Modern Physics Letters B
Vol. 30, No. 29 (2016) 1650358 (11 pages)

DOI: 10.1142/S0217984916503589

Global explicit solutions with n double spectral parameters for the Myrzakulov-I equation

Hai Chen* and Zi-Xiang Zhou†
School of Mathematical Sciences,
Fudan University, Shanghai 200433, China
**12110180003@fudan.edu.cn*
†zxzhou@fudan.edu.cn

Received 30 July 2016
Accepted 24 August 2016
Published 14 October 2016

The Darboux transformation with n double spectral parameters for the Myrzakulov-I equation is obtained by taking suitable limit of the spectral parameters. Global explicit solutions are obtained by using this Darboux transformation with n double spectral parameters.

Keywords: Darboux transformation; n double spectral parameters; global explicit solution.

1. Introduction

The Myrzakulov-I equation is a (2+1)-dimensional generalization of the Heisenberg ferromagnetic equation. It is gauge equivalent to the (2+1)-dimensional nonlinear Schrödinger equation and has a non-isospectral Lax pair. It has been studied in many papers,[1–4] including those constructed by Darboux transformation of degree one and degree two.[5,6] By taking suitable limit of the spectral parameters, the solutions are obtained explicitly and the globalness of the derived solutions is proved.[6]

If there are n pairs of spectral parameters, and the limit is taken in each pair so that one spectral parameter tends to another one, then we get a Darboux transformation with n double spectral parameters. Sometimes, the solutions derived in this way can be used in constructing rogue waves.[7–9] Although this is the way to get positon solutions,[10,11] the behavior of the solutions here is different from positons because of the varying spectral parameters.

The Darboux transformations of higher degree and their limits can be constructed in a standard way.[12] However, the limit solutions usually have singularities, especially in non-isospectral cases. The global solutions are much more important both mathematically and physically.

In this paper, we construct the Darboux transformation with n double spectral parameters for the Myrzakulov-I equation. Then we show that the solutions are globally defined in quite general cases. In Sec. 2, the Darboux transformation with m spectral parameters is constructed. In Sec. 3, the limit Darboux transformation with n pairs of spectral parameters is obtained and the expressions of the derived solutions are given. In Sec. 4, the globalness of the solutions is proved under quite general conditions. An example is shown in Sec. 5.

The Myrzakulov-I equation is

$$S_t = \frac{i}{2}[S, S_y]_x + (uS)_x\,, \tag{1}$$

where S is an unknown 2×2 traceless Hermitian matrix with $S^2 = I$, u is an unknown scalar function. Note that from (1), $\mathrm{tr}(S_t S)$ gives

$$u_x = -\frac{i}{4}\mathrm{tr}(S[S_x, S_y])\,. \tag{2}$$

Sometimes, it is written in the Myrzakulov-I equation, although it is a consequence of (1).

Equation (1) has a Lax pair

$$\begin{aligned}\Phi_x &= \frac{i}{2}\lambda S\Phi\,,\\ \Phi_t &= -\lambda\Phi_y + \frac{i}{2}\lambda uS\Phi - \frac{1}{2}\lambda SS_y\Phi\,,\end{aligned} \tag{3}$$

where the "spectral parameter" λ satisfies

$$\lambda_x = 0, \quad \lambda_t = -\lambda\lambda_y\,. \tag{4}$$

Although λ is not a constant in general, we still call it spectral parameter.

2. Darboux Transformation with m Spectral Parameters

From Ref. 5, we know that the Darboux matrix of degree one for (3) is in the form

$$\begin{aligned}G(x,y,t,\lambda) &= \sqrt{y^2+\sigma^2}(\lambda N - I),\\ N(x,y,t) &= H\Lambda^{-1}H^{-1}\,,\end{aligned} \tag{5}$$

where

$$\Lambda = \begin{pmatrix}\mu(y,t) & 0\\ 0 & \bar{\mu}(y,t)\end{pmatrix}, \quad H = \begin{pmatrix}h_1(x,y,t) & -\bar{h}_2(x,y,t)\\ h_2(x,y,t) & \bar{h}_1(x,y,t)\end{pmatrix}, \tag{6}$$

$$\mu(y,t) = \frac{yt+\sigma\tau\cos\theta}{t^2+\tau^2} \pm i\frac{\sqrt{\tau^2y^2 - 2\sigma\tau yt\cos\theta + \sigma^2t^2 + \sigma^2\tau^2\sin^2\theta}}{t^2+\tau^2}\,, \tag{7}$$

and $\binom{h_1(x,y,t)}{h_2(x,y,t)}$ is a column solution of (3) with $\lambda = \mu(y,t)$. Here σ, θ, τ are real constants with $\sigma \neq 0$, $\tau \neq 0$, $0 < \theta < \pi$. It is easy to verify that $\binom{-\bar{h}_2}{\bar{h}_1}$ is a solution of (3) with $\lambda = \bar{\mu}(y,t)$ provided that $\binom{h_1}{h_2}$ is a solution of (3) with $\lambda = \mu(y,t)$.

Now we consider the Darboux matrix of degree m for (3).

According to Ref. 12, let $\Lambda_i = \begin{pmatrix} \mu_i & \\ & \bar{\mu}_i \end{pmatrix}$, where μ_i is given by (7) with parameters σ_i, τ_i, θ_i $(i = 1, 2, \ldots, m)$. Let $(h_{i,1}, h_{i,2})^T$ be a solution of (3) with $\lambda = \mu_i$,

$$H_i = \begin{pmatrix} h_{i,1} & -\bar{h}_{i,2} \\ h_{i,2} & \bar{h}_{i,1} \end{pmatrix}, \quad N_i = H_i \Lambda_i^{-1} H_i^{-1}. \tag{8}$$

According to (5), the Darboux transformation of degree m should be

$$G(\lambda) = \gamma(y)(\lambda^m G_0 + \lambda^{m-1} G_1 + \cdots + \lambda G_{m-1} + I), \tag{9}$$

where

$$\gamma(y) = \sqrt{(y^2 + \sigma_1^2)(y^2 + \sigma_2^2) \cdots (y^2 + \sigma_m^2)}, \tag{10}$$

$G_0, \ldots, G_{m-1}$ are obtained by solving the linear algebraic system

$$(G_0, \ldots, G_{m-1}) \begin{pmatrix} H_1 \Lambda_1^m & \cdots & H_m \Lambda_m^m \\ \vdots & & \vdots \\ H_1 \Lambda_1 & \cdots & H_m \Lambda_m \end{pmatrix} = -(H_1, \ldots, H_m). \tag{11}$$

Hence,

$$\begin{aligned} &(G_0, \ldots, G_{m-1}) \\ &\quad = -(H_1, \ldots, H_m) \begin{pmatrix} H_1 \Lambda_1^m & \cdots & H_m \Lambda_m^m \\ \vdots & & \vdots \\ H_1 \Lambda_1 & \ldots & H_m \Lambda_m \end{pmatrix}^{-1} \\ &\quad = -(H_1, \ldots, H_m) \begin{pmatrix} H_1 & & \\ & \ddots & \\ & & H_m \end{pmatrix}^{-1} \begin{pmatrix} N_1^{-m} & \cdots & N_m^{-m} \\ \vdots & & \vdots \\ N_1^{-1} & \cdots & N_m^{-1} \end{pmatrix}^{-1} \\ &\quad = -\underbrace{(I, \ldots, I)}_{m} \begin{pmatrix} N_1^{-m} & \cdots & N_m^{-m} \\ \vdots & & \vdots \\ N_1^{-1} & \cdots & N_m^{-1} \end{pmatrix}^{-1}. \end{aligned} \tag{12}$$

Let $\widetilde{\Phi} = G\Phi$, then after the transformation of G, the Lax pair (3) should be transformed to

$$\begin{aligned} \widetilde{\Phi}_x &= \frac{i}{2} \lambda \widetilde{S} \widetilde{\Phi}, \\ \widetilde{\Phi}_t &= -\lambda \widetilde{\Phi}_y + \frac{i}{2} \lambda \widetilde{u} \widetilde{S} \widetilde{\Phi} - \frac{1}{2} \lambda \widetilde{S} \widetilde{S}_y \widetilde{\Phi}. \end{aligned} \tag{13}$$

By comparing (3) and (13), the equations for G are

$$G_x + \frac{i}{2}\lambda GS = \frac{i}{2}\lambda\widetilde{S}G,$$

$$G_t + \frac{i}{2}\lambda uGS - \frac{1}{2}\lambda GSS_y = -\lambda G_y + \frac{i}{2}\lambda\widetilde{u}\widetilde{S}G - \frac{1}{2}\widetilde{S}\widetilde{S}_y G\,.$$

Substituting (9) into the above equations and comparing the coefficient of λ^{m+1} in both equations, we get

$$\begin{aligned} \widetilde{S} &= G_0 S G_0^{-1}, \\ \widetilde{u} &= u - i\,\mathrm{tr}(G_0^{-1}G_{0,y}S)\,. \end{aligned} \tag{14}$$

This gives a solution of (1) by calculating G_0 from (12).

3. Darboux Transformation with n Double Spectral Parameters

In this section, we will construct the Darboux transformation with n double spectral parameters. Similar to Ref. 6, take n spectral parameters $\mu_i^{(s)}$ $(i = 1, \ldots, n)$, which are given by (7) with parameters $\sigma_i^{(s)}$, $\tau_i^{(s)}$ and $\theta_i^{(s)}$, all of which depend on s smoothly. Let $\mu_i = \mu_i^{(0)}$, $\mu_{n+i} = \mu_i^{(\varepsilon)}$ $(i = 1, \ldots, n)$.

Let

$$\Lambda_i^{(s)} = \begin{pmatrix} \mu_i^{(s)} & \\ & \bar{\mu}_i^{(s)} \end{pmatrix}, \quad \Lambda_i = \Lambda_i^{(0)}, \quad \Lambda_{n+i} = \Lambda_i^{(\varepsilon)} \quad (i = 1, 2, \ldots, n), \tag{15}$$

$$H_i^{(s)} = \begin{pmatrix} h_{i,1}^{(s)} & -\bar{h}_{i,2}^{(s)} \\ h_{i,2}^{(s)} & \bar{h}_{i,1}^{(s)} \end{pmatrix}, \quad H_i = H_i^{(0)}, \quad H_{n+i} = H_i^{(\varepsilon)} \quad (i = 1, 2, \ldots, n)\,, \tag{16}$$

where $(h_{i,1}^{(s)}, h_{i,2}^{(s)})^T$ is a smooth solution of (3) with $\lambda = \mu_i^{(s)}$ $(0 \le s \le \varepsilon)$.

With these $2n$ spectral parameters and the corresponding solutions of the Lax pair, we can construct Darboux transformation as in Sec. 2 with $m = 2n$. Define

$$N_i^{(s)} = H_i^{(s)}(\Lambda_i^{(s)})^{-1}(H_i^{(s)})^{-1}, \quad N_i = N_i^{(0)}, \quad N_{n+i} = N_i^{(\varepsilon)} \quad (i = 1, \ldots, n)\,. \tag{17}$$

The Darboux matrix in the form (9) with $m = 2n$ is

$$G^{(\varepsilon)}(x, y, t, \lambda) = \gamma^{(\varepsilon)}(\lambda^{2n}G_0^{(\varepsilon)} + \lambda^{2n-1}G_1^{(\varepsilon)} + \cdots + \lambda G_{2n-1}^{(\varepsilon)} + I)\,, \tag{18}$$

where

$$\gamma^{(\varepsilon)} = \sqrt{(y^2 + (\sigma_1^{(0)})^2)\cdots(y^2 + (\sigma_n^{(0)})^2)(y^2 + (\sigma_1^{(\varepsilon)})^2)\cdots(y^2 + (\sigma_n^{(\varepsilon)})^2)}\,, \tag{19}$$

$G_i^{(\varepsilon)}$ $(i = 0, 1, \ldots, 2n-1)$ is given by (12).

Write

$$F^{(s)} = \begin{pmatrix} (N_1^{(s)})^{-2n} & \cdots & (N_n^{(s)})^{-2n} \\ \vdots & & \vdots \\ (N_1^{(s)})^{-1} & \cdots & (N_n^{(s)})^{-1} \end{pmatrix}, \qquad F = F^{(0)}, \tag{20}$$

which are $(4n) \times (2n)$ matrices, then (12) becomes

$$(G_0^{(\varepsilon)}, \dots, G_{2n-1}^{(\varepsilon)}) = -\underbrace{(I, \dots, I)}_{2n}(F, F^{(\varepsilon)})^{-1}\,. \tag{21}$$

Let f_i and $f_i^{(\varepsilon)}$ be the ith row of F and $F^{(\varepsilon)}$, respectively. Write F and $F^{(\varepsilon)}$ as block matrices

$$F = \begin{pmatrix} f_1 \\ f_2 \\ \hat{F} \end{pmatrix}, \quad F^{(\varepsilon)} = \begin{pmatrix} f_1^{(\varepsilon)} \\ f_2^{(\varepsilon)} \\ \hat{F}^{(\varepsilon)} \end{pmatrix}. \tag{22}$$

Then $G_0^{(\varepsilon)}$ is solved from (21) by the Cramer's rule as

$$G_0^{(\varepsilon)} = -\frac{1}{\det(F, F^{(\varepsilon)})} \begin{pmatrix} \begin{vmatrix} e_1 & e_1 \\ f_2 & f_2^{(\varepsilon)} \\ \hat{F} & \hat{F}^{(\varepsilon)} \end{vmatrix}, & -\begin{vmatrix} e_1 & e_1 \\ f_1 & f_1^{(\varepsilon)} \\ \hat{F} & \hat{F}^{(\varepsilon)} \end{vmatrix} \\ \begin{vmatrix} e_2 & e_2 \\ f_2 & f_2^{(\varepsilon)} \\ \hat{F} & \hat{F}^{(\varepsilon)} \end{vmatrix}, & -\begin{vmatrix} e_2 & e_2 \\ f_1 & f_1^{(\varepsilon)} \\ \hat{F} & \hat{F}^{(\varepsilon)} \end{vmatrix} \end{pmatrix}, \tag{23}$$

where $e_1 = (1, 0, \dots, 1, 0)$, $e_2 = (0, 1, \dots, 0, 1)$ are $2n$ dimensional row vectors.

As usual, for any smooth function $f^{(\varepsilon)}(x, y, t)$, denote

$$f'(x, y, t) = \frac{\partial}{\partial \varepsilon} f^{(\varepsilon)}(x, y, t)\Big|_{\varepsilon=0}. \tag{24}$$

Expanding $F^{(\varepsilon)}$ with respect to ε, we obtain

$$\begin{aligned} G_0 &\triangleq \lim_{\varepsilon\to 0} G_0^{(\varepsilon)} \\ &= -\lim_{\varepsilon\to 0} \frac{1}{\det(F, F'\varepsilon + O(\varepsilon^2))} \begin{pmatrix} \begin{vmatrix} e_1 & 0 \\ f_2 & f_2'\varepsilon + O(\varepsilon^2) \\ \hat{F} & \hat{F}'\varepsilon + O(\varepsilon^2) \end{vmatrix}, & -\begin{vmatrix} e_1 & 0 \\ f_1 & f_1'\varepsilon + O(\varepsilon^2) \\ \hat{F} & \hat{F}'\varepsilon + O(\varepsilon^2) \end{vmatrix} \\ \begin{vmatrix} e_2 & 0 \\ f_2 & f_2'\varepsilon + O(\varepsilon^2) \\ \hat{F} & \hat{F}'\varepsilon + O(\varepsilon^2) \end{vmatrix}, & -\begin{vmatrix} e_2 & 0 \\ f_1 & f_1'\varepsilon + O(\varepsilon^2) \\ \hat{F} & \hat{F}'\varepsilon + O(\varepsilon^2) \end{vmatrix} \end{pmatrix} \\ &= -\frac{1}{\det(F, F')} \begin{pmatrix} \begin{vmatrix} e_1 & 0 \\ f_2 & f_2' \\ \hat{F} & \hat{F}' \end{vmatrix}, & -\begin{vmatrix} e_1 & 0 \\ f_1 & f_1' \\ \hat{F} & \hat{F}' \end{vmatrix} \\ \begin{vmatrix} e_2 & 0 \\ f_2 & f_2' \\ \hat{F} & \hat{F}' \end{vmatrix}, & -\begin{vmatrix} e_2 & 0 \\ f_1 & f_1' \\ \hat{F} & \hat{F}' \end{vmatrix} \end{pmatrix}. \end{aligned} \tag{25}$$

With (25), a new solution $(\widetilde{S}, \widetilde{u})$ is given by (14) provided that the seed solution (S, u) is known.

4. Globalness of the Solutions

Theorem 1. *Let* (S, u) *be a solution of the Eq.* (1) *such that the Lax pair* (3) *is uniquely solvable. Let* $(h_{i,1}^{(\varepsilon)}, h_{i,2}^{(\varepsilon)})^T$ *be a nonzero global solution of* (3) *with* $\lambda = \mu_i^{(\varepsilon)}$. *Let* $\Lambda_i^{(\varepsilon)}$, $H_i^{(\varepsilon)}$ *and* $N_i^{(\varepsilon)}$ *be defined by* (15), (16) *and* (17), *respectively. The Darboux matrix* $G^{(\varepsilon)}(x, y, t, \lambda)$ *is defined by* (18) *whose coefficients* $G_0^{(\varepsilon)}, \ldots, G_{2n}^{(\varepsilon)}$ *are given by* (21). *If*

$$\begin{aligned} &\mu_i^{(\varepsilon)}(x,y,t) \neq \mu_j^{(\varepsilon)}(x,y,t) \quad (i,j = 1, \ldots, n;\ i \neq j;\ (x,y,t) \in \mathbf{R}^3;\ |\varepsilon| < \varepsilon_0)\,, \\ &\mu_i^{(\varepsilon)}(x,y,t) \neq \bar{\mu}_j^{(\varepsilon)}(x,y,t) \quad (i,j = 1, \ldots, n;\ (x,y,t) \in \mathbf{R}^3;\ |\varepsilon| < \varepsilon_0) \end{aligned} \tag{26}$$

and

$$\mu_i'(x,y,t) \neq 0 \quad (i = 1, 2, \ldots, n;\ (x,y,t) \in \mathbf{R}^3)\,, \tag{27}$$

then (14) *is a global solution of the Eq.* (1).

Proof. When $n = 1$, the result is just that given by Ref. 6. If (26) and (27) hold, a Darboux matrix of degree n $(n > 1)$ can be written as the composition of n Darboux matrices of degree one.[12] Hence the result is also true. The theorem is proved. □

Practically, verifying conditions (26) and (27) in the above theorem is not very easy. Here we give stronger but more applicable conditions than those in Theorem 1. Without loss of generality, suppose $\sigma_i > 0$, $\tau_i > 0$ and $0 < \theta_i < \pi$ $(i = 1, 2, \ldots, n)$.

Theorem 2. *Suppose when* ε *is small enough,*

$$\begin{aligned} &0 < \sigma_1^{(\varepsilon)} \leq \cdots \leq \sigma_n^{(\varepsilon)}, \quad \tau_1^{(\varepsilon)} \geq \cdots \geq \tau_n^{(\varepsilon)} > 0\,, \\ &0 < \frac{\sigma_1^{(\varepsilon)}}{\tau_1^{(\varepsilon)}} < \cdots < \frac{\sigma_n^{(\varepsilon)}}{\tau_n^{(\varepsilon)}}\,, \end{aligned} \tag{28}$$

$$\sigma_i'\tau_i' \leq 0, \quad (\sigma_i')^2 + (\tau_i')^2 > 0 \quad (i = 1, 2, \ldots, n)\,, \tag{29}$$

$$\cos\theta_i \neq 0 \quad (i = 1, 2, \ldots, n)\,, \tag{30}$$

then (26) *and* (27) *hold. Therefore, the solution* (14) *is global.*

Proof. First, we prove (26). Since we are going to compare a pair of μ_i's, it is only necessary to prove the result for $n = 2$. Assume $\mu_2 = \mu_1$ or $\mu_2 = \bar{\mu}_1$ held, then $|\mu_2| = |\mu_1|$ and $\operatorname{Re}\mu_2 = \operatorname{Re}\mu_1$ gave

$$\begin{aligned} \frac{y^2 + \sigma_1^2}{t^2 + \tau_1^2} &= \frac{y^2 + \sigma_2^2}{t^2 + \tau_2^2}\,, \\ \frac{yt + \sigma_1\tau_1\cos\theta_1}{t^2 + \tau_1^2} &= \frac{yt + \sigma_2\tau_2\cos\theta_2}{t^2 + \tau_2^2}\,. \end{aligned} \tag{31}$$

Hence, the solution (14) is global if and only if (31) has no real solution (y, t).

Eliminating y or t, we get the quadratic equations for t^2 and y^2:

$$\begin{aligned} a_1t^4 + b_1\tau_1^2t^2 + c_1\tau_1^4 = 0\,, \\ a_2y^4 + b_2\sigma_2^2y^2 + c_2\sigma_2^4 = 0\,, \end{aligned} \tag{32}$$

where

$$\begin{aligned} a_1 &= a_2 = (1-\kappa^2)(\kappa^2-\rho^2) + (\rho\cos\theta_2 - \kappa^2\cos\theta_1)^2\,, \\ b_1 &= (1-\rho^2)(\kappa^2-\rho^2) + 2\rho^2\kappa^2\cos^2\theta_1 + 2\rho^2\cos^2\theta_2 - 2\rho(\rho^2+\kappa^2)\cos\theta_1\cos\theta_2\,, \\ b_2 &= (1-\rho^2)(1-\kappa^2) + 2\kappa^2\cos^2\theta_1 + 2\rho^2\cos^2\theta_2 - 2\rho(\kappa^2+1)\cos\theta_1\cos\theta_2\,, \\ c_1 &= (\rho\cos\theta_1 - \cos\theta_2)^2\,, \\ c_2 &= (\cos\theta_1 - \rho\cos\theta_2)^2, \\ \rho &= \frac{\sigma_1\tau_2}{\sigma_2\tau_1}, \quad \kappa = \frac{\sigma_1}{\sigma_2}\,. \end{aligned}$$

The conditions (28) lead to $0 < \rho \le \kappa \le 1$ and $\rho < 1$. Since either $c_1 > 0$ or $c_2 > 0$ holds, assume $c_1 > 0$ without loss of generality. We have

$$\begin{aligned} a_1 &\ge (1-\kappa^2)(\kappa^2-\rho^2) \ge 0\,, \\ b_1 &\ge (1-\rho^2)(\kappa^2-\rho^2) - \frac{\rho}{\kappa}(\rho-\kappa)^2(\kappa^2\cos^2\theta_1 + \cos^2\theta_2) \\ &\ge \frac{1}{\kappa}(\kappa-\rho)(1-\rho\kappa)(\rho^2+\kappa^2) \ge 0\,. \end{aligned}$$

Now $a_1 \ge 0$, $b_1 \ge 0$, $c_1 > 0$ imply that the first equation of (32) has no real solution.

Let $\mu^{(\varepsilon)}$ be given by (7) with parameters $\sigma^{(\varepsilon)}$, $\tau^{(\varepsilon)}$ and $\theta^{(\varepsilon)}$. $\mu = 0$ is equivalent to

$$\begin{aligned} \frac{\partial}{\partial\varepsilon}\left(\frac{y^2 + (\sigma^{(\varepsilon)})^2}{t^2 + (\tau^{(\varepsilon)})^2}\right)\bigg|_{\varepsilon=0} = 0\,, \\ \frac{\partial}{\partial\varepsilon}\left(\frac{yt + \sigma^{(\varepsilon)}\tau^{(\varepsilon)}\cos\theta}{t^2 + (\tau^{(\varepsilon)})^2}\right)\bigg|_{\varepsilon=0} = 0\,. \end{aligned} \tag{33}$$

Eliminating y or t, we get the quadratic equations for t^2 and y^2:

$$\begin{aligned} a_1t^4 + b_1\tau^2t^2 + c_1\tau^4 = 0\,, \\ a_2y^4 + b_2\sigma^2y^2 + c_2\sigma^4 = 0\,, \end{aligned} \tag{34}$$

where

$$\begin{aligned} a_1 &= a_2 = ((\sigma\tau' + \sigma'\tau)\cos\theta - \sigma\tau\theta'\sin\theta)^2\,, \\ b_1 &= 2(\sigma'\tau\cos\theta - \sigma\tau\theta'\sin\theta)^2 + 2\sigma^2\tau'^2(1+\sin^2\theta) - 4\sigma\sigma'\tau\tau'\,, \\ b_2 &= 2(\sigma\tau'\cos\theta - \sigma\tau\theta'\sin\theta)^2 + 2\sigma'^2\tau^2(1+\sin^2\theta) - 4\sigma\sigma'\tau\tau'\,, \\ c_1 &= ((\sigma\tau' - \sigma'\tau)\cos\theta + \sigma\tau\theta'\sin\theta)^2\,, \\ c_2 &= ((\sigma\tau' - \sigma'\tau)\cos\theta - \sigma\tau\theta'\sin\theta)^2\,. \end{aligned} \tag{35}$$

With the condition (29), either $c_1 > 0$ or $c_2 > 0$, and obviously $a_1 \geq 0$, $a_2 \geq 0$, $b_1 \geq 0$, $b_2 \geq 0$. Hence at least one of the equations in (34) has no real solution.

By Theorem 1, the solution (14) is global. The theorem is proved. □

Example. The conditions in Theorem 2 hold in quite a lot of cases. Here is one of the simplest cases:

$$\sigma_i^{(\varepsilon)} = \alpha_i + \varepsilon, \quad \tau_i^{(\varepsilon)} = \beta_i - \varepsilon, \quad \theta_i^{(\varepsilon)} \neq \pi/2 \quad (i = 1, \dots, n),$$
$$\text{with constants } 0 < \alpha_1 < \cdots < \alpha_n, \quad \beta_1 > \cdots > \beta_n > 0.$$

Remark. If we suppose that the real parts and imaginary parts of all the entries in (S, u) and $H_i^{(\varepsilon)}$'s are real analytic functions of (x, y, t) and (x, y, t, ε), respectively, then the conditions (26) and (27) can be replaced by much weaker conditions (here we still assume $\sigma_i > 0$, $\tau_i > 0$, $0 < \theta_i < \pi$ for $i = 1, \dots, n$):

$$\begin{aligned} &\text{(i) } |\sigma_i - \sigma_j|^2 + |\tau_i - \tau_j|^2 + |\theta_i - \theta_j|^2 \neq 0 \quad (i, j = 1, \dots, n;\ i \neq j), \\ &\text{(ii) } |\sigma_i'|^2 + |\tau_i'|^2 + |\theta_i'|^2 \neq 0 \quad (i = 1, \dots, n), \\ &\text{(iii) } \cos\theta_i \neq 0 \quad (i = 1, \dots, n). \end{aligned} \tag{36}$$

The reason is as follows.

According to the proof of Theorem 2, we can check that the conditions (36) guarantee that at least one of the equations in (32) and one of the equations in (34) are not identities. Hence, (26) and (27) are violated only for finite number of y or finite number of t. Apart from those y and t, G_0 is well-defined. From the expression $\Lambda_i^{(\varepsilon)}$ and $H_i^{(\varepsilon)}$, $N_i^{(\varepsilon)}$ is of form $\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}$. Since the Darboux matrix of higher degree is the product of Darboux matrices of degree one, G_0 is also of form $\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}$. (5) and (9) imply that $\det G_0 = \prod_{i=1}^n |\mu_i|^{-4} \neq 0$. Obviously, the norm of each entry of G_0 does not exceed $\sqrt{\det G_0}$, hence G_0 defined by (25) is bounded. Although the Darboux transformation may be invalid at the solutions of (32) and (34), the analyticity of both the denominator and numerator in (25) implies that G_0 can be extended to a real analytic matrix for all (x, y, t) with $\det G_0 \neq 0$. Therefore, the solution (14) is global.

5. An Example

Since S is Hermitian and traceless, it can be expanded by Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as $S = s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3$. Here we only show the figures for s_3.

Starting from the trivial solution $S = \sigma_3$, $u = 0$, the solution is constructed by the Darboux transformation with double pairs of spectral parameters. Solving the

Lax pair (3), we can choose

$$\Lambda_j = \begin{pmatrix} \mu_j & 0 \\ 0 & \bar{\mu}_j \end{pmatrix}, \quad H_j = \begin{pmatrix} e^{i\mu_j x/2} & -e^{i\bar{\mu}_j x/2} \\ e^{-i\mu_j x/2} & e^{-i\bar{\mu}_j x/2} \end{pmatrix}, \tag{37}$$

then

$$N_j^{-1} = H_j \Lambda_j^{-1} H_j^{-1} = \frac{1}{1+e^{2\mathrm{Im}(\mu_j)x}} \begin{pmatrix} \mu_j + \bar{\mu}_j\, e^{2\mathrm{Im}(\mu_j)x} & 2i\,\mathrm{Im}(\mu_j) e^{i\bar{\mu}_j x} \\ 2i\,\mathrm{Im}(\mu_j) e^{-i\mu_j x} & \bar{\mu}_j + \mu_j e^{2\mathrm{Im}(\mu_j)x} \end{pmatrix} \quad (j=1,2). \tag{38}$$

Substituting (38) into (25) and (14), we get a new solution of (1).

When x, y are bounded and $t \to \infty$,

$$\mu_j \sim \frac{y + i\sigma_j}{t}, \tag{39}$$

hence

$$N_j^{-1} \sim \frac{1}{t} \begin{pmatrix} y & i\sigma_j \\ i\sigma_j & y \end{pmatrix}. \tag{40}$$

From (25) and (14), we get, with the help of computer, that

$$\begin{aligned} \widetilde{s}_1 &\sim 0\,, \\ \widetilde{s}_2 &\sim -\frac{4(\sigma_1+\sigma_2)y(y^2-\sigma_1\sigma_2)((y^2-\sigma_1\sigma_2)^2-(\sigma_1+\sigma_2)^2y^2)}{(y^2+\sigma_1^2)^2(y^2+\sigma_2^2)^2}\,, \\ \widetilde{s}_3 &\sim \frac{(y^2+\sigma_1^2)^2(y^2+\sigma_2^2)^2 - 8(\sigma_1+\sigma_2)^2y^2(y^2-\sigma_1\sigma_2)^2}{(y^2+\sigma_1^2)^2(y^2+\sigma_2^2)^2}\,. \end{aligned} \tag{41}$$

So the asymptotic solution as $t \to \infty$ is a function of y only.

For example, choose the parameters $\sigma_1(\varepsilon) = 1+\varepsilon$, $\sigma_2(\varepsilon) = 2+\varepsilon$, $\tau_1(\varepsilon) = 2-\varepsilon$, $\tau_2(\varepsilon) = 1-\varepsilon$ and $\theta_1(\varepsilon) = \theta_2(\varepsilon) = \pi/4$. Figures 1–4 show $\widetilde{s}_3$ for $t=0$, $t=12$, $t=100$, $t=500$, respectively. Note that the range of y in Fig. 4 is different from that in the others.

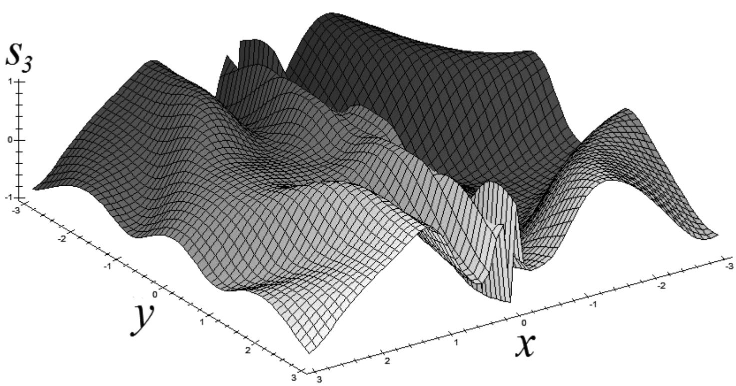

Fig. 1. $\widetilde{s}_3$ for $t = 0$.

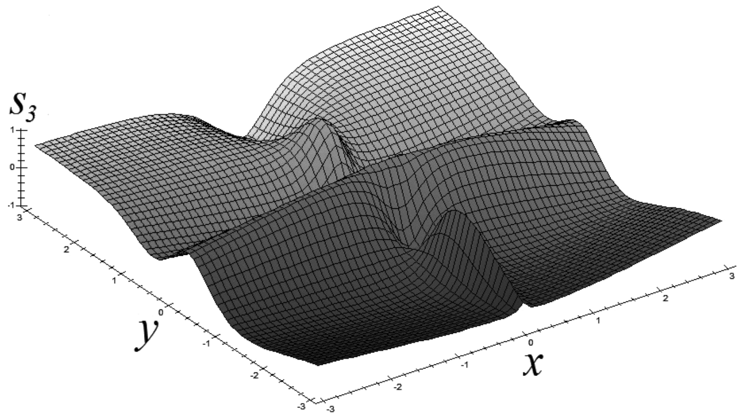

Fig. 2. $\widetilde{s}_3$ for $t = 12$.

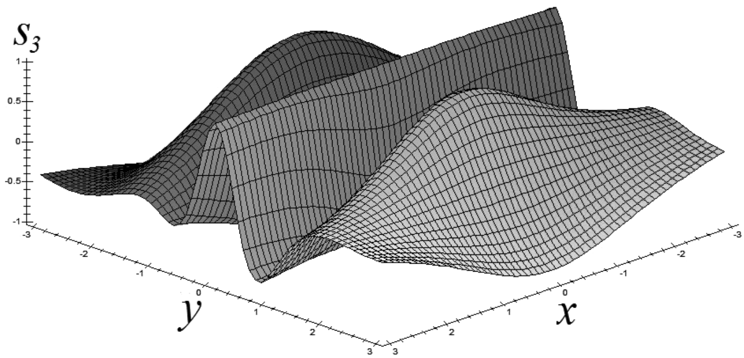

Fig. 3. $\widetilde{s}_3$ for $t = 100$.

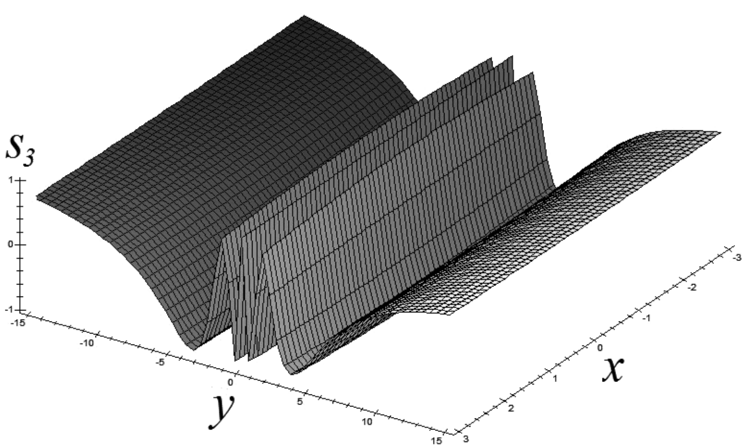

Fig. 4. $\widetilde{s}_3$ for $t = 500$.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (No. 11171073), the Natural Science Foundation of Shanghai (No. 16ZR1402600) and the Key Laboratory of Mathematics for Nonlinear Sciences of Ministry of Education of China.

References

1. S. C. Anco and R. Myrzakulov, *J. Geom. Phys.* **60** (2010) 1576.
2. R. Myrzakulov, G. N. Nugmanova and R. N. Syzdykova, *J. Phys. A* **31** (1998) 9535.
3. R. Myrzakulov, S. Vijayalakshmi, G. N. Nugmanova and M. Lakshmanan, *Phys. Lett. A* **233** (1997) 391.
4. Z. H. Zhang, M. Deng, W. Z. Zhao and K. Wu, *J. Phys. Soc. Jpn.* **75** (2006) 104002.
5. C. Chen and Z. X. Zhou, *Chin. Phys. Lett.* **26** (2009) 080504.
6. H. Chen and Z. X. Zhou, *Chin. Phys. Lett.* **31** (2014) 120504.
7. N. Akhmediev, A. Ankiewicz and M. Taki, *Phys. Lett. A* **373** (2009) 675.
8. J. S. He, S. W. Xu and K. Porseizan, *Phys. Rev. E* **86** (2012) 066603.
9. J. M. Soto-Crespo, N. Devine and N. Akhmediev, *Phys. Rev. Lett.* **116** (2016) 103901.
10. V. B. Matveev, *Phys. Lett. A* **166** (1992) 209.
11. R. Beutler, *J. Math. Phys.* **34** (1993) 3098.
12. C. H. Gu, H. S. Hu and Z. X. Zhou, *Darboux Transformations in Integrable Systems* (Springer, Dordrecht, 2005).