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Section 7: Chaos and Soliton

High-dimensional Integrable Systems and Solitons with Variable Velocities

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ABSTRACT. The Darboux transformation method is used to get explicit solutions of a quite general integral system in $\mathbf{R}^{n+1}$. Soliton solutions are obtained for several integral systems. These solitons can be local or non-local, and move with fixed velocities or variable velocities.

1. Introduction

The solitary waves are very popular in nonlinear problems. For quite a lot of integrable nonlinear systems, various mathematical methods have been established to get the exact solutions. In the strict sense, solitons are solutions to some nonlinear evolution equations and asymptotic to zero at infinity in all directions or all directions except a narrow region in the space. Besides, the interaction of solitons is elastic. Such phenomena were observed in the propagation of water waves in canals and photo-pulse along cables etc. The soliton theory has been studied extensively since sixties. In the early stage the equations considered depend only on two variables t and x, i.e. in the space-time $\mathbf{R}^{1+1}$. Afterwards, a few integrable systems in $\mathbf{R}^{2+1}$ which admit solitons were found, such as Kadomtsev-Petviashvili (KP) equation, Davey-Stewartson (DS) equation etc.

In the present paper, we first introduce a general class of integrable system in $\mathbf{R}^{n+1}$ and give a procedure to get explicit solutions by Darboux transformation. As examples, we discuss the soliton solutions of an integral system in $\mathbf{R}^{n+1}$, the soliton solutions of DS equation and soliton solutions which oscillate and do not move to infinity.

The Darboux transformation method is a powerful method to get explicit solutions of integral systems. It gives an algebraic algorithm to get a series of explicit solutions by solving linear partial differential equations only once. The multi-soliton solutions can be obtained by Darboux transformation of high order from the trivial solutions.

For many equations like KdV equation, the solitary waves have fixed velocity, fixed shape and interact elastically. However, considered from the view of

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Lemma 1: Suppose all the diagonal entries of J_1 are different, then (3)–(7) are equivalent to

$$\partial_1 P_i - \partial_i P_1 + [P_i, P_1] = 0, \tag{9}$$

$$[J_1, V_0] = 0, \quad \partial_i V_0 = 0, \tag{10}$$

$$[J_i, V_{\alpha+1}^{\text{off}}] = \partial_i V_\alpha^{\text{off}} - [P_i, V_\alpha]^{\text{off}}, \tag{11}$$

$$\partial_i V_\alpha^{\text{diag}} = [P_i, V_\alpha^{\text{off}}]^{\text{diag}}, \tag{12}$$

$$\partial_t P_1 - \partial_1 V_m^{\text{off}} + [P_1, V_m]^{\text{off}} = 0. \tag{13}$$

Although (9)–(13) are simplified, generally they are still overdetermined.

For $n = 1$, V_α's can be determined from (10)–(12) recursively and they are polynomials of P and its derivatives.[8] For general n, the existence of P is not obvious, but depends on the integrability of (11), (12). We have

Lemma 2: (1) There exists $\{V_\alpha[P]\}$ which satisfies (10)–(12), and each V_α is a polynomial of P and its derivatives.

(2) For given diagonal matrices $\{V_\alpha^0(t)\}$ (independent of x), there exists uniquely $\{V_\alpha[P]\}$ such that $V_\alpha[0] = V_\alpha^0(t)$ $(\alpha = 0, 1, \cdots, m)$.

Here the notation $\{V_\alpha[P]\}$ implies that V_α depends on P functionally.

From Lemma 2, the evolution equations (13) are differential equations of P.

Example 1: $n = 1$, $N = 2$, $m = 2$,

$$J_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad P = -\frac{1}{2}\begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix}, \qquad V_k^0(t) = -2i\alpha_k(t) J_1,$$

the evolution equation is the generalized nonlinear Schrödinger equation

$$iu_t = \alpha_0(t)(u_{xx} + 2|u|^2 u) + 2\alpha_1(t) u_x.$$

Example 2: $n = 1$, $N = 2$, $m = 3$,

$$J_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad P = -\frac{1}{2}\begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix}, \qquad V_k^0(t) = -4\alpha_k(t) J_1$$

(u is real), the evolution equation is the generalized MKdV equation

$$u_t + \alpha_0(t)(u_{xxx} + 6u^2 u_x) + 2\alpha_1(t) u_{xx} + 4\alpha_2(t) u_x = 0.$$

Example 3: $n = 1$, $N = 2$, $m = 3$,

$$J_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad P = -\frac{1}{2}\begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix}, \qquad V_k^0(t) = -4\alpha_k(t) J_1$$

(u is real), the evolution equation is the generalized KdV equation

$$u_t + \alpha_0(t)(u_{xxx} + 6uu_x) + 2\alpha_1(t)u_{xx} - 4\alpha_2(t)u_x = 0.$$

Example 4: [5] $n = 2$, $N = 2$, $m = 2$,

$$J_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 \\ 0 & -i \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix},$$
$$V_0^0(t) = aJ_1, \qquad V_1^0(t) = V_2^0(t) = 0,$$

the equation is

$$p_t = 2a(p_{zz} + p^2q), \qquad q_t = -2a(q_{zz} + pq^2)$$

with $z = x_1 + ix_2$. This is a generalized nonlinear Schrödinger equation with complex coordinate z and real coordinate t.

Example 5: [10] $n = 2$, $N = 4$, $m = 2$,

$$J_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix},$$
$$P = -\begin{pmatrix} U & Q \\ -Q^* & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix},$$
$$V_0^0 = -2iJ_2, \qquad V_1^0(t) = V_2^0(t) = 0,$$

the equation contains the DSI equation

$$\begin{aligned} iu_t &= u_{xx} + u_{yy} - u(A - D), \\ (\partial_x - \partial_y)A &= (\partial_x + \partial_y)|u|^2, \\ (\partial_x + \partial_y)D &= -(\partial_x - \partial_y)|u|^2 \end{aligned}$$

with constraint

$$u_x = q_{11}q_{21}^* + q_{12}q_{22}^*.$$

Example 6: [3] $n = 3$, $N = 3$, $m = 2$,

$$J_1 = \begin{pmatrix} i & & \\ & 0 & \\ & & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & & \\ & i & \\ & & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & i \end{pmatrix}.$$

We will discuss its solitons later.

3. Darboux transformation

We will use Darboux matrix to construct explicit solutions of the equations (13) with Lax set (1). The general procedure to construct a Darboux matrix is as follows.

Let

$$G(x,t,\lambda) = \lambda I - S, \qquad \Psi' = G\Psi \tag{14}$$

for a solution Ψ of (1). We want that Ψ' satisfies a similar system

$$\begin{aligned} \partial_i \Psi' &= (\lambda J_i + P_i')\Psi', \\ \partial_t \Psi' &= \sum_{\alpha=0}^{m} V_\alpha' \lambda^{m-\alpha} \Psi'. \end{aligned} \tag{15}$$

In this case, we call G a Darboux matrix, and the transformation $(P,\Psi) \to (P',\Psi')$ a Darboux transformation. This is equivalent to that S satisfies

$$\begin{aligned} \partial_i S &= [J_i S + P_i, S], \\ \partial_t S &= [\sum_{\alpha=0}^{m} V_\alpha S^{m-\alpha}, S]. \end{aligned} \tag{16}$$

The explicit solutions of (16) can be constructed as follows.

Theorem 1: [1,7] Suppose P is a solution of (13). Let $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$ where λ_i's are constants, h_i be a column solution of (1) with $\lambda = \lambda_i$, $H = (h_1, \cdots, h_N)$. Then, if $\det H \neq 0$, $\lambda I - S$ with $S = H\Lambda H^{-1}$ is a Darboux matrix.

Remark 2: The S given by the Theorem contains all the diagonizable solutions of (16). The non-diagonizable solutions can be constructed by the limit of diagonizable Darboux matrices. [9]

Remark 3: Suppose P is known and we can get the fundamental solution Ψ of (1), then using the Darboux transformation, (P',Ψ') is obtained by a purely algebraic way. Continuing this process, we can obtain a series of explicit solutions

$$(P,\Psi) \to (P',\Psi') \to (P'',\Psi'') \to \cdots$$

without solving any more differential equations. Multi-solitons can be constructed in this way from the trivial solution $P = 0$.

Remark 4: If J is real, $P^* = -P$, take $\lambda_i = \mu$ or $-\mu$ $(\mu + \mu^* = 0)$ and $\{h_i\}$ satisfies $h_i^* h_j = 0$ $(\lambda_i \neq \lambda_j)$[1], then $P'^* = -P'$ holds.

If J is purely imaginary, $P^* = -P$, take $\lambda_i = \mu$ or μ $(\mu \neq \mu^*)$, $h_i^* h_j = 0$ $(\lambda_i \neq \lambda_j)$ then $P'^* = -P'$ also holds.

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[1] This equality holds identically if it holds at one point $(t^0, x_1^0, \cdots, x_n^0)$.

Remark 5: At finite time, a multi-soliton behaves very complicated. However, as $t \to \pm\infty$, generally, a multi-soliton splits up into several single-solitons. More precisely, when an observer moves in some constant velocities $v_1, \cdots, v_s$, he sees a single-soliton as $t \to \infty$; when he moves with a velocity other than $v_1, \cdots, v_s$, the wave he sees disappears at $t \to \infty$.

As $t \to \infty$, a multi-soliton looks like several single-solitons, therefore we can think of the interaction of solitons. That is, as t evolves from $-\infty$ to $+\infty$, several solitons move together, interact, and move apart again. In many cases (e.g., in many series of integrable equations with constant coefficient, like KdV equation), these interactions are elastic, i.e., the velocity and shape of a soliton does not change after interaction, and there is only a phase shift. Examples can be seen below.

4. Example of solitons

Example 7: Take $\alpha_0 = \alpha_0(t)$, $\alpha_1 = \alpha_2 = 0$ in Example 2. Then the equation is

$$u_t + \alpha_0(t)(u_{xxx} + 6u^2 u_x) = 0. \tag{17}$$

Let $\lambda_1 = \mu$, $\lambda_2 = -\mu$, (μ is a real constant),

$$H = \begin{pmatrix} e^{\mu x + 4\mu^3 A_0(t) + \mu p} & e^{-\mu x - 4\mu^3 A_0(t) - \mu p} \\ e^{-\mu x - 4\mu^3 A_0(t) - \mu p} & e^{\mu x + 4\mu^3 A_0(t) + \mu p} \end{pmatrix}$$

where p is a real constant, $A_0 = -\int \alpha_0(t)\, dt$. Then,

$$S = \frac{\mu}{\operatorname{ch} \sigma} \begin{pmatrix} \operatorname{sh} \sigma & -1 \\ -1 & -\operatorname{sh} \sigma \end{pmatrix}$$

with $\sigma = 2\mu(x + 4\mu^3 A_0(t) + p)$, and

$$u' = -\frac{2\mu}{\operatorname{ch} \sigma}.$$

The center of this soliton locates at $-4\mu^3 A_0(t) - p$. For example, if $\alpha_0(t) = \omega \cos(\omega t)$ (ω is a real constant), the soliton oscillates around $x = -p$ with a fixed frequency. The behavior of the soliton varies a lot according to the coefficients $\alpha_0(t)$ in (17).

Example 8: Let

$$V_0^0 = \begin{pmatrix} a_1 i & 0 & 0 \\ 0 & a_2 i & 0 \\ 0 & 0 & a_3 i \end{pmatrix}, V_1^0 = \begin{pmatrix} b_1 i & 0 & 0 \\ 0 & b_2 i & 0 \\ 0 & 0 & b_3 i \end{pmatrix}, V_2^0 = \begin{pmatrix} c_1 i & 0 & 0 \\ 0 & c_2 i & 0 \\ 0 & 0 & c_3 i \end{pmatrix}.$$

in Example 6. Take the seed solution $P=0$, and

$$\Lambda = \begin{pmatrix} \mu & & \\ & \mu^* & \\ & & \mu^* \end{pmatrix}, \qquad H = \begin{pmatrix} e^{i\zeta_1} & -a^* e^{i\zeta_1^*} & -b^* e^{i\zeta_1^*} \\ a e^{i\zeta_2} & e^{i\zeta_2^*} & 0 \\ b e^{i\zeta_3} & 0 & e^{i\zeta_3^*} \end{pmatrix}$$

with $\zeta_i = \mu x_i + (a_i\mu^2 + b_i\mu + c_i)t$, then the solutions are

$$\begin{aligned} p = P_{23} &= ab^*(\mu^* - \mu)e^{i[(\zeta_3-\zeta_1)-(\zeta_2^*-\zeta_1^*)]}/\Delta \\ q = P_{13} &= (\mu^* - \mu)b^* e^{i(\zeta_1^*-\zeta_3^*)}/\Delta \\ r = P_{12} &= (\mu^* - \mu)a^* e^{i(\zeta_1^*-\zeta_2^*)}/\Delta. \end{aligned}$$

with

$$\Delta = 1 + |a|^2 e^{i[(\zeta_2-\zeta_2^*)-(\zeta_1-\zeta_1^*)]} + |b|^2 e^{i[(\zeta_3-\zeta_3^*)-(\zeta_1-\zeta_1^*)]} > 0.$$

These solutions approach to zero as $x \to \infty$ except for some lower-dimensional subspaces.

Multi-soliton solutions can be obtained by several Darboux transformations. As $t \to \pm\infty$, a k-th multi-soliton solution splits to k separating single solitons asymptotically. Moreover, their shapes and velocities of propagation are unchanged after the interaction.

Example 9: For the DS equation in Example 5, take

$$\Lambda = \begin{pmatrix} \mu & & & \\ & \mu & & \\ & & -\mu^* & \\ & & & -\mu^* \end{pmatrix}, \quad H = \begin{pmatrix} e^{\mu(x+Jy)-2i\mu^2 t} & -e^{-\mu^*(x+Jy)-2i\mu^{*2}t}C^* \\ C & I \end{pmatrix}$$

with $J = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and C is a constant 2×2 matrix, then we get a single soliton solution

$$u = \frac{2bc_{12}e^{2ia(y-4at)+4i(a^2+b^2)t}}{|\det C|\,\mathrm{ch}(2bx+\phi_1) + c_1c_2\,\mathrm{ch}(-2b(y-4at)+\phi_2)}$$

with $b = -\operatorname{Re}(\mu)$, $a = \operatorname{Im}(\mu)$, $c_1 = \sqrt{|C_{11}|^2 + |C_{21}|^2}$, $c_2 = \sqrt{|C_{12}|^2 + |C_{22}|^2}$, $c_{12} = C_{11}^*C_{12} + +C_{21}^*C_{22}$, $\phi_1 = \ln|\det C|$, $\phi_2 = \ln(c_2/c_1)$. This soliton tends to zero exponentially in all directions as $(x,y) \to \infty$.

Multi-solitons can be obtained by repeated Darboux transformations. For example, after twice Darboux transformations, there may be two or four solitons, which also tends to zero exponentially in all directions as $(x,y) \to \infty$. These solitons interact elastically. (See [10])

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