An observation of Relationship between the Fine Structure Constant and the Gibbs Phenomenon in Fourier Analysis *

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(Received 10 November 2003)

A value given by a simple mathematical expression is proposed, which is close to the fine structure constant given by 1998 CODATA recommended values of the fundamental physical constants up to relative accuracy 10^{-7}. *This expression is related closely with the value of the overshoot of the Gibbs phenomenon in the Fourier analysis.*

PACS: 03. 65. −w, 03. 65. Ta, 32. 10. Fn, 06. 20. Dk

The fine structure constant α is the most important dimensionless universal physical constant. Since it is dimensionless and universal, it is very interesting to know whether it can be expressed by a simple expression of universal mathematical constants. This may be helpful for understanding the nature.

As is known, the definition of fine structure constant α is

$$\alpha^{-1} = \frac{2c\epsilon_0 h}{e^2} = \frac{2h}{c\mu_0 e^2}, \tag{1}$$

where c is the speed of light in vacuum, μ_0 and $\epsilon_0 = 1/c^2\mu_0$ are the permeability and permittivity of vacuum respectively, e is the elementary charge and h is the Planck constant.

Here we propose

$$\alpha_z^{-1} = \frac{1}{\sqrt{2}}\Big(\frac{3\pi}{2}\Big)^3 \int_0^\pi \frac{\sin x}{x}\,dx \approx 137.03598260. \tag{2}$$

This value coincides with the value

$$\alpha^{-1} = 137.03599976 \pm 0.00000050 \tag{3}$$

given by the 1998 CODATA recommended values of the fundamental physical constants [1] up to relative accuracy 1.26×10^{-7}.

In the expression (2), the integral

$$\mathrm{Si}(\pi) = \int_0^\pi \frac{\sin x}{x} dx$$

is related to a universal mathematical constant, i.e. the ratio of the overshoot of the Gibbs phenomenon in the Fourier analysis, which is the same to all the jump discontinuities of all the piecewisely smooth functions.

First, let us look at a simple example for the Gibbs phenomenon. Let $f(x)$ be a square wave of period 2π with $f(x) = 1$ for $0 < x < \pi$ and $f(x) = -1$ for $-\pi < x < 0$. Let s_n be the partial sum of its Fourier series, i.e.

$$s_n(x) = \frac{4}{\pi}\sum_{k=1}^{n} \frac{\sin(2k-1)x}{2k-1}. \tag{4}$$

When $n \to \infty$, there are overshoots and undershoots in the graph of s_n (Fig. 1).

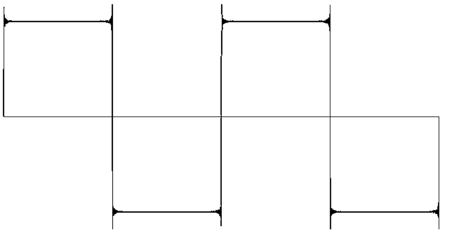

Fig. 1. Partial sum s_n of the square wave for $n = 200$.

The limit of the amplitude of s_n near 0 as $n \to \infty$ is $2C_G$, which is C_G times of the jump of $f(x)$ at 0. Here

$$C_G = \frac{2}{\pi}\mathrm{Si}(\pi) = \frac{2}{\pi}\int_0^\pi \frac{\sin x}{x} dx \approx 1.17897974447217. \tag{5}$$

This is the famous Gibbs phenomenon,[2−5] which was discovered more than one century ago.

The Gibbs phenomenon appears not only in the square wave, but also at all the jump discontinuities generally. For the Fourier series of any piecewisely smooth period function $f(x)$, the limit of the amplitude of the partial sum at a jump discontinuity x_0 of $f(x)$ equals to C_G times of the jump of $f(x)$ at that point. That is, for the partial sum $s_n(x)$,

$$\begin{aligned} &\lim_{\delta\to 0^+}\lim_{n\to\infty}\operatorname*{osc}_{|x-x_0|\le\delta} s_n(x) \\ &= C_G \lim_{\delta\to 0^+}\operatorname*{osc}_{|x-x_0|\le\delta}\lim_{n\to\infty} s_n(x), \end{aligned} \tag{6}$$

where osc refers to the difference of the maximum and the minimum of a function. This fact is also true

* Supported by the Special Funds for Major State Basic Research Programme of China, and the Trans-Century Programme Foundation by the Ministry of Education of China.
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for the Fourier transformation of a non-periodic function, if the function is absolutely integrable and piecewisely smooth.[5] In all the cases, the constant C_G is the same.

Comparing α_z^{-1} in Eq. (2) with C_G, we can see that the expression $\alpha_z^{-1}/\mathrm{Si}(\pi) = (3\pi/2)^3/\sqrt{2}$ is so simple that there may be physical essence behind the relation $\alpha_z \approx \alpha$, although it is unknown presently.

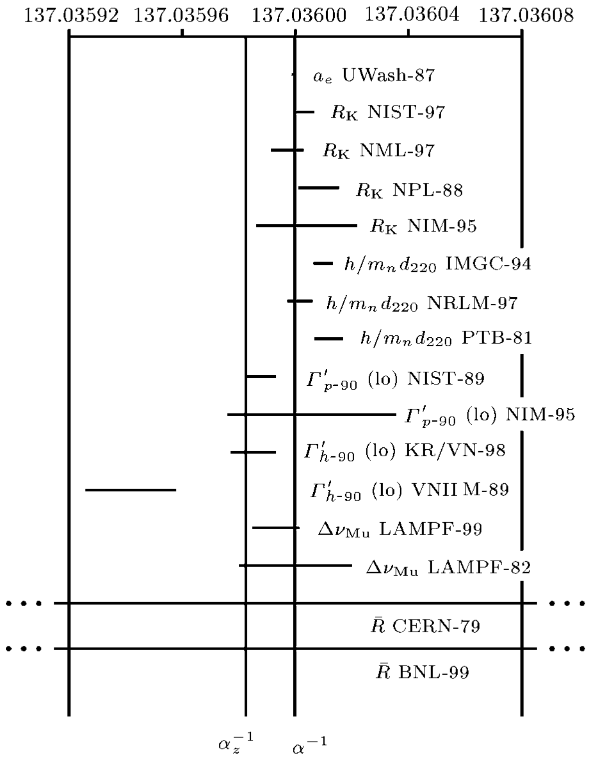

Fig. 2. Comparison of α_z^{-1} given by Eq. (2) with the experimental values used to determine α^{-1}.

There are many experiments for measuring the fine structure constant. Now we compare α_z^{-1} in Eq. (2) with the values of α^{-1} given by the experimental data together with various physical theories. This is shown in Fig. 2. Each line segment in the figure represents a datum in Table XV in Ref. [1]. That table was used to determine the value α in Ref. [1]. For each value x with uncertainty σ, the line segment extends from $x-\sigma$ to $x+\sigma$ in Fig. 2. The abbreviations to the right of the line segments represent the corresponding experiments, which are written in the same way as in Table XV of Ref. [1]. The vertical line marked α_z^{-1} represents the value given by (2) and that marked α^{-1} represents the value given by Ref. [1].

There are other theoretical values of α, such as those given by the string theory.[6,7] For example, the simplest expression is $\alpha^{-1} = \phi^{-10-\phi^3} = 137.7880938$, where $\phi = (\sqrt{5}-1)/2$ is the golden mean. More accurate values with accuracy 3×10^{-7} were also given by a complicated expansion.[7] Although the expression α_z^{-1} in this Letter is simply an observation, it is much simpler than the other theoretical results with the similar accuracy.

There is some possibility that $\alpha_z^{-1} \approx \alpha^{-1}$ is simply an accidental coincidence. Actually it is not difficult to construct complicated relations of $2, 3, \pi$ etc. to approximate α^{-1} because any real number can be approximated by a rational number to any accuracy. For example, $2^{-19/6}3^{157/24}\pi^{-1/16} = 137.0360046$ (there is also the number $2^{19/4}3^{-7/4}5^{1/4}\pi^{11/4} = 137.036082$ given by Wyler [6]). However, the probability is very small to obtain accidentally a simple relation such as $\alpha_z^{-1}/\mathrm{Si}(\pi) = (3\pi/2)^3/\sqrt{2}$ in this Letter. The internal relation between α_z and the fine structure constant is to be revealed.

The author is very grateful to Professor S. Y. Lou for his helpful suggestions.

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