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A Method to Obtain Explicit Solutions of 1+2 Dimensional AKNS System[1]

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1.Introduction.

The 1+2 dimensional AKNS system is a generalization of usual AKNS system in 1+1 dimensions. It includes some of the important equations such as Davey-Stewartson (DS) equation, N-wave equation etc. It takes the form

$$\Psi_y = J\Psi_x + U\Psi, \qquad \Psi_t = \sum_{j=0}^{n} V_j \partial^{n-j}\Psi \tag{1}$$

where $\partial = \partial/\partial x$, J is a constant diagonal matrix with mutually different diagonal entries, $U(x,y,t)$ is off-diagonal (i.e. all the diagonal entries of U are zero), V_j's are $r \times r$ matrices.

The integrability conditions of (1) are

$$[J, V_{j+1}^A] = V_{j,y}^A - JV_{j,x}^A - [U,V_j]^A + \sum_{k=0}^{j-1} C_{n-k}^{n-j}(V_k\partial^{j-k}U)^A \tag{2}$$

$$V_{j,y}^D - JV_{j,x}^D = [U,V_j^A]^D - \sum_{k=0}^{j-1} C_{n-k}^{n-j}(V_k\partial^{j-k}U)^D \tag{3}$$

$$U_t = V_{n,y}^A - JV_{n,x}^A - [U,V_n]^A + \sum_{k=0}^{n-1}(V_k\partial^{n-k}U)^A \tag{4}$$

where the superscript D and A refer to the diagonal and off-diagonal parts of a matrix.

We look (3) and (4) as a system of partial differential equations of unknowns (U, V_j^D) $(j = 1, \cdots, n)$, in which V_j^A's are defined inductively by (2). So (1) is the Lax pair of (3) and (4).

The Lax pair (1) has been studied by various methods. In this paper, we shall study the relation between the 1+2 dimensional AKNS system and a linear system with spectral parameter, and apply it to DS equation. This gives a way to get explicit solutions of AKNS equation and especially, DS equation. This kind of relations have been studied in [1,2,4,6] for KP hierarchy and N-wave equation.

2. A Lax pair with spectral parameter connecting with the 1+2 dimensional AKNS system.

We introduce the following linear system

$$\begin{aligned} \Phi_x &= \begin{pmatrix} \lambda I & p \\ q & 0 \end{pmatrix}\Phi, \qquad \Phi_y = \begin{pmatrix} \lambda J + U & Jp \\ qJ & 0 \end{pmatrix}\Phi \\ \Phi_t &= \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}\Phi = \sum_{j=0}^{n}\begin{pmatrix} W_j & X_j \\ Y_j & Z_j \end{pmatrix}\lambda^{n-j}\Phi \end{aligned} \tag{5}$$

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where λ is a complex parameter, and J, U are as before. W_j's are $r \times r$ matrices, p, X_j are $r \times 1$ matrices, q, Y_j are $1 \times r$ matrices, and Z_j are functions ($j = 0, 1, \cdots, n$). All of them are functions of x, y, t, independent of the parameter λ. From the integrability conditions of (5), we have

$$p_y = Jp_x + Up, \qquad q_y = q_x J - qU \tag{6}$$
$$p_t = X_{n,x} + (W_n - Z_n)p, \qquad q_t = Y_{n,x} - q(W_n - Z_n) \tag{7}$$
$$U_x = [pq, J], \qquad W_{j,x} = pY_j - X_j q, \qquad Z_{j,x} = qX_j - Y_j p \tag{8}$$
$$W^D_{j,y} = [U, W_j]^D + J(pY_j)^D - (X_j q)^D J, \qquad Z_{j,y} = qJX_j - Y_j Jp \tag{9}$$

$$\begin{aligned} &X_0 = Y_0 = 0, \quad [J, W_0] = 0 \\ &X_{j+1} = X_{j,x} + (W_j - Z_j)p, \qquad Y_{j+1} = -Y_{j,x} + q(W_j - Z_j) \\ &[J, W^A_{j+1}] = W^A_{j,y} - [U, W_j]^A - J(pY_j)^A + (X_j q)^A J - U_t \delta_{jn} \end{aligned} \tag{10}$$

According to [5], all the W_j, X_j, Y_j, Z_j are functions of the entries of p, q, U and their x, y-derivatives.

From (7), (8), (9) and (10),

$$p_t = \sum_{j=0}^{n} M^{(n)}_{n-j} \partial^j p, \qquad q_t = \sum_{j=0}^{n} (\partial^j q) N^{(n)}_{n-j} \tag{11}$$

where $M^{(n)}_j$, $N^{(n)}_j$ are given inductively by

$$M^{(0)}_0 = W_0 - Z_0, \quad M^{(k+1)}_j = M^{(k)}_j + M^{(k)}_{j-1,x} + (W_{k+1} - Z_{k+1})\delta_{j,k+1} \tag{12}$$
$$N^{(0)}_0 = W_0 - Z_0, \quad N^{(k+1)}_j = -N^{(k)}_j - N^{(k)}_{j-1,x} - (W_{k+1} - Z_{k+1})\delta_{j,k+1} \tag{13}$$

with $M^{(k)}_j = 0$, $N^{(k)}_j = 0$ if $j > k$ or $j < 0$.

Hence p satisfies a similar equation as Ψ in (1). Since V_j^A's are uniquely determined by the integrability condition of (1) if (U, V_j^D) are looked as unknowns, we have

Theorem 1. *If (5) is completely integrable, then $(U, M^{(n)D}_j)$ gives a solution of (3),(4).*

It seems that the equations (6)-(10) are quite complicated. However, since they have a Lax pair (5), we can use various methods in 1+1 dimensions to obtain explicit solutions of (6)-(10). Here we use the Darboux transformation method. By the general method of constructing Darboux matrices[3], we have

Theorem 2. *For given solution (p, q, U) of (6)-(10), let $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_{r+1})$ be a constant diagonal matrix, let h_i be a vector solution of (5) with $\lambda = \lambda_i$, $H = (h_1, \cdots, h_{r+1})$. If $\det H \neq 0$, let $S = H\Lambda H^{-1}$ and denote $S = \begin{pmatrix} T & \xi \\ \eta & \zeta \end{pmatrix}$ where T, ξ, η, ζ are $r \times r$, $r \times 1$, $1 \times r$, 1×1 matrices respectively. Then, for any solution Φ of (5), $\Phi' = (\lambda - S)\Phi$ satisfies (5) with p, q, U, W_j, X_j, Y_j, Z_j, Φ replaced by p', q', U', W'_j, X'_j, Y'_j, Z'_j, Φ' respectively, where*

$$\begin{aligned} &p' = p + \xi, \quad q' = q - \eta, \quad U' = U + [J, T] \\ &W'_{j+1} = W_{j+1} + W'_j T - TW_j + X'_j \eta - \xi Y_j \\ &Z'_{j+1} = Z_{j+1} + Y'_j \xi - \eta X_j + \zeta(Z'_j - Z_j) \\ &X'_{j+1} = X_{j+1} + W'_j \xi - TX_j + \zeta X'_j - Z_j \xi \\ &Y'_{j+1} = Y_{j+1} + Y'_j T - \eta W_j + Z'_j \eta - \zeta Y_j. \end{aligned} \tag{14}$$

The proof is direct. The matrix $\lambda - S$ here is called a Darboux matrix.

Therefore, for any solution of (6)-(10), if we can solve the corresponding Lax pair (5), then (14) gives a new solution (p', q', U') of (6)-(10), and the corresponding solution of the Lax pair is $\Phi' = (\lambda - S)\Phi$. Moreover, by considering Theorem 1 and [3], we know that a series of solutions of (3),(4) are obtained provided one solution of (6)-(10) and the corresponding solution of (5) are known. Especially, we can obtain infinite number of nontrivial solutions of (3),(4) from the zero solution of (6)-(10).

3. Solutions of DS equation.

The DS equation

$$\begin{aligned} &iu_t = u_{xx} + u_{yy} - u(A - D) \\ &A_x - A_y = (\partial_x + \partial_y)|u|^2 \\ &D_x + D_y = -(\partial_x - \partial_y)|u|^2. \end{aligned} \tag{15}$$

has a Lax pair

$$\begin{aligned} &\Psi_y = J\Psi_x + U\Psi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi_x + \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix} \Psi \\ &\Psi_t = -2iJ\Psi_{xx} - 2iU\Psi_x + Q\Psi \\ &\quad = -2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi_{xx} - 2i \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix} \Psi_x + \begin{pmatrix} iA & -i(u_x + u_y) \\ i(u^*_x - u^*_y) & iD \end{pmatrix} \Psi. \end{aligned} \tag{16}$$

We can take the system (5) as

$$\begin{aligned} &\Phi_x = \begin{pmatrix} \lambda I & p \\ -p^* & 0 \end{pmatrix} \Phi \\ &\Phi_y = \begin{pmatrix} \lambda J + U & Jp \\ -p^* J & 0 \end{pmatrix} \Phi \\ &\Phi_t = \begin{pmatrix} -2iJ\lambda^2 - 2iU\lambda + W_2 & -2iJp\lambda - 2iJp_x - 2iUp \\ 2ip^* J\lambda - 2ip^*_x J + 2ip^* U & 2ip^* Jp \end{pmatrix} \Phi, \end{aligned} \tag{17}$$

whose integrability conditions are

$$\begin{aligned} &p_y = Jp_x + Up, \quad p_t = -2iJp_{xx} - 2iUp_x + (W_2 - \alpha_2 - 2iJpp^*)p \\ &U_x = [J, pp^*], \quad U_y = \frac{i}{2}[J, W_2] \\ &U_t = V_{2,y} - [U, W_2] + 2iJ(pp^*)_x J - 2iJpp^* U + 2iUpp^* J \\ &W_{2,x} = -2i(pp^*_x J + Jp_x p^*) + 2i[pp^*, U]. \end{aligned} \tag{18}$$

The Darboux matrix for (17) can be constructed as in Theorem 1, provided that we take $\lambda_1 = \lambda_0$, $\lambda_2 = \lambda_3 = -\lambda_0^*$ for any given $\lambda_0 \in \mathbb{C}$, and $h_i^* h_j = 0$ if $\lambda_i + \lambda_j^* = 0$. (This is possible since $h_i^* h_j$ are constants by the Lax pair (17).) Let $h_1 = (\phi_1, \phi_2, \phi_3)^T$, direct calculation shows

$$S_{ij} = -\lambda_0^* \delta_{ij} + (\lambda_0 + \lambda_0^*) \frac{\phi_i \phi_j^*}{|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2}. \tag{19}$$

As in (16), denote iA, iD to be the diagonal entries of $Q = W_2 - \alpha_2 - 2iJpp^*$, then, from Theorem 2, we have

Theorem 3. *Suppose (p, U) satisfies (18), then (u, A, D) gives a solution of (15). Moreover, for given $\lambda_0 \in \mathbb{C}$, let $(\phi_1, \phi_2, \phi_3)^T$ be a solution of (17) with $\lambda = \lambda_0$, then*

$$\begin{aligned} u' &= u + 2\kappa\phi_1\phi_2^* \\ A' &= A - 2\kappa(u\phi_1^*\phi_2 + u^*\phi_1\phi_2^*) - 2\kappa(p_1\phi_1^*\phi_3 + p_1^*\phi_1\phi_3^*) - 4\kappa^2|\phi_1|^2(|\phi_2|^2 + |\phi_3|^2) \\ D' &= D + 2\kappa(u\phi_1^*\phi_2 + u^*\phi_1\phi_2^*) + 2\kappa(p_2\phi_2^*\phi_3 + p_2^*\phi_2\phi_3^*) + 4\kappa^2|\phi_2|^2(|\phi_1|^2 + |\phi_3|^2) \end{aligned} \tag{20}$$

where

$$\kappa = \frac{\lambda_0 + \lambda_0^*}{|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2} \tag{21}$$

give a new solution of (15). The solution p' of the corresponding Lax pair (16) is given by

$$p' = p + \kappa\phi_3^* \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{22}$$

From this theorem, we can obtain a series of solutions of DS equation from the zero solutions of (18).

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