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Finite dimensional integrable systems related to two dimensional $A_{2l}^{(2)}$, $C_l^{(1)}$ and $D_{l+1}^{(2)}$ Toda equations

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Abstract

The finite dimensional Hamiltonian systems related to two dimensional $A_{2l}^{(2)}$, $C_l^{(1)}$ and $D_{l+1}^{(2)}$ Toda equations are obtained and their Liouville integrability is proved. Any solution of these Hamiltonian systems will give a solution of the corresponding two dimensional Toda equations.

Keywords: 2+1 dimensional Toda equations; Finite dimensional Hamiltonian systems; Nonlinear constraints

1. Introduction

The two dimensional Toda equations are important integrable systems which have been studied widely. A two dimensional Toda equation corresponding to a Kac–Moody algebra g of affine type can be written as

$$w_{k,xt} = A_k \exp\left(\sum_{i=1}^{n} c_{ki} w_i\right) - A_0 v_k \exp\left(\sum_{i=1}^{n} c_{0i} w_i\right) \quad (k = 1, \ldots, n) \tag{1}$$

where $C = (c_{ij})_{0\le i,j\le n}$ is the generalized Cartan matrix of the Kac–Moody algebra, $v = (v_0, v_1, \ldots, v_n)^{\mathrm{T}}$ is a non-zero vector satisfying $Cv = 0$, and $A_0, A_1, \ldots, A_n$ are real constants [1,2].

The two dimensional periodic Toda equation ($A_l^{(1)}$ Toda equation) has been studied in various ways, such as by the Darboux transformation method [3–5], loop group method [2], nonlinear constraint method [6,7] etc. The Toda equations with other boundary conditions have also been studied [8–10]. In particular, the Darboux transformations for two dimensional $A_{2l}^{(2)}$, $C_l^{(1)}$ and $D_{l+1}^{(2)}$ Toda equations were obtained in [11,12].

As one of the useful methods, the nonlinear constraint method is effective in finding quasi-periodic solutions, and has been applied to many integrable nonlinear partial differential equations, especially to the equations with 2×2 Lax

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pairs in $1+1$ dimensions [13–17] or $2+1$ dimensions [18–22]. When $g = A_l^{(1)}$, the Lax pair of the two dimensional Toda equation has a reality symmetry and a cyclic symmetry of order l, and the finite dimensional integrable systems related to it were studied by [6,7].

The one dimensional Toda equations are finite dimensional Hamiltonian systems, while the two dimensional Toda equations are infinite dimensional Hamiltonian systems. In this paper, we shall give finite dimensional Hamiltonian systems related to the two dimensional Toda equations. Any solution of these Hamiltonian systems will give a solution of the corresponding two dimensional Toda equations. We consider an $N \times N$ integrable system which corresponds to the two dimensional Toda equations with Kac–Moody algebras $g = A_{2l}^{(2)}$, $C_l^{(1)}$ and $D_{l+1}^{(2)}$. It has a unitary symmetry, a reality symmetry and a cyclic symmetry of order N. The number of independent functions in these $N \times N$ systems is only $[N/2]$ or $[N/2-1]$. To get the Lax operator of the finite dimensional Hamiltonian systems related to this two dimensional system, we need to consider all these symmetries in the construction.

In Section 2, the linear system containing the two dimensional $A_{2l}^{(2)}$, $C_l^{(1)}$ and $D_{l+1}^{(2)}$ Toda equations is presented. In Section 3, the finite dimensional Hamiltonian systems are derived in terms of the Lax operator $L(\lambda)$. Then the involution of the set $\{\mathrm{tr}(L^k(\lambda))\}$, which will generate the conserved integrals, is proved in Section 4. The expressions for the conserved integrals are written down in Section 5 and the Hamiltonians are expressed in terms of them. Finally, in Section 6, the functional independence of sufficiently many conserved integrals for Liouville integrability is proved. This gives the Liouville integrability of these finite dimensional Hamiltonian systems, and each solution of these Hamiltonian systems is a solution of the corresponding two dimensional Toda equation.

2. Linear system

Let N be a given integer with $N \geq 2$. Throughout this paper, for any $N \times N$ matrix A or any N dimensional vector v, and for any integers j and k, we define $A_{jk} = A_{j'k'}$ and $v_j = v_{j'}$ where $j \equiv j' \bmod N$ and $k \equiv k' \bmod N$. Hence the indices in this paper can always be arbitrary integers. In particular,

$$\delta_{jk} = \begin{cases} 1 & \text{if } j-k \equiv 0 \bmod N, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Let $\omega = \mathrm{e}^{2\pi \mathrm{i}/N}$, $\Omega = \mathrm{diag}(1, \omega^{-1}, \ldots, \omega^{-N+1})$. Let m be an integer, and $K = (K_{jk})_{N\times N}$ with $K_{jk} = \delta_{m-j,k}$; then K is symmetric and

$$\Omega^* K = \omega^{m-2} K \Omega. \tag{3}$$

Consider the Lax pair

$$\begin{aligned} \partial_x \Phi &= U(x,t,\lambda)\Phi = (\mathrm{i}\lambda J + P(x,t))\Phi, \\ \partial_t \Phi &= V(x,t,\lambda)\Phi = (\mathrm{i}\lambda)^{-1} Q(x,t)\Phi \end{aligned} \tag{4}$$

and its integrability conditions

$$Q_x = [P, Q], \qquad P_t + [J, Q] = 0. \tag{5}$$

Here

$$J = (\delta_{i,j-1})_{N\times N}, \qquad P = (p_i(x,t)\delta_{ij})_{N\times N}, \qquad Q = (q_j(x,t)\delta_{i,j+1})_{N\times N}, \tag{6}$$

and $U(x,t,\lambda)$, $V(x,t,\lambda)$ satisfy the relations

$$\begin{aligned} &\overline{U(x,t,\lambda)} = U(x,t,-\bar{\lambda}), \qquad \overline{V(x,t,\lambda)} = V(x,t,-\bar{\lambda}), \\ &\Omega U(x,t,\lambda)\Omega^{-1} = U(x,t,\omega\lambda), \qquad \Omega V(x,t,\lambda)\Omega^{-1} = V(x,t,\omega\lambda), \\ &K U(x,t,\lambda) K^{-1} = -(U(x,t,\bar{\lambda}))^*, \qquad K V(x,t,\lambda) K^{-1} = -(V(x,t,\bar{\lambda}))^*. \end{aligned} \tag{7}$$

These relations are equivalent to

$$\begin{aligned} &\Omega P \Omega^{-1} = P, \qquad \Omega J \Omega^{-1} = \omega J, \qquad \Omega Q \Omega^{-1} = \omega^{-1} Q, \\ &K P K^{-1} = -P^{\mathrm{T}}, \qquad K J K^{-1} = J^{\mathrm{T}}, \qquad K Q K^{-1} = Q^{\mathrm{T}}. \end{aligned} \tag{8}$$

Written in terms of the components of the matrices, (8) becomes

$$p_{m-j} = -p_j, \qquad q_{m-j-1} = q_j. \tag{9}$$

The integrability condition (5) can also be written in terms of the components as

$$q_{j,x} = (p_{j+1} - p_j)q_j, \qquad p_{j,t} = q_{j-1} - q_j. \tag{10}$$

By direct calculation according to (7), we have the following lemma.

Lemma 1. *Suppose $\mu \in \mathbf{C}$.*

(i) *If $\Phi(x,t,\mu)$ is a solution of (4) for $\lambda = \mu$, then $\overline{\Phi(x,t,\mu)}$ is a solution of (4) for $\lambda = -\bar{\mu}$.*

(ii) *If $\Phi(x,t,\mu)$ is a solution of (4) for $\lambda = \mu$, then for any integer k, $\Omega^k \Phi(x,t,\mu)$ is a solution of (4) for $\lambda = \omega^k \mu$.*

(iii) *If $\Phi(x,t,\mu)$ is a solution of (4) for $\lambda = \mu$, then $\Psi(x,t) = K\overline{\Phi(x,t,\mu)}$ is a solution of the adjoint Lax pair for $\lambda = \bar{\mu}$:*

$$\partial_x \Psi = -(\mathrm{i}\bar{\mu} J + P)^{\mathrm{T}} \Psi, \qquad \partial_t \Psi = -(\mathrm{i}\bar{\mu})^{-1} Q^{\mathrm{T}} \Psi. \tag{11}$$

Remark 1. For any $N \times N$ matrix $A = (A_{ij})$, let

$$\hat{A} = (\hat{A}_{ij})_{1 \le i,j \le N} = (A_{i+1,j+1})_{1 \le i,j \le N}.$$

Since $\hat{\Omega}_{ij} = \Omega_{i+1,j+1} = \omega^{-1}\Omega_{ij}$, $\hat{\Omega}^* \hat{K} = \omega^{m-2} \hat{K} \hat{\Omega}$ implies $\Omega^* \hat{K} = \omega^{m-4} \hat{K} \Omega$. Under the transformation $(\Omega, K, J, P, Q, m) \to (\Omega, \hat{K}, \hat{J}, \hat{P}, \hat{Q}, m-2)$, (3), (5) and (8) still hold. Therefore, we only need to consider the cases $m = 0$ if N is odd or $m = 0, 1$ if N is even.

According to this remark, there are essentially three kinds of equations in the system (4).

(I) $N = 2n+1$ is odd, $m = 0$

From the symmetries (9) and the evolution equations (10), there exist $u_1, \ldots, u_n$ such that

$$\begin{aligned} &p_i = -p_{2n+1-i} = u_{i,x} \quad (1 \le i \le n), \ \ p_{2n+1} = 0, \\ &q_i = q_{2n-i} = A_i \mathrm{e}^{u_{i+1} - u_i} \quad (1 \le i \le n-1), \\ &q_n = A_n \mathrm{e}^{-2u_n}, \qquad q_{2n} = q_{2n+1} = A_0 \mathrm{e}^{u_1} \end{aligned} \tag{12}$$

where $A_0, A_1, \ldots, A_n$ are real constants. If $n \ge 2$, the evolution equations are

$$\begin{aligned} &u_{1,xt} = A_0 \mathrm{e}^{u_1} - A_1 \mathrm{e}^{u_2 - u_1}, \qquad u_{n,xt} = A_{n-1} \mathrm{e}^{u_n - u_{n-1}} - A_n \mathrm{e}^{-2u_n}, \\ &u_{j,xt} = A_{j-1} \mathrm{e}^{u_j - u_{j-1}} - A_j \mathrm{e}^{u_{j+1} - u_j} \quad (2 \le j \le n-1). \end{aligned} \tag{13}$$

Let $w_j = -(u_1 + \cdots + u_j)$ $(j = 1, \ldots, n)$, then $(w_1, \ldots, w_n)$ satisfies (1) with $g = A_{2n}^{(2)}$. If $n = 1$, the evolution equation is $u_{1,xt} = A_0 \mathrm{e}^{u_1} - A_1 \mathrm{e}^{-2u_1}$, which corresponds to $g = A_2^{(2)}$.

(II) $N = 2n+2$ is even, $m = 0$

$$\begin{aligned} &p_i = -p_{2n+2-i} = u_{i,x}, \quad (1 \le i \le n), \qquad p_{n+1} = p_{2n+2} = 0, \\ &q_i = q_{2n+1-i} = A_i \mathrm{e}^{u_{i+1} - u_i} \quad (1 \le i \le n-1), \\ &q_n = q_{n+1} = A_n \mathrm{e}^{-u_n}, \qquad q_{2n+1} = q_{2n+2} = A_0 \mathrm{e}^{u_1}. \end{aligned} \tag{14}$$

If $n \ge 2$, the evolution equations are

$$\begin{aligned} &u_{1,xt} = A_0 \mathrm{e}^{u_1} - A_1 \mathrm{e}^{u_2 - u_1}, \qquad u_{n,xt} = A_{n-1} \mathrm{e}^{u_n - u_{n-1}} - A_n \mathrm{e}^{-u_n}, \\ &u_{j,xt} = A_{j-1} \mathrm{e}^{u_j - u_{j-1}} - A_j \mathrm{e}^{u_{j+1} - u_j} \quad (2 \le j \le n-1). \end{aligned} \tag{15}$$

Let $w_j = -(u_1 + \cdots + u_j)$ $(j = 1, \ldots, n-1)$ and $w_n = -\frac{1}{2}(u_1 + \cdots + u_n)$; then $(w_1, \ldots, w_n)$ satisfies (1) with $g = C_n^{(1)}$. If $n = 1$, the evolution equation is $u_{1,xt} = A_0 \mathrm{e}^{u_1} - A_1 \mathrm{e}^{-u_1}$, which corresponds to $g = A_1^{(1)}$.

(III) $N = 2n$ is even, $m = 1$

$$
\begin{aligned}
&p_i = -p_{2n+1-i} = u_{i,x}, \quad (1 \le i \le n),\\
&q_i = q_{2n-i} = A_i \mathrm{e}^{u_{i+1}-u_i} \quad (1 \le i \le n-1), \qquad q_n = A_n \mathrm{e}^{-2u_n}, \qquad q_{2n} = A_0 \mathrm{e}^{2u_1}.
\end{aligned}
\tag{16}
$$

If $n \ge 2$, the evolution equations are

$$
\begin{aligned}
&u_{1,xt} = A_0 \mathrm{e}^{2u_1} - A_1 \mathrm{e}^{u_2-u_1}, \qquad u_{n,xt} = A_{n-1}\mathrm{e}^{u_n-u_{n-1}} - A_n \mathrm{e}^{-2u_n},\\
&u_{j,xt} = A_{j-1}\mathrm{e}^{u_j-u_{j-1}} - A_j \mathrm{e}^{u_{j+1}-u_j} \quad (2 \le j \le n-1).
\end{aligned}
\tag{17}
$$

Let $w_j = -(u_1 + \cdots + u_j)$ $(j = 1, \ldots, n)$, then $(w_1, \ldots, w_n)$ satisfies (1) with $g = D^{(2)}_{n+1}$. If $n = 1$, the evolution equation is $u_{1,xt} = A_0 \mathrm{e}^{2u_1} - A_1 \mathrm{e}^{-2u_1}$, which corresponds to $g = A^{(1)}_1$.

3. Finite dimensional Hamiltonian systems

Now take r non-zero real numbers $\lambda_1, \ldots, \lambda_r$ such that $\lambda_1^2, \ldots, \lambda_r^2$ are distinct. Let $H_\alpha = (\phi_{1\alpha}, \ldots, \phi_{N\alpha})^{\mathrm{T}}$ be a column solution of the Lax pair (4) with $\lambda = \lambda_\alpha$. Then the $\phi_{j\alpha}$'s satisfy

$$
\partial_x \phi_{j\alpha} = \mathrm{i}\lambda_\alpha \phi_{j+1,\alpha} + p_j \phi_{j\alpha}, \qquad \partial_t \phi_{j\alpha} = (\mathrm{i}\lambda_\alpha)^{-1} q_{j-1} \phi_{j-1,\alpha}. \tag{18}
$$

Write $\{\phi_{k\alpha}\}$ as the ordered set $\{\phi_{11}, \ldots, \phi_{1r}, \phi_{21}, \ldots, \phi_{2r}, \ldots, \phi_{N1}, \ldots, \phi_{Nr}\}$. We shall rewrite (18) as Hamiltonian equations by finding certain constraints

$$
u_j = u_j(\{\phi_{k\alpha}\}) \quad (j = 1, \ldots, n). \tag{19}
$$

Then p_j's and q_j's are also represented as functions of $\{\phi_{k\alpha}\}$ and their derivatives.

First, considering Lemma 1, assume that under constraints (19), the systems of ordinary differential equations in (18) have a Lax operator $L(\lambda)$ in the form

$$
L(\lambda) = J + \sum_{\alpha=1}^{r} \sum_{l=0}^{N-1} \left(\frac{a_l}{\lambda - \omega^l \lambda_\alpha} \Omega^l H_\alpha H_\alpha^* \Omega^l K - \frac{b_l}{\lambda + \omega^l \lambda_\alpha} \Omega^l \overline{H}_\alpha \overline{H}_\alpha^* \Omega^l K \right) \tag{20}
$$

where the a_l's and b_l's are complex constants to be determined, that is, L satisfies

$$
L_x = [\mathrm{i}\lambda J + P, L], \qquad L_t = \frac{1}{\mathrm{i}\lambda}[Q, L] \tag{21}
$$

for certain P and Q expressed by (19).

The first term of $L(\lambda)$ is J. Comparing with the expression for $U(\lambda)$, we want that $\mathrm{i}\lambda L(\lambda)$ satisfies the same relations as $U(\lambda)$ does in (7). That is,

$$
\overline{L(\lambda)} = L(-\bar{\lambda}), \qquad \Omega L(\lambda) \Omega^{-1} = \omega L(\omega\lambda), \qquad K L(\lambda) K^{-1} = L(\bar{\lambda})^*. \tag{22}
$$

Substituting (20) into them, we get $a_l = \omega^{m-2} a_{l-1}$, $a_{N-l} = \bar{a}_l$, $b_l = a_l$. Hence $b_l = a_l = \omega^{(m-2)l}\kappa/N$ where κ is a real constant.

Now we prove that (21) holds.

For the monomials in (20), we have

$$
(\Omega^l H_\alpha H_\alpha^* \Omega^l K)_x = \mathrm{i}\omega^l \lambda_\alpha [J, \Omega^l H_\alpha H_\alpha^* \Omega^l K] + [P, \Omega^l H_\alpha H_\alpha^* \Omega^l K]. \tag{23}
$$

Here we have used the relations (8). Similarly,

$$
(\Omega^l \overline{H}_\alpha \overline{H}_\alpha^* \Omega^l K)_x = -\mathrm{i}\omega^l \lambda_\alpha [J, \Omega^l \overline{H}_\alpha \overline{H}_\alpha^* \Omega^l K] + [P, \Omega^l \overline{H}_\alpha \overline{H}_\alpha^* \Omega^l K]. \tag{24}
$$

Hence,

$$
L_x - [\mathrm{i}\lambda J + P, L] = [J, P - \hat{P}] \tag{25}
$$

where

$$\hat{P} = \frac{\mathrm{i}\kappa}{N}\sum_{\alpha=1}^{r}\sum_{l=0}^{N-1}(\omega^{(m-2)l}\Omega^l H_\alpha H_\alpha^*\Omega^l K - \omega^{(m-2)l}\Omega^l \bar{H}_\alpha \bar{H}_\alpha^*\Omega^l K). \tag{26}$$

Using the identity

$$\sum_{l=0}^{N-1}\omega^{kl} = N\delta_{k0}, \tag{27}$$

we get the entries of $\hat{P}$ as

$$\begin{aligned}\hat{P}_{jk} &= \frac{\mathrm{i}\kappa}{N}\sum_{\alpha=1}^{r}\sum_{l=0}^{N-1}\sum_{s=1}^{N}\omega^{(m-j-s)l}((H_\alpha H_\alpha^*)_{js} - (H_\alpha H_\alpha^*)_{sj})K_{sk}\\ &= \mathrm{i}\kappa\sum_{\alpha=1}^{r}(H_\alpha H_\alpha^*)_{j,m-j}\delta_{jk} - \mathrm{i}\kappa\sum_{\alpha=1}^{r}(H_\alpha H_\alpha^*)_{m-j,j}\delta_{jk}\\ &= \mathrm{i}\kappa\left(\langle\Phi_{m-j},\Phi_j\rangle - \langle\Phi_j,\Phi_{m-j}\rangle\right)\delta_{jk}.\end{aligned} \tag{28}$$

Hereafter, we use the symbol $\langle v_1, v_2\rangle = v_1^* v_2$ for two vectors.

Therefore, the first equation of (21) holds under the constraint

$$\partial_x u_j = p_j = \mathrm{i}\kappa\left(\langle\Phi_{m-j},\Phi_j\rangle - \langle\Phi_j,\Phi_{m-j}\rangle\right). \tag{29}$$

Now we consider the second equation of (21). Like for (25), we have

$$L_t - \frac{1}{\mathrm{i}\lambda}[Q, L] = -\frac{1}{\mathrm{i}\lambda}[Q, J - \hat{J}] \tag{30}$$

where

$$\begin{aligned}\hat{J} = (\hat{J}_{jk}) &= \frac{\kappa}{N}\sum_{\alpha=1}^{r}\sum_{l=0}^{N-1}\frac{\omega^{(m-3)l}}{\lambda_\alpha}(\Omega^l H_\alpha H_\alpha^*\Omega^l K + \Omega^l\overline{H}_\alpha\overline{H}_\alpha^*\Omega^l K),\\ \hat{J}_{jk} &= \kappa\sum_{\alpha=1}^{r}\frac{1}{\lambda_\alpha}\left((H_\alpha H_\alpha^*)_{j,m-1-j} + (H_\alpha H_\alpha^*)_{m-1-j,j}\right)\delta_{j,k-1}.\end{aligned} \tag{31}$$

Write $\hat{J}_{jk} = \theta_j\delta_{j,k-1}$; then

$$\begin{aligned}\theta_j &= \kappa\sum_{\alpha=1}^{r}\frac{1}{\lambda_\alpha}\left((H_\alpha H_\alpha^*)_{j,m-1-j} + (H_\alpha H_\alpha^*)_{m-1-j,j}\right)\\ &= \kappa\left(\langle\Phi_{m-1-j},\Lambda^{-1}\Phi_j\rangle + \langle\Phi_j,\Lambda^{-1}\Phi_{m-1-j}\rangle\right).\end{aligned} \tag{32}$$

$[Q, J - \hat{J}] = 0$ is equivalent to $(1-\theta_j)q_j = (1-\theta_{j+1})q_{j+1}$. Hence, $[Q, J - \hat{J}] = 0$ implies that

$$q_j = (1-\theta_j)^{-1}F \tag{33}$$

for a certain function F. If this is true, then the second equation of (21) holds.

Remark 2. Since $\theta_{m-1-j} = \theta_j$, the identity $q_{m-1-j} = q_j$ holds for all $j = 1, \ldots, N$.

From (12), (14) and (16), $q_1 q_2 \cdots q_N = A_1 A_2 \cdots A_N$ holds if we define $A_j = A_{N+m-1-j}$ for $j \geq n+1$. Hence

$$F = \left(\prod_{j=1}^{N} A_j(1-\theta_j)\right)^{1/N}. \tag{34}$$

Hereafter, we always consider the problem in the region $A_i(1-\theta_j) > 0$ for $j = 1, 2, \ldots, N$.

The integrability of (21) is just (5), which implies that (29) and (33) are compatible.

In summary, we have

Theorem 1. *Suppose p_j, q_j $(j = 1, \ldots, N)$ are given by (29) and (33) respectively where the θ_j's are defined by (32) and F is defined by (34). Let*

$$L(\lambda) = J + \frac{\kappa}{N}\sum_{\alpha=1}^{r}\sum_{l=0}^{N-1}\left(\frac{\omega^{(m-2)l}}{\lambda - \omega^l\lambda_\alpha}\Omega^l H_\alpha H_\alpha^* \Omega^l K - \frac{\omega^{(m-2)l}}{\lambda + \omega^l\lambda_\alpha}\Omega^l \overline{H}_\alpha \overline{H}_\alpha^* \Omega^l K\right) \tag{35}$$

where κ is a real constant; then the Lax equations (21) hold.

Note that $L(\lambda)$ is real when λ is purely imaginary.

Now we write down the expressions for the u_i's in terms of $\{\phi_{j\alpha}, \bar{\phi}_{j\alpha}\}$ according to (12), (14) and (16). They are:

$$\text{Case (I): } \mathrm{e}^{u_j} = F^j \prod_{k=0}^{j-1} A_k^{-1}(1-\theta_k)^{-1}, \tag{36}$$

$$\text{Case (II): } \mathrm{e}^{u_j} = F^j \prod_{k=0}^{j-1} A_k^{-1}(1-\theta_k)^{-1}, \tag{37}$$

$$\text{Case (III): } \mathrm{e}^{u_j} = F^{j-\frac{1}{2}} A_0^{-\frac{1}{2}}(1-\theta_0)^{-\frac{1}{2}} \prod_{k=1}^{j-1} A_k^{-1}(1-\theta_k)^{-1} \quad (j = 1, \ldots, n). \tag{38}$$

Here the product is 1 if its upper bound is smaller than its lower bound. Under the above constraints, the system (18) becomes

$$\partial_x \phi_{j\alpha} = \mathrm{i}\lambda_\alpha \phi_{j+1,\alpha} + \mathrm{i}\kappa\left(\langle \Phi_{m-j}, \Phi_j\rangle - \langle \Phi_j, \Phi_{m-j}\rangle\right)\phi_{j\alpha}, \tag{39}$$

$$\partial_t \phi_{j\alpha} = \frac{\left(\prod_{l=1}^{N} A_l\left(1 - \kappa\langle \Phi_{m-1-l}, \Lambda^{-1}\Phi_l\rangle - \kappa\langle \Phi_l, \Lambda^{-1}\Phi_{m-1-l}\rangle\right)\right)^{\frac{1}{N}}}{\mathrm{i}\lambda_\alpha\left(1 - \kappa\langle \Phi_{m-j}, \Lambda^{-1}\Phi_{j-1}\rangle - \kappa\langle \Phi_{j-1}, \Lambda^{-1}\Phi_{m-j}\rangle\right)}\phi_{j-1,\alpha}. \tag{40}$$

Theorem 2. *If $\{\phi_{j\alpha}\}$ is a solution of (39) and (40), $P = (p_i\delta_{ij})$, $Q = (q_i\delta_{i,j+1})$ are given by (29) and (33) respectively, then (P, Q) satisfies the nonlinear partial differential equations (5).*

Proof. By (29) and (39),

$$\begin{aligned} p_{j,t} &= q_{j-1}\theta_{j-1} - q_j\theta_j = q_j(1-\theta_j) - q_{j-1}(1-\theta_{j-1}) + q_{j-1} - q_j \\ &= q_{j-1} - q_j. \end{aligned} \tag{41}$$

On the other hand, by (32) and (40),

$$\theta_{j,x} = (p_{j+1} - p_j)(1-\theta_j). \tag{42}$$

Hence

$$\begin{aligned} (\ln q_j)_t &= \frac{\theta_{j,x}}{1-\theta_j} + \frac{1}{N}\sum_{k=1}^{N}\frac{\theta_{k,x}}{1-\theta_k} \\ &= p_{j+1} - p_j + \frac{1}{N}\sum_{k=1}^{N}(p_{k+1} - p_k) = p_{j+1} - p_j. \end{aligned} \tag{43}$$

This leads to (10), which is equivalent to (5). The theorem is proved. □

From this theorem, we know that any solution of (39) and (40) gives a solution of the two dimensional $A_{2n}^{(2)}$, $C_n^{(1)}$ or $D_{n+1}^{(2)}$ Toda equations according to the suitable choice of N and m.

Now we consider the Hamiltonian structure of (39) and (40). Considering (iii) of Lemma 1, we can take the symplectic form in $\mathbf{R}^{2Nr}$ with coordinates $\{\phi_{j\alpha}, \bar{\phi}_{j\alpha}\}$ as

$$\omega = \mathrm{i}\sum_{\alpha=1}^{r}\sum_{j=1}^{N}\mathrm{d}\bar{\phi}_{j\alpha}\wedge \mathrm{d}\phi_{m-j,\alpha}, \tag{44}$$

and then the Poisson bracket of two functions f and g on $\mathbf{R}^{2Nr}$ is

$$\{f, g\} = \frac{1}{\mathrm{i}}\sum_{\alpha=1}^{r}\sum_{j=1}^{N}\left(\frac{\partial f}{\partial\phi_{j\alpha}}\frac{\partial g}{\partial\bar{\phi}_{m-j,\alpha}} - \frac{\partial g}{\partial\phi_{j\alpha}}\frac{\partial f}{\partial\bar{\phi}_{m-j,\alpha}}\right). \tag{45}$$

Suppose the systems in (18) are the Hamiltonian equations

$$\mathrm{i}\partial_x\phi_{j\alpha} = \frac{\partial H^x}{\partial\bar{\phi}_{m-j,\alpha}}, \qquad -\mathrm{i}\partial_x\bar{\phi}_{j\alpha} = \frac{\partial H^x}{\partial\phi_{m-j,\alpha}}, \tag{46}$$

and

$$\mathrm{i}\partial_t\phi_{j\alpha} = \frac{\partial H^t}{\partial\bar{\phi}_{m-j,\alpha}}, \qquad -\mathrm{i}\partial_t\bar{\phi}_{j\alpha} = \frac{\partial H^t}{\partial\phi_{m-j,\alpha}} \tag{47}$$

$(j = 1, \ldots, N)$ respectively; then we can integrate them to get the Hamiltonians in the following theorem.

Theorem 3. *The systems of ordinary differential equations (39) and (40) are Hamiltonian systems with the Hamiltonians*

$$H^x = -\sum_{j=1}^{N}\langle\Phi_{m-j}, \Lambda\Phi_{j+1}\rangle - \frac{\kappa}{4}\sum_{j=1}^{N}\big(\langle\Phi_{m-j}, \Phi_j\rangle - \langle\Phi_j, \Phi_{m-j}\rangle\big)^2, \tag{48}$$

$$\begin{aligned} H^t &= -\frac{N}{2\kappa}F \\ &= -\frac{N}{2\kappa}\left(\prod_{j=1}^{N}A_j\left(1-\kappa\langle\Phi_{m-1-j}, \Lambda^{-1}\Phi_j\rangle - \kappa\langle\Phi_j, \Lambda^{-1}\Phi_{m-1-j}\rangle\right)\right)^{\frac{1}{N}} \end{aligned} \tag{49}$$

respectively.

Note that both H^x and H^t are real-valued functions. We shall show in Section 5 that H^x and H^t are in involution. Therefore, the two Hamiltonian flows given by H^x and H^t are compatible.

4. Involution of conserved integrals

We shall show that $\mathrm{tr}(L^k(\lambda))$ $(k = 1, 2, \ldots)$ generate a series of quantities which are independent of λ and will serve as conserved integrals of the Hamiltonian systems given by H^x and H^t.

For any integer k, denote as $\{k\}$ the remainder of k divided by N.

Lemma 2. *For any integer p, complex numbers λ and ζ with $\zeta^N \neq \lambda^N$,*

$$\sum_{j=0}^{N-1}\frac{\omega^{-pj}}{\zeta-\omega^j\lambda} = \frac{N\lambda^{\{p\}}\zeta^{N-1-\{p\}}}{\zeta^N-\lambda^N}. \tag{50}$$

Proof. When $|\zeta| > |\lambda|$,

$$\sum_{j=0}^{N-1} \frac{\omega^{-pj}}{\zeta - \omega^j \lambda} = \sum_{l=0}^{\infty} \sum_{j=0}^{N-1} \frac{1}{\zeta} \left(\frac{\lambda}{\zeta}\right)^l \omega^{(l-p)j} = \frac{N}{\zeta} \sum_{\substack{l \geq 0 \\ l-p \equiv 0 \bmod N}} \left(\frac{\lambda}{\zeta}\right)^l = \frac{N\lambda^{\{p\}} \zeta^{N-1-\{p\}}}{\zeta^N - \lambda^N}. \tag{51}$$

Both sides of (50) are meromorphic functions of ζ. Hence (50) holds identically outside the poles. □

Theorem 4. *For any complex numbers λ, μ and any positive integers k, l,*

$$\{\mathrm{tr}(L^k(\lambda)), \mathrm{tr}(L^l(\mu))\} = 0. \tag{52}$$

Proof. In this proof, we use a, b, j, p, q, s for the indices from 1 to N and σ for the index from 1 to r. By the definition of the Poisson bracket,

$$\begin{aligned} \frac{\mathrm{i}}{kl} \{\mathrm{tr}(L^k(\lambda)), \mathrm{tr}(L^l(\mu))\} = & \sum_{j,\sigma} \mathrm{tr}\left(L^{k-1}(\lambda) \frac{\partial L(\lambda)}{\partial \phi_{j\sigma}}\right) \mathrm{tr}\left(L^{l-1}(\mu) \frac{\partial L(\mu)}{\partial \bar{\phi}_{m-j,\sigma}}\right) \\ & - \sum_{j,\sigma} \mathrm{tr}\left(L^{l-1}(\mu) \frac{\partial L(\mu)}{\partial \phi_{j\sigma}}\right) \mathrm{tr}\left(L^{k-1}(\lambda) \frac{\partial L(\lambda)}{\partial \bar{\phi}_{m-j,\sigma}}\right). \end{aligned} \tag{53}$$

By the expression (35),

$$\begin{aligned} \mathrm{tr}\left(L^{k-1}(\lambda) \frac{\partial L(\lambda)}{\partial \phi_{j\sigma}}\right) = & \sum_{a,b,p,s} \frac{\kappa}{N} (L^{k-1}(\lambda))_{ba} \left(\frac{\omega^{(m-2)p} \omega^{-(a-1)p}}{\lambda - \omega^p \lambda_\sigma} \delta_{ja} \bar{\phi}_{s\sigma} \omega^{-(s-1)p} \right. \\ & \left. - \frac{\omega^{(m-2)p} \omega^{-(a-1)p}}{\lambda + \omega^p \lambda_\sigma} \bar{\phi}_{a\sigma} \delta_{js} \omega^{-(s-1)p} \right) \delta_{s,m-b} \\ = & \frac{\kappa}{N} \sum_{b,p} (L^{k-1}(\lambda))_{bj} \frac{\omega^{(b-j)p}}{\lambda - \omega^p \lambda_\sigma} \bar{\phi}_{m-b,\sigma} - \frac{\kappa}{N} \sum_{a,p} (L^{k-1}(\lambda))_{m-j,a} \frac{\omega^{(m-j-a)p}}{\lambda + \omega^p \lambda_\sigma} \bar{\phi}_{a\sigma} \\ = & \frac{\kappa}{N} \sum_{a,p} (L^{k-1}(\lambda))_{m-a,j} \frac{\omega^{(m-a-j)p}}{\lambda - \omega^p \lambda_\sigma} \bar{\phi}_{a\sigma} - \frac{\kappa}{N} \sum_{a,p} (L^{k-1}(\lambda))_{m-j,a} \frac{\omega^{(m-a-j)p}}{\lambda + \omega^p \lambda_\sigma} \bar{\phi}_{a\sigma} \\ \equiv & A_1^{(j\sigma)} - B_1^{(j\sigma)}. \end{aligned} \tag{54}$$

Likewise,

$$\begin{aligned} \mathrm{tr}\left(L^{l-1}(\mu) \frac{\partial L(\mu)}{\partial \bar{\phi}_{m-j,\sigma}}\right) = & \frac{\kappa}{N} \sum_{b,q} (L^{l-1}(\mu))_{jb} \frac{\omega^{(j-b)q}}{\mu - \omega^q \lambda_\sigma} \phi_{b\sigma} \\ & - \frac{\kappa}{N} \sum_{b,q} (L^{l-1}(\mu))_{m-b,m-j} \frac{\omega^{(j-b)q}}{\mu + \omega^q \lambda_\sigma} \phi_{b\sigma} \equiv A_2^{(j\sigma)} - B_2^{(j\sigma)}, \end{aligned} \tag{55}$$

$$\begin{aligned} \mathrm{tr}\left(L^{l-1}(\mu) \frac{\partial L(\mu)}{\partial \phi_{j\sigma}}\right) = & \frac{\kappa}{N} \sum_{b,q} (L^{l-1}(\mu))_{m-b,j} \frac{\omega^{(m-b-j)q}}{\mu - \omega^q \lambda_\sigma} \bar{\phi}_{b\sigma} \\ & - \frac{\kappa}{N} \sum_{b,q} (L^{l-1}(\mu))_{m-j,b} \frac{\omega^{(m-b-j)q}}{\mu + \omega^q \lambda_\sigma} \bar{\phi}_{b\sigma} \equiv A_3^{(j\sigma)} - B_3^{(j\sigma)}, \end{aligned} \tag{56}$$

$$\begin{aligned} \mathrm{tr}\left(L^{k-1}(\lambda) \frac{\partial L(\lambda)}{\partial \bar{\phi}_{m-j,\sigma}}\right) = & \frac{\kappa}{N} \sum_{a,p} (L^{k-1}(\lambda))_{ja} \frac{\omega^{(j-a)p}}{\lambda - \omega^p \lambda_\sigma} \phi_{a\sigma} \\ & - \frac{\kappa}{N} \sum_{a,p} (L^{k-1}(\lambda))_{m-a,m-j} \frac{\omega^{(j-a)p}}{\lambda + \omega^p \lambda_\sigma} \phi_{a\sigma} \equiv A_4^{(j\sigma)} - B_4^{(j\sigma)}. \end{aligned} \tag{57}$$

Writing

$$
\begin{aligned}
L^{(1)} &= \frac{\kappa}{N}\sum_{\alpha=1}^{r}\sum_{l=0}^{N-1}\frac{\omega^{(m-2)l}}{\lambda-\omega^l\lambda_\alpha}\Omega^l H_\alpha H_\alpha^*\Omega^l K,\\
L^{(2)} &= \frac{\kappa}{N}\sum_{\alpha=1}^{r}\sum_{l=0}^{N-1}\frac{\omega^{(m-2)l}}{\lambda+\omega^l\lambda_\alpha}\Omega^l \overline{H}_\alpha \overline{H}_\alpha^*\Omega^l K,
\end{aligned}
\tag{58}
$$

then

$$
L(\lambda) = J + L^{(1)} - L^{(2)}. \tag{59}
$$

From (22), $\mathrm{tr}(L^k(\omega\lambda)) = \omega^{-k}\mathrm{tr}(L^k(\lambda))$, $\mathrm{tr}(L^k(-\lambda)) = \mathrm{tr}(L^k(\lambda))$. Hence the theorem holds for $\mu^N = (\pm\lambda)^N$. We only need to prove the theorem for $\mu^N \neq (\pm\lambda)^N$.

Using the identity

$$
\frac{1}{(\lambda-\epsilon_1\omega^p\lambda_\sigma)(\mu-\epsilon_2\omega^q\lambda_\sigma)} = \frac{1}{\epsilon_1\omega^p\mu-\epsilon_2\omega^q\lambda}\left(\frac{\epsilon_1\omega^p}{\lambda-\epsilon_1\omega^p\lambda_\sigma}-\frac{\epsilon_2\omega^q}{\mu-\epsilon_2\omega^q\lambda_\sigma}\right), \tag{60}
$$

for $\epsilon_1, \epsilon_2 = \pm 1$ and Lemma 2, we have

$$
\begin{aligned}
\sum_\sigma A_1^{(j\sigma)}A_2^{(j\sigma)} &= \frac{\kappa}{\mu^N-\lambda^N}\sum_b (L^{l-1}(\mu))_{jb}(L^{(1)}(\lambda)L^{k-1}(\lambda))_{bj}\lambda^{\{b-j\}}\mu^{N-1-\{b-j\}}\\
&\quad -\frac{\kappa}{\mu^N-\lambda^N}\sum_a (L^{k-1}(\lambda))_{aj}(L^{l-1}(\mu)L^{(1)}(\mu))_{ja}\lambda^{N-1-\{j-a\}}\mu^{\{j-a\}},\\
\sum_\sigma B_3^{(m-j,\sigma)}B_4^{(m-j,\sigma)} &= \frac{\kappa}{\mu^N-\lambda^N}\sum_b (L^{l-1}(\mu))_{jb}(L^{(2)}(\lambda)L^{k-1}(\lambda))_{bj}\lambda^{\{b-j\}}\mu^{N-1-\{b-j\}}\\
&\quad -\frac{\kappa}{\mu^N-\lambda^N}\sum_a (L^{k-1}(\lambda))_{aj}(L^{l-1}(\mu)L^{(2)}(\mu))_{ja}\lambda^{N-1-\{j-a\}}\mu^{\{j-a\}}.
\end{aligned}
\tag{61}
$$

Hence

$$
\begin{aligned}
\sum_{j,\sigma}(A_1^{(j\sigma)}A_2^{(j\sigma)} - B_3^{(j\sigma)}B_4^{(j\sigma)}) &= \frac{\kappa}{\mu^N-\lambda^N}\sum_{b,j}((L(\lambda)-J)L^{k-1}(\lambda))_{bj}(L^{l-1}(\mu))_{jb}\lambda^{\{b-j\}}\mu^{N-1-\{b-j\}}\\
&\quad -\frac{\kappa}{\mu^N-\lambda^N}\sum_{a,j}(L^{k-1}(\lambda))_{aj}(L^{l-1}(\mu)(L(\mu)-J))_{ja}\lambda^{N-1-\{j-a\}}\mu^{\{j-a\}}.
\end{aligned}
\tag{62}
$$

Likewise, we have

$$
\begin{aligned}
\sum_{j,\sigma}(A_1^{(j\sigma)}B_2^{(j\sigma)} - A_3^{(j\sigma)}B_4^{(j\sigma)}) &= \frac{\kappa}{\mu^N-(-\lambda)^N}\sum_{b,j}((L(\lambda)-J)L^{k-1}(\lambda))_{bj}(L^{l-1}(\mu))_{m-b,m-j}\\
&\quad \cdot(-\lambda)^{\{b-j\}}\mu^{N-1-\{b-j\}}\\
&\quad -\frac{\kappa}{\mu^N-(-\lambda)^N}\sum_{a,j}(L^{k-1}(\lambda))_{aj}((L(\mu)-J)L^{l-1}(\mu))_{m-a,m-j}\\
&\quad \cdot(-\lambda)^{N-1-\{j-a\}}\mu^{\{j-a\}}\\
\sum_{j,\sigma}(A_2^{(j\sigma)}B_1^{(j\sigma)} - A_4^{(j\sigma)}B_3^{(j\sigma)}) &= -\frac{\kappa}{\mu^N-(-\lambda)^N}\sum_{b,j}(L^{k-1}(\lambda)(L(\lambda)-J))_{jb}(L^{l-1}(\mu))_{m-j,m-b}\\
&\quad \cdot(-\lambda)^{\{j-b\}}\mu^{N-1-\{j-b\}}\\
&\quad +\frac{\kappa}{\mu^N-(-\lambda)^N}\sum_{a,j}(L^{k-1}(\lambda))_{ja}(L^{l-1}(\mu)(L(\mu)-J))_{m-j,m-a}\\
&\quad \cdot(-\lambda)^{N-1-\{a-j\}}\mu^{\{a-j\}}
\end{aligned}
$$

$$\begin{aligned}\sum_{j,\sigma}(B_1^{(j\sigma)}B_2^{(j\sigma)}-A_3^{(j\sigma)}A_4^{(j\sigma)}) &= -\frac{\kappa}{\mu^N-\lambda^N}\sum_{b,j}(L^{k-1}(\lambda)(L(\lambda)-J))_{jb}(L^{l-1}(\mu))_{bj}\lambda^{\{j-b\}}\mu^{N-1-\{j-b\}}\\ &\quad+\frac{\kappa}{\mu^N-\lambda^N}\sum_{a,j}(L^{k-1}(\lambda))_{ja}((L(\mu)-J)L^{l-1}(\mu))_{aj}\lambda^{N-1-\{a-j\}}\mu^{\{a-j\}}.\end{aligned}$$

Defining

$$\Gamma_1=\frac{\lambda^N-\mu^N}{\kappa}\sum_{j,\sigma}(A_1^{(j\sigma)}A_2^{(j\sigma)}-B_3^{(j\sigma)}B_4^{(j\sigma)}+B_1^{(j\sigma)}B_2^{(j\sigma)}-A_3^{(j\sigma)}A_4^{(j\sigma)}), \tag{63}$$

then

$$\begin{aligned}\Gamma_1 &= \sum_{b,j}[J,L^{k-1}(\lambda)]_{bj}(L^{l-1}(\mu))_{jb}\lambda^{\{b-j\}}\mu^{N-1-\{b-j\}}\\ &\quad+\sum_{a,j}(L^{k-1}(\lambda))_{aj}[J,L^{l-1}(\mu)]_{ja}\lambda^{N-1-\{j-a\}}\mu^{\{j-a\}}\\ &= \sum_{b,j}(L^{k-1}(\lambda))_{b+1,j}(L^{l-1}(\mu))_{jb}\lambda^{\{b-j\}}\mu^{N-1-\{b-j\}}\\ &\quad-\sum_{a,j}(L^{k-1}(\lambda))_{aj}(L^{l-1}(\mu))_{j,a-1}\lambda^{N-1-\{j-a\}}\mu^{\{j-a\}}\\ &\quad-\sum_{b,j}(L^{k-1}(\lambda))_{b,j-1}(L^{l-1}(\mu))_{jb}\lambda^{\{b-j\}}\mu^{N-1-\{b-j\}}\\ &\quad+\sum_{a,j}(L^{k-1}(\lambda))_{aj}(L^{l-1}(\mu))_{j+1,a}\lambda^{N-1-\{j-a\}}\mu^{\{j-a\}}.\end{aligned} \tag{64}$$

Hence

$$\begin{aligned}\Gamma_1 &= \sum_{b,j}\left((L^{k-1}(\lambda))_{b+1,j}-(L^{k-1}(\lambda))_{b,j-1}\right)(L^{l-1}(\mu))_{jb}\\ &\quad\cdot\left(\lambda^{\{b-j\}}\mu^{N-1-\{b-j\}}-\lambda^{N-1-\{j-b-1\}}\mu^{\{j-b-1\}}\right).\end{aligned} \tag{65}$$

Noticing that $\{k\}+\{-k-1\}=N-1$ holds for any integer k, we get $\Gamma_1=0$.

Similarly,

$$\begin{aligned}\Gamma_2 &\equiv \frac{(-\lambda)^N-\mu^N}{\kappa}\sum_{j,\sigma}\left(A_1^{(j\sigma)}B_2^{(j\sigma)}-A_3^{(j\sigma)}B_4^{(j\sigma)}+A_2^{(j\sigma)}B_1^{(j\sigma)}-A_4^{(j\sigma)}B_3^{(j\sigma)}\right)\\ &= \sum_{b,j}\left((L^{k-1}(\lambda))_{b+1,j}-(L^{k-1}(\lambda))_{b,j-1}\right)(L^{l-1}(\mu))_{m-b,m-j}\\ &\quad\cdot\left((-\lambda)^{\{b-j\}}\mu^{N-1-\{b-j\}}-(-\lambda)^{N-1-\{j-b-1\}}\mu^{\{j-b-1\}}\right)=0.\end{aligned} \tag{66}$$

Therefore,

$$\frac{\mathrm{i}}{kl}\{\mathrm{tr}(L(\lambda)^k),\mathrm{tr}(L(\mu))^l\}=\frac{\kappa}{\lambda^N-\mu^N}\Gamma_1-\frac{\kappa}{(-\lambda)^N-\mu^N}\Gamma_2=0. \tag{67}$$

The theorem is proved. □

5. Expressions for conserved integrals

Expanding $\mathrm{tr}(L^k(\lambda))$ as a Laurent series in λ like

$$\mathrm{tr}(L^k(\lambda))=\sum_{j=0}^{\infty}g_j^{(k)}\lambda^{-j}, \tag{68}$$

then Theorem 4 implies that $\{g_i^{(k)}, g_j^{(k)}\} = 0$ for all positive integers i, j, k.

Supposing the eigenvalues of $L(\lambda)$ are $\nu_1(\lambda), \ldots, \nu_N(\lambda)$, then

$$\mathrm{tr}(L^k(\lambda)) = \sum_{j=1}^{N} \nu_j^k(\lambda). \tag{69}$$

On the other hand,

$$\det(\mu I - L(\lambda)) = \sum_{j=0}^{N} (-1)^j s_j(\lambda) \mu^{N-j} \tag{70}$$

where $s_0 = 1$,

$$s_k(\lambda) = \sum_{1 \le j_1 < \cdots < j_k \le N} \nu_{j_1}(\lambda) \cdots \nu_{j_k}(\lambda) \tag{71}$$

is the sum of all the determinants of the principal submatrices of $L(\lambda)$ of order k.

Using the fact $\lim_{\lambda\to\infty} \det(\mu I - L(\lambda)) = \det(\mu I - J) = \mu^N - 1$, we can perform the expansion

$$s_k(\lambda) = (-1)^{k-1} \sum_{j=1}^{+\infty} \lambda^{-j} \mathfrak{E}_j^{(k)} + (-1)^{N-1} \delta_{kN} \quad (k = 1, 2, \ldots, N). \tag{72}$$

Since the set of symmetric polynomials

$$\left\{ \sum_{j=1}^{N} \nu_j^k \mid k = 1, 2, \ldots, N \right\} \quad \text{and} \quad \left\{ \sum_{1 \le j_1 < \cdots < j_k \le N} \nu_{j_1} \cdots \nu_{j_k} \mid k = 1, 2, \ldots, N \right\}$$

can be represented by each other, $\{\mathfrak{E}_i^{(k)}, \mathfrak{E}_j^{(k)}\} = 0$ holds for all positive integers i, j, k.

Lemma 3. (i) $s_k(-\lambda) = s_k(\lambda)$ *for any positive integer* k. (ii) $\mathfrak{E}_j^{(k)} = 0$ *unless* $j \equiv k \bmod N$. *Moreover, when* N *is even,* $\mathfrak{E}_j^{(k)} = 0$ *if* k *is odd.*

Proof. By (22),

$$L(\lambda) = K^{-1} L(\bar{\lambda})^* K = K^{-1} L(-\lambda)^{\mathrm{T}} K. \tag{73}$$

Hence $(\nu_1(-\lambda), \ldots, \nu_N(-\lambda))$ is a full set of eigenvalues of $L(\lambda)$, which implies that (i) holds.

Again, from (22), $L(\omega\lambda) = \omega^{-1} \Omega L(\lambda) \Omega^{-1}$. Hence $(\omega^{-1}\nu_1(\lambda), \ldots, \omega^{-1}\nu_N(\lambda))$ is a full set of eigenvalues of $L(\omega\lambda)$. This means that

$$s_k(\omega\lambda) = \omega^{-k} s_k(\lambda). \tag{74}$$

By the expression (72), $\mathfrak{E}_j^{(k)} = 0$ unless $j \equiv k \bmod N$.

According to (74), when N is even,

$$s_k(-\lambda) = s_k(\omega^{N/2}\lambda) = \omega^{-kN/2} s_k(\lambda) = (-1)^k s_k(\lambda). \tag{75}$$

Comparing with (i), we get $s_k(\lambda) = 0$ if k is odd. The lemma is proved. □

The $\mathfrak{E}_j^{(k)}$'s are polynomials of $\{\phi_{i\alpha}, \bar{\phi}_{i\alpha} \mid i = 1, \ldots, N; \alpha = 1 \ldots, r\}$. In general, the full expressions are very complicated. Here we first consider their lowest (i.e. quadratic) terms. Then consider the full expressions for special cases which will lead to the Hamiltonians H^x and H^t.

Now write $L(\lambda) = J + M(\lambda)$ where each entry of $M(\lambda)$ is a polynomial of $\{\phi_{i\alpha}, \bar{\phi}_{i\alpha}\}$ of degree ≥ 2. Suppose R is a principal submatrix of $L(\lambda)$ of order k ($k \le N-1$) with non-vanishing quadratic terms of $\{\phi_{i\alpha}, \bar{\phi}_{i\alpha}\}$ in its determinant. Considering the expression for J, the indices of the rows and columns of R in $L(\lambda)$ must be $(j, j+1, \ldots, j+k-1)$

for $j+k \le N+1$ and $(1,\ldots,k+j-N-1,j,\ldots,N)$ for $j+k > N+1$. Let $\hat{R}_{\alpha\beta} = R_{j+\alpha-1,j+\beta-1}$; then $\hat{R} = (\hat{R}_{\alpha\beta})_{k\times k}$ has the form

$$\hat{R} = (M_{j+\alpha-1,j+\beta-1} + \delta_{\alpha+1,\beta})_{1\le\alpha,\beta\le k}, \tag{76}$$

and

$$\det R = \det \hat{R} = (-1)^{k+1} M_{j+k-1,j} + \text{h.o.t.} \tag{77}$$

Here h.o.t. refers to the polynomial of $\{\phi_{i\alpha}, \bar{\phi}_{i\alpha}\}$ of degree > 2. Hence,

$$(-1)^{k-1} s_k = \delta_{kN} + \sum_{j=1}^{N} M_{j+k-1,j} + \text{h.o.t.} \tag{78}$$

for $k = 1, 2, \ldots, N-1$. It is easy to see that (78) is also true for $k = N$ since $s_N = \det(J + M(\lambda))$.

Define

$$S_{j,k}^{(l)} = \langle \Phi_k, \Lambda^l \Phi_j\rangle + \langle \Phi_j, \Lambda^l \Phi_k\rangle, \qquad A_{j,k}^{(l)} = \mathrm{i}(\langle \Phi_k, \Lambda^l \Phi_j\rangle - \langle \Phi_j, \Lambda^l \Phi_k\rangle). \tag{79}$$

From (35),

$$\begin{aligned}\sum_{j=1}^{N} M_{j+k-1,j} &= \frac{\kappa}{N} \sum_{\alpha=1}^{r} \sum_{s=0}^{\infty} \sum_{j=1}^{N} \sum_{l=0}^{N-1} \lambda^{-s-1} \lambda_\alpha^s \omega^{(s-k+1)l} \left(\phi_{j+k-1,\alpha}\bar{\phi}_{m-j,\alpha} - (-1)^s \bar{\phi}_{j+k-1,\alpha}\phi_{m-j,\alpha}\right) \\ &= \kappa \sum_{\substack{p \\ k-1+pN\ge 0}} \lambda^{-k-pN} \sum_{j=1}^{N} \left(\langle \Phi_{m-j}, \Lambda^{k-1+pN} \Phi_{j+k-1}\rangle + (-1)^{k+pN} \langle \Phi_{j+k-1}, \Lambda^{k-1+pN} \Phi_{m-j}\rangle\right) \\ &= \kappa \sum_{\substack{p \\ k-1+pN\ge 0}} \lambda^{-k-pN} \sum_{j=1}^{N} \left(1 + (-1)^{k+pN}\right) \langle \Phi_{m+k-1-j}, \Lambda^{k-1+pN} \Phi_j\rangle.\end{aligned} \tag{80}$$

When both N and k are odd, $1 + (-1)^{k+pN} \ne 0$ if and only if p is odd. Let $k = 2l-1$, $p = 2q-1$; then

$$s_{2l-1} = \kappa \sum_{\substack{q \\ 2l-2+(2q-1)N\ge 0}} \lambda^{-2l+1-(2q-1)N} \sum_{j=1}^{N} S_{j,m+2l-2-j}^{(2l-2+(2q-1)N)} + \delta_{2l-1,N} + \text{h.o.t.} \tag{81}$$

Since N is odd, in the first summation, $2l-2+(2q-1)N \ge 1$ should hold, i.e. $l - \frac{N+3}{2} + qN \ge 0$. Supposing $l - \frac{N+3}{2} = (\rho - q)N + \left\{l - \frac{N+3}{2}\right\}$ where $\rho \in \mathbf{Z}$, then ρ should satisfy $\rho \ge 0$, and

$$2l - 2 + (2q-1)N = 2\left(l - \frac{N+3}{2}\right) + 2qN + 1 = 2\rho N + 1 + 2\left\{l - \frac{N+3}{2}\right\}. \tag{82}$$

Rewriting l as k, ρ as p, we get

$$s_{2k-1} = \kappa \sum_{p=0}^{\infty} \lambda^{-2-2\left\{k-\frac{N+3}{2}\right\}-2pN} \sum_{j=1}^{N} S_{j,m+2k-2-j}^{(2\left\{k-\frac{N+3}{2}\right\}+1+2pN)} + \delta_{2k-1,N} + \text{h.o.t.} \tag{83}$$

Likewise, when N is odd, the even term is

$$s_{2k} = -\kappa \sum_{p=0}^{\infty} \lambda^{-2-2\{k-1\}-2pN} \sum_{j=1}^{N} S_{j,m+2k-1-j}^{(2\{k-1\}+1+2pN)} + \text{h.o.t.} \tag{84}$$

When N is even,

$$s_{2k-1} = 0,$$

$$s_{2k} = -\kappa \sum_{p=0}^{\infty} \lambda^{-2-\{2k-2\}-pN} \sum_{j=1}^{N} S_{j,m+2k-1-j}^{(\{2k-2\}+1+pN)} - \delta_{2k,N} + \text{h.o.t.} \tag{85}$$

Now we redefine the conserved quantities as follows.

For odd N, we define

$$E_p^{(k)} = \frac{1}{2\kappa}\mathfrak{E}_{2+2\{k-1\}+2pN}^{(2k)} = \frac{1}{2}\sum_{j=1}^{N} S_{j,m+2k-1-j}^{(2\{k-1\}+1+2pN)} + \text{h.o.t.} \quad (k = 1, 2, \ldots, (N-1)/2), \tag{86}$$

$$E_p^{(\frac{N-1}{2}+k)} = \frac{1}{2\kappa}\mathfrak{E}_{2+2\left\{k-\frac{N+3}{2}\right\}+2pN}^{(2k-1)} = \frac{1}{2}\sum_{j=1}^{N} S_{j,m+2k-2-j}^{\left(2\left\{k-\frac{N+3}{2}\right\}+1+2pN\right)} + \text{h.o.t.} \quad (k = 1, 2, \ldots, (N+1)/2), \tag{87}$$

and then the above two expressions can be written uniformly as

$$E_p^{(k)} = \frac{1}{2}\sum_{j=1}^{N} S_{j,m+2k-1-j}^{(2\{k-1\}+1+2pN)} + \text{h.o.t.} \quad (k = 1, 2, \ldots, N). \tag{88}$$

For even N, we define

$$E_p^{(k)} = \frac{1}{2\kappa}\mathfrak{E}_{2+\{2k-2\}+pN}^{(2k)} = \frac{1}{2}\sum_{j=1}^{N} S_{j,m+2k-1-j}^{(\{2k-1\}+pN)} + \text{h.o.t.} \quad (k = 1, 2, \ldots, N). \tag{89}$$

Now we express the Hamiltonians H^x and H^t in terms of the $E_j^{(k)}$'s. Then the $E_j^{(k)}$'s are a set of involutive conserved integrals of the Hamiltonian systems given by H^x and H^t.

Theorem 5. *The Hamiltonians in (48) and (49) are*

$$H^x = -E_0^{(1)}, \qquad H^t = -\frac{N}{2\kappa}\left(\frac{1}{N}\left(\prod_{j=1}^{N} A_j\right)\text{tr}(L^N(0))\right)^{1/N}. \tag{90}$$

Both H^x and H^t commute with all $E_j^{(k)}$'s, and $\{H^x, H^t\} = 0$.

Proof. When $N \geq 3$,

$$\begin{aligned} s_2(\lambda) &= \sum_{j=1}^{N-1}\begin{vmatrix} M_{jj} & 1+M_{j,j+1} \\ M_{j+1,j} & M_{j+1,j+1}\end{vmatrix} + \begin{vmatrix} M_{11} & M_{1N} \\ 1+M_{N1} & M_{NN}\end{vmatrix} + \sum_{\substack{1\le j<k\le N\\ k-j\ge 2,\,(j,k)\ne(1,N)}}\begin{vmatrix} M_{jj} & M_{jk} \\ M_{kj} & M_{kk}\end{vmatrix} \\ &= -\sum_{j=1}^{N} M_{j+1,j} + \sum_{1\le j<k\le N}\begin{vmatrix} M_{jj} & M_{jk} \\ M_{kj} & M_{kk}\end{vmatrix}. \end{aligned} \tag{91}$$

When $N = 2$,

$$s_2(\lambda) = \begin{vmatrix} M_{11} & 1+M_{12} \\ 1+M_{21} & M_{22}\end{vmatrix} = -(M_{12}+M_{21}) + \begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{vmatrix} - 1. \tag{92}$$

By (80),

$$-\sum_{j=1}^{N} M_{j+1,j} = -\kappa\lambda^{-2}\sum_{j=1}^{N} S_{j,m+1-j}^{(1)} + o(\lambda^{-2}), \tag{93}$$

and by (35),

$$\begin{aligned} M_{jk} &= \frac{\kappa}{N}\lambda^{-1}\sum_{\alpha=1}^{r}\sum_{l=0}^{N-1}\omega^{(k-j)l}(\phi_{j\alpha}\bar{\phi}_{m-k,\alpha}-\bar{\phi}_{j\alpha}\phi_{m-k,\alpha})+o(\lambda^{-1}) \\ &= -\mathrm{i}\kappa\lambda^{-1}A^{(0)}_{j,m-j}\delta_{jk}+o(\lambda^{-1}). \end{aligned} \tag{94}$$

Hence, if $j \neq k$,

$$\begin{vmatrix} M_{jj} & M_{jk} \\ M_{kj} & M_{kk} \end{vmatrix} = -\kappa^2\lambda^{-2}\begin{vmatrix} A_{j,m-j} & 0 \\ 0 & A_{k,m-k} \end{vmatrix} + o(\lambda^{-2}) = -\kappa^2\lambda^{-2}A_{j,m-j}A_{k,m-k}+o(\lambda^{-2}). \tag{95}$$

Thus

$$\begin{aligned} \sum_{1\le j<k\le N}\begin{vmatrix} M_{jj} & M_{jk} \\ M_{kj} & M_{kk} \end{vmatrix} &= -\frac{1}{2}\kappa^2\lambda^{-2}\left(\left(\sum_{j=1}^{N}A^{(0)}_{j,m-j}\right)^2-\sum_{j=1}^{N}(A^{(0)}_{j,m-j})^2\right)+o(\lambda^{-2}) \\ &= \frac{1}{2}\kappa^2\lambda^{-2}\sum_{j=1}^{N}(A^{(0)}_{j,m-j})^2+o(\lambda^{-2}). \end{aligned} \tag{96}$$

The last equality holds because $A^{(0)}_{j,k}$ is anti-symmetric for j and k. Using the expansion (72), we get

$$\mathfrak{E}_2^{(2)} = \kappa\sum_{j=1}^{N}S^{(1)}_{j,m+1-j}-\frac{1}{2}\kappa^2\sum_{j=1}^{N}(A^{(0)}_{j,m-j})^2. \tag{97}$$

Comparing with (48), we know that $H^x = -\frac{1}{2\kappa}\mathfrak{E}_2^{(2)} = -E_0^{(1)}$.

Now we prove the t-part. By (35),

$$L(0) = J-\frac{\kappa}{N}\sum_{\alpha=1}^{r}\sum_{l=0}^{N-1}\omega^{(m-3)l}\lambda_\alpha^{-1}\Omega^l(H_\alpha H_\alpha^*+\bar{H}_\alpha\bar{H}_\alpha^*)\Omega^l K, \tag{98}$$

and

$$\begin{aligned} (L(0))_{jk} &= \delta_{j+1,k}-\frac{\kappa}{N}\sum_{\alpha=1}^{r}\sum_{l=0}^{N-1}\omega^{(k-j-1)l}\lambda_\alpha^{-1}(\phi_{j\alpha}\bar{\phi}_{m-k,\alpha}+\bar{\phi}_{j\alpha}\phi_{m-k,\alpha}) \\ &= \delta_{j+1,k}-\kappa\sum_{\alpha=1}^{r}\lambda_\alpha^{-1}(\phi_{j\alpha}\bar{\phi}_{m-k,\alpha}+\bar{\phi}_{j\alpha}\phi_{m-k,\alpha})\delta_{j+1,k} \\ &= \left(1-\kappa S^{(-1)}_{j,m-1-j}\right)\delta_{j+1,k}. \end{aligned} \tag{99}$$

Taking the trace, we get

$$\mathrm{tr}(L^N(0)) = N\prod_{j=1}^{N}\left(1-\kappa S^{(-1)}_{j,m-1-j}\right) = N\prod_{j=1}^{N}(1-\theta_j) = NF^N\left(\prod_{j=1}^{N}A_j\right)^{-1}. \tag{100}$$

Hence, according to (49),

$$H^t = -\frac{N}{2\kappa}\left(\frac{1}{N}\left(\prod_{j=1}^{N}A_j\right)\mathrm{tr}(L^N(0))\right)^{1/N}. \tag{101}$$

According to Theorem 4, H^x and H^t commute with all $E_j^{(k)}$'s. In particular, since $H^x = -E_0^{(1)}$, we get $\{H^x, H^t\} = 0$. The theorem is proved. □

6. Independence of the conserved integrals

Theorem 6. $\{E_j^{(k)} \mid k = 1, \dots, N;\ j = 0, \dots, r-1\}$ *are functionally independent in a dense open subset of* $\mathbf{R}^{2Nr}$.

Proof. Since the $E_j^{(k)}$'s are defined in different ways for odd N and even N, we shall prove the theorem for these two cases separately.

First suppose N is odd. By (88), for $k = 1, \dots, N;\ j = 1, \dots, N;\ p = 0, \dots, r-1;\ \alpha = 1, \dots, r$,

$$\frac{\partial E_p^{(k)}}{\partial \bar{\phi}_{j\alpha}} = \lambda_\alpha^{2k-1+2pN} \phi_{m+2k-1-j,\alpha} + \cdots \tag{102}$$

where "$\cdots$" refers to the high order terms of the $\phi_{j\alpha}$'s.

Let $\mathfrak{J}$ be the Jacobian determinant of

$$(E_0^{(1)}, \dots, E_{r-1}^{(1)}, E_0^{(2)}, \dots, E_{r-1}^{(2)}, \dots, E_0^{(N)}, \dots, E_{r-1}^{(N)})$$

with respect to

$$(\bar{\phi}_{11}, \dots, \bar{\phi}_{1r}, \bar{\phi}_{21}, \dots, \bar{\phi}_{2r}, \dots, \bar{\phi}_{N1}, \dots, \bar{\phi}_{Nr}).$$

Take $P_0 \in \mathbf{R}^{2Nr}$ with coordinates $\phi_{j\alpha} = \epsilon\,\{j-m\}$ where ϵ is a small real number. Then, at P_0, for fixed k and j,

$$\left(\frac{\partial E_p^{(k)}}{\partial \bar{\phi}_{j\alpha}}\right)_{\substack{p=0,\dots,r-1\\ \alpha=1,\dots,r}} = \epsilon\,\{2k-j-1\}\, W + o(\epsilon) \tag{103}$$

where $W_k = (\lambda_\beta^{2k-1+2(\alpha-1)N})_{1\le\alpha,\beta\le r}$ for $k = 1, \dots, N$. Then

$$\begin{aligned} |\mathfrak{J}(P_0)| &= |\det(\epsilon \operatorname{diag}(W_1, \dots, W_N))| \\ &\quad \cdot \left|\det\begin{pmatrix} 0 & (N-1)I_r & (N-2)I_r & \cdots & 2I_r & I_r \\ 2I_r & I_r & 0 & \cdots & 4I_r & 3I_r \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ (N-2)I_r & (N-3)I_r & (N-4)I_r & \cdots & 0 & (N-1)I_r \end{pmatrix}\right| + o(\epsilon^{Nr}) \\ &= |\epsilon|^{Nr} \left|\prod_{l=1}^r \lambda_l\right|^{N^2} \left|\prod_{1\le i<j\le r} (\lambda_i^{2N} - \lambda_j^{2N})\right|^N \left|\frac{1}{2} N^{N-1}(N-1)\right|^r + o(\epsilon^{Nr}) \neq 0 \end{aligned} \tag{104}$$

when ϵ is small enough since $\lambda_1^{2N}, \dots, \lambda_r^{2N}$ are distinct and non-zero. Here I_r is the $r \times r$ identity matrix. Since $\mathfrak{J}$ is a real analytic function of $\{\phi_{j\alpha}, \bar{\phi}_{j\alpha}\}$, $\mathfrak{J} \neq 0$ in a dense open subset of $\mathbf{R}^{2Nr}$. This proves the theorem for odd N.

Now we prove the theorem for even N. By (89), for $k = 1, \dots, N;\ j = 1, \dots, N/2;\ p = 0, \dots, r-1;\ \alpha = 1, \dots, r$, we have

$$\begin{aligned} \frac{\partial E_p^{(k)}}{\partial \bar{\phi}_{j\alpha}} &= \lambda_\alpha^{\{2k-1\}+pN} \phi_{m+2k-1-j,\alpha} + \dots, \\ \frac{\partial E_p^{(k)}}{\partial \phi_{j\alpha}} &= \lambda_\alpha^{\{2k-1\}+pN} \bar{\phi}_{m+2k-1-j,\alpha} + \cdots. \end{aligned} \tag{105}$$

Take $P_0 \in \mathbf{R}^{2Nr}$ with coordinates $\phi_{j\alpha} = \epsilon \beta_{j-m}$ where

$$\beta_k = \begin{cases} \{k-1\}+1, & \text{if } 1 \le k \le N/2, \\ \mathrm{i}(\{k-1\}+1), & \text{if } N/2+1 \le k \le N, \end{cases} \tag{106}$$

and ϵ is a small real number. At P_0, the Jacobian determinant of

$$(E_0^{(1)}, \dots, E_{r-1}^{(1)}, E_0^{(2)}, \dots, E_{r-1}^{(2)}, \dots, E_0^{(N)}, \dots, E_{r-1}^{(N)})$$

with respect to

$$(\bar{\phi}_{11}, \ldots, \bar{\phi}_{1r}, \ldots, \bar{\phi}_{N/2,1}, \ldots, \bar{\phi}_{N/2,r}, \phi_{11}, \ldots, \phi_{1r}, \ldots, \phi_{N/2,1}, \ldots, \phi_{N/2,r})$$

is

$$|\mathfrak{J}(P_0)| = |\det\left(\epsilon \operatorname{diag}(Z_1, \ldots, Z_{N/2}, Z_1, \ldots, Z_{N/2})(b_{kj} I_r)_{k,j=1,\ldots,N}\right)| + o(\epsilon^{Nr}) \tag{107}$$

where $Z_k = (\lambda_\beta^{2k-1+(\alpha-1)N})_{\alpha,\beta=1,\ldots,r}$ for $k = 1, \ldots, N/2$ and $b_{kj} = \beta_{2k-j-1}$. Let $B = (b_{kj})$; then

$$B = \begin{pmatrix} \beta_N & \beta_{N-1} & \beta_{N-2} & \cdots & \beta_2 & \beta_1 \\ \beta_2 & \beta_1 & \beta_N & \cdots & \beta_4 & \beta_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \beta_{N-2} & \beta_{N-3} & \beta_{N-4} & \cdots & \beta_N & \beta_{N-1} \\ \beta_N & -\beta_{N-1} & \beta_{N-2} & \cdots & \beta_2 & -\beta_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \beta_{N-2} & -\beta_{N-3} & \beta_{N-4} & \cdots & \beta_N & -\beta_{N-1} \end{pmatrix} \tag{108}$$

since $\bar{\beta}_j = (-1)^j \beta_j$ for $j = 1, 2, \ldots, N$. Hence

$$\begin{aligned} |\det B| &= 2^{N/2} \left\| \begin{matrix} \beta_{N-1} & \beta_{N-3} & \cdots & \beta_1 \\ \beta_1 & \beta_{N-1} & \cdots & \beta_3 \\ \vdots & \vdots & & \vdots \\ \beta_{N-3} & \beta_{N-5} & \cdots & \beta_{N-1} \end{matrix} \right\| \left\| \begin{matrix} \beta_N & \beta_{N-2} & \cdots & \beta_2 \\ \beta_2 & \beta_N & \cdots & \beta_4 \\ \vdots & \vdots & & \vdots \\ \beta_{N-2} & \beta_{N-4} & \cdots & \beta_N \end{matrix} \right\| \\ &= 2^{N/2} \cdot \frac{1}{2} N^{N/2} \cdot \frac{1}{2} N^{N/2-1}(N+2) = 2^{N/2-2} N^{N-1}(N+2) \neq 0. \end{aligned} \tag{109}$$

Again, as for odd N, $\mathfrak{J}$ is not zero in a dense open subset of $\mathbf{R}^{2Nr}$ for even N. This means that $\{E_j^{(k)} \mid k = 1, \ldots, N;\ j = 0, \ldots, r-1\}$ are functionally independent in a dense open subset of $\mathbf{R}^{2Nr}$. The theorem is proved. □

The main results in this paper are summarized as follows.

Theorem 7. *The systems of ordinary differential equations (39) and (40) are Hamiltonian systems with Hamiltonians H^x and H^t given by (48) and (49), and $\{H^x, H^t\} = 0$ under the Poisson bracket (45). They are Liouville integrable in the sense that there are Nr involutive conserved integrals which are functionally independent in a dense open subset of the phase space $\mathbf{R}^{2Nr}$. Each solution of (39) and (40) gives a solution of the two dimensional $A_{2n}^{(2)}$, $C_n^{(1)}$ or $D_{n+1}^{(2)}$ Toda equations. When $N = 2n+1$, $m = 0$, (36) gives a solution of the $A_{2n}^{(2)}$ Toda equation. When $N = 2n+2$, $m = 0$, (37) gives a solution of the $C_n^{(1)}$ Toda equation. When $N = 2n$, $m = 1$, (38) gives a solution of the $D_{n+1}^{(2)}$ Toda equation.*

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