathfrak{D}} c_I \partial^I u\Big|_{u=2a\partial^\rho \ln f}$, *where* $|\lambda| \geq 4$ *is even* ($c_I = 0$ *if* $\rho = 0$ *and* $I \notin \mathfrak{D}^*$). *Let* $r = \max_{I\in\mathfrak{D},\, c_I\neq 0} \deg(I)$,

$$\begin{aligned}\widetilde{\mathfrak{D}}_r = \{I = [\alpha^{(1)}, \cdots, \alpha^{(r)}] \in \mathfrak{D}_r \,\Big|\, & c_I \neq 0,\ and\ there\ exists\ l \\ & such\ that\ |\alpha^{(j)}| + |\rho| = 2\ for\ j \neq l\ and\ |\alpha^{(l)}| + |\rho| > 2\},\end{aligned} \tag{72}$$

$$\widetilde{\Pi} = \bigcup_{[\alpha^{(1)},\cdots,\alpha^{(r)}]\in\widetilde{\mathfrak{D}}_r}\ \bigcup_{j=1}^{r}\{\alpha^{(j)}\}, \tag{73}$$

$$\begin{aligned}\widetilde{\Pi}_0 &= \{\phi \in \widetilde{\Pi}\,\Big|\, |\phi| + |\rho| = 2\},\\ \widetilde{\Pi}_1 &= \{\phi \in \widetilde{\Pi}\,\Big|\, |\phi| + |\rho| > 2\}.\end{aligned} \tag{74}$$

Then

$$\omega = d(\widetilde{\Pi}_1) - d(\widetilde{\Pi}_0) \tag{75}$$

if $\widetilde{\mathfrak{D}}_r$ *is not empty and* $\omega = (0, \cdots, 0)$ *if* $\widetilde{\mathfrak{D}}_r$ *is empty.*

Proof. Clearly, the theorem is true if all ($I = [\alpha^{(1)}, \cdots, \alpha^{(r)}]$, ρ) are replaced by ($[\alpha^{(1)} + \rho, \cdots, \alpha^{(r)} + \rho]$, 0). Hence, without loss of generality, suppose $\rho = 0$.

According to Lemma 5, we can write $\dfrac{D^\lambda f\cdot f}{f^2} = \sum_{J\in\mathfrak{D}^*} b_J \partial^J u\Big|_{u=2a\ln f}$, and $b_J > 0$ if and only if $J \in \mathfrak{D}_k^{0,\lambda} \cap \mathfrak{D}^*$ with $1 \leq k \leq |\lambda|/2$.

By Corollary 1, there exists $\widetilde{J} = [\widetilde{\gamma}^{(1)}, \cdots, \widetilde{\gamma}^{(|\lambda|/2)}]$ such that $b_{\widetilde{J}} > 0$ and $|\widetilde{\gamma}^{(j)}| = 2$ for all $j = 1, \cdots, \dfrac{|\lambda|}{2}$. Hence,

$$r = \max_{I\in\mathfrak{D},\, c_I\neq 0} \deg(I) = \max_{J\in\mathfrak{D},\, b_J\neq 0} \deg(J) = \frac{|\lambda|}{2}. \tag{76}$$

Suppose $J = [\gamma^{(1)}, \cdots, \gamma^{(r)}] \in \mathfrak{D}_r^*$ with $b_J \neq 0$, then $|\gamma^{(j)}| \geq 2$ and is even for $1 \leq j \leq r$. We have

$$|\lambda| = |J| = \sum_{j=1}^{r} |\gamma^{(j)}| \geq 2r = |\lambda|. \tag{77}$$

The equality holds if and only if $|\gamma^{(j)}| = 2$ for all $1 \leq j \leq r$. Therefore, $|\gamma^{(j)}| = 2$ for all $1 \leq j \leq r$ if $b_J \neq 0$ and $\deg(J) = r$.

Denote

$$\widetilde{\mathfrak{D}}'_r = \{J = [\gamma^{(1)}, \cdots, \gamma^{(r)}] \in \mathfrak{D}_r \,\Big|\, b_J \neq 0\}, \tag{78}$$

then for any $J = [\gamma^{(1)}, \cdots, \gamma^{(r)}] \in \widetilde{\mathfrak{D}}'_r$, $|\gamma^{(j)}| = 2$ for all $1 \leq j \leq r$. $\mathfrak{D}'_r \neq \emptyset$ since $\widetilde{J} \in \mathfrak{D}'_r$. When $c_I \neq 0$ with $\deg(I) = r$, Leibniz rule implies that I is of form $[\gamma^{(1)} + \omega^{(1)}, \cdots, \gamma^{(r)} + \omega^{(r)}]$ with $\omega^{(j)} \in \Pi$ ($j = 1, \cdots, r$) and $\omega^{(1)} + \cdots + \omega^{(r)} = \omega$. Then, by the definition of $\widetilde{\mathfrak{D}}_r$,

$$\widetilde{\mathfrak{D}}_r = \bigcup_{j=1}^{r}\left\{[\gamma^{(1)}, \cdots, \gamma^{(j-1)}, \gamma^{(j)} + \omega, \gamma^{(j+1)}, \cdots, \gamma^{(r)}]\,\Big|\, [\gamma^{(1)}, \cdots, \gamma^{(r)}] \in \widetilde{\mathfrak{D}}'_r\right\} \tag{79}$$

is not empty if $\omega \neq 0$, and $\widetilde{\mathfrak{D}}_r = \emptyset$ if $\omega = 0$.

When $\widetilde{\mathfrak{D}}_r \neq \emptyset$, denote

$$\widetilde{\Pi}' = \bigcup_{[\gamma^{(1)},\cdots,\gamma^{(r)}]\in\widetilde{\mathfrak{D}}'_r} \bigcup_{j=1}^{r} \{\gamma^{(j)}\}, \tag{80}$$

then

$$\widetilde{\Pi}_0 = \widetilde{\Pi}', \quad \widetilde{\Pi}_1 = \{\phi + \omega \,|\, \phi \in \widetilde{\Pi}'\}. \tag{81}$$

Therefore,

$$\widetilde{\Pi}_1 = \{\phi + \omega \,|\, \phi \in \widetilde{\Pi}_0\}. \tag{82}$$

Suppose $d(\widetilde{\Pi}_0) = (\beta_1, \cdots, \beta_m)$, $d(\widetilde{\Pi}_1) = (\beta'_1, \cdots, \beta'_m)$, then

$$\beta'_j = \min_{(\psi_1,\cdots,\psi_m)\in\widetilde{\Pi}_1} \psi_j = \min_{(\phi_1,\cdots,\phi_m)\in\widetilde{\Pi}_0} (\phi_j + \omega_j) = \beta_j + \omega_j. \tag{83}$$

The theorem is proved.

Example. For Eq. (64), consider the top class $\mathfrak{D}^{\rho,\theta+\rho} = \mathfrak{D}^{(2,0),(7,0)}$ which corresponds to one linear term and three nonlinear terms. In this class, $r = \max_{\substack{I\in\mathfrak{D},\, c_I\neq 0\\ s(I)+\deg(I)\rho=(7,0)}} \deg(I) = 3$. Since $|\rho| = 2$, we have $\widetilde{\mathfrak{D}}_3 = \{[(0,0),(0,0),(1,0)]\}$. Then $\widetilde{\Pi}_0 = \{(0,0)\}$, $\widetilde{\Pi}_1 = \{(1,0)\}$, $\omega = d(\widetilde{\Pi}_1) - d(\widetilde{\Pi}_0) = (1,0) - (0,0) = (1,0)$.

D. Determination of *a*

Theorem 5. *If*

$$\partial^\omega \left(\frac{D^\lambda f \cdot f}{f^2} \right) = \sum_{I\in\mathfrak{D}} c_I \partial^I u\big|_{u=2a\partial^\rho \ln f}, \tag{84}$$

where $|\lambda| \geq 4$ *is even* ($c_I = 0$ *if* $\rho = 0$ *and* $I \notin \mathfrak{D}^*$), *then*

$$a = \frac{2^{|\lambda|+|\omega|-2} - 2^{|\omega|}}{\sum_{I\in\mathfrak{D}_2^{\rho,\omega+\lambda}} c_I} c_{[\mu]}, \tag{85}$$

where $[\mu]$ *is the unique maximal element in* $\mathfrak{D}_1$ *with* $c_{[\mu]} \neq 0$.

Proof. $c_I \neq 0$ only if $I \in \mathfrak{D}^{\rho,\omega+\lambda}$. According to Lemma 2,

$$S_2 \left(\sum_{I\in\mathfrak{D}^{\rho,\omega+\lambda}} c_I \partial^I u\big|_{u=2a\partial^\rho \ln f} \right) = 2a \sum_{I\in\mathfrak{D}_1^{\rho,\omega+\lambda}} (1 - 2^{|I|+|\rho|-1}) c_I + 4a^2 \sum_{I\in\mathfrak{D}_2^{\rho,\omega+\lambda}} c_I, \tag{86}$$

and Lemma 3 implies

$$S_2 \left(\partial^\omega \left(\frac{D^\lambda f \cdot f}{f^2} \right) \right) = 2(1 - 2^{|\omega|+1}). \tag{87}$$

Moreover, by Lemma 4, if $I \in \mathfrak{D}_1$, then $c_I \neq 0$ if and only if $I = [\mu] = [\omega + \lambda - \rho]$ and $c_I = a^{-1}$. Hence we have

$$1 - 2^{|\omega|+1} = 1 - 2^{|\lambda|+|\omega|-1} + 2a^2 \sum_{I\in\mathfrak{D}_2} c_I, \tag{88}$$

i.e.,

$$2a^2 \sum_{I\in\mathfrak{D}_2} c_I = 2^{|\lambda|+|\omega|-1} - 2^{|\omega|+1}. \tag{89}$$

Theorem 2 implies that $\sum_{I\in\mathfrak{D}_2} c_I > 0$. Since $c_{[\mu]} = a^{-1}$, the theorem is proved.

Remark 4. For the general equation (15), we can determine the constant a only in terms of the top class $\mathfrak{D}^{\rho,\rho+\theta}$. In this case, $\lambda + \omega = \rho + \theta$. Therefore,

$$a = \frac{2^{|\rho|+|\theta|-2} - 2^{|\omega|}}{\sum\limits_{I \in \mathfrak{D}_2^{\rho,\rho+\theta}} w_I} w_{[\theta]}. \tag{90}$$

Example. For Eq. (64), the linear term and quadratic terms corresponding to $I \in \mathfrak{D}^{(2,0),(7,0)}$ are

$$u_{xxxxx} \quad \text{and} \quad -15u_{xx}u_x - 15uu_{xxx}, \tag{91}$$

respectively. Hence,

$$a = \frac{2^{|\theta|+|\rho|-2} - 2^{|\omega|}}{(-15)+(-15)} = \frac{2^5 - 2^1}{(-15)+(-15)} = -1. \tag{92}$$

The coefficients of the linear terms give

$$p_{(6,0)} = a w_{(6,0)+\omega-\rho} = -w_{(5,0)} = -1, \quad p_{(1,1)} = -w_{(0,1)} = -1. \tag{93}$$

Hence $P(\tau) = -\tau_1^6 - \tau_1\tau_2$, where $\tau = (\tau_1, \tau_2)$. If the equation can be transformed to bilinear form, it should be

$$-\partial_x \left(\frac{(D_x^6 + D_x D_t) f \cdot f}{f^2} \right) = 0 \tag{94}$$

with $u = -2(\ln f)_{xx}$. Finally, we can check that it is exactly the same as the Sawada-Kotera equation.

E. Procedure of determining all the parameters

In summary, we have the procedure to determine all the parameters for transforming a nonlinear PDE to bilinear form of KdV type.

Step 1 (Theorem 3). Find the maximal element $\theta \in \mathfrak{D}_1$. Then find all the maximal elements $J \in \mathfrak{D}_2$. (They always exist, but may not be unique.) Let $\rho = \theta - s(J)$, which is independent of the choice of J.

Step 2 (Theorem 4). Determine r and $\widetilde{\mathfrak{D}}_r$. If $\widetilde{\mathfrak{D}}_r = \emptyset$, then $\omega = 0$. Otherwise, compute $\widetilde{\Pi}_0$, $\widetilde{\Pi}_1$, and $\omega = d(\widetilde{\Pi}_1) - d(\widetilde{\Pi}_0)$.

Step 3 (Remark 4 after Theorem 5).

$$a = \frac{2^{|\rho|+|\theta|-2} - 2^{|\omega|}}{\sum\limits_{I \in \mathfrak{D}_2^{\rho,\rho+\theta}} w_I} w_{[\theta]}. \tag{95}$$

Step 4. According to Lemma 4, for each λ,

$$p_\lambda = a w_{[\omega+\lambda-\rho]}. \tag{96}$$

Then compute

$$\partial^\omega \left(\frac{P(D) f \cdot f}{f^2} \right), \tag{97}$$

where $P(\tau) = \sum\limits_{\lambda \in \Pi} p_\lambda \tau^\lambda$. (97) equals $W[u]|_{u=2a\partial^\rho \ln f}$ if and only if the original nonlinear PDE $W[u] = 0$ can be transformed to bilinear form of KdV type.

Remark 5. For a given nonlinear PDE, we have presented an explicit way to transform it to bilinear form of KdV type. What is more, we can solve the following problem using the same method. Suppose there is a nonlinear PDE $\sum\limits_{I \in S} w_I \partial^I u = 0$ where $S \subset \mathfrak{D}$ and the coefficients w_I's of all linear terms are fixed, while the coefficients w_I's of all nonlinear terms are undetermined. Then Step

1 and 2 determine the bilinear form completely if it exists. The coefficients w_I's of the nonlinear terms are determined up to the arbitrarily chosen parameter a.

VII. EXAMPLES

Apart from the Sawada-Kotera equation (64), here we give more examples to show the above-mentioned method.

Example 1. Boussinesq equation

$$u_{tt} - u_{xx} - 3u_{xx}^2 - u_{xxxx} = 0. \tag{98}$$

Step 1.

	$I \in \mathfrak{D}$	$\lvert I\rvert$	$\deg(I)$		$s(I)+\deg(I)\rho$
u_{tt}	$[(0, 2)]$	2	1	$\theta = (4, 0)$	$(0, 2)$
u_{xx}	$[(2, 0)]$	2	1	$\rho = (0, 0)$	$(2, 0)$
u_{xx}^2	$[(2, 0), (2, 0)]$	4	2		$(4, 0)$
u_{xxxx}	$[(4, 0)]$	4	1		$(4, 0)$

Step 2. $r = \max\limits_{\substack{I\in\mathfrak{D},\, c_I\neq 0\\ s(I)+\deg(I)\rho=(4,0)}} \deg(I)=2$, $\widetilde{\mathfrak{D}}_2 = \emptyset$. Hence $\omega = (0, 0)$.

Step 3.

$$a = \frac{2^{|(4,0)|+|(0,0)|-2} - 2^{|(0,0)|}}{-3}(-1) = 1. \tag{99}$$

Step 4. $P(\tau) = \tau_2^2 - \tau_1^2 - \tau_1^4$, where $\tau = (\tau_1, \tau_2)$. The equation of bilinear form should be

$$\frac{(D_t^2 - D_x^2 - D_x^4) f \cdot f}{f^2} = 0 \tag{100}$$

with $u = 2\ln f$, which can be checked directly.

Example 2. The shallow water waves equation

$$u_{xt} - u_{xxxt} - 3u_x u_{xt} - 3u_{xx} u_t + u_{xx} = 0, \tag{101}$$

	$I \in \mathfrak{D}$	$\lvert I\rvert$	$\deg(I)$		$s(I)+\deg(I)\rho$
u_{xt}	$[(1, 1)]$	2	1	$\theta = (3, 1)$	$(2, 1)$
u_{xxxt}	$[(3, 1)]$	4	1	$\rho = (1, 0)$	$(4, 1)$
$u_x u_{xt}$	$[(1, 0), (1, 1)]$	3	2		$(4, 1)$
$u_{xx} u_t$	$[(2, 0), (0, 1)]$	3	2		$(4, 1)$
u_{xx}	$[(2, 0)]$	2	1		$(3, 0)$

$$\begin{aligned} &r = 2, \quad \widetilde{\mathfrak{D}}_2 = \{[(1, 0), (1, 1)], [(2, 0), (0, 1)]\},\\ &\widetilde{\Pi}_0 = \{(1, 0), (0, 1)\}, \quad \widetilde{\Pi}_1 = \{(1, 1), (2, 0)\},\\ &d(\widetilde{\Pi}_0) = (0, 0), \quad d(\widetilde{\Pi}_1) = (1, 0),\\ &\omega = (1, 0), \quad a = 1. \end{aligned} \tag{102}$$

The equation of bilinear form is

$$\left(\frac{(-D_x^3 D_t + D_x^2 + D_x D_t) f \cdot f}{f^2}\right)_x = 0, \tag{103}$$

with $u = 2(\ln f)_x$.

Example 3. Kadomtsev-Petviashvili equation

$$u_{xt} + 6uu_{xx} + 6u_x^2 + u_{xxxx} + 3u_{yy} = 0, \tag{104}$$

	$I \in \mathfrak{D}$	$\lvert I\rvert$	$\deg(I)$		$s(I)+\deg(I)\rho$
u_{xt}	$[(1,0,1)]$	2	1	$\theta=(4,0,0)$	$(3,0,1)$
uu_{xx}	$[(0,0,0),(2,0,0)]$	2	2	$\rho=(2,0,0)$	$(6,0,0)$
u_x^2	$[(1,0,0),(1,0,0)]$	2	2		$(6,0,0)$
u_{xxxx}	$[(4,0,0)]$	4	1		$(6,0,0)$
u_{yy}	$[(0,2,0)]$	2	1		$(2,2,0)$

$$\begin{aligned} &r=2, \quad \widetilde{\mathfrak{D}}_2=\{[(0,0,0),(2,0,0)]\},\\ &\widetilde{\Pi}_0=\{(0,0,0)\}, \quad \widetilde{\Pi}_1=\{(2,0,0)\},\\ &d(\widetilde{\Pi}_0)=(0,0,0), \quad d(\widetilde{\Pi}_1)=(2,0,0),\\ &\omega=(2,0,0), \quad a=1. \end{aligned} \tag{105}$$

The equation of bilinear form is

$$\left(\frac{(D_x^4+D_xD_t+3D_y^2)f\cdot f}{f^2}\right)_{xx}=0, \tag{106}$$

with $u=2(\ln f)_{xx}$.

Example 4. (3+1)-dimensional KdV equation

$$u_t+6u_xu_y+u_{xxy}+u_{xxxxz}+60u_x^2u_z+10u_{xxx}u_z+20u_xu_{xxz}=0\,, \tag{107}$$

	$I \in \mathfrak{D}$	$\lvert I\rvert$	$\deg(I)$	$s(I)+\deg(I)\rho$
u_t	$[(0,0,0,1)]$	1	1	$(1,0,0,1)$
u_xu_y	$[(1,0,0,0),(0,1,0,0)]$	2	2	$(3,1,0,0)$
u_{xxy}	$[(2,1,0,0)]$	3	1	$(3,1,0,0)$
u_{xxxxz}	$[(4,0,1,0)]$	5	1	$(5,0,1,0)$
$u_x^2u_z$	$[(1,0,0,0),(1,0,0,0),(0,0,1,0)]$	3	3	$(5,0,1,0)$
$u_{xxx}u_z$	$[(3,0,0,0),(0,0,1,0)]$	4	2	$(5,0,1,0)$
u_xu_{xxz}	$[(1,0,0,0),(2,0,1,0)]$	4	2	$(5,0,1,0)$

$$\begin{aligned} &\theta=(4,0,1,0), \quad \rho=(1,0,0,0),\\ &r=3, \quad \widetilde{\mathfrak{D}}_3=\emptyset,\\ &\omega=(0,0,0,0), \quad a=\tfrac{1}{2}. \end{aligned} \tag{108}$$

The equation of bilinear form is

$$\frac{1}{2}\frac{(D_xD_t+D_x^5D_z+D_x^3D_y)f\cdot f}{f^2}=0, \tag{109}$$

with $u=(\ln f)_x$.

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