DARBOUX TRANSFORMATIONS IN INTEGRABLE SYSTEMS

Theory and their Applications to Geometry

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Preface

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The soliton theory is an important branch of nonlinear science. On one hand, it describes various kinds of stable motions appearing in nature, such as solitary water wave, solitary signals in optical fibre etc., and has many applications in science and technology (like optical signal communication). On the other hand, it gives many effective methods of getting explicit solutions of nonlinear partial differential equations. Therefore, it has attracted much attention from physicists as well as mathematicians.

Nonlinear partial differential equations appear in many scientific problems. Getting explicit solutions is usually a difficult task. Only in certain special cases can the solutions be written down explicitly. However, for many soliton equations, people have found quite a few methods to get explicit solutions. The most famous ones are the inverse scattering method, Bäcklund transformation etc.. The inverse scattering method is based on the spectral theory of ordinary differential equations. The Cauchy problem of many soliton equations can be transformed to solving a system of linear integral equations. Explicit solutions can be derived when the kernel of the integral equation is degenerate. The Bäcklund transformation gives a new solution from a known solution by solving a system of completely integrable partial differential equations. Some complicated "nonlinear superposition formula" arise to substitute the superposition principle in linear science.

However, if the kernel of the integral equation is not degenerate, it is very difficult to get the explicit expressions of the solutions via the inverse scattering method. For the Bäcklund transformation, the nonlinear superposition formula is not easy to be obtained in general. In late 1970s, it was discovered by V. B. Matveev that a method given by G. Darboux a century ago for the spectral problem of second order ordinary differential equations can be extended to some important soliton equations. This method was called Darboux transformation. After that, it was found that this method is very effective for many partial differential equations. It is now playing an important role in mechanics, physics and differential geometry. V. B. Matveev and M. A. Salle published an important monograph [80] on this topic in 1991. Besides, an interesting monograph of C. Rogers and W. K. Schief [90] with many recent results was published in 2002.

The present monograph contains the Darboux transformations in matrix form and provides purely algebraic algorithms for constructing explicit solutions. Consequently, a basis of using symbolic calculations to obtain explicit exact solutions for many integrable systems is established. Moreover, the behavior of simple and multi-solutions, even in multi-dimensional cases, can be elucidated clearly. The method covers a series of important topics such as varies kinds of AKNS systems in $\mathbf{R}^{n+1}$, the construction of Bäcklund congruences and surfaces of constant Gauss curvature in $\mathbf{R}^3$ and $\mathbf{R}^{2,1}$, harmonic maps from two dimensional manifolds to the Lie group $U(n)$, self-dual Yang-Mills fields and the generalizations to higher dimensional case, Yang-Mills-Higgs fields in $2+1$ dimensional Minkowski and anti-de Sitter space, Laplace sequences of surfaces in projective spaces and two dimensional Toda equations. All these cases are stated in details. In geometric problems, the Lax pair is not only a tool, but also a geometric object to be studied. Many results in this monograph are obtained by the authors in recent years.

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Chapter 1

1+1 DIMENSIONAL INTEGRABLE SYSTEMS

Starting from the original Darboux transformation, we first discuss the classical form of the Darboux transformations for the KdV and the MKdV equation, then discuss the Darboux transformations for the AKNS system and more general systems. The coefficients in the evolution equations discussed here may depend on t. The Darboux matrices are constructed algebraically and the algorithm is purely algebraic and universal to whole hierarchies. The Darboux transformations for reduced systems are also concerned. We also present the relations between Darboux transformation and the inverse scattering theory, and show that the number of solitons (the number of eigenvalues) increases or decreases after the action of a Darboux transformation.

1.1 KdV equation, MKdV equation and their Darboux transformations

1.1.1 Original Darboux transformation

In 1882, G. Darboux [18] studied the eigenvalue problem of a linear partial differential equation of second order (now called the one-dimensional Schrödinger equation)

$$-\phi_{xx} - u(x)\phi = \lambda\phi. \tag{1.1}$$

Here $u(x)$ is a given function, called potential function; λ is a constant, called spectral parameter. He found out the following fact. If $u(x)$ and $\phi(x,\lambda)$ are two functions satisfying (1.1) and $f(x) = \phi(x,\lambda_0)$ is a solution of the equation (1.1) for $\lambda = \lambda_0$ where λ_0 is a fixed constant, then the functions u' and ϕ' defined by

$$u' = u + 2(\ln f)_{xx}, \quad \phi'(x, \lambda) = \phi_x(x, \lambda) - \frac{f_x}{f}\phi(x, \lambda) \tag{1.2}$$

satisfy

$$-\phi'_{xx} - u'\phi' = \lambda\phi', \tag{1.3}$$

which is of the same form as (1.1). Therefore, the transformation (1.2) transforms the functions (u, ϕ) to (u', ϕ') which satisfy the same equations. This transformation

$$(u, \phi) \longrightarrow (u', \phi'), \tag{1.4}$$

is the original Darboux transformation, which is valid for $f \neq 0$.

1.1.2 Darboux transformation for KdV equation

In 1885, the Netherlandish applied mathematicians Korteweg and de Vries introduced a nonlinear partial differential equation describing the motion of water wave, which is now called the Korteweg-de Vries equation (KdV equation)

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.5}$$

In the middle of 1960's, this equation was found out to be closely related to the Schrödinger equation mentioned above [87]. KdV equation (1.5) is the integrability condition of the linear system

$$\begin{aligned} &-\phi_{xx} - u\phi = \lambda\phi, \\ &\phi_t = -4\phi_{xxx} - 6u\phi_x - 3u_x\phi \end{aligned} \tag{1.6}$$

which is called the Lax pair of the KdV equation. Here u and ϕ are functions of x and t. (1.6) is the integrability condition of (1.5). In other words, (1.5) is the necessary and sufficient condition for $(\phi_{xx})_t = (\phi_t)_{xx}$ being an identity for all λ, where $(\phi_{xx})_t$ is computed from $\phi_{xx} = (-\lambda - u)\phi$ (the first equation of (1.6)) and $(\phi_t)_{xx}$ is given by the second equation of (1.6).

Since the first equation of the Lax pair of the KdV equation is just the Schrödinger equation, the Darboux transformation (1.2) can also be applied to the KdV equation, where the functions depend on t. Obviously the transformation keeps the first equation of (1.6) invariant, i.e., (u', ϕ') satisfies

$$-\phi'_{xx} - u'\phi' = \lambda\phi'. \tag{1.7}$$

Moreover, it is easily seen that (u', ϕ') satisfies the second equation of (1.6) as well. Therefore, u' satisfies the KdV equation, which is the integrability condition of (1.6). In summary, suppose one knows a solution u of the KdV equation, solving the linear equations (1.6) one gets $\phi(x,t,\lambda)$. Take λ to be a special value λ_0 and let $f(x,t) = \phi(x,t,\lambda_0)$, then $u' = u + 2(\ln f)_{xx}$ gives a new solution of the KdV equation, and ϕ' given by (1.2) is a solution of the Lax pair corresponding to u'. This gives a way to obtain new solutions of the KdV equation.

This process can be done successively as follows. For a known solution u of (1.5), first solve a system of linear differential equations (1.6) and get ϕ. Then explicit calculation from (1.2) gives new special solutions of the KdV equation. Since ϕ' is known, it is not necessary to solve any linear differential equations again to obtain (u'', ϕ''). That is, we only need algebraic calculation to get (u'', ϕ'') etc.:

$$(u,\phi) \longrightarrow (u',\phi') \longrightarrow (u'',\phi'') \longrightarrow \cdots. \tag{1.8}$$

Therefore, we have extended the Darboux transformation for the Schrödinger equation to the KdV equation. The basic idea here is to get the new solutions of the nonlinear equation and the corresponding solutions of the Lax pair simultaneously from a known solution of the nonlinear equation and a solution of its Lax pair by using algebraic and differential computation. Note that the formula is valid only for $f \neq 0$. If $f = 0$, the Darboux transformation will have singularities.

Remark 1 Let $\psi_1 = \phi$, $\psi_2 = \phi_x$, $\Psi = (\psi_1, \psi_2)^T$, then the Lax pair (1.6) can be written in matrix form as

$$\begin{aligned} \Psi_x &= \begin{pmatrix} 0 & 1 \\ -\lambda - u & 0 \end{pmatrix} \Psi, \\ \Psi_t &= \begin{pmatrix} u_x & 4\lambda - 2u \\ -4\lambda^2 - 2\lambda u + u_{xx} + 2u^2 & -u_x \end{pmatrix} \Psi. \end{aligned} \tag{1.9}$$

The transformation $\phi \to \phi'$ in (1.2) can also be rewritten as a transformation of Ψ, which can be realized via algebraic algorithm only. We shall discuss this Darboux transformation in matrix form later.

1.1.3 Darboux transformation for MKdV equation

The method of Darboux transformation can be applied to many other equations such as the MKdV equation, the sine-Gordon equation etc. [105]. We first take the MKdV equation as an example. General cases will be considered in the latter sections.

MKdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0 \tag{1.10}$$

is the integrability condition of the over-determined linear system [2, 119]

$$\begin{aligned}
\Phi_x &= U\Phi = \begin{pmatrix} \lambda & u \\ -u & -\lambda \end{pmatrix} \Phi, \\
\Phi_t &= V\Phi \\
&= \begin{pmatrix} -4\lambda^3 - 2u^2\lambda & -4u\lambda^2 - 2u_x\lambda - 2u^3 - u_{xx} \\ 4u\lambda^2 - 2u_x\lambda + 2u^3 + u_{xx} & 4\lambda^3 + 2u^2\lambda \end{pmatrix} \Phi,
\end{aligned} \tag{1.11}$$

that is, (1.10) is the necessary and sufficient condition for $\Phi_{xt} = \Phi_{tx}$ being an identity. The system (1.11) is called a Lax pair of (1.10) and λ a spectral parameter. Here Φ may be regarded as a column solution or a 2×2 matrix solution of (1.11).

There are several ways to derive the Darboux transformation for the MKdV equation. Here we use the Darboux matrix method.

For a given solution u of the MKdV equation, suppose we know a fundamental solution of (1.11)

$$\Phi(x,t,\lambda) = \begin{pmatrix} \Phi_{11}(x,t,\lambda) & \Phi_{12}(x,t,\lambda) \\ \Phi_{21}(x,t,\lambda) & \Phi_{22}(x,t,\lambda) \end{pmatrix} \tag{1.12}$$

which composes two linearly independent column solutions of (1.11).

Let λ_1, μ_1 be arbitrary real numbers and

$$\sigma = \frac{\Phi_{22}(x,t,\lambda_1) + \mu_1 \Phi_{21}(x,t,\lambda_1)}{\Phi_{12}(x,t,\lambda_1) + \mu_1 \Phi_{11}(x,t,\lambda_1)} \tag{1.13}$$

be the ratio of the two entries of a column solution of the Lax pair (1.11). Construct the matrix

$$D(x,t,\lambda) = \lambda I - \frac{\lambda_1}{1+\sigma^2} \begin{pmatrix} 1-\sigma^2 & 2\sigma \\ 2\sigma & \sigma^2 - 1 \end{pmatrix} \tag{1.14}$$

and let $\Phi'(x,t,\lambda) = D(x,t,\lambda)\Phi(x,t,\lambda)$. Then it is easily verified that $\Phi'(x,t,\lambda)$ satisfies

$$\Phi'_x = U'\Phi', \qquad \Phi'_t = V'\Phi', \tag{1.15}$$

where

$$
\begin{aligned}
U' &= \begin{pmatrix} \lambda & u' \\ -u' & -\lambda \end{pmatrix}, \\
V' &= \begin{pmatrix} -4\lambda^3 - 2u'^2\lambda & -4u'\lambda^2 - 2u'_x\lambda - 2u'^3 - u'_{xx} \\ 4u'\lambda^2 - 2u'_x\lambda + 2u'^3 + u'_{xx} & 4\lambda^3 + 2u'^2\lambda \end{pmatrix}
\end{aligned}
\tag{1.16}
$$

with

$$
u' = u + \frac{4\lambda_1\sigma}{1+\sigma^2}. \tag{1.17}
$$

(1.15) and (1.16) are similar to (1.11). The only difference is that the u in (1.11) is replaced by u' defined by (1.17). For any solution Φ of (1.11), $D\Phi$ is a solution of (1.15), hence (1.15) is solvable for any given initial data (the value of Φ' at some point (x_0, t_0)). In other words, (1.15) is integrable. The integrability condition of (1.15) implies that u' is also a solution of the MKdV equation. Using this method, we obtain a new solution of the MKdV equation together with the corresponding fundamental solution of its Lax pair from a known one.

The above conclusions can be summarized as follows. Let u be a solution of the MKdV equation and Φ be a fundamental solution of its Lax pair. Take λ_1, μ_1 to be two arbitrary real constants, and let σ be defined by (1.13), then (1.17) gives a new solution u' of the MKdV equation, and the corresponding solution to the Lax pair can be taken as $D\Phi$. The transformation $(u, \Phi) \to (u', \Phi')$ is the Darboux transformation for the MKdV equation. This Darboux transformation in matrix form can be done successively and purely algebraically as

$$
(u, \Phi) \longrightarrow (u', \Phi') \longrightarrow (u'', \Phi'') \longrightarrow \cdots. \tag{1.18}
$$

Remark 2 Both (1.9) and (1.11) are of the form

$$
\Phi_x = U\Phi, \qquad \Phi_t = V\Phi \tag{1.19}
$$

where U and V are $N \times N$ matrices and independent of Φ. The integrability condition of (1.19) is

$$
U_t - V_x + [U, V] = 0 \quad ([U, V] = UV - VU). \tag{1.20}
$$

According to the elementary theory of linear partial differential equations, the solution of (1.19) exists uniquely for given initial data $\Phi(x_0, t_0) = \Phi_0$ if and only if (1.20) holds identically. Here Φ is an $N \times N$ matrix, or a *vector with N entries. In this case $\Phi(x,t)$ is determined by the ordinary differential equation*

$$d\Phi = (U\,\mathrm{d}x + V\,\mathrm{d}y)\Phi \tag{1.21}$$

along an arbitrary path connecting (x_0, t_0) and (x,t). (1.20) is also called a zero-curvature condition.

1.1.4 Examples: single and double soliton solutions

Starting with the trivial solution $u = 0$ of the MKdV equation, one can use the Darboux transformation to obtain the soliton solutions. For $u = 0$, the fundamental solution of the Lax pair can be obtained as

$$\Phi(x,t,\lambda) = \begin{pmatrix} \exp(\lambda x - 4\lambda^3 t) & 0 \\ 0 & \exp(-\lambda x + 4\lambda^3 t) \end{pmatrix} \tag{1.22}$$

by integrating (1.11). Take $\lambda_1 \neq 0$ and $\mu_1 = \exp(2\alpha_1) > 0$, then (1.13) gives

$$\sigma = \sigma_1 = \exp(-2\lambda_1 x + 8\lambda_1^3 t - 2\alpha_1), \tag{1.23}$$

hence

$$D = \lambda I - \frac{\lambda_1}{\cosh v_1} \begin{pmatrix} \sinh v_1 & 1 \\ 1 & -\sinh v_1 \end{pmatrix}, \tag{1.24}$$

where

$$v_1 = 2\lambda_1 x - 8\lambda_1^3 t + 2\alpha_1. \tag{1.25}$$

(1.17) gives the single soliton solution

$$u' = 2\lambda_1 \ \mathrm{sech}(2\lambda_1 x - 8\lambda_1^3 t + 2\alpha_1), \tag{1.26}$$

of the MKdV equation.

(1.26) is called the single soliton solution because it has the following properties: (i) It is a travelling wave solution, i.e., it is in the form $u' = f(x - ct)$; (ii) For any t, $\lim_{x\to\pm\infty} u' = 0$. Speaking intuitively, u' is near 0 outside a small region, i.e., $|u| < 2|\lambda_1|\,\mathrm{sech}\,K$ when $|2\lambda_1 x - 8\lambda_1^3 t + 2\alpha_1| > K$.

The solution of the corresponding Lax pair is

$$\begin{aligned} \Phi'(x,t,\lambda) &= (\Phi'_{ij}(x,t,\lambda)) \\ &= D(x,t,\lambda) \begin{pmatrix} \exp(\lambda x - 4\lambda^3 t) & 0 \\ 0 & \exp(-\lambda x + 4\lambda^3 t) \end{pmatrix} \end{aligned} \tag{1.27}$$

where D is given by (1.24).

If we take u' as a seed solution, a new Darboux matrix can be constructed from Φ' and a series of new solutions of the MKdV equation can be obtained.

We write down the second Darboux transformation explicitly. Suppose u is a solution of the MKdV equation (1.10), Φ is a fundamental solution of the corresponding Lax pair (1.11). Construct the Darboux matrix $D = (D_{ij})$ according to (1.13) and (1.14) and let $\sigma = \sigma_1$. Moreover, take constants $\lambda_2 \neq 0$ $(\lambda_2 \neq \lambda_1)$ and $\mu_2 = \exp(2\alpha_2)$. According to (1.13),

$$\sigma_2' = \frac{\Phi_{22}'(x,t,\lambda_2) + \mu_2 \Phi_{21}'(x,t,\lambda_2)}{\Phi_{12}'(x,t,\lambda_2) + \mu_2 \Phi_{11}'(x,t,\lambda_2)}. \tag{1.28}$$

Substituting $\Phi' = D\Phi$ into it, we have

$$\begin{aligned} \sigma_2' &= \left.\frac{D_{21}(\Phi_{12} + \mu_2\Phi_{11}) + D_{22}(\Phi_{22} + \mu_2\Phi_{21})}{D_{11}(\Phi_{12} + \mu_2\Phi_{11}) + D_{12}(\Phi_{22} + \mu_2\Phi_{21})}\right|_{\lambda=\lambda_2} \\ &= \frac{D_{21}(\lambda_2) + D_{22}(\lambda_2)\sigma_2}{D_{11}(\lambda_2) + D_{12}(\lambda_2)\sigma_2}, \end{aligned} \tag{1.29}$$

where

$$\sigma_2 = \frac{\Phi_{22}(x,t,\lambda_2) + \mu_2 \Phi_{21}(x,t,\lambda_2)}{\Phi_{12}(x,t,\lambda_2) + \mu_2 \Phi_{11}(x,t,\lambda_2)}. \tag{1.30}$$

Starting from $u = 0$, (1.26) and (1.27) are the single soliton solution and the corresponding fundamental solution of the Lax pair. Substituting (1.24), the expression of D, into (1.27), we have

$$\Phi'(x,t,\lambda) = \begin{pmatrix} (\lambda - \lambda_1 \tanh v_1) \mathrm{e}^{\lambda x - 4\lambda^3 t} & -\lambda_1 \operatorname{sech} v_1 \mathrm{e}^{-\lambda x + 4\lambda^3 t} \\ -\lambda_1 \operatorname{sech} v_1 \mathrm{e}^{\lambda x - 4\lambda^3 t} & (\lambda + \lambda_1 \tanh v_1) \mathrm{e}^{-\lambda x + 4\lambda^3 t} \end{pmatrix}, \tag{1.31}$$

hence

$$\sigma_2' = \frac{-\lambda_1 \operatorname{sech} v_1 + (\lambda_2 + \lambda_1 \tanh v_1)\exp(-v_2)}{(\lambda_2 - \lambda_1 \tanh v_1) - \lambda_1 \operatorname{sech} v_1 \exp(-v_2)}, \tag{1.32}$$

where

$$v_2 = 2\lambda_2 x - 8\lambda_2^3 t + 2\alpha_2, \quad (i = 1, 2). \tag{1.33}$$

According to (1.17),

$$\begin{aligned} u'' &= \frac{4\lambda_1\sigma_1}{1+\sigma_1^2} + \frac{4\lambda_2\sigma_2'}{1+\sigma_2'^2} \\ &= \frac{2(\lambda_2^2 - \lambda_1^2)(\lambda_2 \cosh v_1 - \lambda_1 \cosh v_2)}{(\lambda_1^2 + \lambda_2^2)\cosh v_1 \cosh v_2 - 2\lambda_1\lambda_2(1 + \sinh v_1 \sinh v_2)} \end{aligned} \tag{1.34}$$

Figure 1.1. Double soliton solutions of the MKdV equation, $t=-1$

is a new solution of the MKdV equation. This is called the double soliton solution of the MKdV equation. This name follows from the following asymptotic property of the solutions. We shall show that a double soliton solution is asymptotic to two single soliton solutions as $t \to \infty$.

Suppose $\lambda_2 > \lambda_1 > 0$, M is a fixed positive number. Let v_1 be bounded by $|v_1| \le M$, then $x \sim \infty$ as $t \sim \infty$. Since

$$v_2 = \frac{\lambda_2}{\lambda_1} v_1 - 8\lambda_2(\lambda_2^2 - \lambda_1^2)t + 2\alpha_2 - \frac{2\lambda_2\alpha_1}{\lambda_1}, \tag{1.35}$$

$v_2 \sim +\infty$ as $t \sim -\infty$, and

$$u'' \sim -2\lambda_1 \operatorname{sech}(v_1 - v_0) \tag{1.36}$$

as $t \to -\infty$ where

$$v_0 = \tanh^{-1} \frac{2\lambda_1\lambda_2}{\lambda_1^2 + \lambda_2^2}. \tag{1.37}$$

If $t \sim +\infty$, then $v_2 \sim -\infty$, and

$$u'' \sim -2\lambda_1 \operatorname{sech}(v_1 + v_0). \tag{1.38}$$

Hence, for fixed v_1 (i.e., the observer moves in the velocity $4\lambda_1^2$), the solution is asymptotic to one single soliton solution (corresponding to the parameter λ_1) as $t \sim -\infty$ or $t \sim +\infty$. However, there is a phase shift between the asymptotic solitons as $t \sim -\infty$ and $t \sim +\infty$. That is, the center of the soliton (the peak) moves from $v_1 = v_0$ to $v_1 = -v_0$.

Similarly, if $|v_2| \le M$, then

$$v_1 = \frac{\lambda_1}{\lambda_2} v_2 + 8\lambda_1(\lambda_2^2 - \lambda_1^2)t + 2\alpha_1 - \frac{2\lambda_1\alpha_2}{\lambda_2} \tag{1.39}$$

implies that $v_1 \sim \pm\infty$ as $t \sim \pm\infty$, and

$$\begin{aligned} u'' &\sim 2\lambda_2 \operatorname{sech}(v_2 + v_0), \quad t \sim -\infty, \\ u'' &\sim 2\lambda_2 \operatorname{sech}(v_2 - v_0), \quad t \sim +\infty. \end{aligned} \tag{1.40}$$

Finally, if $t \sim \pm\infty$ and both v_1, v_2 tend to $\pm\infty$ (i.e., the observer moves in the velocity $\ne 4\lambda_1^2, 4\lambda_2^2$), then $u'' \sim 0$. Therefore, whenever $t \sim +\infty$ or $t \sim -\infty$, u'' is asymptotic to two single soliton solutions (see Figures 1.1 – 1.3.

Figure 1.2. Double soliton solutions of the MKdV equation, $t = 0.1$

Figure 1.3. Double soliton solutions of the MKdV equation, $t = 1$

This fact means that: (i) a double soliton solution is asymptotic to two single soliton solutions as $t \to \pm\infty$; (ii) if two single solitons (the asymptotic behavior as $t \to -\infty$) interact, they will almost recover later ($t \to +\infty$). Both the shape and the velocity do not change. The only change is the phase shift. Physically speaking, there is elastic scattering between solitons. This is the most important character of solitons. The discovery of this property (first to the KdV equation) greatly promotes the progress of the soliton theory.

Remark 3 Starting from the trivial solution $u = 0$, we can also obtain the single and double soliton solutions of the KdV equation by using the original Darboux transformation mentioned at the beginning of this section. The computation is simpler and is left for the reader.

The Darboux transformation for the MKdV equation can be used to get not only the single and double soliton solutions, but also the multi-soliton solutions. Moreover, this method can be applied to many other nonlinear equations. We shall discuss the general problem in the next section.

1.1.5 Relation between Darboux transformations for KdV equation and MKdV equation

The Darboux transformation for the MKdV equation can also be derived from the "complexification" of the Schrödinger equation (1.1) directly. That is why the transformation given by the matrix D is also called a Darboux transformation.

Take a solution $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ of (1.11), then the first equation of (1.11) is

$$\begin{aligned} \phi_{1,x} &= \lambda\phi_1 + u\phi_2, \\ \phi_{2,x} &= -u\phi_1 - \lambda\phi_2. \end{aligned} \tag{1.41}$$

Let $\psi = \phi_1 + \mathrm{i}\phi_2$ and suppose λ is a real parameter, u is a real function, then ψ satisfies

$$\psi_{xx} = \lambda^2\psi - (\mathrm{i}u_x + u^2)\psi. \tag{1.42}$$

This is a complex Schrödinger equation with potential $(\mathrm{i}u_x + u^2)$. It can be checked directly that if u is a solution of the MKdV equation, then

$w = \mathrm{i}u_x + u^2$ is a complex solution of the KdV equation $w_t + 6ww_x + w_{xxx} = 0$. The transformation from the solution u of the MKdV equation to the solution w of the KdV equation is called a Miura transformation.

Remark 4 *Let $v = \mathrm{i}u$, then (1.10) is*

$$v_t - 6v^2v_x + v_{xxx} = 0, \tag{1.10$'$}$$

and the Miura transformation becomes $w = v_x - v^2$. If v is a real solution of (1.10)$'$, then w is a real solution of the KdV equation.

Take a real number λ_0 and a solution $f = f_1 + \mathrm{i}f_2$ of the equation (1.42) for $\lambda = \lambda_0$, then $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ is a solution of (1.41) for $\lambda = \lambda_0$. Using the conclusion to the KdV equation, we know that

$$\psi' = \psi_x - (f_x/f)\psi, \quad w' = w + 2(\ln f)_{xx} \tag{1.43}$$

satisfy (1.42) and w' is a solution of the KdV equation. Moreover, there is a corresponding u' satisfying the MKdV equation. Now we write down the explicit expression of u'.

Considering (1.41), the first equation of (1.43) can be rewritten in terms of the components as

$$\phi_1' + \mathrm{i}\phi_2' = \lambda\phi_1 - \mathrm{i}\lambda\phi_2 - \lambda_0\frac{\bar{f}}{f}(\phi_1 + \mathrm{i}\phi_2). \tag{1.44}$$

If λ and λ_0 are real numbers, then ϕ_1 and ϕ_2 can be chosen as real functions. (1.44) becomes

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \lambda - \lambda_0\dfrac{f_1^2 - f_2^2}{f_1^2 + f_2^2} & -\lambda_0\dfrac{2f_1f_2}{f_1^2 + f_2^2} \\ \lambda_0\dfrac{2f_1f_2}{f_1^2 + f_2^2} & -\lambda - \lambda_0\dfrac{f_1^2 - f_2^2}{f_1^2 + f_2^2} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{1.45}$$

It should be noted that the matrix in the right hand side of (1.45) is the counterpart of the Darboux matrix defined by (1.24).

It can be checked that $\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix}$ satisfies

$$\begin{aligned} \phi_{1,x}' &= \lambda\phi_1' + u'\phi_2', \\ \phi_{2,x}' &= -u'\phi_1' - \lambda\phi_2' \end{aligned} \tag{1.46}$$

where

$$u' = -u - \frac{4\lambda_0 f_1 f_2}{f_1^2 + f_2^2}. \tag{1.47}$$

The integrability condition of (1.46) implies that u' is a solution of the MKdV equation.

Remark 5 The matrix given by (1.14) and that given by (1.45) are different by a left-multiplied factor $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *Notice that if* (u, ϕ_1, ϕ_2) *is a solution of (1.41), then* $(-u, \phi_1, -\phi_2)$ *is also a solution of (1.41). Therefore, the solution (1.17) given by (1.14) is the minus of the solution given by (1.47). Both* u *and* $-u$ *satisfy the MKdV equation. We can take any one transformation as the Darboux transformation.*

The matrix D *is very important hereafter. It is called a Darboux matrix.*

1.2 AKNS system

1.2.1 2×2 AKNS system

In order to generalize the Lax pair of the MKdV equation, V. E. Zakharov, A. B. Shabat and M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur introduced independently a more general system [2, 119] which is now called the AKNS system. For simplicity, we first discuss the 2×2 AKNS system (i.e., the AKNS system of 2×2 matrices), and then the more general $N \times N$ AKNS system.

2×2 AKNS system is the linear system of differential equations

$$\begin{aligned} \Phi_x &= U\Phi = \lambda J\Phi + P\Phi, \\ \Phi_t &= V\Phi = \sum_{j=0}^{n} V_j \lambda^{n-j} \Phi, \end{aligned} \tag{1.48}$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}, \quad V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{1.49}$$

$$\begin{aligned} A &= \sum_{j=0}^{n} a_j(x,t)\lambda^{n-j}, \\ B &= \sum_{j=0}^{n} b_j(x,t)\lambda^{n-j}, \\ C &= \sum_{j=0}^{n} c_j(x,t)\lambda^{n-j}, \end{aligned} \tag{1.50}$$

p, q, a_j, b_j, c_j are complex or real functions of x and t, λ is a real or complex parameter, called the spectral parameter. As mentioned in Remark 2, the integrability condition of (1.48)

$$U_t - V_x + [U, V] = 0 \tag{1.51}$$

should hold for all λ. In terms of the components, (1.51) becomes

$$\begin{aligned} A_x &= pC - qB, \\ B_x &= p_t + 2\lambda B - 2pA, \\ C_x &= q_t - 2\lambda C + 2qA. \end{aligned} \tag{1.52}$$

Both sides of the above equations are polynomials of λ. Expanding them in terms of the powers of λ, we have

$$\begin{aligned} & b_0 = c_0 = 0, \\ & a_{j,x} = pc_j - qb_j \quad (0 \le j \le n), \\ & b_{j+1} = \frac{1}{2} b_{j,x} + pa_j \quad (0 \le j \le n-1), \\ & c_{j+1} = -\frac{1}{2} c_{j,x} + qa_j \quad (0 \le j \le n-1), \end{aligned} \tag{1.53}$$

and the evolution equations

$$\begin{aligned} p_t &= b_{n,x} + 2pa_n, \\ q_t &= c_{n,x} - 2qa_n. \end{aligned} \tag{1.54}$$

(1.53) can be regarded as the equations to determine A, B, C, and (1.54) is a system of evolution equations of p and q. In (1.53), a_j, b_j, c_j can be derived through algebraic calculation, differentiation and integration. We can see later that they are actually polynomials of p, q and their derivatives with respect to x (without any integral expressions of p and q), the coefficients of which are arbitrary functions of t. After solving a_j, b_j, c_j from (1.53), we get the system of nonlinear evolution equations of p and q from (1.54).

For $j = 0, 1, 2, 3$,

$$\begin{aligned}
a_0 &= \alpha_0(t), \quad b_0 = c_0 = 0, \\
a_1 &= \alpha_1(t), \quad b_1 = \alpha_0(t)p, \quad c_1 = \alpha_0(t)q, \\
a_2 &= -\frac{1}{2}\alpha_0(t)pq + \alpha_2(t), \\
b_2 &= \frac{1}{2}\alpha_0(t)p_x + \alpha_1(t)p, \\
c_2 &= -\frac{1}{2}\alpha_0(t)q_x + \alpha_1(t)q, \\
a_3 &= \frac{1}{4}\alpha_0(t)(pq_x - qp_x) - \frac{1}{2}\alpha_1(t)pq + \alpha_3(t), \\
b_3 &= \frac{1}{4}\alpha_0(t)(p_{xx} - 2p^2q) + \frac{1}{2}\alpha_1(t)p_x + \alpha_2(t)p, \\
c_3 &= \frac{1}{4}\alpha_0(t)(q_{xx} - 2pq^2) - \frac{1}{2}\alpha_1(t)q_x + \alpha_2(t)q.
\end{aligned} \tag{1.55}$$

Here $\alpha_0(t)$, $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$ are arbitrary functions of t, which are the integral constants in integrating a_0, a_1, a_2, a_3 from the second equation of (1.53).

Here are some simplest and most important examples.

EXAMPLE 1.1 *$n = 3$, $p = u$, $q = -1$, $\alpha_0 = -4$, $\alpha_1 = \alpha_2 = \alpha_3 = 0$. In this case, $a_3 = -u_x$, $b_3 = -u_{xx} - 2u^2$, $c_3 = 2u$. (1.54) becomes the KdV equation*

$$u_t + u_{xxx} + 6uu_x = 0. \tag{1.56}$$

EXAMPLE 1.2 *$n = 3$, $p = u$, $q = -u$, $\alpha_0 = -4$, $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then $a_3 = 0$, $b_3 = -u_{xx} - 2u^3$, $c_3 = u_{xx} + 2u^3$. The equation becomes the MKdV equation*

$$u_t + u_{xxx} + 6u^2u_x = 0. \tag{1.57}$$

EXAMPLE 1.3 *$n = 2$, $p = u$, $q = -\bar{u}$, $\alpha_0 = -2\mathrm{i}$, $\alpha_1 = \alpha_2 = 0$, $a_2 = -\mathrm{i}|u|^2$, $b_2 = -\mathrm{i}u_x$, $c_2 = -\mathrm{i}\bar{u}_x$. (1.54) is the nonlinear Schrödinger equation*

$$\mathrm{i}u_t = u_{xx} + 2|u|^2u. \tag{1.58}$$

We have seen that for $j = 0, 1, 2, 3$, a_j, b_j, c_j are differential polynomials of p and q, i.e., polynomials of p, q and their derivatives with respect to x, whose coefficients are constants or arbitrary functions of t.

LEMMA 1.4 *a_j, b_j, c_j given by (1.53) are differential polynomials of p and q.*

Proof. Use induction. The conclusion is obviously true for $j = 0$.

Suppose a_j, b_j, c_j are differential polynomials of p and q for $j < l$, we will prove that a_l, b_l, c_l are also differential polynomials of p and q.

(1.53) implies that b_l, c_l are differential polynomials of p and q. Hence it is only necessary to prove that a_l is a differential polynomial of p and q. For $1 \leq j \leq l-1$,

$$
\begin{aligned}
& b_j c_{l+1-j} - c_j b_{l+1-j} \\
= \; & b_j(q a_{l-j} - \frac{1}{2} c_{l-j,x}) - c_j(p a_{l-j} + \frac{1}{2} b_{l-j,x}) \\
= \; & (q b_j - p c_j) a_{l-j} - \frac{1}{2}(b_j c_{l-j,x} + c_j b_{l-j,x}) \\
= \; & -(a_j a_{l-j} + \frac{1}{2} b_j c_{l-j} + \frac{1}{2} c_j b_{l-j})_x + a_j a_{l-j,x} \\
& + \frac{1}{2}(b_{j,x} c_{l-j} + c_{j,x} b_{l-j}) \\
= \; & -(a_j a_{l-j} + \frac{1}{2} b_j c_{l-j} + \frac{1}{2} c_j b_{l-j})_x + (p a_j + \frac{1}{2} b_{j,x}) c_{l-j} \\
& -(q a_j - \frac{1}{2} c_{j,x}) b_{l-j} \\
= \; & -(a_j a_{l-j} + \frac{1}{2} b_j c_{l-j} + \frac{1}{2} c_j b_{l-j})_x + b_{j+1} c_{l-j} - c_{j+1} b_{l-j}.
\end{aligned}
\tag{1.59}
$$

Summarize for j from 1 to $l-1$, we have

$$
b_1 c_l - c_1 b_l = -\sum_{j=1}^{l-1} (a_j a_{l-j} + \frac{1}{2} b_j c_{l-j} + \frac{1}{2} c_j b_{l-j})_x - (b_1 c_l - c_1 b_l), \tag{1.60}
$$

i.e.,

$$
p c_l - q b_l = -\sum_{j=1}^{l-1} \frac{1}{4a_0} (2 a_j a_{l-j} + b_j c_{l-j} + c_j b_{l-j})_x. \tag{1.61}
$$

Hence

$$
a_l = -\sum_{j=1}^{l-1} \frac{1}{4a_0} (2 a_j a_{l-j} + b_j c_{l-j} + c_j b_{l-j}) + \alpha_l(t) \tag{1.62}
$$

is a differential polynomial of p and q. The lemma is proved.

Since $\{a_j, b_j, c_j\}$ are differential polynomials of p and q, we can define $\{a_j^0[p,q]\}$, $\{b_j^0[p,q]\}$, $\{c_j^0[p,q]\}$ recursively so that they satisfy the recursion relations (1.53) and the conditions $a_0^0[0,0] = 1$, $a_j^0[0,0] = 0$ $(1 \leq j \leq n)$. Clearly, these $\{a_j^0, b_j^0, c_j^0\}$ are uniquely determined as certain polynomials of p, q and their derivatives with respect to x. From (1.53), we have

LEMMA 1.5

$$\begin{aligned} &b_j^0[0,q]=0, \quad c_j^0[p,0]=0, \\ &a_j^0[p,0]=a_j^0[0,q]=0 \quad (1 \le j \le n) \end{aligned} \tag{1.63}$$

for any p and q. Moreover, for any $\{a_j, b_j, c_j\}$ satisfying (1.53), there exist $\alpha_j(t)$ $(0 \le j \le n)$ such that

$$\begin{aligned} a_k[p,q] &= \sum_{j=0}^{k} \alpha_{k-j}(t) a_j^0[p,q], \\ b_k[p,q] &= \sum_{j=0}^{k} \alpha_{k-j}(t) b_j^0[p,q], \\ c_k[p,q] &= \sum_{j=0}^{k} \alpha_{k-j}(t) c_j^0[p,q]. \end{aligned} \tag{1.64}$$

Remark 6 *For any positive integer n, the first equation of (1.48) (x-equation) is fixed, but the second one depends on the choice of $\alpha_0(t)$, $\cdots$, $\alpha_n(t)$. Therefore, the evolution equations (1.54) also depend on the choice of $\alpha_0(t)$, $\cdots$, $\alpha_n(t)$. This means that (1.54) is a series of equations, which is called the AKNS hierarchy. If $\alpha_0(t)$, $\cdots$, $\alpha_n(t)$ are all constants, then the evolution equations have the coefficients independent of t and form a series of infinite dimensional dynamical systems. Especially, if $\alpha_0 = \cdots = \alpha_{n-1} = 0$, $\alpha_n = 1$, then we obtain the normalized AKNS hierarchy, written as*

$$\begin{pmatrix} p \\ q \end{pmatrix}_t = K_n \begin{bmatrix} p \\ q \end{bmatrix}, \tag{1.65}$$

where K_n is a nonlinear differential operator defined by

$$K_n \begin{bmatrix} p \\ q \end{bmatrix} = \begin{pmatrix} b_{n,x}^0 + 2pa_n^0 \\ c_{n,x}^0 - 2qa_n^0 \end{pmatrix}. \tag{1.66}$$

From this definition and (1.53), we know that K_n is derived from K_{n-1} by recursive algorithm.

1.2.2 $N \times N$ AKNS system

In the last subsection, we introduced the 2×2 AKNS system. In order to obtain more nonlinear partial differential equations, the 2×2

Lax pair should be generalized naturally to the problems of $N \times N$ matrices. Therefore, we discuss the Lax pair

$$\begin{aligned} \Phi_x &= U\Phi = \lambda J\Phi + P(x,t)\Phi, \\ \Phi_t &= V\Phi = \sum_{j=0}^{n} V_j(x,t)\lambda^{n-j}\Phi \end{aligned} \tag{1.67}$$

where J is an $N \times N$ constant diagonal matrix, $P(x,t)$, $V_j(x,t)$ are $N \times N$ matrices and $P(x,t)$ is off-diagonal (i.e., its diagonal entries are all zero), λ is a spectral parameter. We assume that all the entries of J are distinct, though the assumption can be released with restrictions on P and V.

The integrability condition of (1.67) is still

$$U_t - V_x + [U,V] = 0. \tag{1.68}$$

Using the expressions of U and V in (1.67), we have

$$P_t - \sum_{j=0}^{n} V_{j,x}\lambda^{n-j} + \sum_{j=-1}^{n-1}[J,V_{j+1}]\lambda^{n-j} + \sum_{j=0}^{n}[P,V_j]\lambda^{n-j} = 0. \tag{1.69}$$

The coefficients of each power of λ on the left hand side should be zero. This leads to

$$\begin{aligned} &[J,V_0] = 0, \\ &[J,V_{j+1}] - V_{j,x} + [P,V_j] = 0 \qquad (0 \le j \le n-1), \\ &P_t - V_{n,x} + [P,V_n] = 0. \end{aligned} \tag{1.70}$$

For any $N \times N$ matrix M, we divide it as $M = M^{\text{diag}} + M^{\text{off}}$, where M^{diag} is the diagonal part of M and $M^{\text{off}} = M - M^{\text{diag}}$ (hence M^{off} is off-diagonal). Since J is diagonal with distinct diagonal entries and P is off-diagonal, (1.70) is divided into

$$\begin{aligned} &V_0^{\text{off}} = 0, \\ &V_{j,x}^{\text{diag}} = [P,V_j^{\text{off}}]^{\text{diag}} \qquad (0 \le j \le n), \\ &[J,V_{j+1}^{\text{off}}] = V_{j,x}^{\text{off}} - [P,V_j]^{\text{off}} \qquad (0 \le j \le n-1), \end{aligned} \tag{1.71}$$

and

$$P_t = V_{n,x}^{\text{off}} - [P,V_n]^{\text{off}}. \tag{1.72}$$

We can solve V_j $(j = 0, \cdots, n)$ from (1.71) by differentiation and integration. In fact, similar to the 2×2 case, V_j can be obtained by differentiation and integration for $n = 0, 1, 2, 3$. They are differential polynomials of the entries of P. For general n, it can be proved by induction that each entry of V_j is a differential polynomial of the entries of P whose coefficients may depend on t. (The proof is omitted here. See [111]). Therefore, (1.72) gives a system of partial differential equations of the entries of P. We shall write $V_j[P]$ for the V_j to specify the dependence on P.

EXAMPLE 1.6 *Let* $n = 1$, $J = A = \mathrm{diag}(a_1, \cdots, a_N)$, $V_0 = B = \mathrm{diag}(b_1, \cdots, b_N)$ *with* $a_i \neq a_j$ *and* $b_i \neq b_j$ $(i \neq j)$. *Take* $V_1 = Q(x,t)$ *whose diagonal entries are all* 0, *then, from (1.71),*

$$Q_{ij} = \frac{b_i - b_j}{a_i - a_j} P_{ij} \quad (i \neq j), \tag{1.73}$$

and the equation (1.72) becomes

$$P_t = Q_x - [P, Q]^{\mathrm{off}}. \tag{1.74}$$

Written in terms of the components, it becomes

$$P_{ij,t} = c_{ij} P_{ij,x} + \sum_{k \neq i,j} (c_{ik} - c_{kj}) P_{ik} P_{kj} \tag{1.75}$$

where

$$c_{ij} = \frac{b_i - b_j}{a_i - a_j}. \tag{1.76}$$

This is a system of nonlinear partial differential equations of P_{ij} $(i \neq j)$, *called the* N *wave equation.*

Similar to the discussion in Lemma 1.5, let $V_j^0[P]$ be the solution of (1.71) satisfying $V_0[0] = I$, $V_l[0] = 0$ $(1 \leq l \leq n)$, then the following lemma holds.

Lemma 1.5′. The general solution of (1.71) can be expressed as

$$V_k[P] = \sum_{j=0}^{k} \alpha_j V_{k-j}^0[P]. \tag{1.77}$$

where $\alpha_0, \cdots, \alpha_n$ are the corresponding integral constants of $\{V_j[P]\}$, which are diagonal matrices independent of x but may depend on t.

1.3 Darboux transformation

1.3.1 Darboux transformation for AKNS system

Let

$$F(u, u_x, u_t, u_{xx}, \cdots) = 0, \tag{1.78}$$

be a system of partial differential equations where u is a function or a vector valued function. Consider the AKNS system

$$\begin{aligned} \Phi_x &= U\Phi = (\lambda J + P)\Phi, \\ \Phi_t &= V\Phi = \sum_{j=0}^{n} V_j \lambda^{n-j}\Phi, \end{aligned} \tag{1.79}$$

where J, P, V_j satisfy the condition in the last section and P is a differential polynomial of u. Suppose (1.78) is equivalent to (1.68), the integrability condition of (1.79), then (1.79) is called the Lax pair of (1.78). In this case, (1.78) is the evolution equation (1.72). The non-degenerate $N \times N$ matrix solution of (1.79) is called a fundamental solution of the Lax pair.

In this section, we suppose that the (off-diagonal) entries of P are independent and (1.78) is the system of differential equations (1.72) in which all off-diagonal entries of P are unknown functions. This system is called unreduced.

We first discuss the Darboux transformation for the unreduced AKNS system.

Definition 1.7 *Suppose $D(x, t, \lambda)$ is an $N \times N$ matrix. If for given P and any solution Φ of (1.79), $\Phi' = D\Phi$ satisfies a linear system*

$$\begin{aligned} \Phi'_x &= U'\Phi' = (\lambda J + P')\Phi', \\ \Phi'_t &= V'\Phi' = \sum_{j=0}^{n} V'_j \lambda^{n-j}\Phi', \end{aligned} \tag{1.80}$$

where P' is an off-diagonal $N \times N$ matrix function, then the transformation $(P, \Phi) \to (P', \Phi')$ is called a Darboux transformation for the unreduced AKNS system, $D(x, t, \lambda)$ is called a Darboux matrix. A Darboux matrix is of degree k if it is a polynomial of λ of degree k.

According to this definition, P' satisfies the equation

$$P'^{\text{off}}_t - V'^{\text{off}}_{n,x} + [P', V'_n]^{\text{off}} = 0, \tag{1.72$'$}$$

where the entries of V'_n are differential polynomials of P'. Later, we will see that $V'_n = V_n[P']$ when the Darboux matrix is a polynomial of λ. In this case $(1.72)'$ and (1.72) are the same partial differential equations.

Substituting $\Phi' = D\Phi$ into (1.80), we get

$$\begin{aligned} U' &= DUD^{-1} + D_x D^{-1}, \\ V' &= DVD^{-1} + D_t D^{-1}. \end{aligned} \tag{1.81}$$

Proposition 1 *If D is a Darboux matrix for (1.79) and D' is a Darboux matrix for (1.80), then $D'D$ is a Darboux matrix for (1.79).*

Proof. Since D' is a Darboux matrix for (1.80), there exists $U'' = \lambda J + P''$ (P'' is off-diagonal) and $V'' = \sum_{j=0}^{n} V_j'' \lambda^{n-j}$ such that $\Phi'' = D'\Phi' = D'D\Phi$ satisfies

$$\Phi_x'' = U''\Phi'', \qquad \Phi_t'' = V''\Phi''. \tag{1.82}$$

Hence, by definition, $D'D$ is a Darboux matrix for (1.79).

Remark 7 Any constant diagonal matrix K independent of λ is a Darboux matrix of degree 0*, since under its action according to (1.81),*

$$\begin{aligned} &\lambda J + P \to \lambda J + KPK^{-1}, \\ &\sum_{j=0}^{n} V_j \lambda^{n-j} \to \sum_{j=0}^{n} KV_jK^{-1}\lambda^{n-j}. \end{aligned} \tag{1.83}$$

If we do not consider the relations among the entries of P, this kind of Darboux matrices are trivial.

Now we first consider the Darboux matrix of degree one, which is linear in λ. Suppose it has the form $\lambda I - S$ where S an $N \times N$ matrix function, I is the identity matrix. According to Proposition 1 and Remark 7, the discussion on the Darboux matrix $K(\lambda I - S)$ (K is a non-degenerate constant matrix which must be diagonal in order to get the first equation of (1.80)) can be reduced to the discussion on the Darboux matrix $\lambda I - S$. Therefore, to construct the Darboux matrix, it is only necessary to construct S.

The differential equations of S are derived as follows. From the first equation of (1.80),

$$(\lambda J + P')(\lambda I - S)\Phi = ((\lambda I - S)\Phi)_x = (\lambda I - S)(\lambda J + P)\Phi - S_x\Phi. \tag{1.84}$$

It must hold for any solution of (1.79). Comparing the coefficients of the powers of λ, we have

$$P' = P + [J, S], \tag{1.85}$$

This is the expression of P'.

The term independent of λ in (1.84) gives

$$S_x = P'S - SP = PS - SP + JS^2 - SJS, \tag{1.86}$$

i.e.,

$$S_x + [S, JS + P] = 0. \tag{1.87}$$

This is the first equation which S satisfies.

The second equation of (1.80) leads to

$$\begin{aligned} & \sum_{j=0}^{n} V_j' \lambda^{n-j} (\lambda I - S)\Phi = ((\lambda I - S)\Phi)_t \\ = \quad & (\lambda I - S) \sum_{j=0}^{n} V_j \lambda^{n-j} \Phi - S_t \Phi. \end{aligned} \tag{1.88}$$

Comparing the coefficients of λ^{n+1}, λ^n, $\cdots$, λ, we can determine $\{V_j'\}$ recursively by

$$V_0' = V_0, \qquad V_{j+1}' = V_{j+1} + V_j' S - S V_j, \tag{1.89}$$

and get the second equation of S

$$S_t = V_n' S - S V_n. \tag{1.90}$$

From (1.89) V_j's can be expressed as

$$\begin{aligned} V_0' &= V_0, \\ V_j' &= V_j + \sum_{k=1}^{j} [V_{j-k}, S] S^{k-1} \qquad (1 \le j \le n), \end{aligned} \tag{1.91}$$

and (1.90) becomes

$$S_t + [S, \sum_{j=0}^{n} V_j S^{n-j}] = 0. \tag{1.92}$$

THEOREM 1.8 *$\lambda I - S$ is a Darboux matrix for (1.79) if and only if S satisfies*

$$\begin{aligned} & S_x + [S, JS + P] = 0, \\ & S_t + [S, \sum_{j=0}^{n} V_j S^{n-j}] = 0. \end{aligned} \tag{1.93}$$

Moreover, under the action of the Darboux matrix $\lambda I - S$, $P' = P + [J, S]$.

Proof. Suppose $\lambda I - S$ is a Darboux matrix, then (1.93) is just (1.87) and (1.92) derived above. Conversely, if (1.87) and (1.92) hold, then for any solution Φ of (1.79), there are the relations (1.84) and (1.88). Hence (1.80) holds for the P' determined by (1.85) and the $\{V_j'\}$ determined by (1.89), which means that $\lambda I - S$ is a Darboux matrix.

This theorem implies that we need to solve S from the system of nonlinear partial differential equations (1.93) to get the Darboux matrix. Fortunately, most of the solutions of (1.93) can be constructed explicitly. The following theorem gives the explicit construction of the Darboux matrix of degree one.

Suppose P is a solution of (1.72). Take complex numbers $\lambda_1, \cdots, \lambda_N$ such that they are not all the same. Let $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$. Let h_i be a column solution of (1.79) for $\lambda = \lambda_i$. $H = (h_1, \cdots, h_N)$. When $\det H \neq 0$, let

$$S = H\Lambda H^{-1}, \tag{1.94}$$

then we have the following theorem.

THEOREM 1.9 *The matrix $\lambda I - S$ defined by (1.94) is a Darboux matrix for (1.79).*

Proof. h_i is a solution of (1.79) for $\lambda = \lambda_i$, that is, it satisfies

$$\begin{aligned} h_{i,x} &= \lambda_i J h_i + P h_i, \\ h_{i,t} &= \sum_{j=0}^{n} V_j \lambda^{n-j} h_i. \end{aligned} \tag{1.95}$$

By taking the derivatives of $H = (h_1, h_2, \cdots, h_N)$ with respect to x and t, (1.95) is equivalent to

$$\begin{aligned} H_x &= JH\Lambda + PH, \\ H_t &= \sum_{j=0}^{n} V_j H \Lambda^{n-j}. \end{aligned} \tag{1.96}$$

Hence

$$\begin{aligned} S_x \; &= H_x \Lambda H^{-1} - H\Lambda H^{-1} H_x H^{-1} = [H_x H^{-1}, S] \\ &= [JS + P, S] \\ S_t \; &= H_t \Lambda H^{-1} - H\Lambda H^{-1} H_t H^{-1} = [H_t H^{-1}, S] \\ &= [\sum_{j=0}^{n} V_j S^{n-j}, S]. \end{aligned} \tag{1.97}$$

Therefore, the matrix S defined by (1.94) is a solution of (1.93). Theorem 1.8 implies that $\lambda I - S$ is a Darboux matrix for (1.79). The theorem is proved.

THEOREM 1.10 *For any given (x_0, t_0) and the matrix S_0, (1.93) has a solution satisfying $S(x_0, t_0) = S_0$. That is, the system is integrable.*

Proof. First suppose the Jordan form of S_0 is a diagonal matrix. Suppose its eigenvalues are $\lambda_1, \cdots, \lambda_N$ and the corresponding eigenvectors are h_{0i}. Let h_i be a solution of (1.95) satisfying $h_i(x_0, t_0) = h_{0i}$. Then these h_i are linearly independent in a neighborhood of (x_0, t_0). Theorem 1.9 implies that the Darboux matrix exists, i.e., (1.93) has a solution. If the Jordan form of S_0 is not diagonal, then there is a series of matrices $\{S_0^{(k)}\}$ such that the Jordan form of $S_0^{(k)}$ is diagonal and $S_0^{(k)} \to S_0$ as $k \to \infty$. Construct S according to (1.94) with initial value $S_0^{(k)}$, then $S^{(k)}$ solves (1.93). The smooth dependency of the solution of (1.93) to the initial value implies that $S^{(k)} \to S$ and $S_x^{(k)}$, $S_t^{(k)}$ converge in a neighborhood of (x_0, t_0). Therefore S is a solution of (1.93) with initial value S_0. Thus (1.93) is solvable for any given initial value, which means that it is integrable. The theorem is proved.

We can also prove this theorem by direct but tedious calculation.

This theorem implies that a Darboux matrix of degree one can be obtained either by (1.94) or the limit of such Darboux matrices.

Remark 8 h_i can be expressed as $h_i = \Phi(\lambda_i) l_i$ $(i = 1, 2, \cdots, N)$, where l_1, l_2, $\cdots$, l_N are N linearly independent constant column matrices. Hence H in (1.94) can be written as

$$H = (\Phi(\lambda_1) l_1, \Phi(\lambda_2) l_2, \cdots, \Phi(\lambda_N) l_N). \tag{1.98}$$

This construction of Darboux matrix was given by [33, 94]. Theorem 1.9 and 1.10 implies that (1.94) contains all the matrices S which are similar to diagonal matrices and $\lambda I - S$ are Darboux matrices of degree one. A Darboux matrix expressed by (1.94) is called a diagonalizable Darboux matrix or Darboux matrix with explicit expressions. It is useful in constructing the solutions because it is expressed explicitly. Hereafter, we mostly use the diagonalizable Darboux matrices and the word "diagonalizable" is omitted.

The "single soliton solution" can be obtained by the Darboux transformation from the seed solution $P = 0$.

For $P = 0$, the fundamental solution of (1.67) is $\Phi = \mathrm{e}^{\lambda J x + \Omega(\lambda, t)}$ where $\Omega(t) = \int \sum_{j=0}^{n} V_j[0](t) \lambda^{n-j}\, dt$ is a diagonal matrix. For any constants

$\lambda_1, \cdots, \lambda_N$ and column matrices $l_1, \cdots, l_N$, let

$$H = \left(e^{\lambda_1 Jx+\Omega(\lambda_1,t)}l_1, \cdots, e^{\lambda_N Jx+\Omega(\lambda_N,t)}l_N\right), \tag{1.99}$$

then

$$P' = [J, H\Lambda H^{-1}] \tag{1.100}$$

is a solution of (1.72) and the fundamental solution of the Lax pair (1.80) is $\Phi' = (\lambda I - H\Lambda H^{-1})\Phi$.

Remark 9 If $V_j[0]$ depends on t, then $\Omega(\lambda_i, t)$ is not a linear function of t, hence the velocities of the solitons are not constants [45].

The double soliton solution can be obtained from P' by applying further Darboux transformation. Since Φ' is known in this process, P'' and Φ'' can be obtained by a purely algebraic algorithm. The multi-soliton solutions are obtained similarly.

For general AKNS system, $\det H \neq 0$ may not hold for all (x, t). Therefore, the solutions given by Darboux transformations may not be regular for all (x, t).

1.3.2 Invariance of equations under Darboux transformations

We have known that (P', V_j') and (P, V_j) satisfy the same recursion relations (1.71) and (1.72) holds true for the two sets of functions. V_j' is a differential polynomial of P' which is expressed by a similar equality as (1.77), i.e.,

$$V_k'[P'] = \sum_{j=0}^{k} \alpha_j'(t) V_{k-j}^0[P']. \tag{1.101}$$

Here we prove that actually the coefficients $\alpha_0'(t), \cdots, \alpha_n'(t)$ are the same as $\alpha_0(t), \cdots, \alpha_n(t)$ respectively. Therefore P' and P satisfy the same evolution equation (1.72).

THEOREM 1.11 *Suppose V_j's are differential polynomials of P satisfying (1.71). S is a matrix satisfying (1.87), V_i's are defined by (1.89) and $P' = P + [J, S]$. Then V_i's are differential polynomials of P' and*

$$V_j'[P'] = V_j[P'] \qquad (j = 1, 2, \cdots, n). \tag{1.102}$$

Proof. At first we see that the equation $P' = P + [J, S]$ is equivalent to $P = P' - [J, S]$ and the equation (1.87) is equivalent to

$$S_x = [P' + SJ, S]. \tag{1.103}$$

Moreover, for arbitrary x-function P'. This equation admits solutions in a neighborhood around any given point $x = x_0$. Thus we may consider P' as an arbitrary off-diagonal matrix-valued function of x.

From (1.89) we have

$$\begin{aligned}&[J, V'_{j+1}] - V'_{j,x} + [P', V'_j]\\ &= [J, V_{j+1}] - V_{j,x} + [P, V_j] + ([J, V'_j] - V'_{j-1,x} + [P', V'_{j-1}])S \\ &\quad -S([J, V_j] - V_{j-1,x} + [P, V_{j-1}]) \quad (j = 0, \cdots, n-1).\end{aligned} \tag{1.104}$$

Using induction, we know that

$$[J, V'_{j+1}] - V'_{j,x} + [P', V'_j] = 0 \quad (j = 0, \cdots, n-1) \tag{1.105}$$

from the equations (1.71) for $V_j[P]$. Moreover, we can prove

$$V'^{\text{diag}}_{n,x} - [P', V'_n]^{\text{diag}} = 0. \tag{1.106}$$

This means that V'_j and P' also satisfy (1.71). Therefore, as mentioned above, V'_j can be expressed as a differential polynomial of P': $V'_j = V'_j[P']$.

Let

$$\Delta_j[P'] = V'_j[P'] - V_j[P'], \tag{1.107}$$

(1.89) implies $\Delta_0 = 0$. Suppose $\Delta_k = 0$, then (1.71) leads to

$$[J, \Delta_{k+1}] = \Delta^{\text{off}}_{k,x} - [P', \Delta_k]^{\text{off}} = 0, \tag{1.108}$$

hence $\Delta^{\text{off}}_{k+1} = 0$. From (1.71),

$$\Delta^{\text{diag}}_{k+1,x} = [P', \Delta_{k+1}]^{\text{off}} = 0, \tag{1.109}$$

which means that $\Delta^{\text{diag}}_{k+1}[P']$ is independent of x. We should prove that $\Delta^{\text{diag}}_{k+1}[P']$ is independent of P'. Denote P'_{ij} be the entries of P', $P'^{(\alpha)}_{ij}$ be the αth derivative of P'_{ij} with respect to x. Suppose the order of the highest derivatives of P' in $\Delta^{\text{diag}}_{k+1}$ is r, then

$$0 = \frac{\partial \Delta^{\text{diag}}_{k+1}}{\partial x} = \sum_{i,j} \sum_{\alpha=0}^{r} \frac{\partial \Delta^{\text{diag}}_{k+1}}{\partial P'^{(\alpha)}_{ij}} P'^{(\alpha+1)}_{ij}. \tag{1.110}$$

In this equation, the coefficient of $P'^{(r+1)}_{ij}$ should be 0. Hence $\Delta^{\text{diag}}_{k+1}$ does not contain the rth derivative of P', which means that it is independent of P'. Especially, let $S = 0$, $P = P'$, then (1.89) implies $\Delta^{\text{diag}}_{k+1} = 0$. Thus (1.102) is proved

Theorem 1.11 implies that for the evolution equations (1.72) in the AKNS system, the Darboux transformation transforms a solution of an equation to a new solution of the same equation.

Note that the Darboux transformation

$$(P,\Phi) \longrightarrow (P',\Phi') \tag{1.111}$$

defined by

$$P' = P + [J,S], \quad \Phi' = (\lambda I - S)\Phi \tag{1.112}$$

can be taken successively in a purely algebraic algorithm and leads to an infinite series of solutions of the AKNS system:

$$(P,\Phi) \longrightarrow (P',\Phi') \longrightarrow (P'',\Phi'') \longrightarrow \cdots \tag{1.113}$$

1.3.3 Darboux transformations of higher degree and the theorem of permutability

The Darboux matrices discussed above are all of degree one. In this subsection, we construct Darboux matrices with explicit expressions which are the polynomials of λ of degree > 1. Then we derive the theorem of permutability from the Darboux matrices of degree two.

Clearly, the composition of r Darboux transformations of degree one gives a Darboux transformation of degree r. On the other hand, we can also construct the Darboux transformations of degree r directly.

As known above, a Darboux matrix of degree one is $D(x,t,\lambda) = \lambda I - S$ where S is given by (1.94) if it can be diagonalized (Theorem 1.9). Then, $SH = H\Lambda$ is equivalent to $D(x,t,\lambda_i)h_i = 0$ where h_i is a column solution of the Lax pair for $\lambda = \lambda_i$ such that $\det H = \det(h_1,\cdots,h_N) \neq 0$. This fact can be generalized to the Darboux matrix of degree r, that is, we can consider an $N\times N$ Darboux matrix in the form

$$D(x,t,\lambda) = \sum_{j=0}^{r} D_{r-j}(x,t)\lambda^j, \quad D_0 = I. \tag{1.114}$$

Take Nr complex numbers λ_1, λ_2, $\cdots$, λ_{Nr} and the column solution h_i of the Lax pair for $\lambda = \lambda_i$ $(i = 1,\cdots,Nr)$. Let

$$F_r = \begin{pmatrix} h_1 & h_2 & \cdots & h_{Nr} \\ \lambda_1 h_1 & \lambda_2 h_2 & \cdots & \lambda_{Nr} h_{Nr} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{r-1} h_1 & \lambda_2^{r-1} h_2 & \cdots & \lambda_{Nr}^{r-1} h_{Nr} \end{pmatrix} \tag{1.115}$$

which is an $Nr \times Nr$ matrix. The system $D(x,t,\lambda_i)h_i = 0$ is equivalent to

$$\sum_{j=0}^{r-1} D_{r-j}(x,t)\lambda_i^j h_i = -\lambda_i^r h_i \qquad (i = 1, \cdots, Nr) \tag{1.116}$$

and can be written as

$$(D_r, D_{r-1}, \cdots, D_1)F_r = -(\lambda_1^r h_1, \cdots, \lambda_{Nr}^r h_{Nr}). \tag{1.117}$$

This is a system of linear algebraic equations for $(D_r, D_{r-1}, \cdots, D_1)$. When $\det F_r \neq 0$, it has a unique solution $(D_r, D_{r-1}, \cdots, D_1)$. Therefore, when $\det F_r \neq 0$, there exists a unique $N \times N$ matrix $D(x,t,\lambda)$ satisfying $D(x,t,\lambda_i)h_i = 0$ $(i = 1, \cdots, Nr)$. We write it as $D(h_1, \cdots, h_{Nr}, \lambda)$ to indicate that D is constructed from $h_1, \cdots, h_{Nr}$.

The next theorem shows that it is a Darboux matrix and decomposable as a product of two Darboux matrices of lower degree [52, 74].

THEOREM 1.12 *Given Nr complex numbers $\lambda_1, \cdots, \lambda_{Nr}$. Let h_i be a column solution of the Lax pair (1.79) for $\lambda = \lambda_i$ $(i = 1, \cdots, Nr)$, F_r be defined by (1.115). Suppose* $\det F_r \neq 0$, *then the following conclusions hold.*

(1) There exists a unique matrix $D(h_1, \cdots, h_{Nr}, \lambda)$ in the form (1.114) such that

$$D(h_1, \cdots, h_{Nr}, \lambda_i)h_i = 0 \quad (i = 1, 2, \cdots, Nr). \tag{1.118}$$

In this case, $D(h_1, \cdots, h_{Nr}, \lambda)$ is a Darboux matrix of degree r for (1.79).

(2) If $\det F_{r-1} \neq 0$, *then the above Darboux matrix of degree r can be decomposed as*

$$\begin{aligned}
& D(h_1, \cdots, h_{Nr}, \lambda) \\
= \; & D\Big(D(h_1, \cdots, h_{N(r-1)}, \lambda_{N(r-1)+1})h_{N(r-1)+1}, \cdots, \\
& D(h_1, \cdots, h_{N(r-1)}, \lambda_{Nr})h_{Nr}, \lambda\Big) \cdot \\
& \cdot D(h_1, \cdots, h_{N(r-1)}, \lambda).
\end{aligned} \tag{1.119}$$

On the right hand side of this equality, the first term is a Darboux matrix of degree one and the second term is a Darboux matrix of degree $(r-1)$.

(3) The Darboux matrix $D(h_1, \cdots, h_{Nr}, \lambda)$ of degree r can be decomposed into the product of r Darboux matrices of degree one.

(4) $\widetilde{P} = P - [J, D_1]$ is a solution of (1.72).

Proof. We first prove (2). Let

$$\Lambda_k = \operatorname{diag}(\lambda_{N(k-1)+1}, \cdots, \lambda_{Nk}), \tag{1.120}$$

$$H_k = (h_{N(k-1)+1}, \cdots, h_{Nk}), \tag{1.121}$$

then

$$F_r = \begin{pmatrix} H_1 & H_2 & \cdots & H_r \\ H_1\Lambda_1 & H_2\Lambda_2 & \cdots & H_r\Lambda_r \\ \vdots & \vdots & \ddots & \vdots \\ H_1\Lambda_1^{r-1} & H_2\Lambda_2^{r-1} & \cdots & H_r\Lambda_r^{r-1} \end{pmatrix}. \tag{1.122}$$

Since

$$F_{r-1} = \begin{pmatrix} H_1 & H_2 & \cdots & H_{r-1} \\ H_1\Lambda_1 & H_2\Lambda_2 & \cdots & H_{r-1}\Lambda_{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ H_1\Lambda_1^{r-2} & H_2\Lambda_2^{r-2} & \cdots & H_{r-1}\Lambda_{r-1}^{r-2} \end{pmatrix} \tag{1.123}$$

is non-degenerate, there is a matrix $D(h_1, \cdots, h_{N(r-1)}, \lambda)$ of degree $(r-1)$ with respect to λ such that

$$D(h_1, \cdots, h_{N(r-1)}, \lambda_i)h_i = 0 \qquad (i = 1, 2, \cdots, N(r-1)). \tag{1.124}$$

Let

$$h_i' = D(h_1, \cdots, h_{N(r-1)}, \lambda_i)h_i \qquad (i = N(r-1)+1, \cdots, Nr). \tag{1.125}$$

Construct a Darboux matrix $D(h'_{N(r-1)+1}, \cdots, h'_{Nr}, \lambda)$ from h_i' and let

$$D'(\lambda) = D(h'_{N(r-1)+1}, \cdots, h'_{Nr}, \lambda)D(h_1, \cdots, h_{N(r-1)}, \lambda), \tag{1.126}$$

then $D'(\lambda_i)h_i = 0$ $(i = 1, 2, \cdots, N(r-1))$. Moreover, for $i = N(r-1)+1, \cdots, Nr$,

$$D'(\lambda_i)h_i = D(h'_{N(r-1)+1}, \cdots, h'_{Nr}, \lambda_i)h_i' = 0. \tag{1.127}$$

Hence $D'(\lambda) = D(\lambda)$.

Thus, we have decomposed the matrix of degree r determined by (1.116) to the product of a matrix of degree $(r-1)$ and a matrix of degree one, expressed by (1.119). This proves (2).

For $D(h_1, \cdots, h_{N(r-1)}, \lambda)$, if all the determinants of $F_{r-2}, F_{r-3}, \cdots$ are non-zero, then $D(h_1, \cdots, h_N, \lambda)$ can be decomposed to r matrices of degree one by repeating the above procedure. For $r = 1$, $D(h_1, \cdots, h_N, \lambda)$ is a Darboux matrix. Hence $D(h_1, \cdots, h_{Nr}, \lambda)$ is also a Darboux matrix and it can be decomposed to the product of r Darboux matrices of degree one:

$$D = (\lambda I - S_r) \cdots (\lambda I - S_1). \tag{1.128}$$

Since $\det F_r \neq 0$, we can always permute the subscripts of Λ_i and H_i so that all the determinants of F_{r-2}, F_{r-3}, $\cdots$ are non-zero. (3) is proved.

Since D is the product of r Darboux matrices of degree one, D itself is a Darboux matrix. Hence (1) holds.

Note that

$$D_1 = -(S_1 + \cdots + S_r). \tag{1.129}$$

After the transformation of $\lambda I - S_1$, $P \to P' = P + [J, S_1]$. Then after the transformation of $\lambda I - S_2$, $P' \to P'' = P' + [J, S_2]$, and so on. Hence, after the transformation of D,

$$P \to P + [J, S_1 + \cdots + S_r] = P - [J, D_1]. \tag{1.130}$$

Therefore, $P - [J, D_1]$ is a solution of (1.72). (4) is proved. This proves the lemma.

Darboux transformation has an important property — the theorem of permutability. This theorem originated from the Bäcklund transformation of the sine-Gordon equation and there are a lot of generalizations and various proofs. The proof here is given by [52] (2×2 case) and [33] ($N \times N$ case). This proof does not depend on any boundary conditions and the permutation of the parameters is expressed definitely.

From the solution $(P, \Phi(\lambda))$, we can construct the Darboux transformation with parameters $\lambda_1^{(1)}$, $\cdots$, $\lambda_N^{(1)}$ and the solutions $h_i^{(1)} = \Phi(\lambda_i) l_i^{(1)}$ of the Lax pair. Then the solution $(P^{(1)}, \Phi^{(1)}(\lambda))$ is obtained. Here $l_i^{(1)}$'s are N constant vectors. Next, construct a Darboux matrix for $(P^{(1)}, \Phi^{(1)}(\lambda))$ with parameters $\lambda_1^{(2)}$, $\cdots$, $\lambda_N^{(2)}$ and $l_i^{(2)}$ to get $(P^{(1,2)}, \Phi^{(1,2)}(\lambda))$. On the other hand, construct the Darboux transformation for $(P, \Phi(\lambda))$ with parameters $\lambda_i^{(2)}$ and $l_i^{(2)}$ to get $(P^{(2)}, \Phi^{(2)}(\lambda))$. Then construct the Darboux transformation with parameters $\lambda_i^{(1)}$ and $l_i^{(1)}$ to get $(P^{(2,1)}, \Phi^{(2,1)}(\lambda))$. The following theorem holds.

THEOREM 1.13 *(Theorem of permutability) Suppose*

$$\det \begin{pmatrix} H_1 & H_2 \\ H_1\Lambda_1 & H_2\Lambda_2 \end{pmatrix} \neq 0, \tag{1.131}$$

then

$$(P^{(1,2)}, \Phi^{(1,2)}(\lambda)) = (P^{(2,1)}, \Phi^{(2,1)}(\lambda)). \tag{1.132}$$

Proof. Theorem 1.12 implies that $\Phi^{(1,2)}(\lambda)$ and $\Phi^{(2,1)}(\lambda)$ are both obtained from $\Phi(\lambda)$ by the action of the Darboux transformation of degree two, and are expressed by

$$
\begin{aligned}
\Phi^{(1,2)}(\lambda) &= D(h_1^{(1)}, \cdots, h_N^{(1)}, h_1^{(2)}, \cdots, h_N^{(2)}, \lambda), \\
\phi^{(2,1)}(\lambda) &= D(h_1^{(2)}, \cdots, h_N^{(2)}, h_1^{(1)}, \cdots, h_N^{(1)}, \lambda).
\end{aligned}
\tag{1.133}
$$

From (1) of Theorem 1.12, we know that the right hand side of the above equations are equal. Hence the theorem of permutability holds.

The theorem of permutability can be expressed by the following Bianchi diagram:

$$
\begin{tikzcd}
 & (P^{(1)}, \Phi^{(1)}) \arrow[dr, "\Lambda^{(2)}, L^{(2)}"] & \\
(P, \Phi) \arrow[ur, "\Lambda^{(1)}, L^{(1)}"] \arrow[dr, "\Lambda^{(2)}, L^{(2)}"'] & & (P^{(1,2)}, \Phi^{(1,2)}) = (P^{(2,1)}, \Phi^{(2,1)}) \\
 & (P^{(2)}, \Phi^{(2)}) \arrow[ur, "\Lambda^{(1)}, L^{(1)}"'] &
\end{tikzcd}
\tag{1.134}
$$

Here $L^{(1)}$ and $L^{(2)}$ denote the sets $\{l_i^{(1)}\}$ and $\{l_i^{(2)}\}$ respectively.

Remark 10 *The Darboux transformation of higher degree is much more complicated than the Darboux transformation of degree one. The theorem of decomposition implies that Darboux transformations of degree one can generate Darboux transformations of higher degree. Therefore, we can use Darboux transformations of degree one successively instead of a Darboux transformation of higher degree so as to avoid the calculation of the determinant of a matrix of very high order (of order Nr). Since the algorithm for the Darboux transformation of degree one is purely algebraic and independent of the seed solution P, it is quite convenient to calculate the solutions using symbolic calculation with computer. However, some solutions, e.g., multi-solitons can be expressed by an explicit formulae by using Darboux transformations of higher degree [80].*

Remark 11 *The proof for Theorem 1.13 is for the Darboux transformations with explicit expressions. Since any Darboux transformation without explicit expression is a limit of Darboux transformations with explicit expressions, the theorem of permutability also holds for the Darboux transformations without explicit expressions.*

Now we compute the more explicit expression of the Darboux matrix of degree two. Suppose it is constructed from (Λ_1, H_1) and (Λ_2, H_2)

which satisfy (1.131). Let $S_j = H_j \Lambda_j H_j^{-1}$ and denote

$$\Lambda_\alpha = \operatorname{diag}(\lambda_1^{(\alpha)}, \cdots, \lambda_N^{(\alpha)}), \quad H_\alpha = (h_1^{(\alpha)}, \cdots, h_N^{(\alpha)}).$$

After the action of $\lambda I - S_1$, $h_j^{(2)}$ is transformed to $(\lambda_j^{(2)} I - S_1) h_j^{(2)}$. Hence H_2 is transformed to $\widetilde{H}_2 = H_2 \Lambda_2 - S_1 H_2 = (S_2 - S_1) H_2$. The second Darboux matrix of degree one is $\lambda I - \widetilde{S}_2$ where

$$\widetilde{S}_2 = \widetilde{H}_2 \Lambda_2 \widetilde{H}_2^{-1} = (S_2 - S_1) S_2 (S_2 - S_1)^{-1}. \tag{1.135}$$

According to (1.131), $S_2 - S_1$ is non-degenerate. The Darboux matrix of degree two is

$$\begin{aligned} D(\lambda) &= (\lambda I - \widetilde{S}_2)(\lambda I - S_1) \\ &= \lambda^2 I - \lambda (S_2^2 - S_1^2)(S_2 - S_1)^{-1} + (S_2 - S_1) S_2 (S_2 - S_1)^{-1} S_1. \end{aligned} \tag{1.136}$$

It is easy to check that $D(\lambda)$ is symmetric to S_1 and S_2. Therefore, we can also obtain the theorem of permutability by this symmetry.

1.3.4 More results on the Darboux matrices of degree one

In this subsection, we show that the Darboux matrix method in Theorem 1.9 can be applied not only to the AKNS system, but also to many other evolution equations, especially to the Lax pairs whose U and V are polynomials of λ. On the other hand, we also show that those Darboux matrices include all the diagonalizable Darboux matrices of the form $\lambda I - S$, and any non-diagonalizable Darboux matrix can be obtained from the limit of diagonalizable Darboux matrices.

We generalize the Lax pair (1.79) to

$$\begin{aligned} \Phi_x &= U \Phi, \\ \Phi_t &= V \Phi, \end{aligned} \tag{1.137}$$

where U and V are polynomials of the spectral parameter λ:

$$\begin{aligned} U(x,t,\lambda) &= \sum_{j=0}^{m} U_j(x,t) \lambda^{m-j}, \\ V(x,t,\lambda) &= \sum_{j=0}^{n} V_j(x,t) \lambda^{n-j}, \end{aligned} \tag{1.138}$$

U_j's and V_j's are $N \times N$ matrices.

Clearly, the integrability condition of (1.137) is

$$U_t - V_x + [U, V] = 0. \tag{1.139}$$

In this subsection, we still discuss the Darboux matrices for the Lax pairs without reductions. That is, we suppose that all the entries of U_j's and V_j's are independent except for the partial differential equations (1.139). This is to say that apart from the integrability condition (1.139), there is no other constraint. Therefore, the nonlinear partial differential equation to be studied is just (1.139), i.e., the equations given by the coefficients of each power of λ in (1.139) and the unknowns are the $N \times N$ matrices U_j and V_j ($j = 0, 1, \cdots, n$). Compared with Subsection 1.3.1, $D = \lambda I - S$ is a Darboux matrix if and only if there exist

$$\begin{aligned} U'(x,t,\lambda) &= \sum_{j=0}^{m} U'_j(x,t)\lambda^{m-j}, \\ V'(x,t,\lambda) &= \sum_{j=0}^{n} V'_j(x,t)\lambda^{n-j} \end{aligned} \tag{1.140}$$

such that $\Phi' = (\lambda I - S)\Phi$ satisfies

$$\Phi'_x = U'\Phi', \qquad \Phi'_t = V'\Phi' \tag{1.141}$$

where Φ is a fundamental solution of (1.137).

Clearly, U' and V' have the expressions

$$\begin{aligned} U' &= DUD^{-1} + D_x D^{-1}, \\ V' &= DVD^{-1} + D_t D^{-1} \end{aligned} \tag{1.142}$$

and they satisfy

$$U'_t - V'_x + [U', V'] = 0. \tag{1.143}$$

The remaining problem is to obtain S so that (1.141) holds. If S is obtained, we have the Darboux transformation

$$(U, V, \Phi) \to (U', V', \Phi'). \tag{1.144}$$

Comparing to Theorem 1.8, 1.9 and 1.10, we have

Theorem 1.8′. $\lambda I - S$ is a Darboux matrix of degree one for (1.137) if and only if S satisfies

$$S_x + [S, U(S)] = 0, \qquad S_t + [S, V(S)] = 0. \tag{1.145}$$

Here

$$U(S)=\sum_{j=0}^{m}U_jS^{m-j},\qquad V(S)=\sum_{j=0}^{n}V_jS^{n-j}. \tag{1.146}$$

Now suppose (U,V) satisfies the integrability condition (1.139). For given constant diagonal matrix $\Lambda=\operatorname{diag}(\lambda_1,\cdots,\lambda_N)$, let h_i be a column solution of (1.137) for $\lambda=\lambda_i$, $H=(h_1,\cdots,h_N)$. If $\det H\neq 0$, let $S=H\Lambda H^{-1}$, then the following theorems holds.

Theorem 1.9′. The matrix $\lambda I-S$ is a Darboux matrix for (1.137).

Theorem 1.10′. The system (1.145) is integrable.

The proofs are omitted since they are similar to the proofs for the corresponding theorems above.

Note that for the AKNS system, we can solve $V_i[P]$'s from a system of differential equations by choosing "integral constants" and these $V_i[P]$'s are differential polynomials of P. The remaining equation is only the equation (1.72) for P. In the present case, all the entries of U_i and V_i are regarded as independent unknowns satisfying the partial differential equations (1.139).

The inverse of Theorem 1.9′ also holds.

THEOREM 1.14 *(1) If $\lambda I-S$ is a Darboux matrix for (1.137) and S is diagonalized at one point, then there exists a constant diagonal matrix $\Lambda=\operatorname{diag}(\lambda_1,\cdots,\lambda_N)$ and column solutions h_i's of the Lax pair (1.137) for $\lambda=\lambda_i$ $(i=1,2,\cdots,N)$ such that $H=(h_1,\cdots,h_N)$ and $S=H\Lambda H^{-1}$.*

(2) If $\lambda I-S$ is a Darboux matrix for (1.137) but it can not be diagonalized at any points, then there exist a series of Darboux matrices $\lambda I-S_k$ such that S_k's and their derivatives with respect to x and t converge to S and its derivatives respectively.

The proof is similar to that for Theorem 1.10.

EXAMPLE 1.15 *An example of a Darboux matrix which is not diagonalizable everywhere.*

Consider the Lax pair

$$\begin{aligned}\Phi_x &= \begin{pmatrix}\lambda & p\\ q & -\lambda\end{pmatrix}\Phi,\\ \Phi_t &= \begin{pmatrix}-2\mathrm{i}\lambda^2+\mathrm{i}pq & -2\mathrm{i}\lambda p-\mathrm{i}p_x\\ -2\mathrm{i}\lambda q+\mathrm{i}q_x & 2\mathrm{i}\lambda^2-\mathrm{i}pq\end{pmatrix}\Phi\end{aligned} \tag{1.147}$$

whose integrability condition leads to the nonlinear evolution equations

$$\mathrm{i}p_t = p_{xx} - 2p^2 q, \qquad -\mathrm{i}q_t = q_{xx} - 2pq^2. \tag{1.148}$$

This system of equations has a solution

$$p = \alpha\ \mathrm{sech}(\alpha x) e^{-\mathrm{i}\alpha^2 t}, \qquad q = -\alpha\ \mathrm{sech}(\alpha x) e^{\mathrm{i}\alpha^2 t}, \tag{1.149}$$

which is derived from the trivial solution $p = q = 0$ by the Darboux matrix $D = \lambda I - H\Lambda H^{-1}$ with

$$\begin{aligned} \Lambda &= \frac{\alpha}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ H &= \begin{pmatrix} e^{-\mathrm{i}\alpha^2 t/2} & 0 \\ 0 & e^{\mathrm{i}\alpha^2 t/2} \end{pmatrix}\begin{pmatrix} e^{\alpha x/2} & -e^{-\alpha x/2} \\ e^{-\alpha x/2} & e^{\alpha x/2} \end{pmatrix}. \end{aligned} \tag{1.150}$$

Now we take (1.149) as a seed solution, whose corresponding fundamental solution of the Lax pair (1.147) is

$$(\lambda I - H\Lambda H^{-1})\begin{pmatrix} e^{\lambda x - 2\mathrm{i}\lambda^2 t} & 0 \\ 0 & e^{-\lambda x + 2\mathrm{i}\lambda^2 t} \end{pmatrix}. \tag{1.151}$$

Take

$$\Lambda^{(\epsilon)} = \begin{pmatrix} \epsilon & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.152}$$

then we can choose

$$H^{(\epsilon)} = \begin{pmatrix} h_{11}^{(\epsilon)} & h_{12}^{(\epsilon)} \\ h_{21}^{(\epsilon)} & h_{22}^{(\epsilon)} \end{pmatrix}, \tag{1.153}$$

where

$$\begin{aligned} h_{11}^{(\epsilon)} &= \Big(\frac{2\epsilon^2}{\alpha} - \epsilon\tanh(\alpha x)\Big)e^{\theta} - \frac{\alpha}{2}\ \mathrm{sech}(\alpha x) e^{-\mathrm{i}\alpha^2 t - \theta}, \\ h_{12}^{(\epsilon)} &= -\epsilon\tanh(\alpha x) + \ \mathrm{sech}(\alpha x) e^{-\mathrm{i}\alpha^2 t}, \\ h_{21}^{(\epsilon)} &= -\epsilon\ \mathrm{sech}(\alpha x) e^{\mathrm{i}\alpha^2 t + \theta} + \Big(\epsilon + \frac{\alpha}{2}\tanh(\alpha x)\Big)e^{-\theta}, \\ h_{22}^{(\epsilon)} &= -\epsilon\ \mathrm{sech}(\alpha x) e^{\mathrm{i}\alpha^2 t} - \tanh(\alpha x), \\ \theta &= \epsilon x - 2\mathrm{i}\epsilon^2 t. \end{aligned} \tag{1.154}$$

When $\epsilon \to 0$,

$$H^{(\epsilon)}\Lambda^{(\epsilon)}(H^{(\epsilon)})^{-1} \to S, \tag{1.155}$$

$$S = \frac{\alpha}{\Delta}\begin{pmatrix} \sinh(\alpha x)e^{-\mathrm{i}\alpha^2 t} & e^{-2\mathrm{i}\alpha^2 t} \\ -\sinh^2(\alpha x) & -\sinh(\alpha x)e^{-\mathrm{i}\alpha^2 t} \end{pmatrix}, \tag{1.156}$$

$$p' = \frac{2\alpha e^{-2\mathrm{i}\alpha^2 t}}{\Delta}, \qquad q' = \frac{2\alpha\sinh^2(\alpha x)}{\Delta} \tag{1.157}$$

where

$$\Delta = (\alpha + 2 - 2\,\mathrm{sech}(\alpha x)e^{-\mathrm{i}\alpha^2 t})\cosh^2(\alpha x). \tag{1.158}$$

Note that both eigenvalues of S are zero, but $S \neq 0$. Hence S is not diagonalizable. However, from the construction of S, we know that $\lambda I - S$ is a Darboux matrix, i.e., it satisfies (1.145).

Finally, the conclusions for the Darboux transformations of higher degree and the theorem of permutability in Subsection 1.3.3 also hold for the general Lax pair (1.137). Moreover, when U and V in (1.137) are generalized to rational functions of λ, similar conclusions hold [121].

1.4 KdV hierarchy, MKdV-SG hierarchy, NLS hierarchy and AKNS system with $u(N)$ reduction

In the last section, we discussed the Darboux transformations for the AKNS system and more general systems. In those cases, we supposed that there were no reductions. In particular, there were no restrictions among the off-diagonal entries of P. However, in many cases, there are constraints on P and the Darboux transformation should keep those constraints. This problem is solved in many cases. Nevertheless, it should be very interesting to establish a systematic method to treat with reduced problems.

In this section, we first discuss some equations when $N = 2$ and there is certain relation between p and q. They are the important special cases of the 2×2 AKNS system: (1) KdV hierarchy: p is real and $q = -1$; (2) MKdV-SG hierarchy: $q = -p$ is real; (3) Nonlinear Schrödinger hierarchy: $q = -\bar{p}$. These special cases were studied widely (e.g. [82, 88, 91, 105, 117, 118]). Here we use a unified method to deal with the whole hierarchy, and the coefficients may depend on t [32, 45]. At the end of this section, we discuss the general AKNS system with $u(N)$ reduction. This is a generalization of the nonlinear Schrödinger hierarchy and has many applications to other problems.

1.4.1 KdV hierarchy

Consider the Lax pair [45]

$$\Phi_x = U\Phi, \qquad \Phi_t = V\Phi \tag{1.159}$$

where

$$U = \begin{pmatrix} 0 & 1 \\ \zeta - u & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{1.160}$$

A, B and C are polynomials of the spectral parameter ζ.

Compared with Section 1.2, the integrability condition (1.68) leads to

$$\begin{aligned} & -A_x + C - B(\zeta - u) = 0, \qquad B_x + 2A = 0, \\ & -u_t - C_x + 2A(\zeta - u) = 0. \end{aligned} \tag{1.161}$$

The first two equations imply

$$\begin{aligned} & A = -\frac{1}{2}B_x, \\ & C = \zeta B - uB - \tfrac{1}{2}B_{xx}. \end{aligned} \tag{1.162}$$

Substituting (1.162) into (1.161) we get

$$u_t = -2(\zeta - u)B_x + u_x B + \frac{1}{2}B_{xxx}. \tag{1.163}$$

Let

$$B = \sum_{j=0}^{n} b_j(x,t)\zeta^{n-j}, \tag{1.164}$$

then (1.163) leads to

$$\begin{aligned} & b_{0,x} = 0, \\ & b_{j+1,x} = ub_{j,x} + \frac{1}{2}u_x b_j + \tfrac{1}{4}b_{j,xxx} \quad (0 \le j \le n-1), \end{aligned} \tag{1.165}$$

$$u_t = 2ub_{n,x} + u_x b_n + \frac{1}{2}b_{n,xxx}. \tag{1.166}$$

(1.166) is the equation of u. When $n \geq 2$, it is called a KdV equation of higher order.

Similar to Lemma 1.5, (1.165) leads to

$$b_k = \sum_{j=0}^{k} \alpha_{k-j}(t) b_j^0[u], \tag{1.167}$$

where $b_j^0[u]$'s satisfy the recursion relations (1.165) and $b_0^0[0] = 1$, $b_j^0[0] = 0$ $(j \geq 1)$. Clearly $b_j^0[u]$'s are determined by (1.165) uniquely.

The first few b_j^0's are

$$\begin{aligned} &b_0^0 = 1, \quad b_1^0 = \frac{1}{2}u, \\ &b_2^0 = \frac{1}{8}u_{xx} + \frac{3}{8}u^2, \cdots. \end{aligned} \tag{1.168}$$

The corresponding equations are

$n = 0$: Linear equation

$$u_t = \alpha_0(t)u_x. \tag{1.169}$$

$n = 1$:

$$u_t = \alpha_0(t)\Big(\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x\Big) + \alpha_1(t)u_x. \tag{1.170}$$

If $\alpha_0 =$ constant, $\alpha_1 = 0$, (1.170) is the standard KdV equation.

$n = 2$:

$$\begin{aligned} u_t = \;& \alpha_0(t)\Big(\frac{1}{16}u_{xxxxx} + \frac{5}{8}uu_{xxx} + \frac{5}{4}u_x u_{xx} + \frac{15}{8}u^2 u_x\Big) \\ &+\alpha_1(t)\Big(\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x\Big) + \alpha_2(t)u_x, \end{aligned} \tag{1.171}$$

which is called the KdV equation of 5th-order.

Next we discuss the Darboux transformation for the KdV hierarchy by using the general results for the AKNS system. It seems that the calculation is tedious. However, we can see the application of the general results more clearly. The method here is valid to the whole hierarchy comparing to the special method in Section 1.1.

The U and V given by (1.160) are different from those of the AKNS system. However, the Lax pair can be transformed to a Lax pair in the AKNS system by a similar transformation given by a constant matrix depending on ζ.

Let

$$\begin{aligned} &R = \begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix} \qquad (\lambda^2 = \zeta), \\ &\Psi = R\Phi, \\ &\widetilde{U} = RUR^{-1} = \begin{pmatrix} \lambda & u \\ -1 & -\lambda \end{pmatrix}, \\ &\widetilde{V} = RVR^{-1} = \begin{pmatrix} \lambda B - A & \lambda^2 B - 2\lambda A - C \\ -B & A - \lambda B \end{pmatrix}, \end{aligned} \tag{1.172}$$

then Ψ satisfies

$$\Psi_x = \widetilde{U}\Psi, \qquad \Psi_t = \widetilde{V}\Psi. \tag{1.173}$$

This is the Lax pair for the KdV equation in the AKNS form. The Darboux transformation can be constructed based on the discussion in Section 1.3. Take two constants λ_1, λ_2 and column solutions h_1, h_2 of the Lax pair when $\lambda = \lambda_1, \lambda_2$ respectively. Moreover, we want that the matrices given by the Darboux transformation are still of the form of (1.172). That is,

$$\begin{aligned} \widetilde{U}' &= \begin{pmatrix} \lambda & u' \\ -1 & -\lambda \end{pmatrix}, \\ \widetilde{V}' &= \begin{pmatrix} \lambda B[u'] - A[u'] & \lambda^2 B[u'] - 2\lambda A[u'] - C[u'] \\ -B[u'] & A[u'] - \lambda B[u'] \end{pmatrix}. \end{aligned} \tag{1.174}$$

This condition (especially that the $(2,1)$ entry of $\widetilde{U}'$ is -1) holds only when λ_2, λ_1, h_2 and h_1 are specified.

Suppose $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a solution of the Lax pair (1.173) for $\lambda = \lambda_0$, then $\begin{pmatrix} \alpha + 2\lambda_0\beta \\ \beta \end{pmatrix}$ is a solution of (1.173) for $\lambda = -\lambda_0$. Thus we choose

$$\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{pmatrix}, \quad H = \begin{pmatrix} \alpha & \alpha + 2\lambda_0\beta \\ \beta & \beta \end{pmatrix}. \tag{1.175}$$

Let

$$S = H\Lambda H^{-1} = \begin{pmatrix} -\lambda_0 - \dfrac{1}{\tau} & \dfrac{1}{\tau^2} + \dfrac{2\lambda_0}{\tau} \\ -1 & \dfrac{1}{\tau} + \lambda_0 \end{pmatrix} \tag{1.176}$$

where $\tau = \beta/\alpha$, and

$$\widetilde{D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\lambda I - S), \tag{1.177}$$

then after the action of the Darboux transformation given by the Darboux matrix $\widetilde{D}$,

$$\widetilde{U}' = \widetilde{D}\widetilde{U}\widetilde{D}^{-1} + \widetilde{D}_x\widetilde{D}^{-1} = \begin{pmatrix} \lambda & u' \\ -1 & -\lambda \end{pmatrix} \tag{1.178}$$

where

$$u' = -u - 2\left(\frac{1}{\tau^2} + \frac{2\lambda_0}{\tau}\right). \tag{1.179}$$

According to the general discussion to the AKNS system, V' is given by the second equation of (1.174). Therefore, the Darboux transformation given by $\widetilde{D}$ in (1.177) is a Darboux transformation from any equation in the KdV hierarchy to the same equation.

Next we compare the results here with those in Section 1.1. If $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a solution of (1.173) for $\lambda = \lambda_0$, then the corresponding solution of (1.159) is

$$R^{-1}(\lambda_0)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\beta \\ \alpha + \lambda_0\beta \end{pmatrix}. \tag{1.180}$$

Let σ be the ratio of the second and the first components, i.e.,

$$\sigma = \frac{\alpha + \lambda_0\beta}{-\beta} = -\frac{1}{\tau} - \lambda_0, \tag{1.181}$$

then σ satisfies

$$\sigma_x = \lambda_0^2 - u - \sigma^2, \tag{1.182}$$

and

$$S = \begin{pmatrix} \sigma & \sigma^2 - \lambda_0^2 \\ -1 & -\sigma \end{pmatrix}. \tag{1.183}$$

In order to get the Darboux matrix for the Lax pair in the form (1.159), let

$$\begin{aligned} D &= R^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(\lambda I - S)R = \begin{pmatrix} -\sigma & 1 \\ \lambda^2 - \lambda_0^2 + \sigma^2 & -\sigma \end{pmatrix} \\ &= \begin{pmatrix} -\sigma & 1 \\ \zeta - \zeta_0 + \sigma^2 & -\sigma \end{pmatrix} \qquad (\zeta_0 = \lambda_0^2). \end{aligned} \tag{1.184}$$

Then

$$\begin{aligned} U' &= DUD^{-1} + D_xD^{-1} \\ &= \begin{pmatrix} 0 & 1 \\ \zeta - 2\zeta_0 + u + 2\sigma^2 & 0 \end{pmatrix}. \end{aligned} \tag{1.185}$$

Hence the action of D keeps the x part of the Lax pair invariant, and transforms u to

$$u' = 2\zeta_0 - u - 2\sigma^2 \tag{1.186}$$

(the same as (1.179)). Theorem 1.11 implies that $V'[u] = V[u']$, i.e., the Darboux transformation keeps the t part invariant. Therefore, we have

THEOREM 1.16 *Suppose u is a solution of (1.166), ζ_0 is a non-zero real constant, $\begin{pmatrix} a \\ b \end{pmatrix}$ is a solution of the Lax pair (1.159) for $\zeta = \zeta_0$, $\sigma = b/a$, then*

$$D = \begin{pmatrix} -\sigma & 1 \\ \zeta - \zeta_0 + \sigma^2 & -\sigma \end{pmatrix} \tag{1.187}$$

is a Darboux matrix for (1.159). It transforms a solution u of (1.166) to a new solution

$$u' = 2\zeta_0 - u - 2\sigma^2 \tag{1.188}$$

of the same equation.

Remark 12 *In order to let $\widetilde{U}'$ and $\widetilde{U}$ have the same $(2,1)$-entry -1, the Darboux matrix (1.184) is chosen as $D = R^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(\lambda I - S)R$, not $R^{-1}(\lambda I - S)R$. This guarantees that the transformation transforms a solution of (1.166) to a solution of the same equation (1.166).*

Remark 13 *Let $\Phi = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$, then ϕ satisfies*

$$\begin{aligned} \phi_{xx} &= (\zeta - u)\phi, \\ \phi_t &= A\phi + B\phi_x. \end{aligned} \tag{1.189}$$

It is similar to the system in Section 1.1. However, here A and B can be polynomials of ζ of arbitrary degrees, whose coefficients are differential polynomials of u. The problem discussed in Section 1.1 was a special case.

In Theorem 1.16, $b = a_x$, hence $\sigma = a_x/a$. The transformation D in (1.184) gives

$$\phi \to \phi' = \phi_x - \sigma\phi = \phi_x - \frac{a_x}{a}\phi, \tag{1.190}$$

and (1.182), (1.186) give the original Darboux transformation

$$u' = u + 2(\ln a)_{xx}. \tag{1.191}$$

Remark 14 *From (1.165), we can get b_0, b_1, $\cdots$ recursively, whose integral constants can be functions of t. Therefore, the coefficients of the* *nonlinear equations can be functions of t, as in the examples (1.170) and (1.171). The solutions of the equations whose coefficients depending on t differ a lot from the solutions of the equations whose coefficients independent of t. In the latter case, each soliton moves in a fixed velocity and the soliton with larger amplitude moves faster. However, in the former case, each soliton can have varying velocity (e.g. oscillates), and the soliton with larger amplitude may move slower.*

1.4.2 MKdV-SG hierarchy

Consider the Lax pair [32]

$$\begin{aligned} \Phi_x = U\Phi = \begin{pmatrix} \lambda & p \\ -p & -\lambda \end{pmatrix} \Phi, \\ \Phi_t = V\Phi = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \Phi, \end{aligned} \tag{1.192}$$

where A, B and C are polynomials of λ and λ^{-1} satisfying

$$A(-\lambda) = -A(\lambda), \qquad B(-\lambda) = -C(\lambda). \tag{1.193}$$

Moreover, suppose

$$A = \sum_{j=0}^{n+m} a_j \lambda^{2n-2j+1} \tag{1.194}$$

($m \geq 0$, $n \geq 0$). Unlike the AKNS system, here A, B and C are not restricted to the polynomials of λ, but the negative powers of λ are allowed. The term with lowest power of λ in (1.194) is $a_{n+m}\lambda^{-2m+1}$.

The integrability condition $U_t - V_x + [U, V] = 0$ leads to

$$\begin{aligned} & A_x = p(B + C), \\ & p_t - B_x - 2pA + 2\lambda B = 0, \\ & p_t + C_x + 2pA + 2\lambda C = 0. \end{aligned} \tag{1.195}$$

Hence

$$\begin{aligned} B + C &= \frac{A_x}{p} = \sum_{j=0}^{n+m} \frac{a_{j,x}}{p} \lambda^{2n-2j+1}, \\ B - C &= \frac{(B + C)_x + 4pA}{2\lambda} \\ &= \sum_{j=0}^{n+m} \left(\frac{1}{2} \left(\frac{a_{j,x}}{p} \right)_x + 2pa_j \right) \lambda^{2n-2j}, \end{aligned} \tag{1.196}$$

(thus $B(-\lambda) = -C(\lambda)$ holds automatically) and

$$p_t = \frac{1}{2}(B-C)_x - \lambda(B+C). \tag{1.197}$$

Comparing the coefficients of λ in (1.197), we can obtain the recursion relations among a_j's. They include two parts. The first part

$$\begin{aligned} &a_{0,x} = 0, \\ &a_{j+1,x} = \frac{1}{4}p\left(\left(\frac{a_{j,x}}{p}\right)_x + 4a_jp\right)_x \qquad (j = 0, 1, \cdots, n-1) \end{aligned} \tag{1.198}$$

are obtained from the coefficients of positive powers of λ and the second part

$$\begin{aligned} &\left(\left(\frac{a_{n+m,x}}{p}\right)_x + 4a_{n+m}p\right)_x = 0, \\ &\left(\left(\frac{a_{j,x}}{p}\right)_x + 4a_jp\right)_x = 4\frac{a_{j+1,x}}{p} \\ &\qquad (j = n+m-1, \cdots, n+1) \end{aligned} \tag{1.199}$$

are obtained from the coefficients of negative powers of λ. Moreover, the term without λ leads to the equation

$$p_t - \frac{1}{4}\left(\left(\frac{a_{n,x}}{p}\right)_x + 4a_np\right)_x + \frac{a_{n+1,x}}{p} = 0. \tag{1.200}$$

The first few a_j's ($0 \le j \le n$) are

$$\begin{aligned} &a_0 = \alpha_0(t), \\ &a_1 = \frac{1}{2}\alpha_0(t)p^2 + \alpha_1(t), \\ &a_2 = \alpha_0(t)\left(\frac{1}{4}pp_{xx} - \frac{1}{8}p_x^2 + \frac{3}{8}p^4\right) + \frac{1}{2}\alpha_1(t)p^2 + \alpha_2(t), \\ &\cdots. \end{aligned} \tag{1.201}$$

If V does not contain negative powers of λ, i.e., $m = 0$, then from the general conclusion to the AKNS system, all a_j's are differential polynomials of p. The equation (1.200) becomes

$$p_t - \frac{1}{4}\left(\left(\frac{a_{n,x}}{p}\right)_x + 4a_np\right)_x = 0. \tag{1.202}$$

This is called the MKdV hierarchy. By using the notion a_j^0 in Section 1.2, these equations can be written as

$$p_t + \sum_{j=0}^{n}\alpha_j(t)M_{n-j}[p] = 0, \tag{1.203}$$

where

$$M_l[p] = -\frac{1}{4}\left(\left(\frac{a^0_{l,x}}{p}\right)_x + 4a^0_l p\right)_x \quad (l = 0, 1, \cdots, n). \tag{1.204}$$

Especially, if $n = 1$ and $\alpha_0 = -4$, $\alpha_1 = 0$, then (1.202) becomes the MKdV equation

$$p_t + p_{xxx} + 6p^2 p_x = 0. \tag{1.205}$$

Next, we consider the negative powers of λ in V. Take $p = -u_x/2$ and suppose it satisfies the boundary condition: $u - k\pi$ and its derivatives tend to 0 fast enough as $x \to -\infty$ (k is an integer).

The first equation of (1.199) gives

$$\left(\left(\frac{a_{n+m,x}}{u_x}\right)_x + a_{n+m}u_x\right)_x = 0. \tag{1.206}$$

Write a_{n+m} as a function of u, then the above equation becomes

$$((a_{n+m,uu} + a_{n+m})u_x)_x = 0. \tag{1.207}$$

The boundary condition as $x \to -\infty$ gives $a_{n+m,uu} + a_{n+m} = 0$. Hence

$$a_{n+m} = \alpha \cos(u + \beta) \tag{1.208}$$

where α and β are constants.

Now take a special a_{n+m}: $a^0_{n+m} = \frac{1}{4}\cos u$. a_{n+j} $(j = 1, 2, \cdots, m-1)$ can be determined as follows. Let $g_{n+j} = a^0_{n+j,x}/p$, then $g_{n+m} = \frac{1}{2}\sin u$, and

$$a^0_{j-1} = \int_{-\infty}^{x} p g_{j-1}\, dx + a^-_{j-1} \quad (n + 2 \le j \le n + m) \tag{1.209}$$

where a^-_{j-1} is the limit of a^0_{j-1} as $x \to -\infty$. From the boundary condition $\lim\limits_{x\to-\infty}(g_{j-1})_x = 0$, the recursion relations (1.199) become

$$\frac{1}{4}(g_{j-1})_x + p\left(a^-_{j-1} + \int_{-\infty}^{x} p g_{j-1}\, dx\right) = \int_{-\infty}^{x} g_j\, dx. \tag{1.210}$$

Moreover, suppose

$$\lim_{x\to-\infty} g_{j-1} = 0, \tag{1.211}$$

then

$$\begin{aligned} &g_{j-1} + 4\int_{-\infty}^{x} p(\xi)\left(a^-_{j-1} + \int_{-\infty}^{\xi} p(\zeta)g_{j-1}(\zeta)\, d\zeta\right) d\xi \\ &= 4\int_{-\infty}^{x}\int_{-\infty}^{\xi} g_j(\zeta)\, d\zeta\, d\xi. \end{aligned} \tag{1.212}$$

This is an integral equation of Volterra type. It has a unique solution in the class of functions which tend to zero fast enough together with its derivatives as $x \to -\infty$.

Take $a_{j-1}^- = 0$ and write the solution of (1.212) as

$$g_{j-1} = Q(g_j) = Q^2(g_{j+1}) = \cdots = \frac{1}{2}Q^{n+m-j+1}[\sin u]. \tag{1.213}$$

Here Q is the operator to determine g_{j-1} from g_j defined by (1.212). g_{j-1} is not a differential polynomial of g_j.

If $n = 0$, $\alpha_0 = 0$, then we obtain the SG hierarchy

$$p_t + \frac{1}{2}\sum_{j=0}^{m-1} \beta_j(t) Q^{m-j-1}[\sin u] = 0 \tag{1.214}$$

where $\beta_j(t)$'s are arbitrary functions of t.

Generally, we have the compound MKdV-SG hierarchy

$$\begin{aligned} &p_t + \sum_{j=0}^{n} \alpha_j(t) M_{n-j}[p] + \frac{1}{2}\sum_{j=0}^{m-1} \beta_j(t) Q^{m-j-1}[\sin u] = 0, \\ &(p = -\tfrac{u_x}{2}). \end{aligned} \tag{1.215}$$

Example 1.17 *$n = 0$, $m = 2$, $\beta_0 = 0$, $\beta_1 = 1$, then, $g_2 = \frac{1}{2}\sin u$, and the equation becomes the sine-Gordon equation*

$$u_{xt} = \sin u. \tag{1.216}$$

Example 1.18 *$n = 1$, $m = 2$, $\alpha_0 = -4$, $\alpha_1 = 0$, $\beta_0 = 0$, $\beta_1 = 1$, then the equation becomes the equation describing one-dimensional nonlinear lattice of atoms [70]*

$$u_{xt} + \frac{3}{2}u_x^2 u_{xx} + u_{xxxx} - \sin u = 0. \tag{1.217}$$

Now we consider the Darboux transformation. If $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a solution of (1.192) for $\lambda = \lambda_0$, then $\begin{pmatrix} -\beta \\ \alpha \end{pmatrix}$ is a solution of (1.192) for $\lambda = -\lambda_0$. Therefore, we can choose

$$\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{pmatrix}, \qquad H = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \tag{1.218}$$

where $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a solution of (1.192) for $\lambda = \lambda_0$. Let $\sigma = \beta/\alpha$,

$$S = H\Lambda H^{-1} = \frac{\lambda_0}{1+\sigma^2}\begin{pmatrix} 1-\sigma^2 & 2\sigma \\ 2\sigma & \sigma^2 - 1 \end{pmatrix} \tag{1.219}$$

and denote $\tan\frac{\theta}{2} = \sigma$, then

$$S = \lambda_0 \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{1.220}$$

From $\sigma_x = -p(1+\sigma^2) - 2\lambda_0\sigma$, we have

$$\theta_x = -2p - 2\lambda_0 \sin\theta. \tag{1.221}$$

By direct calculation,

$$(\lambda I - S)U(\lambda I - S)^{-1} - S_x(\lambda I - S)^{-1} = \begin{pmatrix} \lambda & p' \\ -p' & -\lambda \end{pmatrix} \tag{1.222}$$

where

$$p' = p + 2\lambda_0 \sin\theta = -p - \theta_x, \tag{1.223}$$

or equivalently,

$$u' = -u + 2\theta \tag{1.224}$$

for suitable choice of the integral constant.

It remains to prove that the Darboux matrix $\lambda I - S$ keeps the reduction of MKdV-SG hierarchy. This includes (1) the transformed A', B' and C' still satisfy $A'(-\lambda) = -A'(\lambda)$ and $B'(-\lambda) = -C(\lambda)$; (2) the coefficients $\alpha_j(t)$'s keeps invariant.

Since $V^T(-\lambda) = -V(\lambda)$, $S^T = S$ and $(\lambda I + S)^T(\lambda I - S) = \lambda^2 I - S^2 = (\lambda^2 - \lambda_0^2)I$, it can be verified by direct calculation that $V'^T(-\lambda) = -V'(\lambda)$ holds. This proves (1).

(2) is proved as follows. For a_j ($j \le n$), this has been proved for the AKNS system; for a_j ($j \ge n+1$), the conclusion follows from the boundary condition at infinity.

Therefore, the following theorem holds.

THEOREM 1.19 *Suppose u is a solution of (1.200), λ_0 is a non-zero real number, $\begin{pmatrix} a \\ b \end{pmatrix}$ is a solution of the Lax pair (1.192) for $\lambda = \lambda_0$. Let*

$\theta = 2\tan^{-1}(b/a)$,

$$S = \lambda_0 \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \tag{1.225}$$

then $\lambda I - S$ is a Darboux matrix for (1.192). It transforms a solution p of (1.200) to the solution $p' = p + 2\lambda_0 \sin\theta$ of the same equation. Moreover, $u' = -u + 2\theta$ with suitable boundary condition, where $p = -u_x/2$, $p' = -u'_x/2$.

Remark 15 For the sine-Gordon equation, the Bäcklund transformation is a kind of method to get explicit solutions, which was known in the nineteenth century. In that method, to obtain a new solution from a known solution, there is an integrable system of differential equations to be solved (moreover, one can obtain explicit expression by using the theorem of permutability and the nonlinear superposition formula). Using Darboux transformation, that explicit expression can be obtained directly. This will be discussed in Chapter 4 together with the related geometric problems.

1.4.3 NLS hierarchy

The Lax pair for the nonlinear Schrödinger hierarchy (NLS hierarchy) is

$$\begin{aligned} \Phi_x = U\Phi &= \begin{pmatrix} \lambda & p \\ -\bar{p} & -\lambda \end{pmatrix} \Phi, \\ \Phi_t = V\Phi &= \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \Phi, \end{aligned} \tag{1.226}$$

where A, B and C are polynomials of λ (λ, p, A, B and C are complex-valued) satisfying

$$A(-\bar{\lambda}) = -\overline{A(\lambda)}, \qquad B(-\bar{\lambda}) = -\overline{C(\lambda)} \tag{1.227}$$

(i.e., $V^*(-\bar{\lambda}) = -V(\lambda)$ where $*$ refers to the complex conjugate transpose of a matrix). This is also a special case of the AKNS system. We shall construct a Darboux matrix keeping this reduction.

The integrability condition $U_t - V_x + [U, V] = 0$ is

$$\begin{aligned} A_x &= pC + \bar{p}B, \\ B_x &= p_t + 2\lambda B - 2pA, \\ C_x &= -\bar{p}_t - 2\lambda C - 2\bar{p}A. \end{aligned} \tag{1.228}$$

We can use (1.55) to write down the coefficients of the powers of λ in A, B and C. They depend on $p, p_x, \cdots$ and the integral constants $\alpha_j(t)$. Moreover, there is a nonlinear evolution equation

$$p_t = b_{n,x} + 2pa_n. \tag{1.229}$$

Especially, for $n = 2$, $\alpha_0 = -2\mathrm{i}$, $\alpha_1 = \alpha_2 = 0$, the equation is the nonlinear Schrödinger equation

$$\mathrm{i}p_t = p_{xx} + 2|p|^2 p. \tag{1.230}$$

The Darboux transformation for the nonlinear Schrödinger hierarchy is also constructed from the choice of Λ and H. Suppose $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a solution of (1.226) for $\lambda = \lambda_0$, then $\begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix}$ is a solution of (1.226) for $\lambda = -\bar{\lambda}_0$. Hence, we can choose

$$\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\bar{\lambda}_0 \end{pmatrix}, \qquad H = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \tag{1.231}$$

$$S = H\Lambda H^{-1} = \frac{1}{1+|\sigma|^2} \begin{pmatrix} \lambda_0 - \bar{\lambda}_0|\sigma|^2 & (\lambda_0 + \bar{\lambda}_0)\bar{\sigma} \\ (\lambda_0 + \bar{\lambda}_0)\sigma & -\bar{\lambda}_0 + \lambda_0|\sigma|^2 \end{pmatrix}, \tag{1.232}$$

where $\sigma = \beta/\alpha$. Since $\det H \neq 0$, S can be defined globally. It can be checked that H satisfies

$$H^* H = |\alpha|^2 + |\beta|^2, \tag{1.233}$$

hence S satisfies

$$\begin{aligned} &S^* S = |\lambda_0|^2, \\ &S - S^* = \lambda_0 - \bar{\lambda}_0. \end{aligned} \tag{1.234}$$

Therefore, under the action of the Darboux matrix $\lambda I - S$, U is transformed to

$$U' = (\lambda I - S)U(\lambda I - S)^{-1} - S_x(\lambda I - S)^{-1} = \begin{pmatrix} \lambda & p' \\ -\bar{p}' & -\lambda \end{pmatrix} \tag{1.235}$$

where

$$p' = p + 2S_{12} = p + \frac{2(\lambda_0 + \bar{\lambda}_0)\bar{\sigma}}{1+|\sigma|^2}. \tag{1.236}$$

From the discussion on the AKNS system (Theorem 1.11), we know that $V' = (\lambda I - S)V(\lambda I - S)^{-1} - S_t(\lambda I - S)^{-1}$ is also a polynomial of λ and ${V'}^*(-\bar{\lambda}) = -V'(\lambda)$ holds. Moreover, $\lambda I - S$ gives a Darboux transformation from an equation in the nonlinear Schrödinger hierarchy to the same equation. This leads to the following theorem.

THEOREM 1.20 *Suppose p is a solution of (1.229), λ_0 is a non-real complex number, $\begin{pmatrix} a \\ b \end{pmatrix}$ is a solution of the Lax pair (1.226) for $\lambda = \lambda_0$. Let $\sigma = b/a$,*

$$S = \frac{1}{1+|\sigma|^2}\begin{pmatrix} \lambda_0 - \bar{\lambda}_0|\sigma|^2 & (\lambda_0 + \bar{\lambda}_0)\bar{\sigma} \\ (\lambda_0 + \bar{\lambda}_0)\sigma & -\bar{\lambda}_0 + \lambda_0|\sigma|^2 \end{pmatrix}, \tag{1.237}$$

then $\lambda I - S$ is a Darboux matrix for (1.226). It transforms a solution p of (1.229) to a solution

$$p' = p + \frac{2(\lambda_0 + \bar{\lambda}_0)\bar{\sigma}}{1+|\sigma|^2} \tag{1.238}$$

of the same equation.

1.4.4 AKNS system with $u(N)$ reduction

For the nonlinear Schrödinger hierarchy, (1.226) and (1.227) imply

$$U(-\bar{\lambda}) = -U(\lambda)^*, \quad V(-\bar{\lambda}) = -V(\lambda)^*. \tag{1.239}$$

Here we generalize it to the AKNS system.

For the AKNS system (1.226), if U and V satisfy (1.239), then we say that (1.226) has $u(N)$ reduction, because $U(\lambda)$ and $V(\lambda)$ are in the Lie algebra $u(N)$ when λ is purely imaginary. This is a very popular reduction.

We want to construct Darboux matrix which keeps $u(N)$ reduction. That is, after the action of the Darboux matrix, the derived potentials $U'(\lambda)$ and $V'(\lambda)$ must satisfy

$$U'(-\bar{\lambda}) = -U'(\lambda)^*, \quad V'(-\bar{\lambda}) = -V'(\lambda)^*. \tag{1.240}$$

With this additional condition, Λ and H in (1.94) can not be arbitrary. They should satisfy the following two conditions:

(1) $\lambda_1, \cdots, \lambda_N$ can only be μ or $-\bar{\mu}$ where μ is a complex number (μ is not real).

(2) If $\lambda_j \neq \lambda_k$, then

$$h_j^* h_k = 0 \tag{1.241}$$

holds at one point (x_0, t_0).

In fact, if (1.241) holds at one point, then it holds everywhere. This is proved as follows.

When $\lambda_j \neq \lambda_k$, $\lambda_k = -\bar{\lambda}_j$, hence

$$\begin{aligned} &h_{k,x} = U(\lambda_k)h_k, \quad h_{k,t} = V(\lambda_k)h_k, \\ &h_{j,x}^* = h_j^* U(\lambda_j)^* = -h_j^* U(-\bar{\lambda}_j) = -h_j^* U(\lambda_k), \\ &h_{j,t}^* = h_j^* V(\lambda_j)^* = -h_j^* V(-\bar{\lambda}_j) = -h_j^* V(\lambda_k). \end{aligned} \tag{1.242}$$

This implies that

$$(h_j^* h_k)_x = 0, \quad (h_j^* h_k)_t = 0. \tag{1.243}$$

Therefore, $h_j^* h_k = 0$ holds everywhere if it holds at one point.

THEOREM 1.21 *If λ_j's, h_j's satisfy the above conditions (1) and (2), $H = (h_1, \cdots, h_N)$, then $\det H \neq 0$ holds everywhere if it holds at one point. Moreover, U' and V' given by (1.81) satisfy*

$$U'(-\bar{\lambda}) = -U'(\lambda)^*, \quad V'(-\bar{\lambda}) = -V'(\lambda)^*. \tag{1.244}$$

Proof. Let (x_0, t_0) be a fixed point. Then by the property of linear differential equation, all $\{h_\alpha\}$ with $\lambda_\alpha = \mu$ are linearly independent if they are linearly independent at (x_0, t_0). Likewise, all $\{h_\alpha\}$ with $\lambda_\alpha = \bar{\mu}$ are also linearly independent if they are linearly independent at (x_0, t_0). Moreover, (1.241) implies that all $\{h_1, \cdots, h_N\}$ are linearly independent if they are linearly independent at (x_0, t_0). Therefore, $\det H \neq 0$ and $S = H\Lambda H^{-1}$ is globally defined.

According to the definition of S,

$$Sh_j = \lambda_j h_j, \quad h_k^* S^* = h_k^* \bar{\lambda}_k. \tag{1.245}$$

Hence

$$h_k^*(S - S^*)h_j = (\lambda_j - \bar{\lambda}_k)h_k^* h_j. \tag{1.246}$$

If $\lambda_j = \mu$, $\lambda_k = -\bar{\mu}$, then

$$h_k^*(S - S^*)h_j = 0. \tag{1.247}$$

If $\lambda_j = \lambda_k = \mu$ (or $\lambda_j = \lambda_k = -\bar{\mu}$), then

$$h_k^*(S - S^*)h_j = (\mu - \bar{\mu})h_k^* h_j. \tag{1.248}$$

Hence

$$S - S^* = (\mu - \bar{\mu})I. \tag{1.249}$$

On the other hand, from (1.245), we have

$$h_k^* S^* S h_j = \lambda_j \bar{\lambda}_k h_k^* h_j. \tag{1.250}$$

If $\lambda_j = \mu$, $\lambda_k = -\bar{\mu}$, then

$$h_k^* S^* S h_j = 0. \tag{1.251}$$

If $\lambda_j = \lambda_k = \mu$ (or $\lambda_j = \lambda_k = -\bar{\mu}$), then

$$h_k^* S^* S h_j = |\mu|^2 h_k^* h_j, \tag{1.252}$$

Therefore,

$$S^* S = |\mu|^2 I. \tag{1.253}$$

From (1.249) and (1.253), we obtain

$$(\bar{\lambda} I + S)^* (\lambda I - S) = (\lambda - \mu)(\lambda + \bar{\mu}) I. \tag{1.254}$$

According to the action of the Darboux transformation on V_j,

$$\begin{aligned}
&\sum_{j=0}^{m} V_j' \lambda^{m-j} \\
&= (\lambda I - S) \sum_{j=0}^{m} V_j \lambda^{m-j} (\lambda I - S)^{-1} + (\lambda I - S)_t (\lambda I - S)^{-1},
\end{aligned} \tag{1.255}$$

$$\begin{aligned}
&(\sum_{j=0}^{m} V_j' \lambda^{m-j})^* \\
&= (\lambda I - S)^{*-1} \sum_{j=0}^{m} V_j^* \bar{\lambda}^{m-j} (\lambda I - S)^* + (\lambda I - S)^{*-1} (\lambda I - S)_t^* \\
&= -(\bar{\lambda} I + S) \sum_{j=0}^{m} V_j (-\bar{\lambda})^{m-j} (\bar{\lambda} I + S)^{-1} - (\bar{\lambda} I + S)_t (\bar{\lambda} I + S)^{-1} \\
&= -\sum_{j=0}^{m} V_j' (-\bar{\lambda})^{m-j}.
\end{aligned} \tag{1.256}$$

Hence $V'(-\bar{\lambda}) = -V'(\lambda)^*$. Likewise, $U'(-\bar{\lambda}) = -U'(\lambda)^*$. The theorem is proved.

As in Section 1.3, a Darboux transformation of higher degree can be derived by the composition of Darboux transformations of degree one. However, with the $u(N)$ reduction, we have also the following special and more direct construction [117, 17].

Suppose we take l times of Darboux transformations of degree one. Each Darboux transformation is constructed from Λ_α, H_α ($\alpha = 1, \cdots, l$).

In each $\Lambda_\alpha = \operatorname{diag}(\lambda_1^{(\alpha)}, \cdots, \lambda_k^{(\alpha)})$, suppose $\lambda_1^{(\alpha)} = \cdots = \lambda_k^{(\alpha)} = \mu_\alpha$, $\lambda_{k+1}^{(\alpha)} = \cdots = \lambda_N^{(\alpha)} = -\bar{\mu}_\alpha$. Here k is the same for all α. For each $\lambda_j^{(\alpha)}$, solve the Lax pair and get a solution $h_j^{(\alpha)}$ satisfying the orthogonal relations (1.241).

Denote $H_\alpha = (h_1^{(\alpha)}, \cdots, h_N^{(\alpha)})$, $\mathring{H}_\alpha = (h_1^{(\alpha)}, \cdots, h_k^{(\alpha)})$. Let

$$\Gamma_{\alpha\beta} = \frac{\mathring{H}_\alpha^* \mathring{H}_\beta}{\mu_\beta + \bar{\mu}_\alpha}, \tag{1.257}$$

$$D(\lambda) = \prod_{\gamma=1}^{l} (\lambda + \bar{\mu}_\gamma) \left(1 - \sum_{\alpha,\beta=1}^{l} \frac{\mathring{H}_\alpha (\Gamma^{-1})_{\alpha\beta} \mathring{H}_\beta^*}{\lambda + \bar{\lambda}_\beta} \right). \tag{1.258}$$

Now we prove that $D(\lambda)$ is a Darboux matrix.

Let $\hat{H}_\alpha = (h_{k+1}^{(\alpha)}, \cdots, h_N^{(\alpha)})$. Then $H_\alpha = (\mathring{H}_\alpha, \hat{H}_\alpha)$ and $\mathring{H}_\alpha^* \hat{H}_\alpha = 0$ for all $\alpha = 1, \cdots, l$. Hence

$$D(\mu_\alpha) \mathring{H}_\alpha = 0, \quad D(-\bar{\mu}_\alpha) \hat{H}_\alpha = 0. \tag{1.259}$$

According to Theorem 1.12, $D(\lambda)$ is a Darboux matrix.

Moreover, the inverse of $D(\lambda)$ can be written out explicitly as

$$D(\lambda)^{-1} = \prod_{\gamma=1}^{l} (\lambda + \bar{\mu}_\gamma)^{-1} \left(1 + \sum_{\alpha,\beta=1}^{l} \frac{\mathring{H}_\alpha (\Gamma^{-1})_{\alpha\beta} \mathring{H}_\beta^*}{\lambda - \lambda_\alpha} \right). \tag{1.260}$$

(1.258) gives a compact form of Darboux matrix of higher degree. Although it is special, it is very useful.

1.5 Darboux transformation and scattering, inverse scattering theory

The scattering and inverse scattering theory is an important part of the soliton theory. It transforms the problem of solving the Cauchy problem of a nonlinear partial differential equation to the problem of describing the spectrum and eigenfunctions of the Lax pair. Here we consider the 2×2 AKNS system as an example to show the outline of the scattering and inverse scattering theory (see [23] for details). Moreover, we discuss the change of the scattering data under Darboux transformation for $su(2)$ reduction. For the KdV equation, the problem can be solved similarly, but the scattering and inverse scattering theory is simpler.

1.5.1 Outline of the scattering and inverse scattering theory for the 2×2 AKNS system

First, we give the definition of the scattering data for the 2×2 complex AKNS system. In order to coincide with the usual scattering theory, let $\lambda=-\mathrm{i}\zeta$, then the first equation of the 2×2 AKNS system (1.48) becomes

$$\Phi_x=\begin{pmatrix}-\mathrm{i}\zeta & p\\ q & \mathrm{i}\zeta\end{pmatrix}\Phi. \tag{1.261}$$

Suppose p, q and their derivatives with respect to x decay fast enough at infinity. Let $\mathbf{C}$ be the complex plane and $\mathbf{R}$ be the real line. Besides $\mathbf{C}_+$ and $\mathbf{C}_-$ are the upper and lower half plane of $\mathbf{C}$ respectively, i.e., $\mathbf{C}_+=\{z\in\mathbf{C}\mid\ \operatorname{Im}\zeta>0\}$, $\mathbf{C}_-=\{z\in\mathbf{C}\mid\ \operatorname{Im}\zeta<0\}$.

Property 1. For each one of the following boundary conditions, the equation (1.261) has a unique column solution

$$\text{(1)}\qquad \psi_r(x,\zeta)=R(x,\zeta)\mathrm{e}^{-\mathrm{i}\zeta x},\quad \lim_{x\to-\infty}R(x,\zeta)=\begin{pmatrix}1\\0\end{pmatrix},\quad (\operatorname{Im}\zeta\geq 0), \tag{1.262}$$

$$\text{(2)}\qquad \widetilde{\psi}_r(x,\zeta)=\widetilde{R}(x,\zeta)\mathrm{e}^{\mathrm{i}\zeta x},\quad \lim_{x\to-\infty}\widetilde{R}(x,\zeta)=\begin{pmatrix}0\\1\end{pmatrix},\quad (\operatorname{Im}\zeta\leq 0), \tag{1.263}$$

$$\text{(3)}\qquad \psi_l(x,\zeta)=L(x,\zeta)\mathrm{e}^{\mathrm{i}\zeta x},\quad \lim_{x\to+\infty}L(x,\zeta)=\begin{pmatrix}0\\1\end{pmatrix},\quad (\operatorname{Im}\zeta\geq 0), \tag{1.264}$$

$$\text{(4)}\qquad \widetilde{\psi}_l(x,\zeta)=\widetilde{L}(x,\zeta)\mathrm{e}^{-\mathrm{i}\zeta x},\quad \lim_{x\to+\infty}\widetilde{L}(x,\zeta)=\begin{pmatrix}1\\0\end{pmatrix},\quad (\operatorname{Im}\zeta\leq 0). \tag{1.265}$$

Moreover, ψ_r, ψ_l (resp. $\widetilde{\psi}_r$, $\widetilde{\psi}_l$) are continuous for $\zeta\in\mathbf{C}_+\cup\mathbf{R}$ (resp. $z\in\mathbf{C}_-\cup\mathbf{R}$), and holomorphic with respect to ζ in $\mathbf{C}_+$ (resp. $\mathbf{C}_-$). These solutions are called Jost solutions.

If $\zeta \in \mathbf{R}$, then ψ_l and $\widetilde{\psi}_l$ are linearly independent. Hence, there exist functions $r_+(\zeta)$, $r_-(\zeta)$, $\widetilde{r}_+(\zeta)$ and $\widetilde{r}_-(\zeta)$ such that

$$\begin{aligned} \psi_r &= r_+\psi_l + r_-\widetilde{\psi}_l, \\ \widetilde{\psi}_r &= \widetilde{r}_+\psi_l + \widetilde{r}_-\widetilde{\psi}_l. \end{aligned} \tag{1.266}$$

Considering the Wronskian determinant between ψ_r, ψ_l and the Wronskian determinant between $\widetilde{\psi}_r$, $\widetilde{\psi}_l$, we have

Property 2. For $\zeta \in \mathbf{R}$,

$$\begin{aligned} r_-(\zeta) &= R_1(x,\zeta)L_2(x,\zeta) - R_2(x,\zeta)L_1(x,\zeta), \\ \widetilde{r}_+(\zeta) &= \widetilde{R}_2(x,\zeta)\widetilde{L}_1(x,\zeta) - \widetilde{R}_1(x,\zeta)\widetilde{L}_2(x,\zeta). \end{aligned} \tag{1.267}$$

$r_-(\zeta)$ can be holomorphically extended to $\mathbf{C}_+ \cup \mathbf{R}$, and $\widetilde{r}_+(\zeta)$ can be holomorphically extended to $\mathbf{C}_- \cup \mathbf{R}$. Here R_1 and R_2 are two components of the vector R, i.e., $R = (R_1, R_2)^T$. L_1, L_2, $\widetilde{R}_1$, $\widetilde{R}_2$, $\widetilde{L}_1$, $\widetilde{L}_2$ have the similar meanings.

The asymptotic properties of the four Jost solutions in Property 1 as $x \to \pm\infty$ are listed in the next property.

Property 3. (1) The following limits hold uniformly for ζ:

$$\begin{aligned} &\lim_{x\to-\infty} R(x,\zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \zeta \in \mathbf{C}_+ \cup \mathbf{R}, \\ &\lim_{x\to-\infty} \widetilde{R}(x,\zeta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \zeta \in \mathbf{C}_- \cup \mathbf{R}, \\ &\lim_{x\to+\infty} L(x,\zeta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \zeta \in \mathbf{C}_+ \cup \mathbf{R}, \\ &\lim_{x\to+\infty} \widetilde{L}(x,\zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \zeta \in \mathbf{C}_- \cup \mathbf{R}. \end{aligned} \tag{1.268}$$

(2) The following limits hold uniformly for ζ in a compact subset:

$$\begin{aligned}
\lim_{x\to+\infty} R(x,\zeta) &= \begin{pmatrix} r_-(\zeta) \\ 0 \end{pmatrix}, \qquad \zeta \in \mathbf{C}_+, \\
\lim_{x\to+\infty} \widetilde{R}(x,\zeta) &= \begin{pmatrix} 0 \\ \widetilde{r}_+(\zeta) \end{pmatrix}, \qquad \zeta \in \mathbf{C}_-, \\
\lim_{x\to-\infty} L(x,\zeta) &= \begin{pmatrix} 0 \\ r_-(\zeta) \end{pmatrix}, \qquad \zeta \in \mathbf{C}_+, \\
\lim_{x\to-\infty} \widetilde{L}(x,\zeta) &= \begin{pmatrix} \widetilde{r}_+(\zeta) \\ 0 \end{pmatrix}, \qquad \zeta \in \mathbf{C}_-.
\end{aligned} \tag{1.269}$$

(3) The following limits hold uniformly for real $\zeta \in \mathbf{R}$:

$$\begin{aligned}
\lim_{x\to+\infty} \left| R(x,\zeta) - \begin{pmatrix} r_-(\zeta) \\ r_+(\zeta)\mathrm{e}^{2\mathrm{i}\zeta x} \end{pmatrix} \right| &= 0, \qquad \zeta \in \mathbf{R}, \\
\lim_{x\to+\infty} \left| \widetilde{R}(x,\zeta) - \begin{pmatrix} \widetilde{r}_-(\zeta)\mathrm{e}^{-2\mathrm{i}\zeta x} \\ \widetilde{r}_+(\zeta) \end{pmatrix} \right| &= 0, \qquad \zeta \in \mathbf{R}, \\
\lim_{x\to-\infty} \left| L(x,\zeta) - \begin{pmatrix} -\widetilde{r}_-(\zeta)\mathrm{e}^{-2\mathrm{i}\zeta x} \\ r_-(\zeta) \end{pmatrix} \right| &= 0, \qquad \zeta \in \mathbf{R}, \\
\lim_{x\to-\infty} \left| \widetilde{L}(x,\zeta) - \begin{pmatrix} \widetilde{r}_+(\zeta) \\ -r_+(\zeta)\mathrm{e}^{2\mathrm{i}\zeta x} \end{pmatrix} \right| &= 0, \qquad \zeta \in \mathbf{R}.
\end{aligned}$$

Rewrite (1.261) as

$$\mathcal{L}\Phi = \zeta\Phi, \tag{1.270}$$

where

$$\mathcal{L} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \left(\frac{\mathrm{d}}{\mathrm{d}x} - \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \right), \tag{1.271}$$

then (1.261) becomes a spectral problem of a linear ordinary differential operator. We consider its spectrum in $L^2(\mathbf{R}) \times L^2(\mathbf{R})$.

If $\zeta \in \mathbf{C}_+$ and $r_-(\zeta) = 0$, then (1.267) implies that ψ_r and ψ_l are linearly dependent. Hence $\psi_r \to 0$ as $x \to \pm\infty$. Similarly, if $\zeta \in \mathbf{C}_-$ and $\widetilde{r}_+(\zeta) = 0$, then $\widetilde{\psi}_r$ and $\widetilde{\psi}_l$ are linearly dependent. Hence $\widetilde{\psi}_r \to 0$ as $x \to \pm\infty$. Since $r_-(\zeta)$ and $\widetilde{r}_+(\zeta)$ are holomorphic in $\mathbf{C}_+$ and $\mathbf{C}_-$

respectively, their zeros are discrete. These zeros are the eigenvalues of $\mathcal{L}$. The set of all eigenvalues of $\mathcal{L}$ is denoted by $IP\sigma(\mathcal{L})$. If $\zeta \in \mathbf{R}$, then it can be proved that (1.270) has a nontrivial bounded solution. $\sigma(\mathcal{L}) = \mathbf{R} \cup IP\sigma(\mathcal{L})$ is called the spectrum of the operator $\mathcal{L}$. Its compliment $\mathbf{C} - \sigma(\mathcal{L})$ is called the regular set of $\mathcal{L}$.

Property 4. If $r_-(\zeta) \neq 0$ and $\widetilde{r}_+(\zeta) \neq 0$ hold for $\zeta \in \mathbf{R}$, then $IP\sigma(\mathcal{L})$ is a finite set.

Hereafter, we always suppose $r_-(\zeta) \neq 0$ and $r_+(\zeta) \neq 0$ when $\zeta \in \mathbf{R}$. First we consider the eigenvalues.

If $\zeta \in IP\sigma(\mathcal{L})$, then ψ_r and ψ_l are linearly dependent. Suppose

$$\begin{aligned} &\psi_r(x,\zeta) = \alpha(\zeta)\psi_l(x,\zeta) \quad (\zeta \in \mathbf{C}_+ \cap IP\sigma(\mathcal{L})), \\ &\widetilde{\psi}_r(x,\zeta) = \widetilde{\alpha}(\zeta)\widetilde{\psi}_l(x,\zeta) \quad (\zeta \in \mathbf{C}_- \cap IP\sigma(\mathcal{L})). \end{aligned} \tag{1.272}$$

Denote $IP\sigma(\mathcal{L}) \cap \mathbf{C}_+ = \{\zeta_1, \cdots \zeta_d\}$ and $IP\sigma(\mathcal{L}) \cap \mathbf{C}_- = \{\widetilde{\zeta}_1, \cdots \widetilde{\zeta}_{\widetilde{d}}\}$ to be the set of eigenvalues in $\mathbf{C}_+$ and $\mathbf{C}_-$ respectively. Moreover, suppose $\zeta_1, \cdots, \widetilde{\zeta}_{\widetilde{d}}$ are all simple zeros. Corresponding to each eigenvalue, there is a constant

$$\begin{aligned} &C_k = \alpha(\zeta_k) \Big/ \frac{\mathrm{d}r_-(\zeta_k)}{\mathrm{d}\zeta} \quad (k = 1, \cdots, d), \\ &\widetilde{C}_k = \widetilde{\alpha}(\zeta_k) \Big/ \frac{\mathrm{d}\widetilde{r}_+(\zeta_k)}{\mathrm{d}\zeta} \quad (k = 1, \cdots, \widetilde{d}). \end{aligned} \tag{1.273}$$

Using these data, we define the functions

$$B_d(y) = -\mathrm{i}\sum_{k=1}^{d} C_k \mathrm{e}^{\mathrm{i}\zeta_k y}, \quad \widetilde{B}_d(y) = \mathrm{i}\sum_{k=1}^{\widetilde{d}} \widetilde{C}_k \mathrm{e}^{-\mathrm{i}\zeta_k y}. \tag{1.274}$$

Next, we consider the continuous spectrum $\zeta \in \mathbf{R}$. As is known, the Fourier transformation of a Schwarz function ϕ is

$$F(\phi)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(s)\mathrm{e}^{-\mathrm{i}ks}\, ds. \tag{1.275}$$

It can be extended to $L^2(\mathbf{R})$ and becomes a bounded map from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$.

Property 5.

$$L(x,\cdot) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in L^2(\mathbf{R}), \quad \widetilde{L}(x,\cdot) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in L^2(\mathbf{R}). \tag{1.276}$$

Denote

$$\begin{aligned} N(x,s) &= \frac{1}{\sqrt{2\pi}} F\left(L(x,\cdot) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)(s), \\ \widetilde{N}(x,s) &= \tfrac{1}{\sqrt{2\pi}} F\left(\widetilde{L}(x,\cdot) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)(s), \quad (s \geq 0), \end{aligned} \tag{1.277}$$

then

$$\begin{aligned} L(x,\zeta) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^{+\infty} N(x,s) \mathrm{e}^{\mathrm{i}\zeta s}\, ds, \quad \forall \zeta \in \mathbf{C}_+ \cup \mathbf{R}, \\ \widetilde{L}(x,\zeta) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \textstyle\int_0^{+\infty} \widetilde{N}(x,s) \mathrm{e}^{-\mathrm{i}\zeta s}\, ds, \quad \forall \zeta \in \mathbf{C}_- \cup \mathbf{R}, \end{aligned} \tag{1.278}$$

and the above integrals converge absolutely. Moreover,

$$p(x) = -2N_1(x,0), \qquad q(x) = -2\widetilde{N}_2(x,0), \tag{1.279}$$

where the subscripts refer to the components.

For $\zeta \in \mathbf{R}$, denote

$$b(\zeta) = \frac{r_+(\zeta)}{r_-(\zeta)}, \quad \widetilde{b}(\zeta) = \frac{\widetilde{r}_-(\zeta)}{\widetilde{r}_+(\zeta)}. \tag{1.280}$$

It can be proved that

$$b, \widetilde{b} \in L^2(\mathbf{R}) \cap L^1(\mathbf{R}) \cap C^0(\mathbf{R}). \tag{1.281}$$

Hence, we can define

$$\begin{aligned} B_c(y) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} b(\zeta) \mathrm{e}^{\mathrm{i}\zeta y}\, d\zeta, \\ \widetilde{B}_c(y) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widetilde{b}(\zeta) \mathrm{e}^{-\mathrm{i}\zeta y}\, d\zeta. \end{aligned} \tag{1.282}$$

The data

$$\{\zeta_k, C_k\, (k = 1, \cdots, d),\ \widetilde{\zeta}_k, \widetilde{C}_k\, (k = 1, \cdots, \widetilde{d}),\ b(\zeta), \widetilde{b}(\zeta)\, (\zeta \in \mathbf{R})\} \tag{1.283}$$

are called the scattering data corresponding to (p,q), denoted by $\Sigma(p,q)$. We can also call

$$\{r_-(\zeta)\ (\zeta \in \mathbf{C}_+ \cup \mathbf{R}),\ \widetilde{r}_+(\zeta)\ (\zeta \in \mathbf{C}_- \cup \mathbf{R})\} \tag{1.284}$$

the scattering data, since the data in (1.283) can be obtained from the data in (1.284).

Define $B = B_c + B_d$ and $\widetilde{B} = \widetilde{B}_c + \widetilde{B}_d$ according to (1.274) and (1.282).

Property 6. N and $\widetilde{N}$ satisfy the follow system of linear integral equations (Gelfand-Levitan-Marchenko equations)

$$\begin{aligned} &N(x,s) + \widetilde{B}(2x+s)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^{+\infty} \widetilde{N}(x,\sigma)\widetilde{B}(2x+s+\sigma)\,d\sigma = 0, \\ &\widetilde{N}(x,s) + B(2x+s)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^{+\infty} N(x,\sigma)B(2x+s+\sigma)\,d\sigma = 0. \end{aligned} \tag{1.285}$$

If the scattering data $\{\zeta_k, C_k, \widetilde{\zeta}_k, \widetilde{C}_k, b(\zeta), \widetilde{b}(\zeta)\}$ are known, N and $\widetilde{N}$ are solved from the above integral equations and (1.279) gives (p,q).

The process to get scattering data from (p,q) is called the scattering process. It needs to solve the spectral problem of ordinary differential equations. The process to get (p,q) from the scattering data is called the inverse scattering process. It needs to solve linear integral equations.

Now we consider the evolution of the scattering data. In the AKNS system, p and q are functions of (x,t). Therefore, we should consider the full AKNS system (with time t)

$$\Phi_x = \begin{pmatrix} -\mathrm{i}\zeta & p \\ q & \mathrm{i}\zeta \end{pmatrix}\Phi, \quad \Phi_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\Phi, \tag{1.286}$$

where A, B and C are polynomials of ζ,

$$\begin{aligned} A &= \sum_{j=0}^{n} a_j(-\mathrm{i}\zeta)^{n-j}, \\ B &= \sum_{j=0}^{n} b_j(-\mathrm{i}\zeta)^{n-j}, \\ C &= \sum_{j=0}^{n} c_j(-\mathrm{i}\zeta)^{n-j}. \end{aligned} \tag{1.287}$$

We also suppose

$$A|_{p=q=0} = \mathrm{i}\omega(\zeta,t). \tag{1.288}$$

Lemma 1.5 implies $B|_{p=q=0} = C|_{p=q=0} = 0$.

Property 7. Suppose (p,q) satisfies the equations

$$p_t = b_{n,x} + 2pa_n, \qquad q_t = c_{n,x} - 2qa_n \tag{1.289}$$

given by the integrability condition, then the evolution of the corresponding scattering data is given by

$$\begin{aligned}
&r_-(\zeta,t) = r_-(\zeta,0) \qquad \zeta \in \mathbf{C}_+ \cup \mathbf{R},\\
&\widetilde{r}_+(\zeta,t) = \widetilde{r}_+(\zeta,0) \qquad \zeta \in \mathbf{C}_- \cup \mathbf{R},\\
&r_+(\zeta,t) = r_+(\zeta,0)\exp(-2\mathrm{i}\int_0^t \omega(\zeta,\tau)\,d\tau) \qquad \zeta \in \mathbf{R},\\
&\widetilde{r}_-(\zeta,t) = \widetilde{r}_-(\zeta,0)\exp(2\mathrm{i}\int_0^t \omega(\zeta,\tau)\,d\tau) \qquad \zeta \in \mathbf{R},
\end{aligned} \tag{1.290}$$

and

$$\begin{aligned}
&\zeta_k(t) = \zeta_k(0),\\
&\widetilde{\zeta}_k(t) = \widetilde{\zeta}_k(0),\\
&C_k(t) = C_k(0)\exp(-2\mathrm{i}\int_0^t \omega(\zeta,\tau)\,d\tau),\\
&\widetilde{C}_k(t) = \widetilde{C}_k(0)\exp(2\mathrm{i}\int_0^t \omega(\zeta,\tau)\,d\tau),\\
&b(\zeta,t) = b(\zeta,0)\exp(-2\mathrm{i}\int_0^t \omega(\zeta,\tau)\,d\tau),\\
&\widetilde{b}(\zeta,t) = \widetilde{b}(\zeta,0)\exp(2\mathrm{i}\int_0^t \omega(\zeta,\tau)\,d\tau).
\end{aligned} \tag{1.291}$$

(1.290) or (1.291) gives the evolution of the scattering data explicitly.

In summary, the process of solving the initial value problem of nonlinear evolution equations (1.289) of (p,q) is as follows. Here the initial condition is $t=0: p=p_0,\ q=q_0$.

For given (p_0,q_0), first solve the x-part of the Lax pair (1.286) for $p=p_0,\ q=q_0$ and get the scattering data corresponding to p_0 and q_0. Then, using the evolution of the scattering data (1.291), the scattering data corresponding to $(p(t),q(t))$ are obtained. Finally, solve the integral equations (1.285) to get $(p(t),q(t))$. Therefore, the inverse scattering method changes the initial value problem of nonlinear partial differential equations to the problem of solving systems of linear integral equations. This gives an effective way to solve the initial value problem. Especially, when $b_r=\widetilde{b}_r=0$, $B_c=\widetilde{B}_c=0$, (1.285) has a degenerate kernel. Hence it can be solved algebraically and the soliton solutions can be obtained. Please see [23] for details.

Remark 16 Denote $\Sigma(p,q)$ to be the scattering data corresponding to $(p(x,t),q(x,t))$, $p_0(x)$ and $q_0(x)$ to be the initial values of p and q at $t=0$, then the procedure of inverse scattering method can be shown in *the following diagram:*

$$\begin{tikzcd} t=0:\ (p_0,q_0) \arrow[r, "\text{scattering}"] & \Sigma(p_0,q_0) \arrow[d] \\ t=t:\quad (p,q) & \Sigma(p,q) \arrow[l, "\text{inverse scattering}"'] \end{tikzcd} \tag{1.292}$$

For a linear equation, if "scattering" is changed to "Fourier transformation" and "inverse scattering" is changed to "inverse Fourier transformation" in the above diagram, then it becomes the diagram for solving the initial value problem by Fourier transformations which has been used extensively for linear problems. Therefore, the scattering and inverse scattering method can be regarded as a kind of Fourier method for nonlinear problems.

1.5.2 Change of scattering data under Darboux transformations for $su(2)$ AKNS system

For the AKNS system, the scattering data include

$$\{\zeta_k, C_k, \widetilde{\zeta}_k, \widetilde{C}_k, b(\zeta), \widetilde{b}(\zeta)\}. \tag{1.293}$$

The $su(2)$ AKNS system means that $U, V \in su(2)$ for $\zeta \in \mathbf{R}$, i.e., $q = -\bar{p}$, $\bar{A} = -A$, $C = -\bar{B}$. Therefore, it is just the nonlinear Schrödinger hierarchy. The Lax pair is

$$\begin{aligned} \Phi_x &= \begin{pmatrix} -\mathrm{i}\zeta & p \\ -\bar{p} & \mathrm{i}\zeta \end{pmatrix} \Phi, \\ \Phi_t &= \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \Phi \end{aligned} \tag{1.294}$$

where $\overline{A(\bar{\zeta})} = -A(\zeta)$, $\overline{B(\bar{\zeta})} = -C(\zeta)$. Here we consider the $su(2)$ AKNS system instead of general 2×2 AKNS system because the Darboux transformation may exist globally in this case.

If $\begin{pmatrix} \alpha(\zeta) \\ \beta(\zeta) \end{pmatrix}$ is a solution of (1.294), then $\begin{pmatrix} -\bar{\beta}(\bar{\zeta}) \\ \bar{\alpha}(\bar{\zeta}) \end{pmatrix}$ is also its solution. This leads to the following property.

Property 8. For the Lax pair (1.294), if $\zeta \in \mathbf{R}$, then there are following relations among the Jost solutions and the scattering data:

$$\begin{aligned} &\widetilde{R}_1 = -\bar{R}_2, \quad \widetilde{R}_2 = \bar{R}_1, \\ &\widetilde{L}_1 = \bar{L}_2, \quad \widetilde{L}_2 = -\bar{L}_1, \end{aligned} \tag{1.295}$$

$$\widetilde{r}_+(\zeta) = \bar{r}_-(\zeta), \qquad \widetilde{r}_-(\zeta) = \bar{r}_+(\zeta). \tag{1.296}$$

By reordering the eigenvalues,

$$\widetilde{d} = d, \quad \widetilde{\zeta}_k = \bar{\zeta}_k, \quad \widetilde{C}_k = -\bar{C}_k, \quad \widetilde{b}(\zeta) = \bar{b}(\zeta) \quad (\zeta \in \mathbf{R}). \tag{1.297}$$

Therefore, for the $su(2)$ AKNS system, the scattering data can be reduced to $\zeta_k \in \mathbf{C}_+$, C_k $(k = 1, 2, \cdots, d)$ and $b(\zeta)$ $(\zeta \in \mathbf{R})$.

Now we consider the change of the scattering data under Darboux transformations.

From the discussion on the nonlinear Schrödinger hierarchy, we know that if p is defined globally on $(-\infty, +\infty)$, so is the Darboux matrix. In order to use the scattering theory, we want that p and its derivatives tend to 0 fast enough at infinity.

Take a constant μ and a column solution of the Lax pair

$$\psi_r(\zeta_0) - \mu\psi_l(\zeta_0) = \begin{pmatrix} R_1(\zeta_0)\mathrm{e}^{-\mathrm{i}\zeta_0 x} - \mu L_1(\zeta_0)\mathrm{e}^{\mathrm{i}\zeta_0 x} \\ R_2(\zeta_0)\mathrm{e}^{-\mathrm{i}\zeta_0 x} - \mu L_2(\zeta_0)\mathrm{e}^{\mathrm{i}\zeta_0 x} \end{pmatrix} \tag{1.298}$$

$(\zeta_0 \in \mathbf{C}_+)$.

Let

$$\sigma = \frac{R_2(\zeta_0) - \mu L_2(\zeta_0)\mathrm{e}^{2\mathrm{i}\zeta_0 x}}{R_1(\zeta_0) - \mu L_1(\zeta_0)\mathrm{e}^{2\mathrm{i}\zeta_0 x}} \tag{1.299}$$

be the ratio of the second and the first components. Then the Darboux matrix is

$$\begin{aligned} &-\mathrm{i}\zeta I - S \\ &= -\mathrm{i}\zeta I - \frac{1}{1+|\sigma|^2}\begin{pmatrix} -\mathrm{i}\zeta_0 - \mathrm{i}\bar{\zeta}_0|\sigma|^2 & (-\mathrm{i}\zeta_0 + \mathrm{i}\bar{\zeta}_0)\bar{\sigma} \\ (-\mathrm{i}\zeta_0 + \mathrm{i}\bar{\zeta}_0)\sigma & -\mathrm{i}\bar{\zeta}_0 - \mathrm{i}\zeta_0|\sigma|^2 \end{pmatrix}, \end{aligned} \tag{1.300}$$

and the solution is transformed by

$$p' = p + 2\mathrm{i}\frac{(\bar{\zeta}_0 - \zeta_0)\bar{\sigma}}{1+|\sigma|^2}. \tag{1.301}$$

The change of the scattering data under Darboux transformation is given by the following theorem [75].

THEOREM 1.22 *If the scattering data for (1.294) are* $r_-(\zeta)$ ($\zeta \in \mathbf{C}_+ \cup \mathbf{R}$), $r_+(\zeta)$ ($\zeta \in \mathbf{R}$) *and* $\alpha(\zeta_k)$ ($k = 1, \cdots, d$), *then, under the action of the Darboux matrix (1.300)* ($\mu \neq 0$, $\zeta_0 \in \mathbf{C}_+$), *the scattering data are changed as follows:*

(1) If ζ_0 *is not an eigenvalue, then, after the action of the Darboux transformation, the number of eigenvalues increase one. All the original eigenvalues are not changed, and* ζ_0 *is a unique additional eigenvalue. Moreover,*

$$\begin{aligned}
&r'_-(\zeta) = \frac{\zeta - \zeta_0}{\zeta - \bar{\zeta}_0} r_-(\zeta) \quad (\zeta \in \mathbf{C}_+ \cup \mathbf{R}), \\
&r'_+(\zeta) = r_+(\zeta) \quad (\zeta \in \mathbf{R}), \\
&\alpha'(\zeta_k) = \alpha(\zeta_k) \quad (k = 1, \cdots, d), \\
&\alpha'(\zeta_0) = 1/\mu,
\end{aligned} \tag{1.302}$$

hence

$$\begin{aligned}
&b'(\zeta) = \frac{\zeta - \bar{\zeta}_0}{\zeta - \zeta_0} b(\zeta) \quad (\zeta \in \mathbf{R}), H \\
&C'_k = \frac{\zeta_k - \bar{\zeta}_0}{\zeta_k - \zeta_0} C_k \quad (k = 1, \cdots, d), H \\
&C'_0 = \frac{\zeta_0 - \bar{\zeta}_0}{\mu r_-(\zeta_0)}.
\end{aligned} \tag{1.303}$$

(2) If ζ_0 *is an eigenvalue:* $\zeta_0 = \zeta_j$, *and* $\mu \neq \alpha(\zeta_j)$, *then, after the action of the Darboux transformation,* ζ_0 *is no longer an eigenvalue. Moreover,*

$$\begin{aligned}
&r'_-(\zeta) = \frac{\zeta - \bar{\zeta}_0}{\zeta - \zeta_0} r_-(\zeta) \quad (\zeta \in \mathbf{C}_+ \cup \mathbf{R}), \\
&r'_+(\zeta) = r_+(\zeta) \quad (\zeta \in \mathbf{R}), \\
&\alpha'(\zeta_k) = \alpha(\zeta_k) \quad (k = 1, \cdots, d,\ k \neq j),
\end{aligned} \tag{1.304}$$

and

$$\begin{aligned}
&b'(\zeta) = \frac{\zeta - \zeta_0}{\zeta - \bar{\zeta}_0} b(\zeta) \quad (\zeta \in \mathbf{R}), H \\
&C'_k = \frac{\zeta_k - \zeta_0}{\zeta_k - \bar{\zeta}_0} C_k \quad (k = 1, \cdots, d,\ k \neq j).
\end{aligned} \tag{1.305}$$

Proof. (1) $\zeta_0 \notin IP\sigma(L)$.

Then, both the numerator and denominator of (1.299) are not 0. Property 3 implies

$$\lim_{x\to-\infty}\sigma=\infty,\qquad \lim_{x\to+\infty}\sigma=0. \tag{1.306}$$

Hence

$$\begin{aligned}\lim_{x\to-\infty}(-\mathrm{i}\zeta I-S)&=\begin{pmatrix}-\mathrm{i}\zeta+\mathrm{i}\bar{\zeta}_0 & 0\\ 0 & -\mathrm{i}\zeta+\mathrm{i}\zeta_0\end{pmatrix},\\ \lim_{x\to+\infty}(-\mathrm{i}\zeta I-S)&=\begin{pmatrix}-\mathrm{i}\zeta+\mathrm{i}\zeta_0 & 0\\ 0 & -\mathrm{i}\zeta+\mathrm{i}\bar{\zeta}_0\end{pmatrix}.\end{aligned} \tag{1.307}$$

Under the action of the Darboux transformation, the Jost solutions are changed to

$$\begin{aligned}\psi_r'(x,t,\zeta)&=\frac{1}{-\mathrm{i}\zeta+\mathrm{i}\bar{\zeta}_0}(-\mathrm{i}\zeta I-S)\psi_r(x,t,\zeta),\\ \psi_l'(x,t,\zeta)&=\frac{1}{-\mathrm{i}\zeta+\mathrm{i}\bar{\zeta}_0}(-\mathrm{i}\zeta I-S)\psi_l(x,t,\zeta).\end{aligned} \tag{1.308}$$

Hence

$$R'=\frac{1}{-\mathrm{i}\zeta+\mathrm{i}\bar{\zeta}_0}(-\mathrm{i}\zeta I-S)R. \tag{1.309}$$

If $\zeta\in\mathbf{C}_+$,

$$\lim_{x\to+\infty}R'=\begin{pmatrix}\dfrac{\zeta-\zeta_0}{\zeta-\bar{\zeta}_0} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}r_-(\zeta)\\ 0\end{pmatrix}. \tag{1.310}$$

Thus

$$r_-'(\zeta)=\frac{\zeta-\zeta_0}{\zeta-\bar{\zeta}_0}r_-(\zeta), \tag{1.311}$$

$r_-'(\zeta)$ has an additional zero ζ_0 than $r_-(\zeta)$. This means that ζ_0 is a new eigenvalue. For $\zeta\in\mathbf{R}$,

$$R'\sim\begin{pmatrix}\dfrac{\zeta-\zeta_0}{\zeta-\bar{\zeta}_0} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}r_-(\zeta)\\ r_+(\zeta)\mathrm{e}^{2\mathrm{i}\zeta x}\end{pmatrix}. \tag{1.312}$$

Hence $r_+'(\zeta)=r_+(\zeta)$, and

$$b'(\zeta)=\frac{\zeta-\bar{\zeta}_0}{\zeta-\zeta_0}b(\zeta). \tag{1.313}$$

If ζ_k is a zero of $r_-(\zeta)$, then (1.308) implies $\alpha'(\zeta_k) = \alpha(\zeta_k)$, and

$$C'_k = \alpha'(\zeta_k) \Bigg/ \frac{\mathrm{d}r'_-(\zeta_k)}{\mathrm{d}\zeta} = \frac{\zeta_k - \bar{\zeta}_0}{\zeta_k - \zeta_0} C_k. \tag{1.314}$$

When $\zeta = \zeta_0$,

$$\begin{aligned} \psi'_r(x,t,\zeta_0) &= \frac{1}{1+|\sigma|^2} \begin{pmatrix} |\sigma|^2 & -\bar{\sigma} \\ -\sigma & 1 \end{pmatrix} \psi_r(x,t,\zeta_0), \\ \psi'_l(x,t,\zeta_0) & \frac{1}{1+|\sigma|^2} \begin{pmatrix} |\sigma|^2 & -\bar{\sigma} \\ -\sigma & 1 \end{pmatrix} \psi_l(x,t,\zeta_0), \end{aligned} \tag{1.315}$$

$$\alpha'(\zeta_0) = \frac{\sigma L_1 \exp(\mathrm{i}\zeta_0 x) - L_2 \exp(\mathrm{i}\zeta_0 x)}{\sigma R_1 \exp(-\mathrm{i}\zeta_0 x) - R_2 \exp(-\mathrm{i}\zeta_0 x)} = \frac{1}{\mu}, \tag{1.316}$$

$$C'_0 = \alpha'(\zeta_0) / \frac{\mathrm{d}r'_-(\zeta_0)}{\mathrm{d}\zeta} = \frac{\zeta_0 - \bar{\zeta}_0}{\mu r_-(\zeta_0)}. \tag{1.317}$$

(1) is proved.

(2) $\zeta_0 = \zeta_j \in IP\sigma(L)$, $\mu \neq \alpha(\zeta_j)$.

Now

$$\sigma = \frac{R_2(\zeta_j)}{R_1(\zeta_j)} = \frac{L_2(\zeta_j)}{L_1(\zeta_j)}, \tag{1.318}$$

hence

$$\lim_{x \to -\infty} \sigma = 0, \quad \lim_{x \to +\infty} \sigma = \infty, \tag{1.319}$$

$$\begin{aligned} \lim_{x \to -\infty} (-\mathrm{i}\zeta I - S) &= \begin{pmatrix} -\mathrm{i}\zeta + \mathrm{i}\zeta_0 & 0 \\ 0 & -\mathrm{i}\zeta + \mathrm{i}\bar{\zeta}_0 \end{pmatrix}, \\ \lim_{x \to +\infty} (-\mathrm{i}\zeta I - S) &= \begin{pmatrix} -\mathrm{i}\zeta + \mathrm{i}\bar{\zeta}_0 & 0 \\ 0 & -\mathrm{i}\zeta + \mathrm{i}\zeta_0 \end{pmatrix}. \end{aligned} \tag{1.320}$$

Under the action of the Darboux transformation, the Jost solutions become

$$\begin{aligned} \psi'_r(x,t,\zeta) &= \frac{1}{-\mathrm{i}\zeta + \mathrm{i}\zeta_0} (-\mathrm{i}\zeta I - S) \psi_r(x,t,\zeta), \\ \psi'_l(x,t,\zeta) &= \frac{1}{-\mathrm{i}\zeta + \mathrm{i}\zeta_0} (-\mathrm{i}\zeta I - S) \psi_l(x,t,\zeta). \end{aligned} \tag{1.321}$$

For $\zeta \in \mathbf{C}_+$,

$$\lim_{x \to +\infty} R' = \begin{pmatrix} \dfrac{\zeta - \bar{\zeta}_0}{\zeta - \zeta_0} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_-(\zeta) \\ 0 \end{pmatrix}, \tag{1.322}$$

and for $\zeta \in \mathbf{R}$,

$$R' \sim \begin{pmatrix} \dfrac{\zeta - \bar{\zeta}_0}{\zeta - \zeta_0} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_-(\zeta) \\ r_+(\zeta)\mathrm{e}^{2\mathrm{i}\zeta x} \end{pmatrix}. \tag{1.323}$$

Hence

$$\begin{aligned} r'_-(\zeta) &= \frac{\zeta - \bar{\zeta}_0}{\zeta - \zeta_0} r_-(\zeta) \quad (\zeta \in \mathbf{C}_+ \cup \mathbf{R}), \\ r'_+(\zeta) &= r_+(\zeta) \quad (\zeta \in \mathbf{R}), \end{aligned} \tag{1.324}$$

and

$$b'(\zeta) = \frac{\zeta - \zeta_0}{\zeta - \bar{\zeta}_0} b(\zeta). \tag{1.325}$$

From (1.324) we know that the Darboux transformation removes the eigenvalue ζ_0 $(= \zeta_j)$.

If $\zeta = \zeta_k$ $(k \neq j)$, then $\psi'_r = \alpha(\zeta_k)\psi'_l$, hence

$$\begin{aligned} &\alpha'(\zeta_k) = \alpha(\zeta_k), \\ &C'_k = \alpha'(\zeta_k) \Big/ \frac{\mathrm{d}r'_-(\zeta_k)}{\mathrm{d}\zeta} = \frac{\zeta_k - \zeta_0}{\zeta_k - \bar{\zeta}_0} C_k. \end{aligned} \tag{1.326}$$

The theorem is proved.

We have given the formulae for the change of the scattering data under Darboux transformation in the $su(2)$ case. A Darboux transformation increase or decrease the number of eigenvalues (number of solitons). However, it does not affect the scattering data related to the continuous spectrum. Thus we can use the Darboux transformation to change a general inverse scattering problem to an inverse scattering problem without eigenvalues.

Remark 17 For the KdV equation, $q = 1$ *(or* -1*) in the Lax pair (1.286). Since* q *does not tend to zero at infinity, the above conclusions can not be applied directly. However, the inverse scattering theory for the KdV equation is actually simpler than the AKNS system (see [23]). The conclusions similar to Theorem 1.22 for the KdV equation holds true as well [29].*

Chapter 2

2+1 DIMENSIONAL INTEGRABLE SYSTEMS

This chapter is devoted to the Darboux transformations of 2+1 dimensional integrable systems. Starting from the KP equation, we discuss the Darboux transformation for 2+1 dimensional AKNS system and more general systems. Unlike the Darboux matrices in 1+1 dimensions, the Darboux transformations here are given by differential operators (called Darboux operators). The construction of the Darboux operators is uniform to all the equations in the system, as in the 1+1 dimensional case. The binary Darboux transformation, which is a kind of Darboux transformation in integral form, is introduced briefly. Explicit solutions of the DSI equation can be obtained by the combination of Darboux transformation and binary Darboux transformation. Moreover, the nonlinear constraint method is used to separate the differentials in the 2+1 dimensional AKNS system so that the Darboux transformation in 1+1 dimensions can be used to get the localized soliton solutions.

2.1 KP equation and its Darboux transformation

A 2+1 dimensional integrable system has three independent variables (x, y, t) where x and y usually refer to space variables and t refers to time variable. A typical 2+1 dimensional integrable partial differential equation is the Kadomtsev-Petviashvili equation (KP equation) [68]

$$u_{xt} = (u_{xxx} + 6uu_x)_x + 3\alpha^2 u_{yy}, \tag{2.1}$$

where $\alpha = \pm 1$ or $\pm \mathrm{i}$. (2.1) is called the KPI equation if $\alpha = \pm 1$, and the KPⅡ equation if $\alpha = \pm \mathrm{i}$. The KP equation is the natural generalization of the KdV equation, which describes the motion of two dimensional water wave. (2.1) can also be written as

$$v_{xt} = v_{xxxx} + 6v_x v_{xx} + 3\alpha^2 v_{yy} \tag{2.2}$$

where v satisfies $v_x = u$. The KP equation has a Lax pair

$$\phi_y = \alpha^{-1}\phi_{xx} + \alpha^{-1}u\phi, \tag{2.3}$$

$$\phi_t = 4\phi_{xxx} + 6u\phi_x + 3(\alpha v_y + u_x)\phi. \tag{2.4}$$

We first derive ϕ_{yt} by differentiating (2.3) with respect to t and inserting the expression of ϕ_t. Also, we can derive ϕ_{ty} by differentiating (2.4) with respect to y and inserting the expression of ϕ_y. The equality $\phi_{yt} = \phi_{ty}$ is equivalent to (2.1) when $\phi \neq 0$. The proof of this fact is direct, which is left for the reader. Therefore, (2.1) is the integrability condition of the overdetermined system (2.3) and (2.4).

The Darboux transformation for the KP equation is similar with that for the KdV equation. It can be constructed as follows. Let h be a solution of the Lax pair (2.3) and (2.4). For any solution ϕ of (2.3) and (2.4), define

$$\phi' = \phi_x - (h_x/h)\phi, \tag{2.5}$$

then ϕ' is a solution of

$$\begin{aligned} &\phi'_y = \alpha^{-1}\phi'_{xx} + \alpha^{-1}u'\phi', \\ &\phi'_t = 4\phi'_{xxx} + 6u'\phi'_x + 3(\alpha v'_y + u'_x)\phi' \end{aligned} \tag{2.6}$$

where

$$u' = u + 2(h_x/h)_x, \qquad v' = v + 2h_x/h. \tag{2.7}$$

Comparing (2.6) with (2.3) and (2.4), the only difference is that (u, ϕ) is changed to (u', ϕ'). Hence (2.7) gives a new solution u' of the KP equation [77].

Similar to 1+1 dimensions, if the seed solution u is simple enough, we can solve the Lax pair (2.3) and (2.4) to get h, then (2.5) gives a more complicated solution of the KP equation. Especially, if $u = v = 0$, then (2.3) and (2.4) becomes

$$\begin{aligned} &\phi_y = \alpha^{-1}\phi_{xx} \\ &\phi_t = 4\phi_{xxx}. \end{aligned} \tag{2.8}$$

Therefore, for any solution h of (2.8) with $h \neq 0$, $u' = 2(h_x/h)_x$ gives a solution of the KP equation.

EXAMPLE 2.1 *For $\alpha = 1$, h can be chosen as*

$$h = e^{\lambda x + \lambda^2 y + 4\lambda^3 t} + 1, \tag{2.9}$$

where λ is a real constant, then

$$u' = \frac{\lambda^2}{2}\,\mathrm{sech}^2\Big(\frac{1}{2}(\lambda x + \lambda^2 y + 4\lambda^3 t)\Big) \tag{2.10}$$

is a solution of the KPI equation.

EXAMPLE 2.2 *For $\alpha = -\mathrm{i}$, let*

$$h = e^{\lambda x + \mathrm{i}\lambda^2 y + 4\lambda^3 t} + e^{-\bar\lambda x + \mathrm{i}\bar\lambda^2 y - 4\bar\lambda^3 t}, \tag{2.11}$$

where $\lambda = a + b\mathrm{i}$ is a complex constant, then we obtain a solution of the KPII equation:

$$u' = 2a^2\,\mathrm{sech}^2(ax - 2aby + 4(a^3 - 3ab^2)t). \tag{2.12}$$

These two solutions are both travelling waves, i.e., they are of form $u' = f(t + a_1 x + a_2 y)$ and u' is invariant along the line $t + a_1 x + a_2 y =$ constant on the (x, y) plane. For fixed t, u' is a non-zero constant along certain lines (for KPI, they are $\lambda x + \lambda^2 y =$ constant, while for KPII, they are $ax - 2aby =$ constant), and u' tends to zero exponentially at infinity along other lines. Hence the region where u' is far from zero forms a band on the (x, y) plane. This kind of solutions are call "line-solitons". This does not happen in 1+1 dimensions.

Suppose we have known a solution u of the KP equation and a set of solutions $\{\phi\}$ of the corresponding Lax pair. Let h be a special ϕ, then $u' = u + 2(\ln h)_{xx}$ is a solution of the KP equation. Moreover, $\phi' = \phi_x - (h_x/h)\phi$ gives the set of solutions of the Lax pair for u'. Now we take a special ϕ' as h', then we can obtain another solution $u'' = u' + 2(\ln h')_{xx}$ of the KP equation and the solution $\phi'' = \phi'_x - (h'_x/h')\phi'$ of the corresponding Lax pair by constructing Darboux transformation with h'. Continuing this procedure, we obtain a series of solutions of the KP equation without solving differential equations.

Except the first step, this algorithm can be realized by algebraic computation and differentiations. Therefore, it can be done by symbolic calculation. The solutions are global for all (x, y, t) if $h, h', h'' \cdots$ do not equal zero. This process can be expressed as

$$(u, \phi) \longrightarrow (u', \phi') \longrightarrow (u'', \phi'') \longrightarrow \cdots. \tag{2.13}$$

The differential operator of order three on the right hand side of (2.4) can be changed to differential operators of arbitrary order, then we get the KP hierarchy

$$\begin{aligned}\phi_y &= \alpha^{-1}\phi_{xx} + \alpha^{-1}u\phi,\\ \phi_t &= \sum_{j=0}^{n} v_{n-j}\partial^j\phi,\end{aligned} \tag{2.14}$$

$(\partial = \partial/\partial x)$. Computing the integrability condition of (2.14) and letting all the coefficients of the derivatives of ϕ with respect to x be zero, we have

$$2v_{j+1,x} = \alpha v_{j,y} - v_{j,xx} + \sum_{k=0}^{j-1} C_{n-k}^{n-j} v_k \partial^{j-k}u, \tag{2.15}$$

$$u_t = \alpha v_{n,y} - v_{n,xx} + \sum_{k=0}^{n-1} v_k \partial^{n-k}u. \tag{2.16}$$

In (2.15), v_{j+1} can be solved by integration. Unlike the 1+1 dimensional systems such as the KdV hierarchy, here, in general, v_j's can not be expressed as differential polynomials of u. Therefore, $(u, v_1, \cdots, v_n)$ are regarded as a set of unknowns of (2.15) – (2.16). The Darboux transformation is still valid for this system. In practical problems, some additional relations among $(v_1, \cdots, v_n, u)$ should be satisfied. This is called a reduction of the original one. In that case, we should choose proper h so that the relations among $v_1, \cdots, v_n, u$ keeps after the Darboux transformation. Usually this is a difficult problem and some special cases can be solved by certain techniques.

2.2 2+1 dimensional AKNS system and DS equation

2+1 dimensional AKNS system is

$$\Phi_y = J\Phi_x + P\Phi, \qquad \Phi_t = \sum_{j=0}^{n} V_{n-j}\partial^j\Phi, \tag{2.17}$$

where J is an $N \times N$ constant diagonal matrix, $P(x, y, t)$ is an off-diagonal $N \times N$ matrix, $V_j(x, y, t)$'s are also $N \times N$ matrices, $\partial = \partial/\partial x$. For simplicity, we assume that the diagonal entries of J are distinct. Moreover, we consider the non-degenerate Φ only.

The integrability condition of (2.17) leads to

$$[J, V_{j+1}^{\mathrm{off}}] = V_{j,y}^{\mathrm{off}} - JV_{j,x}^{\mathrm{off}} - [P, V_j]^{\mathrm{off}} + \sum_{k=0}^{j-1} C_{n-k}^{n-j}(V_k\partial^{j-k}P)^{\mathrm{off}}, \tag{2.18}$$

$$V_{j,y}^{\rm diag} - JV_{j,x}^{\rm diag} = [P, V_j^{\rm off}]^{\rm diag} - \sum_{k=0}^{j-1} C_{n-k}^{n-j} (V_k \partial^{j-k} P)^{\rm diag}, \qquad (2.19)$$

$$P_t = V_{n,y}^{\rm off} - JV_{n,x}^{\rm off} - [P, V_n]^{\rm off} + \sum_{k=0}^{n-1} (V_k \partial^{n-k} P)^{\rm off}. \qquad (2.20)$$

Here the superscripts "diag" and "off" refer to the diagonal and off-diagonal part of a matrix respectively.

Usually V_j's are not differential polynomials of P. But they can be generated from P by differentiation and integration with respect to x. (2.19) and (2.20) are regarded as a system of partial differential equations for P and $V_j^{\rm diag}$'s $(j = 0, 1, \cdots, n)$ where $V_j^{\rm off}$'s $(j = 1, \cdots, n)$ are determined by (2.18). (2.17) is the Lax pair of this system of equations.

A typical equation in 2+1 dimensional AKNS system is the Davey-Stewartson equation (DS equation), which is the natural generalization of the nonlinear Schrödinger equation in 2+1 dimensions.

Take $N = 2$, $n = 2$ in (2.17) and let

$$\begin{aligned}
&J = \alpha^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & u \\ -\epsilon \bar{u} & 0 \end{pmatrix}, \\
&\alpha = \pm 1 \text{ or } \pm \mathrm{i}, \qquad \epsilon = \pm 1, \\
&V_0 = 2\mathrm{i}\alpha J, \qquad V_1 = 2\mathrm{i}\alpha P, \\
&V_2 = \mathrm{i}\alpha \begin{pmatrix} w_1 & u_x + \alpha u_y \\ -\epsilon \bar{u}_x + \alpha \epsilon \bar{u}_y & w_2 \end{pmatrix}.
\end{aligned} \qquad (2.21)$$

where u, w_1, w_2 are complex-valued functions, $\bar{u}$ is the complex conjugate of u. Then, (2.17) becomes

$$\begin{aligned}
\Phi_y = &\ \alpha^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_x + \begin{pmatrix} 0 & u \\ -\epsilon \bar{u} & 0 \end{pmatrix} \Phi, \\
\Phi_t = &\ 2\mathrm{i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_{xx} + 2\mathrm{i}\alpha \begin{pmatrix} 0 & u \\ -\epsilon \bar{u} & 0 \end{pmatrix} \Phi_x \\
&+ \mathrm{i}\alpha \begin{pmatrix} w_1 & u_x + \alpha u_y \\ -\epsilon \bar{u}_x + \alpha \epsilon \bar{u}_y & w_2 \end{pmatrix} \Phi.
\end{aligned} \qquad (2.22)$$

(2.19) and (2.20) lead to

$$\begin{aligned} &\mathrm{i}u_t = -u_{xx} - \alpha^2 u_{yy} - \alpha u(w_1 - w_2),\\ &w_{1,y} - \alpha^{-1} w_{1,x} = \epsilon(|u|^2)_x + \alpha\epsilon(|u|^2)_y,\\ &w_{2,y} + \alpha^{-1} w_{2,x} = \epsilon(|u|^2)_x - \alpha\epsilon(|u|^2)_y, \end{aligned} \tag{2.23}$$

and $\overline{w_2 - w_1} = \alpha^{-2}(w_2 - w_1)$. Denote

$$v = -\epsilon|u|^2 + \frac{1}{2\alpha}(w_1 - w_2), \tag{2.24}$$

then (2.23) becomes

$$\begin{aligned} &\mathrm{i}u_t = -u_{xx} - \alpha^2 u_{yy} - 2\epsilon\alpha^2|u|^2 u - 2\alpha^2 uv,\\ &v_{xx} - \alpha^2 v_{yy} + 2\epsilon(|u|^2)_{xx} = 0. \end{aligned} \tag{2.25}$$

(2.25) is called the DSI equation if $\epsilon = 1$, $\alpha = \pm 1$, and DSII equation if $\epsilon = 1$, $\alpha = \pm\mathrm{i}$. They describe the motion of long wave and short wave in the water of finite depth [20].

2.3 Darboux transformation

2.3.1 General Lax pair

Similar to the KP equation, we also want to construct the Darboux transformation for the AKNS system. Here we first discuss the Darboux transformation for the following more general Lax pair without any reductions.

Consider Lax pair

$$\Phi_y = U(x,y,t,\partial)\Phi, \qquad \Phi_t = V(x,y,t,\partial)\Phi, \tag{2.26}$$

where

$$\begin{aligned} U(x,y,t,\partial) &= \sum_{j=0}^{m} U_{m-j}(x,y,t)\partial^j,\\ V(x,y,t,\partial) &= \sum_{j=0}^{n} V_{n-j}(x,y,t)\partial^j \end{aligned} \tag{2.27}$$

are differential operators with respect to x whose coefficients U_j's and V_j's are $N \times N$ matrices. For simplicity, we write $U(x,y,t,\partial) = U(\partial)$, $V(x,y,t,\partial) = V(\partial)$.

$\Phi_{yt} = \Phi_{ty}$ can be obtained by differentiating the first equation of (2.26) with respect to t or by differentiating the second equation with respect to x. Let these two equal, we get

$$U_t(\partial) - V_y(\partial) + [U(\partial), V(\partial)] = 0. \tag{2.28}$$

(2.26) is called integrable if (2.28) holds. (2.28) is the generalization of the zero-curvature equations in 1+1 dimensions. It gives a system of partial differential equations by equating all the coefficients of ∂ to be zero.

Remark 18 The existence and uniqueness of the solutions of a system of partial differential equations are very difficult problems. The local solvability of a system of linear partial differential equations have been studied by many authors. In the present case, even (2.28) holds, local solution of (2.26) near $t = t_0$, $y = y_0$ with initial data $\Phi(t_0, x, y_0) = \Phi_0(x)$ may not exist. However, if each set of equations in (2.26) is locally solvable and the solutions are smooth enough with respect to the parameters y and t, $U(\partial)$ and $V(\partial)$ satisfy (2.28), then (2.26) is locally solvable. This follows from the following consideration. Suppose that the initial data $(x_0, y_0, t_0, \Phi_0(x))$ are given. First solve the first set of equations of (2.26) at $t = t_0$ with initial value $\Phi_1(x, y_0) = \Phi_0(x)$ and get the solution $\Phi_1(x, y)$. Using $\Phi_1(x, y)$ as the initial value, solve the second set of equations of (2.26) for fixed y and get the solution $\Phi(x, y, t)$. Using (2.28) and the second equation of (2.26), we have

$$(\Phi_y - U(\partial)\Phi)_t = V(\partial)(\Phi_y - U(\partial)\Phi). \tag{2.29}$$

Therefore, $\Phi_y = U(\partial)\Phi$ holds identically near (x_0, y_0, t_0) by the uniqueness of the solution.

No matter whether the existence and uniqueness hold, (2.28) is called the integrability condition of (2.26). It gives a system of nonlinear partial differential equations of $U(\partial)$ and $V(\partial)$. (2.26) is called the Lax pair of this system of nonlinear partial differential equations. It is interesting to see that we can apply Darboux transformation as well provided that the set of solutions of (2.26) is not empty.

2.3.2 Darboux transformation of degree one

Similar to 1+1 dimensional case, we can define Darboux operator for the integrable nonlinear partial differential equations (2.28) and there Lax pair (2.26).

Definition 2.3 *A differential operator $D(x, y, t, \partial)$ with respect to x is called a Darboux operator for (2.26) if there exist differential operators $U'(\partial)$ and $V'(\partial)$ with respect to x such that for any solution Φ of (2.26), $\Phi' = D(\partial)\Phi$ satisfies*

$$\Phi'_y = U'(\partial)\Phi', \qquad \Phi'_t = V'(\partial)\Phi'. \tag{2.30}$$

The transformation $(\Phi, U(\partial), V(\partial)) \to (\Phi', U'(\partial), V'(\partial))$ *given by* $D(\partial)$ *is called a Darboux transformation.*

Substituting $\Phi' = D\Phi$ into (2.30), we have

$$\begin{aligned} D_y(\partial) &= U'(\partial)D(\partial) - D(\partial)U(\partial), \\ D_t(\partial) &= V'(\partial)D(\partial) - D(\partial)V(\partial), \end{aligned} \tag{2.31}$$

and

$$U'_t(\partial) - V'_y(\partial) + [U'(\partial), V'(\partial)] = 0. \tag{2.32}$$

(2.31) is the necessary and sufficient condition for $D(\partial)$ being a Darboux operator. Hence, if $(U(\partial), V(\partial))$ is a solution of (2.28), so is $(U'(\partial), V'(\partial))$. This means that the Darboux transformation gives a new solution of (2.28). Our main task is to construct the solution D of (2.31).

We first discuss the most fundamental Darboux operator, the Darboux operator of degree one. This is the Darboux operator in the form $D(x, y, t, \partial) = \partial - S(x, y, t)$. The Darboux operator of higher degree will be discussed later. In order to get the general construction of S, we first derive the equations that S should satisfy.

For a matrix $M(x)$, we define a sequence of matrices $M^{(j)}$ by $M^{(0)} = I$ and

$$M^{(j+1)} = M_x^{(j)} + M^{(j)}M, \tag{2.33}$$

then, for any solution Φ of the equation $\Phi_x = M\Phi$, $\partial^j \Phi = M^{(j)}\Phi$ holds.

For any differential operator

$$U(\partial) = \sum_{j=0}^{k} U_{k-j}\partial^j, \qquad V(\partial) = \sum_{j=0}^{k} V_{k-j}\partial^j \tag{2.34}$$

and an $N \times N$ matrix S, we define

$$U(S) = \sum_{j=0}^{k} U_{k-j}S^{(j)}, \qquad V(S) = \sum_{j=0}^{k} V_{k-j}S^{(j)}. \tag{2.35}$$

Suppose that Φ satisfies $\Phi_x = S\Phi$, then $U(\partial)\Phi = U(S)\Phi$, $V(\partial)\Phi = V(S)\Phi$. Notice that $U(S)$ and $V(S)$ are not given by replacing ∂ in $U(\partial)$ and $V(\partial)$ with S. Actually they are obtained by replacing ∂^j with $S^{(j)}$.

THEOREM 2.4 $\partial - S$ *is a Darboux operator for (2.26) if and only if* S *satisfies*

$$\begin{aligned} S_y + [S, U(S)] &= (U(S))_x, \\ S_t + [S, V(S)] &= (V(S))_x. \end{aligned} \tag{2.36}$$

Proof. Suppose $\partial - S$ is a Darboux operator for (2.26), then the first equation of (2.31) is

$$S_y - (\partial - S)U(\partial) + U'(\partial)(\partial - S) = 0. \tag{2.37}$$

Let Ψ be the fundamental solution of $\Psi_x = S\Psi$, then

$$S_y\Psi = (\partial - S)U(S)\Psi = (U(S))_x\Psi - [S, U(S)]\Psi. \tag{2.38}$$

This gives the first equation of (2.36). The second equation is derived similarly. The necessity of (2.36) is proved.

Conversely, suppose S is a solution of (2.36). Define

$$U'(\partial) = \sum_{j=0}^{m} U'_{m-j}\partial^j, \tag{2.39}$$

where U'_j's are determined recursively by

$$\begin{aligned} &U'_0 = U_0, \\ &U'_{j+1} = U_{j+1} + U_{j,x} - SU_j + \sum_{k=0}^{j} C_{m-k}^{m-j} U'_k \partial^{j-k} S. \end{aligned} \tag{2.40}$$

Then

$$S_y - (\partial - S)U(\partial) + U'(\partial)(\partial - S) \tag{2.41}$$

does not contain any terms with ∂, i.e., it is a matrix-valued function of x, y and t. On the other hand, for any fundamental solution Φ of $\Psi_x = S\Psi$, (2.36) leads to

$$(S_y - (\partial - S)U(\partial) + U'(\partial)(\partial - S))\Psi = 0. \tag{2.42}$$

Hence, as a matrix,

$$S_y - (\partial - S)U(\partial) + U'(\partial)(\partial - S) = 0. \tag{2.43}$$

This shows that $\partial - S$ satisfies the first equation of (2.31). The second one can be proved similarly. Therefore, $\partial - S$ is a Darboux operator for (2.26). The theorem is proved.

THEOREM 2.5 *$\partial - S$ is a Darboux operator for (2.26) if and only if there exists an $N \times N$ non-degenerate matrix solution H of (2.26) such that $S = H_x H^{-1}$ [122].*

Proof. First we prove the sufficiency, i.e., to show the S satisfies (2.36). From (2.26),

$$\begin{aligned} S_y &= H_{xy}H^{-1} - SH_yH^{-1} = (U(S)H)_xH^{-1} - SU(S) \\ &= [U(S), S] + (U(S))_x, \end{aligned} \tag{2.44}$$

which is the first equation of (2.36). The second one is derived in the same way.

Now suppose S satisfies (2.36). We shall show that the system of equations

$$H_x = SH, \qquad H_y = U(\partial)H, \qquad H_t = V(\partial)H, \tag{2.45}$$

has a solution. Clearly (2.45) is equivalent to

$$H_x = SH, \qquad H_y = U(S)H, \qquad H_t = V(S)H. \tag{2.46}$$

Hence, we only need to verify the integrability conditions of (2.46).

Let Ψ be a fundamental solution of $\Psi_x = S\Psi$. (2.37) implies

$$(\Psi_y - U(\partial)\Psi)_x = (S\Psi)_y - \partial U(\partial)\Psi = S(\Psi_y - U(\partial)\Psi). \tag{2.47}$$

Hence

$$\begin{aligned} & (V_y(\partial) + V(\partial)U(\partial))\Psi \\ = \; & (V(\partial)\Psi)_y - V(\partial)(\Psi_y - U(\partial)\Psi) \\ = \; & (V(S)\Psi)_y - V(S)(\Psi_y - U(S)\Psi) \\ = \; & V(S)_y\Psi + V(S)U(S)\Psi. \end{aligned} \tag{2.48}$$

Similarly,

$$(U_t(\partial) + U(\partial)V(\partial))\Psi = U(S)_t\Psi + U(S)V(S)\Psi. \tag{2.49}$$

Since $\det\Psi \neq 0$, the integrability condition (2.28) gives

$$U(S)_t - V(S)_y + [U(S), V(S)] = 0. \tag{2.50}$$

Hence the integrability condition $H_{yt} = H_{ty}$ for (2.46) holds.

Theorem 2.4 gives the other two integrability conditions $H_{xy} = H_{yx}$ and $H_{xt} = H_{tx}$. Hence (2.46) is integrable. For given initial value $H = H_0$ at $(t, x, y) = (t_0, x_0, y_0)$, (2.46) has a solution H. If H_0 is non-degenerate, H is also non-degenerate in a neighborhood of (t_0, x_0, y_0). That is, (2.46) has a non-degenerate matrix solution H such that $S = H_xH^{-1}$. The theorem is proved.

If there is no reduction, this theorem shows that any Darboux operator in the form $\partial - S$ can be expressed explicitly by the solutions of the Lax pair. Darboux transformation exists as long as the Lax pair has a non-degenerate $N \times N$ matrix solution. Under the Darboux transformation, U_j is transformed to U_j' given by (2.40). V_j''s have similar expressions.

Thus, we have constructed the Darboux transformation

$$(U, V, \Phi) \longrightarrow (U', V', \Phi'). \tag{2.51}$$

This process can be continued by algebraic and differential operations to get infinite number of solutions provided that the set of solutions of the Lax pair for the seed solution is big enough.

For the AKNS system (2.17), the action of the Darboux operator $\partial - S$ gives

$$(\partial - S)(J\partial + P) - S_x = (J\partial + P')(\partial - S). \tag{2.52}$$

The coefficients of ∂^2 on both sides are equal. Comparing the coefficient of ∂, we have

$$P' = P + [J, S]. \tag{2.53}$$

For practical problems, the entries of U and V often have some constraint relations. In that case, H in the theorem should also satisfy certain conditions so that (U', V') and (U, V) satisfy the same constraints. If so, we can obtain a transformation from a solution of a nonlinear partial differential equation to a solution of the same equation.

Remark 19 For the KP equation, the construction for the Darboux operator is completely the same as in Section 2.1. However, for the Davey-Stewartson equation, it is more difficult because we should consider the relations among the entries of P. We shall discuss it in Section 2.4.

Similar with the 1+1 dimensional case, we can also compose several Darboux transformations of degree one to a Darboux transformation of higher degree. However, they can be constructed directly with explicit formulae.

2.3.3 Darboux transformation of higher degree and the theorem of permutability

Now we discuss a Darboux operator of higher degree. It is a differential operator in the form

$$D(\partial) = \sum_{j=0}^{r} D_{r-j}\partial^j, \qquad D_0 = I \tag{2.54}$$

such that

$$\begin{aligned} D_y(\partial) &= U'(\partial)D(\partial) - D(\partial)U(\partial), \\ D_t(\partial) &= V'(\partial)D(\partial) - D(\partial)V(\partial). \end{aligned} \tag{2.55}$$

Here $U'(\partial)$ and $V'(\partial)$ are differential operators with respect to x.

For simplicity, we only discuss the Darboux operator of degree two. When $r > 2$, the Darboux operator can also be written down explicitly, but is more complicated.

THEOREM 2.6 *Let H_1 and H_2 be two $N \times N$ non-degenerate matrix solutions of (2.26). Let F be the block matrix*

$$\begin{pmatrix} H_1 & H_2 \\ \partial H_1 & \partial H_2 \end{pmatrix}. \tag{2.56}$$

Suppose $\det F \neq 0$*, then the following conclusions hold:*

(1) There is a unique differential operator of degree two

$$D(H_1, H_2, \partial) = \partial^2 + D_1\partial + D_2 \tag{2.57}$$

satisfying

$$D(H_1, H_2, \partial)H_i = 0 \qquad (i = 1, 2). \tag{2.58}$$

It is a Darboux operator.

(2) The theorem of permutability holds:

$$D(H_1, H_2, \partial) = D(H_2, H_1, \partial). \tag{2.59}$$

(3) There is a decomposition

$$D(H_1, H_2, \partial) = D(D(H_1, \partial)H_2, \partial)D(H_1, \partial). \tag{2.60}$$

Proof. Since $\det F \neq 0$, the linear algebraic system

$$D_1\partial H_1 + D_2 H_1 = -\partial^2 H_1, \quad D_1\partial H_2 + D_2 H_2 = -\partial^2 H_2 \tag{2.61}$$

for D_1, D_2 has a unique solution, which determines $D(\partial)$ uniquely and $D(\partial)$ satisfies (2.58). Since (2.61) is symmetric with respect to H_1 and H_2, (2) holds.

By the definitions of $D(D(H_1, \partial)H_2, \partial)$ and $D(H_1, \partial)$,

$$\begin{aligned} D(D(H_1, \partial)H_2, \partial)D(H_1, \partial)H_1 = 0, \\ D(D(H_1, \partial)H_2, \partial)D(H_1, \partial)H_2 = 0. \end{aligned} \tag{2.62}$$

Hence (2.60) holds. From (2.60) it is seen that $D(H_1, H_2, \partial)$ is a Darboux operator because it is the composition of two Darboux operators of degree one.

Similar to (1.134), the theorem of permutability can be expressed by the following diagram:

$$\begin{tikzcd}[column sep=small]
 & (U^{(1)}, V^{(1)}, \Phi^{(1)}) \arrow[dr, "H_2"] & \\
(U, V, \Phi) \arrow[ur, "H_1"] \arrow[dr, "H_2"'] & & (U^{(1,2)}, V^{(1,2)}, \Phi^{(1,2)}) \\
 & & \| \\
 & (U^{(2)}, V^{(2)}, \Phi^{(2)}) \arrow[r, "H_1"'] & (U^{(2,1)}, V^{(2,1)}, \Phi^{(2,1)})
\end{tikzcd} \tag{2.63}$$

EXAMPLE 2.7 *For the KP equation, $N = 1$, we can get the expression of u after the Darboux transformation. Denote $H_i = h_i$. Suppose the Darboux operator is*

$$\sum_{j=0}^{r} D_{r-j}\partial^j, \tag{2.64}$$

then Theorem 2.6 implies

$$\sum_{j=0}^{r-1} D_{r-j}\partial^j h_i = -\partial^r h_i, \tag{2.65}$$

i.e.,

$$(D_r, \cdots, D_1)F_r = -(\partial^r h_1, \cdots, \partial^r h_r). \tag{2.66}$$

Solving this system, we have

$$\begin{aligned}
D_1 &= -\det \begin{pmatrix} h_1 & \partial h_1 & \cdots & \partial^{r-2} h_1 & \partial^r h_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_r & \partial h_r & \cdots & \partial^{r-2} h_r & \partial^r h_r \end{pmatrix} \cdot \\
&\quad \cdot \left(\det \begin{pmatrix} h_1 & \partial h_1 & \cdots & \partial^{r-2} h_1 & \partial^{r-1} h_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_r & \partial h_r & \cdots & \partial^{r-2} h_r & \partial^{r-1} h_r \end{pmatrix} \right)^{-1} \\
&= -(\ln \det F_r)_x.
\end{aligned} \tag{2.67}$$

Therefore, for the KP equation, the transformation between two solutions is

$$u' = u - 2D_{1,x} = u + 2(\ln\det F_r)_{xx}. \tag{2.68}$$

Many solutions can be obtained in this way [77, 80, 89].

2.4 Darboux transformation and binary Darboux transformation for DS equation

2.4.1 Darboux transformation for DSII equation

In Section 2.2 we introduced the DSI and DSII equations (2.25) and their Lax pairs (2.17) and (2.21). Since the reductions in DSI equation and in DSII equation are different, the method of solving these two equations are also quite different.

First, consider the DSII equation, i.e., $\epsilon = 1$, $\alpha = -\mathrm{i}$ [120].

In this case, we should have $w_2 = \bar{w}_1$. Hence $v = -|u|^2 + \dfrac{\mathrm{i}}{2}(w_1 - \bar{w}_1)$, and J, P, V_j $(j = 0, 1, 2)$ are

$$\begin{aligned} &J = \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}, \\ &V_0 = 2J, \quad V_1 = 2P, \quad V_2 = \begin{pmatrix} w_1 & u_x - \mathrm{i}u_y \\ -\bar{u}_x - \mathrm{i}\bar{u}_y & \bar{w}_1 \end{pmatrix}. \end{aligned} \tag{2.69}$$

J, P and V_j have the properties

$$\bar{J} = \sigma J \sigma^{-1}, \quad \bar{P} = \sigma P \sigma^{-1}, \quad \bar{V}_j = \sigma V_j \sigma^{-1} \quad (j = 0, 1, 2), \tag{2.70}$$

where

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.71}$$

$\bar{P}$ is the matrix each of whose entry is the complex conjugate of the corresponding entry of P. Now (2.25) becomes

$$\begin{aligned} &\mathrm{i}u_t = -u_{xx} + u_{yy} + 2|u|^2 u + 2uv, \\ &v_{xx} + v_{yy} + 2(|u|^2)_{xx} = 0. \end{aligned} \tag{2.72}$$

Its Lax pair is

$$
\begin{aligned}
\Phi_y = & \ \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Phi_x + \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}\Phi, \\
\Phi_t = & \ 2\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Phi_{xx} + 2\begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}\Phi_x \\
& + \begin{pmatrix} w_1 & u_x - \mathrm{i}u_y \\ -\bar{u}_x - \mathrm{i}\bar{u}_y & \bar{w}_1 \end{pmatrix}\Phi,
\end{aligned} \tag{2.73}
$$

with

$$
v = -|u|^2 + \frac{i}{2}(w_1 - \bar{w}_1). \tag{2.74}
$$

The Darboux operator for (2.73) is constructed as follows.

Suppose $(\xi,\, \eta)^T$ is a solution of (2.73), then $(-\bar{\eta},\, \bar{\xi})^T$ is also its solution. Hence we can choose

$$
H = \begin{pmatrix} \xi & -\bar{\eta} \\ \eta & \bar{\xi} \end{pmatrix}, \tag{2.75}
$$

$$
S = H_x H^{-1} = \frac{1}{|\xi|^2 + |\eta|^2}\begin{pmatrix} \bar{\xi}\xi_x + \eta\bar{\eta}_x & \bar{\eta}\xi_x - \xi\bar{\eta}_x \\ \bar{\xi}\eta_x - \eta\bar{\xi}_x & \xi\bar{\xi}_x + \bar{\eta}\eta_x \end{pmatrix}. \tag{2.76}
$$

Since $\bar{H} = \sigma H \sigma^{-1}$, we have $\bar{S} = \sigma S \sigma^{-1}$. The equations

$$
\begin{aligned}
U'(\partial)(\partial - S) &= (\partial - S)U(\partial) - S_y, \\
V'(\partial)(\partial - S) &= (\partial - S)V(\partial) - S_t
\end{aligned} \tag{2.77}
$$

imply

$$
\begin{aligned}
\bar{U}' &= \sigma U' \sigma^{-1}, \\
\bar{V}' &= \sigma V' \sigma^{-1}.
\end{aligned} \tag{2.78}
$$

This means that the Darboux transformation keeps the reduction relations (2.70) invariant.

After the action of the Darboux operator $\partial - S$,

$$
\begin{aligned}
P' &= P + [J, S], \\
V_2' &= V_2 + V_{1,x} + 2V_0 S_x + [V_0, S]S + [V_1, S].
\end{aligned} \tag{2.79}
$$

Figure 2.1. Single line-soliton, $t = 0$

Figure 2.2. Double line-soliton, $t = 0$

Figure 2.3. Double line-soliton, $t = 0.5$

Hence the new solution of the DSII equation is

$$\begin{aligned} u' &= u + 2\mathrm{i}S_{12} = u + 2\mathrm{i}\frac{\bar{\eta}\xi_x - \xi\bar{\eta}_x}{|\xi|^2 + |\eta|^2}, \\ v' &= v - 2(\operatorname{Re} S_{11})_x = v - (\ln(|\xi|^2 + |\eta|^2))_{xx}. \end{aligned} \tag{2.80}$$

EXAMPLE 2.8 *Take the seed solution $u = 0$, then we can choose $v = 0$ ($w_1 = 0$), $\xi = \xi(x + \mathrm{i}y, t)$, $\eta = \eta(x - \mathrm{i}y, t)$ (i.e., ξ is analytic with respect to $x + \mathrm{i}y$ and η is analytic with respect to $x - \mathrm{i}y$) satisfying $\xi_t = 2\mathrm{i}\xi_{xx}$, $\eta_t = -2\mathrm{i}\eta_{xx}$. For these (ξ, η), (u, v) given by (2.80) are all solutions of DSII equation. Especially, let $\xi = e^{\alpha x + \mathrm{i}\alpha y + 2\mathrm{i}\alpha^2 t}$, $\eta = e^{\bar{\beta}x - \mathrm{i}\bar{\beta}y - 2\mathrm{i}\bar{\beta}^2 t}$, then*

$$\begin{aligned} u &= \frac{2\mathrm{i}(\alpha - \beta)e^{(\alpha+\beta)x + \mathrm{i}(\alpha+\beta)y + 2\mathrm{i}(\alpha^2+\beta^2)t}}{e^{2\mathrm{Re}\alpha x - 2\mathrm{Im}\alpha y - 2\mathrm{Im}(\alpha^2)t} + e^{2\mathrm{Re}\beta x - 2\mathrm{Im}\beta y - 2\mathrm{Im}(\beta^2)t}}, \\ v &= -\frac{4(\operatorname{Re}\alpha - \operatorname{Re}\beta)^2 e^{2\mathrm{Re}(\alpha+\beta)x - 2\mathrm{Im}(\alpha+\beta)y - 2\mathrm{Im}(\alpha^2+\beta^2)t}}{(e^{2\mathrm{Re}\alpha x - 2\mathrm{Im}\alpha y - 2\mathrm{Im}(\alpha^2)t} + e^{2\mathrm{Re}\beta x - 2\mathrm{Im}\beta y - 2\mathrm{Im}(\beta^2)t})^2}. \end{aligned} \tag{2.81}$$

When t is fixed, the solution u is a constant along the line with slope $x : y = \operatorname{Im}(\beta - \alpha) : \operatorname{Re}(\beta - \alpha)$, and tends to zero in any other directions. This kind of solution also belongs to "line-soliton".

Multi-line-solitons can be obtained by successive Darboux transformations. They tend to zero at infinity except for finitely many directions.

Figures 2.1 – 2.4 show the single line-soliton and multi-line-solitons, where the parameters are $\alpha_1 = 3 + 2\mathrm{i}$, $\beta_1 = 1 + \mathrm{i}$, $\alpha_2 = \mathrm{i}$, $\beta_2 = (2 + \mathrm{i})/4$. (For the single line-soliton, only (α_1, β_1) is used.)

Remark 20 Comparing to the general method discussed in the last section, the key point in the construction of Darboux transformation for the DSII equation is the choice of H in (2.75). Although it is successful to the DSII equation, it can not be applied to the DSI equation.

Figure 2.4. Double line-soliton, $t = 1$

2.4.2 Darboux transformation and binary Darboux transformation for DSI equation

When $\epsilon = 1$ and $\alpha = 1$, (2.25) becomes

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} + u_{yy} + 2|u|^2u + 2uv = 0,\\ &v_{xx} - v_{yy} + 2(|u|^2)_{xx} = 0, \end{aligned} \tag{2.82}$$

and its Lax pair is

$$\begin{aligned} \Phi_y = &\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Phi_x + \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}\Phi,\\ \Phi_t = &\,2\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Phi_{xx} + 2\mathrm{i}\begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}\Phi_x\\ &+\mathrm{i}\begin{pmatrix} w_1 & u_x + u_y \\ -\bar{u}_x + \bar{u}_y & w_2 \end{pmatrix}\Phi, \end{aligned} \tag{2.83}$$

where

$$v = -|u|^2 + \frac{1}{2}(w_1 - w_2), \tag{2.84}$$

w_1 and w_2 are real functions.

Since we can not find the solution H of (2.83) like that of (2.75) to construct the Darboux transformation for (2.82), we should use the binary Darboux transformation. The binary Darboux transformation was first introduced by V. B. Matveev et al and has many applications [4, 81, 80, 72, 125, 126]. Here we show its application to DSI equation for constructing new solutions. For the general case, please refer to [80].

For simplicity, rewrite (2.83) as

$$\begin{aligned} &\Phi_y = J\Phi_x + P\Phi,\\ &\Phi_t = 2\mathrm{i}J\Phi_{xx} + 2\mathrm{i}P\Phi_x + \mathrm{i}V_2\Phi. \end{aligned} \tag{2.85}$$

Apart from this Lax pair, consider its adjoint equations

$$\begin{aligned} &\Psi_y = \Psi_x J - \Psi P,\\ &\Psi_t = -2\mathrm{i}\Psi_{xx}J + 2\mathrm{i}(\Psi P)_x - \mathrm{i}\Psi V_2. \end{aligned} \tag{2.86}$$

With $P^* = -P$ and $V_2^* = V_2 - 2P_x$, we know that if Φ is a solution of (2.85), then $\Psi = \Phi^*$ is a solution of (2.86), and vice versa. Therefore, as soon as a solution of (2.85) or (2.86) is known, a solution of its adjoint equations is also known.

Similar to Section 2.3, we first take Darboux transformation for the adjoint equation (2.86):

$$\begin{aligned}&\Psi' = \Psi_x - \Psi S,\\ &P' = P + [J, S],\\ &V_2' = V_2 - 2P_x + 2[P, S] - 2S[J, S] + 2(JS_x + S_x J)\end{aligned} \tag{2.87}$$

where

$$S = \Psi_0^{-1}\Psi_{0,x}, \tag{2.88}$$

Ψ_0 is a non-degenerate 2×2 matrix solution of (2.86). Notice that P' does not satisfy $P'^* = -P'$, and Ψ'^* is not a solution of (2.85) with P replaced by P'. In order to preserve the reduction, the binary Darboux transformation is a useful tool. To get a new solution of the DSI equation, it needs the following steps:

Step 1: For a solution Φ of (2.85) and a solution Ψ of the adjoint equations (2.86), define 1-form

$$\omega(\Psi, \Phi) = \Psi\Phi\,\mathrm{d}x + \Psi J\Phi\,\mathrm{d}y + 2\mathrm{i}(\Psi P\Phi + \Psi J\Phi_x - \Psi_x J\Phi)\,\mathrm{d}t. \tag{2.89}$$

It can be verified that $\omega(\Psi, \Phi)$ is a closed 1-form, that is, its exterior differential $\mathrm{d}\omega(\Psi, \Phi) = 0$. Hence, in a simply connected region, the integral of ω along any closed curve is zero. In $\mathbf{R}^{2,1}$, define

$$\Omega(\Psi, \Phi)(x, y, t) = \int_{(x_0,y_0,t_0)}^{(x,y,t)} \omega(\Psi, \Phi), \tag{2.90}$$

which is independent of the path of integration, and $\omega(\Psi, \Phi) = \mathrm{d}\Omega(\Psi, \Phi)$.

Step 2: Let $\Phi' = \Psi_0^{-1}\Omega(\Psi_0, \Phi)$, then we can verify that Φ' is a solution of (2.85) with (P, V_2) replaced by (P', V_2').

Step 3: Let $\Phi_0' = \Psi_0^{-1}\Omega(\Psi_0, \Psi_0^*)$. Acting the Darboux operator $\partial - \Phi_0'\Phi_0'^{-1}$ on Φ', we get the Darboux transformation

$$\begin{aligned}&\Phi'' = \Phi_x' - \Phi_{0,x}'\Phi_0'^{-1}\Phi' = \Phi - \Psi_0^*\Omega(\Psi_0, \Psi_0^*)^{-1}\Omega(\Psi_0, \Phi),\\ &P'' = P' + [J, \Psi_{0,x}'\Psi_0'^{-1}] = P + [J, \Psi_0^*\Omega(\Psi_0, \Psi_0^*)^{-1}\Psi_0].\end{aligned} \tag{2.91}$$

$P^* = -P$ leads to $P''^* = -P''$. Therefore, we get a new solution of the DSI equation.

The process in Step 1 and Step 2 is called a binary Darboux transformation. For the DSI equation, a new solution is obtained by the composition of a Darboux transformation and a binary Darboux transformation. It needs differentiation and integration in this procedure.

2.5 Application to 1+1 dimensional Gelfand-Dickey system

In this section, we use Theorem 2.5 to discuss the Darboux transformation for the (1+1 dimensional) Gelfand-Dickey system

$$\begin{aligned} \lambda\Phi &= U(x,t,\partial)\Phi, \\ \Phi_t &= V(x,t,\partial)\Phi \end{aligned} \tag{2.92}$$

where

$$\begin{aligned} U(\partial) &= \sum_{j=0}^{m} U_{m-j}(x,t)\partial^j, \\ V(\partial) &= \sum_{j=0}^{n} V_{n-j}(x,t)\partial^j. \end{aligned} \tag{2.93}$$

From the first equation of (2.92), we can compute Φ_t and it should be the same as that given by the second equation of of (2.92). This gives the integrability condition

$$U_t(\partial) + [U(\partial), V(\partial)] = 0 \tag{2.94}$$

of (2.92).

Let $D(x,t,\partial)$ be a differential operator. If for any solution Φ of (2.92), $\Phi' = D(\partial)\Phi$ satisfies

$$\begin{aligned} \lambda\Phi' &= U'(x,t,\partial)\Phi', \\ \Phi'_t &= V'(x,t,\partial)\Phi', \end{aligned} \tag{2.95}$$

where U' and V' are differential operators of the form

$$\begin{aligned} U'(\partial) &= \sum_{j=0}^{m} U'_{m-j}(x,t)\partial^j, \\ V'(\partial) &= \sum_{j=0}^{n} V'_{n-j}(x,t)\partial^j, \end{aligned} \tag{2.96}$$

then $D(x,t,\partial)$ is called a Darboux operator for (2.92).

For a differential operator $D(x,t,\partial) = \partial - S(x,t)$, we have the following theorem.

THEOREM 2.9 *$\partial - S(x,t)$ is a Darboux operator for (2.92) if and only if $S = H_x H^{-1}$, where H is an $N \times N$ non-degenerate matrix solution of*

$$\begin{aligned} H\Lambda &= U(\partial)H, \\ H_t &= V(\partial)H, \end{aligned} \tag{2.97}$$

and Λ is a constant upper-triangular matrix.

Proof. Introduce a new variable y and consider the system

$$\begin{aligned}\Psi_y &= U(x,t,\partial)\Psi,\\ \Psi_t &= V(x,t,\partial)\Psi.\end{aligned} \tag{2.98}$$

If $\partial - S(x,t)$ is a Darboux operator for (2.92), then there exist $U'(\partial)$ and $V'(\partial)$ such that

$$\begin{aligned}0 &= (\partial - S)U(\partial) - U'(\partial)(\partial - S),\\ S_t &= (\partial - S)V(\partial) - V'(\partial)(\partial - S).\end{aligned} \tag{2.99}$$

Since S is independent of y and (2.37) holds, $\partial - S$ is a Darboux operator for (2.98) which is independent of y. According to Theorem 2.5, there exists an $N\times N$ non-degenerate matrix solution H_0 of (2.98) such that $S = H_{0,x}H_0^{-1}$. Here H_0 may depend on y.

Let $L_0 = H_0^{-1}H_{0,y}$, (2.36) leads to

$$\begin{aligned}L_0 = &\ H_0^{-1}U(S)H_0,\\ L_{0,x} = &\ -H_0^{-1}H_{0,x}H_0^{-1}U(S)H_0 + H_0^{-1}(U(S))_xH_0\\ &+H_0^{-1}U(S)H_{0,x}\\ = &\ H_0^{-1}\{(U(S))_x - [S,U(S)]\}H_0 = 0,\end{aligned}$$

and (2.50) leads to

$$\begin{aligned}L_{0,y} = &\ -H_0^{-1}H_{0,y}H_0^{-1}U(S)H_0 + H_0^{-1}U(S)H_{0,y} = 0,\\ L_{0,t} = &\ -H_0^{-1}H_{0,t}H_0^{-1}U(S)H_0 + H_0^{-1}(U(S))_tH_0\\ &+H_0^{-1}U(S)H_{0,t}\\ = &\ H_0^{-1}\{(U(S))_t + [U(S),V(S)]\}H_0 = 0.\end{aligned}$$

Hence L_0 is a constant matrix. Therefore, there exists a constant upper-triangular matrix Λ and a constant matrix T such that $L_0 = T\Lambda T^{-1}$. According to the definition of L_0,

$$H_{0,y} = H_0T\Lambda T^{-1}.$$

Hence

$$H_0(x,y,t) = H(x,t)\exp(\Lambda y)T^{-1}$$

where H satisfies (2.97) and $S = H_xH^{-1}$.

Conversely, if H is a solution (2.97) and $S = H_x H^{-1}$, then S satisfies (2.99), i.e., $\partial - S$ is a Darboux operator for (2.92). The theorem is proved.

Remark 21 (1) When $N = 1$, H satisfies the Lax pair (2.92).

(2) If L_0 in the above theorem is diagonalizable, then each column in H satisfies the Lax pair (2.92) for specific λ.

Example 2.10 *The original Darboux transformation for the KdV equation can also be deduced from the above theorem.*

The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{2.100}$$

has the Lax pair

$$\begin{aligned} &\lambda\phi = -\phi_{xx} - u\phi, \\ &\phi_t = 2(2\lambda - u)\phi_x + u_x\phi. \end{aligned} \tag{2.101}$$

From Theorem 2.9, the Darboux transformation is

$$\phi' = \phi_x - \frac{f_x}{f}\phi \tag{2.102}$$

where f is a solution of the Lax pair for $\lambda = \lambda_0$. The new solution given by this Darboux transformation is

$$u' = u + 2(\ln f)_{xx}. \tag{2.103}$$

Example 2.11 *The Boussinesq equation*

$$(u_{xxx} + 6uu_x)_x + 3\epsilon u_{tt} = 0 \quad (\epsilon = \pm 1)$$

has the Lax pair

$$\begin{aligned} &\lambda\phi = \phi_{xxx} + \frac{3u}{2}\phi_x + w\phi, \\ &\phi_t = \sigma\phi_{xx} + \sigma u\phi \end{aligned} \tag{2.104}$$

($\sigma^2 = \epsilon$, $w_x = 3(\sigma u_{xx} + u_t)/4\sigma$). Theorem 2.9 also gives the Darboux transformation [73]

$$\phi' = \phi_x - \frac{f_x}{f}\phi \tag{2.105}$$

where f is a solution of the Lax pair for $\lambda = \lambda_0$. The new solution of the Boussinesq equation is

$$\begin{aligned} &u' = u + 2(\ln f)_{xx}, \\ &w' = w + \frac{3}{2}u_x + 3((\ln f)_{xxx} + (\ln f)_x(\ln f)_{xx}). \end{aligned} \tag{2.106}$$

2.6 Nonlinear constraints and Darboux transformation in 2+1 dimensions

Now we come back to the 2+1 dimensional AKNS system. In this section we will use the nonlinear constraint method and the Darboux transformation method to solve this system.

The basic idea of the nonlinear constraint method is:

(1) Find a suitable nonlinear relation between U and Ψ and express U as a nonlinear matrix function of Ψ: $U = f(\Psi)$.

(2) Substitute $U = f(\Psi)$ into the Lax pair so that the original Lax pair becomes a system of nonlinear partial differential equations of Ψ. In each equation, the derivative with respect to only one of x, y, t is concerned.

(3) The constraint $U = f(\Psi)$ is suitable so that the new system of nonlinear equations has a Lax set (generalized Lax pair).

Then by solving the new system of nonlinear equations and its Lax set, we can get solutions of the original problem.

This idea was first applied in 1+1 dimensional integrable systems [11] and was generalized to the (2+1 dimensional) KP equation [14, 71]. Here we pay our attention to the 2+1 dimensional AKNS system so that we can get localized soliton solutions. With this method, we can also get a lot of non-localized solutions [123, 124]. However, since localized solutions are more interesting, here we only consider localized solutions [127, 128].

In order to use the nonlinear constraint method, here we add some conditions on the 2+1 dimensional AKNS system. As in Section 2.2, the 2+1 dimensional AKNS system is

$$\begin{aligned} \Psi_y &= J\Psi_x + U(x,y,t)\Psi, \\ \Psi_t &= \sum_{j=0}^{n} V_j(x,y,t)\partial^{n-j}\Psi \end{aligned} \tag{2.107}$$

where $\partial = \partial/\partial x$, $J = \mathrm{diag}(J_1, \cdots, J_N)$ is a constant diagonal $N \times N$ matrix with distinct entries. $U(x,y,t)$ is off-diagonal. Moreover, here we want that all J_j's are real and $U^* = -U$. In this case, we call (2.107) a hyperbolic $u(N)$ AKNS system. The condition $U^* = -U$ will imply that the solutions are globally defined, and the condition J_j's are real will guarantee that there exist localized solutions.

As in Section 2.2, the integrability conditions of (2.107) are given by (2.20).

Now we introduce a new linear system

$$\begin{aligned} &\Phi_x = \begin{pmatrix} \mathrm{i}\lambda I & \mathrm{i}F \\ \mathrm{i}F^* & 0 \end{pmatrix}\Phi, \qquad \Phi_y = \begin{pmatrix} \mathrm{i}\lambda J + U & \mathrm{i}JF \\ \mathrm{i}F^*J & 0 \end{pmatrix}\Phi, \\ &\Phi_t = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}\Phi = \sum_{j=0}^{n} \begin{pmatrix} W_j & X_j \\ -X_j^* & Z_j \end{pmatrix}\lambda^{n-j}\Phi \end{aligned} \tag{2.108}$$

where F, W_j, X_j, Z_j are $N\times K$, $N\times N$, $N\times K$, $K\times K$ matrices respectively ($K\geq 1$) and satisfy $W_j^* = -W_j$, $Z_j^* = -Z_j$.

The integrability conditions of (2.108) consists of

$$\begin{aligned} &F_y = JF_x + UF, \\ &\mathrm{i}F_t = X_{n,x} + \mathrm{i}W_nF - \mathrm{i}FZ_n, \end{aligned} \tag{2.109}$$

$$\begin{aligned} &\mathrm{i}X_{j+1} = X_{j,x} + \mathrm{i}W_jF - \mathrm{i}FZ_j \quad (j=0,1,\cdots,n-1) \\ &W_{j,x} = -\mathrm{i}FX_j^* - \mathrm{i}X_jF^* \quad (j=0,1,\cdots,n) \\ &Z_{j,x} = \mathrm{i}F^*X_j + \mathrm{i}X_j^*F \quad (j=0,1,\cdots,n) \\ &\mathrm{i}[J,W_{j+1}] = W_{j,y} - [U,W_j] + \mathrm{i}JFX_j^* + \mathrm{i}X_jF^*J \\ &\quad (j=0,1,\cdots,n-1) \\ &Z_{j,y} = \mathrm{i}F^*JX_j + \mathrm{i}X_j^*JF \quad (j=0,1,\cdots,n), \end{aligned} \tag{2.110}$$

$$U_x = [J, FF^*], \tag{2.111}$$

$$U_t = W_{n,y} - [U,W_n] + \mathrm{i}JFX_n^* + \mathrm{i}X_nF^*J. \tag{2.112}$$

For $U=0$, $F=0$, (2.110) implies that $W_j(\lambda) = \mathrm{i}\Omega_j(t)$, $X_j=0$, $Z_j = \mathrm{i}Z_j^0(t)$ where $\Omega_j(t)$'s are real diagonal matrices and $Z_j^0(t)$'s are real matrices.

When $Z_j^0(t) = \zeta_j(t)I_K$ (I_K is the $K\times K$ identity matrix) where $\zeta_j(t)$ is a real function of t, (2.109) is just the Lax pair (2.107) for $n=1,2,3$. (2.110) and (2.112) give the recursion relations to determine W_j, X_j, Z_j, together with the evolution equations corresponding to (2.18)–(2.20), which are the integrability conditions of (2.107). (2.111) gives a nonlinear constraint between U and F.

This system includes the DSI equation and the 2+1 dimensional N-wave equation as special cases.

In order to consider the asymptotic behavior of the solution U, here we suppose Ω_j is independent of t and $\zeta_j = 0$. Moreover, denote $\Omega = \sum_{j=0}^{n}\Omega_j\lambda^{n-j}$ and write $\Omega = \mathrm{diag}(\omega_1,\cdots,\omega_N)$.

Remark 22 (2.108) is a special case of the high-dimensional generalized AKNS system (3.1). Here we only consider this special system to find localized solutions. The general theory will be discussed in the next chapter.

The soliton solutions are obtained by Darboux transformations from $U = 0$, $F = 0$. In the present case, the Darboux transformation can be constructed as in Subsection 1.4.4 with $u(n)$ reduction. However, in order to get localized solutions, there should be more restrictions on the parameters of Darboux transformations.

Let λ_α ($\alpha = 1, 2, \cdots, r$) be r non-real complex numbers such that $\lambda_\alpha \neq \lambda_\beta$ for $\alpha \neq \beta$ and $\lambda_\alpha \neq \bar{\lambda}_\beta$ for all α, β. Let

$$\Lambda_\alpha = \operatorname{diag}(\underbrace{\lambda_\alpha, \cdots, \lambda_\alpha}_{N}, \underbrace{\bar{\lambda}_\alpha, \cdots, \bar{\lambda}_\alpha}_{K}). \tag{2.113}$$

Considering the orthogonal relation (1.241), we can always take

$$H_\alpha = \begin{pmatrix} \exp(Q_\alpha) & -\exp(-Q_\alpha^*)C_\alpha^* \\ C_\alpha & I_K \end{pmatrix}, \tag{2.114}$$

where C_α's are $K \times N$ constant matrices,

$$Q_\alpha = \operatorname{diag}(q_1, \cdots, q_N), \quad q_j = \mathrm{i}\lambda_\alpha x + \mathrm{i}\lambda_\alpha J_j y + \mathrm{i}\omega_j(\lambda_\alpha, t). \tag{2.115}$$

According to Section 1.4, the derived solutions are always global. However, in order to get localized solutions, we choose special

$$C_\alpha = (0, \cdots, 0, \underset{l_\alpha}{\kappa_\alpha}, 0, \cdots, 0) \tag{2.116}$$

where κ_α is a constant $K \times 1$ non-zero vector being the l_α's column of C_α.

The Darboux matrices for such $\{\Lambda_\alpha, H_\alpha\}$ can be constructed as follows. Let

$$\begin{aligned}
&D^{(1)}(\lambda) = \lambda - H_1\Lambda_1 H_1^{-1}, \quad H_\alpha^{(1)} = D^{(1)}(\lambda_\alpha)H_\alpha \\
&\qquad (\alpha = 2, 3, \cdots, r), \\
&D^{(2)}(\lambda) = \lambda - H_2^{(1)}\Lambda_2 H_2^{(1)-1}, \quad H_\alpha^{(2)} = D^{(2)}(\lambda_\alpha)H_\alpha^{(1)} \\
&\qquad (\alpha = 3, 4, \cdots, r), \\
&\cdots \\
&D^{(r)}(\lambda) = \lambda - H_r^{(r-1)}\Lambda_r H_r^{(r-1)-1},
\end{aligned} \tag{2.117}$$

$$D(\lambda) = D^{(r)}(\lambda)D^{(r-1)}(\lambda)\cdots D^{(1)}(\lambda), \tag{2.118}$$

then $D(\lambda)$ is a polynomial of λ of degree r. The permutability (Theorem 1.12) implies that if $(\Lambda_\alpha, H_\alpha)$ and (Λ_β, H_β) are interchanged, $D(\lambda)$ is invariant.

Let

$$m_j = \#\{\,\alpha \,|\, 1 \le \alpha \le r,\, l_\alpha = j\,\} \qquad m = (m_1, \cdots, m_N) \tag{2.119}$$

then $m_1 + \cdots + m_N = r$.

Suppose

$$D(\lambda) = \lambda^r - D_1\lambda^{r-1} + \cdots + (-1)^r D_r. \tag{2.120}$$

The solution given by this Darboux matrix is

$$U^{[m]} = \mathrm{i}[J, (D_1)_{B_N}]. \tag{2.121}$$

Here $(D_1)_{B_N}$ denotes the first $N \times N$ principal submatrix of D_1.

In order to consider the localization, the asymptotic behavior as $t \to \infty$ and the asymptotic behavior as the phase difference tends to infinity uniformly, we write

$$q_j = a_{\alpha j}s + b_{\alpha j} \tag{2.122}$$

where $a_{\alpha j}$ and $b_{\alpha j}$ are independent of s. Here s can be a linear parameter of a straight line in (x, y) plane, or time t, or any other parameters.

Moreover, denote

$$\begin{aligned} \rho_\alpha &= \mathrm{Re}(a_{\alpha,l_\alpha}), \qquad & \phi_\alpha &= \mathrm{Im}(a_{\alpha,l_\alpha}),\\ \pi_\alpha &= \mathrm{Re}(b_{\alpha,l_\alpha}), \qquad & \psi_\alpha &= \mathrm{Im}(b_{\alpha,l_\alpha}). \end{aligned} \tag{2.123}$$

In order to prove the following theorem, we need some symbols and simple facts.

If M_1, M_2 are $j \times k$ matrices, we write $M_1 \doteq M_2$ if there is a non-degenerate diagonal $k \times k$ matrix A such that $M_2 = M_1 A$.

If L is a $k \times k$ diagonal matrix, M_1 and M_2 are $k \times k$ matrices with $M_1 \doteq M_2$ and $\det M_1 \ne 0$, then $M_1 L M_1^{-1} = M_2 L M_2^{-1}$.

Let

$$M = \begin{pmatrix} a & -v^*/\bar{a} \\ v & I_K \end{pmatrix} \tag{2.124}$$

where $v \ne 0$ is an $K \times 1$ vector, $a \ne 0$ is a number. Let

$$\Lambda = \begin{pmatrix} \lambda_0 & \\ & \bar{\lambda}_0 I_K \end{pmatrix}. \tag{2.125}$$

Then we have

$$M^{-1} = \frac{1}{\Delta}\begin{pmatrix} \bar{a} & v^* \\ -\bar{a}v & \Delta I_K - vv^* \end{pmatrix}, \tag{2.126}$$

$$M\Lambda M^{-1} = \frac{1}{\Delta}\begin{pmatrix} \bar{\lambda}_0\Delta + (\lambda_0 - \bar{\lambda}_0)|a|^2 & (\lambda_0 - \bar{\lambda}_0)av^* \\ (\lambda_0 - \bar{\lambda}_0)\bar{a}v & \bar{\lambda}_0\Delta I_K + (\lambda_0 - \bar{\lambda}_0)vv^* \end{pmatrix} \tag{2.127}$$

where $\Delta = v^*v + |a|^2$. Moreover,

$$\lim_{a\to\infty} M\Lambda M^{-1} = \begin{pmatrix} \lambda_0 & \\ & \bar{\lambda}_0 I_K \end{pmatrix}, \tag{2.128}$$

$$\lim_{a\to 0} M\Lambda M^{-1} = \begin{pmatrix} \bar{\lambda}_0 & \\ & \bar{\lambda}_0 I_K + (\lambda_0 - \bar{\lambda}_0)\dfrac{vv^*}{v^*v} \end{pmatrix}. \tag{2.129}$$

THEOREM 2.12 *(1) If there is at most one α ($1 \le \alpha \le r$) such that $\rho_\alpha = 0$, then $\lim_{s\to\infty} U^{[m]} = 0$.*

(2) If $\rho_{\alpha_j} = 0$ ($j = 1, 2, \cdots, q$) with $\alpha_j \ne \alpha_k$ for $j \ne k$, $\rho_\gamma \ne 0$ for all $\gamma \ne \alpha_j$ ($j = 1, \cdots, q$) and $l_{\alpha_1} = \cdots = l_{\alpha_q}$, then $\lim_{s\to\infty} U^{[m]} = 0$.

(3) If $\rho_\alpha = 0$, $\rho_\beta = 0$ ($\alpha \ne \beta$), $\rho_\gamma \ne 0$ for all $\gamma \ne \alpha, \beta$, and $l_\alpha \ne l_\beta$, then

$$\lim_{s\to\infty} U^{[m]}_{ab} = 0 \quad \textit{for } (a, b) \ne (l_\alpha, l_\beta) \tag{2.130}$$

and as $s \to \infty$,

$$U^{[m]}_{l_\alpha,l_\beta} \sim \frac{B_{\alpha\beta}\exp(\mathrm{i}(\psi_\alpha - \psi_\beta) + \mathrm{i}(\phi_\beta - \phi_\alpha)s)}{A_{\alpha\beta}\cosh(\pi_\alpha + \pi_\beta - \delta^{(1)}_{\alpha\beta}) + \cosh(\pi_\alpha - \pi_\beta - \delta^{(2)}_{\alpha\beta})} \tag{2.131}$$

where $A_{\alpha\beta}$, $\delta^{(1)}_{\alpha\beta}$, $\delta^{(2)}_{\alpha\beta}$ are real constants, $A_{\alpha\beta} > 0$, and $B_{\alpha\beta}$ are complex constants. Moreover, if $K = 1$, then $B_{\alpha\beta} \ne 0$ if and only if $\kappa_\alpha \ne 0$ and $\kappa_\beta \ne 0$.

Proof. First suppose $\rho_\alpha \ne 0$. By (2.128) and (2.129),

$$\lim_{\rho_\alpha s\to\pm\infty} H_\alpha \Lambda_\alpha H_\alpha^{-1} = S_\alpha^{\pm\infty} \tag{2.132}$$

where

$$
\begin{aligned}
S_\alpha^{+\infty} &= \begin{pmatrix} \lambda_\alpha I_N & \\ & \bar{\lambda}_\alpha I_K \end{pmatrix}, \\
S_\alpha^{-\infty} &= \begin{pmatrix} \lambda_\alpha I_N + (\bar{\lambda}_\alpha - \lambda_\alpha) E_{l_\alpha l_\alpha} & \\ & \bar{\lambda}_\alpha I_K + (\lambda_\alpha - \bar{\lambda}_\alpha) \dfrac{\kappa_\alpha \kappa_\alpha^*}{\kappa_\alpha^* \kappa_\alpha}, \end{pmatrix},
\end{aligned}
\tag{2.133}
$$

E_{jk} is an $N \times N$ matrix whose (j,k)th entry is 1 and the rest entries are zero.

For $\beta \neq \alpha$,

$$
(\lambda_\beta - S_\alpha^{\pm\infty}) H_\beta \doteq \begin{pmatrix} \exp(Q_\beta(s)) & -\exp(-Q_\beta(s)^*) \widetilde{C}_\beta^{\pm *} \\ \widetilde{C}_\beta^{\pm} & I_K \end{pmatrix} \tag{2.134}
$$

where

$$
\begin{aligned}
\widetilde{C}_\beta^{\pm} &= (0, \cdots, 0, \underset{l_\beta}{\widetilde{\kappa}_\beta^{\pm}}, 0, \cdots, 0), \\
\widetilde{\kappa}_\beta^{+} &= \frac{\lambda_\beta - \bar{\lambda}_\alpha}{\lambda_\beta - \lambda_\alpha} \kappa_\beta, \\
\widetilde{\kappa}_\beta^{-} &= \begin{cases} \dfrac{\lambda_\beta - \bar{\lambda}_\alpha}{\lambda_\beta - \lambda_\alpha} \kappa_\beta - \dfrac{\lambda_\alpha - \bar{\lambda}_\alpha}{\lambda_\beta - \lambda_\alpha} \dfrac{\kappa_\alpha^* \kappa_\beta}{\kappa_\alpha^* \kappa_\alpha} \kappa_\alpha & \text{if } l_\beta \neq l_\alpha, \\ \kappa_\beta - \dfrac{\lambda_\alpha - \bar{\lambda}_\alpha}{\lambda_\beta - \bar{\lambda}_\alpha} \dfrac{\kappa_\alpha^* \kappa_\beta}{\kappa_\alpha^* \kappa_\alpha} \kappa_\alpha & \text{if } l_\beta = l_\alpha. \end{cases}
\end{aligned}
\tag{2.135}
$$

Therefore, if $\rho_\alpha \neq 0$, the action of the limit Darboux matrix $\lambda - S_\alpha^{\pm\infty}$ on H_β ($\beta \neq \alpha$) does not change the form of H_β, but only changes the constant vector κ_β.

If $K = 1$, then $\kappa_\beta^{\pm *} \kappa_\gamma^{\pm} \neq 0$ implies $\widetilde{\kappa}_\beta^{\pm *} \widetilde{\kappa}_\gamma^{\pm} \neq 0$. When $K > 1$, this does not hold in general.

Now suppose $\rho_\alpha = 0$. Without loss of generality, suppose $l_\alpha = 1$. Then

$$
H_\alpha \doteq \begin{pmatrix} \exp(\pi_\alpha) & & & & -\exp(-\bar{\pi}_\alpha) \kappa_\alpha^* \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ \kappa_\alpha & 0 & \cdots & 0 & I_K \end{pmatrix}. \tag{2.136}
$$

By (2.127),

$$H_\alpha \Lambda_\alpha H_\alpha^{-1} = \frac{1}{\Delta} \cdot \\ \cdot \begin{pmatrix} \bar{\lambda}_\alpha \Delta + (\lambda_\alpha - \bar{\lambda}_\alpha) \exp(\pi_\alpha + \bar{\pi}_\alpha) & & & & (\lambda_\alpha - \bar{\lambda}_\alpha) \exp(\pi_\alpha) \kappa_\alpha^* \\ & \lambda_\alpha & & & 0 \\ & & \ddots & & \vdots \\ & & & \lambda_\alpha & 0 \\ (\lambda_\alpha - \bar{\lambda}_\alpha) \exp(\bar{\pi}_\alpha) \kappa_\alpha & 0 & \cdots & 0 & \bar{\lambda}_\alpha \Delta I_K + (\lambda_\alpha - \bar{\lambda}_\alpha) \kappa_\alpha \kappa_\alpha^* \end{pmatrix} \tag{2.137}$$

where $\Delta = \exp(\pi_\alpha + \bar{\pi}_\alpha) + \kappa_\alpha^* \kappa_\alpha$.

Part (1) of the theorem is derived as follows. Owing to the permutability of Darboux transformations, we can suppose $\rho_1 \neq 0$, $\cdots$, $\rho_{r-1} \neq 0$, $\rho_r = 0$. Then, as $s \to \infty$, $D^{(\alpha)}$ tends to a diagonal matrix for $\alpha \leq r-1$. Considering (2.137), the limit of $(D^{(r)}(\lambda))_{B_N}$ is also diagonal, hence

$$U^{[m]} = \mathrm{i}[J, (D_1)_{B_N}] \to 0. \tag{2.138}$$

Now we turn to prove part (2). We use the construction of Darboux matrices in (1.258). However, the λ in (1.258) should be replaced by $\mathrm{i}\lambda$ because of its appearance in (2.108).

Let

$$\mathring{H}_\alpha = \begin{pmatrix} \exp(Q_\alpha(s)) \\ C_\alpha \end{pmatrix}, \quad \Gamma_{\alpha\beta} = \frac{\mathring{H}_\alpha^* \mathring{H}_\beta}{\lambda_\beta - \bar{\lambda}_\alpha}, \tag{2.139}$$

then the Darboux matrix is

$$D(\lambda) = \prod_{\alpha=1}^{r} (\lambda - \bar{\lambda}_\alpha) \left(1 - \sum_{\alpha,\beta=1}^{r} \frac{\mathring{H}_\alpha (\Gamma^{-1})_{\alpha\beta} \mathring{H}_\beta^*}{\lambda - \bar{\lambda}_\beta} \right) \tag{2.140}$$

and the new solution is

$$U^{[m]} = \mathrm{i} \left[J, \sum_{\alpha,\beta=1}^{r} \left(\mathring{H}_\alpha (\Gamma^{-1})_{\alpha\beta} \mathring{H}_\beta^* \right)_{B_N} \right]. \tag{2.141}$$

First, suppose $q = r$ and $\alpha_j = j$ $(j = 1, 2, \cdots, r)$.

Since

$$\overset{\circ}{H}_j \doteq \begin{array}{c}\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \exp(\pi_j) & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \\ 0 & \cdots & 0 & \kappa_j & 0 & \cdots & 0 \end{pmatrix} \\ l_j \end{array} \tag{2.142}$$

and $l_1 = \cdots = l_q$, we have

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \cdots & \Gamma_{1q} \\ \vdots & \ddots & \vdots \\ \Gamma_{q1} & \cdots & \Gamma_{qq} \end{pmatrix} \tag{2.143}$$

where Γ_{jk}'s are $N \times N$ diagonal matrices. Therefore,

$$\Gamma^{-1} = \begin{pmatrix} \Sigma_{11} & \cdots & \Sigma_{1q} \\ \vdots & \ddots & \vdots \\ \Sigma_{q1} & \cdots & \Sigma_{qq} \end{pmatrix} \tag{2.144}$$

where Σ_{jk}'s are also $N \times N$ diagonal matrices. This implies that $U^{[m]} = 0$.

When $r > q$, we use the permutability of Darboux transformations and suppose $\rho_1 \neq 0, \cdots, \rho_{r-q} \neq 0, \rho_{r-q+1} = \cdots = \rho_r = 0$. Then, after the action of $D^{(r-q)}(\lambda) \cdots D^{(1)}(\lambda)$, the derived $H_{r-q+1}^{(r-q)}$ and $H_r^{(r-q)}$ have the same asymptotic form as H_{r-q+1} and H_r respectively, provided that the constant vectors κ_{r-q+1} and κ_r are changed to $\kappa_{r-q+1}^{(r-q)}$ and $\kappa_r^{(r-q)}$. Therefore, as in the case $r = q$, the limits of the components of $U^{[m]}$ are all zero. This proved part (2).

Now we prove part (3). First consider the case $r = 2$. Suppose $\overset{\circ}{H}_j$ is given by (2.142) ($j = 1, 2$) and $l_1 \neq l_2$. Denote

$$\theta_{12} = \frac{\kappa_1^* \kappa_2}{\sqrt{\kappa_1^* \kappa_1 \kappa_2^* \kappa_2}}, \tag{2.145}$$

$$g_{12} = 1 - \frac{4 \operatorname{Im}\lambda_1 \operatorname{Im}\lambda_2}{|\lambda_2 - \bar{\lambda}_1|^2}|\theta_{12}|^2 > 0, \tag{2.146}$$

then, by direct calculation, we have

$$\begin{aligned}
&\lim_{s\to\infty} U^{[m]}_{l_1,l_2} \exp(\mathrm{i}(\phi_2 - \phi_1)s) = \frac{2\mathrm{i}(J_{l_2} - J_{l_1}) \operatorname{Im}\lambda_1 \operatorname{Im}\lambda_2\theta_{12}}{\bar{\lambda}_2 - \lambda_1} \cdot \\
&\cdot \frac{\exp(\mathrm{i}(\psi_1 - \psi_2))}{\sqrt{g_{12}}\cosh(\pi_1 + \pi_2 - \delta_1) + \cosh(\pi_1 - \pi_2 - \delta_2)}, \\
&\delta_1 = \frac{1}{2}\ln g_{12} + \frac{1}{2}\ln(\kappa_1^*\kappa_1\kappa_2^*\kappa_2) + 2\ln\left|\frac{\lambda_2 - \lambda_1}{\lambda_2 - \bar{\lambda}_1}\right|, \\
&\delta_2 = \frac{1}{2}\ln\frac{\kappa_1^*\kappa_1}{\kappa_2^*\kappa_2},
\end{aligned} \tag{2.147}$$

and $U^{[m]}_{\mu\nu} \to 0$ if $(\mu,\nu) \neq (l_1,l_2)$.

When $r > 2$, we still use the permutability of Darboux transformations and suppose $\rho_1 \neq 0, \cdots, \rho_{r-2} \neq 0$, $\rho_{r-1} = \rho_r = 0$. As in the proof of part (2), after the action of $D^{(r-2)}(\lambda)\cdots D^{(1)}(\lambda)$, the derived $H^{(r-2)}_{r-1}$ and $H^{(r-2)}_r$ have the same asymptotic form as H_{r-1} and H_r respectively, provided that the constant vectors κ_{r-1} and κ_r are changed to $\kappa^{(r-2)}_{r-1}$ and $\kappa^{(r-2)}_r$. Therefore, as in the case $r = 2$, the limit of U_{l_{r-1},l_r} has the desired form, and the limits of the other components of $U^{[m]}$ are all zero. The theorem is proved.

Now we can discuss the properties of the solution $U^{[m]}$.

(1) Localization of the solutions

For the Lax pair (2.108),

$$Q_\alpha = \mathrm{i}\lambda_\alpha(x + Jy) + \mathrm{i}\omega(\lambda_\alpha)t. \tag{2.148}$$

Consider the limit of the solution as $(x,y) \to \infty$ along a straight line $x = \xi + v_x s$, $y = \eta + v_y s$ $(v_x^2 + v_y^2 > 0)$, then

$$Q_\alpha = \mathrm{i}\lambda_\alpha(\xi + J\eta) + \mathrm{i}\omega(\lambda_\alpha)t + \mathrm{i}\lambda_\alpha(v_x + Jv_y)s. \tag{2.149}$$

Now

$$\rho_\alpha = \operatorname{Re}(\mathrm{i}\lambda_\alpha(v_x + J_{l_\alpha}v_y)) = -\operatorname{Im}\lambda_\alpha(v_x + J_{l_\alpha}v_y). \tag{2.150}$$

If there is at most one $\rho_\alpha = 0$, then part (1) of Theorem 2.12 implies that $U^{[m]} \to 0$ as $s \to \infty$. If $\rho_\alpha = 0$, $\rho_\beta = 0$ $(\alpha \neq \beta)$, then $l_\alpha = l_\beta$ since $J_{l_\alpha} \neq J_{l_\beta}$ if $l_\alpha \neq l_\beta$. Hence, part (2) of Theorem 2.12 also implies $U^{[m]} \to 0$ as $s \to \infty$. Therefore, we have

THEOREM 2.13 *$U^{[m]} \to 0$ as $(x, y) \to \infty$ in any directions.*

(2) Asymptotic behavior of the solutions as $t \to \infty$

Now we use a frame (ξ, η) which moves in a fixed velocity (v_x, v_y), that is, let $x = \xi + v_x t$, $y = \eta + v_y t$, then

$$Q_\alpha = \mathrm{i}\lambda_\alpha(\xi + J\eta) + (\mathrm{i}\lambda_\alpha(v_x + Jv_y) + \mathrm{i}\omega(\lambda_\alpha))t, \tag{2.151}$$

$$\rho_\alpha = -\operatorname{Im}\lambda_\alpha(v_x + J_{l_\alpha}v_y) - \operatorname{Im}(\omega_{l_\alpha}(\lambda_\alpha)). \tag{2.152}$$

Suppose that for distinct α, β, γ,

$$\det\begin{pmatrix} 1 & J_{l_\alpha} & \sigma_\alpha \\ 1 & J_{l_\beta} & \sigma_\beta \\ 1 & J_{l_\gamma} & \sigma_\gamma \end{pmatrix} \neq 0 \tag{2.153}$$

where

$$\sigma_\alpha = \operatorname{Im}(\omega_{l_\alpha}(\lambda_\alpha))/\operatorname{Im}(\lambda_\alpha). \tag{2.154}$$

Then there are at most two $\rho_\alpha = 0$ $(\alpha = 1, \cdots, r)$. By Theorem 2.12, $U^{[m]}_{l_\alpha, l_\beta} \not\to 0$ only if $\rho_\alpha = 0$, $\rho_\beta = 0$. This leads to

$$v_x = \frac{J_{l_\alpha}\sigma_\beta - J_{l_\beta}\sigma_\alpha}{J_{l_\beta} - J_{l_\alpha}}, \qquad v_y = \frac{\sigma_\alpha - \sigma_\beta}{J_{l_\beta} - J_{l_\alpha}}. \tag{2.155}$$

For $U^{[m]}_{jk} \not\to 0$, α, β can take m_j and m_k values respectively, hence there are at most $m_j m_k$ velocities (v_x, v_y) such that $U^{[m]}_{jk} \not\to 0$. Therefore, we have

THEOREM 2.14 *Suppose (2.153) is satisfied. Then as $t \to \infty$, the asymptotic solution of $U^{[m]}_{jk}$ has at most $m_j m_k$ peaks whose velocities are given by (2.155) ($l_\alpha = j$, $l_\beta = k$). If a peak has velocity (v_x, v_y), then, in the coordinate $\xi = x - v_x t$, $\eta = y - v_y t$, $\lim_{t\to\infty} U_{ab} = 0$ for all $(a, b) \neq (j, k)$, and as $t \to \infty$*

$$\begin{aligned} U^{[m]}_{jk} &\sim \frac{B_{\alpha\beta}\exp(\mathrm{i}\operatorname{Re}(\lambda_\alpha - \lambda_\beta)\xi + \mathrm{i}(\lambda_\alpha J_j - \lambda_\beta J_k)\eta + \mathrm{i}(\phi_\alpha - \phi_\beta)t)}{\Delta}, \\ \Delta &= A_{\alpha\beta}\cosh(\operatorname{Im}(\lambda_\alpha + \lambda_\beta)\xi + \operatorname{Im}(\lambda_\alpha J_j + \lambda_\beta J_k)\eta + \delta^{(1)}_{\alpha\beta}) \\ &\quad + \cosh(\operatorname{Im}(\lambda_\alpha - \lambda_\beta)\xi + \operatorname{Im}(\lambda_\alpha J_j - \lambda_\beta J_k)\eta + \delta^{(2)}_{\alpha\beta}) \end{aligned} \tag{2.156}$$

where $A_{\alpha\beta}$, $\delta^{(1)}_{\alpha\beta}$, $\delta^{(2)}_{\alpha\beta}$ are real constants, $A_{\alpha\beta} > 0$, and $B_{\alpha\beta}$ are complex constants,

$$\phi_\gamma = \operatorname{Re}\lambda_\gamma(v_x + J_{l_\gamma}v_y) + \operatorname{Re}(\omega_{l_\gamma}(\lambda_\gamma)) \quad (\gamma = \alpha, \beta). \tag{2.157}$$

Figure 2.5. $u^{[1,3]}$ of the DSI equation

Remark 23 *The condition (2.153) implies that the velocities of the solitons are all different. This is true for the DSI equation. However, for the 2+1 dimensional N-wave equation, all the solitons move in the same velocity. We shall discuss this problem later.*

Example 2.15 *DSI equation*

Let $n = 2$, $N = 2$,

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}, \qquad \omega = -2\mathrm{i}J\lambda^2, \tag{2.158}$$

then we have

$$\begin{aligned} &F_y = JF_x + UF, \\ &F_t = 2\mathrm{i}JF_{xx} + 2\mathrm{i}UF_x + \mathrm{i}\begin{pmatrix} |u|^2 + 2q_1 & u_x + u_y \\ -\bar{u}_x + \bar{u}_y & -|u|^2 - 2q_2 \end{pmatrix} F, \end{aligned} \tag{2.159}$$

$$\begin{aligned} &-\mathrm{i}u_t = u_{xx} + u_{yy} + 2|u|^2u + 2(q_1 + q_2)u, \\ &q_{1,x} - q_{1,y} = q_{2,x} + q_{2,y} = -(|u|^2)_x, \end{aligned} \tag{2.160}$$

$$(FF^*)^D = \frac{1}{2}\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \qquad [J, FF^*] = U_x. \tag{2.161}$$

(2.160) is the DSI equation.

If we construct the solution $U^{[m]}$ as above, then Theorem 2.13 implies that $U^{[m]} \to 0$ as $(x, y) \to \infty$ in any directions. If $\operatorname{Re}\lambda_\alpha \neq \operatorname{Re}\lambda_\beta$ for $\alpha \neq \beta$ and $l_\alpha = l_\beta$, then, Theorem 2.14 implies that as $t \to \infty$, the derived solution u has at most m_1m_2 peaks ($m_1 + m_2 = r$). From (2.154), $\sigma_\alpha = -4J_{l_\alpha}\operatorname{Re}\lambda_\alpha$, hence (2.155) implies that each peak has the velocity $v_x = 2\operatorname{Re}(\lambda_\alpha - \lambda_\beta)$, $v_y = 2\operatorname{Re}(\lambda_\alpha + \lambda_\beta)$ ($l_\alpha = 1$, $l_\beta = 2$). This is the (m_1, m_2) solitons [30]. When $K = 1$, these peaks do not vanish if and only if all κ_α's are non-zero.

Figures 2.5 – 2.7 show the solitons $u^{[1,3]}$, $u^{[2,3]}$ and $u^{[3,3]}$ respectively. The parameters are $K = 1$, $t = 2$, $\lambda_1 = 1 - 2\mathrm{i}$, $\lambda_2 = -3 - \mathrm{i}$, $\lambda_3 = 2 + \mathrm{i}$, $\lambda_4 = -1 + 3\mathrm{i}$, $\lambda_5 = 2 + 1.5\mathrm{i}$, $\lambda_6 = -0.5 - 1.5\mathrm{i}$, $C_1 = (1, 0)$, $C_2 = (0, 1)$, $C_3 = (0, 2)$, $C_4 = (0, -2)$, $C_5 = (2, 0)$, $C_6 = (-2, 0)$.

Figure 2.6. $u^{[2,3]}$ of the DSI equation

Figure 2.7. $u^{[3,3]}$ of the DSI equation

(3) Asymptotic solutions as the phases differences tend to infinity

For the equations whose solitons move in the same speed, like the 2+1 dimensional N-wave equation, the peaks do not separate as $t \to \infty$. However, we can still see some peaks in the figures. Here we will get the corresponding asymptotic properties of the solitons.

THEOREM 2.16 *Let p_α $(\alpha = 1.\cdots, r)$ be constant real numbers satisfying*

$$\det \begin{pmatrix} 1 & J_{l_\alpha} & p_\alpha/\operatorname{Im}\lambda_\alpha \\ 1 & J_{l_\beta} & p_\beta/\operatorname{Im}\lambda_\beta \\ 1 & J_{l_\gamma} & p_\gamma/\operatorname{Im}\lambda_\gamma \end{pmatrix} \neq 0 \tag{2.162}$$

for distinct α, β, γ. Let d_α be complex constant $K \times 1$ vectors, $\kappa_\alpha = d_\alpha \exp(p_\alpha \tau)$ and construct the Darboux transformations as above. Let $x = \xi + v_x\tau$, $y = \eta + v_y\tau$, then, for any j, k with $1 \le j, k \le N$, $j \neq k$, $\lim_{\tau\to\infty} U_{jk}^{[m]} \neq 0$ only if (v_x, v_y) takes specific $m_j m_k$ values.

Proof. Here

$$Q_\alpha = \mathrm{i}\lambda_\alpha(\xi + J\eta) + \mathrm{i}\omega(\lambda_\alpha)t + \mathrm{i}\lambda_\alpha(v_x + Jv_y)\tau. \tag{2.163}$$

Hence

$$\overset{\circ}{H}_\alpha \doteq \begin{pmatrix} \exp(\widetilde{Q}_\alpha(\tau)) \\ D_\alpha \end{pmatrix} \tag{2.164}$$

where

$$D_\alpha = (0, \cdots, 0, \underset{l_\alpha}{d_\alpha}, 0, \cdots, 0), \tag{2.165}$$

$$\widetilde{Q}_\alpha(\tau) = \mathrm{i}\lambda_\alpha(\xi + J\eta) + \mathrm{i}\omega(\lambda_\alpha)t + (\mathrm{i}\lambda_\alpha(v_x + Jv_y) - p_\alpha)\tau. \tag{2.166}$$

The real part of the coefficient of τ in $\widetilde{Q}_\alpha(\tau)$ is

$$\widetilde{\rho}_\alpha = -\operatorname{Im}\lambda_\alpha(v_x + Jv_y) - p_\alpha. \tag{2.167}$$

Condition (2.162) implies that there are at most two $\widetilde{\rho}_\alpha$'s such that $\widetilde{\rho}_\alpha = 0$. According to Theorem 2.12, as $\tau \to \infty$, $U_{jk}^{[m]} \not\to 0$ only if there exist $\widetilde{\rho}_\alpha = 0$, $\widetilde{\rho}_\beta = 0$, $\alpha \neq \beta$, $l_\alpha = j$, $l_\beta = k$. Therefore, the theorem is verified.

When the condition (2.153) holds, this theorem is useless, because the evolution will always separate the peaks. However, when (2.153) does not hold, especially when it is never satisfied, this theorem reveals a fact of the separation of the peaks.

EXAMPLE 2.17 *2+1 dimensional N-wave equation*
Let $n = 1$, $\omega = L\lambda$ where $L = \mathrm{diag}(L_1, \cdots, L_N)$ is a constant real diagonal matrix such that $L_j \neq L_k$ for $j \neq k$. Then, the integrability conditions (2.109) – (2.112) imply

$$F_y = JF_x + UF, \quad F_t = LF_x + VF, \tag{2.168}$$
$$[J, V] = [L, U], \quad U_t - V_y + [U.V] + JV_x - LU_x = 0, \tag{2.169}$$
$$U_x = [J, FF^*]. \tag{2.170}$$

(2.169) is just the 2+1 dimensional N-wave equation.

Suppose $U^{[m]}$ is constructed as above, then Theorem 2.13 implies that $U^{[m]} \to 0$ as $(x, y) \to \infty$ in any directions. Theorem 2.14 cannot be applied here. The reason is: the condition (2.153) holds only if $l_\alpha \neq l_\beta$ for $\alpha \neq \beta$. Hence for any j, $m_j = 0$ or 1. This implies that (2.153) does not hold generally unless $m_j = 0$ or 1 for all $1 \leq j \leq N$. Therefore, we apply Theorem 2.16 to the previous problem. Theorem 2.16 implies that if we choose $\{p_\alpha\}$ such that (2.162) is satisfied, then, for each (j, k), $\lim_{\tau\to\infty} U_{jk}^{[m]}$ has at most $m_j m_k$ peaks. When $K = 1$, these peaks do not vanish if and only if all κ_α's are non-zero.

Remark 24 *Here $\tau \to \infty$ means that the phase differences of different peaks tend to infinity. Therefore, the peaks are separated by enlarging the phase differences.*

Here are the figures describing the solutions $U^{[0,1,2]}$ and $U^{[1,1,2]}$ of the 3-wave equation. The vertical axis is $(|u_{12}|^2 + |u_{13}|^2 + |u_{23}|^2)^{1/4}$ so that all the components are shown in one figure. The parameters are

$$J = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \qquad L = \begin{pmatrix} 2 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

$K = 1$, $t = 10$, $\lambda_1 = 1 - 2\mathrm{i}$, $\lambda_2 = -3 - \mathrm{i}$, $\lambda_3 = 2 + i$, $\lambda_4 = -1 + 3\mathrm{i}$, $C_1 = (0, 1, 0)$, $C_2 = (0, 0, 1)$, $C_3 = (0, 0, 4096)$, $C_4 = (1, 0, 0)$. Note that for $U^{[0,1,2]}$, only U_{23} has two peaks, and for $U^{[1,1,2]}$, U_{12}, U_{13}, U_{23} have one, two, two peaks respectively.

Figure 2.8. $U^{[0,1,2]}$ of the 3-wave equation

Figure 2.9. $U^{[1,1,2]}$ of the 3-wave equation

Chapter 3

$N + 1$ DIMENSIONAL INTEGRABLE SYSTEMS

In this chapter, we discuss the soliton theory in higher dimensions and generalize the AKNS system in $1 + 1$ dimensions to the AKNS system in $n + 1$ dimensions. The Darboux transformation method can still be used and the algorithm is universal and purely algebraic. The collision of solitons in higher dimensions is also elastic as in $1+1$ dimensions. For the Cauchy problems, the initial data of this generalized AKNS system is given on a straight line. This theory covers many problems, especially some differential geometric problems in higher dimensions. It can also be applied to $2 + 1$ dimensional problems (like the KP, DSI and 2+1 dimensional N-wave equations) as in Section 2.6. However, for $n = 2$, the contents do not cover those in Chapter 2.

3.1 $n + 1$ dimensional AKNS system

3.1.1 $n + 1$ dimensional AKNS system

The physical space-time is $3 + 1$ dimensional. On the other hand, the soliton theory is developed extensively in $1 + 1$ and $2 + 1$ dimensions. Here we will pay attention to the integrable systems in higher dimensions and introduce a kind of higher dimensional solitons with the property of elastic collision [34, 35, 39, 53].

First, we generalize the AKNS system in Section 1.2 to arbitrary $n + 1$ dimensions. When $n = 2$, the system here is different from that in Chapter 2. Here the system still has spectral parameter while in the generalized AKNS system the spectral parameter is replaced by a differential operator.

Suppose that $(t, x_1, x_2, \cdots, x_n)$ are the coordinates of $\mathbf{R}^{n+1}$ where $(x_1, \cdots, x_n)$ denotes spatial coordinates and t denotes the time coordinate. Consider the following linear system:

$$
\begin{aligned}
&\partial_i \Psi = U_i \Psi = (\lambda J_i + P_i)\Psi \qquad (i = 1, \cdots, n), \\
&\partial_t \Psi = V \Psi = \sum_{\alpha=0}^{m} V_\alpha \lambda^{m-\alpha} \Psi,
\end{aligned}
\tag{3.1}
$$

where ∂_i are ∂_t are the partial derivatives with respect to x_i and t respectively, P_i's and V_α's are $N \times N$ matrix functions, P_i's are off-diagonal, J_i's are constant diagonal matrices. Since (3.1) contains several equations, we call it Lax set.

The integrability conditions of the first part of (3.1) (its spatial part) are

$$\partial_j U_i - \partial_i U_j + [U_i, U_j] = 0. \tag{3.2}$$

Considering the coefficients of the powers of λ, we have

$$[J_i, J_j] = 0, \tag{3.3}$$

$$[J_i, P_j] = [J_j, P_i], \tag{3.4}$$

$$\partial_j P_i - \partial_i P_j + [P_i, P_j] = 0. \tag{3.5}$$

Since J_i is diagonal, the equations in (3.3) are identities.

In what follows, we need two more assumptions: (1) J_i's are linearly independent. (2) If a matrix A satisfies $[A, J_i] = 0$ for all i, then A is a diagonal matrix. The first condition implies that the system is really an $n + 1$ dimensional problem and cannot be reduced to a lower dimensional system in some sense. In fact, if not, there would be a linear combination of J_i's and a linear transformation of spatial variables so that $J_n = 0$, that is, the essential part of the system is $(n - 1) + 1$ dimensional. The second condition is equivalent to that there is a linear combination $J = \sum a_i J_i$ such that the diagonal entries of J are distinct. Therefore, there is a linear transformation in x_i's so that the diagonal entries of J_1 are distinct.

From (3.4) and (3.5), we have:

(i) There is an $N \times N$ off-diagonal matrix function P such that

$$P_i = [P, J_i] \quad (i = 1, \cdots, n). \tag{3.6}$$

(ii) P satisfies the spatial constraints

$$\partial_j P_i - \partial_i P_j + [P_i, P_j] = 0. \tag{3.7}$$

(iii) The integrability conditions of (3.1) give the equations for V_α's:

$$[J_i, V_0] = 0, \quad \partial_i V_0 = 0, \tag{3.8}$$
$$[J_i, V_{\alpha+1}^{\mathrm{off}}] = \partial_i V_\alpha^{\mathrm{off}} - [P_i, V_\alpha]^{\mathrm{off}}, \tag{3.9}$$
$$\partial_i V_{\alpha+1}^{\mathrm{diag}} = [P_i, V_{\alpha+1}^{\mathrm{off}}]^{\mathrm{diag}}, \tag{3.10}$$
$$(\alpha = 0, 1, \cdots, m-1),$$

and the evolution equations

$$\partial_t P_i - \partial_i V_m^{\mathrm{off}} + [P_i, V_m]^{\mathrm{off}} = 0. \tag{3.11}$$

There are many redundant equations in (3.7) – (3.11). In fact, with the assumptions on J, we can always assume that all the diagonal entries of J_1 are distinct, which can be realized by a linear transformation of $(x_1, \cdots, x_n)$. (3.6) implies that the off-diagonal entries of P are determined by P_1. The following lemma shows that (3.7) – (3.10) can be replaced by part of the equations in it.

Lemma 3.1 *If the diagonal entries of J_1 are distinct, then (3.7) – (3.11) are equivalent to*

$$\partial_1 P_i - \partial_i P_1 + [P_i, P_1] = 0, \tag{3.12}$$
$$[J_i, V_{\alpha+1}^{\mathrm{off}}] = \partial_i V_\alpha^{\mathrm{off}} - [P_i, V_\alpha]^{\mathrm{off}}, \tag{3.13}$$
$$\partial_i V_\alpha^{\mathrm{diag}} = [P_i, V_\alpha^{\mathrm{off}}]^{\mathrm{diag}}, \tag{3.14}$$
$$\partial_t P_1 - \partial_1 V_m^{\mathrm{off}} + [P_1, V_m]^{\mathrm{off}} = 0. \tag{3.15}$$

Proof. (3.13) and (3.14) are the same as (3.9) and (3.10). Hence we only need to prove that (3.7) and (3.11) are consequence of (3.12) and (3.15) with the help of (3.13) and (3.14). Let

$$\Delta_{ij} = \partial_j P_i - \partial_i P_j + [P_i, P_j]. \tag{3.16}$$

By Jacobi identity, we have

$$\begin{aligned} [J_1, \Delta_{ij}] &= \partial_j [J_i, P_1] - \partial_i [J_j, P_1] + [J_1, [P_i, P_j]] \\ &= [J_i, \Delta_{1j}] - [J_j, \Delta_{1i}]. \end{aligned} \tag{3.17}$$

$P_i = [P, J_i]$ and $P_j = [P, J_j]$ imply that all the diagonal entries of $[P_i, P_j]$ are 0. Hence all the diagonal entries of Δ_{ij} are 0. Using $\Delta_{1i} = 0$ $(i \geq 2)$, we know that $[J_1, \Delta_{ij}] = 0$ for any i and j. Hence $\Delta_{ij}^{\mathrm{off}} = 0$, i.e., $\Delta_{ij} = 0$. (3.7) is proved.

(3.11) is proved as follows. For any $i > 1$, using $[P_i, J_j] = [P_j, J_i]$, we have

$$\begin{aligned} & [\partial_t P_i - \partial_i V_m + [P_i, V_m], J_1] \\ = & [\partial_1 V_m - [P_1, V_m], J_i] - [\partial_i V_m - [P_i, V_m], J_1] \\ = & \partial_1 [V_m, J_i] - \partial_i [V_m, J_1] + [[P_i, V_m], J_1] - [[P_1, V_m], J_i]. \end{aligned} \tag{3.18}$$

With (3.7), (3.9), (3.10) and the Jacobi identity, the last equality equals

$$\begin{aligned} & \partial_1 [P_i, V_{m-1}] - \partial_i [P_1, V_{m-1}] - [[V_m, J_1], P_i] + [[V_m, J_i], P_1] \\ = & [\partial_1 P_i - \partial_i P_1, V_{m-1}] + [P_i, [P_1, V_{m-1}]] - [[P_1, [P_i, V_{m-1}]] = 0. \end{aligned} \tag{3.19}$$

Hence (3.11) holds. The lemma is proved.

From this lemma, the spatial constraints are simplified to (3.12) and the evolution equations are simplified to (3.15). In general, (3.12) and (3.15) are still over-determined partial differential equations.

In Chapter 1, we have known that V_α's do exist when $n = 1$, and they are differential polynomials of P. When $n > 1$, V_α's should satisfy many more equations (3.13) and (3.14). Hence we need to show that they do exist and are still differential polynomials of P.

LEMMA 3.2 *(1) There exist $\{V_\alpha\}$'s satisfying (3.13) and (3.14). These $\{V_\alpha\}$'s are polynomials of P and its derivatives with respect to x, and are denoted as $V_\alpha = V_\alpha[P]$. (2) For given diagonal matrices $V_\alpha^0(t)$'s which are independent of x_i's, there exist unique $\{V_\alpha[P]\}$ such that $V_\alpha[0] = V_\alpha^0(t)$.*

Sketch of proof. V_0 and V_1 can be obtained from (3.13) and (3.14) by direct calculation. For general V_α, this can be derived inductively. First, from the result in [111], V_α's are polynomials of P, $\partial_1 P$, $\cdots$, $\partial_1^\alpha P$ if x_2, $\cdots$, x_n are regarded as parameters. Then from the spatial constraints (3.12) we can prove that these differential polynomials satisfy all the equations in (3.13) and (3.14). The complete proof is formal and tedious. For the details, see [53].

By this lemma, all the equations derived from the Lax set (3.1) are partial differential equations of x_1, $\cdots$, x_n and t.

3.1.2 Examples

In (3.1), m, n, N ($N \geq n$) and J_i are quite arbitrary. Hence (3.1) leads to a lot of nonlinear partial differential equations. Here we show two simple examples.

EXAMPLE 3.3 *$n = N = 2$, $m = 2$,*

$$J_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad P = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix},$$

p, q are complex-valued functions. Then,

$$P_1 = [P, J_1] = \begin{pmatrix} 0 & -p \\ q & 0 \end{pmatrix}, \quad P_2 = [P, J_2] = \mathrm{i} \begin{pmatrix} 0 & -p \\ q & 0 \end{pmatrix},$$

and the Lax set is

$$\begin{aligned} &\partial_1 \Psi = (\lambda J_1 + P_1)\Psi, \quad \partial_2 \Psi = (\lambda J_2 + P_2)\Psi, \\ &\partial_t \Psi = (V_0 \lambda^2 + V_1 \lambda + V_2)\Psi. \end{aligned}$$

The spatial constraints

$$\partial_2 P_1 - \partial_1 P_2 = 0$$

becomes

$$\partial_1 p + i\partial_2 p = 0, \quad \partial_1 q + i\partial_2 q = 0,$$

i.e., p, q are holomorphic functions of the complex variable $z = x_1 + ix_2$, and $\partial_1 p = p_z$, $\partial_1 q = q_z$. From the recursion relations,

$$\begin{aligned} V_0 &= V_0^0 = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \\ V_1 &= \begin{pmatrix} b_1 & (a_2 - a_1)p \\ (a_1 - a_2)q & b_2 \end{pmatrix}, \\ V_2 &= \begin{pmatrix} (a_1 - a_2)pq + c_1 & (a_2 - a_1)p_x + (b_2 - b_1)p \\ (a_2 - a_1)q_x + (b_1 - b_2)q & (a_2 - a_1)pq + c_2 \end{pmatrix}, \end{aligned}$$

where a_1, b_1, c_1, a_2, b_2, c_2 are functions of t only. Moreover, we get the nonlinear evolution equations

$$\begin{aligned} &\frac{\partial p}{\partial t} + (a_2 - a_1)p_{zz} + (b_2 - b_1)p_z \\ &\quad + 2(a_2 - a_1)p^2 q + (c_2 - c_1)p = 0, \\ &\frac{\partial q}{\partial t} + (a_1 - a_2)q_{zz} + (b_2 - b_1)q_z \\ &\quad + 2(a_1 - a_2)pq^2 + (c_1 - c_2)q = 0. \end{aligned}$$

EXAMPLE 3.4 *$n = N = 3$, $m = 2$,*

$$J_1 = \begin{pmatrix} \mathrm{i} & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & & \\ & \mathrm{i} & \\ & & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & \mathrm{i} \end{pmatrix}.$$

Now

$$\begin{aligned} V_0[P] = & \; V_0^0(t), \\ V_1[P] = & \; [P, V_0] + V_1^0(t), \\ V_2[P] = & \; -\sum J_j \left[\frac{\partial P}{\partial x_j}, V_0\right] + [P,[P,V_0]]^{\text{off}} + [P, V_1^0]^{\text{off}} \\ & -([P,V_0]P)^{\text{diag}} + V_2^0(t). \end{aligned}$$

We get a system of second order quasi-linear partial differential equations. Its soliton solutions will be discussed later.

3.2 Darboux transformation and soliton solutions

3.2.1 Darboux transformation

In this subsection we discuss the Darboux transformation for the higher dimensional AKNS system.

We have known in Chapter 1 that in all Darboux matrices those of degree one are the most fundamental ones. The Darboux matrices of higher degree can be produced from the Darboux matrices of degree one inductively. Hence we only discuss the Darboux matrix in the form

$$D(x,t,\lambda) = \lambda I - S(x,t). \tag{3.20}$$

Let $\Psi' = (\lambda I - S)\Psi$ and substitute it into

$$\partial_i \Psi' = (\lambda J_i + P_i')\Psi', \tag{3.21}$$

we get

$$P_i' = P_i + [J_i, S], \tag{3.22}$$

$$\partial_i S + S P_i = P_i' S. \tag{3.23}$$

From (3.22) and $P_i' = [P', J_i]$, we have

$$P' = P - S + \text{arbitrary diagonal matrix.}$$

For simplicity, let

$$P' = P - S. \tag{3.24}$$

Then (3.23) becomes

$$\partial_i S = [P_i, S] + [J_i, S]S. \tag{3.25}$$

Substitute $\Psi' = (\lambda I - S)\Psi$ into

$$\partial_t \Psi' = \sum_{\alpha=0}^{m} V'_\alpha \lambda^{m-\alpha} \Psi', \tag{3.26}$$

and expand both sides in terms of the powers of λ, we get

$$\begin{aligned} &V'_0 = V_0, \\ &V'_{\alpha+1} = V_{\alpha+1} - SV_\alpha + V'_\alpha S \quad (\alpha = 0, 1, \cdots, m-1), \end{aligned} \tag{3.27}$$

and

$$\partial_t S = \left[\sum_{\alpha=0}^{m} V_\alpha S^{m-\alpha}, S\right]. \tag{3.28}$$

(3.23) and (3.28) are partial differential equations for S. It can be verified directly that they are completely integrable. Therefore, for the initial value of S at $t = t_0$, $x_i = x_{i0}$, the solution S exists uniquely in some region around $t = t_0$, $x_i = x_{i0}$.

What is more, we need to prove

$$V'_\alpha = V_\alpha[P']. \tag{3.29}$$

so that P' and P satisfy the same evolution equation (3.11) (or (3.15)). For $\alpha = 0, 1, 2\cdots$, (3.29) can be verified directly. For general α, the proof is quite tedious. It is similar to that for Theorem 1.11 and can be found in [53].

From the above discussion and the general method of constructing Darboux matrix, we have

THEOREM 3.5 *Suppose P is a solution of (3.15). For given constant diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$, let h_i be a column solution of (3.1) with $\lambda = \lambda_i$, $H = (h_1, \cdots, h_N)$. When $\det H \neq 0$, let $S = H\Lambda H^{-1}$. Then $\lambda I - S$ is a Darboux matrix for (3.1).*

Proof. The proof is similar to that for Theorem 1.9.

Therefore, with Darboux transformation, we can get a series of solutions

$$(P^{(0)}, \Psi^{(0)}) \longrightarrow (P^{(1)}, \Psi^{(1)}) \longrightarrow (P^{(2)}, \Psi^{(2)}) \longrightarrow \cdots$$

from a given solution $P^{(0)}$ and the corresponding solution $\Psi^{(0)}$ of the Lax set (3.1). We have also seen that the formula

$$S = H\Lambda H^{-1} \tag{3.30}$$

appears in many cases.

3.2.2 $u(N)$ case

If P satisfies some extra constraints, usually the Darboux transformation constructed above can not keep these constraints. In this case, corresponding constraints should be added on the Darboux transformation. This would be quite complicated in general. Here we will discuss an interesting and simple case in which

$$J_j = \mathrm{i}e_j = \mathrm{i}\ \mathrm{diag}(0, \cdots, 0, \underset{j}{1}, 0, \cdots, 0), \tag{3.31}$$

and $P \in u(N)$ (i.e. $P^* = -P$). Suppose a solution $P \in u(N)$ of (3.5) is known, let Φ be the fundamental solution of its Lax set. Now

$$P_i^* = [P, J_i]^* = [J_i^*, P^*] = [J_i, P] = -P_i. \tag{3.32}$$

In solving (3.8) – (3.10), take the "integral constant" (function of t) for V_α to be a diagonal matrix $V_\alpha^0(t)$ in $u(N)$, i.e., $V_\alpha^0(t)$ are purely imaginary diagonal matrices, then

$$V_\alpha \in u(N), \quad \alpha = 0, 1, \cdots, m. \tag{3.33}$$

In fact, $V_0 = V_0^0(t) \in u(N)$. Suppose $V_l \in u(N)$, then from (3.9) for $\alpha = l$, we have $[J_j, V_{l+1}^{\mathrm{off}}] \in u(N)$. This implies $V_{l+1}^{\mathrm{off}} \in u(N)$. From (3.10) for $\alpha = l$, we have $\partial_i V_{l+1}^{\mathrm{diag}} \in u(N)$. Therefore, $V_{l+1}^{\mathrm{diag}} \in u(N)$ if it holds at one point. (3.33) is proved by induction.

Now we construct the Darboux transformation $(P, \Phi) \to (P', \Phi')$ so that $P' \in u(N)$. As in Subsection 1.4.4, let λ_α ($\alpha = 1, \cdots, N$) take only two mutually conjugate complex numbers:

$$\lambda_\alpha = \mu \quad \text{or} \quad \bar{\mu} \quad (\mu \neq \bar{\mu}). \tag{3.34}$$

Let h_α^0's satisfy

$$h_\beta^{0*} h_\alpha^0 = 0 \quad (\mu_\beta \neq \mu_\alpha).$$

Solve

$$\partial_i h_\alpha = (\lambda_\alpha J_i + P_i) h_\alpha, \quad \partial_t h_\alpha = \sum_{l=0}^{m} V_l \lambda_\alpha^{m-l} h_\alpha$$

for initial data $h_\alpha = h_\alpha^0$ at some point to get the column solution h_α. Then when $\lambda_\alpha \neq \lambda_\beta$ (i.e. $\bar{\lambda}_\beta = \lambda_\alpha$), $\partial_i(h_\beta^* h_\alpha) = 0$ and $\partial_t(h_\beta^* h_\alpha) = 0$ hold. Hence

$$h_\beta^* h_\alpha = 0 \tag{3.35}$$

holds everywhere. Moreover, suppose that all the solutions $\{h_\alpha^0\}$ corresponding to $\lambda_\alpha = \mu$ are linearly independent, and all the solutions $\{h_\beta^0\}$

corresponding to $\lambda_\beta = \bar{\mu}$ are linearly independent. Then all these solutions are linearly independent. Hence (3.35) implies that $\{h_1, \cdots, h_N\}$ are linearly independent. This implies $\det H \neq 0$, and $S = H\Lambda H^{-1}$ is defined globally on $\mathbf{R}^{n+1}$.

According to the discussion in Subsection 1.4.4, we have

$$S + S^* = (\mu + \bar{\mu})I, \tag{3.36}$$

and

$$S^* S = |\mu|^2 I. \tag{3.37}$$

We have also

$$(\bar{\lambda} I - S)^*(\lambda I - S) = (\lambda - \mu)(\lambda - \bar{\mu})I.$$

Moreover, from Subsection 1.4.4, $V'_\alpha \in u(N)$. Therefore, we get the construction of Darboux transformation for the AKNS system with $u(N)$ reduction. The Darboux transformations for many other systems with $u(N)$ reduction can be constructed similarly.

3.2.3 Soliton solutions

Now we construct single and multi-soliton solutions. For simplicity, only the case $n = 3$, $N = 3$ and $m = 2$ will be considered. It will also be proved that the soliton interaction in multi-soliton solution is elastic.

Take

$$V_0^0 = \begin{pmatrix} a_1\mathrm{i} & 0 & 0 \\ 0 & a_2\mathrm{i} & 0 \\ 0 & 0 & a_3\mathrm{i} \end{pmatrix}, \quad V_1^0 = \begin{pmatrix} b_1\mathrm{i} & 0 & 0 \\ 0 & b_2\mathrm{i} & 0 \\ 0 & 0 & b_3\mathrm{i} \end{pmatrix},$$
$$V_2^0 = \begin{pmatrix} c_1\mathrm{i} & 0 & 0 \\ 0 & c_2\mathrm{i} & 0 \\ 0 & 0 & c_3\mathrm{i} \end{pmatrix},$$

where a_i, b_i, c_i $(i = 1, 2, 3)$ are real constants. As before, take the seed solution $P = 0$, then

$$\Psi_0 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(\lambda x_1 + \phi_1(\lambda)t)} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}(\lambda x_2 + \phi_2(\lambda)t)} & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}(\lambda x_3 + \phi_3(\lambda)t)} \end{pmatrix}, \tag{3.38}$$

where

$$\phi_i(\lambda) = a_i\lambda^2 + b_i\lambda + c_i \qquad (i = 1, 2, 3). \tag{3.39}$$

Let

$$\lambda_1 = \mu, \qquad \lambda_2 = \lambda_3 = \bar{\mu} \qquad (\mu \neq \bar{\mu}), \tag{3.40}$$

$$\zeta_i = \mu x_i + \phi_i(\mu)t. \tag{3.41}$$

Let

$$H = \begin{pmatrix} e^{i\zeta_1} & -\bar{a}e^{i\bar{\zeta}_1} & -\bar{b}e^{i\bar{\zeta}_1} \\ ae^{i\zeta_2} & e^{i\bar{\zeta}_2} & 0 \\ be^{i\zeta_3} & 0 & e^{i\bar{\zeta}_3} \end{pmatrix}, \tag{3.42}$$

then λ_i's and H satisfy the conditions in (3.34) and (3.35), and

$$\det H = e^{i(\zeta_1+\bar{\zeta}_2+\bar{\zeta}_3)}\Delta, \tag{3.43}$$

$$\Delta = 1 + |a|^2 e^{i[(\zeta_2-\bar{\zeta}_2)-(\zeta_1-\bar{\zeta}_1)]} + |b|^2 e^{i[(\zeta_3-\bar{\zeta}_3)-(\zeta_1-\bar{\zeta}_1)]} > 0. \tag{3.44}$$

The derived solution consists of three waves. They are

$$\begin{aligned} p &= S_{23} = a\bar{b}(\mu-\bar{\mu})e^{i[(\zeta_2-\zeta_1)-(\bar{\zeta}_3-\bar{\zeta}_1)]}/\Delta, \\ q &= S_{13} = (\mu-\bar{\mu})\bar{b}e^{i(\bar{\zeta}_1-\bar{\zeta}_3)}/\Delta, \\ r &= S_{12} = (\mu-\bar{\mu})\bar{a}e^{i(\bar{\zeta}_1-\bar{\zeta}_2)}/\Delta. \end{aligned} \tag{3.45}$$

According to (3.41) and (3.45), $|p|$, $|q|$ and $|r|$ are travelling waves with velocities

$$v_i = -\frac{\phi_i(\bar{\mu}) - \phi_i(\mu)}{\bar{\mu} - \mu} \qquad (i = 1, 2, 3) \tag{3.46}$$

respectively. In fact,

$$\begin{aligned} |p| &= |ab|\,|\mu-\bar{\mu}|e^{i(\mu-\bar{\mu})[(\xi_2-\xi_1)+(\xi_3-\xi_1)]/2}/\Delta, \\ |q| &= |b|\,|\mu-\bar{\mu}|e^{i(\mu-\bar{\mu})(\xi_3-\xi_1)/2}/\Delta, \\ |r| &= |a|\,|\mu-\bar{\mu}|e^{i(\mu-\bar{\mu})(\xi_2-\xi_1)/2}/\Delta, \end{aligned} \tag{3.47}$$

where

$$\Delta = 1 + |a|^2 e^{i(\mu-\bar{\mu})(\xi_2-\xi_1)} + |b|^2 e^{i(\mu-\bar{\mu})(\xi_3-\xi_1)}, \tag{3.48}$$

$$\xi_i = x_i - v_i t \qquad (i = 1, 2, 3). \tag{3.49}$$

For fixed t, $|p|$, $|q|$ and $|r|$ are all bounded. Moreover, when $|x| \to \infty$, they tend to zero rapidly along all directions except the following directions:

$$\begin{aligned} &\text{(i)} \quad \text{for } |p|: \quad |x_2 - x_3| \not\to +\infty,\ 2x_1 - x_2 - x_3 \not\to +\infty, \\ &\text{(ii)} \quad \text{for } |q|: \quad |x_3 - x_1| \not\to +\infty,\ 2x_2 - x_1 - x_3 \not\to +\infty, \\ &\text{(iii)} \quad \text{for } |r|: \quad |x_2 - x_1| \not\to +\infty,\ 2x_3 - x_1 - x_2 \not\to +\infty. \end{aligned} \tag{3.50}$$

Here we suppose $i(\mu-\bar{\mu})>0$. The solution we derived here is called a single-soliton solution. Each solution consists of three waves p, q and r. If $\mathrm{i}(\mu-\bar{\mu})<0$, the inequalities $2x_1-x_2-x_3 \not\to +\infty \cdots$ in (3.50) should be replaced by $2x_1-x_2-x_3 \not\to -\infty \cdots$.

These solutions are not localized, since p, q and r are constants along the straight line $x_2-x_1=$ constant, $x_3-x_1=$ constant. They are a kind of line solitons appeared in Chapter 2.

The other components of S are

$$\begin{aligned}
S_{11} &= \tfrac{1}{\Delta}(\mu+\bar{\mu}|a|^2\mathrm{e}^{\mathrm{i}(\mu-\bar{\mu})(\xi_2-\xi_1)}+\bar{\mu}|b|^2\mathrm{e}^{\mathrm{i}(\mu-\bar{\mu})(\xi_3-\xi_1)}),\\
S_{22} &= \tfrac{1}{\Delta}(\bar{\mu}+\mu|a|^2\mathrm{e}^{\mathrm{i}(\mu-\bar{\mu})(\xi_2-\xi_1)}+\bar{\mu}|b|^2\mathrm{e}^{\mathrm{i}(\mu-\bar{\mu})(\xi_3-\xi_1)}),\\
S_{33} &= \tfrac{1}{\Delta}(\bar{\mu}+\bar{\mu}|a|^2\mathrm{e}^{\mathrm{i}(\mu-\bar{\mu})(\xi_2-\xi_1)}+\mu|b|^2\mathrm{e}^{\mathrm{i}(\mu-\bar{\mu})(\xi_3-\xi_1)}).
\end{aligned} \tag{3.51}$$

These expressions will be useful in considering multi-soliton solutions.

Remark 25 If a_i's, b_i's and c_i's are functions of t rather than constants, then the soliton solution has variable velocities. The term $\phi_i(\lambda)t$ in (3.41) should be changed to

$$\omega_i(\lambda,t)=\int_{t_0}^{t}\phi_i(\lambda,\tau)\,d\tau=\int_{t_0}^{t}[a_i(\tau)\lambda^2+b_i(\tau)\lambda+c_i(\tau)]\,d\tau. \tag{3.52}$$

Especially, $|p|$, $|q|$ and $|r|$ are constants along the curve

$$x_i+\frac{\omega_i(\bar{\mu},t)-\omega_i(\mu,t)}{\bar{\mu}-\mu}=\text{constant}.$$

Different choice of the functions a_i's, b_i's and c_i's leads to different behavior of the waves. For instance, if the integrals of a_i's, b_i's and c_i's are periodic functions, we get the waves which oscillate periodically. If a_i's, b_i's and c_i's increase fast at infinity, the waves have large acceleration and will move apart soon.

k-multi-soliton solution can be obtained by k times of Darboux transformation of degree one from the seed solution $P=0$. Take the parameters μ_0, μ_1, $\cdots$, μ_{k-1} ($\mu_a\neq\mu_b$, $\mu_a\neq\bar{\mu}_b$ when $a\neq b$), then

$$\Psi_l(\lambda,x,t)=(\lambda I-S_{l-1})\cdots(\lambda I-S_0)\Psi_0 \quad (k=1,\cdots,l-1). \tag{3.53}$$

Here the matrices S_k ($k=0,1,\cdots,l-1$) are constructed from Φ_k by using (3.30) with (3.34) and (3.35) successively. Then we obtain the solution

$$P^l=-(S_0+S_1+\cdots+S_{l-1}) \tag{3.54}$$

of the nonlinear equation. Moreover, for considering the behavior of the multi-soliton solutions, we assume

$$v_i(\mu_l) \qquad (i = 1, 2, \cdots, n;\ l = 0, 1, \cdots, k-1) \tag{3.55}$$

are distinct.

Multi-soliton solutions have the following property.

THEOREM 3.6 *When $t \to \pm\infty$, a k-multi-soliton is asymptotic to k single solitons. These single solitons for $t \to -\infty$ and $t \to +\infty$ have the same amplitude.*

Proof. We first consider the case $k = 2$. Choose the moving coordinate with velocity $\{v_i(\mu_1)\}$, that is, keep $\xi_i^1 = x_i - v_i(\mu_1)t$ finite and $t \to \pm\infty$. Then

$$\xi_i^0 - \xi_j^0 = x_i - x_j - (v_i(\mu_0) - v_j(\mu_0))t \to \pm\infty \quad (i \neq j), \tag{3.56}$$

$$S_0 \sim \begin{pmatrix} \sigma_1^{\pm} & & \\ & \sigma_2^{\pm} & \\ & & \sigma_3^{\pm} \end{pmatrix} \qquad (t \to \pm\infty) \tag{3.57}$$

where $\sigma_i^{\pm} = \mu_0$ or $\bar{\mu}_0$. Hence,

$$\Psi_1 \sim \begin{pmatrix} (\lambda - \sigma_1^{\pm})\mathrm{e}^{\mathrm{i}(\lambda x_1 + \phi_1(\lambda)t)} & & \\ & (\lambda - \sigma_2^{\pm})\mathrm{e}^{\mathrm{i}(\lambda x_2 + \phi_2(\lambda)t)} & \\ & & (\lambda - \sigma_3^{\pm})\mathrm{e}^{\mathrm{i}(\lambda x_3 + \phi_3(\lambda)t)} \end{pmatrix}. \tag{3.58}$$

Choose H_1 so that

$$H_1 \sim \begin{pmatrix} \mathrm{e}^{\mathrm{i}\zeta_1^1} & -\bar{a}_1\mathrm{e}^{\mathrm{i}\bar{\zeta}_1^1} & -\bar{b}_1\mathrm{e}^{\mathrm{i}\bar{\zeta}_1^1} \\ a_1\mathrm{e}^{\mathrm{i}\zeta_2^1} & \mathrm{e}^{\mathrm{i}\bar{\zeta}_2^1} & 0 \\ b_1\mathrm{e}^{\mathrm{i}\zeta_3^1} & 0 & \mathrm{e}^{\mathrm{i}\bar{\zeta}_3^1} \end{pmatrix}, \tag{3.59}$$

where

$$\zeta_i^1 = \mu_1 x_i + \phi_i(\mu_1)t - \ln(\mu_1 - \sigma_i^{\pm})\mathrm{i}. \tag{3.60}$$

(3.59) and (3.60) are similar to (3.42) and (3.41). For the asymptotic behavior of S_1, the equalities similar to (3.45), (3.47) and (3.51) hold, provided that a, b, μ and ζ_i's are changed to a_1, b_1, μ_1 and ζ_i^1's respectively. Hence, when $t \to \pm\infty$ and ξ_i^1's keep finite,

$$P^2 = -S_0^{\mathrm{off}} - S_1^{\mathrm{off}}$$

Figure 3.1. p wave of the double soliton solution, $t=-1$

Figure 3.2. p wave of the double soliton solution, $t=-0.5$

Figure 3.3. p wave of the double soliton solution, $t=0$

is asymptotic to a single soliton solution.

The asymptotic behavior of p_1, q_1 and r_1 are similar when $t\to+\infty$ and $t\to-\infty$ respectively. However, the extra term $-\ln[(\mu_1-\sigma_i^{\pm})\mathrm{i}$ in (3.60) are different for $t\to+\infty$ and $t\to-\infty$. The difference of these two terms represents the phase shift. This means that the "center" of the wave $\xi_i^1=0$, $x_i-v_i(\mu_1)t=0$ shifts differently for $t\to+\infty$ and $t\to-\infty$.

When $t\to\pm\infty$, $\xi_i^0=x_i-v(\mu_0)t$ keep finite (then $\xi_i^1\to\pm\infty$), the same result can be obtained by the theorem of permutability. Hence the theorem is proved for $k=2$.

Similar to Chapter 1, multi-soliton solution can be obtained by several Darboux transformations of degree one. Using induction, we can prove that when $t\to\pm\infty$, a multi-soliton solution splits up into several single soliton solutions. The interactions of these single solitons are elastic. That is, the shape and velocity of the norm of these solitons will not change from $t\to-\infty$ to $t\to+\infty$, but there is a phase shift. The phase shift is given by the difference of $\sigma_i^{\pm}$ in (3.60) for $t\to-\infty$ and $t\to+\infty$.

In summary, the interactions of the solitons are elastic, and the phase shifts of the solitons can be computed explicitly. This implies that the higher dimensional solitons also have this basic property of 1+1 dimensional solitons. However, these solitons are not local. They do not approach to zero along every direction.

Figures 3.1 – 3.5 describe the p wave of the double soliton solution.

Figure 3.4. p wave of the double soliton solution, $t=0.5$

Figure 3.5. p wave of the double soliton solution, $t=1$

3.3 A reduced system on $\boldsymbol{R}^n$

As an example, we consider a special AKNS system on $\mathbf{R}^n$. Let

$$e_1 = \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix},$$

$$\cdots \quad e_n = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \tag{3.61}$$

be n diagonal matrices. Let

$$E_i = \begin{pmatrix} e_i & 0 \\ 0 & -e_i \end{pmatrix} \tag{3.62}$$

be n $2n \times 2n$ matrices. Consider the system

$$\frac{\partial \Psi}{\partial x_i} = (\lambda E_i + P_i)\Psi, \tag{3.63}$$

where P_i's are real $2n \times 2n$ matrices. Note that if a_i $(i = 1, 2, \cdots, n)$ are n real numbers so that $|a_i|$ are distinct, then the entries of $\sum a_i E_i$ are distinct. Hence there exists $2n \times 2n$ matrix P such that

$$P_i = [P, E_i]. \tag{3.64}$$

What is more, we want that P satisfies the following extra conditions. This is a reduction of (3.63). Let P be written as

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \tag{3.65}$$

where P_{11}, P_{12}, P_{21} and P_{22} are $n \times n$ matrices. It is required that

$$\begin{aligned} &P_{11} = -P_{22} \text{ is symmetric,} \\ &P_{12} = -P_{21} \text{ is anti-symmetric.} \end{aligned} \tag{3.66}$$

This is an AKNS system with strong reduction, which is derived from a geometric problem [7].

The Darboux transformation will be constructed as follows [47]. By the general theory (Section 3.2), let $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_{2n})$, ψ_a be a column solution of (3.63) for the eigenvalue $\lambda = \lambda_a$. Let

$$H = (\psi_1, \cdots, \psi_{2n}), \tag{3.67}$$

$$S = H\Lambda H^{-1}. \tag{3.68}$$

From the general method for constructing the Darboux matrix, we know that $\lambda I - S$ is a Darboux matrix, and the Darboux transformation $(P, \Psi) \to (P', \Psi')$ is given by

$$\Psi' = (\lambda I - S)\Psi, \quad P' = P - S. \tag{3.69}$$

In order that P' still satisfies the conditions (3.65) and (3.66), we need special choice of λ_a's and ψ_a's. First, we prove the following lemma.

LEMMA 3.7 *Suppose P satisfies (3.66), and*

$$\psi(\lambda_0) = \begin{pmatrix} \psi^1(\lambda_0) \\ \psi^2(\lambda_0) \end{pmatrix} \tag{3.70}$$

is a column solution of (3.63) with $\lambda = \lambda_0$ where ψ^1, ψ^2 are n-vectors, then,

$$\widetilde{\psi}(\lambda_0) = \begin{pmatrix} \psi^2(\lambda_0) \\ \psi^1(\lambda_0) \end{pmatrix} \tag{3.71}$$

is a column solution of (3.63) with $\lambda = -\lambda_0$.

Proof. Let $P_{11} = F$, $P_{21} = G$, then

$$P_i = \begin{pmatrix} [F, e_i] & \{G, e_i\} \\ \{G, e_i\} & [F, e_i] \end{pmatrix} = \begin{pmatrix} F_i & G_i \\ G_i & F_i \end{pmatrix}, \tag{3.72}$$

where

$$F_i = [F, e_i], \quad G_i = \{G, e_i\} = Ge_i + e_iG. \tag{3.73}$$

(3.63) can be written as

$$\begin{aligned} \frac{\partial \psi^1}{\partial x_i} &= \lambda e_i \psi^1 + F_i \psi^1 + G_i \psi^2, \\ \frac{\partial \psi^2}{\partial x_i} &= -\lambda e_i \psi^2 + G_i \psi^1 + F_i \psi^2 \end{aligned} \tag{3.74}$$

which is invariant under the transformation

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \longrightarrow \widetilde{\psi} = \begin{pmatrix} \psi^2 \\ \psi^1 \end{pmatrix}, \quad \lambda \to -\lambda \tag{3.75}$$

The lemma is proved.

Now take a non-zero real number μ. Let

$$\lambda_i = \mu, \quad \lambda_{n+i} = -\mu, \quad (i = 1, 2, \cdots, n). \tag{3.76}$$

Let $\psi_i = \psi_i(\mu)$ $(i = 1, 2, \cdots, n)$ be n linearly independent column solutions of (3.63) with $\lambda = \mu$, and

$$\psi_{n+i} = \widetilde{\psi}_i \tag{3.77}$$

where $\widetilde{\psi}_i$ $(i = 1, 2, \cdots, 2n)$ are defined by (3.72). Using (3.74) and $P_i^* = -P_i$ we know

$$\frac{\partial}{\partial x_i}(\psi_\alpha^* \psi_\beta) = 0 \quad (\lambda_\alpha \neq \lambda_\beta) \quad (\alpha, \beta = 1, 2, \cdots, 2n). \tag{3.78}$$

Hence if

$$\psi_\alpha^* \psi_\beta = 0 \quad (\lambda_\alpha \neq \lambda_\beta) \tag{3.79}$$

hold at one point, they hold everywhere. We shall prove later that there are linearly independent ψ_α's so that (3.77) and (3.79) hold.

From the construction of S, $SH - H\Lambda = 0$, i.e.,

$$S\psi_\beta = \lambda_\beta \psi_\beta, \quad \psi_\gamma^* S^* = \lambda_\gamma \psi_\gamma^*. \tag{3.80}$$

Hence

$$\psi_\gamma^* S^* S \psi_\beta = \lambda_\beta \lambda_\gamma \psi_\gamma^* \psi_\beta. \tag{3.81}$$

When $\lambda_\beta \neq \lambda_\gamma$, $\psi_\gamma^* \psi_\beta = 0$. When $\lambda_\beta = \lambda_\gamma$, $\lambda_\beta \lambda_\gamma = \mu^2$. Thus,

$$\psi_\gamma^* S^* S \psi_\beta = \mu^2 \psi_\gamma^* \psi_\beta \tag{3.82}$$

for all β and γ. Therefore, $S^* S = \mu^2 I$. On the other hand,

$$\psi_\gamma^* (S^* - S) \psi_\beta = (\lambda_\gamma - \lambda_\beta) \psi_\gamma^* \psi_\beta = 0. \tag{3.83}$$

So S satisfies

$$S^* = S, \quad S^* S = \mu^2 I. \tag{3.84}$$

Rewrite S as

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \quad (S_{ab}\ (a, b = 1, 2) \text{ are } n \times n \text{ matrices}). \tag{3.85}$$

Since $S^* = S$, S_{11} and S_{22} are symmetric matrices, and $S_{12} = S_{21}^*$. Moreover, since $\Lambda = \begin{pmatrix} \mu I & 0 \\ 0 & -\mu I \end{pmatrix}$, by the construction of H we know that

$$S_{11} = -S_{22}, \quad S_{12} = -S_{21}. \tag{3.86}$$

This implies that the matrix S satisfies the conditions (3.65) and (3.66) for P. Hence $P' = P - S$ also satisfies these conditions. It is remained to prove that there are linearly independent column vectors $\psi_\alpha(\lambda)$ satisfying (3.79). In fact, it is sufficient to choose their initial value to be linearly independent and to satisfy (3.79). From $S^*S = \mu^2 I$,

$$S_{11}^2 - S_{12}^2 = \mu^2 I, \quad [S_{12}, S_{11}] = 0. \tag{3.87}$$

Let

$$\sigma = \mu^{-1}(S_{11} + S_{12}), \tag{3.88}$$

then

$$\sigma^* = \mu^{-1}(S_{11} - S_{12}), \tag{3.89}$$

and $\sigma^*\sigma = I$, i.e., σ is an $n \times n$ orthogonal matrix.

Take σ^0 to be an arbitrary constant orthogonal matrix. Let

$$S_{11}^0 = \frac{\mu}{2}(\sigma^0 + \sigma^{0*}), \quad S_{12}^0 = \frac{\mu}{2}(\sigma^0 - \sigma^{0*}), \tag{3.90}$$

then

$$(S_{11}^0)^2 - (S_{12}^0)^2 = \mu^2 I, \quad [S_{12}^0, S_{11}^0] = 0. \tag{3.91}$$

Let

$$S^0 = \begin{pmatrix} S_{11}^0 & S_{12}^0 \\ -S_{12}^0 & -S_{11}^0 \end{pmatrix}. \tag{3.92}$$

S^0 is a symmetric matrix. Since $S^{0*}S^0 = (S^0)^2 = \mu^2 I$, the eigenvalues of S^0 are $\pm\mu$. If ψ^0 is the eigenvector with eigenvalue μ, then $\widetilde{\psi}^0$ is the eigenvector with eigenvalue $-\mu$. Therefore, ψ_i^0's and $\widetilde{\psi}_i^0$ $(i = 1, \cdots, n)$ form a set of $2n$ linearly independent vectors ψ_α^0 $(\alpha = 1, \cdots, 2n)$ with $\psi_{n+i} = \psi_i^0$ $(i = 1, \cdots n)$. These are the initial values of ψ_α's to be required and the Darboux matrix S which keeps the reduction is constructed.

THEOREM 3.8 *The Darboux transformation which keeps the reduction (3.66) exists and is obtained from the above construction of S.*

Remark 26 (3.63) can be generalized to

$$\frac{\partial \Psi}{\partial x_i} = (\lambda E_i + P_i + \frac{Q_i}{\lambda})\Psi. \tag{3.93}$$

In this case the above construction of Darboux transformation is still valid. The transformation of Q_i *is given by*

$$Q_i' = SQ_iS^{-1}. \tag{3.94}$$

Using the Darboux transformation in (3.66) and (3.93), we can solve the geometric problem in [7] and give the explicit solution of the Bäcklund transformation presented there.

Chapter 4

SURFACES OF CONSTANT CURVATURE, BÄCKLUND CONGRUENCES AND DARBOUX TRANSFORMATION

There are many important partial differential equations originating from classical differential geometry. The famous sine-Gordon equation is one of them.

The non-Euclidean geometry was initiated in the early nineteenth century. Afterwards, it was found that the surfaces of constant negative Gauss curvature realize the non-Euclidean geometry locally. Therefore, these surfaces were studied extensively. The sine-Gordon equation and its Bäcklund transformation appeared in that period. A surface of constant negative Gauss curvature corresponds to a non-zero solution of the sine-Gordon equation, and the Bäcklund transformation provides a way to construct a new solution of the sine-Gordon equation from a known one, and a way to construct a new surface of constant negative Gauss curvature from a known surface of constant negative Gauss curvature too. Since the middle of the twentieth century, the transformations of the solutions of some partial differential equations become effective methods in the soliton theory. The role of differential geometry in the soliton theory becomes more and more important.

Since the Bäcklund transformation depends on solving a system of integrable nonlinear partial differential equations, usually its solutions can not be expressed explicitly, except for some very special cases. However, as in the previous chapters, the Darboux transformation is a way to obtain explicit expressions. In this chapter, we combine the Darboux transformation and the classical Bäcklund transformation to realize the construction of the Bäcklund congruences and the surfaces of constant negative Gauss curvature. Hence a serious of such congruences and surfaces can be obtained explicitly by purely algebraic algorithm. Besides, we also discuss the surfaces of constant Gauss curvature in the Minkowski space $\mathbf{R}^{2,1}$. The existence and explicit construction of the generalized Bäcklund congruence are studied as well. However, the situation becomes more complicated since a surface or a line congruence may be time-like or surface-like. Especially, we give the geometric meaning of the Bäcklund transformation between $\triangle\alpha = \sin\alpha$ and $\triangle\alpha = \sinh\alpha$, the explicit construction of the time-like and space-like surfaces of constant negative Gauss curvature and the Bäcklund congruences. The Darboux transformations are also used to construct surfaces of constant mean curvature.

In summary, in these geometric problems, the Lax pair is not only a tool for solving related partial differential equations, but also the geometric objects that we want to study. With the idea of Darboux transformation, some of the geometric objects can be constructed by purely algebraic algorithm. So it becomes much easier to constructing them by computer.

There are also other geometric problems related with integrable systems which we will not discuss in this book. For example, the soliton surfaces introduced by A. Sym [96, 97, 17] and the topics in [90].

For the convenience of the readers we give a sketch of the basic facts of the theory of surfaces in $\mathbf{R}^3$ and $\mathbf{R}^{2,1}$ respectively by using the differential forms.

4.1 Theory of surfaces in the Euclidean space $\mathbf{R}^3$

We use the moving frame method to introduce the basic theory of surfaces in Euclidean space. In $\mathbf{R}^3$, $\mathbf{r}$ represents the position vector of a point. The length of $\mathbf{r}$ is $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$. As is known, a surface in $\mathbf{R}^3$ is a two dimensional differential manifold embedded (or immersed) in $\mathbf{R}^3$. It can be covered by some open subsets (surface charts) which are homeomorphic to connected open regions of a plane. In each surface chart, $\mathbf{r}$ can be represented by a vector-valued function of two parameters as $\mathbf{r} = \mathbf{r}(u^1, u^2)$ and in the intersection of two surface charts, the parametric representations (u^1, u^2) and (v^1, v^2) are linked by differentiable relations $v^a = \phi^a(u^1, u^2)$ $(a = 1, 2)$ with $\dfrac{\partial(\phi^1,\phi^2)}{\partial(u^1,u^2)} \neq 0$. $\mathbf{r}_a = \dfrac{\partial\mathbf{r}}{\partial u^a}$ $(a = 1, 2)$ form a basis of the tangent plane at $\mathbf{r}$, and $\mathbf{n} = \dfrac{\mathbf{r}_1\times\mathbf{r}_2}{|\mathbf{r}_1\times\mathbf{r}_2|}$ is the normal vector of the surface. $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{n}\}$ form a frame at $\mathbf{r}$. As a basis of the global theory of surfaces, the local theory of surfaces discusses the properties of the surface charts with the above-mentioned parametrization. We use the term surface to indicate a surface chart, since we only consider the local theory. The frame $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{n})$ is called a natural frame because it is obtained naturally from the parametrization of the surface. Differentiate $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{n}$ and write them as linear combinations of $\mathbf{r}_a, \mathbf{n}$, we obtained the fundamental equations of a surface

$$\begin{aligned} d\mathbf{r} &= \mathrm{d}u^a \mathbf{r}_a, \\ d\mathbf{r}_a &= \omega_a^b \mathbf{r}_b + \omega_a^3 \mathbf{n}, \\ d\mathbf{n} &= \omega_3^b \mathbf{r}_b, \quad (a, b = 1, 2), \end{aligned} \tag{4.1}$$

where $\omega_a^b, \omega_a^3, \omega_3^b$ are 1-forms of u^1, u^2. In (4.1) and hereafter, the summation convention is used, i.e., the symbol Σ is omitted for double indices.

The first fundamental form of the surface is

$$I = \mathrm{d}s^2 = \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} = g_{ab}\, \mathrm{d}u^a\, \mathrm{d}u^b, \tag{4.2}$$

where

$$g_{ab} = \mathbf{r}_a \cdot \mathbf{r}_b = g_{ba}. \tag{4.3}$$

Since

$$\mathbf{n} \cdot \mathrm{d}\mathbf{r}_a + \mathbf{r}_a \cdot \mathrm{d}\mathbf{n} = 0,$$

ω_3^b and ω_3^a are related by

$$\omega_a^3 = -g_{ab}\omega_3^b. \tag{4.4}$$

Let

$$\omega_a^3 = b_{ab} \mathrm{d}u^b. \tag{4.5}$$

From $\mathrm{d}^2\mathbf{r} = 0$, $\omega_a^3 \wedge \mathrm{d}u^a = 0$. We have $b_{ab} = b_{ba}$. The quadratic form

$$II = -\mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{n} = -g_{ab}\omega_3^b\, \mathrm{d}u^a = b_{ab}\, \mathrm{d}u^a\, \mathrm{d}u^b \tag{4.6}$$

is the second fundamental form of the surface with coefficients

$$b_{ab} = -\frac{\partial \mathbf{r}}{\partial u^a} \cdot \frac{\partial \mathbf{n}}{\partial u^b} = \frac{\partial^2 \mathbf{r}}{\partial u^a \partial u^b} \cdot \mathbf{n}. \tag{4.7}$$

Two principal curvatures are eigenvalues of the second fundamental form with respect to the first fundamental form, i.e., they are two roots of

$$\det(b_{\alpha\beta} - \lambda g_{\alpha\beta}) = 0.$$

Since the first fundamental form is positive definite, two principal curvatures are both real.

Now ω_a^b can be written as

$$\omega_a^b = \Gamma_{ac}^b\, \mathrm{d}u^c, \tag{4.8}$$

where Γ^b_{ac}'s are the Christoffel symbols of the surface. From $\mathrm{d}^2\mathbf{r} = 0$, we obtain $\mathrm{d}u^a \wedge \omega^b_a = 0$, hence $\Gamma^c_{ab} = \Gamma^c_{ba}$. Moreover, $\mathbf{r}_a \cdot \mathbf{r}_b = g_{ab}$ and $\mathbf{r}_a \cdot \mathrm{d}\mathbf{r}_b + \mathrm{d}\mathbf{r}_a \cdot \mathbf{r}_b = \mathrm{d}g_{ab}$ imply

$$\mathrm{d}g_{ab} = g_{ac}\omega^c_b + g_{cb}\omega^c_a.$$

Therefore,

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}\Big(\frac{\partial g_{ad}}{\partial u^b} + \frac{\partial g_{bd}}{\partial u^a} - \frac{\partial g_{ab}}{\partial u^d}\Big). \tag{4.9}$$

Exterior differentiating the fundamental equations (4.1) gives the integrability conditions of (4.1). Using $\mathrm{d}^2\mathbf{r} = 0$ and the exterior differentiations of the second and third equations of (4.1), we obtain

$$\mathrm{d}\omega^b_a + \omega^b_c \wedge \omega^c_a = \omega^3_a \wedge \omega^b_3, \tag{4.10}$$

$$\mathrm{d}\omega^3_a + \omega^3_b \wedge \omega^b_a = 0. \tag{4.11}$$

(4.10) and (4.11) are called Gauss equations and Codazzi equations of the surface respectively. The left hand side of (4.10) are usually written as

$$\mathrm{d}\omega^b_a + \omega^b_c \wedge \omega^c_a = \frac{1}{2} R^b{}_{acd} \mathrm{d}u^c \wedge \mathrm{d}u^d, \tag{4.12}$$

where

$$R^b{}_{acd} = \frac{\partial \Gamma^b_{ad}}{\partial u^c} - \frac{\partial \Gamma^b_{ac}}{\partial u^d} + \Gamma^b_{ec}\Gamma^e_{ad} - \Gamma^b_{ed}\Gamma^e_{ac} \tag{4.13}$$

is the Riemannian curvature tensor. It is fully determined by the coefficients of the fundamental form g_{ab} and their derivatives of first and second order. From (4.10), (4.4) and (4.5),

$$\frac{1}{2} R^b{}_{acd}\, \mathrm{d}u^c \wedge \mathrm{d}u^d = -b_{ac}\, \mathrm{d}u^c \wedge g^{be} b_{ed}\, \mathrm{d}u^d. \tag{4.14}$$

Hence

$$R^b{}_{acd} = -g^{be}(b_{ac}b_{ed} - b_{ad}b_{ec}). \tag{4.15}$$

Denote

$$R_{bacd} = g_{be} R^e{}_{acd}, \tag{4.16}$$

then

$$R_{bacd} = b_{bc}b_{ad} - b_{ac}b_{bd},$$

where the subscripts a, b, c, d take the values 1 or 2. In fact, there is only one independent equation, i.e.,

$$R_{1212} = b_{11}b_{22} - b_{12}^2. \tag{4.17}$$

This is another form of the Gauss equation. It can also been written as

$$\frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} = K \tag{4.18}$$

where K is called the total curvature or Gauss curvature of the surface. The left hand side of (4.18) is determined by the first fundamental form. Before Gauss' work, K is expressed by the first fundamental form together with the second fundamental form (second equality of (4.18)). The first equality in (4.18) is the famous Gauss Theorem. It implies that the Gauss curvature is actually determined by the first fundamental form of the surface. The properties which can be determined by the first fundamental form are call intrinsic properties.

The above discussion is summarized as follows. For a given surface S, its first and second fundamental forms satisfy the Gauss-Codazzi equations. Conversely, given two differential forms $I = g_{ab}\,\mathrm{d}u^a\,\mathrm{d}u_b$ ($a, b = 1, 2$, g_{ab} is positive definite), $II = b_{ab}\,\mathrm{d}u^a\,\mathrm{d}u^b$, and suppose that the Gauss-Codazzi equations hold, then there exists a surface chart whose first and second fundamental forms are I and II respectively. This surface chart is uniquely determined in a simply connected region up to rigid motions and reflections. This is the fundamental theorem of surfaces. In fact, the surface chart $\mathbf{r} = \mathbf{r}(u, v)$ is determined by solving the system of linear equations (4.1). Since the integrability condition of (4.1) is just the Gauss-Codazzi equations, (4.1) is completely integrable. Hence the fundamental theorem of surfaces holds.

The tangent vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ can be replaced by their linear combinations. Suppose $\mathbf{e}_1$ and $\mathbf{e}_2$ are linear combinations of $\mathbf{r}_1, \mathbf{r}_2$ and $\mathbf{e}_1, \mathbf{e}_2$ are orthogonal with each other, then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$ form an orthogonal frame of the surface at $\mathbf{r}$. The fundamental equations of the surface can be written as

$$\begin{aligned}
&\mathrm{d}\mathbf{r} = \omega^a \mathbf{e}_a,\\
&\mathrm{d}\mathbf{e}_a = \omega_a^b \mathbf{e}_b + \omega_a^3 \mathbf{n},\\
&\mathrm{d}\mathbf{n} = \omega_3^a \mathbf{e}_a.
\end{aligned} \tag{4.19}$$

There are also the following relations among ω^a and ω_a^b:

$$\mathrm{d}\omega^a + \omega_b^a \wedge \omega^b = 0 \quad (a, b = 1, 2), \tag{4.20}$$

$$\omega_i^j + \omega_j^i = 0 \quad (i, j = 1, 2, 3). \tag{4.21}$$

ω_b^a $(a, b = 1, 2)$ are uniquely determined by (4.20) and the relation $\omega_b^a + \omega_a^b = 0$. In the orthogonal frame, the Gauss-Codazzi equations are simplified as

$$\mathrm{d}\omega_2^1 = \omega_2^3 \wedge \omega_3^1 \tag{4.22}$$

and

$$\mathrm{d}\omega_a^3 + \omega_b^3 \wedge \omega_a^b = 0. \tag{4.23}$$

Especially, the Gauss equation can also be written as

$$\mathrm{d}\omega_2^1 = R_{1212}\omega^1 \wedge \omega^2 = K\omega^1 \wedge \omega^2. \tag{4.24}$$

Hereafter, we will mainly use the orthogonal frames. Moreover, the orientation of the frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$ will not always be fixed, that is, $\mathbf{n}$ may be replaced by $-\mathbf{n}$.

4.2 Surfaces of constant negative Gauss curvature, sine-Gordon equation and Bäcklund transformations

4.2.1 Relation between sine-Gordon equation and surface of constant negative Gauss curvature in $\mathbf{R}^3$

Suppose S is a surface of constant negative Gauss curvature in $\mathbf{R}^3$. By using a scaling transformation of S, we can also suppose $K = -1$. Take the lines of curvature as coordinate curves and let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the unit tangent vectors of the lines of curvature. Denote

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}\,\mathrm{d}u + \frac{\partial \mathbf{r}}{\partial v}\,\mathrm{d}v = A\,\mathrm{d}u\mathbf{e}_1 + B\,\mathrm{d}v\mathbf{e}_2, \tag{4.25}$$

$$\omega^1 = A\,\mathrm{d}u, \quad \omega^2 = B\,\mathrm{d}v. \tag{4.26}$$

The first fundamental form of the surface is

$$\mathrm{d}s^2 = A^2\,\mathrm{d}u^2 + B^2\,\mathrm{d}v^2, \tag{4.27}$$

and the second fundamental form is

$$I\!I = k_1 A^2\,\mathrm{d}u^2 + k_2 B^2\,\mathrm{d}v^2 = k_1(\omega^1)^2 + k_2(\omega^2)^2, \tag{4.28}$$

where k_1 and k_2 are principal curvatures. The Gauss curvature $K = k_1 k_2$ $(= -1)$. On the other hand, (4.6) implies

$$I\!I = \omega_1^3\omega^1 + \omega_2^3\omega^2. \tag{4.29}$$

Comparing with (4.28), we get

$$\omega_1^3 = k_1\omega^1 = k_1 A\,\mathrm{d}u, \quad \omega_2^3 = k_2\omega^2 = k_2 B\,\mathrm{d}v. \tag{4.30}$$

From (4.20),

$$\mathrm{d}\omega^a + \omega_b^a \wedge \omega^b = 0,$$

hence

$$\omega_1^2 = -\omega_2^1 = -\frac{A_v}{B}\,\mathrm{d}u + \frac{B_u}{A}\,\mathrm{d}v. \tag{4.31}$$

The Codazzi equation

$$\mathrm{d}\omega_1^3 + \omega_2^3 \wedge \omega_1^2 = 0$$

leads to

$$(k_1 A)_v = k_2 A_v,$$

i.e.,

$$(k_1 - k_2)A_v + k_{1v}A = 0. \tag{4.32}$$

Since $K = k_1 k_2 = -1$, we can set $k_1 = \tan\dfrac{\alpha}{2}$, $k_2 = -\cot\dfrac{\alpha}{2}$, $(0 < \alpha < \pi)$, then

$$k_1 - k_2 = \frac{1}{\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}}.$$

Substituting it into (4.32), we get

$$(\log A)_v - \Big(\log\cos\frac{\alpha}{2}\Big)_v = 0.$$

Hence

$$A = \cos\frac{\alpha}{2}\,U(u).$$

Similarly,

$$B = \sin\frac{\alpha}{2}\,V(v).$$

Here $U(u)$ and $V(v)$ are functions depending on u and v respectively. Let $\mathrm{d}u_1 = U(u)\,\mathrm{d}u,\ \mathrm{d}v_1 = V(v)\,\mathrm{d}v$, then u_1 and v_1 become new parameters and will still be written as u, v. Hence

$$\begin{aligned}
&A = \cos\frac{\alpha}{2}, \quad B = \sin\frac{\alpha}{2},\\
&\omega^1 = \cos\frac{\alpha}{2}\,\mathrm{d}u, \quad \omega^2 = \sin\frac{\alpha}{2}\,\mathrm{d}v,\\
&\omega_1^3 = \sin\frac{\alpha}{2}\,\mathrm{d}u, \quad \omega_2^3 = -\cos\frac{\alpha}{2}\,\mathrm{d}v,\\
&\omega_1^2 = \frac{1}{2}(\alpha_v\,\mathrm{d}u + \alpha_u\,\mathrm{d}v) = -\omega_2^1.
\end{aligned} \tag{4.33}$$

Substituting them into Gauss equation (4.10), we get

$$\mathrm{d}\omega_2^1 = \omega_2^3 \wedge \omega_2^1 = k_1 k_2 \omega^1 \wedge \omega^2 = -\omega^1 \wedge \omega^2,$$

from which it is seen that α satisfies the sine-Gordon equation

$$\alpha_{uu} - \alpha_{vv} = \sin\alpha. \tag{4.34}$$

The coordinates we are using are called the Chebyshev coordinates, and the corresponding frame is called the Chebyshev frame. It can be checked directly that the Codazzi equations are consequence of the sine-Gordon equation (4.34). Hence the fundamental theorem of surfaces leads to the following theorem.

THEOREM 4.1 *For any solution α of the sine-Gordon equation (4.34) $(0 < \alpha < \pi)$, one can construct a surface of constant negative Gauss curvature by solving the fundamental equation of surfaces (4.19), in which the coefficients ω^a, ω_b^a, ω_b^3 are given by (4.33).*

Note that under the Chebyshev coordinates, the first fundamental form of the surface is

$$\mathrm{d}s^2 = \cos^2\frac{\alpha}{2}\,\mathrm{d}u^2 + \sin^2\frac{\alpha}{2}\,\mathrm{d}v^2, \tag{4.35}$$

and the second fundamental form is

$$II = \cos\frac{\alpha}{2}\sin\frac{\alpha}{2}(\mathrm{d}u^2 - \mathrm{d}v^2). \tag{4.36}$$

Hence two systems of asymptotic curves are real and the directions $\mathrm{d}u : \mathrm{d}v = 1 : \pm 1$ are the asymptotic directions of the surface. The cosine of the angle between these two directions is

$$\cos\theta = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \cos\alpha. \tag{4.37}$$

Therefore, the geometric meaning of α is the angle between two asymptotic curves of the surface.

Remark 27 Theorem 4.1 is a local result. On the other hand, the classical Hilbert Theorem says that there is no complete surface of constant negative Gauss curvature in $\mathbf{R}^3$ *[56]. Here a complete surface means a two dimensional open manifold on which each geodesic can be extended infinitely.*

Remark 28 On a surface of constant positive Gauss curvature (not a sphere), there are also Chebyshev coordinates. In this case,

$$\begin{aligned}
&\omega^1 = \cosh\frac{\alpha}{2}\,\mathrm{d}u, \quad \omega^2 = \sinh\frac{\alpha}{2}\,\mathrm{d}v,\\
&\omega_1^3 = \sinh\frac{\alpha}{2}\,\mathrm{d}u, \quad \omega_2^3 = \cosh\frac{\alpha}{2}\,\mathrm{d}v,\\
&\omega_1^2 = -\omega_2^1 = \frac{1}{2}(-\alpha_v\,\mathrm{d}u + \alpha_u\,\mathrm{d}v).
\end{aligned} \tag{4.38}$$

The construction of the surface depends on the solution of the negative sinh-Laplace equation

$$\triangle\alpha = -\sinh\alpha. \tag{4.39}$$

The proof of these facts is similar with the case of constant negative Gauss curvature.

Remark 29 *Suppose a solution of the sine-Gordon equation or the negative sinh-Laplace equation is known. The construction of surface of constant negative Gauss curvature or constant positive Gauss curvature is reduced to solving the fundamental equations of the surface. These are completely integrable systems of linear partial differential equations. The solutions of this system can be obtained by solving ordinary differential equations. However, the construction of explicit solutions is still not easy. The Darboux transformation will provide an efficient method for the explicit construction.*

4.2.2 Pseudo-spherical congruence

Line congruences originated from the study of refraction and reflection of light. A two-parametric family of straight lines is called a line congruence. For example, all the normal lines of a surface constitute a line congruence which is called the normal congruence of the surface. However, in general, a line congruence may not be a normal congruence, that is, there may not exist a surface which is orthonormal to all the straight lines in the congruence.

Locally, a line congruence can be expressed as follows. Suppose S is a surface expressed as

$$\mathbf{X} = \mathbf{X}(u, v).$$

Given a unit vector $\xi(u, v)$ at each point of the surface, let

$$\mathbf{Y}(u, v, \lambda) = \mathbf{X}(u, v) + \lambda\xi(u, v). \tag{4.40}$$

When (u, v) are fixed and λ changes, $\mathbf{Y}$ forms a straight line. Hence $\mathbf{Y}$ represents a line congruence. The surface S is called the reference surface. Clearly, the reference surface of a line congruence is quite arbitrary. For a curve C on the reference surface S, the straight lines in the congruence passing through C form a ruled surface. Suppose the equations of C are $u = u(t)$, $v = v(t)$. Substituting them into (4.40), we can get the equations of the ruled surface parametrized by t and λ. If there exists $\lambda = \lambda(t)$ such that $\dfrac{\mathrm{d}\mathbf{Y}}{\mathrm{d}t}$ and ξ are parallel, then the ruled surface becomes a developable surface, and the curve $\mathbf{Y}(u, v, \lambda) = \mathbf{Y}(u(t), v(t), \lambda(t))$ is the line of regression of the developable surface. Now

$$\mathbf{Y} = \mathbf{Y}(t) = \mathbf{X}(u(t), v(t)) + \lambda(t)\xi(u(t), v(t)),$$

and
$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} + \frac{\mathrm{d}\lambda}{\mathrm{d}t}\xi + \lambda(t)\frac{\mathrm{d}\xi}{\mathrm{d}t} = \mu\xi,$$
that is, ξ, $\dfrac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}\xi}{\mathrm{d}t}$ are linearly dependent. Hence, the ruled surface is developable if and only if $u = u(t), v = v(t)$ satisfy
$$\det\left(\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}, \frac{\mathrm{d}\xi}{\mathrm{d}t}, \xi\right) = 0. \tag{4.41}$$
The differential form of this condition is
$$\det(\mathbf{X}_u\,\mathrm{d}u + \mathbf{X}_v\,\mathrm{d}v,\ \xi_u\,\mathrm{d}u + \xi_v\,\mathrm{d}v,\ \xi\,) = 0.$$
Write this equation as
$$A\,\mathrm{d}u^2 + 2B\,\mathrm{d}u\,\mathrm{d}v + C\,\mathrm{d}v^2 = 0. \tag{4.42}$$
This is the quadratic equation of $\mathrm{d}u : \mathrm{d}v$. When $B^2 - AC > 0$, there are two different real roots for $\mathrm{d}u : \mathrm{d}v$ which correspond to two developable surfaces in the line congruence. Thus for each line in the congruence there are two developable surfaces passing through it.

As is known, a developable surface is generated by the tangent lines of a spatial curve named line of regression. When $B^2 - AC > 0$, each line in the congruence are tangent to a line of regression of each developable surface. The tangent points are called focal points. The set of all focal points form the focal surfaces. A focal surface is also obtained from the lines of regression. Therefore a line congruence can be regarded as the set of all common tangent lines of two focal surfaces.

We only consider the case $B^2 - AC > 0$. Suppose S and S^* are two focal surfaces of a line congruence. Suppose PP^* is a line in the congruence, which is the common tangent line of two focal surfaces, and P, P^* are the tangent points. The correspondence between P and P^* leads to the correspondence between S and S^*. In differential geometry, this is called a Laplace transformation between the surfaces S and S^* (it is different from the terminology Laplace transformation in analysis).

Suppose $\mathbf{n}$ and $\mathbf{n}^*$ are unit normal vectors of S at P and of S^* at P^* respectively. Let τ be the angle between $\mathbf{n}$ and $\mathbf{n}^*$, and l be the distance between P and P^*, i.e.,
$$\mathbf{n}\cdot\mathbf{n}^* = \cos\tau \quad (\sin\tau \neq 0), \tag{4.43}$$
$$d_{pp^*} = l \quad (l \neq 0). \tag{4.44}$$
When τ and l are both constants, the congruence is called a pseudo-spherical congruence.

Historically, a surface of constant negative Gauss curvature is called a pseudo-sphere. This is the origin of the name pseudo-spherical congruence. A pseudo-spherical congruence is also called a Bäcklund congruence.

THEOREM 4.2 *(Bäcklund Theorem) Two focal surfaces S and S^* of a pseudo-spherical congruence are both the surfaces of constant negative Gauss curvature with the same $K = -\sin^2\tau/l^2$.*

Proof. Let P be a general point on S, with position vector $\mathbf{r}(u,v)$. Its corresponding point P^* has position vector $\mathbf{r}^*(u,v)$. $\overrightarrow{PP^*}$ is a common tangent line of S and S^*. Let $\mathbf{e}_1$ be the unit vector along $\overrightarrow{PP^*}$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$ be the orthogonal frame of S at P and $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{n}^*\}$ be the orthogonal frame of S^* at P^* with $\mathbf{e}_1^* = \mathbf{e}_1$. From the above assumptions,

$$\mathbf{r}^* = \mathbf{r} + l\mathbf{e}_1, \tag{4.45}$$

$$\begin{aligned} \mathbf{e}_2^* &= \cos\tau\mathbf{e}_2 + \sin\tau\mathbf{n}, \\ \mathbf{n}^* &= -\sin\tau\mathbf{e}_2 + \cos\tau\mathbf{n}. \end{aligned} \tag{4.46}$$

The fundamental equations of S and S^* are

$$\begin{aligned} \mathrm{d}\mathbf{r} &= \omega^a\mathbf{e}_a, \\ \mathrm{d}\mathbf{e}_a &= \omega_a^b\mathbf{e}_b + \omega_a^3\mathbf{n}, \\ \mathrm{d}\mathbf{n} &= \omega_3^a\mathbf{e}_a, \end{aligned} \tag{4.47}$$

and

$$\begin{aligned} \mathrm{d}\mathbf{r}^* &= \omega^{*a}\mathbf{e}_a^*, \\ \mathrm{d}\mathbf{e}_a^* &= \omega_a^{*b}\mathbf{e}_b^* + \omega_a^{*3}\mathbf{n}, \\ \mathrm{d}\mathbf{n}^* &= \omega_3^{*a}\mathbf{e}_a \end{aligned} \tag{4.48}$$

respectively. From (4.45) – (4.48), we get

$$\begin{aligned} \mathrm{d}\mathbf{r}^* &= \omega^a\mathbf{e}_a + l\mathrm{d}\mathbf{e}_1 = (\omega^a + l\omega_1^a)\mathbf{e}_a + l\omega_1^3\mathbf{n} \\ &= \omega^{*a}\mathbf{e}_a^* = \omega^{*1}\mathbf{e}_1 + \omega^{*2}(\cos\tau\mathbf{e}_2 + \sin\tau\mathbf{n}). \end{aligned}$$

Hence

$$\omega^{*1} = \omega^1, \quad \omega^{*2} = \frac{l}{\sin\tau}\omega_1^3. \tag{4.49}$$

Rewrite (4.45) and (4.46) so that $\mathbf{r}, \mathbf{e}_1, \mathbf{e}_2$ are expressed by $\mathbf{r}^*, \mathbf{e}_1^*, \mathbf{e}_2^*$, then we have

$$\omega^2 = \frac{l}{\sin\tau}\omega_1^{*3}. \tag{4.50}$$

Figure 4.1.

By (4.46) and (4.48),

$$\begin{aligned}\mathrm{d}\mathbf{e}_2^* &= \cos\tau(\omega_2^1\mathbf{e}_1 + \omega_2^3\mathbf{n}) + \sin\tau(\omega_3^1\mathbf{e}_1 + \omega_3^2\mathbf{e}_2) \\ &= \omega_2^{*1}\mathbf{e}_1^* + \omega_2^{*3}\mathbf{n}^* = \omega_2^{*1}\mathbf{e}_1 + \omega_2^{*3}(-\sin\tau\mathbf{e}_2 + \cos\tau\mathbf{n}),\end{aligned}$$

which leads to

$$\omega_2^{*3} = \omega_2^3.$$

The Gauss curvature K^* of S^* is determined by

$$\mathrm{d}\omega_1^{*2} = \omega_1^{*3} \wedge \omega_3^{*2} = -K^*\omega^{*1} \wedge \omega^{*2}. \tag{4.51}$$

Denote $\omega_2^{*3} = \omega_2^3 = b\omega^1 + c\omega^2$, then

$$\omega_1^{*3} \wedge \omega_3^{*2} = \frac{\sin\tau}{l}\omega^2 \wedge (-b\omega^1) = b\frac{\sin\tau}{l}\omega^1 \wedge \omega^2. \tag{4.52}$$

On the other hand, $\omega_a^3 \wedge \omega^a = 0$ implies that ω_1^3 has the form

$$\omega_1^3 = a\omega^1 + b\omega^2.$$

Hence

$$\omega^1 \wedge \omega^2 = \frac{1}{b}\omega^1 \wedge \omega_1^3 = \frac{1}{b}\frac{\sin\tau}{l}\omega^{*1} \wedge \omega^{*2}. \tag{4.53}$$

Substituting (4.52) and (4.53) into (4.51), we get $K^* = -\dfrac{\sin^2\tau}{l^2}$. Similarly, $K = -\dfrac{\sin^2\tau}{l^2}$ holds. The theorem is proved.

4.2.3 Bäcklund transformation

Theorem 4.1 implies that a solution α ($\alpha \neq 0$) of the sine-Gordon equation leads to a surface of constant negative Gauss curvature. The Bäcklund theorem implies that two focal surfaces of a pseudo-spherical congruence have the same constant negative Gauss curvature, and these two focal surfaces correspond to two solutions of the sine-Gordon equation. However, the above proof of the Bäcklund theorem does not imply the existence of the Bäcklund congruences. We are going to construct the Bäcklund congruence from a given surface S of constant negative Gauss curvature with $K = -1$.

Suppose $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ is the Chebyshev frame of S at $\mathbf{r}$, and (u, v) are the Chebyshev coordinates. Let

$$\mathbf{r}^* = \mathbf{r} + l\mathbf{e} = \mathbf{r} + l(\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2) \tag{4.54}$$

be the transformation from the surface S to the surface S^*. Let P and P^* be the corresponding points on S and S^* with position vectors $\mathbf{r}$ and $\mathbf{r}^*$ respectively. It is required that the lines PP^* form a pseudo-spherical congruence. Here $\mathbf{e} = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$, or θ, is to be determined.

Suppose S corresponds to a solution α of the sine-Gordon equation. Differentiating (4.54) and using (4.33), we have

$$\begin{aligned} \mathrm{d}\mathbf{r}^* = \ & \mathrm{d}\mathbf{r} + l(\cos\theta\mathrm{d}\mathbf{e}_1 + \sin\theta\mathrm{d}\mathbf{e}_2) + l(-\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2)\mathrm{d}\theta \\ = \ & \Big[\cos\frac{\alpha}{2}\,\mathrm{d}u - l\sin\theta\mathrm{d}\theta - l\sin\theta\Big(\frac{\alpha_v}{2}\,\mathrm{d}u + \frac{\alpha_u}{2}\,\mathrm{d}v\Big)\Big]\mathbf{e}_1 \\ & +\Big[\sin\frac{\alpha}{2}\,\mathrm{d}v + l\cos\theta\mathrm{d}\theta + l\cos\theta\Big(\frac{\alpha_v}{2}\,\mathrm{d}u + \frac{\alpha_u}{2}\,\mathrm{d}v\Big)\Big]\mathbf{e}_2 \\ & +\Big[l\sin\frac{\alpha}{2}\cos\theta\,\mathrm{d}u - l\cos\frac{\alpha}{2}\sin\theta\,\mathrm{d}v\Big]\mathbf{n}. \end{aligned} \tag{4.55}$$

Since the angle τ between the normal vector $\mathbf{n}^*$ of S^* at P^* and the normal vector $\mathbf{n}$ of S at P is a constant, and PP^* is tangent to S^* at P^*, we have

$$\mathbf{n}^* = \sin\tau\sin\theta\mathbf{e}_1 - \sin\tau\cos\theta\mathbf{e}_2 + \cos\tau\mathbf{n}. \tag{4.56}$$

$\mathbf{n}^*$ is the normal vector of S^*, hence it must satisfy $\mathrm{d}\mathbf{r}^*\cdot\mathbf{n}^* = 0$. Thus

$$\begin{aligned} & l\sin\tau\mathrm{d}\theta - \sin\tau\Big(\cos\frac{\alpha}{2}\sin\theta\,\mathrm{d}u - \sin\frac{\alpha}{2}\cos\theta\,\mathrm{d}v\Big) \\ & \quad +l\sin\tau\Big(\frac{\alpha_v}{2}\,\mathrm{d}u + \frac{\alpha_u}{2}\,\mathrm{d}v\Big) \\ & \quad -l\cos\tau\Big(\sin\frac{\alpha}{2}\cos\theta\,\mathrm{d}u - \cos\frac{\alpha}{2}\sin\theta\,\mathrm{d}v\Big) = 0. \end{aligned} \tag{4.57}$$

If (4.57) can be solved for θ, then PP^* generates exactly a pseudo-spherical congruence. From the Bäcklund theorem, the surfaces S and S^* have the same $K = K^* = -\dfrac{\sin^2\tau}{l^2}$. Without loss of generality, suppose $l = \sin\tau$. Let $\theta = \alpha_1/2$, then $\mathrm{d}\theta = \dfrac{1}{2}\Big(\dfrac{\partial\alpha_1}{\partial u}\,\mathrm{d}u + \dfrac{\partial\alpha_1}{\partial v}\,\mathrm{d}v\Big)$. (4.57) can be written as a system of partial differential equations:

$$\begin{aligned} \frac{1}{2}\sin\tau\Big(\frac{\partial\alpha_1}{\partial u} + \frac{\partial\alpha}{\partial v}\Big) &= \sin\tau\sin\frac{\alpha_1}{2}\cos\frac{\alpha}{2} + \cos\tau\cos\frac{\alpha_1}{2}\sin\frac{\alpha}{2} \\ \frac{1}{2}\sin\tau\Big(\frac{\partial\alpha_1}{\partial v} + \frac{\partial\alpha}{\partial u}\Big) &= -\sin\tau\cos\frac{\alpha_1}{2}\sin\frac{\alpha}{2} - \cos\tau\sin\frac{\alpha_1}{2}\cos\frac{\alpha}{2}. \end{aligned} \tag{4.58}$$

This system is the original form of the Bäcklund transformation of sine-Gordon equation. Regarded as a system of α_1, its integrability condition is the sine-Gordon equation

$$\alpha_{uu} - \alpha_{vv} = \sin\alpha.$$

Since it holds already, the solution α_1 of (4.58) exists uniquely for a given value α_{10} ($0 < \alpha_{10} < \pi$) of α_1 at (u_0, v_0). From $\alpha_{uv} = \alpha_{vu}$, we see that α_1 also satisfies the sine-Gordon equation. This transformation from α to α_1 is called a Bäcklund transformation between two solutions of the sine-Gordon equation. Thus we have obtained

THEOREM 4.3 *Let S be a surface of constant negative Gauss curvature $K = -\sin^2\tau/l^2$ in $\mathbf{R}^3$, l and $\sin\tau$ be non-zero constants and $\mathbf{e}_0$ be a unit tangent vector of S at $P \in S$. Then there exists a surface S^* of $K = -\sin^2\tau/l^2$ such that the common tangent lines of SS^* constitute a pseudo-spherical congruence with parameters $(l, \sin\tau)$ and the line from P in the direction $\mathbf{e}_0$ belongs to the congruence.*

This theorem indicates the method of the construction of pseudo-spherical congruences.

In differential geometry, the surface S^* is called a Bäcklund transformation of the surface S. For S^*, (4.55) can be rewritten as

$$\mathrm{d}\mathbf{r}^* = A^*\,\mathrm{d}u\mathbf{e}_1^* + B^*\,\mathrm{d}v\mathbf{e}_2^*. \tag{4.59}$$

We shall prove that the Chebyshev coordinates (u, v) of S are also the Chebyshev coordinates of S^*. Since $\mathbf{e}_1^*$ and $\mathbf{e}_2^*$ are unit vectors, (4.48) and (4.55) lead to

$$\begin{aligned}
&A^* = \cos\frac{\alpha_1}{2}, \qquad B^* = \sin\frac{\alpha_1}{2},\\
&\mathbf{e}_1^* = (\cos\frac{\alpha}{2}\cos\frac{\alpha_1}{2} - \cos\tau\sin\frac{\alpha}{2}\sin\frac{\alpha_1}{2})\mathbf{e}_1\\
&\qquad +(\cos\frac{\alpha}{2}\sin\frac{\alpha_1}{2} + \sin\tau\sin\frac{\alpha}{2}\cos\frac{\alpha_1}{2})\mathbf{e}_2 + \sin\tau\sin\frac{\alpha}{2}\mathbf{n},\\
&\mathbf{e}_2^* = (\sin\frac{\alpha}{2}\cos\frac{\alpha_1}{2} + \cos\tau\cos\frac{\alpha}{2}\sin\frac{\alpha_1}{2})\mathbf{e}_1\\
&\qquad +(\sin\frac{\alpha}{2}\sin\frac{\alpha_1}{2} - \cos\tau\cos\frac{\alpha}{2}\cos\frac{\alpha_1}{2})\mathbf{e}_2 - \sin\tau\cos\frac{\alpha}{2}\mathbf{n}.
\end{aligned} \tag{4.60}$$

Hence

$$\omega_1^* = \cos\frac{\alpha_1}{2}\,\mathrm{d}u, \quad \omega_2^* = \sin\frac{\alpha_1}{2}\,\mathrm{d}v. \tag{4.61}$$

Computing $\mathrm{d}\mathbf{n}^*$ from (4.56) and expanding it in terms of $\mathbf{e}_1^*$, $\mathbf{e}_2^*$, we get

$$\begin{aligned}
&\mathrm{d}\mathbf{n}^* = \omega_3^{*1}\mathbf{e}_1^* + \omega_3^{*2}\mathbf{e}_2^*,\\
&\omega_3^{*1} = -\sin\frac{\alpha_1}{2}\,\mathrm{d}u, \quad \omega_3^{*2} = \cos\frac{\alpha_1}{2}\,\mathrm{d}v.
\end{aligned} \tag{4.62}$$

Comparing with (4.33), we conclude that (u, v) are the Chebyshev coordinates of S^*. The difference of sign can be eliminated by using $-\mathbf{n}^*$ instead of $\mathbf{n}^*$.

THEOREM 4.4 *Suppose S^* is the Bäcklund transformation of a surface S of constant negative Gauss curvature, and (u, v) are the Chebyshev coordinates of S, then (u, v) are also the Chebyshev coordinates of S^*.*

Thus one can apply the Bäcklund transformation successively to obtain a sequence of pseudo-spherical congruences and surfaces of constant negative Gauss curvature.

4.2.4 Darboux transformation

In the last subsection, we have got the following well-known facts. Suppose α is a solution of the sine-Gordon equation and S is the corresponding surface of constant negative Gauss curvature. If α_1 is a solution of (4.58), then (4.54) gives an explicit expression of the Bäcklund transformation of S where $\theta = \alpha_1/2$. In order to get the explicit expression of S^*, we need to use Darboux transformation to get the explicit expression of α_1. Let

$$\xi = \frac{u+v}{2}, \quad \eta = \frac{u-v}{2}, \tag{4.63}$$

then (4.58) becomes

$$\begin{aligned} (\alpha_1 + \alpha)_\xi &= 2\beta \sin \frac{\alpha_1 - \alpha}{2}, \\ (\alpha_1 - \alpha)_\eta &= \frac{2}{\beta} \sin \frac{\alpha_1 + \alpha}{2}, \end{aligned} \tag{4.64}$$

where

$$\beta = \frac{1 - \cos \tau}{\sin \tau} \neq 0. \tag{4.65}$$

The sine-Gordon equation becomes

$$\alpha_{\xi\eta} = \sin \alpha. \tag{4.66}$$

In Chapter 1, the Darboux transformation for the sine-Gordon equation has already been introduced. The construction is as follows. Let α

be a solution of the sine-Gordon equation, then the Lax pair

$$\Phi_\eta = \begin{pmatrix} \lambda & -\dfrac{\alpha_\eta}{2} \\ \dfrac{\alpha_\eta}{2} & -\lambda \end{pmatrix} \Phi, \qquad\qquad \Phi_\xi = \frac{1}{4\lambda} \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \Phi \tag{4.67}$$

is completely integrable. It can be checked that if $\begin{pmatrix} \phi_1(\lambda) \\ \phi_2(\lambda) \end{pmatrix}$ is a column solution of the Lax pair (4.67), then $\begin{pmatrix} \phi_2(\lambda) \\ -\phi_1(\lambda) \end{pmatrix}$ is a column solution of the Lax pair with λ replaced by $-\lambda$. Hence $\begin{pmatrix} \phi_2(-\lambda) \\ -\phi_1(-\lambda) \end{pmatrix}$ is also a column solution of the Lax pair (4.67). $\Phi(\lambda)$ can be chosen as

$$\Phi(\lambda) = \begin{pmatrix} \phi_1(\lambda) & \phi_2(-\lambda) \\ \phi_2(\lambda) & -\phi_1(-\lambda) \end{pmatrix}. \tag{4.68}$$

Take $\lambda = \lambda_1 \neq 0$. Let $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ be a column solution of the Lax pair (4.67) with $\lambda = \lambda_1$, and

$$h_1 = \phi_1(\lambda_1) + b_1\phi_2(-\lambda_1), \quad h_2 = \phi_2(\lambda_1) - b_1\phi_1(-\lambda_1),$$

where b_1 is a constant. Then $\begin{pmatrix} -h_2 \\ h_1 \end{pmatrix}$ is a solution of the Lax pair (4.67) with $\lambda = -\lambda_1$. Let

$$H = \begin{pmatrix} h_1 & -h_2 \\ h_2 & h_1 \end{pmatrix},$$

then $\det H = h_1^2 + h_2^2$, which is not zero provided that $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ is not a trivial solution. (From the property of linear ordinary differential equations, $h_1 = h_2 = 0$ holds everywhere if it holds at one point.) Let

$$\begin{aligned} S &= H \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix} H^{-1} = \frac{\lambda_1}{h_1^2 + h_2^2} \begin{pmatrix} h_1^2 - h_2^2 & 2h_1 h_2 \\ 2h_1 h_2 & h_2^2 - h_1^2 \end{pmatrix} \\ &= \frac{\lambda_1}{1+\sigma^2} \begin{pmatrix} 1-\sigma^2 & 2\sigma \\ 2\sigma & -1+\sigma^2 \end{pmatrix} = \lambda_1 \begin{pmatrix} \cos\dfrac{\psi}{2} & \sin\dfrac{\psi}{2} \\ \sin\dfrac{\psi}{2} & -\cos\dfrac{\psi}{2} \end{pmatrix}, \end{aligned} \tag{4.69}$$

where

$$\sigma = \frac{h_2}{h_1}, \quad \psi = 4\tan^{-1}\sigma, \tag{4.70}$$

$$\cos\frac{\psi}{2} = \frac{1-\sigma^2}{1+\sigma^2}, \quad \sin\frac{\psi}{2} = \frac{2\sigma}{1+\sigma^2}. \tag{4.71}$$

The matrix

$$D(\lambda) = \lambda I - S \tag{4.72}$$

is the Darboux matrix. Let

$$\Phi_1(\lambda) = D(\lambda)\Phi(\lambda), \tag{4.73}$$

then $\Phi_1(\lambda)$ satisfies the Lax pair (4.67) when α is replaced by certain α_1. This can be verified directly as follows. From

$$\begin{aligned} \Phi_{1\eta} &= -S_\eta \Phi + (\lambda I - S)\Phi_\eta \\ &= \left(-S_\eta + (\lambda I - S) \begin{pmatrix} \lambda & -\alpha_\eta/2 \\ \alpha_\eta/2 & -\lambda \end{pmatrix} \right) \Phi(\lambda) \\ &= \begin{pmatrix} \lambda & -\alpha_{1\eta}/2 \\ \alpha_{1\eta}/2 & -\lambda \end{pmatrix} (\lambda I - S)\Phi \end{aligned}$$

and $\det \Phi(\lambda) \neq 0$, we obtain

$$-S_\eta + (\lambda I - S) \begin{pmatrix} \lambda & -\alpha_\eta/2 \\ \alpha_\eta/2 & -\lambda \end{pmatrix} = \begin{pmatrix} \lambda & -\alpha_{1\eta}/2 \\ \alpha_{1\eta}/2 & -\lambda \end{pmatrix} (\lambda I - S).$$

Now compare the coefficients of the powers of λ in both sides. The terms with λ^2 are equal identically. The terms with λ lead to

$$\alpha_{1\eta} - \alpha_\eta = -4\lambda_1 \sin\frac{\psi}{2}. \tag{4.74}$$

The term with λ^0 leads to

$$\alpha_{1\eta} = -\alpha_\eta + \psi_\eta. \tag{4.75}$$

Similarly, considering the part of the Lax pair with derivative to ξ, we get

$$\alpha_1 = \psi - \alpha, \tag{4.76}$$

$$\psi_\xi = \frac{1}{\lambda_1} \sin \frac{\alpha - \alpha_1}{2}. \tag{4.77}$$

(4.76) gives the new solution of the sine-Gordon equation explicitly by Darboux transformation. Let $\beta = -\dfrac{1}{2\lambda_1}$, then (4.75) – (4.77) lead to (4.64). Therefore, α_1 is a solution of the partial differential equations (4.64) in the Bäcklund transformation. More precisely, we have

THEOREM 4.5 *Take* $\lambda_1 = -\dfrac{1}{2\beta}$, *then* α_1 *derived by the Darboux transformation* $(\Phi, \alpha) \to (\Phi_1, \alpha_1)$ *is a solution of (4.64).*

The explicit construction of the surface S^* of constant negative Gauss curvature is as follows.

THEOREM 4.6 *Suppose* α $(0 < \alpha < \pi)$ *is a solution of the sine-Gordon equation, and* Φ *is the fundamental solution of its Lax pair. Let* S *be the surface of constant negative Gauss curvature corresponding to* α *(expressed in Chebyshev coordinates). Then the Darboux transformation gives a new solution* α_1 *of the sine-Gordon equation, and the transformation (4.54) gives the surface of constant negative Gauss curvature corresponding to* α_1*.*

According to this theorem, we can get a series of surfaces of constant negative Gauss curvature by applying Darboux transformation (4.54) to a solution α of the sine-Gordon equation, the fundamental solution Φ of its Lax pair and the corresponding surface S of constant negative Gauss curvature with Chebyshev coordinates. This can be illustrated by the following diagram.

$$\begin{array}{ccccccccc} & \nearrow & \text{solution } \Phi \text{ of Lax pair} & \longrightarrow & (\alpha_1, \Phi_1) & \longrightarrow & (\alpha_2, \Phi_2) & \longrightarrow & \\ \alpha & & & & \downarrow & & \downarrow & & \cdots \\ & \searrow & \text{surface } S & \longrightarrow & S^* & \longrightarrow & S^{**} & \longrightarrow & \end{array}$$

In the diagram, the arrows from α to Φ and from α to S are realized by solving a system of linear integrable partial differential equations. Other arrows can be realized by purely algebraic operations. Here the algebraic operations include exponential, trigonometric and inverse trigonometric functions. In the construction of Bäcklund transformation, we only need to express $\cos \dfrac{\alpha_1}{2}$ and $\sin \dfrac{\alpha_1}{2}$ in terms of σ, $\cos \dfrac{\alpha}{2}$ and $\sin \dfrac{\alpha}{2}$. These expressions are purely algebraic.

4.2.5 Example

Starting from $\alpha = 0$, a series of non-trivial solutions of the sine-Gordon equation can be obtained by using Darboux transformation. With $\alpha = 0$, the Lax pair is

$$\Phi_\eta = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \Phi, \quad \Phi_\xi = \frac{1}{4\lambda} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi,$$

hence the fundamental solution is

$$\Phi = \begin{pmatrix} \mathrm{e}^{\lambda\eta + \frac{\xi}{4\lambda}} & 0 \\ 0 & \mathrm{e}^{-\lambda\eta - \frac{\xi}{4\lambda}} \end{pmatrix}.$$

Take $\lambda = \lambda_1$. From (4.70) we have

$$\sigma = -b\mathrm{e}^{-2\lambda_1\eta - \frac{\xi}{2\lambda_1}}.$$

For simplicity, take $b = -1$, then

$$\alpha_1 = 4\tan^{-1}\sigma = 4\tan^{-1}(\mathrm{e}^{-2\lambda_1\eta - \frac{\xi}{2\lambda_1}}).$$

The Darboux matrix is

$$D(\lambda) = \begin{pmatrix} \lambda + \lambda_1 \tanh\gamma & -\lambda_1 \operatorname{sech}\gamma \\ -\lambda_1 \operatorname{sech}\gamma & \lambda - \lambda_1 \tanh\gamma \end{pmatrix},$$

where

$$\gamma = -2\lambda_1\eta - \frac{\xi}{2\lambda_1}.$$

Then

$$\begin{aligned} \Phi_1(\lambda) &= D(\lambda)\Phi(\lambda) \\ &= \begin{pmatrix} (\lambda + \lambda_1 \tanh\gamma)\mathrm{e}^{\lambda\eta + \frac{\xi}{4\lambda}} & -\lambda_1 \operatorname{sech}\gamma \mathrm{e}^{-\lambda\eta - \frac{\xi}{4\lambda}} \\ -\lambda_1 \operatorname{sech}\gamma \mathrm{e}^{\lambda\eta + \frac{\xi}{4\lambda}} & (\lambda - \lambda_1 \tanh\gamma)\mathrm{e}^{-\lambda\eta - \frac{\xi}{4\lambda}} \end{pmatrix}. \end{aligned}$$

When $\lambda = \lambda_2$,

$$\begin{aligned} \phi_1'(\lambda_2) &= (\lambda_2 + \lambda_1 \tanh\gamma)\mathrm{e}^{\lambda_2\eta + \frac{\xi}{4\lambda_2}}, \\ \phi_2'(\lambda_2) &= (-\lambda_1 \operatorname{sech}\gamma)\mathrm{e}^{\lambda_2\eta + \frac{\xi}{4\lambda_2}}, \end{aligned}$$

$$\sigma' = \frac{(-\lambda_1 \operatorname{sech}\gamma)\mathrm{e}^{\lambda_2\eta + \frac{\xi}{4\lambda_2}} + b(\lambda_2 - \lambda_1 \tanh\gamma)\mathrm{e}^{-\lambda_2\eta - \frac{\xi}{4\lambda_2}}}{(\lambda_2 + \lambda_1 \tanh\gamma)\mathrm{e}^{\lambda_2\eta + \frac{\xi}{4\lambda_2}} + b(-\lambda_1 \operatorname{sech}\gamma)\mathrm{e}^{-\lambda_2\eta - \frac{\xi}{4\lambda_2}}}$$

and the new solution

$$\alpha_2 = 4\tan^{-1}\sigma' - \alpha_1.$$

On the other hand, by the fundamental equations of the surface for $\alpha = 0$,

$$\mathrm{d}\mathbf{r} = \mathbf{e}_1\,\mathrm{d}u, \quad \mathrm{d}\mathbf{e}_1 = 0, \quad \mathrm{d}\mathbf{e}_2 = -\mathbf{n}\,\mathrm{d}v, \quad \mathrm{d}\mathbf{n} = \mathbf{e}_2\,\mathrm{d}v,$$

we have

$$\begin{aligned} &\mathbf{r} = (u, 0, 0), \quad \mathbf{e}_1 = (1, 0, 0), \\ &\mathbf{e}_2 = (0, \cos v, \sin v), \quad \mathbf{n} = (0, \sin v, -\cos v). \end{aligned}$$

It is not a surface, but a line together with a two-parametric family of orthonormal frames along the line. However, since the fundamental equations hold, we can still use it to construct surface of constant negative Gauss curvature. Now

$$\begin{aligned} &\theta = \frac{\alpha_1}{2} = 2\tan^{-1}\sigma, \\ &\cos\theta = -\tanh\gamma, \quad \sin\theta = \operatorname{sech}\gamma. \end{aligned}$$

According to (4.54), the equation for the surface S^* is

$$\mathbf{r}^* = \mathbf{r} + l\Big(\cos\frac{\alpha_1}{2}\mathbf{e}_1 + \sin\frac{\alpha_1}{2}\mathbf{e}_2\Big),$$

or

$$\begin{aligned} &x_1 = u - l\tanh\gamma, \\ &x_2 = l\operatorname{sech}\gamma\cos v, \\ &x_3 = l\operatorname{sech}\gamma\sin v, \\ &\Big(l = \frac{-4\lambda_1}{1+\lambda_1^2}, \quad \gamma = -2\lambda_1\eta - \frac{\xi}{2\lambda_1}\Big). \end{aligned}$$

In order to construct the second surface S^{**}, we first write down the expressions of $\mathbf{e}_1^*$ and $\mathbf{e}_2^*$. From (4.60) and $\alpha = 0$,

$$\begin{aligned} &\mathbf{e}_1^* = \cos\frac{\alpha_1}{2}\mathbf{e}_1 + \sin\frac{\alpha_1}{2}\mathbf{e}_2, \\ &\mathbf{e}_2^* = \cos\tau\Big(\sin\frac{\alpha_1}{2}\mathbf{e}_1 - \cos\frac{\alpha_1}{2}\mathbf{e}_2\Big) - \sin\tau\mathbf{n}. \end{aligned}$$

Hence the equation for the second surface is

$$\begin{aligned} \mathbf{r}^{**} = \quad & u\mathbf{e}_1 - \frac{2\lambda_1}{1+\lambda_1^2}\Big(\cos\frac{\alpha_1}{2}\mathbf{e}_1 + \sin\frac{\alpha_1}{2}\mathbf{e}_2\Big) \\ & - \frac{2\lambda_2}{1+\lambda_2^2}\Big(\cos\frac{\alpha_2}{2}\mathbf{e}_1^* + \sin\frac{\alpha_2}{2}\mathbf{e}_2^*\Big). \end{aligned}$$

These surfaces of constant negative Gauss curvature can be plotted by computer. In [85], there are figures for some interesting surfaces including several surfaces of constant negative Gauss curvature.

4.3 Surface of constant Gauss curvature in the Minkowski space $\mathrm{R}^{2,1}$ and pseudo-spherical congruence

4.3.1 Theory of surfaces in the Minkowski space $\mathbf{R}^{2,1}$

As the Euclidean space $\mathbf{R}^3$, the Minkowski space $\mathbf{R}^{2,1}$ is a three dimensional flat space. A vector $\mathbf{l}$ in $\mathbf{R}^{2,1}$ has three coordinates l_1, l_2 and l_3. In an orthonormal coordinate system, the inner product of two vectors $\mathbf{l} = (l_1, l_2, l_3)$ and $\mathbf{m} = (m_1, m_2, m_3)$ in $\mathbf{R}^{2,1}$ is given by

$$\mathbf{l} \cdot \mathbf{m} = l_1 m_1 + l_2 m_2 - l_3 m_3, \tag{4.78}$$

and the square of the norm of a vector $\mathbf{l}$ is

$$\mathbf{l}^2 = l_1^2 + l_2^2 - l_3^2, \tag{4.79}$$

which is not positive definite. According to the sign of $\mathbf{l}^2$, there are three types of non-zero vectors:

$$\begin{aligned} &\mathbf{l}^2 > 0 \qquad \text{space-like,} \\ &\mathbf{l}^2 < 0 \qquad \text{time-like,} \\ &\mathbf{l}^2 = 0 \qquad \text{light-like.} \end{aligned} \tag{4.80}$$

A light-like vector is also called a null vector. All the null vectors form a light cone $l_1^2 + l_2^2 - l_3^2 = 0$. A space-like vector points to the exterior of the light cone, whereas a time-like vector points to the interior of the light cone. Moreover, according the sign of l_3, time-like and light-like vectors are divided into past-oriented and future-oriented vectors. All the vectors orthogonal to a time-like vector are space-like. A vector orthogonal to a space-like vector may be space-like, time-like or light-like. The vectors orthogonal to a light-like vector may be light-like or space-like, and all these vectors form the tangent planes of the light cone. A light-like vector is always orthogonal to itself.

A point (x, y, z) in $\mathbf{R}^{2,1}$ can be expressed by a position vector $\mathbf{r}$. In the special theory of relativity, the space and time are expressed uniformly by a four dimensional Minkowski space-time. The study of the geometry of $\mathbf{R}^{2,1}$ may help understanding the space-time in the relativity. The terms space-like, time-like, light-like etc. also originate from the relativity.

For a surface in $\mathbf{R}^{2,1}$, normal vector $\mathbf{n}$ can also be defined as the unit vector orthogonal to the tangent plane. When $\mathbf{n}$ is time-like, all the vectors on the tangent plane are space-like, and the surface is called space-like. When $\mathbf{n}$ is space-like, the vectors on the tangent plane may be space-like, time-like or light-like, and the surface is called time-like. There is also light-like surface, whose normal vector is light-like. In this case, the normal vector locates in the tangent plane. A mixed surface is a connected surface which includes space-like, time-like and light-like parts. In this book, we only consider the space-like surfaces and time-like surfaces.

Similar to $\mathbf{R}^3$, the fundamental equation of a space-like or time-like surface can be written as

$$\begin{aligned} \mathrm{d}\mathbf{r} &= \omega^a \mathbf{e}_a, \\ \mathrm{d}\mathbf{e}_a &= \omega_a^b \mathbf{e}_b + \omega_a^3 \mathbf{n}, \\ \mathrm{d}\mathbf{n} &= \omega_3^a \mathbf{e}_a. \end{aligned} \tag{4.81}$$

The integrability conditions are

$$\begin{aligned} &\mathrm{d}\omega^a + \omega_b^a \wedge \omega^b = 0, \quad \omega_a^3 \wedge \omega^a = 0, \\ &\mathrm{d}\omega_b^a + \omega_c^a \wedge \omega_a^c = -\omega_3^a \wedge \omega_b^3 \quad \text{(Gauss equation)}, \\ &\mathrm{d}\omega_a^3 + \omega_b^3 \wedge \omega_a^b = 0 \quad \text{(Codazzi equation)}. \end{aligned} \tag{4.82}$$

Space-like and time-like surfaces should be considered separately.

(1) space-like surface

For the orthogonal frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$ at $\mathbf{r}$,

$$\mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab}, \quad \mathbf{e}_a \cdot \mathbf{n} = 0, \quad \mathbf{n}^2 = -1, \tag{4.83}$$

and

$$\omega_1^1 = \omega_2^2 = 0, \quad \omega_2^1 = -\omega_2^1, \quad \omega_a^3 = \omega_3^a. \tag{4.84}$$

The first fundamental form of the surface is

$$I = \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} = (\omega^1)^2 + (\omega^2)^2 = g_{ab}\omega_a\omega_b, \tag{4.85}$$

where

$$g_{11} = g_{22} = 1, \quad g_{12} = g_{21} = 0.$$

Let

$$\omega_a^3 = b_{ac}\omega^c \quad (b_{ac} = b_{ca}), \tag{4.86}$$

then the second fundamental form of the surface is

$$II = -\mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{n} = -\omega_a^3 \omega^a = -b_{ac}\omega^a\omega^c. \tag{4.87}$$

The Gauss equation is

$$\begin{aligned} \mathrm{d}\omega_b^a &= \tfrac{1}{2} R^a{}_{bcd}\,\omega^c \wedge \omega^d = -\omega_3^a \wedge \omega_b^3 \\ &= -\tfrac{1}{2}(b_{ac}b_{bd} - b_{ad}b_{bc})\omega^c \wedge \omega^d. \end{aligned}$$

As before, it contains only one independent equality

$$R_{1212} = -(b_{11}b_{22} - b_{12}^2) \quad (R_{1212} = R^1{}_{212}). \tag{4.88}$$

The intrinsic definition of Gauss curvature is

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = R_{1212}. \tag{4.89}$$

Hence the Gauss equation can be written as

$$K = -(b_{11}b_{22} - b_{12}^2), \tag{4.90}$$

while

$$\mathrm{d}\omega_2^1 = -\omega_3^1 \wedge \omega_2^3 = R_{1212}\omega^1 \wedge \omega^2 = K\omega^1 \wedge \omega^2. \tag{4.91}$$

(4.90) differs from the corresponding formula in Euclidean space by a sign. The principal curvatures of the surface are still the eigenvalues of the second fundamental form with respect to the first fundamental form.

(2) Time-like surface

Take the orthogonal frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$ at $\mathbf{r}$, so that $\mathbf{e}_1$ is space-like, $\mathbf{e}_2$ is time-like, and

$$\mathbf{e}_1^2 = 1, \quad \mathbf{e}_2^2 = -1, \quad \mathbf{n}^2 = 1. \tag{4.92}$$

Then

$$\begin{aligned} &\omega_1^1 = \omega_2^2 = 0, \quad \omega_2^1 = \omega_1^2, \\ &\omega_3^1 = -\omega_1^3, \quad \omega_2^3 = \omega_2^3. \end{aligned} \tag{4.93}$$

The first fundamental form

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} = (\omega^1)^2 - (\omega^2)^2 = g_{ab}\omega^a\omega^b \tag{4.94}$$

is not positive definite. Here $g_{11} = 1$, $g_{22} = -1$, $g_{12} = 0$. Write

$$\omega_a^3 = b_{ac}\omega^c \quad (b_{ac} = b_{ca}), \tag{4.95}$$

then the second fundamental form is

$$\begin{aligned} II &= -\mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{n} = -\omega^1\omega_3^1 + \omega^2\omega_3^2 \\ &= \omega^1\omega_1^3 + \omega^2\omega_2^3 = b_{ac}\omega^a\omega^c. \end{aligned} \tag{4.96}$$

The intrinsic definition of Gauss curvature is

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = -R_{1212}, \tag{4.97}$$

The Gauss equation is

$$\begin{aligned} \mathrm{d}\omega_2^1 &= R^1_{212}\omega^1 \wedge \omega_2 = R_{1212}\omega^1 \wedge \omega^2 = -\omega_3^1 \wedge \omega_2^3 \\ &= (b_{11}b_{22} - b_{12}^2)\omega^1 \wedge \omega^2 = -K\omega^1 \wedge \omega^2. \end{aligned} \tag{4.98}$$

The right hand side of the above formula differs from (4.91) by a sign. Note that if the frame satisfies $\mathbf{e}_1^2 = -1$, $\mathbf{e}_2^2 = 1$, then (4.98) will be changed to

$$\begin{aligned} \mathrm{d}\omega_2^1 &= R^1{}_{212}\omega^1 \wedge \omega_2 = -R_{1212}\omega^1 \wedge \omega^2 = -\omega_3^1 \wedge \omega_2^3 \\ &= -(b_{11}b_{22} - b_{12}^2)\omega^1 \wedge \omega^2 = K\omega^1 \wedge \omega^2. \end{aligned} \tag{4.98$'$}$$

Comparing the formulae for the surfaces in Euclidean space with the space-like or time-like surfaces in Minkowski space, there are differences on sign (especially for the expressions of K).

4.3.2 Chebyshev coordinates for surfaces of constant Gauss curvature

The surfaces of constant Gauss curvature in $\mathbf{R}^{2,1}$ may be space-like, time-like or light-like, also may have positive or negative curvature ($K = \pm 1$). Moreover, for time-like surface of constant positive Gauss curvature, the eigenvalues of the second fundamental form with respect to the first fundamental form (i.e., the principal curvatures) may be real, imaginary, or be repeated eigenvalues with only one dimensional eigenspace. Therefore, we need to consider these cases separately. In each case, appropriate frame and coordinates are chosen and the surfaces are constructed with the help of the solutions of the corresponding integrable equations. First of all, we write down the fundamental equations and the Gauss equations under Chebyshev coordinates in each case.

(1) space-like surface of constant positive Gauss curvature ($K = +1$)

$\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{n}$ satisfy (4.83), and $\mathbf{e}_1$, $\mathbf{e}_2$ are tangent to the lines of curvature. Take

$$\omega^1 = \cos\frac{\alpha}{2}\,\mathrm{d}u, \quad \omega^2 = \sin\frac{\alpha}{2}\,\mathrm{d}v. \tag{4.99}$$

Similar to the surface of constant negative Gauss curvature in the Euclidean space, we still have

$$\omega_1^2 = -\omega_2^1 = \frac{1}{2}(\alpha_v\,\mathrm{d}u + \alpha_u\,\mathrm{d}v). \tag{4.100}$$

Take

$$\omega_1^3 = \sin\frac{\alpha}{2}\,\mathrm{d}u, \quad \omega_2^3 = -\cos\frac{\alpha}{2}\,\mathrm{d}v, \tag{4.101}$$

then it can be verified easily that the Codazzi equation

$$\mathrm{d}\omega_1^3 + \omega_2^3 \wedge \omega_1^2 = 0, \quad \mathrm{d}\omega_2^3 + \omega_1^3 \wedge \omega_2^1 = 0 \tag{4.102}$$

holds, and

$$b_{11} = \tan\frac{\alpha}{2}, \quad b_{22} = -\cot\frac{\alpha}{2}, \quad b_{12} = 0. \tag{4.103}$$

Hence

$$K = -b_{11}b_{22} = 1. \tag{4.104}$$

The first fundamental form is

$$\mathrm{d}s^2 = (\omega^1)^2 + (\omega^2)^2 = \cos^2\frac{\alpha}{2}\,\mathrm{d}u^2 + \sin^2\frac{\alpha}{2}\,\mathrm{d}v^2, \tag{4.105}$$

and the second fundamental form is

$$II = \cos\frac{\alpha}{2}\sin\frac{\alpha}{2}(\mathrm{d}u^2 - \mathrm{d}v^2). \tag{4.106}$$

The Gauss equation becomes the negative sine-Gordon equation

$$\alpha_{uu} - \alpha_{vv} = -\sin\alpha. \tag{4.107}$$

By exchanging the parameters u and v, it is the same as the usual sine-Gordon equation.

(2) space-like surface of constant negative Gauss curvature ($K = -1$)

Under the assumption that there are no umbilic points, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{n}$ satisfy (4.83), and $\mathbf{e}_1$, $\mathbf{e}_2$ are tangent to the lines of curvature. Take

$$\omega^1 = \cosh\frac{\alpha}{2}\,\mathrm{d}u, \quad \omega^2 = \sinh\frac{\alpha}{2}\,\mathrm{d}v, \tag{4.108}$$

then

$$\omega_2^1 = -\omega_1^2 = \frac{1}{2}(\alpha_v\,\mathrm{d}u - \alpha_u\,\mathrm{d}v). \tag{4.109}$$

Take

$$\omega_1^3 = \sinh\frac{\alpha}{2}\,\mathrm{d}u, \quad \omega_2^3 = \cosh\frac{\alpha}{2}\,\mathrm{d}v, \tag{4.110}$$

then

$$b_{11} = \tanh\frac{\alpha}{2}, \quad b_{22} = \coth\frac{\alpha}{2}, \quad b_{12} = 0. \tag{4.111}$$

Hence

$$K = -b_{11}b_{22} = -1. \tag{4.112}$$

The first fundamental form is

$$\mathrm{d}s^2 = (\omega^1)^2 + (\omega^2)^2 = \cosh^2\frac{\alpha}{2}\,\mathrm{d}u^2 + \sinh^2\frac{\alpha}{2}\,\mathrm{d}v^2, \tag{4.113}$$

and the second fundamental form is

$$II = \cosh\frac{\alpha}{2}\sinh\frac{\alpha}{2}(\mathrm{d}u^2 + \mathrm{d}v^2). \tag{4.114}$$

The Gauss-Codazzi equations become

$$\triangle\alpha = \sinh\alpha. \tag{4.115}$$

(3) Time-like surface of constant positive Gauss curvature ($K = +1$)

Under the assumption that there are no umbilic points, this case is divided into the following three subcases.

(3a) Principal curvatures are real

Let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{n}$ satisfy (4.92). Take

$$\omega^1 = \cosh\frac{\alpha}{2}\,\mathrm{d}u, \quad \omega^2 = \sinh\frac{\alpha}{2}\,\mathrm{d}v, \tag{4.116}$$

then

$$\omega_2^1 = \omega_1^2 = \frac{1}{2}(\alpha_v\,\mathrm{d}u + \alpha_u\,\mathrm{d}v). \tag{4.117}$$

Let

$$\omega_1^3 = \sinh\frac{\alpha}{2}\,\mathrm{d}u, \quad \omega_2^3 = -\cosh\frac{\alpha}{2}\,\mathrm{d}v, \tag{4.118}$$

then

$$b_{11} = \tanh\frac{\alpha}{2}, \quad b_{22} = -\coth\frac{\alpha}{2}, \tag{4.119}$$

$$K = -b_{11}b_{22} = +1. \tag{4.120}$$

The Codazzi equation holds. The first fundamental form is

$$\mathrm{d}s^2 = (\omega^1)^2 - (\omega^2)^2 = \cosh^2\frac{\alpha}{2}\,\mathrm{d}u^2 - \sinh^2\frac{\alpha}{2}\,\mathrm{d}v^2, \tag{4.121}$$

and the second fundamental form is

$$II = \cosh\frac{\alpha}{2}\sinh\frac{\alpha}{2}(\mathrm{d}u^2 - \mathrm{d}v^2). \tag{4.122}$$

The Gauss equation becomes

$$\alpha_{uu} - \alpha_{vv} = -\sinh\alpha. \tag{4.123}$$

(3b) Principal curvatures are imaginary

Take the frame so that

$$\mathbf{e}_1^2 = 0, \quad \mathbf{e}_2^2 = 0, \quad \mathbf{n}\cdot\mathbf{e}_a = 0 \quad (a = 1, 2),$$

and

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \frac{1}{2}\mathrm{e}^{\alpha} > 0.$$

From the structure equations (4.81), we get

$$\begin{aligned} &\omega_2^1 = \omega_1^2 = 0, \\ &\omega_1^3 + \frac{1}{2}\mathrm{e}^{\alpha}\omega_3^2 = 0, \quad \omega_2^3 + \frac{1}{2}\mathrm{e}^{\alpha}\omega_3^1 = 0. \end{aligned}$$

Choose coordinates (u, v) [51] so that

$$\omega^1 = \mathrm{d}u - \mathrm{e}^{-\alpha}\mathrm{d}v, \quad \omega^2 = -\mathrm{e}^{-\alpha}\mathrm{d}u - \mathrm{d}v,$$

and

$$\omega_3^1 = \frac{1}{2}(\mathrm{d}u - \mathrm{e}^{\alpha}\mathrm{d}v), \quad \omega_2^3 = \frac{1}{2}(\mathrm{e}^{\alpha}\mathrm{d}u + \mathrm{d}v).$$

Then

$$\omega_1^3 = \frac{1}{2}(\mathrm{d}u - \mathrm{e}^{\alpha}\mathrm{d}v), \quad \omega_2^3 = \frac{1}{2}(\mathrm{e}^{\alpha}\mathrm{d}u + \mathrm{d}v).$$

$\mathrm{d}^2\mathbf{r} = 0$ leads to

$$\omega_1^1 = \alpha_u \mathrm{d}u, \quad \omega_2^2 = \alpha_v \mathrm{d}v.$$

Hence the first fundamental form and the second fundamental form are

$$\begin{aligned} &I = \mathrm{d}\mathbf{r}^2 = -\mathrm{d}u^2 - 2\sinh\alpha\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}v^2, \\ &I\!I = -\mathrm{d}\mathbf{r}\cdot\mathrm{d}\mathbf{n} = -2\cosh\alpha\,\mathrm{d}u\,\mathrm{d}v. \end{aligned}$$

Therefore, $K = 1$ and two principal curvatures are imaginary. From the Gauss equation $\mathrm{d}\omega_b^a + \omega_c^a \wedge \omega_b^c = -\omega_3^a \wedge \omega_b^3$, we get

$$\alpha_{uv} = \cosh\alpha.$$

All the Codazzi equations are also the consequences of the cosh-Gordon equation. Therefore, from a solution of the cosh-Gordon, a time-like surface with Gauss curvature 1 and two imaginary principal curvatures can be obtained by solving the fundamental equations of the surface. The parameters (u, v) are called asymptotic Chebyshev coordinates, since u and v are asymptotic lines.

(3c) Repeated principal curvature with only one principal direction

Take $\mathbf{e}_1$, $\mathbf{e}_2$ satisfying

$$\mathbf{e}_1^2 = 0, \quad \mathbf{e}_2^2 = 0, \quad \mathbf{e}_1 \cdot \mathbf{e}_2 = \frac{1}{2}\mathrm{e}^{\alpha}.$$

Take the parameters (u, v) so that

$$\omega^1 = \mathrm{d}u - \mathrm{e}^{-\alpha}\mathrm{d}v, \quad \omega^2 = -\mathrm{d}v,$$
$$\omega_3^1 = -\mathrm{d}u - \mathrm{e}^{-\alpha}\mathrm{d}v, \quad \omega_3^2 = \mathrm{d}v.$$

The first and second fundamental forms are

$$I = -\mathrm{e}^{\alpha}\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}v^2,$$
$$II = -\mathrm{e}^{\alpha}\,\mathrm{d}u\,\mathrm{d}v.$$

The Gauss-Codazzi equations become the Liouville equation

$$\alpha_{uv} = \frac{1}{2}\mathrm{e}^{\alpha}.$$

From any solution of this equation, one can get a surface of this kind by solving the fundamental equations of surface.

(4) Time-like surface of constant negative Gauss curvature ($K = -1$)

Let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{n}$ satisfy (4.92),

$$\omega^1 = \cos\frac{\alpha}{2}\,\mathrm{d}u, \quad \omega^2 = \sin\frac{\alpha}{2}\,\mathrm{d}v, \tag{4.124}$$

then

$$\omega_2^1 = \omega_1^2 = \frac{1}{2}(-\alpha_v\,\mathrm{d}u + \alpha_u\,\mathrm{d}v). \tag{4.125}$$

Take

$$\omega_1^3 = \sin\frac{\alpha}{2}\,\mathrm{d}u, \quad \omega_2^3 = \cos\frac{\alpha}{2}\,\mathrm{d}v, \tag{4.126}$$

then the first fundamental form is

$$\mathrm{d}s^2 = (\omega^1)^2 - (\omega^2)^2 = \cos^2\frac{\alpha}{2}\,\mathrm{d}u^2 - \sin^2\frac{\alpha}{2}\,\mathrm{d}v^2, \tag{4.127}$$

and the second fundamental form is

$$II = \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}(\mathrm{d}u^2 + \mathrm{d}v^2), \tag{4.128}$$

and

$$b_{11} = \tan\frac{\alpha}{2}, \quad b_{22} = \cot\frac{\alpha}{2}.$$

The Gauss curvature $K = -1$. The Gauss-Codazzi equation

$$\mathrm{d}\omega_2^1 = -\omega_3^1 \wedge \omega_2^3$$

becomes

$$\triangle\alpha = \sin\alpha. \tag{4.129}$$

In summary, we have the following theorem.

THEOREM 4.7 *The Gauss-Codazzi equations of various kinds of surfaces (space-like or time-like) of constant Gauss curvature ($K = \pm 1$) can be sine-Gordon equation, sinh-Gordon equation, cosh-Gordon equation or Liouville equation in a suitable parametrization. The construction of these surfaces is reduced to solve these equations and to integrate the fundamental equations of surfaces.*

This result can be listed as follows:

Space-like	$K = 1$	First fundamental form: $\cos^2 \frac{\alpha}{2} \mathrm{d}u^2 + \sin^2 \frac{\alpha}{2} \mathrm{d}v^2$ Second fundamental form: $\cos \frac{\alpha}{2} \sin \frac{\alpha}{2} (\mathrm{d}u^2 - \mathrm{d}v^2)$ Gauss equation: $\alpha_{uu} - \alpha_{vv} = -\sin \alpha$
Space-like	$K = -1$	First fundamental form: $\cosh^2 \frac{\alpha}{2} \mathrm{d}u^2 + \sinh^2 \frac{\alpha}{2} \mathrm{d}v^2$ Second fundamental form: $\cosh \frac{\alpha}{2} \sinh \frac{\alpha}{2} (\mathrm{d}u^2 + \mathrm{d}v^2)$ Gauss equation: $\triangle \alpha = \sinh \alpha$
Time-like	$K = 1$	(a) First fundamental form: $\cosh^2 \frac{\alpha}{2} \mathrm{d}u^2 - \sinh^2 \frac{\alpha}{2} \mathrm{d}v^2$ Second fundamental form: $\cosh \frac{\alpha}{2} \sinh \frac{\alpha}{2} (\mathrm{d}u^2 - \mathrm{d}v^2)$ Gauss equation: $\alpha_{uu} - \alpha_{vv} = -\sinh \alpha$
Time-like	$K = 1$	(b) First fundamental form: $-\mathrm{d}u^2 - 2 \sinh \alpha \mathrm{d}u \, \mathrm{d}v + \mathrm{d}v^2$ Second fundamental form: $-2 \cosh \alpha \, \mathrm{d}u \, \mathrm{d}v$ Gauss equation: $\alpha_{uv} = \cosh \alpha$
Time-like	$K = 1$	(c) First fundamental form: $-\mathrm{e}^{\alpha} \, \mathrm{d}u \mathrm{d}v + \mathrm{d}v^2$ Second fundamental form: $-\mathrm{e}^{-\alpha} \, \mathrm{d}u \, \mathrm{d}v$ Gauss equation: $\alpha_{uv} = \frac{1}{2} \mathrm{e}^{\alpha}$
Time-like	$K = -1$	First fundamental form: $\cos^2 \frac{\alpha}{2} \, \mathrm{d}u^2 - \sin^2 \frac{\alpha}{2} \mathrm{d}v^2$ Second fundamental form: $\cos \frac{\alpha}{2} \sin \frac{\alpha}{2} (\mathrm{d}u^2 + \mathrm{d}v^2)$ Gauss equation: $\triangle \alpha = \sin \alpha$

There are still two problems left. (1) How to get explicit solutions of these partial differential equations? (2) How to get the explicit expressions of these surfaces. In order to answer these problems, we need to consider the Bäcklund transformation and pseudo-spherical congruences in $\mathbf{R}^{2,1}$ and use the Darboux transformation.

4.3.3 Pseudo-spherical congruence in $\mathrm{R}^{2,1}$

In $\mathbf{R}^{2,1}$, a line congruence may be space-like, time-like, light-like or mixed. Here we want to study time-like line congruences (i.e., all the lines in it are time-like) and space-like line congruences (i.e., all the lines in it are space-like). Under the assumption that they have two focal surfaces, these line congruences can be classified into following four types.

(a) The line congruence is space-like, and two focal surfaces are space-like;

(b) The line congruence is space-like, and two focal surfaces are time-like;

(c) The line congruence is time-like, and two focal surfaces are also time-like;

(d) The line congruence is space-like, and one focal surface is space-like, another focal surface is time-like.

Let S and S^* be two focal surfaces of a line congruence Σ, PP^* be the lines in Σ which are common tangent lines of S and S^*, and $P \in S$, $P^* \in S^*$ are the tangent points. Let $\mathbf{n}$ and $\mathbf{n}^*$ be the normal vectors of S and S^* at P and P^* respectively. If

$$|PP^*| = l, \quad \mathbf{n} \cdot \mathbf{n}^* = k \tag{4.130}$$

are non-zero constants, then Σ is called a pseudo-spherical congruence. In [66, 84], the Bäcklund transformation for various pseudo-spherical congruences has been discussed. However, the discussion there is to be completed and the method of explicit construction is insufficient.

Here we shall consider all the possible cases and use the Darboux transformation to get explicit expressions of the surfaces and congruences.

THEOREM 4.8 *(Generalized Bäcklund theorem) Two focal surfaces of a pseudo-spherical congruence in $\mathbf{R}^{2,1}$ are surfaces of the same constant Gauss curvature K. In cases (a), (b) and (c), K is positive. In case (d), K is negative.*

Proof. The theorem should be proved for all four cases.

Case (a): The congruence is space-like, and two focal surfaces S and S^* are space-like

Take the orthogonal frames $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$ and $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{n}^*\}$ of S at $\mathbf{r}$ and of S^* at $\mathbf{r}^*$ respectively so that $\mathbf{e}_1 = -\mathbf{e}_1^*$ is the unit vector parallel to PP^*. Here $\mathbf{r}$ and $\mathbf{r}^*$ are the position vectors of P and P^* respectively. Then

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_1^{*2} = \mathbf{e}_2^{*2} = 1, \quad \mathbf{n}^2 = \mathbf{n}^{*2} = -1,$$

and

$$\begin{aligned} \mathbf{e}_2^* &= (\cosh\tau)\mathbf{e}_2 + (\sinh\tau)\mathbf{n}, \\ \mathbf{n}^* &= (\sinh\tau)\mathbf{e}_2 + (\cosh\tau)\mathbf{n}. \end{aligned} \tag{4.131}$$

Here $\tau = \text{constant} \neq 0$. Moreover,

$$\mathbf{r}^* = \mathbf{r} + l\mathbf{e}_1 \quad (l = \text{constant} \neq 0). \tag{4.132}$$

Differentiating this equation and comparing it with $\mathrm{d}\mathbf{r}^* = \omega^{*1}\mathbf{e}_1^* + \omega^{*2}\mathbf{e}_2^*$, we get

$$\begin{aligned} &\omega^1 = -\omega^{*1}, \\ &\omega^2 + l\omega_1^2 = (\cosh\tau)\omega^{*2}, \\ &l\omega_1^3 = (\sinh\tau)\omega^{*2}. \end{aligned} \tag{4.133}$$

Hence

$$(\cosh\tau)\omega_1^3 - (\sinh\tau)\omega_1^2 = \frac{\sinh\tau}{l}\omega^2. \tag{4.134}$$

From

$$\begin{aligned} &\omega_1^{*3} = (-\mathrm{d}\mathbf{e}_1^*)\cdot\mathbf{n}^* = -\frac{\omega^2}{l}\sinh\tau, \\ &\omega_2^{*3} = (-\mathrm{d}\mathbf{e}_2^*)\cdot\mathbf{n}^* = \omega_2^3, \end{aligned} \tag{4.135}$$

we have

$$\begin{aligned} &-\omega_3^{*1}\wedge\omega_2^{*3} = \omega_2^3\wedge(-\frac{\omega^2}{l}\sinh\tau) = \frac{\sinh\tau}{l}\omega_1^3\wedge\omega^1 \\ &= \frac{\sinh^2\tau}{l^2}\omega^{*1}\wedge\omega^{*2}, \end{aligned} \tag{4.136}$$

Hence the Gauss curvatures of S and S^* are the same constant

$$K^* = K = \frac{\sinh^2\tau}{l^2}.$$

Case (b): The congruence is space-like, and two focal surfaces S and S^* are time-like

Now

$$\mathbf{e}_1^2 = \mathbf{e}_1^{*2} = 1, \quad \mathbf{e}_2^2 = \mathbf{e}_2^{*2} = -1, \quad \mathbf{n}^2 = \mathbf{n}^{*2} = 1, \tag{4.137}$$

and

$$\begin{aligned} &\mathbf{e}_1^* = -\mathbf{e}_1, \\ &\mathbf{e}_2^* = \cosh\tau\mathbf{e}_2 + \sinh\tau\mathbf{n}, \\ &\mathbf{n}^* = \sinh\tau\mathbf{e}_2 + \cosh\tau\mathbf{n}, \quad (\mathbf{n}\cdot\mathbf{n}^* = \cosh\tau = \text{constant}). \end{aligned} \tag{4.138}$$

From $\mathbf{r}^* = \mathbf{r} + l\mathbf{r}_1$ ($l = \text{constant}$),

$$\begin{aligned} &\omega^1 = -\omega^{*1}, \\ &\omega^2 + l\omega_1^2 = \cosh\tau\omega^{*2}, \\ &l\omega_1^3 = \sinh\tau\omega^{*2}. \end{aligned} \tag{4.139}$$

(4.138) and (4.139) are the same as (4.131) and (4.133). Moreover,

$$\omega_1^{*3} = (\mathrm{d}\mathbf{e}_1^*)\cdot\mathbf{n}^* = -\frac{\omega^2}{l}\sinh\tau, \quad \omega_2^{*3} = (\mathrm{d}\mathbf{e}_2^*)\cdot\mathbf{n} = \omega_2^3. \tag{4.140}$$

Hence

$$\begin{aligned} -\omega_3^{*1}\wedge\omega_2^{*3} &= \frac{\sinh\tau}{l}\omega_2^3\wedge\omega^2 = -\frac{\sinh\tau}{l}\omega_1^3\wedge\omega^1 \\ &= -\frac{\sinh^2\tau}{l^2}\omega^{*1}\wedge\omega^{*2}. \end{aligned} \tag{4.141}$$

This means that S and S^* are surfaces of the same constant positive Gauss curvature

$$K^* = K = \frac{\sinh^2\tau}{l^2}. \tag{4.142}$$

Case (c): The congruence is time-like, and two focal surfaces S and S^* are time-like

Now

$$\mathbf{e}_1^2 = \mathbf{e}_1^{*2} = -1, \quad \mathbf{e}_2^2 = \mathbf{e}_2^{*2} = 1, \quad \mathbf{n}^2 = \mathbf{n}^{*2} = 1, \tag{4.143}$$

and

$$\begin{aligned} &\mathbf{e}_1^* = -\mathbf{e}_1, \\ &\mathbf{e}_2^* = (\cos\tau)\mathbf{e}_2 + (\sin\tau)\mathbf{n}, \\ &\mathbf{n}^* = -(\sin\tau)\mathbf{e}_2 + (\cos\tau)\mathbf{n} \\ &(\mathbf{n}\cdot\mathbf{n}^* = \cos\tau = \text{constant} \neq 0). \end{aligned} \tag{4.144}$$

Differentiating $\mathbf{r}^* = \mathbf{r} + l\mathbf{e}_1$, we get

$$\begin{aligned} &\omega^1 = -\omega^{*1}, \\ &\omega^2 + l\omega_1^2 = (\cos\tau)\omega^{*2}, \\ &l\omega_1^3 = (\sin\tau)\omega^{*2}. \end{aligned} \tag{4.145}$$

(4.144) implies

$$\begin{aligned} \omega_1^{*3} &= -\frac{\omega^2}{l}\sin\tau, \\ \omega_2^{*3} &= \omega_2^3, \end{aligned} \tag{4.146}$$

which leads to

$$\begin{aligned} -\omega_3^{*1}\wedge\omega_2^{*3} &= \omega_2^3\wedge\left(-\frac{\omega^2}{l}\sin\tau\right) = \frac{\sin\tau}{l}\omega_1^3\wedge\omega^1 \\ &= \frac{\sin^2\tau}{l^2}\omega^{*1}\wedge\omega^{*2}. \end{aligned} \tag{4.147}$$

$(4.98)'$ implies that

$$K^* = \frac{\sin^2\tau}{l^2}, \quad K = \frac{\sin^2\tau}{l^2}. \tag{4.148}$$

Hence S and S^* are surfaces of the same constant positive Gauss curvature.

Case (d): The congruence is space-like, and one focal surface S is time-like, another focal surface S^* is space-like

Now

$$\mathbf{e}_1^2 = \mathbf{e}_1^{*2} = 1, \quad \mathbf{e}_2^2 = -1, \quad \mathbf{e}_2^{*2} = 1, \quad \mathbf{n}^2 = 1, \quad \mathbf{n}^{*2} = -1, \tag{4.149}$$

and

$$\begin{aligned}
&\mathbf{e}_1^* = -\mathbf{e}_1,\\
&\mathbf{e}_2^* = (\sinh\tau)\mathbf{e}_2 + (\cosh\tau)\mathbf{n},\\
&\mathbf{n}^* = (\cosh\tau)\mathbf{e}_2 + (\sinh\tau)\mathbf{n}\\
&(\mathbf{n}\cdot\mathbf{n}^* = \sinh\tau = \text{constant}).
\end{aligned} \tag{4.150}$$

Differentiating $\mathbf{r}^* = \mathbf{r} + l\mathbf{e}_1$ ($l = \text{constant}$), we get

$$\begin{aligned}
&\omega^1 = -\omega^{*1},\\
&\omega^2 + l\omega_1^2 = (\sinh\tau)\omega^{*2},\\
&l\omega_1^3 = (\cosh\tau)\omega^{*2}.
\end{aligned} \tag{4.151}$$

Moreover,

$$\omega_1^{*3} = \frac{\cosh\tau}{l}\omega^2, \quad \omega_2^{*3} = \omega_2^3 \tag{4.152}$$

leads to

$$\begin{aligned}
-\omega_3^{*1}\wedge\omega_2^{*3} &= \frac{\cosh\tau}{l}\omega_2^3\wedge\omega^2 = -\frac{\cosh\tau}{l}\omega_1^3\wedge\omega^1\\
&= -\frac{\cosh^2\tau}{l^2}\omega^{*1}\wedge\omega^{*2}.
\end{aligned} \tag{4.153}$$

Hence, from (4.91),

$$K^* = -\frac{\cosh^2\tau}{l^2}. \tag{4.154}$$

On the other hand,

$$\begin{aligned}
-\omega_3^1\wedge\omega_2^3 &= -\omega_2^{*3}\wedge\frac{\cosh\tau}{l}\omega^{*2} = \frac{\cosh\tau}{l}\omega_1^{*3}\wedge\omega^{*1}\\
&= \frac{\cosh^2\tau}{l}\omega^1\wedge\omega^2.
\end{aligned} \tag{4.155}$$

Hence, from (4.98),

$$K = -\frac{\cosh^2\tau}{l^2}. \tag{4.156}$$

The theorem is proved.

This theorem does not confirm that all four cases can occur, because it does not contain the proof of existence. This will be completed in the next subsection.

4.3.4 Bäcklund transformation and Darboux transformation for surfaces of constant Gauss curvature in $\mathbf{R}^{2,1}$

Now we shall give the method for constructing new surfaces of constant Gauss curvature in $\mathbf{R}^{2,1}$ from a known one by using pseudo-spherical congruence [61, 63]. This also gives the proof of existence of various pseudo-spherical congruences. It is the generalization of the classical Bäcklund transformation. Using Darboux transformation, this analytical process can be realized algebraically. The method is similar with that in Section 4.2. However, in each case the method has some special feature.

(1) S is a space-like surface of constant positive Gauss curvature, the congruence is space-like

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$ be the Chebyshev frames at $\mathbf{r}$. If there exists a Bäcklund transformation such that its another focal surface S^* is space-like, then

$$\mathbf{r}^* = \mathbf{r} + l(\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2), \quad (l = \text{constant} \neq 0). \tag{4.157}$$

The normal vector $\mathbf{n}^*$ of S^* is time-like and orthogonal to $\overrightarrow{PP^*}$. Hence

$$\mathbf{n}^* = \sinh\tau(-\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2) + \cosh\tau\mathbf{n}, \quad (\tau = \text{constant} \neq 0). \tag{4.158}$$

Differentiating $\mathbf{r}^*$, using the fundamental equations of surface, and using the equations (4.99) – (4.101) on space-like surfaces of constant positive Gauss curvature, we get

$$\begin{aligned}
\mathrm{d}\mathbf{r}^* = \ & \Big\{\Big[\cos\frac{\alpha}{2} - l\sin\theta\Big(\theta_u + \frac{\alpha_v}{2}\Big)\Big]\mathbf{e}_1 + l\cos\theta\Big(\theta_u + \frac{\alpha_v}{2}\Big)\mathbf{e}_2 \\
& + l\cos\theta\sin\frac{\alpha}{2}\mathbf{n}\Big\}\,\mathrm{d}u + \Big\{-l\sin\theta\Big(\theta_v + \frac{\alpha_u}{2}\Big)\mathbf{e}_1 \\
& + \Big[\sin\frac{\alpha}{2} + l\cos\theta\Big(\theta_v + \frac{\alpha_u}{2}\Big)\Big]\mathbf{e}_2 - l\sin\theta\cos\frac{\alpha}{2}\mathbf{n}\Big\}\,\mathrm{d}v.
\end{aligned} \tag{4.159}$$

Let $\theta = \alpha_1/2$ and suppose $K = 1$, then $l = \sinh\tau$. The condition $\mathrm{d}\mathbf{r}^* \cdot \mathbf{n}^* = 0$ implies

$$\begin{aligned} l(\alpha'_u + \alpha_v) &= 2\sin\frac{\alpha_1}{2}\cos\frac{\alpha}{2} + 2\cosh\tau\cos\frac{\alpha_1}{2}\sin\frac{\alpha}{2}, \\ l(\alpha'_v + \alpha_u) &= -2\cos\frac{\alpha_1}{2}\sin\frac{\alpha}{2} - 2\cosh\tau\sin\frac{\alpha_1}{2}\cos\frac{\alpha}{2}. \end{aligned} \tag{4.160}$$

(4.160) is a system of partial differential equations for α_1. Its integrability condition is exactly (4.107)

$$\alpha_{uu} - \alpha_{vv} = -\sin\alpha. \tag{4.161}$$

Therefore, (4.160) is solvable. This gives the following theorem.

THEOREM 4.9 *For a given space-like surface S of constant positive Gauss curvature, there exists a space-like pseudo-spherical congruence whose focal surfaces are S and another space-like surface S^* of the same constant Gauss curvature.*

Using the method in Section 4.2, we can prove that (u, v) are also the Chebyshev coordinates of S^*. Using the Darboux transformation for the sine-Gordon equation, a series of surfaces of constant positive Gauss curvature can be obtained. If a solution α of the sine-Gordon equation corresponding to the given surface of constant positive Gauss curvature together with a fundamental solution of its Lax pair are known, then the construction is purely algebraic. Note that in this case u and v should be interchanged (i.e. $\xi = \dfrac{u-v}{2}$, $\eta = \dfrac{u+v}{2}$) so that (4.161) can be changed to the standard sine-Gordon equation (4.66).

(2) S is a time-like surface of constant positive Gauss curvature

There are following three subcases.

(2a) Two principal curvatures of S are real and distinct

Take the Chebyshev coordinates, then

$$\begin{aligned} &\omega^1 = \cosh\frac{\alpha}{2}\,\mathrm{d}u, \quad \omega^2 = \sinh\frac{\alpha}{2}\,\mathrm{d}v, \\ &\omega_2^1 = \omega_1^2 = \frac{1}{2}(\alpha_v\,\mathrm{d}u + \alpha_u\,\mathrm{d}v), \\ &\omega_1^3 = \sinh\frac{\alpha}{2}\,\mathrm{d}u, \quad \omega_2^3 = -\cosh\frac{\alpha}{2}\,\mathrm{d}v, \end{aligned} \tag{4.162}$$

and

$$\mathbf{e}_1^2 = 1, \quad \mathbf{e}_2^2 = -1, \quad \mathbf{n}^2 = 1.$$

($2a_1$) Space-like Bäcklund transformation

The equation of S^* is

$$\mathbf{r}^* = \mathbf{r} + l(\cosh\theta\mathbf{e}_1 + \sinh\theta\mathbf{e}_2) \tag{4.163}$$

and the normal vector is

$$\mathbf{n}^* = \sinh\tau(\sinh\theta\mathbf{e}_1 + \cosh\theta\mathbf{e}_2) + \cosh\tau\mathbf{n}. \tag{4.164}$$

Suppose $K = 1$, then the proof of Theorem 4.8 leads to $l = \sinh\tau$. We have

$$\begin{aligned}\mathrm{d}\mathbf{r}^* = & \Big\{\Big[\cosh\frac{\alpha}{2} + l\sinh\theta\Big(\theta_u + \frac{\alpha_v}{2}\Big)\Big]\mathbf{e}_1 + l\cosh\theta\Big(\theta_u + \frac{\alpha_v}{2}\Big)\mathbf{e}_2 \\ & + l\cosh\theta\sinh\frac{\alpha}{2}\mathbf{n}\Big\}\,\mathrm{d}u + \Big\{l\sinh\theta\Big(\theta_v + \frac{\alpha_u}{2}\Big)\mathbf{e}_1 \\ & + \Big[\sinh\frac{\alpha}{2} + l\cosh\theta\Big(\theta_v + \frac{\alpha_u}{2}\Big)\Big]\mathbf{e}_2 - l\sinh\theta\cosh\frac{\alpha}{2}\mathbf{n}\Big\}\,\mathrm{d}v.\end{aligned} \tag{4.165}$$

From $\mathrm{d}\mathbf{r}^* \cdot \mathbf{n}^* = 0$, we have

$$\begin{aligned} l\Big(\theta_u + \frac{\alpha_v}{2}\Big) &= \sinh\theta\cosh\frac{\alpha}{2} + \cosh\tau\cosh\theta\sinh\frac{\alpha}{2}, \\ l\Big(\theta_v + \frac{\alpha_u}{2}\Big) &= -\cosh\theta\sinh\frac{\alpha}{2} - \cosh\tau\sinh\theta\cosh\frac{\alpha}{2}.\end{aligned} \tag{4.166}$$

Regarded as a system of partial differential equations for $\theta = \alpha_1/2$, its integrability condition is

$$\alpha_{vv} - \alpha_{uu} = \sinh\alpha. \tag{4.167}$$

This equation holds because it is the Gauss equation of S. It is easy to see that S^* is also time-like. This proves the existence of the pseudo-spherical congruence. By Theorem 4.8, the curvature of S^* is also $+1$.

Therefore, we have the following theorem.

THEOREM 4.10 *There exist space-like pseudo-spherical congruences with two time-like focal surfaces of constant positive Gauss curvature.*

Similarly, we can prove that (u, v) are also the Chebyshev coordinates of S^*, and can write down the explicit expressions of $\mathbf{e}_1^*$, $\mathbf{e}_2^*$.

In order to construct S^* explicitly in terms of the Darboux transformation, we want to discuss the Darboux transformation for the sinh-Gordon equation. Let

$$\xi = \frac{u+v}{2}, \quad \eta = \frac{v-u}{2}, \tag{4.168}$$

then (4.123) becomes

$$\alpha_{\xi\eta} = \sinh\alpha. \tag{4.169}$$

It has a Lax pair

$$\Phi_\xi = \frac{\lambda}{2}\begin{pmatrix} 0 & \mathrm{e}^{-\alpha} \\ \mathrm{e}^{\alpha} & 0 \end{pmatrix}\Phi, \quad \Phi_\eta = \frac{1}{2}\begin{pmatrix} -\alpha_\eta & \dfrac{1}{\lambda} \\ \dfrac{1}{\lambda} & \alpha_\eta \end{pmatrix}\Phi, \tag{4.170}$$

that is, the integrability condition of (4.170) is just (4.169).

Let $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ be a column solution of the Lax pair (4.170) with $\lambda = \lambda_1$, then $\begin{pmatrix} h_1 \\ -h_2 \end{pmatrix}$ is a column solution of (4.170) with $\lambda = -\lambda_1$. Let $H = \begin{pmatrix} h_1 & h_1 \\ h_2 & -h_2 \end{pmatrix}$ and suppose $h_1, h_2 \neq 0$, then the Darboux matrix is

$$D(\lambda) = I - \lambda H \begin{pmatrix} \dfrac{1}{\lambda_1} & 0 \\ 0 & -\dfrac{1}{\lambda_1} \end{pmatrix} H^{-1} = I - \frac{\lambda}{\lambda_1}\begin{pmatrix} 0 & \dfrac{h_1}{h_2} \\ \dfrac{h_2}{h_1} & 0 \end{pmatrix}. \tag{4.171}$$

It can be checked directly that (α_1, Φ_1) defined by

$$\mathrm{e}^{\alpha_1} = \mathrm{e}^{-\alpha}\Big(\frac{h_2}{h_1}\Big)^2, \quad \Phi_1 = D(\lambda)\Phi \tag{4.172}$$

still satisfy the Lax pair (4.170). Hence α_1 is a solution of (4.169), and $(\alpha, \Phi) \longrightarrow (\alpha_1, \Phi_1)$ is the Darboux transformation for (4.169).

Now we prove that α_1 obtained by (4.172) provides a solution $\theta = \alpha_1/2$ of (4.166).

Let $\theta = \alpha_1/2$. Adding and subtracting (4.168) and (4.166), we get

$$\begin{aligned} l\Big(\frac{\alpha_{1\xi}}{2} + \frac{\alpha_\xi}{2}\Big) &= (1 - \cosh\tau)\sinh\frac{\alpha_1 - \alpha}{2}, \\ l\Big(\frac{\alpha_{1\eta}}{2} - \frac{\alpha_\eta}{2}\Big) &= -(1 + \cosh\tau)\sinh\frac{\alpha_1 + \alpha}{2}. \end{aligned} \tag{4.173}$$

Since $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ satisfies the Lax pair (4.170),

$$\begin{aligned} (\ln|h_1|)_\xi &= \frac{\lambda_1}{2}\mathrm{e}^{-\alpha}\frac{h_2}{h_1}, \quad (\ln|h_1|)_\eta = -\frac{\alpha_\eta}{2} + \frac{1}{2\lambda_1}\frac{h_2}{h_1}, \\ (\ln|h_2|)_\xi &= \frac{\lambda_1}{2}\mathrm{e}^{\alpha}\frac{h_1}{h_2}, \quad (\ln|h_2|)_\eta = \frac{1}{2\lambda_1}\frac{h_1}{h_2} + \frac{1}{2}\alpha_\eta. \end{aligned} \tag{4.174}$$

Choose $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ so that $\dfrac{h_2}{h_1}$ is positive (if it is negative, the discussion is similar), then (4.172) leads to

$$\begin{aligned}\alpha_{1\xi}+\alpha_\xi &= (2\ln|h_2|)_\xi-(2\ln|h_1|)_\xi \\ &= \lambda_1\Big(\mathrm{e}^{\alpha}\frac{h_1}{h_2}-\mathrm{e}^{-\alpha}\frac{h_2}{h_1}\Big)=-2\lambda_1\sinh\frac{\alpha_1-\alpha}{2}.\end{aligned}\tag{4.175}$$

Likewise,

$$\alpha_{1\eta}+\alpha_\eta=2\alpha_\eta+\frac{1}{\lambda_1}(\mathrm{e}^{-(\alpha_1+\alpha)/2}-\mathrm{e}^{(\alpha+\alpha_1)/2}),$$

or equivalently

$$\alpha_{1\eta}-\alpha_\eta=-\frac{1}{\lambda_1}\sinh\frac{\alpha_1+\alpha}{2},\tag{4.176}$$

where $\lambda_1=\dfrac{-1+\cosh\tau}{l}$. Since $l=\sinh\tau$, $\lambda_1^{-1}=\dfrac{1+\cosh\tau}{\sinh\tau}$, (4.175) and (4.176) become (4.173). Thus by using Darboux transformation we can obtain the solution α_1 of (4.176) explicitly. Besides, in realizing the Bäcklund transformation (4.163), we only need to use $\cosh\dfrac{\alpha_1}{2}$ and $\sinh\dfrac{\alpha_1}{2}$, hence the algorithm is purely algebraic.

Now we turn to examples. Starting from $\alpha=0$ and solving the Lax pair

$$\Phi_\xi=\frac{\lambda}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\Phi,\quad \Phi_\eta=\frac{1}{2\lambda}\begin{pmatrix}0&1\\1&0\end{pmatrix}\Phi,\tag{4.177}$$

we get

$$\Phi(\lambda)=\begin{pmatrix}\cosh\gamma&\sinh\gamma\\ \sinh\gamma&\cosh\gamma\end{pmatrix},\quad \Big(\gamma=\frac{\lambda}{2}\xi+\frac{1}{2\lambda}\eta\Big).\tag{4.178}$$

Let

$$h_1=\cosh\gamma_1,\quad h_2=\sinh\gamma_1,\quad \Big(\gamma_1=\frac{\lambda_1}{2}\xi+\frac{1}{2\lambda_1}\eta\Big).\tag{4.179}$$

According to (4.171), the Darboux matrix is

$$D(\lambda)=I-\frac{\lambda}{\lambda_1}\begin{pmatrix}0&\coth\gamma_1\\ \tanh\gamma_1&0\end{pmatrix}.\tag{4.180}$$

Hence

$$\Phi_1(\lambda) = D(\lambda)\Phi(\lambda)$$
$$= \begin{pmatrix} \cosh\gamma - \dfrac{\lambda}{\lambda_1}\coth\gamma_1\sinh\gamma & \sinh\gamma - \dfrac{\lambda}{\lambda_1}\coth\gamma_1\cosh\gamma \\ \sinh\gamma - \dfrac{\lambda}{\lambda_1}\tanh\gamma_1\cosh\gamma & \cosh\gamma - \dfrac{\lambda}{\lambda_1}\tanh\gamma_1\cosh\gamma \end{pmatrix}. \tag{4.181}$$

Take $\lambda_2 > 0$ $(\lambda_2 \neq \lambda_1)$, $\gamma_2 = \dfrac{\lambda_2}{2}\xi + \dfrac{1}{2\lambda_2}\eta$, and let

$$\begin{aligned} \widetilde{h}_1 &= \cosh\gamma_2 - \frac{\lambda_2}{\lambda_1}\coth\gamma_1\sinh\gamma_2, \\ \widetilde{h}_2 &= \sinh\gamma_2 - \frac{\lambda_2}{\lambda_1}\tanh\gamma_1\cosh\gamma_2, \end{aligned} \tag{4.182}$$

then

$$\mathrm{e}^{\alpha_2} = \mathrm{e}^{-\alpha_1}\Big(\frac{\widetilde{h}_2}{\widetilde{h}_1}\Big)^2$$

gives a new solution α_2. This process can be done successively to get $\alpha_1, \alpha_2, \alpha_3, \cdots$.

The surfaces are constructed as follows. Starting from $\alpha = 0$ and solving the fundamental equations of the surface, we obtain a family of frames along a line. In suitable coordinates, they are

$$\begin{aligned} &\mathbf{r} = (u, 0, 0), \quad \mathbf{e}_1 = (1, 0, 0), \\ &\mathbf{e}_2 = (0, \sinh v, \cosh v), \quad \mathbf{n} = (0, -\cosh v, -\sinh v). \end{aligned} \tag{4.183}$$

Then by Theorem 4.8 and formula (4.163), α_1 gives a time-like surface of Gauss curvature $+1$. Using $\alpha_2, \alpha_3, \cdots$, we can construct a series of surfaces in the same way.

($2a_2$) Time-like Bäcklund transformation

From the same S, the time-like Darboux transformation is

$$\begin{aligned} \mathbf{r}^* &= \mathbf{r} + l(\sinh\frac{\alpha_1}{2}\mathbf{e}_1 + \cosh\frac{\alpha_1}{2}\mathbf{e}_2), \\ \mathbf{n}^* &= \sin\tau(\cosh\frac{\alpha_1}{2}\mathbf{e}_1 + \sinh\frac{\alpha_1}{2}\mathbf{e}_2) + \cos\tau\mathbf{n}, \end{aligned} \tag{4.184}$$

where l and τ are constants, $\sin\tau = l \neq 0$.

Computing $\mathrm{d}\mathbf{r}^*$ and using the condition $\mathbf{n}^* \cdot \mathrm{d}\mathbf{r}^* = 0$, we get the equations

$$\begin{aligned} &l\Big(\frac{\alpha_{1u}}{2} + \frac{\alpha_v}{2}\Big) = -\cosh\frac{\alpha_1}{2}\cosh\frac{\alpha}{2} + \cos\tau\sinh\frac{\alpha_1}{2}\sinh\frac{\alpha}{2}, \\ &l\Big((\frac{\alpha_{1v}}{2} + \frac{\alpha_u}{2}\Big) = \sinh\frac{\alpha_1}{2}\sinh\frac{\alpha}{2} + \cos\tau\cosh\frac{\alpha_1}{2}\cosh\frac{\alpha}{2}, \end{aligned} \tag{4.185}$$

which is similar with (4.166). This is the Bäcklund transformation from α to α_1. Its integrability condition

$$\alpha_{uu} - \alpha_{vv} = -\sinh\alpha \tag{4.186}$$

already holds. Hence the Bäcklund transformation from S to S^* can be realized by solving this integrable system. If the expression of S and the solution of the Lax pair corresponding to the sinh-Gordon equation are known, then S^* can be obtained by Darboux transformation in algebraic algorithm.

(2b) Two principal curvatures of S are imaginary

When two principal curvatures of S are imaginary, the frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$, parameters (u, v) and ω^1, ω^2, ω^i_j $(i, j = 1, 2, 3)$ have been chosen in the case (3b) of Subsection 4.3.2.

($2\mathrm{b}_1$) Time-like Bäcklund transformation

Suppose that the expression of S^* is

$$\mathbf{r}^* = \mathbf{r} + l(a\mathbf{e}_1 + b\mathbf{e}_2),$$

where l is a non-zero constant and $ab\mathrm{e}^\alpha = -1$. The last condition means that PP^* is a time-like straight line and $(\mathbf{r}^* - \mathbf{r})^2 = -l^2$. If we want that

$$\mathbf{n}^* = (a\mathbf{e}_1 - b\mathbf{e}_2)\cos\tau + \mathbf{n}\sin\tau$$

is the normal vector of S^* where $\sin\tau = \mathbf{n}\cdot\mathbf{n}^* =$ constant, then by $\mathbf{n}^*\cdot \mathrm{d}\mathbf{r}^* = 0$, we get

$$\begin{aligned}
\mathbf{r}^*_u &= (1 - la\frac{b_u}{b})\mathbf{e}_1 + (-\mathrm{e}^{-\alpha} + lb_u)\mathbf{e}_2 + \frac{l}{2}(a - a^{-1})\mathbf{n},\\
\mathbf{r}^*_v &= (-\mathrm{e}^{-\alpha} + la_v)\mathbf{e}_1 + (-1 - lb\frac{a_v}{a})\mathbf{e}_2 + \frac{l}{2}(b + b^{-1})\mathbf{n}
\end{aligned} \tag{4.187}$$

where $l = \sin\tau$. From $\mathbf{n}^*\cdot \mathrm{d}\mathbf{r}^* = 0$,

$$2b^{-1}b_u = -\mu(a - a^{-1}), \quad 2a^{-1}a_v = -\frac{1}{\mu}(b + b^{-1}), \tag{4.188}$$

with $\mu = \sec\tau - \tan\tau$.

Let α_1 be a function such that

$$a = \exp\frac{\alpha_1 - \alpha}{2}, \quad b = -\exp\frac{-\alpha_1 - \alpha}{2},$$

then

$$(\alpha_1 + \alpha)_u = 2\mu\sinh\frac{\alpha_1 - \alpha}{2}, \quad (\alpha_1 - \alpha)_v = \frac{2}{\mu}\cosh\frac{\alpha_1 - \alpha}{2}. \tag{4.189}$$

This is the Bäcklund transformation for the cosh-Gordon equation. The integrability condition for α_1 is that α satisfies the cosh-Gordon equation which is the Gauss equation of S and hence is satisfied by α. Therefore, α_1 exists and S^* is obtained via α_1. From the Bäcklund theorem we know that S^* is time-like and has constant positive Gauss curvature $K = 1$. Moreover, by tedious calculation we can verify that (u, v) are the asymptotic Chebyshev coordinates of S^*.

Now we want to get the expression of α_1 by using Darboux transformation.

LEMMA 4.11 *The Lax pair of cosh-Gordon equation is*

$$\Phi_u = U\Phi = \frac{\lambda}{2}\begin{pmatrix} 0 & e^{-\alpha} \\ e^{\alpha} & 0 \end{pmatrix}\Phi, \quad \Phi_v = V\Phi = \frac{1}{2}\begin{pmatrix} -\alpha_v & \lambda^{-1} \\ \lambda^{-1} & \alpha_v \end{pmatrix}\Phi. \tag{4.190}$$

Proof. The proof follows from direct calculations, showing that the integrability condition of (4.190)

$$U_v - V_u + [U, V] = 0$$

is equivalent to the cosh-Gordon equation.

LEMMA 4.12 *(i) If* $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ *is a solution of the Lax pair for* $\lambda = \lambda_0$, *then* $\begin{pmatrix} -h_1 \\ h_2 \end{pmatrix}$ *is a solution of the Lax pair for* $\lambda = -\lambda_0$.

(ii) Suppose λ is purely imaginary and h_2/h_1 is purely imaginary at one point (u_0, v_0), then h_2/h_1 is purely imaginary in a neighborhood of (u_0, v_0).

Proof. (i) is obvious. Now we prove (ii).

From the Lax pair (4.190),

$$\begin{aligned}
\left(\frac{h_2}{h_1}\right)_u &= \frac{\lambda_0}{2}\left(\mathrm{e}^{\alpha} + \mathrm{e}^{-\alpha}\left(\frac{h_2}{h_1}\right)^2\right), \\
\left(\frac{h_2}{h_1}\right)_v &= \frac{1}{2\lambda_0} + \alpha_v\left(\frac{h_2}{h_1}\right) - \frac{1}{2\lambda_0}\left(\frac{h_2}{h_1}\right)^2.
\end{aligned}$$

Let $A = \dfrac{h_2}{h_1} + \dfrac{\bar{h}_2}{\bar{h}_1}$, then

$$\begin{aligned} A_u &= \frac{\bar{\lambda}_0}{2}\mathrm{e}^{-\alpha}\left(\frac{\bar{h}_2}{\bar{h}_1} - \frac{h_2}{h_1}\right)A, \\ A_v &= \alpha_v A + \frac{1}{2\bar{\lambda}_0}\left(\frac{h_2}{h_1} - \frac{\bar{h}_2}{\bar{h}_1}\right)A. \end{aligned}$$

Hence if $A = 0$ at (u_0, v_0), then it holds in a neighborhood of (u_0, v_0). The lemma is proved.

By the general formulae for constructing Darboux matrix, we have

$$\begin{aligned} \Phi_1 &= \left(I - \frac{\lambda}{\lambda_0}H\begin{pmatrix} \lambda_0^{-1} & 0 \\ 0 & -\lambda_0^{-1} \end{pmatrix}H^{-1}\right)\Phi \\ &= \left(I + \frac{\lambda}{\lambda_0}\begin{pmatrix} 0 & h_1/h_2 \\ -h_2/h_1 & 0 \end{pmatrix}\right)\Phi, \end{aligned}$$

where

$$H = \begin{pmatrix} h_1 & -h_1 \\ h_2 & h_2 \end{pmatrix}.$$

From

$$\Phi_{1u} = \frac{\lambda}{2}\begin{pmatrix} 0 & -\mathrm{e}^{\alpha_1} \\ \mathrm{e}^{\alpha_1} & 0 \end{pmatrix}\Phi_1,$$

we get

$$\mathrm{e}^{\alpha_1} = -\left(\frac{h_2}{h_1}\right)^2\mathrm{e}^{-\alpha}.$$

This is the explicit expression of α_1. The explicit expression of S^* can be obtained from $\mathbf{r}^* = \mathbf{r} + l(a\mathbf{e}_1 + b\mathbf{e}_2)$ by using this α_1. It can also be proved that α_1 is a solution of the Bäcklund transformation.

Therefore, the explicit formula for the Bäcklund transformation of S is obtained.

($2\mathrm{b}_2$) Space-like Bäcklund transformation

The expression of S^* is still

$$\mathbf{r}^* = \mathbf{r} + l(a\mathbf{e}_1 + b\mathbf{e}_2),$$

but $ab\mathrm{e}^{\alpha} = 1$ now. The normal vector of S^* is

$$\mathbf{n}^* = (a\mathbf{e}_1 - b\mathbf{e}_2)\sinh\tau + \mathbf{n}\cosh\tau.$$

Then

$$\begin{aligned}\mathbf{r}^*_u &= (1 - la\frac{b_u}{b})\mathbf{e}_1 + (-\mathrm{e}^{-\alpha} + lb_u)\mathbf{e}_2 + \frac{l}{2}(a + a^{-1})\mathbf{n},\\ \mathbf{r}^*_v &= (-\mathrm{e}^{-\alpha} + la_v)\mathbf{e}_1 + (-1 - lb\frac{a_v}{a})\mathbf{e}_2 + \frac{l}{2}(b - b^{-1})\mathbf{n}.\end{aligned} \tag{4.191}$$

Let $l = \sinh\tau$. $\mathbf{n}^* \cdot \mathrm{d}\mathbf{r}^* = 0$ leads to

$$\frac{2b_u}{b} = \mu(a + a^{-1}), \quad \frac{2a_u}{a} = -\frac{1}{\mu}(b - b^{-1}),$$

where $\mu = \operatorname{cosech}\tau - \coth\tau$. Let

$$a = \exp\frac{-\alpha_1 - \alpha}{2}, \quad b = \exp\frac{\alpha_1 - \alpha}{2},$$

then we get

$$(\alpha_1 - \alpha)_u = 2\mu\cosh\frac{\alpha_1 + \alpha}{2}, \quad (\alpha_1 + \alpha)_v = \frac{2}{\mu}\sinh\frac{\alpha_1 - \alpha}{2}, \tag{4.192}$$

which is the Bäcklund transformation for the cosh-Gordon equation. If μ is changed to μ^{-1}, this is just (4.189). Using Darboux transformation, we can get the explicit expression of α_1, which gives the explicit Bäcklund transformation from S to S^*.

(2c) Two principal curvatures of S are equal and there is only one principal direction

Choose the frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ and the parameters (u, v) as the case (3c) in Subsection 4.3.2, then

$$\begin{aligned}&\mathbf{e}_1^2 = \mathbf{e}_2^2 = 0, \quad \mathbf{e}_1 \cdot \mathbf{e}_2 = \frac{1}{2}\mathrm{e}^{\alpha},\\ &\omega^1 = \mathrm{d}u - \mathrm{e}^{-\alpha}\mathrm{d}v, \quad \omega^2 = -\mathrm{d}v,\\ &\omega_3^1 = -\mathrm{d}u - \mathrm{e}^{-\alpha}\mathrm{d}v, \quad \omega_3^2 = \mathrm{d}v.\end{aligned}$$

Hence

$$I = -\mathrm{e}^{\alpha}\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}v^2, \quad II = -\mathrm{e}^{\alpha}\,\mathrm{d}u\,\mathrm{d}v.$$

The Gauss-Codazzi equations become the Liouville equation

$$\alpha_{uv} = \frac{1}{2}\mathrm{e}^{\alpha}.$$

Take the time-like Bäcklund transformation for S as

$$\mathbf{r}^* = \mathbf{r} + l(a\mathbf{e}_1 + b\mathbf{e}_2), \quad ab\mathrm{e}^{\alpha} = -1,$$

then

$$\mathbf{n}^* = (a\mathbf{e}_1 - b\mathbf{e}_2)\cos\tau + \mathbf{n}\sin\tau.$$

From $\mathbf{n}^* \cdot \mathrm{d}\mathbf{r}^* = 0$, we get

$$(\alpha_1 + \alpha)_u = -\mu \exp\frac{\alpha - \alpha_1}{2}, \quad (\alpha_1 - \alpha)_v = \frac{2}{\mu}\cosh\frac{\alpha_1 + \alpha}{2}.$$

Considering the Lax pair

$$\Phi_u = \frac{\lambda}{2}\begin{pmatrix} 0 & 0 \\ \mathrm{e}^{\alpha} & 0 \end{pmatrix}\Phi, \quad \Phi_v = \frac{1}{2}\begin{pmatrix} -\alpha_v & \lambda^{-1} \\ \lambda^{-1} & \alpha_v \end{pmatrix}\Phi \tag{4.193}$$

of the Liouville equation, we can use Darboux transformation to get the explicit expression of S^*.

The space-like Bäcklund transformation is given by

$$\mathbf{r}^* = \mathbf{r} + l(a\mathbf{e}_1 + b\mathbf{e}_2), \quad ab\mathrm{e}^{\alpha} = 1,$$

and S^* can be obtained by Darboux transformation in a similar way.

In summary, we have the following theorem.

THEOREM 4.13 *Suppose S is a time-like surface of constant positive Gauss curvature without umbilic points, then there are time-like and space-like Bäcklund congruences such that S is one of its focal surface, and another focal surface is also a time-like surface of constant positive Gauss curvature.*

(3) Space-like pseudo-spherical congruence with one space-like focal surface and one time-like focal surface

Suppose that S is a time-like surface of constant negative Gauss curvature -1. Choose the Chebyshev coordinates and the corresponding frame as (4) in Subsection 4.3.2, then (4.92) and (4.124) – (4.126) hold. Apply the Bäcklund transformation

$$\begin{aligned} &\mathbf{r}^* = \mathbf{r} + l(\cosh\theta\mathbf{e}_1 + \sinh\theta\mathbf{e}_2), \quad (l = \cosh\tau), \\ &\mathbf{n}^* = \cosh\tau(\sinh\theta\mathbf{e}_1 + \cosh\theta\mathbf{e}_2) + \sinh\tau\mathbf{n}. \end{aligned} \tag{4.194}$$

By the fundamental equations of S, the integrability condition, together with (4.124) – (4.126), we have

$$\begin{aligned} \mathrm{d}\mathbf{r}^* = \ & \Big\{\Big[\cos\frac{\alpha}{2} + l\sinh\theta\Big(\theta_u - \frac{\alpha_v}{2}\Big)\Big]\mathbf{e}_1 + l\cosh\theta\Big(\theta_u - \frac{\alpha_v}{2}\Big)\mathbf{e}_2 \\ & + l\cosh\theta\sin\frac{\alpha}{2}\mathbf{n}\Big\}\,\mathrm{d}u + \Big\{l\sinh\theta(\theta_v + \frac{\alpha_u}{2}\Big)\mathbf{e}_1 \\ & + \Big[\sin\frac{\alpha}{2} + l\cosh\theta\Big(\theta_v + \frac{\alpha_u}{2}\Big)\Big]\mathbf{e}_2 + l\sinh\theta\cos\frac{\alpha}{2}\mathbf{n}\Big\}\,\mathrm{d}v. \end{aligned} \tag{4.195}$$

From $\mathbf{n}^* \cdot \mathrm{d}\mathbf{r}^* = 0$, we have

$$\begin{aligned}
&\sinh\theta\Big(\cos\frac{\alpha}{2} + l\sinh\theta\Big(\theta_u - \frac{\alpha_v}{2}\Big)\Big) \\
&\quad -l\cosh^2\theta\Big(\theta_u - \frac{\alpha_v}{2}) + \sinh\tau\cosh\theta\sin\frac{\alpha}{2} = 0, \\
&l\sinh^2\theta\Big(\theta_v + \frac{\alpha_u}{2}\Big) - \cosh\theta\Big(\sin\frac{\alpha}{2} + l\cosh\theta\Big(\theta_v + \frac{\alpha_u}{2}\Big)\Big) \\
&\quad +\sinh\tau\sinh\theta\cos\frac{\alpha}{2} = 0,
\end{aligned} \tag{4.196}$$

i.e.,

$$\begin{aligned}
l\Big(\theta_u - \frac{\alpha_v}{2}\Big) &= \sinh\theta\cos\frac{\alpha}{2} + \sinh\tau\cosh\theta\sin\frac{\alpha}{2}, \\
l\Big(\theta_v + \frac{\alpha_u}{2}\Big) &= -\cosh\theta\sin\frac{\alpha}{2} + \sinh\tau\sinh\theta\cos\frac{\alpha}{2},
\end{aligned} \tag{4.197}$$

where $l = \cosh\tau$. The integrability condition for θ is

$$\triangle\alpha = \sin\alpha, \tag{4.198}$$

which is satisfied already. Hence for any given initial condition $\theta = \theta_0$ at $u = u_0$ and $v = v_0$, the solution θ of (4.197) exists, and it satisfies

$$\triangle\alpha_1 = \sinh\alpha_1,$$

where $\alpha_1 = \theta/2$. This implies that the corresponding pseudo-spherical congruence exists, and $\mathbf{r}^*$ is a space-like surface of constant negative Gauss curvature.

Theorem 4.14 *There exists a space-like pseudo-spherical congruence with one time-like and one space-like focal surfaces of constant negative Gauss curvature.*

We shall prove that (u, v) are also the Chebyshev coordinates of the space-like surface of constant negative Gauss curvature S^*. In fact, according to (4.195),

$$\mathrm{d}\mathbf{r}^* = \omega^{*1}\mathbf{e}_1^* + \omega^{*2}\mathbf{e}_2^*, \tag{4.199}$$

$$\omega^{*1} = \cosh\theta, \quad \omega^{*2} = \sinh\theta, \tag{4.200}$$

where

$$\begin{aligned}
\mathbf{e}_1^* = &\ \frac{1}{\cosh\theta}\Big\{\Big[\cos\frac{\alpha}{2} + l\sinh\theta\Big(\theta_u - \frac{\alpha_v}{2}\Big)\mathbf{e}_1 \\
&+l\sinh\theta\Big(\theta_u - \frac{\alpha_v}{2}\Big)\mathbf{e}_2 + l\sinh\theta\sin\frac{\alpha}{2}\mathbf{n}\Big\}, \\
\mathbf{e}_2^* = &\ \frac{1}{\sinh\theta}\Big\{l\sinh\theta\Big(\theta_v + \frac{\alpha_u}{2}\Big)\mathbf{e}_1 \\
&+\Big[\sin\frac{\alpha}{2} + l\cosh\theta\Big(\theta_v + \frac{\alpha_u}{2}\Big)\Big]\mathbf{e}_2 + l\sinh\theta\cos\frac{\alpha}{2}\mathbf{n}\Big\}.
\end{aligned} \tag{4.201}$$

From (4.197),

$$\mathbf{e}_1^{*2} = 1, \quad \mathbf{e}_2^{*2} = 1, \quad \mathbf{e}_1^* \cdot \mathbf{e}_2^* = 0. \tag{4.202}$$

Hence

$$\mathrm{d}\mathbf{n}^* = \omega_1^{*3}\mathbf{e}_1^* + \omega_2^{*3}\mathbf{e}_2^*, \tag{4.203}$$

and

$$\omega_1^{*3} = \sinh\theta\,\mathrm{d}u, \quad \omega_2^{*3} = \cosh\theta\,\mathrm{d}v. \tag{4.204}$$

Let $\theta = \alpha_1/2$, then (u, v) are the Chebyshev coordinates of S^*. Moreover, α_1 satisfies

$$\triangle\alpha_1 = \sinh\alpha_1. \tag{4.205}$$

This can also be derived from the differentiation of (4.197).

We can do the same procedure conversely, i.e., starting from a space-like surface of constant negative Gauss curvature S^*, the reverse procedure leads to a time-like surface of constant negative Gauss curvature S.

The Bäcklund transformations described above are not transformations between two solutions of the same sine-Gordon equation or the same sine-Laplace equation. In fact, they transform a solution of the sine-Laplace equation to a solution of the sinh-Laplace equation, and vice versa (the differential form of these transformations has been given in [91]). Applying the transformations twice, we can get a transformation of solutions of the same equation. Darboux transformation will give an explicit form of these transformations.

Introduce the complex variables

$$\zeta = \frac{u + \mathrm{i}v}{2}, \quad \bar{\zeta} = \frac{u - \mathrm{i}v}{2}, \tag{4.206}$$

then

$$\begin{aligned} &\frac{\partial}{\partial\zeta} = \frac{\partial}{\partial u} - \mathrm{i}\frac{\partial}{\partial v}, \quad \frac{\partial}{\partial\bar{\zeta}} = \frac{\partial}{\partial u} + \mathrm{i}\frac{\partial}{\partial v}, \\ &\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}} = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} = \triangle. \end{aligned} \tag{4.207}$$

The integrability condition of the Lax pair

$$\begin{aligned} \Phi_\zeta &= \frac{\lambda}{2}\begin{pmatrix} 0 & -\mathrm{e}^{-\alpha} \\ \mathrm{e}^{\alpha} & 0 \end{pmatrix}\Phi, \\ \Phi_{\bar{\zeta}} &= \frac{1}{2}\begin{pmatrix} -\alpha_{\bar{\zeta}} & 1/\lambda \\ -1/\lambda & \alpha_{\bar{\zeta}} \end{pmatrix}\Phi \end{aligned} \tag{4.208}$$

is

$$\triangle\alpha = \alpha_{\zeta\bar{\zeta}} = \sinh\alpha. \tag{4.209}$$

If α can be complex, this equation is called the complex sinh-Laplace equation. If α is real, it is the usual sinh-Laplace equation. If α is purely imaginary ($\alpha = \mathrm{i}\beta$, β is real), it becomes the sine-Laplace equation

$$\triangle\beta = \beta_{\zeta\bar{\zeta}} = \sin\beta.$$

For the complex sinh-Laplace equation, the Darboux transformation $(\alpha, \Phi) \longrightarrow (\alpha_1, \Phi_1)$ can be constructed similarly as the real sinh-Laplace equation. However, the Lax pair (4.208) is slightly different from (4.170) for simplifying the calculations.

Take $\lambda_1 \neq 0$. Let $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ be a column solution of the Lax pair for $\lambda = \lambda_1$, then $\begin{pmatrix} -h_1 \\ h_2 \end{pmatrix}$ is a column solution of the Lax pair for $\lambda = -\lambda_1$. Let

$$H = \begin{pmatrix} h_1 & -h_1 \\ h_2 & h_2 \end{pmatrix},$$

then

$$S = H \begin{pmatrix} 1/\lambda_1 & 0 \\ 0 & -1/\lambda_1 \end{pmatrix} H^{-1} = \frac{1}{\lambda_1} \begin{pmatrix} 0 & \dfrac{h_1}{h_2} \\ \dfrac{h_2}{h_1} & 0 \end{pmatrix},$$

$$D(\lambda) = I - \lambda S = \begin{pmatrix} 1 & -\dfrac{\lambda}{\lambda_1}\dfrac{h_1}{h_2} \\ -\dfrac{\lambda}{\lambda_1}\dfrac{h_2}{h_1} & 1 \end{pmatrix}.$$

Hence the Darboux transformation is

$$\Phi_1(\lambda) = D(\lambda)\Phi(\lambda).$$

Here α_1 is determined by

$$\mathrm{e}^{\alpha_1} = -\mathrm{e}^{-\alpha} \left(\frac{h_2}{h_1}\right)^2. \tag{4.210}$$

Suppose α is real, then α_1 is purely imaginary if and only if

$$\left|\frac{h_2}{h_1}\right|^2 \mathrm{e}^{-\alpha} = 1. \tag{4.211}$$

It is remained to find h_1 and h_2 such that (4.211) is satisfied. Take $\lambda = \lambda_1$ with $|\lambda_1| = 1$. From the Lax pair,

$$\begin{aligned} &h_{1\zeta} = -\frac{\lambda_1}{2}\mathrm{e}^{-\alpha}h_2, \quad h_{2\zeta} = \frac{\lambda_1}{2}\mathrm{e}^{\alpha}h_1, \\ &h_{1\bar\zeta} = -\frac{1}{2}\alpha_{\bar\zeta}h_1 + \frac{1}{2\lambda_1}h_2, \quad h_{2\bar\zeta} = -\frac{1}{2\lambda_1}h_1 + \frac{\alpha_{\bar\zeta}}{2}h_2. \end{aligned} \tag{4.212}$$

Take the complex conjugation, we get

$$\begin{aligned} &\bar h_{1\zeta} = -\frac{1}{2}\alpha_\zeta \bar h_1 + \frac{1}{2\bar\lambda_1}\bar h_2, \quad \bar h_{2\zeta} = -\frac{1}{2\bar\lambda_1}\bar h_1 + \frac{\alpha_\zeta}{2}\bar h_2, \\ &\bar h_{1\bar\zeta} = -\frac{\bar\lambda_1}{2}\mathrm{e}^{-\alpha}\bar h_2, \quad \bar h_{2\bar\zeta} = \frac{\bar\lambda_1}{2}\mathrm{e}^{\alpha}\bar h_1. \end{aligned} \tag{4.213}$$

Let

$$A = \left|\frac{h_2}{h_1}\right|^2 \mathrm{e}^{-\alpha} - 1 = \frac{h_2\bar h_2}{h_1\bar h_1}\mathrm{e}^{-\alpha} - 1, \tag{4.214}$$

then

$$A_\zeta = -\frac{h_2\bar h_2}{h_1\bar h_1}\mathrm{e}^{-\alpha}\alpha_\zeta + \mathrm{e}^{-\alpha}\left[\frac{h_2\bar h_{2\zeta} + h_{2\zeta}\bar h_2}{h_1\bar h_1} - \frac{h_2\bar h_2(h_1\bar h_{1\zeta} + h_{1\zeta}\bar h_1)}{(h_1\bar h_1)^2}\right].$$

Using (4.212) and (4.213),

$$A_\zeta = \frac{\lambda_1}{2}\left(\frac{h_2}{h_1}\mathrm{e}^{-\alpha} - \frac{\bar h_2}{\bar h_1}\right)A. \tag{4.215}$$

Similarly,

$$A_{\bar\zeta} = \frac{\bar\lambda_1}{2}\left(\frac{h_2}{h_1} - \frac{\bar h_2}{\bar h_1}\mathrm{e}^{-\alpha}\right)A. \tag{4.216}$$

If we choose h_1 and h_2 so that $A = 0$ holds at one point (initial value), then $A = 0$ holds identically. Hence the following theorem is true.

THEOREM 4.15 *Suppose α is real, λ_1 is a complex number with $|\lambda_1| = 1$. Take h_1 and h_2 satisfying (4.211) at some point. Then α_1 derived by the Darboux transformation is purely imaginary.*

This is the Darboux transformation from a solution of the sinh-Laplace equation to a solution of the sine-Laplace equation.

Now suppose α is purely imaginary. From (4.211), α_1 is real if and only if

$$\overline{\left(\frac{h_2}{h_1}\right)^2 \mathrm{e}^{-\alpha}} = \left(\frac{h_2}{h_1}\right)^2 \mathrm{e}^{-\alpha} < 0,$$

since $\mathrm{e}^{\alpha_1} > 0$. From the above equality,

$$\left(\frac{h_2\bar{h}_1}{h_1\bar{h}_2}\mathrm{e}^{-\alpha}\right)^2 = 1,$$

hence

$$\frac{h_2\bar{h}_1}{h_1\bar{h}_2}\mathrm{e}^{-\alpha} = \pm 1.$$

On the other hand,

$$\left(\frac{h_2}{h_1}\right)^2 \mathrm{e}^{-\alpha} = \frac{h_2\bar{h}_2}{h_1\bar{h}_1}\frac{h_2\bar{h}_1}{h_1\bar{h}_2}\mathrm{e}^{-\alpha}.$$

Hence we should choose h_1 and h_2 such that

$$\frac{h_2\bar{h}_1}{h_1\bar{h}_2}\mathrm{e}^{-\alpha} = -1. \tag{4.217}$$

Similar to the proof of Theorem 4.15, by taking λ_1 such that $|\lambda_1| = 1$, (4.217) holds everywhere if it holds at one point. This leads to the following theorem.

THEOREM 4.16 *Suppose α is a purely imaginary solution of (4.209), λ_1 is a complex number with $|\lambda_1| = 1$. Take h_1 and h_2 satisfying (4.217) at some point. Then the Darboux transformation provides a real α_1 which is a solution of the sinh-Laplace equation.*

Using the above two kinds of Darboux transformations, we get the following series of Darboux transformations

$$(\alpha, \Phi) \longrightarrow (\alpha_1, \Phi_1) \longrightarrow (\alpha_2, \Phi_2) \longrightarrow \cdots.$$

If α is purely imaginary and $\beta = -\mathrm{i}\alpha$ is a solution of the sine-Laplace equation, then α_1, α_3, $\cdots$, α_{2n+1}, $\cdots$ are solutions of the sinh-Laplace equation, and $\beta_2 = -\mathrm{i}\alpha_2$, $\beta_4 = -\mathrm{i}\alpha_4$, $\cdots$, $\beta_{2n} = -\mathrm{i}\alpha_{2n}$, $\cdots$ are solutions of the sine-Laplace equation. On the other hand, if α is a solution of the sinh-Laplace equation, then all α_{2n} $(n = 1, 2, 3, \cdots)$ are solutions of the sinh-Laplace equation, and all $\beta_{2n+1} = -\mathrm{i}\alpha_{2n+1}$ $(n = 0, 1, 2, \cdots)$ are solutions of the sine-Laplace equation. This can be shown by the following figure:

$$\begin{array}{ccccccccccc}
\alpha_0 & & \widetilde{\alpha}_1 & & \alpha_2 & & \widetilde{\alpha}_3 & & \alpha_4 & & \cdots \\
 & \times & & \times & & \times & & \times & & \times & \\
\widetilde{\alpha}_0 & & \alpha_1 & & \widetilde{\alpha}_2 & & \alpha_3 & & \widetilde{\alpha}_4 & & \cdots
\end{array}$$

α_i and $\widetilde{\alpha}_{i-1}$ are real for odd i and purely imaginary for even i.

Now we prove that this Darboux transformation is consistent with the Bäcklund transformation (4.197).

Rewrite (4.210) as

$$\mathrm{e}^{(\alpha_1+\alpha)/2} = \frac{h_2}{h_1}. \tag{4.218}$$

Differentiating (4.218) with respect to ζ and using (4.212), we get

$$\frac{(\alpha_1+\alpha)_\zeta}{2} = \lambda_1 \sinh(\alpha-\alpha_1). \tag{4.219}$$

Similarly, differentiating (4.218) with respect to $\bar{\zeta}$ and using (4.212), we get

$$\frac{(\alpha_1-\alpha)_\zeta}{2} = \frac{1}{\lambda_1}\sinh\frac{\alpha+\alpha_1}{2}. \tag{4.220}$$

Let $\theta = \dfrac{\alpha_1}{2}$, $\lambda_1 = \dfrac{1+\mathrm{i}\sinh\tau}{\cosh\tau}$, then from (4.197), we obtain (4.219) and (4.220). Therefore, the function α_1 derived from the Darboux transformation is a solution of (4.219) and the construction of the Bäcklund congruence from a time-like surface of constant negative Gauss curvature is completed. This gives the following theorem.

THEOREM 4.17 *Starting from one focal surface, the Bäcklund congruence of type (d) together with its another focal surface can be constructed by using Darboux transformation (4.210) and Bäcklund transformation (4.194),*

Example:

Take the trivial solution $\alpha = 0$. The Lax pair is

$$\Phi_\zeta = \frac{\lambda}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\Phi, \quad \Phi_{\bar{\zeta}} = \frac{1}{2}\begin{pmatrix} 0 & 1/\lambda \\ -1/\lambda & 0 \end{pmatrix}\Phi. \tag{4.221}$$

Its fundamental solution is

$$\Phi(\lambda) = \begin{pmatrix} \mathrm{e}^{\gamma} & \mathrm{e}^{-\gamma} \\ -\mathrm{i}\mathrm{e}^{\gamma} & \mathrm{i}\mathrm{e}^{-\gamma} \end{pmatrix}, \quad \gamma = \Big(\frac{\lambda}{2}\zeta - \frac{1}{2\lambda}\bar{\zeta}\Big)\mathrm{i}. \tag{4.222}$$

Using the Darboux transformation, we can get α_1. Since $\alpha = 0$ can be regarded as a real or a purely imaginary solution, we can get purely imaginary or real α_1 from $\alpha = 0$.

First we want to find real α_1. Let $\lambda = \lambda_1$ with $|\lambda_1| = 1$, then

$$\gamma_1 = \Big(\frac{\lambda_1}{2}\zeta - \frac{1}{2\lambda_1}\bar{\zeta}\Big)\mathrm{i} \tag{4.223}$$

is real. Let $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ be a column solution of the Lax pair for $\lambda = \lambda_1$, i.e.,

$$h_1 = \mathrm{e}^{\gamma_1} + b\mathrm{e}^{-\gamma_1}, \quad h_2 = -\mathrm{i}\mathrm{e}^{\gamma_1} + b\mathrm{i}\mathrm{e}^{-\gamma_1}. \tag{4.224}$$

Take b to be real, then

$$\frac{h_2\bar{h}_1}{h_1\bar{h}_2} = -1.$$

Suppose b is positive, then it can be taken as 1 by adding a constant on γ_1. From (4.210),

$$\mathrm{e}^{\alpha_1/2} = \mathrm{i}\frac{h_2}{h_1} = \frac{\mathrm{e}^{\gamma_1} - \mathrm{e}^{-\gamma_1}}{\mathrm{e}^{\gamma_1} + \mathrm{e}^{-\gamma_1}} = \tanh\gamma_1, \quad (\gamma_1 > 0),$$

and

$$\mathrm{e}^{-\alpha_1/2} = \coth\gamma_1,$$

$$\cosh\frac{\alpha_1}{2} = \coth(2\gamma_1), \quad \sinh\frac{\alpha_1}{2} = -\ \mathrm{cosech}(2\gamma_1). \tag{4.225}$$

When $\gamma_1 < 0$, the right hand sides of (4.225) should change signs.

$\alpha = 0$ does not correspond to a time-like surface, but only a system of orthogonal frames along a straight line. In fact, from (4.124) – (4.126),

$$\mathrm{d}\mathbf{r} = \mathrm{d}u\mathbf{e}_1, \quad \mathrm{d}\mathbf{e}_1 = 0, \quad \mathrm{d}\mathbf{e}_2 = \mathrm{d}v\mathbf{n}, \quad \mathrm{d}\mathbf{n} = \mathrm{d}v\mathbf{e}_2, \tag{4.226}$$

hence

$$\begin{aligned} &\mathbf{r} = (u, 0, 0), \quad \mathbf{e}_1 = (1, 0, 0), \\ &\mathbf{e}_2 = (0, \sinh v, \cosh v), \quad \mathbf{n} = (0, \cosh v, \sinh v). \end{aligned} \tag{4.227}$$

According to (4.194), the space-like surface of constant negative Gauss curvature is

$$\begin{aligned} \mathbf{r}^* = \ & (u + l\coth(2\gamma_1), -l\ \mathrm{cosech}(2\gamma_1)\sinh v, \\ & -l\ \mathrm{cosech}(2\gamma_1)\cosh v). \end{aligned} \tag{4.228}$$

According to the general theory, $\alpha_1 = -2\tanh^{-1}(\,\mathrm{sech}(2\gamma_1))$ is a solution of the sinh-Laplace equation. It is defined on the (u, v) plane except for the straight line $\gamma_1 = 0$. When $\gamma_1 \to 0$, the point on the surface tends to infinity.

Now we apply the Darboux transformation again.

$$\begin{aligned}\Phi_1(\lambda) &= D(\lambda)\Phi(\lambda)\\ &= \left(I - \frac{\lambda}{\lambda_1}\begin{pmatrix} 0 & \dfrac{h_1}{h_2} \\ \dfrac{h_2}{h_1} & 0 \end{pmatrix}\right)\begin{pmatrix} \mathrm{e}^{\gamma} & \mathrm{e}^{-\gamma} \\ -\mathrm{i}\mathrm{e}^{\gamma} & \mathrm{i}\mathrm{e}^{-\gamma} \end{pmatrix}\\ &= \begin{pmatrix} \left(1+\dfrac{\lambda}{\lambda_1}\coth\gamma_1\right)\mathrm{e}^{\gamma} & \left(1-\dfrac{\lambda}{\lambda_1}\coth\gamma_1\right)\mathrm{e}^{-\gamma} \\ -\mathrm{i}\left(1+\dfrac{\lambda}{\lambda_1}\tanh\gamma_1\right)\mathrm{e}^{\gamma} & \mathrm{i}\left(1-\dfrac{\lambda}{\lambda_1}\tanh\gamma_1\right)\mathrm{e}^{-\gamma} \end{pmatrix}.\end{aligned} \tag{4.229}$$

Take λ_2 with $|\lambda_2| = 1$ and let $\lambda_2/\lambda_1 = \mathrm{e}^{\mathrm{i}\mu}$ (μ is real), then

$$\begin{aligned} h_1' &= (1+\mathrm{e}^{\mathrm{i}\mu}\coth\gamma_1)\mathrm{e}^{\gamma_2} + b(1-\mathrm{e}^{\mathrm{i}\mu}\coth\gamma_1)\mathrm{e}^{-\gamma_2},\\ h_2' &= i(-1-\mathrm{e}^{\mathrm{i}\mu}\tanh\gamma_1)\mathrm{e}^{\gamma_2} + \mathrm{i}b(1-\mathrm{e}^{\mathrm{i}\mu}\tanh\gamma_1)\mathrm{e}^{-\gamma_2}, \end{aligned} \tag{4.230}$$

where

$$\gamma_2 = \left(\frac{\lambda_2}{2}\zeta - \frac{1}{2\lambda_2}\bar{\zeta}\right)\mathrm{i} \tag{4.231}$$

is real. In order that the derived solution α_2 is purely imaginary and $\beta = -\mathrm{i}\alpha_2$ satisfies the sine-Laplace equation, (4.211) should hold. This can be done if b is chosen to be a real number. In fact, when b is real,

$$\bar{h}_1' = ih_2'\coth\gamma_1\mathrm{e}^{\mathrm{i}\mu}, \tag{4.232}$$

$$\left|\frac{h_2'}{h_1'}\right|^2 \mathrm{e}^{-\alpha_1} = \frac{1}{\coth^2\gamma_1}\coth^2\gamma_1 = 1.$$

In (4.210), if α is replaced by α_1, $\dfrac{h_2}{h_1}$ is replaced by $\dfrac{h_2'}{h_1'}$, then $\beta = -\mathrm{i}\alpha_2$ is a solution of the sine-Laplace equation. From the explicit expressions of $\cos\dfrac{\beta}{2}$, $\sin\dfrac{\beta}{2}$, the Bäcklund transformation can be constructed explicitly and the time-like surface of constant negative Gauss curvature is obtained.

Now we construct purely imaginary α_1 from $\alpha = 0$. By the fundamental solution (4.222) of the Lax pair corresponding to $\alpha = 0$, take the column solution $\begin{pmatrix} h_1 \\ h_1 \end{pmatrix}$ such that

$$\left|\frac{h_2}{h_1}\right|^2 = 1, \tag{4.233}$$

i.e.,

$$\frac{e^{2\gamma_1}+|b|^2e^{-2\gamma_1}-(b+\bar{b})}{e^{2\gamma_1}+|b|^2e^{-2\gamma_1}+(b+\bar{b})}=1. \tag{4.234}$$

Hence b must be purely imaginary. Since $|b|$ can be transformed to 1 by adding a constant to γ_1, we can choose $b=\pm\mathrm{i}$. Suppose $b=\mathrm{i}$, then

$$h_1=e^{\gamma_1}+\mathrm{i}e^{-\gamma_1},\quad h_2=e^{\gamma_1}-\mathrm{i}e^{-\gamma_1}. \tag{4.235}$$

Hence

$$\begin{aligned}e^{\alpha_1/2}&=\mathrm{i}\frac{h_2}{h_1}=\operatorname{sech}(2\gamma_1)+\mathrm{i}\tanh(2\gamma_1),\\ e^{-\alpha_1/2}&=-\mathrm{i}\frac{h_1}{h_2}=\operatorname{sech}(2\gamma_1)-\mathrm{i}\tanh(2\gamma_1).\end{aligned}$$

Let $\alpha_1=\mathrm{i}\beta_1$, then

$$\begin{aligned}&\cos\frac{\beta_1}{2}=\operatorname{sech}(2\gamma_1),\quad \sin\frac{\beta_1}{2}=\tanh(2\gamma_1),\\ &\frac{\beta_1}{2}=\cos^{-1}(\operatorname{sech}(2\gamma_1)),\end{aligned} \tag{4.236}$$

β_1 is a solution of the sine-Laplace equation. The Darboux matrix is

$$D(\lambda)=I-\frac{\lambda}{\lambda_1}\begin{pmatrix}0 & \mathrm{i}e^{-\alpha_1/2}\\ -\mathrm{i}e^{\alpha_1/2} & 0\end{pmatrix} \tag{4.237}$$

and the fundamental solution is

$$\begin{aligned}\Phi_1(\lambda)&=D(\lambda)\Phi(\lambda)\\ &=\begin{pmatrix}e^{\gamma}(1-\frac{\lambda}{\lambda_1}e^{-\alpha_1/2}) & e^{-\gamma}(1+\frac{\lambda}{\lambda_1}e^{-\alpha_1/2})\\ -\mathrm{i}e^{\gamma}(1-\frac{\lambda}{\lambda_1}e^{\alpha_1/2}) & \mathrm{i}e^{-\gamma}(1+\frac{\lambda}{\lambda_1}e^{\alpha_1/2})\end{pmatrix}.\end{aligned} \tag{4.238}$$

Take λ_2 such that $|\lambda_2|=1$. Let $\lambda_2/\lambda_1=e^{\mathrm{i}\mu}$ (μ is real) and γ_2 as before. Let

$$\begin{aligned}h_1'&=e^{\gamma_2}\left(1-e^{\mathrm{i}\mu}e^{-\alpha_1/2}\right)+be^{-\gamma_2}\left(1+e^{\mathrm{i}\mu}e^{-\alpha_1/2}\right),\\ h_2'&=-\mathrm{i}e^{\gamma_2}\left(1-e^{\mathrm{i}\mu}e^{\alpha_1/2}\right)+ibe^{-\gamma_2}\left(1+e^{\mathrm{i}\mu}e^{\alpha_1/2}\right).\end{aligned} \tag{4.239}$$

Take b to be purely imaginary, then

$$\bar{h}_1'=-h_1'e^{-\mathrm{i}\mu}e^{\alpha_1/2},\quad \bar{h}_2'=h_2'e^{-\mathrm{i}\mu}e^{-\alpha_1/2}, \tag{4.240}$$

hence

$$\frac{h_2'\bar{h}_1'}{h_1'\bar{h}_2'}e^{-\alpha_1}=-1.$$

This implies that

$$e^{\alpha_2/2} = i\frac{h_2'}{h_1'}e^{-\alpha_1/2}, \quad e^{-\alpha_2/2} = i\frac{h_1'}{h_2'}e^{\alpha_1/2} \tag{4.241}$$

are real positive functions, and α_2 satisfies the sinh-Laplace equation. From α_1 and α, one can use the similar method as in the last subsection to construct Bäcklund congruences as well as the space-like and time-like surfaces of constant negative Gauss curvature.

4.4 Orthogonal frame and Lax pair

In this section, we consider the relation between the orthogonal frame of a surface of constant negative Gauss curvature in $\mathbf{R}^3$ and the Lax pair of the sine-Gordon equation. The geometric meaning of the Lax pair is elucidated clearly.

The group $SU(2)$ consists of all 2×2 matrices $\mathcal{A}$ satisfying $\mathcal{A}^*\mathcal{A} = I$ and $\det\mathcal{A} = 1$. Its general element is of form

$$\mathcal{A} = \begin{pmatrix} a_0 + a_1 i & a_2 + a_3 i \\ -a_2 + a_3 i & a_0 - a_1 i \end{pmatrix} \tag{4.242}$$

where a_0, a_1, a_2, a_3 are real numbers with $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$. The Lie algebra $su(2)$ consists of all 2×2 matrices A satisfying $A^* + A = 0$ and $\operatorname{tr} A = 0$. Its general element is of form

$$A = \begin{pmatrix} \beta i & \gamma + \delta i \\ -\gamma + \delta i & -\beta i \end{pmatrix} \tag{4.243}$$

where β, γ, δ are real numbers.

The group $SO(3)$ consists of all 3×3 real orthogonal matrices with determinant 1. Its Lie algebra $so(3)$ consists of all 3×3 anti-symmetric real matrices. The correspondence

$$\sigma : \frac{1}{2}\begin{pmatrix} \beta i & \gamma + \delta i \\ -\gamma + \delta i & -\beta i \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0 & \beta & \delta \\ -\beta & 0 & \gamma \\ -\delta & -\gamma & 0 \end{pmatrix} \tag{4.244}$$

is an isomorphism between $su(2)$ and $so(3)$, i.e.,

$$[\sigma A, \sigma B] = \sigma[A, B] \tag{4.245}$$

for all $A, B \in su(2)$. This isomorphism of Lie algebras can be lifted to an isomorphism of the group $SU(2)$ to the double covering of $SO(3)$ (see Lemma 4.18 and Lemma 4.19).

We turn to discuss the geometrical meaning of the fundamental solution Φ to the Lax pair of the sine-Gordon equation. According to (4.33), under the Chebyshev coordinates, there exist orthogonal frames of a surface S of constant negative Gauss curvature satisfying

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix}_u = \begin{pmatrix} 0 & \frac{\alpha_v}{2} & \sin\frac{\alpha}{2} \\ -\frac{\alpha_v}{2} & 0 & 0 \\ -\sin\frac{\alpha}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix}_v = \begin{pmatrix} 0 & \frac{\alpha_u}{2} & 0 \\ -\frac{\alpha_u}{2} & 0 & -\cos\frac{\alpha}{2} \\ 0 & \cos\frac{\alpha}{2} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix}. \tag{4.246}$$

These frames are called Chebyshev frames. Using the isomorphism σ^{-1}, we derive a system of equations in 2×2 matrices

$$\Psi_u = \frac{1}{2}\begin{pmatrix} \frac{\alpha_v}{2}\mathrm{i} & \mathrm{i}\sin\frac{\alpha}{2} \\ \mathrm{i}\sin\frac{\alpha}{2} & -\frac{\alpha_v}{2}\mathrm{i} \end{pmatrix}\Psi,$$
$$\Psi_v = \frac{1}{2}\begin{pmatrix} \frac{\alpha_u}{2}\mathrm{i} & -\cos\frac{\alpha}{2} \\ \cos\frac{\alpha}{2} & -\frac{\alpha_u}{2}\mathrm{i} \end{pmatrix}\Psi, \tag{4.247}$$

where Ψ is a 2×2 matrix in $SU(2)$.

Let

$$\xi = \frac{u+v}{2}, \quad \eta = \frac{u-v}{2}. \tag{4.248}$$

We have

$$\Psi_\xi = \Psi_u + \Psi_v = \frac{1}{2}\begin{pmatrix} \frac{\alpha_\xi}{2}\mathrm{i} & -\mathrm{e}^{-\alpha\mathrm{i}/2} \\ \mathrm{e}^{\alpha\mathrm{i}/2} & -\frac{\alpha_\xi}{2}\mathrm{i} \end{pmatrix}\Psi,$$
$$\Psi_\eta = \Psi_u - \Psi_v = \frac{1}{2}\begin{pmatrix} -\frac{\alpha_\eta}{2}\mathrm{i} & \mathrm{e}^{\alpha\mathrm{i}/2} \\ -\mathrm{e}^{-\alpha\mathrm{i}/2} & \frac{\alpha_\eta}{2}\mathrm{i} \end{pmatrix}\Psi. \tag{4.249}$$

Define

$$\Phi(\xi,\eta) = \begin{pmatrix} \mathrm{e}^{-\alpha\mathrm{i}/4} & 0 \\ 0 & \mathrm{e}^{\alpha\mathrm{i}/4} \end{pmatrix}\Psi, \tag{4.250}$$

(4.249) becomes

$$
\begin{aligned}
\Phi_\xi &= \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{e}^{-\alpha\mathrm{i}} \\ \mathrm{e}^{\alpha\mathrm{i}} & 0 \end{pmatrix}\Phi, \\
\Phi_\eta &= \frac{1}{2}\begin{pmatrix} -\alpha_\eta\mathrm{i} & 1 \\ -1 & \alpha_\eta\mathrm{i} \end{pmatrix}\Phi.
\end{aligned}
\tag{4.251}
$$

Let $\xi = \mu\xi_1$, $\eta = \dfrac{1}{\mu}\eta_1$ and denote ξ_1 and η_1 by ξ and η again, we can introduce a spectral parameter μ in (4.251) formally. Thus we have an alternative form of the Lax pair of the sine-Gordon equation with spectral parameter μ:

$$
\begin{aligned}
\Phi_\xi &= \frac{\mu}{2}\begin{pmatrix} 0 & -\mathrm{e}^{-\alpha\mathrm{i}} \\ \mathrm{e}^{\alpha\mathrm{i}} & 0 \end{pmatrix}\Phi, \\
\Phi_\eta &= \frac{1}{2}\begin{pmatrix} -\alpha_\eta\mathrm{i} & 1/\mu \\ -1/\mu & \alpha_\eta\mathrm{i} \end{pmatrix}\Phi.
\end{aligned}
\tag{4.252}
$$

Remark 30 By using the isomorphism of $su(2)$

$$
\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix},
$$
$$
\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix},
$$

and changing λ *to* $\dfrac{\mathrm{i}}{2\mu}$, *the Lax pair (4.67) becomes (4.252).*

In applying Darboux transformation we can use the Lax pair (4.252) as well as (4.67), but one should note that the λ in (4.67) and μ in (4.252) are related by $\lambda = \dfrac{\mathrm{i}}{2\mu}$, i.e., if we use real λ_1 to construct the Darboux matrix for (4.67), then we should use purely imaginary μ_1 for (4.252).

In order to find out the relation between the Chebyshev frames and the fundamental solution of the Lax pair, we need the map from $SU(2)$ to $SO(3)$ corresponding to the isomorphism (4.244) between their Lie algebras.

LEMMA 4.18 *Let $\mathcal{A} \in SU(2)$ be defined by (4.242). The map*

$$\tau(\mathcal{A}) = \begin{pmatrix} a_0^2+a_2^2-a_3^2-a_1^2 & 2(a_0a_1-a_2a_3) & 2(a_0a_3+a_1a_2) \\ -2(a_0a_1+a_2a_3) & a_0^2+a_3^2-a_2^2-a_1^2 & 2(a_0a_2-a_1a_3) \\ 2(-a_0a_3+a_1a_2) & 2(-a_0a_2-a_3a_1) & a_0^2-a_3^2-a_2^2+a_1^2 \end{pmatrix} \tag{4.253}$$

is an isomorphism from $SU(2)$ to the double covering of $SO(3)$. If Ψ satisfies (4.249), then $\tau(\Psi)$ satisfies (4.246).

Proof. It is necessary to prove:
(i) $\tau(\mathcal{A}) \in SO(3)$;
(ii) τ is a homomorphism, i.e., $\tau(\mathcal{A}\mathcal{B}) = \tau(\mathcal{A})\tau(\mathcal{B})$;
(iii) $\tau(\mathcal{A}) = \tau(\mathcal{B})$ if and only if $\mathcal{B} = \pm\mathcal{A}$;
(iv) $\tau : SU(2) \to SO(3)$ induces an isomorphism σ between their Lie algebras. Moreover, if $\mathcal{A}$ is a matrix function of certain parameters, then $(\mathrm{d}\tau(\mathcal{A}))(\tau(\mathcal{A}))^{-1} = \sigma((\mathrm{d}\mathcal{A})\mathcal{A}^{-1})$.

These are the basic facts on the relationship between $SU(2)$ and $SO(3)$. They can all be verified by direct calculations.

LEMMA 4.19 *Suppose $\tau(\mathcal{A}) = (a_{ij}) \in SO(3)$, then $\mathcal{A}$ is given by*

$$\begin{aligned} &a_0^2 = \frac{1}{4}(1 + a_{11} + a_{22} + a_{33}), \quad a_1^2 = \frac{1}{4}(1 + a_{33} - a_{11} - a_{22}), \\ &a_2^2 = \frac{1}{4}(1 + a_{11} - a_{22} - a_{33}), \quad a_3^2 = \frac{1}{4}(1 + a_{22} - a_{11} - a_{33}), \\ &a_0a_1 = \frac{1}{4}(a_{12} - a_{21}), \quad a_0a_2 = \frac{1}{4}(a_{23} - a_{32}), \\ &a_0a_3 = \frac{1}{4}(a_{13} - a_{31}). \end{aligned} \tag{4.254}$$

If the sign of any one of a_i (say a_0) is fixed, the signs of the other a_i's are also fixed. Hereafter, unless otherwise stated, $\tau^{-1}(A)$ is referred to one branch of this double covering.

Now we can discuss the relation between the solution of the Lax pair related to the solution α of the sine-Gordon equation and the Chebyshev frame of the surface S of constant negative Gauss curvature corresponding to α.

Let S be the surface with $K = -1$ and related with the solution $\alpha(\xi, \eta)$ of the sine-Gordon equation, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ be the Chebyshev frames and $\Phi(1, \xi, \eta)$ be the fundamental solution of the Lax pair with $\mu = 1$.

From the above discussion, we have

$$\Phi(1,\xi,\eta) = \begin{pmatrix} \mathrm{e}^{-\alpha\mathrm{i}/4} & 0 \\ 0 & \mathrm{e}^{\alpha\mathrm{i}/4} \end{pmatrix} \tau^{-1}\left(\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix}\right). \tag{4.255}$$

and

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix} = \tau\left(\begin{pmatrix} \mathrm{e}^{-\alpha\mathrm{i}/4} & 0 \\ 0 & \mathrm{e}^{\alpha\mathrm{i}/4} \end{pmatrix} \Phi(1,\xi,\eta)\right). \tag{4.256}$$

(4.255) and (4.256) give the algebraic relation between $\Phi(1,\xi,\eta)$ and the Chebyshev frame, so we can say $\Phi(1,\xi,\eta)$ is an $SU(2)$ representation of the Chebyshev frame.

We turn to the geometrical meaning of the fundamental solution $\Phi(\mu,\xi,\eta)$. From (4.252) it is easily seen that $\widetilde{\Phi} = \Phi\Big(\mu, \dfrac{\xi}{\mu}, \mu\eta\Big)$ (μ is real and non-zero) satisfies

$$\widetilde{\Phi}_\xi = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{e}^{-\alpha_\mu \mathrm{i}} \\ \mathrm{e}^{\alpha_\mu \mathrm{i}} & 0 \end{pmatrix}\widetilde{\Phi}, \quad \widetilde{\Phi}_\eta = \frac{1}{2}\begin{pmatrix} -\alpha_{\mu\eta}\mathrm{i} & 1 \\ -1 & \alpha_{\mu\eta}\mathrm{i} \end{pmatrix}\widetilde{\Phi}. \tag{4.257}$$

Here $\alpha_\mu = \alpha\Big(\mu\xi, \dfrac{\eta}{\mu}\Big)$ is another solution of the sine-Gordon equation. The corresponding surface $S_{\mu^{-1}}$ with $K=-1$ is called the Lie transformation of S [26]. Comparing (4.257) with (4.251) we obtain

THEOREM 4.20 $\Phi\Big(\mu, \dfrac{\xi}{\mu}, \mu\eta\Big)$ *is an* $SU(2)$ *representation of the surface* $S_{\mu^{-1}}$.

Remark 31 When α *and* $\Phi(1,\xi,\eta)$*, or equivalently the Chebyshev frame, is known, the surface* $\mathbf{r} = \mathbf{r}(\xi,\eta)$ *can be determined by direct integration. In fact, the right hand side of the equation*

$$\mathrm{d}\mathbf{r} = \cos\frac{\alpha}{2}\,\mathrm{d}u\mathbf{e}_1 + \sin\frac{\alpha}{2}\,\mathrm{d}v\mathbf{e}_2, \tag{4.258}$$

is known already and its exterior derivative is zero. Hence $\mathbf{r}$ *can be determined by the integration of (4.258).*

In conclusion, if a solution α of the sine-Gordon equation is known, then a surface S with $K=-1$ and the fundamental solution of the Lax pair together with the Lie transformation S_μ can be obtained by solving a system of linear partial differential equations, or simply by integration. Having these data, by using Darboux transformation, a series of surfaces of $K = -1$ together with their Lie transformations can be obtained by purely algebraic algorithm.

Similarly, for various surfaces of constant Gauss curvature in $\mathbf{R}^{2,1}$, we can get the relation between the Chebyshev frames and the solutions of the Lax pair, i.e. the relation between the Lax pair and the Chebyshev frames of the family of surfaces obtained by the Lie transformation from the seed surface. In this case, we need the isomorphism between the Lie algebras $su(1,1)$ and $so(2,1)$. Here $su(1,1)$ is the set of 2×2 matrix A satisfying

$$A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A^* = 0,$$

and $so(2,1)$ is the Lie algebra of the Lorentz group on $\mathbf{R}^{2,1}$ with determinant 1.

4.5 Surface of constant mean curvature

Recently, the surface of constant mean curvature is studied widely. For example, the surface of constant mean curvature immersed in Euclidean space and homeomorphic to torus is constructed which gives an answer to the Hopf conjecture [110]. In this section we will use Darboux transformation to construct a series of surfaces of constant mean curvature from a known surface of constant mean curvature by purely algebraic algorithm.

4.5.1 Parallel surface in Euclidean space

Suppose S is a surface in $\mathbf{R}^3$. By moving each point on S a distance l along the normal direction, we obtain another surface, which is a parallel surface of S. Take the orthogonal frame such that $\mathbf{e}_1$, $\mathbf{e}_2$ are tangent to the lines of curvature, then the fundamental equations of S are

$$\begin{aligned} &\mathrm{d}\mathbf{r} = \omega^a \mathbf{e}_a \quad (a = 1, 2), \\ &\mathrm{d}\mathbf{e}_a = \omega_a^b \mathbf{e}_b + \omega_a^3 \mathbf{n}, \\ &\mathrm{d}\mathbf{n} = \omega_3^a \mathbf{e}_a. \end{aligned} \tag{4.259}$$

Since $\mathbf{e}_1$ and $\mathbf{e}_2$ are tangent to the lines of curvature,

$$\omega_1^3 = b_{11}\omega^1, \quad \omega_2^3 = b_{22}\omega^2, \tag{4.260}$$

where b_{11} and b_{22} are principal curvatures of the lines of curvature.

Suppose $S(l)$ is the parallel surface of S whose distance to S is l, then its position vector is

$$\mathbf{r}^* = \mathbf{r} + l\mathbf{n}. \tag{4.261}$$

By differentiation, we get

$$\mathrm{d}\mathbf{r}^* = \omega^{*a}\mathbf{e}_a, \tag{4.262}$$

where

$$\omega^{*a} = \omega^a + l\omega_3^a = (1 - lb_{aa})\omega^a. \tag{4.263}$$

Hence $\mathbf{e}_1$ and $\mathbf{e}_2$ are still the tangent vectors of $S(l)$, and $\mathbf{n}$ is the normal vector. When $1 - lb_{11} \neq 0$ and $1 - lb_{22} \neq 0$ hold, the surface $S(l)$ is regular, and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$ is an orthonormal frame of $S(l)$ at $\mathbf{r}^*$. Moreover,

$$\omega_2^{*1} = \omega_2^1, \quad \omega_1^{*3} = \omega_1^3 = b_{11}\omega^1, \quad \omega_2^{*3} = \omega_2^3 = b_{22}\omega^2. \tag{4.264}$$

On the other hand, ω_1^{*3} and ω_2^{*3} can be written as the linear combinations of ω^{*1} and ω^{*2}:

$$\begin{aligned}
&\omega_1^{*3} = b_{11}^*\omega^{*1} + b_{12}^*\omega^{*2} = b_{11}^*\omega^1 - lb_{11}^*\omega_1^3 + b_{12}^*\omega^{*2},\\
&\omega_2^{*3} = b_{22}^*\omega^2 - lb_{22}^*\omega_2^3 + b_{21}^*\omega^{*1}.
\end{aligned} \tag{4.265}$$

Hence

$$b_{11}^* - lb_{11}^*b_{11} = b_{11}, \quad b_{22}^* - lb_{22}^*b_{22} = b_{22}, \quad b_{12}^* = b_{21}^* = 0, \tag{4.266}$$

which gives

$$b_{11}^* = \frac{b_{11}}{1 - lb_{11}}, \quad b_{22}^* = \frac{b_{22}}{1 - lb_{22}}. \tag{4.267}$$

This means that $\mathbf{e}_1$ and $\mathbf{e}_2$ are still unit vectors tangent to the lines of curvature on $S(l)$, and the principal curvatures are b_{11}^* and b_{22}^*. The Gauss curvature and the mean curvature of the parallel surface $S(l)$ are

$$K^* = \frac{b_{11}b_{22}}{(1 - lb_{11})(1 - lb_{22})} = \frac{K}{1 - 2lH + l^2K}, \tag{4.268}$$

$$H^* = \frac{H - lK}{1 - 2lH + l^2K}, \tag{4.269}$$

respectively, where $H = \frac{1}{2}(b_{11} + b_{22})$ is the mean curvature of the surface S. The most simple surfaces of constant mean curvature are spheres and cylinders. We shall not consider these trivial cases.

(4.269) implies

$$H(-2lH^* - 1) = -lK - H^*(1 + l^2K). \tag{4.270}$$

Suppose that K is a constant, H is not a constant. We want to find $S(l)$ such that H^* is a constant. The right hand side of (4.270) is a constant. Hence

$$H^* = -\frac{1}{2l}, \quad 1 - l^2 K = 0. \tag{4.271}$$

Therefore, $K > 0$, $l = \pm\sqrt{\dfrac{1}{K}}$. Conversely, if the mean curvature H of the surface S is a constant, then from (4.268),

$$K(1 - l^2 K^*) = K^*(1 - 2lH). \tag{4.272}$$

If K is not a constant and K^* is a constant, then the right hand side of (4.272) is a constant. However, since K is not a constant, we should have $l = \dfrac{1}{2H}$, $K^* = \dfrac{1}{l^2}$. This gives the proof of the well-known facts:

THEOREM 4.21 *If the surface S of constant positive Gauss curvature $K = \dfrac{1}{l^2}$ is not a sphere, then on each side of S, there is a parallel surface of constant mean curvature $H^* = \pm\dfrac{1}{2l}$ whose distance to S is l. If the surface S of constant mean curvature $H = \dfrac{1}{2l}$ is neither sphere nor cylinder, then there is a parallel surface of constant positive Gauss curvature $K = \dfrac{1}{l^2}$ whose distance to S is l.*

Therefore, in the Euclidean space, the construction of the surface of constant mean curvature and the construction of surface of constant positive Gauss curvature are equivalent in local sense.

4.5.2 Construction of surfaces

We shall consider the construction of surfaces of constant positive Gauss curvature. Although there is no Bäcklund congruence, Darboux transformation can still be used.

As mentioned before, if we choose the Chebyshev coordinates of a surface of constant positive Gauss curvature, then (4.38) holds, hence

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix}_u = \begin{pmatrix} 0 & \dfrac{\alpha_v}{2} & \sinh\dfrac{\alpha}{2} \\ -\dfrac{\alpha_v}{2} & 0 & 0 \\ -\sinh\dfrac{\alpha}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix}, \tag{4.273}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix}_v = \begin{pmatrix} 0 & -\dfrac{\alpha_u}{2} & 0 \\ \dfrac{\alpha_u}{2} & 0 & \cosh\dfrac{\alpha}{2} \\ 0 & -\cosh\dfrac{\alpha}{2} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix}. \tag{4.274}$$

Its integrability condition is

$$\triangle\alpha = -\sinh\alpha.$$

Using the isomorphism (4.244) between $so(3)$ and $su(2)$, the system (4.273) and (4.274) is equivalent to

$$\begin{aligned} \Psi_u &= \frac{1}{2}\begin{pmatrix} -\dfrac{\mathrm{i}\alpha_v}{2} & \mathrm{i}\sinh\dfrac{\alpha}{2} \\ \mathrm{i}\sinh\dfrac{\alpha}{2} & \dfrac{\mathrm{i}\alpha_v}{2} \end{pmatrix}\Psi, \\ \Psi_v &= \frac{1}{2}\begin{pmatrix} \dfrac{\mathrm{i}\alpha_u}{2} & \cosh\dfrac{\alpha}{2} \\ -\cosh\dfrac{\alpha}{2} & -\dfrac{\mathrm{i}\alpha_u}{2} \end{pmatrix}\Psi. \end{aligned} \tag{4.275}$$

Let

$$\zeta = \frac{u+iv}{2}, \quad \bar{\zeta} = \frac{u-iv}{2}, \tag{4.276}$$

then

$$\begin{aligned} \Psi_\zeta &= \Psi_u - \mathrm{i}\Psi_v = \frac{1}{2}\begin{pmatrix} \dfrac{\alpha_\zeta}{2} & -\mathrm{i}e^{-\alpha/2} \\ \mathrm{i}e^{\alpha/2} & -\dfrac{\alpha_\zeta}{2} \end{pmatrix}\Psi, \\ \Psi_{\bar{\zeta}} &= \Psi_u + \mathrm{i}\Psi_v = \frac{1}{2}\begin{pmatrix} -\dfrac{\alpha_{\bar{\zeta}}}{2} & \mathrm{i}e^{\alpha/2} \\ -\mathrm{i}e^{-\alpha/2} & \dfrac{\alpha_{\bar{\zeta}}}{2} \end{pmatrix}\Psi. \end{aligned} \tag{4.277}$$

Write

$$\Phi = \begin{pmatrix} e^{-\alpha/4} & 0 \\ 0 & e^{\alpha/4} \end{pmatrix}\Psi, \tag{4.278}$$

then

$$\begin{aligned} \Phi_\zeta &= \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i}e^{-\alpha} \\ \mathrm{i}e^{\alpha} & 0 \end{pmatrix}\Phi, \\ \Phi_{\bar{\zeta}} &= \frac{1}{2}\begin{pmatrix} -\alpha_{\bar{\zeta}} & \mathrm{i} \\ -i & \alpha_{\bar{\zeta}} \end{pmatrix}\Phi. \end{aligned} \tag{4.279}$$

Take the constant λ such that $|\lambda| = 1$, then $1/\lambda = \bar{\lambda}$. Rewrite ζ as $\dfrac{\zeta}{\lambda}$ and $\bar{\zeta}$ as $\lambda\bar{\zeta}$, then we get the Lax pair with spectral parameter

$$\begin{aligned} \Phi_\zeta &= \frac{\lambda}{2}\begin{pmatrix} 0 & -\mathrm{i}e^{-\alpha} \\ \mathrm{i}e^{\alpha} & 0 \end{pmatrix}\Phi, \\ \Phi_{\bar{\zeta}} &= \frac{1}{2}\begin{pmatrix} -\alpha_{\bar{\zeta}} & \mathrm{i}/\lambda \\ -\mathrm{i}/\lambda & \alpha_{\bar{\zeta}} \end{pmatrix}\Phi, \end{aligned} \tag{4.280}$$

whose integrability condition is still the negative sinh-Laplace equation (4.39). Thus we have the Lax pair for the sinh-Laplace equation.

The surface of constant positive Gauss curvature can be constructed explicitly as follows.

For any complex α, $\sinh(\pi\mathrm{i} + \alpha) = -\sinh\alpha$. By changing λ to $-\mathrm{i}\lambda$, α to $\pi\mathrm{i} + \alpha$, the Lax pair (4.208) of the sinh-Laplace equation becomes the Lax pair of the negative sinh-Laplace equation. Thus the solution of the sinh-Laplace equation (and the fundamental solution of its Lax pair) and the solution of the negative sinh-Laplace equation (and the fundamental solution of its Lax pair) can be changed with each other. We shall use the Darboux transformation for the sinh-Laplace equation to construct the solution of the negative sinh-Laplace equation.

Suppose α is a solution of the negative sinh-Laplace equation and Φ is the fundamental solution of its Lax pair, the algorithm is shown in the following diagram.

$$\begin{tikzcd}
(\alpha, \Phi(\lambda)) \arrow[d, "\substack{\alpha=\mathrm{i}\pi+\alpha' \\ \lambda=\mathrm{i}\lambda'}"] & (\alpha_2, \Phi_2(\lambda)) \\
(\alpha', \Phi'(\lambda')) \arrow[r, "\text{twice Darboux transformations}"] & (\alpha_2', \Phi_2'(\lambda')) \arrow[u, "\substack{\lambda'=-\mathrm{i}\pi \\ \alpha'=\alpha-\mathrm{i}\pi}"']
\end{tikzcd}$$

In this diagram, $(\alpha', \Phi'(\lambda'))$ and $(\alpha_2', \Phi_2'(\lambda'))$ are solutions of the sinh-Laplace equation and the fundamental solutions of the corresponding Lax pairs, $(\alpha, \Phi(\lambda))$ and $(\alpha_2, \Phi_2(\lambda))$ are solutions of the negative sinh-Laplace equation and the fundamental solutions of the corresponding Lax pairs, the "twice Darboux transformations" are

$$(\alpha', \Phi'(\lambda')) \longrightarrow (\alpha_1', \Phi_1'(\lambda')) \longrightarrow (\alpha_2', \Phi_2'(\lambda'))$$

where α_1' is purely imaginary while α' and α_2' are real. This process has been explained in detail in Section 4.3.

Following the inverse of the procedure from (4.273) and (4.274) to (4.280), we get the frame $\begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{n}' \end{pmatrix}$ from Φ_2. In this case there is no Bäcklund congruence, hence we can not get the expression of the surface algebraically. However,

$$\omega'^1 = \cosh\frac{\alpha_2}{2}\,\mathrm{d}u, \quad \omega'^2 = \sinh\frac{\alpha_2}{2}\,\mathrm{d}v \tag{4.281}$$

are known, and the integrability condition of

$$\mathrm{d}\mathbf{r}' = \omega'^1\mathbf{e}'_1 + \omega'^2\mathbf{e}'_2 \tag{4.282}$$

holds. Hence the surface of constant positive Gauss curvature $\mathbf{r}'$ can be obtained by a direct integration (without solving differential equations).

Therefore, the surfaces of constant mean curvature are parallel with the surfaces of constant positive Gauss curvature. The new surface of constant mean curvature is derived from suitable parallel surface of the surface of constant positive Gauss curvature ($K = 1$, $l = \pm 1$). From $b_{11} \neq b_{22}$, $b_{11}b_{22} = 1$, we have $1 - b_{11} \neq 0$, $1 - b_{22} \neq 0$, hence the parallel surface exists and is regular.

THEOREM 4.22 *Suppose the expression of a non-spherical surface of constant positive Gauss curvature (resp. surface of constant mean curvature) under the Chebyshev coordinates is known and its Lie transformation is also known, then new surface of constant positive Gauss curvature (resp. surface of constant mean curvature) and its Lie transformation can be obtained by algebraic computation together with an integral.*

This process can be continued successively. Whenever we know a family of non-spherical surfaces of constant positive Gauss curvature (resp. surfaces of constant mean curvature) which are Lie transformation with each other, a series of such kinds of surfaces can be obtained. Algebraic computation together with an integral is needed in this process.

4.5.3 The case in Minkowski space

In the Minkowski space, we shall consider both space-like and time-like surfaces.

(a) Space-like surface

Suppose S is a space-like surface of constant Gauss curvature. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the unit vectors tangent to the lines of curvature and $\mathbf{n}$ be the unit normal vector. $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ forms an orthogonal frame, and

$$\omega_1^3 = b_{11}\omega^2, \quad \omega_2^3 = b_{22}\omega^2. \tag{4.283}$$

From $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$, $\mathbf{n}^2 = -1$, we have $\omega_1^3 = \omega_3^1$, $\omega_2^3 = \omega_3^2$.

The position vector of the parallel surface $S(l)$ is

$$\mathbf{r}^* = \mathbf{r} + l\mathbf{n}. \tag{4.284}$$

By differentiation,

$$\mathrm{d}\mathbf{r}^* = \omega^a \mathbf{e}_a + l\omega_3^a \mathbf{e}_a = \omega^{*a}\mathbf{e}_a. \tag{4.285}$$

Hence

$$\begin{aligned} \omega^{*1} &= \omega^1 + l\omega_1^3 = (1 + lb_{11})\omega^1, \\ \omega^{*2} &= \omega^2 + l\omega_2^3 = (1 + lb_{22})\omega^2. \end{aligned} \tag{4.286}$$

Since $\omega_a^{*3} = \omega_a^3$,

$$b_{11}^* = \frac{b_{11}}{1 + lb_{11}}, \quad b_{22}^* = \frac{b_{22}}{1 + lb_{22}}, \quad b_{12}^* = 0, \tag{4.287}$$

we have

$$\begin{aligned} K^* &= -b_{11}^* b_{22}^* = \frac{K}{1 + 2lH - l^2K}, \\ H^* &= \frac{1}{2}(b_{11}^* + b_{22}^*) = \frac{H - lK}{1 + 2lH - l^2K}, \end{aligned} \tag{4.288}$$

and

$$H(2lH^* - 1) = -lK - (1 - l^2K)H^*. \tag{4.289}$$

Suppose that K, H^* are constants and H is not a constant, then

$$H^* = \frac{1}{2l}, \quad K = -\frac{1}{l^2}. \tag{4.290}$$

Alternatively, if H, K^* are constants and K is not a constant, then from

$$K(1 + l^2K^*) = K^*(1 + 2lH),$$

we have

$$H = -\frac{1}{2l}, \quad K^* = -\frac{1}{l^2}. \tag{4.291}$$

This leads to the following theorem.

Theorem 4.23 *For each space-like surface of constant Gauss curvature* $-\dfrac{1}{l^2}$*, two parallel surfaces with distance* $\pm l$ *have constant mean curvature* $H = \pm\dfrac{1}{2l}$ *respectively. For each space-like surface of constant mean* *curvature* $-\dfrac{1}{2l}$ *and non-constant Gauss curvature, the parallel surface with distance* l *is of constant negative Gauss curvature* $-\dfrac{1}{l^2}$.

Therefore, the construction of a space-like surface of constant mean curvature is equivalent to the construction of a space-like surface of constant negative Gauss curvature (see Section 4.2). The following theorem holds.

THEOREM 4.24 *From a known space-like surface of constant mean curvature, a series of space-like surfaces of constant mean curvature can be obtained via constructing parallel surfaces, applying Bäcklund transformation and Darboux transformation.*

Remark 32 In this case, the Bäcklund congruence belongs to the case (d) in Subsection 4.3.3. In the construction of Bäcklund transformation, time-like surface of constant negative Gauss curvature will appear as an intermediate configuration.

(b) Time-like surface

Now suppose S is a time-like surface, $\mathbf{e}_1^2 = 1$, $\mathbf{e}_2^2 = -1$. Similar to the space-like case, $\omega_1^3 = -\omega_3^1$, $\omega_2^3 = \omega_3^2$, and

$$b_{11}^* = \frac{b_{11}}{1 - lb_{11}}, \quad b_{22}^* = \frac{b_{22}}{1 + lb_{22}}, \quad b_{12} = 0. \tag{4.292}$$

$K = -b_{11}b_{22}$, $H = \dfrac{1}{2}(b_{11} - b_{22})$ leads to

$$K^* = \frac{K}{1 - 2lH + l^2K}, \quad H^* = \frac{H - lK}{1 - 2lH + l^2K}. \tag{4.293}$$

If K and H^* are constants, then

$$H^* = -\frac{1}{2l}, \quad K = \frac{1}{l^2}, \tag{4.294}$$

while K^* and H are constants, we have

$$H = \frac{1}{2l}, \quad K^* = \frac{1}{l^2}. \tag{4.295}$$

This gives the following theorem.

THEOREM 4.25 *Two parallel surfaces of a time -like surface of constant positive Gauss curvature* $K = \dfrac{1}{l^2}$ *with "distance"* $\pm l$ *are surfaces of* *constant mean curvature* $\pm\dfrac{1}{2l}$. *On one side of a time-like surface of constant mean curvature* $H = \dfrac{1}{2l}$, *there is a parallel surface with distance* l *and with constant positive Gauss curvature* $\dfrac{1}{l^2}$.

Therefore, the construction of time-like surface of constant mean curvature and the construction of time-like surface of constant positive Gauss curvature are equivalent.

THEOREM 4.26 *From a known time-like surface of constant mean curvature, a series of time-like surfaces of constant mean curvature can be obtained via constructing parallel surfaces, Bäcklund transformation and Darboux transformation.*

In this case the related Bäcklund congruences belong to the case (2) in Subsection 4.3.4.

Chapter 5

DARBOUX TRANSFORMATION AND HARMONIC MAP

Harmonic map is an important subject in differential geometry [25, 24, 115] and is closely related with mathematical physics and the soliton theory [24, 57, 46, 48]. At the beginning of this chapter, we introduce the notion of the harmonic map. Then the harmonic maps from Euclidean plane $\mathbf{R}^2$ or Minkowski plane $\mathbf{R}^{1,1}$ to Euclidean sphere S^2 in $\mathbf{R}^3$, and H^2 or $S^{1,1}$ in Minkowski space $\mathbf{R}^{2,1}$ are elucidated. We shall show that all these harmonic maps can be obtained from the construction of the surfaces of constant Gauss curvature in Chapter 4, and show the relations of these harmonic maps with some special soliton equations. Therefore, we can construct new harmonic maps from known harmonic maps and their extended solutions by using purely algebraic algorithm.

Using Darboux transformation, we can also construct the harmonic maps from $\mathbf{R}^{1,1}$ or $\mathbf{R}^2$ to the Lie group $U(N)$ and get explicit expressions of the solutions. The solitons interact elastically. The unitons are also considered by using Darboux transformation. Comparing with [102], the construction here is more explicit with purely algebraic algorithm by using Darboux transformation.

5.1 Definition of harmonic map and basic equations

Riemannian manifold and Lorentzian manifold are the generalizations of the Euclidean space and Minkowski space. A Riemannian manifold or a Lorentzian manifold is an n dimensional differential manifold M with a metric g. In the local coordinates $(x^1, \cdots, x^n)$ of M, the metric g is expressed as

$$\mathrm{d}s^2 = \sum_{i,j=1}^{n} g_{ij}(x)\mathrm{d}x^i\,\mathrm{d}x^j = g_{ij}(x)\mathrm{d}x^i\,\mathrm{d}x^j.$$

If (g_{ij}) is positive definite, then M is a Riemannian manifold and $\mathrm{d}s$ is the differential of the arc length of a curve. If the eigenvalues of (g_{ij}) have the signs $(+,\cdots,+,-)$, then M is a Lorentzian manifold and g is called a Lorentzian metric.

Let M and N be Riemannian manifolds or Lorentzian manifolds, $\phi : M \to N$ be a C^2-map. The energy (or the action if M is a Lorentzian manifold) of the map ϕ is

$$E(\phi) = \int_M e(\phi)\,\mathrm{d}V_M \tag{5.1}$$

where $\mathrm{d}V_M$ is the volume element of M. $e(\phi)$ is called the energy (action) density. In local coordinates,

$$\begin{aligned} e(\phi) &= \underset{N}{g}_{\alpha\beta}(\phi)\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}\underset{M}{g}^{ij}(x) \\ &(\alpha,\beta = 1,\cdots,n;\ i,j = 1,\cdots,m). \end{aligned} \tag{5.2}$$

Here $n = \dim N$, $m = \dim M$. $\underset{M}{g}^{ij}$'s and $\underset{N}{g}_{\alpha\beta}$'s are contravariant components of $\underset{M}{g}$ and covariant components of $\underset{N}{g}$ respectively, and $(\underset{M}{g}^{ij})$ is the inverse of $(\underset{M}{g}_{ij})$.

The Euler equation of the functional $E(\phi)$ is derived directly as follows. In local coordinates, $\mathrm{d}V_M = \sqrt{g}\,\mathrm{d}^m x$ where $g = |\det(\underset{M}{g}_{ij})|$. Hence the Euler equation of $E(\phi)$ is

$$\frac{\partial e(\phi)\sqrt{g}}{\partial\phi^\gamma} - \frac{\partial}{\partial x^k}\frac{\partial e(\phi)\sqrt{g}}{\partial(\phi^\gamma_{,k})} = 0. \tag{5.3}$$

For simplicity, we write g_{ij} for $\underset{M}{g}_{ij}$, and $g_{\alpha\beta}$ for $\underset{N}{g}_{\alpha\beta}$. The partial derivative $\dfrac{\partial\phi^\gamma}{\partial x^k}$ is denoted by $\phi^\gamma_{,k}$. Then, the first term of (5.3) is

$$\frac{\partial g_{\alpha\beta}(\phi)}{\partial\phi^\gamma}\phi^\alpha_{,i}\phi^\beta_{,j}g^{ij}\sqrt{g},$$

and the second term is

$$\begin{aligned} 2\frac{\partial}{\partial x^k}(g_{\alpha\gamma}g^{ki}\phi^\alpha_{,i}\sqrt{g}) &= 2\Big(\frac{\partial g_{\alpha\gamma}}{\partial\phi^\beta}g^{ki}\phi^\alpha_{,i}\phi^\beta_{,k}\sqrt{g} \\ &+ g_{\alpha\gamma}\frac{\partial g^{ki}}{\partial x^k}\phi^\alpha_{,i}\sqrt{g} + g_{\alpha\gamma}g^{ki}\frac{\partial^2\phi^\alpha}{\partial x^i\partial x^k}\sqrt{g} + g_{\alpha\gamma}g^{ki}\phi^\alpha_{,i}\frac{\partial\sqrt{g}}{\partial x^k}\Big). \end{aligned}$$

Using the Christoffel symbols $\underset{M}{\Gamma}{}^k_{ij}$ and $\underset{N}{\Gamma}{}^\gamma_{\alpha\beta}$ of M, N and the well-known formulae

$$\frac{\partial g_{\alpha\gamma}}{\partial\phi^\beta} = \underset{N}{\Gamma}{}^\lambda_{\alpha\beta} g_{\lambda\gamma} + \underset{N}{\Gamma}{}^\lambda_{\gamma\beta} g_{\alpha\lambda}, \tag{5.4}$$

$$\frac{\partial g^{ij}}{\partial x^k} = -g^{lj}\underset{M}{\Gamma}{}^i_{lk} - g^{il}\underset{M}{\Gamma}{}^j_{lk}, \tag{5.5}$$

$$\frac{\partial\sqrt{g}}{\partial x^k} = \underset{M}{\Gamma}{}^i_{ik}\sqrt{g}, \tag{5.6}$$

we obtain the Euler equation

$$g^{ik}\Big(\frac{\partial^2\phi^\gamma}{\partial x^i\partial x^k} - \underset{M}{\Gamma}{}^j_{ik}\frac{\partial\phi^\gamma}{\partial x^j} + \underset{N}{\Gamma}{}^\gamma_{\alpha\beta}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^k}\Big) = 0. \tag{5.7}$$

Definition 5.1 *A C^2 map ϕ from M to N is called a harmonic map if and only if it satisfies (5.7) [24].*

(5.7) is the system of partial differential equations for the harmonic map ϕ from M to N. If M is a Riemannian manifold, (5.7) is a system of nonlinear elliptic equations. If M is a Lorentzian manifold, (5.7) is a system of nonlinear hyperbolic equations. The harmonic map from a Lorentzian manifold is also called a wave map.

Harmonic map is a very important subject both in mathematics and in physics. In mathematics, there are many known special cases. If N is the line $\mathbf{R}$, a harmonic map is a harmonic function on M (especially, when $M = \mathbf{R}^n$, it is the ordinary harmonic function); if M is the line $\mathbf{R}$ or the circle S^1, the harmonic map is a geodesic or a closed geodesic respectively. A minimal surface is a conformal harmonic map which keeps the angles between two lines invariant.

In physics, there are quite a lot of applications:

(1) Nonlinear σ-model (or chiral field) is a harmonic map $M \to N$ where M is a Minkowski space and N is usually a homogeneous space. Especially, if N is a Lie group, it is called a principal chiral field [86].

(2) Ernst equation describes the static axially symmetric solution of Einstein gravitation in vacuum [27]. It is the Euler equation of the Energy

$$\int \frac{1}{\phi^2}\sum_{i=1}^3 \Big[\Big(\frac{\partial\phi}{\partial x^i}\Big)^2 + \Big(\frac{\partial\psi}{\partial x^i}\Big)^2\Big]\,\mathrm{d}^3x$$

under the axially symmetric constraint. Therefore, the Ernst equation is an axially symmetric harmonic map from $\mathbf{R}^3$ to the hyperbolic plane

H^2 which is equipped with the Poincaré metric

$$\mathrm{d}s^2 = \frac{1}{\phi^2}(\mathrm{d}\phi^2 + \mathrm{d}\psi^2).$$

(3) In particle physics, string is used as a model of hadron. In four dimensional Lorentzian space-time V, the world surface describing the motion of a classical string is a two dimensional time-like surface. It is determined by the equation

$$\phi^\alpha_{\tau\tau} - \phi^\alpha_{\sigma\sigma} + \underset{V}{\Gamma}{}^\alpha_{\beta\gamma}(\phi^\beta_\tau\phi^\gamma_\tau - \phi^\beta_\sigma\phi^\gamma_\sigma) = 0,$$

This is the equation for the harmonic map from two dimensional Minkowski plane $\mathbf{R}^{1,1}$ to V [36].

(4) Some solutions of the Yang-Mills equation in $\mathbf{R}^4$ under R-gauge can also be obtained from harmonic maps.

(5) The simplest model for liquid crystal is a harmonic map from $\mathbf{R}^2$ to S^2.

In this book, we only discuss harmonic maps which can be constructed explicitly by Darboux transformation. The starting manifold M is the Euclidean plane $\mathbf{R}^2 = \{(x, y)\}$ or Minkowski plane $\mathbf{R}^{1,1} = \{(t, x)\}$ (or a part of them). In these two cases, the equations for the harmonic maps are

$$\phi^\gamma_{xx} + \phi^\gamma_{yy} + \underset{N}{\Gamma}{}^\gamma_{\alpha\beta}(\phi^\alpha_x\phi^\beta_x + \phi^\alpha_y\phi^\beta_y) = 0 \tag{5.8}$$

and

$$\phi^\gamma_{tt} - \phi^\gamma_{xx} + \underset{N}{\Gamma}{}^\gamma_{\alpha\beta}(\phi^\alpha_t\phi^\beta_t - \phi^\alpha_x\phi^\beta_x) = 0 \tag{5.9}$$

respectively.

5.2 Harmonic maps from R^2 or $R^{1,1}$ to S^2, H^2 or $S^{1,1}$

Let S^2 be the sphere of Euclidean space $\mathbf{R}^3$, which consists of all the points $\mathbf{l} = (l_1, l_2, l_3)$ with $\mathbf{l}^2 = 1$.

The coordinates in Euclidean plane $\mathbf{R}^2$ are represented by (x, y), and the differential form of the Euclidean metric is $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$.

A map from $\mathbf{R}^2$ to S^2 can be written as $\mathbf{l} = \mathbf{l}(x, y)$ with $\mathbf{l}^2 = 1$. From (5.7) or (5.8), the equation of $\mathbf{l}$ for the harmonic map $\mathbf{R}^2 \to S^2$ is given by

$$\mathbf{l}_{xx} + \mathbf{l}_{yy} + (\mathbf{l}_x^2 + \mathbf{l}_y^2)\mathbf{l} = 0. \tag{5.10}$$

It can be derived as follows:

$$E(\mathbf{l}) = \int_\Omega \Big[\Big(\frac{\partial\mathbf{l}}{\partial x}\Big)^2 + \Big(\frac{\partial\mathbf{l}}{\partial y}\Big)^2\Big]\,\mathrm{d}x\,\mathrm{d}y,$$

where Ω is a region of $\mathbf{R}^2$, $\mathbf{l}$ satisfies $\mathbf{l}^2 = 1$. Introduce the Lagrangian undetermined multipliers λ, we rewrite $E(\mathbf{l})$ as

$$\widetilde{E}(\mathbf{l}) = \int_\Omega \Big[\Big(\frac{\partial \mathbf{l}}{\partial x}\Big)^2 + \Big(\frac{\partial \mathbf{l}}{\partial y}\Big)^2 + \lambda(\mathbf{l}^2 - 1)\Big]\,\mathrm{d}x\,\mathrm{d}y,$$

then the Euler equation becomes

$$\lambda \mathbf{l} = \frac{\partial}{\partial x}\Big(\frac{\partial \mathbf{l}}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial \mathbf{l}}{\partial y}\Big),$$

or

$$\triangle \mathbf{l} = \lambda \mathbf{l}.$$

With $\mathbf{l}^2 = 1$, we have $\mathbf{l}\cdot\mathbf{l}_x = 0$ and $\mathbf{l}\cdot\mathbf{l}_y = 0$, hence $\mathbf{l}\cdot\mathbf{l}_{xx} = -\mathbf{l}_x^2$, $\mathbf{l}\cdot\mathbf{l}_{yy} = -\mathbf{l}_y^2$. This leads to $\lambda = -(\mathbf{l}_x^2 + \mathbf{l}_y^2)$, and (5.10) follows.

As is known, the Minkowski plane $\mathbf{R}^{1,1}$ is the simplest two dimensional Lorentzian manifold, whose metric is $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2$. Instead of (5.7), the equation for the harmonic map from $\mathbf{R}^{1,1}$ to S^2 is

$$\mathbf{l}_{tt} - \mathbf{l}_{xx} + (\mathbf{l}_t^2 - \mathbf{l}_x^2)\mathbf{l} = 0. \tag{5.11}$$

The proof is similar to the Euclidean space.

In the Minkowski space $\mathbf{R}^{2,1}$, we can define two kinds of "spheres": H^2 and $S^{1,1}$. H^2 is defined by

$$\mathbf{l}^2 = -1, \quad l_3 > 0. \tag{5.12}$$

It is the upper branch of the biparted rotational hyperboloid in $\mathbf{R}^{2,1}$ and realizes the Lobachevsky geometry globally, while the surface of constant negative Gauss curvature in $\mathbf{R}^3$ only realizes a certain part of the Lobachevsky plane.

$S^{1,1}$ is given by

$$\mathbf{l}^2 = 1. \tag{5.13}$$

It is a uniparted rotational hyperboloid. It is a time-like surface in the Minkowski space. Its metric is indefinite.

Similar to the above discussions, the equations for the harmonic maps from $\mathbf{R}^2$ and H^2 to $S^{1,1}$ are

$$\mathbf{l}_{xx} + \mathbf{l}_{yy} - (\mathbf{l}_x^2 + \mathbf{l}_y^2)\mathbf{l} = 0 \quad (\mathbf{l}^2 = -1, l_3 > 0) \tag{5.14}$$

and

$$\mathbf{l}_{xx} + \mathbf{l}_{yy} + (\mathbf{l}_x^2 + \mathbf{l}_y^2)\mathbf{l} = 0 \quad (\mathbf{l}^2 = 1), \tag{5.15}$$

respectively. The equations for the harmonic maps from $\mathbf{R}^{1,1}$ to H^2 and $S^{1,1}$ are

$$\mathbf{l}_{tt} - \mathbf{l}_{xx} - (\mathbf{l}_t^2 - \mathbf{l}_x^2)\mathbf{l} = 0 \quad (\mathbf{l}^2 = -1, l_3 > 0) \tag{5.16}$$

and

$$\mathbf{l}_{tt} - \mathbf{l}_{xx} + (\mathbf{l}_t^2 - \mathbf{l}_x^2)\mathbf{l} = 0 \quad (\mathbf{l}^2 = 1) \tag{5.17}$$

respectively. A harmonic map may be defined in a region of $\mathbf{R}^2$ or $\mathbf{R}^{1,1}$ instead of the whole plane.

The harmonic maps from $\mathbf{R}^2$ (or $\mathbf{R}^{1,1}$) are conformal invariant. That is, suppose $h : \Omega \to N$ is a harmonic map from a region Ω in $\mathbf{R}^2$ (or $\mathbf{R}^{1,1}$) to a Riemannian (or Lorentzian) manifold N, ϕ is a conformal map from a region Ω_1 in $\mathbf{R}^2$ (or $\mathbf{R}^{1,1}$) to Ω, then the map $h \circ \phi$ is a harmonic map from Ω_1 to N.

This fact can be derived from the general equation or the energy integral. In fact, when $n = 2$, the energy integral (5.1) is invariant under the conformal transformation of $\mathbf{R}^2$ (or $\mathbf{R}^{1,1}$). Hence its Euler equation has the same property. Here we explain it by two cases.

Case 1: Conformal invariance of the harmonic map from $\mathbf{R}^2$ to S^2 (or H^2, $S^{1,1}$)

First we consider $\mathbf{R}^2$ as a one dimensional complex space $\mathbf{C}^1$ with complex coordinate $\{z = u + \mathrm{i}v\}$ and metric $\mathrm{d}s^2 = \mathrm{d}z\,\mathrm{d}\bar{z}$. Suppose $z = f(w) = u(u_1, v_1) + \mathrm{i}v(u_1, v_1)$ is a conformal map from a region Ω_1 to Ω, l is a harmonic map from Ω to S^2, then $l \circ f : l = l(x(u_1, v_1), y(u_1, v_1))$ is a harmonic map from Ω_1 to S^2 (or H^2, $S^{1,1}$). In fact, (5.10) can be written as

$$\mathbf{l}_{z\bar{z}} + (\mathbf{l}_z \cdot \mathbf{l}_{\bar{z}})\mathbf{l} = 0. \tag{5.18}$$

Under the transformation $z = z(w)$, (5.18) is transformed to

$$\mathbf{l}_{w\bar{w}} + (\mathbf{l}_w \cdot \mathbf{l}_{\bar{w}})\mathbf{l} = 0.$$

If S^2 is changed to H^2 or $S^{1,1}$, the proof is similar.

Case 2: Conformal invariance of the harmonic map from $\mathbf{R}^{1,1}$ to S^2 (or H^2, $S^{1,1}$)

Let

$$\xi = \frac{t+x}{2}, \quad \eta = \frac{t-x}{2} \tag{5.19}$$

be the characteristic coordinates (or light-cone coordinates) of $\mathbf{R}^{1,1}$, then $\mathrm{d}s^2 = 4\mathrm{d}\xi\,\mathrm{d}\eta$. A conformal map ϕ of $\mathbf{R}^{1,1}$ is of form

$$\xi = f(\xi_1), \quad \eta = g(\eta_1). \tag{5.20}$$

(5.11) can be written as

$$\mathbf{l}_{\xi\eta} + (\mathbf{l}_\xi \cdot \mathbf{l}_\eta)\mathbf{l} = 0 \quad (\mathbf{l}^2 = 1). \tag{5.21}$$

$l \circ \phi$ is also a harmonic map, since it satisfies

$$\mathbf{l}_{\xi_1\eta_1} + (\mathbf{l}_{\xi_1} \cdot \mathbf{l}_{\eta_1})\mathbf{l} = 0.$$

Similar result holds if S^2 is changed to H^2 or $S^{1,1}$.

Using the conformal invariance, we can define normalized harmonic map.

First we show that a harmonic map $\mathbf{l}$ from $\mathbf{R}^{1,1}$ to S^2 satisfies

$$(\mathbf{l}_\xi^2)_\eta = 0, \quad (\mathbf{l}_\eta^2)_\xi = 0, \quad \mathbf{l}^2 = 1. \tag{5.22}$$

Suppose (5.21) holds, then take the inner product with $\mathbf{l}_\xi$, we obtain $(\mathbf{l}_\xi^2)_\eta = 0$. Similarly, $(\mathbf{l}_\eta^2)_\xi = 0$ holds. This leads to (5.22). Conversely, suppose (5.22) holds and $\mathbf{l}_\xi$'s, $\mathbf{l}_\eta$'s are linearly independent, then

$$\mathbf{l}_\xi \cdot \mathbf{l}_{\xi\eta} = 0, \quad \mathbf{l}_\eta \cdot \mathbf{l}_{\xi\eta} = 0. \tag{5.23}$$

Hence

$$\mathbf{l}_{\xi\eta} = \sigma \mathbf{l}.$$

The inner product with $\mathbf{l}$ leads to

$$\sigma = \mathbf{l} \cdot \mathbf{l}_{\xi\eta} = (\mathbf{l} \cdot \mathbf{l}_\xi)_\eta - \mathbf{l}_\xi \cdot \mathbf{l}_\eta = -\mathbf{l}_\xi \cdot \mathbf{l}_\eta,$$

Hence (5.21) holds.

From (5.22),

$$\mathbf{l}_\xi^2 = f(\xi), \quad \mathbf{l}_\eta^2 = g(\eta).$$

When $f(\xi) \neq 0$ and $g(\eta) \neq 0$, we can define the transformation $\xi = \xi(\xi_1)$ and $\eta = \eta(\eta_1)$ such that $\left(\dfrac{\mathrm{d}\xi_1}{\mathrm{d}\xi}\right)^2 = f(\xi)$, $\left(\dfrac{\mathrm{d}\eta_1}{\mathrm{d}\eta}\right)^2 = g(\eta)$. Rewrite (ξ_1, η_1) as (ξ, η), we obtain

$$\mathbf{l}_\xi^2 = 1, \quad \mathbf{l}_\eta^2 = 1. \tag{5.24}$$

The harmonic map satisfying condition (5.24) is called a "normalized harmonic map" [38]. When $\mathbf{l}_\xi^2 \neq 0$ and $\mathbf{l}_\eta^2 \neq 0$, a harmonic map can always be transformed to a normalized harmonic map by the transformation $\xi = \xi(\xi_1)$, $\eta = \eta(\eta_1)$. Thus, some problems on harmonic maps can be simplified by using conformal transformations.

For a normalized harmonic map,

$$\mathbf{l}_x^2 + \mathbf{l}_t^2 = 1, \quad \mathbf{l}_t \cdot \mathbf{l}_x = 0 \tag{5.25}$$

holds [58]. In fact, (5.24) is equivalent to (5.25). Hence a normalized harmonic map can also be defined by (5.25).

A harmonic map from $\mathbf{R}^2$ to S^2 (or H^2, $S^{1,1}$) can also be normalized. From (5.18) and $\mathbf{l}^2 = 1$, we have $(\mathbf{l}_\zeta^2)_{\bar\zeta} = 0$ and $(\mathbf{l}_{\bar\zeta}^2)_\zeta = 0$. Hence $\mathbf{l}_\zeta^2$ is a holomorphic function of ζ. If

$$\mathbf{l}_\zeta^2 = (\mathbf{l}_u^2 - \mathbf{l}_v^2) - 2\mathrm{i}\mathbf{l}_u \cdot \mathbf{l}_v = 1, \tag{5.26}$$

i.e.,

$$\mathbf{l}_u^2 - \mathbf{l}_v^2 = 1, \quad \mathbf{l}_u \cdot \mathbf{l}_v = 0, \tag{5.27}$$

this harmonic map is called normalized.

Suppose $\mathbf{l} = \mathbf{l}(u, v)$ is a harmonic map defined in a simply connected region Ω and $\mathbf{l}_\zeta^2 \neq 0$, then $\mathbf{l}$ can be normalized. In fact, take the conformal map $\zeta = \zeta(w)$,

$$\mathbf{l}_w^2 = \mathbf{l}_\zeta^2 \Big(\frac{\mathrm{d}\zeta}{\mathrm{d}w}\Big)^2. \tag{5.28}$$

When $\mathbf{l}_\zeta^2 \neq 0$, the equation $(\dfrac{\mathrm{d}w}{\mathrm{d}\zeta})^2 = \mathbf{l}_\zeta^2$ has a single-valued solution $w = f(\zeta)$ in the simply connected region Ω. w maps Ω to Ω_1. The map $\mathbf{l} \circ f^{-1}$ from Ω_1 to S^2 (or H^2, $S^{1,1}$) has been normalized.

We have seen in Chapter 4 that there are many kinds of surfaces of constant Gauss curvature on which Chebyshev coordinates (u, v) and the corresponding Chebyshev frames $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ exist (under the assumption that there are no umbilical points). The map from a region of $\mathbf{R}^2$ (or $\mathbf{R}^{1,1}$) to the surface defined by Chebyshev coordinates is called a Chebyshev map. In this case, the normal vector $\mathbf{n}$ defines the Gauss map of S^2 (or H^2, $S^{1,1}$).

THEOREM 5.2 *The composition of the Chebyshev map and the Gauss map is a normalized harmonic map, and vise versa [58].*

Proof. We prove this theorem in several cases.

(a) $\mathbf{R}^{1,1} \to S^2$

Suppose $\mathbf{l}$ is a normalized harmonic map. Take $\mathbf{e}_1$ and $\mathbf{e}_2$ so that $\mathbf{e}_1$ is the unit vector of $\mathbf{l}_t$, $\mathbf{e}_2$ is the unit vector of $\mathbf{l}_x$, then (5.25) implies that $\mathbf{e}_1$ is orthogonal to $\mathbf{e}_2$. Moreover, there is a function $\alpha(x, t)$ such that

$$\mathbf{l}_t = -\sin\frac{\alpha}{2}\mathbf{e}_1, \quad \mathbf{l}_x = \cos\frac{\alpha}{2}\mathbf{e}_2. \tag{5.29}$$

Since $\mathbf{l}$, $\mathbf{e}_1$ and $\mathbf{e}_2$ form a orthonormal frame,

$$\begin{aligned}
\mathbf{e}_{1t} &= \sin\tfrac{\alpha}{2}\mathbf{l} + \sigma\mathbf{e}_2,\\
\mathbf{e}_{1x} &= \tau\mathbf{e}_2,\\
\mathbf{e}_{2t} &= -\sigma\mathbf{e}_1,\\
\mathbf{e}_{2x} &= -\cos\tfrac{\alpha}{2}\mathbf{l} - \tau\mathbf{e}_1.
\end{aligned} \tag{5.30}$$

Their integrability condition leads to

$$\sigma = \frac{1}{2}\alpha_x, \quad \tau = \frac{1}{2}\alpha_t \tag{5.31}$$

and

$$\alpha_{tt} - \alpha_{xx} = \sin \alpha. \tag{5.32}$$

Denote $\mathbf{l} = \mathbf{n}$, we have

$$\begin{aligned}
\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix}_t &= \begin{pmatrix} 0 & \dfrac{\alpha_x}{2} & \sin\dfrac{\alpha}{2} \\ -\dfrac{\alpha_x}{2} & 0 & 0 \\ -\sin\dfrac{\alpha}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix}, \\
\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix}_x &= \begin{pmatrix} 0 & \dfrac{\alpha_t}{2} & 0 \\ -\dfrac{\alpha_t}{2} & 0 & -\cos\dfrac{\alpha}{2} \\ 0 & \cos\dfrac{\alpha}{2} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix},
\end{aligned} \tag{5.33}$$

This is just the equation of Chebyshev frame for a surface of constant negative Gauss curvature, $\mathbf{l} = \mathbf{n}$ is its normal vector, $\mathbf{l}(u, v)$ is the Gauss map. Therefore, a normalized harmonic map $\mathbf{R}^{1,1} \to S^2$ is the composition of a Chebyshev map and a Gauss map. Converse conclusion follows from

$$\begin{aligned}
\mathbf{n}_\xi^2 &= (\mathbf{n}_t + \mathbf{n}_x)^2 = \sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} = 1, \\
\mathbf{n}_\eta^2 &= (\mathbf{n}_t - \mathbf{n}_x)^2 = 1
\end{aligned}$$

and the fact that $\mathbf{n}_\xi$ and $\mathbf{n}_\eta$ are linearly independent.

(b) $\mathbf{R}^2 \to H^2$

Suppose $\mathbf{l}(x, y)$ is a normalized harmonic map, then (5.25) holds. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the unit vectors of $\mathbf{l}_u$ and $\mathbf{l}_v$ respectively. We can choose function $\alpha(x, t)$ so that

$$\begin{aligned}
\mathbf{l}_u &= \cosh\frac{\alpha}{2}\mathbf{e}_1, \\
\mathbf{l}_v &= \sinh\frac{\alpha}{2}\mathbf{e}_2.
\end{aligned}$$

Similar to the case (a), they satisfy a system of different equations. The equations are just the equations of the Chebyshev frame for the surface of constant negative Gauss curvature in $\mathbf{R}^{2,1}$ (Section 4.3), $\mathbf{n}$ is the Gauss map from this surface to H^2.

Other cases can be proved similarly. Here we list the results in the following table.

From	To	Constant curvature	Differential equation
$\mathbf{R}^2$	$S^2 \subset \mathbf{R}^3$	$+$	$\triangle\alpha = -\sinh\alpha$
	$H^2 \subset \mathbf{R}^{2,1}$	$-$ (space-like)	$\triangle\alpha = \sinh\alpha$
	$S^{1,1} \subset \mathbf{R}^{2,1}$	$-$ (time-like)	$\triangle\alpha = \sin\alpha$
$\mathbf{R}^{1,1}$	$S^2 \subset \mathbf{R}^3$	$-$	$\frac{\partial^2\alpha}{\partial u^2} - \frac{\partial^2\alpha}{\partial v^2} = \sin\alpha$
	$H^2 \subset \mathbf{R}^{2,1}$	$+$ (space-like)	$\frac{\partial^2\alpha}{\partial u^2} - \frac{\partial^2\alpha}{\partial v^2} = \sin\alpha$
	$S^{1,1} \subset \mathbf{R}^{2,1}$	$+$ (time-like)	$\frac{\partial^2\alpha}{\partial u^2} - \frac{\partial^2\alpha}{\partial v^2} = \sinh\alpha$ $\frac{\partial^2\alpha}{\partial u^2} - \frac{\partial^2\alpha}{\partial v^2} = \cosh\alpha$ $\frac{\partial^2\alpha}{\partial u^2} - \frac{\partial^2\alpha}{\partial v^2} = \mathrm{e}^\alpha$

This theorem implies that the construction of various normalized harmonic maps is closely related to the construction of the Chebyshev frames of various surfaces of constant Gauss curvature. Therefore, the following theorem holds [58–60].

THEOREM 5.3 *Suppose a normalized harmonic map from $\mathbf{R}^2$ or $\mathbf{R}^{1,1}$ to various two dimensional "sphere" is known. Then, using Darboux transformation, we can obtain an infinite series of normalized harmonic maps of the same type. Moreover, normalized harmonic map from $\mathbf{R}^2$ to $S^{1,1}$ (resp. H^2) can also be constructed by Darboux transformation from a known normalized harmonic map from $\mathbf{R}^2 \to H^2$ (resp. $S^{1,1}$).*

Remark 33 It is know that for a surface in $\mathbf{R}^3$ with constant positive Gauss curvature, the Gauss map is a harmonic map [92]. This can be derived from Theorem 5.3 as a corollary. For example, for a surface of constant Gauss curvature 1 in $\mathbf{R}^3$, the corresponding surface of constant mean curvature (see Section 4.5) is

$$\omega^{*1} = \omega^1 \pm \omega_1^3, \quad \omega^{*2} = \omega^2 \pm \omega_2^3,$$

i.e.,

$$\omega^{*1} = (\cosh\frac{\alpha}{2} \pm \sinh\frac{\alpha}{2})\mathrm{d}u, \quad \omega^{*2} = (\cosh\frac{\alpha}{2} \pm \sinh\frac{\alpha}{2})\mathrm{d}v.$$

Its first fundamental form is

$$\begin{aligned} \mathrm{d}s^{*2} &= (\cosh\frac{\alpha}{2} \pm \sinh\frac{\alpha}{2})^2(\mathrm{d}u^2 + \mathrm{d}v^2) \\ &= e^{\pm\alpha}(\mathrm{d}u^2 + \mathrm{d}v^2). \end{aligned}$$

Hence its Chebyshev coordinates are isothermal coordinates. This means that the Chebyshev coordinates give a conformal correspondence of surfaces of constant mean curvature and the Euclidean plane. The above conclusion follows from the conformal invariance of the harmonic map in two dimensional cases.

Remark 34 *We can prove that there is a similar conclusion as Remark 33 for space-like and time-like surfaces of constant mean curvature in* $\mathbf{R}^{2,1}$. *In Section 4.5 we have known that a space-like surface of constant mean curvature is a parallel surface of a space-like surface of constant negative Gauss curvature, while a time-like surface of constant mean curvature is a parallel surface of a time-like surface of constant positive Gauss curvature. Suppose their Gauss curvatures are* -1 *and* $+1$ *respectively, then*

$$\omega^{*1} = \omega^1 \pm \omega_1^3, \quad \omega^{*2} = \omega^2 \pm \omega_2^3.$$

For space-like surface,

$$\omega^{*1} = (\cosh\frac{\alpha}{2} \pm \sinh\frac{\alpha}{2})\mathrm{d}u, \quad \omega^{*2} = (\sinh\frac{\alpha}{2} \pm \cosh\frac{\alpha}{2})\mathrm{d}v,$$

hence

$$\mathrm{d}s^{*2} = (\omega^{*1})^2 + (\omega^{*2})^2 = e^{\pm\alpha}(\mathrm{d}u^2 + \mathrm{d}v^2)$$

where (u, v) *are isothermal coordinates. In the time-like case,*

$$\omega^{*1} = (\cosh\frac{\alpha}{2} \pm \sinh\frac{\alpha}{2})\mathrm{d}u, \quad \omega^{*2} = (\sinh\frac{\alpha}{2} \pm \cosh\frac{\alpha}{2})\mathrm{d}v,$$

hence

$$\mathrm{d}s^{*2} = (\omega^{*1})^2 - (\omega^{*2})^2 = e^{\pm\alpha}(\mathrm{d}u^2 - \mathrm{d}v^2)$$

where (u, v) *are also isothermal coordinates. From the above theorem, we know that a Gauss map of a surface of constant mean curvature is a harmonic map, and a Chebyshev map from* (u, v) *plane to the surface is a conformal map.*

Remark 35 *The harmonic map from the Minkowski plane* $\mathbf{R}^{1,1}$ *was first studied in [37]. The solution of the Cauchy problem for harmonic map exists globally if the target manifold is a complete Riemannian manifold. In [38], it was pointed out that when the target manifold is* $S^{1,1}$, *global solution of the Cauchy problem may not exist.*

5.3 Harmonic maps from $R^{1,1}$ to $U(N)$

5.3.1 Riemannian metric on $U(N)$

The group $U(N)$ is composed of all $N \times N$ unitary matrices, i.e., it consists of all $N \times N$ matrices g satisfying $gg^* = I$. On $U(N)$, there is a Riemannian metric

$$\mathrm{d}s^2 = -\ \mathrm{tr}(\mathrm{d}g\, g^{-1}\mathrm{d}g\, g^{-1}) = -\ \mathrm{tr}(\mathrm{d}g\, g^*\mathrm{d}g\, g^*). \tag{5.34}$$

Here $g \in U(N)$, $\mathrm{d}g$ is its differential, the trace (tr) of a matrix is the sum of all its diagonal entries. This metric $\mathrm{d}s^2$ is invariant under the left and right translation (i.e., the transformation $g \to g_1 = ag$ and $g \to g_2 = ga$ for any fixed element a of $U(N)$) of the group. In fact,

$$\begin{aligned} \mathrm{d}g_1\, g_1^{-1} &= a\, \mathrm{d}g\, g^{-1}a^{-1}, \\ \mathrm{d}g_2\, g_2^{-1} &= \mathrm{d}g\, g^{-1} \end{aligned}$$

imply

$$\mathrm{tr}(\mathrm{d}g\, g^{-1}\mathrm{d}g\, g^{-1}) = \mathrm{tr}(\mathrm{d}g_1\, g_1^{-1}\mathrm{d}g_1\, g_1^{-1}) = \mathrm{tr}(\mathrm{d}g_2\, g_2^{-1}\mathrm{d}g_2\, g_2^{-1}).$$

Next we prove that $\mathrm{d}s^2$ is actually a Riemannian metric, i.e., (5.34) is positive definite. Since $U(N)$ is a differential manifold, we take the local coordinate $\{u^\alpha\}$ of (N) so that

$$g = g(u^\alpha) \quad (\alpha = 1, 2, \cdots, r),$$

where $r = N^2$ is the dimension of $U(N)$. Then

$$\begin{aligned} \mathrm{d}g &= \frac{\partial g}{\partial u^\alpha}\mathrm{d}u^\alpha, \\ \mathrm{d}s^2 &= -\ \mathrm{tr}\Big(\frac{\partial g}{\partial u^\alpha}g^*\frac{\partial g}{\partial u^\beta}g^*\Big)\mathrm{d}u^\alpha\, \mathrm{d}u^\beta = g_{\alpha\beta}\mathrm{d}u^\alpha\, \mathrm{d}u^\beta, \\ g_{\alpha\beta} &= -\ \mathrm{tr}\Big(\frac{\partial g}{\partial u^\alpha}g^*\frac{\partial g}{\partial u^\beta}g^*\Big). \end{aligned} \tag{5.35}$$

Suppose $\{\xi^\alpha\}$ are real and not all zero. Let

$$Q = \frac{\partial g}{\partial u^\alpha}g^{-1}\xi^\alpha.$$

Substituting ξ^α for $\mathrm{d}u^\alpha$ into $\mathrm{d}s^2$, we obtain $g_{\alpha\beta}\xi^\alpha\xi^\beta = -\ \mathrm{tr}(Q^2)$. On the other hand, by differentiating $gg^* = I$, we have

$$\frac{\partial g}{\partial u^\alpha}g^* + g\frac{\partial g^*}{\partial u^\alpha} = 0,$$

which gives $Q + Q^* = 0$. Hence

$$-\ \mathrm{tr}(Q^2) = \mathrm{tr}(QQ^*) \geq 0.$$

Clearly $\mathrm{tr}(QQ^*) = 0$ if and only if $Q = 0$, which means $\xi^\alpha = 0$. Therefore, $g_{\alpha\beta}\xi^\alpha\xi^\beta \geq 0$, and the equality holds if and only if $\xi^\alpha = 0$. This proves that $\mathrm{d}s^2$ is a Riemannian metric.

5.3.2 Harmonic maps from $R^{1,1}$ to $U(N)$

Suppose (x,t) are orthonormal coordinates of $\mathbf{R}^{1,1}$. Let $\xi = x+t$, $\eta = x-t$, then (ξ,η) are characteristic coordinates (light-cone coordinates) of $\mathbf{R}^{1,1}$, and the metric of $\mathbf{R}^{1,1}$ is $\mathrm{d}s^2 = \mathrm{d}\xi\,\mathrm{d}\eta$. The action of the harmonic map $g(\xi,\eta)$ from $\mathbf{R}^{1,1}$ to $U(N)$ is

$$\begin{aligned}\mathcal{A}[g] &= \int g_{\alpha\beta}u_\xi^\alpha u_\eta^\beta \mathrm{d}\xi\,\mathrm{d}\eta\\ &= -\int \mathrm{tr}\Big(\frac{\partial g}{\partial u^\alpha}g^*\frac{\partial g}{\partial u^\beta}g^*\Big)u_\xi^\alpha u_\eta^\beta \mathrm{d}\xi\,\mathrm{d}\eta\\ &= -\int \mathrm{tr}\Big(\frac{\partial g}{\partial \xi}g^*\frac{\partial g}{\partial \eta}g^*\Big)\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned} \tag{5.36}$$

Denote

$$A = g_\eta g^*, \quad B = g_\xi g^*, \tag{5.37}$$

then A and B are valued in the Lie algebra $u(N)$ of the Lie group $U(N)$.

We can write down the Euler equation of $\mathcal{A}[g]$ by using the standard procedure of variation. Denote

$$\mathcal{A}_\Omega[g] = -\int_\Omega \mathrm{tr}(g_\xi\, g^* g_\eta\, g^*)\mathrm{d}\xi\,\mathrm{d}\eta,$$

where Ω is a bounded region. Suppose g depends on another parameter τ, i.e., $g = g(\tau,\xi,\eta)$ and $g(0,\xi,\eta) = g$. Let $\left.\dfrac{\partial g}{\partial\tau}\right|_{\tau=0} = h$ and h is 0 at the boundary of Ω. Then

$$\left.\frac{\mathrm{d}\mathcal{A}_\Omega[g]}{\mathrm{d}\tau}\right|_{\tau=0} = -\int_\Omega \mathrm{tr}(h_\xi g^* A + g_\xi h^* A + B h_\eta g^* + B g_\eta h^*)\mathrm{d}\xi\,\mathrm{d}\eta.$$

Take the partial integration for the first and the third terms, we have

$$\begin{aligned}\left.\frac{\mathrm{d}\mathcal{A}_\Omega[g]}{\mathrm{d}\tau}\right|_{\tau=0} = & -\int_\Omega \mathrm{tr}(-hg^*A_\xi - hg_\xi^*A + g_\xi h^* A\\ & -B_\eta h g^* - Bhg_\eta^* + Bg_\eta h^*)\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned}$$

Differentiating $gg^* = I$ and $g^*g = I$ leads to $hg^* = -gh^*$ and $h^*g = -g^*h$. Hence

$$\begin{aligned}&g_\xi h^* A = g_\xi h^* g g^* g_\eta g^* = -g_\xi g^* h g^* g_\eta g^*,\\ &B g_\eta h^* = g_\xi g^* g_\eta h^* g g^* = -g_\xi g^* g_\eta g^* h g^*,\\ &h g_\xi^* A = h g_\xi^* g g^* g_\eta g^* = -h g^* g_\xi g^* g_\eta g^*,\\ &B h g_\eta^* = g_\xi g^* h g^* g g_\eta^* = -g_\xi g^* h g^* g_\eta g^*.\end{aligned}$$

With the above equation and using the equality

$$\mathrm{tr}(A_1 A_2 \cdots A_n) = \mathrm{tr}(A_n A_1 A_2 \cdots A_{n-1}),$$

we obtain

$$\left.\frac{\mathrm{d}\mathcal{A}_\Omega[g]}{\mathrm{d}\tau}\right|_{\tau=0} = \int_\Omega \mathrm{tr}(hg^*(A_\xi + B_\eta))\mathrm{d}\xi\,\mathrm{d}\eta = 0.$$

Since hg^* can be an arbitrary matrix in the Lie algebra $u(N)$ and $A_\xi + B_\eta$ is also a matrix in the Lie algebra $u(N)$, the Euler equation

$$A_\xi + B_\eta = 0 \tag{5.38}$$

is derived. By direct calculation, A and B defined by (5.37) also satisfy

$$A_\xi - B_\eta + [A, B] = 0. \tag{5.39}$$

Remark 36 (5.37) is actually $g_\xi = Bg$ and $g_\eta = Ag$, and (5.39) is the integrability condition of (5.37). Hence, in $\mathbf{R}^{1,1}$ (or its simply connected region), A and B which satisfy (5.39) can determine $g(\xi, \eta)$ uniquely up to a right-multiplier of a fix element of $U(N)$. Therefore, we can consider A and B as unknown functions and study the partial differential equations (5.38) and (5.39) for A and B.

As in many cases considered before, we need to find the Lax pair with spectral parameter for (5.38) and (5.39). Notice that the integrability condition of

$$\Phi_\eta = \lambda A\Phi, \quad \Phi_\xi = \lambda(2\lambda - 1)^{-1}B\Phi \tag{5.40}$$

is

$$\lambda A_\xi - \lambda(2\lambda - 1)^{-1}B_\eta + \lambda^2(2\lambda - 1)^{-1}[A, B] = 0,$$

i.e.,

$$\lambda^2(A_\xi - B_\eta + [A, B]) + (\lambda^2 - \lambda)(A_\xi + B_\eta) = 0.$$

It should hold true for arbitrary λ, so we get (5.38) and (5.39).

Hence, the following theorem holds.

THEOREM 5.4 *The partial differential equations (5.38) and (5.39) have Lax pair (5.40).*

A non-degenerate $N \times N$ matrix solution $\Phi(\lambda)$ of the Lax pair is called its fundamental solution. A fundamental solution is determined by its value at one point, say $(0, 0)$. When λ is real, if $\Phi(\lambda)|_{(0,0)} \in U(N)$, then $\Phi(\lambda) \in U(N)$ holds everywhere. Hereafter, we suppose $\Phi(\lambda)|_{(0,0)} \in U(N)$ for real λ. Then $\Phi(1)K$ is a harmonic map for any $K \in U(N)$.

Remark 37 We can replace $U(N)$ by any matrix Lie group G. If we use

$$A = g_\eta g^{-1}, \quad B = g_\xi g^{-1}$$

to replace (5.37), then (5.38) and (5.39) are extended as the equations for harmonic maps to G, and the Lax pair is still (5.40). In this case G may not have an invariant positive definite metric.

Now we use the Darboux transformation to construct explicit solutions of (5.38) and (5.39).

First, suppose the group G is a complex (or real) general linear group $GL(N, \mathbf{C})$ (or $GL(N, \mathbf{R})$). $A(\xi, \eta)$ and $B(\xi, \eta)$ are real (or complex) $N \times N$ matrix functions. Suppose $\Phi(\xi, \eta, \lambda)$ satisfies $\det \Phi \neq 0$, and A, B, Φ are solutions of (5.38)–(5.40).

We want to construct a Darboux matrix D in the form

$$D(\lambda) = I - \lambda S \tag{5.41}$$

so that

$$\Phi_1 = D(\lambda)\Phi \tag{5.42}$$

with suitable A_1 and B_1 satisfying (5.38)–(5.39). That is, Φ_1, A_1, B_1 satisfy

$$\Phi_{1\eta} = \lambda A_1 \Phi_1, \quad \Phi_{1\xi} = \lambda(2\lambda - 1)^{-1} B_1 \Phi_1. \tag{5.40$'$}$$

Substituting (5.42) into the above equations and using (5.40), we have

$$A_1 = A - S_\eta, \quad B_1 = B + S_\xi \tag{5.43}$$

and

$$S_\eta S = AS - SA, \quad S_\xi S - 2S_\xi = SB - BS. \tag{5.44}$$

Therefore, as soon as we get a nontrivial solution $S(\xi, \eta)$ of (5.44), then, A_1, B_1 are obtained from (5.43) and Φ_1 is obtained from (5.41), (5.42). Hence, Φ_1 satisfies (5.40), and A_1, B_1 satisfy (5.38), (5.39). This means that A_1 and B_1 give a new solution.

The explicit expression of S is constructed as follows.

Take N real (or complex) constants $\lambda_1, \cdots, \lambda_N$, which are not all the same and $\lambda_\alpha \neq 0, 1/2, 1$; $\alpha = 1, 2, \cdots, N$. Take N constant column vectors $l_1, \cdots, l_N$ so that the $N \times N$ matrix

$$H = (\Phi(\lambda_1)l_1, \cdots, \Phi(\lambda_N)l_N) \tag{5.45}$$

is non-degenerate. (Each column $\Phi(\lambda_\alpha)l_\alpha$ in H is a column solution of the Lax pair when $\lambda = \lambda_\alpha$.) Let

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N), \tag{5.46}$$

then we have the following theorem.

THEOREM 5.5 *The matrix*

$$S = H\Lambda^{-1}H^{-1} \tag{5.47}$$

satisfies (5.44).

Proof. $h_\alpha = \Phi(\lambda_\alpha)l_\alpha$ is a column solution of (5.40) for $\lambda = \lambda_\alpha$, i.e., it satisfies

$$\begin{aligned} &h_{\alpha\eta} = \lambda_\alpha A h_\alpha, \quad h_{\alpha\xi} = \lambda_\alpha(2\lambda_\alpha - 1)^{-1} B h_\alpha, \\ &(\alpha = 1, 2, \cdots, N). \end{aligned} \tag{5.48}$$

Hence

$$H_\eta = AH\Lambda, \quad H_\xi = BH\Lambda(2\Lambda - 1)^{-1},$$

and

$$S_\eta = AH\Lambda\Lambda^{-1}H^{-1} - H\Lambda^{-1}H^{-1}AH\Lambda H^{-1} = A - SAS^{-1}.$$

This is the first equation of (5.44). On the other hand,

$$\begin{aligned} S_\xi \quad &= BH\Lambda(2\Lambda - 1)^{-1}\Lambda^{-1}H^{-1} \\ &\quad - H\Lambda^{-1}H^{-1}BH\Lambda(2\Lambda - 1)^{-1}H^{-1} \\ &= BH(2\Lambda - 1)^{-1}H^{-1} - SBH\Lambda(2\Lambda - 1)^{-1}H^{-1}. \end{aligned}$$

It is easy to check that the second equation of (5.44) is satisfied.

Therefore, we have the following theorem.

THEOREM 5.6 *Suppose* (A, B, Φ) *is a solution of (5.38)–(5.40),* S *is defined by (5.47), then* (A_1, B_1, Φ_1) *defined by (5.43) and (5.42) is also a solution of (5.38)–(5.40).*

Thus, we have the Darboux transformation

$$(A, B, \Phi) \longrightarrow (A_1, B_1, \Phi_1).$$

Moreover, the algorithm is purely algebraic since S, S_ξ, S_η can be obtained from A, B and Φ with explicit algebraic expressions.

In this construction, we cannot guarantee that the condition $\det H \neq 0$ holds globally. That is, if the seed solution (A, B, Φ) is defined on $\mathbf{R}^{1,1}$ (or its simply connected region Ω), it is not always possible that A_1, B_1 and Φ_1 can be defined on $\mathbf{R}^{1,1}$ (or Ω) globally. What we can be sure is that $\det H \neq 0$ holds in a small region since it holds at some point by construction. There is another problem. If G is a group, A and B are valued in the Lie algebra G' of G, whether will A_1 and B_1 be still in G'? These are two related difficult problems. We shall give their answers for $G = U(N)$.

As we have known, the group $U(N)$ is composed of all the $N \times N$ matrices g satisfying $g^* = g^{-1}$. A matrix A belongs to the Lie algebra $u(N)$ of $U(N)$ if and only if

$$A^* + A = 0. \tag{5.49}$$

Now suppose (A, B, Φ) is a solution of (5.38)–(5.40) which is defined on whole $\mathbf{R}^{1,1}$ and $A, B \in u(N)$. (If they are defined in a simply connected region of $\mathbf{R}^{1,1}$, the statements below hold as well.) The Darboux transformation is constructed following (5.43) and (5.42). In order to have $A_1, B_1 \in u(N)$, we need

$$(S + S^*)_\eta = 0, \quad (S + S^*)_\xi = 0. \tag{5.50}$$

Hence we want to make specific S to satisfy (5.50).

Let λ^0 be a non-zero complex number and

$$\lambda_\alpha = \lambda^0 \text{ or } \bar{\lambda}^0 \quad (\alpha = 1, 2, \cdots, N). \tag{5.51}$$

Choose l_α so that

$$h_\alpha^* h_\beta = 0 \quad (\text{if } \lambda_\alpha \neq \lambda_\beta) \tag{5.52}$$

holds at one point (say $\xi = \eta = 0$) and h_α's are linearly independent. We shall show that the S constructed from these λ_α and l_α satisfies (5.50).

First, we prove that (5.52) holds everywhere on $\mathbf{R}^{1,1}$ if it holds at one point. In fact, (5.48) leads to

$$\begin{aligned}
(h_\alpha^* h_\beta)_\eta &= \bar{\lambda}_\alpha h_\alpha^* A^* h_\beta + \lambda_\beta h_\alpha^* A h_\beta, \\
(h_\alpha^* h_\beta)_\xi &= \bar{\lambda}_\alpha (2\bar{\lambda}_\alpha - 1)^{-1} h_\alpha^* B^* h_\beta + \lambda_\beta (2\lambda_\beta - 1)^{-1} h_\alpha^* B h_\beta.
\end{aligned}$$

Hence

$$(h_\alpha^* h_\beta)_\eta = (h_\alpha^* h_\beta)_\xi = 0$$

when $\lambda_\alpha \neq \lambda_\beta$ (i.e. $\bar{\lambda}_\alpha = \lambda_\beta$). This means that (5.52) holds everywhere if it holds at one point.

Next, for the same $\lambda_\alpha = \lambda^0$ (or $\bar{\lambda}^0$), h_α's are linearly independent everywhere if they are linearly independent at one point, since (5.48) is linear. Choose the initial value of $\{h_\alpha\}$ so that they are linearly independent and satisfy (5.52), then $\{h_\alpha\}$ are linearly independent and (5.52) holds everywhere. This means that the solution is globally defined on $\mathbf{R}^{1,1}$.

(5.47) leads to $SH - H\Lambda^{-1} = 0$, i.e.,

$$Sh_\beta - \lambda_\beta^{-1} h_\beta = 0.$$

Its complex conjugate transpose gives

$$h_\beta^* S^* - \bar{\lambda}_\beta^{-1} h_\beta^* = 0.$$

Hence

$$h_\beta^*(S^* + S)h_\gamma = (\bar{\lambda}_\beta^{-1} + \lambda_\gamma^{-1})h_\beta^* h_\gamma.$$

It is zero if $\lambda_\beta \neq \lambda_\gamma$. When $\lambda_\beta = \lambda_\gamma = \lambda^0$ (or $\bar{\lambda}^0$), it equals $\Big(\dfrac{1}{\lambda^0} + \dfrac{1}{\bar{\lambda}^0}\Big)h_\beta^* h_\gamma$. Hence

$$h_\beta^*(S^* + S)h_\gamma = h_\beta^*\Big(\frac{1}{\lambda^0} + \frac{1}{\bar{\lambda}^0}\Big)I h_\gamma.$$

Since $\{h_\gamma\}$ $(\gamma = 1, 2, \cdots, N)$ are linearly independent, we have

$$S^* + S = \Big(\frac{1}{\lambda^0} + \frac{1}{\bar{\lambda}^0}\Big)I. \tag{5.53}$$

Hence (5.50) holds, and $A_1, B_1 \in u(N)$. Therefore, we have the following theorem.

THEOREM 5.7 *Let $G = U(N)$, $A, B \in u(N)$. Suppose (A, B, Φ) is a solution of (5.38)–(5.40) on $\mathbf{R}^{1,1}$, S is defined by (5.47) in which λ_α and l_α's are chosen so that (5.51), (5.52) and* $\det H \neq 0$ *are satisfied. Then (A_1, B_1, Φ_1) defined by (5.42) is also a global solution of (5.38)–(5.40) on $\mathbf{R}^{1,1}$. Moreover, $A_1, B_1 \in u(N)$.*

Now we turn to the Lie group $SU(N)$. A matrix $g \in SU(N)$ if and only if $g \in U(N)$ and $\det g = 1$. The Lie algebra $su(N)$ of $SU(N)$ composes of all matrices A such that $A \in u(N)$ and $\operatorname{tr} A = 0$.

The above Theorem 5.7 also holds for the case of $SU(N)$. In this case, there are additional conditions $\operatorname{tr} A = \operatorname{tr} B = 0$ and $\operatorname{tr} A_1 = \operatorname{tr} B_1 = 0$. Thus, we must have $\operatorname{tr} S_\eta = 0$ and $\operatorname{tr} S_\xi = 0$. In fact, the first equation of (5.44) can be written as

$$S_\eta = -SAS^{-1} + A$$

and $\operatorname{tr} S_\eta = 0$ follows. From the definition of S, $-S + 2I = -H(\Lambda^{-1} - 2I)H^{-1}$. Since λ_0 is not real, $2I - S$ is non-degenerate. The second equation of (5.44) can be written as

$$S_\xi(S - 2I) = (S - 2I)B - B(S - 2I),$$

which implies $\operatorname{tr} S_\xi = 0$.

Remark 38 Let $\lambda = 1$, then $\Phi(1)$ satisfies

$$\Phi(1)_\eta = A\Phi(1), \quad \Phi(1)_\xi = B\Phi(1).$$

Comparing with (5.37), we know that $g = \Phi(1)g_0$ is a harmonic map where g_0 is a constant matrix. This kind of harmonic map has been studied in [8]. Here we use Darboux transformation to get more explicit conclusions.

5.3.3 Single soliton solutions

Take the trivial solution as a seed solution, single soliton and multi-soliton solutions can be obtained by Darboux transformations as mentioned above. In this process, only algebraic algorithm is necessary. For simplicity, we only consider the harmonic map $\mathbf{R}^{1,1} \to SU(2)$. However, the following discussions are essentially the same for $U(N)$.

An element of $SU(2)$ can be written as

$$\begin{pmatrix} \gamma & \beta \\ -\bar{\beta} & \bar{\gamma} \end{pmatrix},$$

where β and γ are complex numbers satisfying $\gamma\bar{\gamma} + \beta\bar{\beta} = 1$. Take A and B to be two non-zero constant elements of $su(2)$ with $[A, B] = 0$. Clearly they satisfy the equation of harmonic map (5.38) and (5.39). Suppose

$$A = \begin{pmatrix} \mathrm{i}p & 0 \\ 0 & -\mathrm{i}p \end{pmatrix}, \quad B = \begin{pmatrix} \mathrm{i}q & 0 \\ 0 & -\mathrm{i}q \end{pmatrix},$$

where p and q are non-zero real numbers. The solution of $\mathrm{d}g_0 = (A\mathrm{d}\eta + B\mathrm{d}\xi)g_0$ is

$$g_0 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(p\eta+q\xi)} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}(p\eta+q\xi)} \end{pmatrix},$$

which is a harmonic map. The corresponding $\Phi_0(\lambda)$ is

$$\Phi_0(\lambda) = \begin{pmatrix} l(\lambda) & 0 \\ 0 & l^{-1}(\lambda) \end{pmatrix},$$

where
$$l(\lambda) = \exp\Big(\mathrm{i}\lambda p\eta + \frac{\mathrm{i}\lambda}{2\lambda - 1}q\xi\Big).$$
g_0, A, B and Φ_0 constitute the seed solution of the Darboux transformation.

Take $\lambda_1 = \lambda_0$, $\lambda_2 = \bar{\lambda}_0$,
$$H = (h_1, h_2) = \begin{pmatrix} l(\lambda_0) & bl(\bar{\lambda}_0) \\ al^{-1}(\lambda_0) & l^{-1}(\bar{\lambda}_0) \end{pmatrix}.$$
Then
$$h_2^* h_1 = \bar{b}\overline{l(\bar{\lambda}_0)}l(\lambda_0) + a\overline{l^{-1}(\bar{\lambda}_0)}l^{-1}(\lambda_0).$$
Since
$$\begin{aligned} &\overline{l(\bar{\lambda}_0)} = \overline{\exp\Big(\mathrm{i}\bar{\lambda}_0 p\eta + \tfrac{\mathrm{i}\bar{\lambda}_0}{2\bar{\lambda}_0 - 1}q\xi\Big)} = l^{-1}(\lambda_0), \\ &l(\lambda_0) = \overline{l^{-1}(\bar{\lambda}_0)}, \end{aligned}$$
the equality $h_2^* h_1 = 0$ holds if and only if $\bar{b} = -a$.

By direct calculations, we have
$$\begin{aligned} &\det H = \mathrm{e}^r + |a|^2\mathrm{e}^{-r} > 0, \\ &S = H\Lambda^{-1}H^{-1} \\ &\quad = \frac{1}{\mathrm{e}^r + |a|^2\mathrm{e}^{-r}} \begin{pmatrix} \dfrac{\mathrm{e}^r}{\lambda_0} + \dfrac{\mathrm{e}^{-r}}{\bar{\lambda}_0}|a|^2 & \Big(\dfrac{1}{\lambda_0} - \dfrac{1}{\bar{\lambda}_0}\Big)\bar{a}\mathrm{e}^{\mathrm{i}s} \\ \Big(\dfrac{1}{\lambda_0} - \dfrac{1}{\bar{\lambda}_0}\Big)a\mathrm{e}^{-\mathrm{i}s} & \dfrac{\mathrm{e}^r}{\bar{\lambda}_0} + \dfrac{\mathrm{e}^{-r}}{\lambda_0}|a|^2 \end{pmatrix}, \end{aligned}$$
where
$$\begin{aligned} &r = \mathrm{i}(\lambda_0 - \bar{\lambda}_0)p\eta + \mathrm{i}\Big(\frac{\lambda_0}{2\lambda_0 - 1} - \frac{\bar{\lambda}_0}{2\bar{\lambda}_0 - 1}\Big)q\xi, \\ &s = (\lambda_0 + \bar{\lambda}_0)p\eta + \Big(\frac{\lambda_0}{2\lambda_0 - 1} + \frac{\bar{\lambda}_0}{2\bar{\lambda}_0 - 1}\Big)q\xi \end{aligned}$$
are real linear functions of ξ and η. The new fundamental solution and the new harmonic map are
$$\begin{aligned} &\Phi_1(\lambda) = (I - \lambda S)\Phi(\lambda), \\ &g_1 = \Phi_1(1)\left|\frac{\lambda_0}{\lambda_0 - 1}\right| \end{aligned}$$
respectively. The above right-multiplier $\left|\dfrac{\lambda_0}{\lambda_0 - 1}\right|$ is used to keep $g_1 \in SU(2)$.

Figure 5.1.

Remark 39 If we use $r - \ln|a|$ *and* $s + \mathrm{i}\ln(a/|a|)$ *instead of* r *and* s*, then* a *in* S *can be changed to* 1*.*

g_1 can be written as

$$g_1 = \begin{pmatrix} \gamma_1 & \beta_1 \\ -\bar{\beta}_1 & \bar{\gamma}_1 \end{pmatrix}$$

with

$$\gamma_1 = \frac{\left(1 - \frac{1}{\lambda_0}\right)\mathrm{e}^r + \left(1 - \frac{1}{\bar{\lambda}_0}\right)\mathrm{e}^{-r}}{\mathrm{e}^r + \mathrm{e}^{-r}} \mathrm{e}^{\mathrm{i}(p\eta + q\xi)} \left|\frac{\lambda_0}{\lambda_0 - 1}\right|,$$

$$\beta_1 = -\left(\frac{1}{\lambda_0} - \frac{1}{\bar{\lambda}_0}\right)\frac{1}{\mathrm{e}^r + \mathrm{e}^{-r}} \mathrm{e}^{\mathrm{i}(s - p\eta - q\xi)} \left|\frac{\lambda_0}{\lambda_0 - 1}\right|.$$

Write

$$\beta_1 = \rho_1 \mathrm{e}^{\mathrm{i}\theta_1}, \quad \gamma_1 = \sigma_1 \mathrm{e}^{\mathrm{i}\tau_1},$$

where $\rho_1 > 0$, $\sigma_1 > 0$, θ_1, τ_1 is real, then

$$\rho_1 = \frac{1}{2}\left|\frac{\lambda_0 - \bar{\lambda}_0}{\lambda_0(\lambda_0 - 1)}\right| \operatorname{sech} r,$$

$$\sigma_1 = (1 - \rho_1^2)^{1/2}.$$

The figures for ρ_1 and σ_1 with respect to r are shown in Figure 5.1.

Both ρ_1 and σ_1 are of the shape of solitons. As is known, $SU(2)$ is a three dimensional manifold, which is actually the three dimensional sphere

$$S^3 = \{(x_1, x_2, x_3, x_4) \,|\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

in $\mathbf{R}^4$. The relation between (x_1, x_2, x_3, x_4) and $\begin{pmatrix} \gamma & \beta \\ -\bar{\beta} & \bar{\gamma} \end{pmatrix}$ is given by

$$\gamma = x_1 + ix_2, \quad \beta = x_3 + ix_4.$$

Unless r depends on x or t only, the image of the map tends to two circles $x_3 = x_4 = 0$ and $x_1^2 + x_2^2 = 1$ as $x \to \pm\infty$ for fixed t, or as $t \to \pm\infty$ for fixed x. Hence these two circles are limiting circles.

Remark 40 For $SU(2)$*, the invariant metric is the standard metric of the sphere* S^3*. Therefore, a harmonic map to* $SU(2)$ *is just a harmonic map to the sphere* S^3*.*

5.3.4 Multi-soliton solutions

Multi-soliton solutions can be obtained by composition of successive Darboux transformations.

Let λ_0, λ_1, $\cdots$, λ_k be $k+1$ non-real complex numbers. Suppose that they are distinct and satisfy

$$\begin{aligned}&\text{(i) } |2\lambda_l - 1| \ (l = 0, 1, \cdots, k) \text{ are distinct,}\\ &\text{(ii) } (\lambda_l - \bar{\lambda}_l)p \pm \Big(\frac{\lambda_l}{2\lambda_l - 1} - \frac{\bar{\lambda}_l}{2\bar{\lambda}_l - 1}\Big)q \neq 0.\end{aligned}$$

Here p and q are the constants appeared in the seed solution

$$g_0 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(p\eta + q\xi)} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}(p\eta + q\xi)} \end{pmatrix}.$$

Let

$$\begin{aligned}r_l &= \mathrm{i}(\lambda_l - \bar{\lambda}_l)p\eta + \mathrm{i}\Big(\frac{\lambda_l}{2\lambda_l - 1} - \frac{\bar{\lambda}_l}{2\bar{\lambda}_l - 1}\Big)q\xi,\\ s_l &= (\lambda_l + \bar{\lambda}_l)p\eta + \Big(\frac{\lambda_l}{2\lambda_l - 1} + \frac{\bar{\lambda}_l}{2\bar{\lambda}_l - 1}\Big)q\xi,\\ &(l = 0, 1, 2, \cdots, k).\end{aligned}$$

They are real linear functions of ξ and η. It can be verified that r_l/r_j's $(l \neq j)$ are not constants if and only if (i) holds. On the other hand, (ii) means that r_l really depends on x and t. Here r_0 and s_0 are just r and s in the construction of the single soliton solutions.

Since we have g_0 and Φ_0, we can define

$$\begin{aligned}&\Phi_j(\lambda) = (I - \lambda S_{j-1}) \cdots (I - \lambda S_0)\Phi_0(\lambda),\\ &g_j = (I - S_{j-1}) \cdots (I - S_0) g_0 \left|\frac{\lambda_0}{\lambda_0 - 1}\right| \cdots \left|\frac{\lambda_{j-1}}{\lambda_{j-1} - 1}\right|\\ &(j = 1, 2, \cdots, l),\end{aligned}$$

recursively. Let S_0 be the matrix S in the construction of the single soliton solutions, S_1, S_2, $\cdots$ be defined by (5.47) for $\Phi = \Phi_1$, Φ_2, $\cdots$. According to the expression of S_0,

$$\lim_{r_0 \to \pm\infty} S_0 = \begin{pmatrix} \mu_0 & 0 \\ 0 & \bar{\mu}_0 \end{pmatrix}$$

where

$$\mu_0 = \begin{cases} 1/\lambda_0 & \text{as } r_0 \to +\infty,\\ 1/\bar{\lambda}_0 & \text{as } r_0 \to -\infty.\end{cases}$$

LEMMA 5.8 *When r_l is bounded and $t \to +\infty$,*

$$\lim_{t\to\pm\infty} S_l \longrightarrow \begin{pmatrix} \mu_l & 0 \\ 0 & \bar{\mu}_l \end{pmatrix}$$
$$(\mu_l = 1/\lambda_l \textit{ or } 1/\bar{\lambda}_l;\ l = 0, 1, 2, \cdots, k-1).$$

Proof. We use induction to prove it. Suppose it holds for $l = 0, 1, 2, \cdots, i-1$. Now we consider the case $l = i$.

Since r_i is finite, $r_0, \cdots, r_{i-1} \to \pm\infty$ when $t \to +\infty$, hence

$$\Phi_i \sim \begin{pmatrix} A(\lambda)l(\lambda) & 0 \\ 0 & \widetilde{A}(\lambda)l^{-1}(\lambda) \end{pmatrix},$$

where $\sim$ represents asymptotic value,

$$A(\lambda) = \prod_{h=0}^{i-1}(1-\lambda\mu_h), \quad \widetilde{A}(\lambda) = \prod_{h=0}^{i-1}(1-\lambda\bar{\mu}_k).$$

Since

$$\overline{A(\lambda)} = \widetilde{A}(\bar{\lambda}),$$

we have

$$H_i \sim \begin{pmatrix} A(\lambda_i)l(\lambda_i) & b_i A(\bar{\lambda}_i)l(\bar{\lambda}_i) \\ a_i\widetilde{A}(\lambda_i)l^{-1}(\lambda_i) & \widetilde{A}(\bar{\lambda}_i)l^{-1}(\bar{\lambda}_i) \end{pmatrix}$$

where

$$b_i = -\bar{a}_i.$$

As before, a_i and b_i can be put into r_i and s_i so that $a_i = 1$, $b_i = -1$. By direct calculation,

$$\begin{aligned} S_i = \ & H_i\Lambda_i^{-1}H_i^{-1} \sim \frac{1}{|A(\lambda_i)|^2\mathrm{e}^{r_i} + |A(\bar{\lambda}_i)|^2\mathrm{e}^{-r_i}} \\ & \cdot \begin{pmatrix} \dfrac{|A(\lambda_i)|^2}{\lambda_i}\mathrm{e}^{r_i} + \dfrac{|A(\bar{\lambda}_i)|^2}{\bar{\lambda}_i}\mathrm{e}^{-r_i} & \Big(\dfrac{1}{\lambda_i} - \dfrac{1}{\bar{\lambda}_i}\Big)A(\lambda_i)A(\bar{\lambda}_i)\mathrm{e}^{\mathrm{i}s_i} \\ \Big(\dfrac{1}{\lambda_i} - \dfrac{1}{\bar{\lambda}_i}\Big)\widetilde{A}(\lambda_i)\widetilde{A}(\bar{\lambda}_i)\mathrm{e}^{-\mathrm{i}s_i} & \dfrac{|A(\lambda_i)|^2}{\bar{\lambda}_i}\mathrm{e}^{r_i} + \dfrac{|A(\bar{\lambda}_i)|^2}{\lambda_i}\mathrm{e}^{-r_i} \end{pmatrix}. \end{aligned}$$

When $t \to +\infty$ and $r_i \to \pm\infty$,

$$S_i \sim \begin{pmatrix} \mu_i & 0 \\ 0 & \bar{\mu}_i \end{pmatrix}, \quad \mu_i = \frac{1}{\lambda_i} \text{ or } \frac{1}{\bar{\lambda}_i}.$$

The lemma is proved.

When r_{k-1} is finite and $t \to +\infty$, $r_0, \cdots, r_{k-2} \to \pm\infty$, hence

$$\begin{aligned} g_k \sim & \; (I - S_{k-1}) \begin{pmatrix} 1-\mu_{k-2} & \\ & 1-\bar{\mu}_{k-2} \end{pmatrix} \cdots \begin{pmatrix} 1-\mu_0 & \\ & 1-\bar{\mu}_0 \end{pmatrix} \\ & \cdot \begin{pmatrix} \mathrm{e}^{\mathrm{i}(p\eta+q\xi)} & \\ & \mathrm{e}^{-\mathrm{i}(p\eta+q\xi)} \end{pmatrix} \left| \frac{\lambda_0}{\lambda_0 - 1} \cdots \frac{\lambda_{k-1}}{\lambda_{k-1}-1} \right|. \end{aligned}$$

According to the asymptotic expression of S_{k-1}, we know that the asymptotic behavior of g_k as r_{k-1} bounded and $t \to +\infty$ is the same as a single soliton solution.

If one of $r_0, \cdots, r_{k-2}$ does not tend to $\pm\infty$ (say, $r_i \not\to \pm\infty$ with certain $i < k-1$) as $t \to +\infty$, then the other $r_j \to \pm\infty$. According to the theorem of permutability of Darboux transformation, we can change the Darboux matrix with λ_i to be the last one. The factors should have some change, but the result is still g_k. The asymptotic behavior of g_k as $t \to +\infty$ along the direction $r_i \not\to \pm\infty$ is still a single soliton.

If $t \to -\infty$, there are similar conclusions, but μ_i should be changed to its complex conjugate. This means that the soliton has "phase shift". Therefore, we have the following theorem.

Theorem 5.9 *A harmonic map from $\mathbf{R}^{1,1}$ to $U(N)$ (or $SU(N)$) derived from the trivial solution by k Darboux transformations has the following property. When $t \to \pm\infty$, the asymptotic solution split up into k single solitons. When $t \to +\infty$ and $t \to -\infty$, these k single solitons are arranged in opposite order and have their own phase shifts.*

Thus, the behavior of the multi-soliton solutions of a harmonic map from $\mathbf{R}^{1,1}$ to $U(N)$ (or $SU(N)$) is quite similar to that of the multi-solitons of KdV equations etc., i.e., all of them have the property of elastic scattering.

5.4 Harmonic maps from R^2 to $U(N)$

5.4.1 Harmonic maps from R^2 to $U(N)$ and their Darboux transformations

Let (x, y) be the orthonormal coordinates of $\mathbf{R}^2$, then the energy of $\phi(x, y)$ from $\mathbf{R}^2$ (or a region Ω in $\mathbf{R}^2$) to $U(N)$ is

$$S[\phi] = -\int \operatorname{tr}(\phi_x\phi^{-1}\phi_x\phi^{-1} + \phi_y\phi^{-1}\phi_y\phi^{-1})\mathrm{d}x\,\mathrm{d}y. \tag{5.54}$$

ϕ is called a harmonic map from $\mathbf{R}^2$ to $U(N)$ if the Euler equation for $S[\phi]$ holds. We suppose Ω to be simply connected.

Denote

$$\zeta = x + \mathrm{i}y, \quad \bar{\zeta} = x - \mathrm{i}y \tag{5.55}$$

and

$$\frac{\partial}{\partial\bar{\zeta}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial\zeta} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\Big). \tag{5.56}$$

Let

$$A = \phi_{\bar{\zeta}}\phi^{-1}, \quad B = \phi_\zeta\phi^{-1}. \tag{5.57}$$

Similar to Section 5.3, ϕ is a harmonic map if and only if A and B satisfy

$$A_\zeta - B_{\bar{\zeta}} + [A, B] = 0, \tag{5.58}$$

$$A_\zeta + B_{\bar{\zeta}} = 0. \tag{5.59}$$

Since $\phi \in U(N)$, i.e., $\phi^* = \phi^{-1}$,

$$A^* = \phi^{*-1}\phi^*_\zeta = \phi(\phi^{-1})_\zeta = -\phi\phi^{-1}\phi_\zeta\phi^{-1} = -B.$$

Hence the constraint on A and B is

$$A^* = -B. \tag{5.60}$$

Conversely, if A and B satisfy the constraint (5.60) and the equations (5.58), (5.59), then ϕ can be solved from

$$\phi_{\bar{\zeta}} = A\phi, \quad \phi_\zeta = B\phi. \tag{5.61}$$

Moreover, ϕ satisfies

$$\begin{aligned}(\phi^*\phi)_\zeta \quad &= \phi^*_\zeta\phi + \phi^*\phi_\zeta = (\phi_{\bar{\zeta}})^*\phi + \phi^*\phi_\zeta \\ &= (A\phi)^*\phi + \phi^*B\phi = \phi^*(A^* + B)\phi = 0.\end{aligned} \tag{5.62}$$

Similarly,

$$(\phi^*\phi)_{\bar{\zeta}} = 0. \tag{5.63}$$

If $\phi^*\phi = I$ holds at one point, it holds identically. Hence, (5.58), (5.59) and (5.60) can be regarded as the equations of the harmonic map from $\mathbf{R}^2$ to $U(N)$. These equations uniquely determine the harmonic map up to a right-multiplied constant matrix in $U(N)$.

Now consider the Lax pair

$$\Phi_{\bar{\zeta}} = \lambda A\Phi, \quad \Phi_\zeta = \frac{\lambda}{2\lambda - 1}B\Phi \tag{5.64}$$

($\lambda \neq 1/2$) whose integrability conditions are (5.58) and (5.59). Suppose the initial condition $\Phi|_{\zeta=0} \in U(N)$, then

$$g = \Phi(1) \tag{5.65}$$

is a harmonic map. g is unique up to a right-multiplied constant matrix in $U(N)$. Here $\Phi(\lambda)$ is called an extended solution of the harmonic map g [102]. Notice that if $|2\lambda - 1| = 1$, then $\bar{\lambda} = \dfrac{\lambda}{2\lambda - 1}$ and $\Phi(\lambda) \in U(N)$.

The harmonic maps from $\mathbf{R}^2$ to $U(N)$ have been discussed in [102]. Here we mainly consider their properties related to the Darboux transformation.

Remark 41 The symbols used here differ from those in [102]. The main differences are:

(i) The A and B are equivalent to $2A_{\bar{\zeta}}$ and $2A_{\zeta}$ in [102].

(ii) In the definition of A, B and the Lax pair (5.64), the order of multiplication of matrices is different from those used in [102]. Therefore, left multiplications in [102] are changed to right multiplications in many places.

(iii) The parameters λ and $\dfrac{\lambda}{2\lambda - 1}$ were written as $\dfrac{1}{2}(1-\lambda)$ and $\dfrac{1}{2}(1-\lambda^{-1})$ in [102]. In our notation we introduce $\mu = 1-2\lambda$ which corresponds to the λ in [102].

As in Section 5.3, we can construct the Darboux matrix $D = I - \lambda S$ and the Darboux transformation

$$(A, B, \Phi) \longrightarrow (A_1, B_1, \Phi_1) \tag{5.66}$$

where

$$\Phi_1 = (I - \lambda S)\Phi, \tag{5.67}$$

$$A_1 = A - S_{\bar{\zeta}}, \quad B_1 = B + S_{\zeta}. \tag{5.68}$$

In order to make (A_1, B_1, Φ_1) satisfy the Lax pair

$$\Phi_{1\bar{\zeta}} = \lambda A_1 \Phi_1, \quad \Phi_{1\zeta} = \frac{\lambda}{2\lambda - 1} B_1 \Phi_1, \tag{5.69}$$

S should satisfy

$$S_{\bar{\zeta}} S = AS - SA, \quad S_{\zeta}(S - 2I) = SB - BS. \tag{5.70}$$

Moreover, $A_1^* = -B_1$ must hold.

Take N complex numbers λ_1, λ_2, $\cdots$, λ_N ($\neq 0, 1/2$). Let

$$h_\rho = \Phi(\lambda_\rho) l_\rho \quad (\rho = 1, 2, \cdots, N) \tag{5.71}$$

where l_ρ's are constant column vectors so that $H = (h_1, h_2, \cdots, h_N)$ is non-degenerate, then

$$S = H\Lambda^{-1}H^{-1} \tag{5.72}$$

with $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$ satisfies (5.70). The proof for this is the same as in $\mathbf{R}^{1,1}$ case.

It is necessary to choose λ_ρ and l_ρ so that $A_1^* = -B_1$ holds. This can be done as follows. Take ω_1 to be a non-real complex number and

$$\omega_2 = \frac{\bar{\omega}_1}{2\bar{\omega}_1 - 1}. \tag{5.73}$$

We choose

$$\lambda_\rho = \begin{cases} \omega_1 & (\rho = 1, \cdots, k) \\ \omega_2 & (\rho = k+1, \cdots, N) \end{cases} \qquad (0 < k < N). \tag{5.74}$$

Moreover, suppose $|2\omega_1 - 1| \neq 1$ so that $\omega_1 \neq \omega_2$. Let $h_1, \cdots, h_k$ be the column solutions of the Lax pair for $\lambda = \omega_1$, $h_{k+1}, \cdots, h_N$ be the column solutions of the Lax pair for $\lambda = \omega_2$, i.e.,

$$\begin{aligned} &h_a = \Phi(\omega_1) l_a, \quad h_p = \Phi(\omega_2) l_p \\ &(a = 1, 2, \cdots, k;\ p = k+1, \cdots, N) \end{aligned} \tag{5.75}$$

where $l_1, \cdots, l_N$ are linearly independent constant column vectors. If $h_1, \cdots, h_N$ are chosen to be orthogonal with each other at one point (say $(0,0)$), then

$$h_p^* h_a = 0 \quad (a = 1, 2, \cdots, k;\ p = k+1, \cdots, N) \tag{5.76}$$

hold everywhere. This follows from

$$\begin{aligned} &(h_p^*)_{\bar\zeta} = (h_{p\zeta})^* = \Big(\frac{\omega_2}{2\omega_2 - 1} B h_p\Big)^* = -\omega_1 h_p^* A, \\ &(h_p^* h_a)_{\bar\zeta} = -\omega_1 h_1^* A h_a + h_1^* \omega_1 A h_a = 0, \end{aligned} \tag{5.77}$$

and similarly,

$$(h_p^* h_a)_\zeta = 0. \tag{5.78}$$

From the definition of S, we have

$$S h_a = \frac{1}{\omega_1} h_a, \quad S h_p = \frac{1}{\omega_2} h_p, \tag{5.79}$$

$$h_a^* S^* = \frac{1}{\bar{\omega}_1} h_a^*, \quad h_p^* S^* = \frac{1}{\bar{\omega}_2} h_p^*. \tag{5.80}$$

Hence

$$\begin{aligned}
&h_a^*(S^* - S)h_b = \Big(\frac{1}{\bar{\omega}_1} - \frac{1}{\omega_1}\Big) h_a^* h_b,\\
&h_p^*(S^* - S)h_q = \Big(\frac{1}{\bar{\omega}_2} - \frac{1}{\omega_2}\Big) h_p^* h_q,\\
&h_p^*(S^* - S)h_a = 0,\\
&h_a^*(S^* - S)h_p = 0\\
&(a, b = 1, \cdots, k;\ p, q = k+1, \cdots, N).
\end{aligned} \tag{5.81}$$

According to (5.73),

$$\frac{1}{\omega_1} - \frac{1}{\bar{\omega}_1} = \frac{1}{\omega_2} - \frac{1}{\bar{\omega}_2}. \tag{5.82}$$

So

$$h_\beta^*(S^* - S)h_\alpha = h_\beta^* \Big(\frac{1}{\bar{\omega}_1} - \frac{1}{\omega_1}\Big) I h_\alpha$$

$$(\alpha, \beta = 1, 2, \cdots, N)$$

and $\dfrac{1}{\omega_1} - \dfrac{1}{\omega_2}$ is real. Since $\{h_\alpha\}$ consists of N linearly independent vectors,

$$S^* - S = \Big(\frac{1}{\bar{\omega}_1} - \frac{1}{\omega_1}\Big) I. \tag{5.83}$$

Then

$$A_1^* + B_1 = (A_1 - S_{\bar{\zeta}})^* + B + S_\zeta = A^* + B - S_\zeta^* + S_\zeta = 0,$$

i.e., A_1 and B_1 satisfy the constraint (5.60) for $U(N)$. Therefore, we have the following theorem.

Theorem 5.10 *Suppose that λ_ρ's and h_ρ's satisfy (5.74) and (5.76), h_ρ's are linearly independent. Then the Darboux transformation (5.66) gives a new harmonic map from $\mathbf{R}^2$ to $U(N)$, which is expressed by*

$$g_1 = \Phi_1(1)(\Phi_1|_0)^{-1}$$

where $(\Phi_1|_0)$ is the value of Φ_1 at a fixed point $\zeta = 0$.

This topic has also been discussed by several authors such as [112] and [54].

Now we use the Darboux transformation to derive the projective operators π and $\pi^\perp$ in $\mathbf{C}^N$ given by [102].

According to (5.83),

$$S^* - \frac{1}{\bar{\omega}_1} I = S - \frac{1}{\omega_1} I,$$

so $S - \dfrac{1}{\omega_1} I$ is Hermitian. On the other hand,

$$S - \frac{1}{\omega_1} I = H\Big(\Lambda^{-1} - \frac{1}{\omega_1} I\Big) H^{-1},$$

so $S - \dfrac{1}{\omega_1} I$ has k eigenvalues 0 and $N-k$ real eigenvalues $\dfrac{1}{\omega_2} - \dfrac{1}{\omega_1}$. Moreover,

$$\begin{aligned}
\pi &= \Big(\frac{1}{\omega_2} - \frac{1}{\omega_1}\Big)^{-1}\Big(S - \frac{1}{\omega_1} I\Big) = \Big(\frac{1}{\omega_2} - \frac{1}{\omega_1}\Big)^{-1} \cdot \\
&\qquad \cdot H \left[\begin{pmatrix} \frac{1}{\omega_1} I_k & 0 \\ 0 & \frac{1}{\omega_2} I_{N-k} \end{pmatrix} - \begin{pmatrix} \frac{1}{\omega_1} I_k & 0 \\ 0 & \frac{1}{\omega_1} I_{N-k} \end{pmatrix} \right] H^{-1} \\
&= H \begin{pmatrix} 0 & 0 \\ 0 & I_{N-k} \end{pmatrix} H^{-1}
\end{aligned} \tag{5.84}$$

is an Hermitian matrix with k eigenvalues 0 and $N-k$ eigenvalues 1. π is a projective matrix, and

$$\pi^\perp = I - \pi = H \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} H^{-1} \tag{5.85}$$

is the complement of π. The invariant subspace of $\pi^\perp$ is orthogonal to that of π. From (5.84) and (5.85),

$$S = \frac{1}{\omega_2}\pi + \frac{1}{\omega_1}\pi^\perp, \tag{5.86}$$

and the Darboux matrix

$$D = I - \lambda S = \Big(1 - \frac{\lambda}{\omega_2}\Big)\pi + \Big(1 - \frac{\lambda}{\omega_1}\Big)\pi^\perp = (\pi + \gamma\pi^\perp)\Big(1 - \frac{\lambda}{\omega_2}\Big) \tag{5.87}$$

where

$$\gamma = \frac{\omega_2(\omega_1 - \lambda)}{\omega_1(\omega_2 - \lambda)}. \tag{5.88}$$

Remark 42 S can also be written as

$$S = \frac{1}{\omega_1} I - \Big(\frac{1}{\omega_1} - \frac{1}{\omega_2}\Big)\pi. \tag{5.89}$$

Substituting it into (5.70), we get

$$\begin{aligned} \pi_{\bar\zeta} &= -\omega_1 \pi A + \omega_2 A\pi + (\omega_1 - \omega_2)\pi A \pi, \\ \pi_\zeta &= \bar\omega_1 B\pi - \bar\omega_2 \pi B + (\bar\omega_2 - \bar\omega_1)\pi B \pi, \end{aligned} \tag{5.90}$$

These equations are first derived in [102] by the decomposition of loop group. Here we use Darboux transformation to give the explicit expressions of the solutions of these equations. Moreover, the new harmonic map is given by

$$g_1 = \Phi_1(1) \Big/ \Big(1 - \frac{1}{\omega_2}\Big) = (\pi + \gamma_1 \pi^\perp)\Phi(1) \tag{5.91}$$

where

$$\gamma_1 = \frac{1 - 1/\omega_1}{1 - 1/\omega_2} \tag{5.92}$$

satisfies $\bar\gamma_1 \gamma_1 = 1$. If we are considering the harmonic map to $SU(N)$, then the new harmonic map is

$$g_1 = \Phi_1(1)(I - \Lambda^{-1})^{-1}. \tag{5.93}$$

5.4.2 Soliton solutions

Similar to Section 5.3, the soliton solutions of the harmonic map from $\mathbf{R}^2 \to U(N)$ can be constructed explicitly. For simplicity, we choose $N = 2$, so the solution is a special harmonic map from $\mathbf{R}^2 \to SU(2)$.

Take the seed solution

$$g_0 = \begin{pmatrix} \mathrm{e}^{\tau\bar\zeta - \bar\tau\zeta} & 0 \\ 0 & \mathrm{e}^{-(\tau\bar\zeta - \bar\tau\zeta)} \end{pmatrix}$$

where τ is a non-zero complex number, then

$$A = g_{0\bar\zeta} g_0^{-1} = \begin{pmatrix} \tau & 0 \\ 0 & -\tau \end{pmatrix}, \quad B = g_{0\zeta} g_0^{-1} = \begin{pmatrix} -\bar\tau & 0 \\ 0 & \bar\tau \end{pmatrix}.$$

The solution of the Lax pair (5.64) is

$$\Phi_0 = \begin{pmatrix} l(\lambda) & 0 \\ 0 & l^{-1}(\lambda) \end{pmatrix} \tag{5.94}$$

where

$$l(\lambda) = \exp\Big(\lambda\tau\bar{\zeta} - \frac{\lambda}{2\lambda - 1}\bar{\tau}\zeta\Big). \tag{5.95}$$

Take a complex number ω_1 satisfying $|2\omega_1 - 1| \neq 1$. Let

$$\omega_2 = \frac{\bar{\omega}_1}{2\bar{\omega}_1 - 1},$$

then

$$l^{-1}(\omega_1) = \overline{l(\omega_2)}, \quad l^{-1}(\omega_2) = \overline{l(\omega_1)}. \tag{5.96}$$

Let

$$H = \begin{pmatrix} l(\omega_1) & -\bar{a}l(\omega_2) \\ al^{-1}(\omega_1) & l^{-1}(\omega_2) \end{pmatrix}, \tag{5.97}$$

then

$$\det H = |l(\omega_1)|^2 + |a|^2|l(\omega_2)|^{-2}. \tag{5.98}$$

By (5.72),

$$S = \frac{1}{\mathrm{e}^r + |a|^2\mathrm{e}^{-r}} \begin{pmatrix} \dfrac{\mathrm{e}^r}{\omega_1} + |a|^2\dfrac{\mathrm{e}^{-r}}{\omega_2} & \Big(\dfrac{1}{\omega_1} - \dfrac{1}{\omega_2}\Big)\bar{a}\mathrm{e}^{\mathrm{i}s} \\ \Big(\dfrac{1}{\omega_1} - \dfrac{1}{\omega_2}\Big)a\mathrm{e}^{-\mathrm{i}s} & \dfrac{\mathrm{e}^r}{\omega_2} + |a|^2\dfrac{\mathrm{e}^{-r}}{\omega_1} \end{pmatrix}, \tag{5.99}$$

where

$$\begin{aligned} r &= (\omega_1 - \omega_2)\tau\bar{\zeta} + (\bar{\omega}_1 - \bar{\omega}_2)\bar{\tau}\zeta, \\ s &= -\mathrm{i}((\omega_1 + \omega_2)\tau\bar{\zeta} - (\bar{\omega}_1 + \bar{\omega}_2)\bar{\tau}\zeta) \end{aligned} \tag{5.100}$$

are real valued functions. $|a|^2$, a and $\bar{a}$ can all be eliminated by adding some constants to r and s. Therefore, we can suppose $a = 1$. The harmonic map is

$$\begin{aligned} g_1 &= \Phi_1(1) = (I - S)\begin{pmatrix} \dfrac{\omega_1}{1-\omega_1} & 0 \\ 0 & \dfrac{\omega_2}{1-\omega_2} \end{pmatrix} g_0(\zeta, \bar{\zeta}) \\ &= \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \end{aligned} \tag{5.101}$$

where

$$\alpha = \Big(\mathrm{e}^r + \frac{1}{\gamma_1}\mathrm{e}^{-r}\Big)\mathrm{e}^{\tau\bar{\zeta} - \bar{\tau}\zeta}, \quad \beta = \frac{\omega_2 - \omega_1}{\omega_1(1-\omega_2)}\mathrm{e}^{\mathrm{i}s - (\tau\bar{\zeta} - \bar{\tau}\zeta)}, \tag{5.102}$$

γ_1 is defined by (5.92). The asymptotic behavior of g_1 is similar to that in $\mathbf{R}^{1,1}$ case. Moreover, we can also construct multi-soliton solutions explicitly, and can prove that a k-soliton solution is asymptotic to k single solitons as $y \to \pm\infty$. The proof is similar to the $\mathbf{R}^{1,1}$ case and is omitted here.

5.4.3 Uniton

Uniton is a special harmonic map from $\mathbf{R}^2$ (or a region $\Omega \subset \mathbf{R}^2$) to $U(N)$ and was introduced by K. Uhlenbeck in [102]. It is an important notion because each harmonic map from S^2 to $U(N)$ is a uniton. In order to be compatible with [102], we first change some parameters which are used before.

For Lax pair (5.64), let $\mu = 1 - 2\lambda$, then

$$\lambda = \frac{1-\mu}{2}, \qquad \frac{\lambda}{2\lambda - 1} = \frac{1-\mu^{-1}}{2}. \tag{5.103}$$

Let

$$\Phi(\lambda) = \Phi\left(\frac{1-\mu}{2}\right) = \Psi(\mu), \tag{5.104}$$

(5.64) becomes

$$\Psi_{\bar\zeta} = \frac{1-\mu}{2} A\Psi, \quad \Psi_\zeta = \frac{1-\mu^{-1}}{2} B\Psi. \tag{5.105}$$

Definition 5.11 *Suppose g is a harmonic map from $\mathbf{R}^2$ (or Ω) to $U(N)$. If its extended solution $\Psi(\mu)$ satisfies the following four conditions:*

$$\text{(a) } \Psi(\mu) = \sum_{a=0}^{n} T_a \mu^a \quad (\text{a polynomial of } \mu), \tag{5.106}$$

$$\text{(b) } \Psi(1) = I, \tag{5.107}$$

$$\text{(c) } \Psi(-1) = g, \tag{5.108}$$

$$\text{(d) } \Psi(\mu)^* = \Psi(\bar\mu^{-1})^{-1} \quad (\mu \neq 0), \tag{5.109}$$

then g is called a uniton, and $\Psi(\mu)$ is called an extended solution of uniton.

The conditions (b), (c) and (d) above are constraints to the initial values. (b) means that $\Phi(\mu)|_0 = I$ for $\mu = 1$. (c) holds if $\Psi(-1)|_0 \in U(N)$. (d) holds everywhere if it holds at $z = 0$. The last statement can be proved by differentiating $\Psi(\mu)^*\Psi(\bar\mu^{-1})$ and using the Lax pair together with the relation $A^* = -B$. (a) is an essential condition, based on the conditions (b), (c) and (d).

For a uniton, the extended solution $\Psi(\mu)$ and its degree are not unique. The minimum degree of the extended solutions is called the degree of the uniton, or uniton number [102].

THEOREM 5.12 *Suppose an $N \times N$ matrix valued polynomial*

$$\Psi(\mu) = \sum_{a=0}^{n} T_a \mu^a$$

satisfies the conditions (b) and (d), then $\Psi(\mu)$ can be written as

$$\Psi(\mu) = (\pi_1 + \mu\pi_1^{\perp}) \cdots (\pi_n + \mu\pi_n^{\perp}), \tag{5.110}$$

where π_a $(a = 1, \cdots, n)$ are Hermitian projections, and $\pi_a^{\perp}$'s are their orthogonal complements.

Proof. First, we consider the case $n = 1$, i.e.,

$$\Psi(\mu) = T_0 + T_1\mu.$$

By $\Psi(1) = I$, $T_0 + T_1 = I$. The condition (d) is

$$(T_0 + (I - T_0)\mu)^*(T_0 + (I - T_0)\bar{\mu}^{-1}) = I,$$

which leads to

$$\begin{aligned} &T_0^*(I - T_0) = (I - T_0)^* T_0 = 0, \\ &T_0^* T_0 + (I - T_0^*)(I - T_0) = I. \end{aligned}$$

Hence

$$T_0^* T_0 = T_0^* = T_0.$$

This means that T_0 is the Hermitian projective operator π_0 and $T_1 = I - T_0 = \pi^{\perp}$.

Now we use mathematical induction to prove the general result. Assume (5.110) is true for n, and

$$\widetilde{\Psi}(\mu) = \sum_{a=0}^{n+1} T_a \mu^a$$

satisfies conditions (b) and (d). Without loss of generality, suppose $T_0 \neq 0$, $T_{n+1} \neq 0$. (Otherwise, the degree of $\Psi(\mu)$ is less than $n + 1$ or $\Psi(\mu)$ can be written as $\mu\Psi_1(\mu)$ with $\deg(\Psi_1(\mu)) \leq n$. The assumption of induction implies that the theorem is true.) $\det \Psi(\mu)$ is a polynomial of

μ. If it has a non-zero root $\mu = \mu_0 \neq 0$, then $\Psi(\mu_0)$ is degenerate, which is contradict to condition (d). Hence $\det \Psi(\mu) = 0$ only when $\mu = 0$. It follows that $\det T_0 = 0$ and $T_0 l = 0$ has non-zero solutions for l. All the solutions l of $T_0 l = 0$ form a proper subspace P. Let $\pi_{n+1}^{\perp}$ be the Hermitian projective operator to P. It is non-trivial and its orthogonal complement is denoted by π_{n+1} ($\pi_{n+1} \neq 0$, $\pi_{n+1}^{\perp} \neq 0$).

Let

$$\Psi(\mu) = \widetilde{\Psi}(\mu)(\pi_{n+1} + \mu^{-1}\pi_{n+1}^{\perp}).$$

Since $T_0 \pi_{n+1}^{\perp} = 0$, $\Psi(\mu)$ is a polynomial of μ of degree n and condition (b) holds. The condition (d) holds for $\Psi(\mu)$ too, since

$$\begin{aligned}
\Psi(\bar{\mu}^{-1})\Psi(\mu)^* &= \widetilde{\Psi}(\bar{\mu}^{-1})(\pi_{n+1} + \bar{\mu}\pi_{n+1}^{\perp})(\pi_{n+1} + \bar{\mu}^{-1}\pi_{n+1}^{\perp})\widetilde{\Psi}(\mu)^* \\
&= \widetilde{\Psi}(\bar{\mu}^{-1})\widetilde{\Psi}(\mu)^* = I.
\end{aligned}$$

By the assumption of induction, $\Psi(\mu) = \prod_{a=1}^{n}(\pi_a + \mu\pi_a^{\perp})$, hence

$$\widetilde{\Psi}(\mu) = \prod_{a=1}^{n+1}(\pi_a + \mu\pi_a^{\perp}).$$

The theorem is proved.

This theorem shows that the extended solution of uniton can be expressed as a product (5.110). However, this decomposition is point-wise. We have not proved the smoothness of π_a and $\pi_a^{\perp}$. This will be obtained later.

In [102], the uniton of degree one is constructed as follows.

Suppose $\Psi(\mu) = \pi + \mu\pi^{\perp}$ is an extended solution of uniton. Let

$$\begin{aligned}
\Psi_{\bar{\zeta}}(\mu)\Psi^{-1}(\mu) &= (1-\mu)\pi_{\bar{\zeta}}(\pi + \mu^{-1}\pi^{\perp}) \\
&= (1-\mu)\pi_{\bar{\zeta}}\pi + \frac{1-\mu}{\mu}\pi_{\bar{\zeta}}\pi^{\perp}, \\
\Psi_{\zeta}(\mu)\Psi^{-1}(\mu) &= (1-\mu)\pi_{\zeta}(\pi + \mu^{-1}\pi^{\perp}) \\
&= (1-\mu^{-1})(-\pi_{\zeta}\pi^{\perp}) + (1-\mu)\pi_{\zeta}\pi.
\end{aligned}$$

Comparing with the Lax pair (5.105), we have

$$A = 2\pi_{\bar{\zeta}}\pi, \quad B = -2\pi_{\zeta}\pi^{\perp} \tag{5.111}$$

and the conditions for π

$$\pi_{\bar{\zeta}}\pi^{\perp} = 0, \quad \pi_{\zeta}\pi = 0. \tag{5.112}$$

The last two conditions are equivalent, since

$$\pi_\zeta \pi = (\pi \pi_{\bar{\zeta}})^* = (-\pi \pi^\perp_{\bar{\zeta}})^* = (\pi_{\bar{\zeta}} \pi^\perp)^*.$$

From $\pi^\perp \pi = 0$, we have $0 = (\pi^\perp \pi)_\zeta = \pi^\perp \pi_\zeta - \pi_\zeta \pi$. Hence (5.112) is equivalent to

$$\pi^\perp \pi_\zeta = 0. \tag{5.112$'$}$$

The projective operator π satisfying this condition is constructed as follows.

If $u_1, \cdots, u_k$ are k linearly independent vector functions satisfying

$$u_{a\zeta} = \sum c_{ab} u_b, \tag{5.113}$$

then for any invertible linear combination

$$v_a = \sum q_{ab} u_b \quad (\det(q_{ab}) \neq 0),$$

whose coefficients are arbitrary functions, $v_{a\zeta}$'s are also linear combinations of $v_1, \cdots, v_k$. Now suppose $v_1, \cdots, v_k$ are Schmidt orthogonalization of $u_1, \cdots, u_k$, i.e., they are linear combinations of u_1, $\cdots$, u_k satisfying

$$v_b^* v_a = \delta_{ba}, \tag{5.114}$$

then

$$\pi = \sum v_a v_a^* \tag{5.115}$$

is an Hermitian projective operator of rank k:

$$\pi^* = \pi, \quad \pi\pi = \pi, \tag{5.116}$$

and v_1, $\cdots$, v_k (also u_1, $\cdots$, u_k) are its invariant vectors. Moreover, from (5.113), $v_{a\zeta} = \sum \widetilde{c}_{ab} v_b$ where $\widetilde{c}_{ab}$'s are suitable functions, and

$$\pi_\zeta = \sum \widetilde{c}_{ab} v_b v_a^* + \sum v_a v_{a\zeta}^*.$$

From $\pi^\perp v_a = (I - \pi) v_a = v_a - v_a = 0$ we know that (5.112)$'$ holds.

Conversely, suppose π is an Hermitian projective operator of rank k satisfying (5.112)$'$, then for any k linearly independent invariant vectors u_a $(k = 1, \cdots, k)$,

$$\pi u_a = u_a, \quad \pi^\perp u_a = 0.$$

By (5.112)$'$,

$$\pi^\perp u_{a\zeta} = \pi^\perp (\pi u_a)_\zeta = \pi^\perp \pi_\zeta u_a + \pi^\perp \pi u_{a\zeta} = 0.$$

Hence $u_{a\zeta}$ is a linear combination of u_1, $\cdots$, u_k.

Moreover, if u_a's satisfy (5.113), then there exist linearly independent vectors $w_1, \cdots, w_k$ which are linear combinations of u_a's and satisfy

$$w_{a\zeta} = 0.$$

In fact, this can be done as follows. Let $U = (u_1, \cdots, u_k)$, $C = (c_{ab})$, then U satisfies

$$U_\zeta = UC.$$

Let $W = (w_1, \cdots, w_k) = UB$ where B is a $k \times k$ matrix, then

$$W_\zeta = U_\zeta B + UB_\zeta = U(CB + B_\zeta).$$

B can be determined by solving the linear equation

$$B_\zeta + CB = 0.$$

Therefore, to construct π, we only need a set of linearly independent vectors w_α whose entries are holomorphic functions of $\bar{\zeta}$ (i.e. anti-holomorphic functions of ζ). This gives the construction of unitons of degree one.

Remark 43 Let the set of all k dimensional subspaces of $\mathbf{C}^N$ be attached at each point of $\mathbf{R}^2$ (or Ω). Then we get a k dimensional plane bundle on $\mathbf{R}^2$. A section of this plane bundle is to assign a k dimensional subspace at each point of $\mathbf{R}^2$ (or Ω). It can also be obtained by assigning k dimensional Hermitian projective operator $\pi = \pi(\zeta, \bar{\zeta})$ at each point. In [102], the section was called holomorphic if π satisfies the condition $\pi^\perp \pi_{\bar{\zeta}} = 0$. Because of the difference of the symbols (See Remark 41), the condition here should be $\pi_{\bar{\zeta}} \pi^\perp = 0$, or equivalently, $\pi^\perp \pi_\zeta = 0$, i.e., the section is anti-holomorphic. As mentioned above, this kind of section can be constructed by k linearly independent anti-holomorphic vectors.

Example 5.13 *Take $N = 2$. Let f, g be two holomorphic functions of $\bar{\zeta}$ without common zero. Let*

$$u = \frac{1}{(|f|^2 + |g|^2)^{1/2}} \begin{pmatrix} f \\ g \end{pmatrix},$$

then

$$\pi = \frac{1}{|f|^2 + |g|^2} \begin{pmatrix} f \\ g \end{pmatrix} (\bar{f}, \bar{g}) = \frac{1}{|f|^2 + |g|^2} \begin{pmatrix} |f|^2 & f\bar{g} \\ g\bar{f} & |g|^2 \end{pmatrix},$$

$$\pi^\perp = I - \pi = \frac{1}{|f|^2 + |g|^2} \begin{pmatrix} |g|^2 & -f\bar{g} \\ -g\bar{f} & |f|^2 \end{pmatrix}.$$

The expression of the uniton is

$$g = \pi - \pi^{\perp} = \frac{1}{|f|^2 + |g|^2} \begin{pmatrix} |f|^2 - |g|^2 & 2f\bar{g} \\ 2g\bar{f} & |g|^2 - |f|^2 \end{pmatrix},$$

and the extended solution is

$$\Psi(\mu) = \frac{1}{|f|^2 + |g|^2} \begin{pmatrix} |f|^2 + \mu|g|^2 & (1-\mu)f\bar{g} \\ (1-\mu)g\bar{f} & |g|^2 + \mu|f|^2 \end{pmatrix}.$$

5.4.4 Darboux transformation and singular Darboux transformation for unitons

We construct the Darboux matrix by using the algorithm in Section 5.1.

Suppose g is a uniton and $\Psi(\mu)$ is its extended solution satisfying the normalizing condition: $\Psi^*(\bar{\mu}^{-1}) = \Psi(\mu)^{-1}$. Let

$$\omega_1 = \frac{1-\epsilon}{2}, \quad \omega_2 = \frac{1-\bar{\epsilon}^{-1}}{2}. \tag{5.117}$$

This is equivalent to putting μ to be ϵ. Choose the basis of $\mathbf{C}^n$ such that

$$L_1 = \begin{pmatrix} l_1 & \cdots & l_k \end{pmatrix} = \begin{pmatrix} I_k \\ 0 \end{pmatrix},$$

$$L_1^{\perp} = L_2 = \begin{pmatrix} l_{k+1} & \cdots & l_N \end{pmatrix} = \begin{pmatrix} 0 \\ I_{N-k} \end{pmatrix}.$$

Then

$$\begin{aligned} H_\epsilon &= \begin{pmatrix} \Psi(\epsilon)L_1 & \Psi(\bar{\epsilon}^{-1})L_2 \end{pmatrix}, \\ \pi_\epsilon &= H_\epsilon \begin{pmatrix} 0 & 0 \\ 0 & I_{N-k} \end{pmatrix} H_\epsilon^{-1}, \quad \pi_\epsilon^{\perp} = H_\epsilon \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} H_\epsilon^{-1}. \end{aligned} \tag{5.118}$$

The Darboux transformation of the extended uniton $\Psi(\mu)$ is

$$\Psi_1(\mu) = (\pi_\epsilon + \gamma\pi_\epsilon^{\perp})\Psi(\mu)(\sigma + \gamma^{-1}\sigma^{\perp}), \tag{5.119}$$

where $(\pi_\epsilon + \gamma\pi_\epsilon^{\perp})$ is the Darboux matrix, $(\sigma + \gamma^{-1}\sigma^{\perp})$ is an additional factor in which σ and $\sigma^{\perp}$ are the constant Hermitian projective operators to $L_1^{\perp}$ and L_1 respectively. Moreover,

$$\gamma = \gamma_\epsilon(\mu) = \frac{\omega_2\left(\omega_1 - \frac{1-\mu}{2}\right)}{\omega_1\left(\omega_2 - \frac{1-\mu}{2}\right)} = \frac{(\mu-\epsilon)(\bar{\epsilon}-1)}{(\bar{\epsilon}\mu - 1)(1-\epsilon)}. \tag{5.120}$$

Clearly,

$$\gamma_\epsilon(\bar{\mu}^{-1}) = \frac{(\bar{\mu}^{-1}-\epsilon)(\bar{\epsilon}-1)}{(\bar{\epsilon}\bar{\mu}^{-1}-1)(1-\epsilon)} = \frac{(1-\epsilon\bar{\mu})(\bar{\epsilon}-1)}{(\bar{\epsilon}-\bar{\mu})(1-\epsilon)} = (\overline{\gamma_\epsilon(\mu)})^{-1}. \quad (5.121)$$

Hence

$$\begin{aligned} &\Psi_1(\bar{\mu}^{-1}) = (\pi_\epsilon + \bar{\gamma}^{-1}\pi_\epsilon^\perp)\Psi(\bar{\mu}^{-1})(\sigma + \gamma\sigma^\perp), \\ &\Psi_1^*(\bar{\mu}^{-1})^{-1} = (\pi_\epsilon + \gamma\pi_\epsilon^\perp)\Psi(\mu)(\sigma + \gamma^{-1}\sigma^\perp) = \Psi_1(\mu). \end{aligned} \quad (5.122)$$

This means that Ψ_1 also satisfies the normalizing condition.

Now consider

$$\Psi_1(\mu) = \pi_\epsilon\Psi(\mu)\sigma + \pi_\epsilon\Psi(\mu)\gamma^{-1}(\mu)\sigma^\perp + \pi_\epsilon^\perp\gamma(\mu)\Psi(\mu)\sigma + \pi_\epsilon^\perp\Psi(\mu)\sigma^\perp. \quad (5.123)$$

On the right hand side, the first and fourth terms are polynomials of μ, the second and third terms are rational functions of μ. $\mu = \epsilon$ might be a pole of the second term and $\mu = \bar{\epsilon}^{-1}$ might be a pole of the third term. However, we can prove that the denominators $\mu - \epsilon$ and $\bar{\epsilon}\mu - 1$ in these two terms can be cancelled with the corresponding enumerators. Hence all the terms are polynomials of μ. This fact is proved as follows.

Notice that

$$H_\epsilon^{-1} = \begin{pmatrix} C_1(\epsilon)L_1^*\Psi(\epsilon)^* \\ C_2(\epsilon)L_2^*\Psi(\bar{\epsilon}^{-1})^* \end{pmatrix}, \quad (5.124)$$

where $C_1(\epsilon)$ is a $k \times k$ matrix and $C_2(\epsilon)$ is an $(N-k)\times(N-k)$ matrix, which are determined by

$$\begin{aligned} &C_1(\epsilon)L_1^*\Psi(\epsilon)^*\Psi(\epsilon)L_1 = I_k, \\ &C_2(\epsilon)L_2^*\Psi(\bar{\epsilon}^{-1})^*\Psi(\bar{\epsilon}^{-1})L_2 = I_{N-k} \end{aligned} \quad (5.125)$$

respectively. In (5.125), $L_1^*\Psi(\epsilon)^*\Psi(\epsilon)L_1$ is invertible. In fact, suppose l is a k dimensional vector such that $L_1^*\Psi(\epsilon)^*\Psi(\epsilon)L_1 l = 0$, then from $l^*L_1^*\Psi(\epsilon)^*\Psi(\epsilon)L_1 l = 0$ we obtain $\Psi(\epsilon)L_1 l = 0$. It follows that $L_1 l = 0$ and hence $l = 0$. Consequently, $C_1(\epsilon)$ is determined by (5.125) uniquely. Similarly, $C_2(\epsilon)$ also exists uniquely. Therefore,

$$\begin{aligned} &\pi_\epsilon = H_\epsilon \begin{pmatrix} 0 & 0 \\ 0 & I_{N-k} \end{pmatrix} H_\epsilon^{-1} = \Psi(\bar{\epsilon}^{-1})L_2C_2(\epsilon)L_2^*\Psi(\bar{\epsilon}^{-1})^*, \\ &\pi_\epsilon^\perp = H_\epsilon \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} H_\epsilon^{-1} = \Psi(\epsilon)L_1C_1(\epsilon)L_1^*\Psi(\epsilon)^*, \end{aligned} \quad (5.126)$$

and

$$\pi_\epsilon\Psi(\mu)\gamma^{-1}(\mu)\sigma^\perp = \Psi(\bar\epsilon^{-1})L_2C_2(\epsilon)L_2^*\Psi(\bar\epsilon^{-1})^*\Psi(\mu)\gamma^{-1}\sigma^\perp. \tag{5.127}$$

Denote the right hand side of the last equation by $\gamma^{-1}F(\mu)$. When $\mu = \epsilon$,

$$F(\epsilon) = \Psi(\bar\epsilon^{-1})L_2C_2(\epsilon)L_2^*\Psi(\bar\epsilon^{-1})^*\Psi(\epsilon)\sigma^\perp. \tag{5.128}$$

Using the normalizing condition $\Psi(\bar\epsilon^{-1})^*\Psi(\epsilon) = I$ and

$$\sigma^\perp = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix},$$

we get $F(\epsilon) = 0$. This means that $\mu = \epsilon$ is actually not a pole of the second term of (5.123). Similar conclusion holds for the third term.

Thus, we have the following theorem.

THEOREM 5.14 *A Darboux transformation with right-multiplied normalizing factor $(\sigma + \mu^{-1}\sigma^\perp)$ transforms an extended solution $\Psi(\mu)$ to an extended solution $\Psi_1(\mu)$. The degree of $\Psi_1(\mu)$ with respect to μ cannot exceed the degree of $\Psi(\mu)$ with respect to μ.*

We have mentioned before that the extended solutions of a uniton are not unique, and their degrees are not definite either. However, the uniton number is defined as the minimum of the degrees of all possible extended solutions. The above theorem implies that a Darboux transformation cannot increase uniton number.

In [102], singular Bäcklund transformation was introduced. For a set of usual Bäcklund transformations with parameter ϵ, let $\epsilon \to 0$, then the derived transformation is called a singular Bäcklund transformation. Since Bäcklund transformation can be solved explicitly by Darboux transformation, singular Darboux transformation (limit of Darboux transformations with parameter ϵ as $\epsilon \to 0$) should give singular Bäcklund transformation explicitly. Clearly, when $\epsilon \to 0$, $\gamma_\epsilon \to \mu$. Hence if $\pi_\epsilon \to \pi$, $\pi_\epsilon^\perp \to \pi^\perp$, the Darboux transformations will converge to a singular Darboux transformation

$$\Psi_1(\mu) = (\pi + \mu\pi^\perp)\Psi(\mu).$$

Here the factor $(\sigma + \mu^{-1}\sigma^\perp)$ is neglected.

LEMMA 5.15 *Suppose $\Psi(\mu)$ is an extended solution of a uniton, then $\Psi_1(\mu) = (\pi + \mu\pi^\perp)\Psi(\mu)$ is an extended solution of a uniton if and only if*

$$(2\pi_{\bar\zeta} + \pi A)\pi^\perp = 0, \tag{5.129}$$

$$\pi^\perp A\pi = 0. \tag{5.130}$$

Proof. By direct calculation,

$$\begin{aligned}\Psi_{1\bar{\zeta}}\Psi_1^{-1} &= \frac{1-\mu}{2}(2\pi_{\bar{\zeta}}\pi + \pi A\pi + \pi^\perp A\pi^\perp) \\ &\quad + \frac{1-\mu}{2\mu}(2\pi_{\bar{\zeta}}\pi^\perp + \pi A\pi^\perp) + \frac{\mu(1-\mu)}{2}\pi^\perp A\pi \\ &= \frac{1-\mu}{2}A_1.\end{aligned}$$

This leads to (5.129) and (5.130).

Alternatively,

$$\begin{aligned}\Psi_{1\zeta}\Psi_1^{-1} &= \frac{1-\mu^{-1}}{2}(-2\pi_\zeta\pi^\perp + \pi B\pi + \pi^\perp B\pi^\perp) \\ &\quad + \frac{1-\mu}{2}(2\pi_\zeta\pi - \pi^\perp B\pi) + \frac{1-\mu^{-1}}{2\mu}\pi B\pi^\perp \\ &= \frac{1-\mu^{-1}}{2}B_1.\end{aligned}$$

Hence we have

$$2\pi_\zeta\pi - \pi^\perp B\pi = 0, \tag{5.129$'$}$$

$$\pi B\pi^\perp = 0. \tag{5.130$'$}$$

However, $(5.129)'$ and $(5.130)'$ are equivalent to (5.129) and (5.130) respectively. In fact, complex conjugate of (5.130) is $(5.130)'$. Taking the complex conjugate of $(5.129)'$, we have

$$0 = 2\pi\pi_{\bar{\zeta}} + \pi A\pi^\perp = -2\pi\pi^\perp_{\bar{\zeta}} + \pi A\pi^\perp = 2\pi_{\bar{\zeta}}\pi^\perp + \pi A\pi^\perp.$$

This is just (5.129). The lemma is proved.

Remark 44 Let $\epsilon \to 0$ in the Bäcklund transformation (5.90), then we obtain (5.129) and (5.130). Hence (5.129) and (5.130) are called the equations of singular Bäcklund transformation in [102].

Notice that when $\epsilon \to 0$, $C_1(\epsilon)$ and $C_2(\epsilon)$ may not converge, and $L_1^*\Psi(\epsilon)^*\Psi(\epsilon)L_1$ and $L_2^*\Psi(\bar{\epsilon}^{-1})^*\Psi(\bar{\epsilon}^{-1})L_2$ may not converge to invertible matrices. However, by the following method we can eliminate the term which tends to infinity as $\epsilon \to 0$ so that π_ϵ tends to π. This method can be regarded as a method of renormalization [49].

The rank of $\Psi(\epsilon)L_1$ is k. From the second equation of (5.126) and the first equation of (5.125), we have

$$\pi_\epsilon^\perp\Psi(\epsilon)L_1 = \Psi(\epsilon)L_1.$$

Hence each column of $\Psi(\epsilon)L_1$ is an invariant vector of $\pi_\epsilon^\perp$ and its entries are polynomials of ϵ whose degrees do not exceed n. Consider the terms of the zeroth order of ϵ in the columns of $\Psi(\epsilon)L_1$ and choose a maximal linearly independent set in it. The corresponding column vectors of $\Psi(\epsilon)L_1$ constitute an ϵ-dependent matrix

$$\chi^0 = \chi_0^0 + \chi_1^0\epsilon + \cdots + \chi_n^0\epsilon^n,$$

where χ_0^0 is of full rank. The coefficients of ϵ^0 in other columns of $\Psi(\epsilon)L_1$ are linear combinations of the columns in χ_0^0. Subtract these other columns by suitable linear combination of the columns in χ^0 so that all the coefficients of ϵ^0 are eliminated. After divided by ϵ, we obtain a set of columns in the form

$$b + b_1\epsilon + \cdots + b_{n-1}\epsilon^{n-1}.$$

Combining these columns to the matrix χ^0, we obtain a matrix $M_1(\epsilon)$. Evidently, the columns of M_1 span the same bundle of subspaces P_ϵ over $\mathbf{R}^2$ (or Ω) as the columns of $\Psi(\epsilon)L_1$. We can transform the matrix $M_1(\epsilon)$ to $M_2(\epsilon)$ in the same way and so on. Finally we obtain a matrix

$$M(\epsilon) = \tau_0 + \tau_1\epsilon + \cdots + \tau_n\epsilon^n \tag{5.131}$$

such that τ_0 is of rank k. In fact, if $\mathrm{rank}\tau_0 < k$, we can make the above transformation further, since the columns of M span the bundle P_ϵ.

Let

$$W_\epsilon = M(\epsilon)(M^*(\epsilon)M(\epsilon))^{-1}M^*(\epsilon). \tag{5.132}$$

It is easily seen that W_ϵ is Hermitian and $W_\epsilon^2 = W_\epsilon$. Moreover, $W_\epsilon M(\epsilon) = M(\epsilon)$, i.e., $W(\epsilon)$ is the Hermitian projective operator to P_ϵ. Hence

$$W_\epsilon = \pi_\epsilon^\perp. \tag{5.133}$$

Let $\epsilon \to 0$, we have $\pi_\epsilon^\perp \to \pi^\perp$ with

$$\pi^\perp = \tau_0(\tau_0^*\tau_0)^{-1}\tau_0^* \tag{5.134}$$

and $\pi_\epsilon \to \pi = I - \pi^\perp$.

Thus we have proved the convergence of π_ϵ (and $\tau_\epsilon^\perp$) as $\epsilon \to 0$ and obtained the explicit formulae for $\pi^\perp$ and π. The convergence of $\pi_{\epsilon,\zeta}$ and $\pi_{\epsilon,\bar{\zeta}}$ as $\epsilon \to 0$ can be obtained from some properties of harmonic maps (see [50]).

We have the following theorem.

THEOREM 5.16 *Let $\pi^\perp$ be defined by (5.134), then*

$$\Psi_1 = (\pi + \mu\pi^\perp)\Psi \tag{5.135}$$

is a new extended solution of uniton. Moreover, π *and* $\pi^\perp$ *satisfy (5.129) and (5.130).*

Thus we have established the singular Darboux transformation which realizes the singular Bäcklund transformation. According to the Theorem 5.14, the extended solution of uniton

$$(\pi + \mu\pi^\perp)\Psi(\sigma + \mu^{-1}\sigma^\perp) \tag{5.136}$$

is of degree $\le n$. Hence the uniton number cannot be increased by singular Bäcklund transformation.

We shall take $n = 1$ as the example to see the limiting process of π_ϵ and $\pi_\epsilon^\perp$ as $\epsilon \to 0$.

When $n = 1$,

$$\Psi(\mu) = \pi_1 + \mu\pi_1^\perp,$$

$$\pi_\epsilon = (\pi_1 + \bar\epsilon^{-1}\pi_1^\perp)L_2C_2(\epsilon)L_2^*(\pi_1 + \epsilon^{-1}\pi_1^\perp), \tag{5.137}$$

$$\pi_\epsilon^\perp = (\pi_1 + \epsilon\pi_1^\perp)L_1C_1(\epsilon)L_1^*(\pi_1 + \bar\epsilon\pi_1^\perp), \tag{5.138}$$

$$C_1(\epsilon)L_1^*(\pi_1 + |\epsilon|^2\pi_1^\perp)L_1 = I_k, \tag{5.139}$$

$$C_2(\epsilon)L_2^*(\pi_1 + |\epsilon|^{-2}\pi_1^\perp)L_2 = I_{N-k}. \tag{5.140}$$

The existence of the limits of π_ϵ and $\pi_\epsilon^\perp$ as $\epsilon \to 0$ is discussed as follows. By using a transformation of the basis, we have

$$L_1 = \begin{pmatrix} I_k \\ 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 \\ I_{N-k} \end{pmatrix}. \tag{5.141}$$

Let

$$\pi_1 = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \tag{5.142}$$

as a block matrix where π_{11}, π_{12}, π_{21}, π_{22} are $k \times k$, $k \times (N-k)$, $(N-k) \times k$, $(N-k) \times (N-k)$ matrices respectively. Since $\pi_1^* = \pi_1$, we have

$$\pi_{11}^* = \pi_{11}, \quad \pi_{12}^* = \pi_{21}, \quad \pi_{21}^* = \pi_{12}, \quad \pi_{22}^* = \pi_{22}. \tag{5.143}$$

From $\pi_1^2 = \pi_1$,

$$\begin{aligned} &\pi_{11}^2 + \pi_{12}\pi_{21} = \pi_{11}, \quad \pi_{21}\pi_{11} + \pi_{22}\pi_{21} = \pi_{21}, \\ &\pi_{11}\pi_{12} + \pi_{12}\pi_{22} = \pi_{12}, \quad \pi_{21}\pi_{12} + \pi_{22}^2 = \pi_{22}. \end{aligned} \tag{5.144}$$

Since π_{11} is an Hermitian matrix, there exists a $k \times k$ unitary matrix β such that

$$\pi_{11} = \beta\ \mathrm{diag}(0, \cdots, 0, \lambda_{r+1}, \cdots, \lambda_k)\beta^* \quad (\lambda_{r+1}, \cdots, \lambda_k \neq 0). \qquad (5.145)$$

Hence (5.139) becomes

$$\begin{aligned} &C_1(\epsilon)\beta\Big(\ \mathrm{diag}(0, \cdots, 0, \lambda_{r+1}, \cdots, \lambda_k) \\ &\quad +|\epsilon|^2\ \mathrm{diag}(1, \cdots, 1, 1-\lambda_{r+1}, \cdots, 1-\lambda_k)\Big)\beta^* = I_k, \end{aligned}$$

or equivalently,

$$\begin{aligned} C_1(\epsilon) = \beta\bigg(\ &\mathrm{diag}\Big(|\epsilon|^{-2}, \cdots, |\epsilon|^{-2}, \\ &\frac{1}{\lambda_{r+1} + |\epsilon|^2(1-\lambda_{r+1})}, \cdots, \frac{1}{\lambda_k + |\epsilon|^2(1-\lambda_k)}\Big)\bigg)\beta^*, \end{aligned} \qquad (5.146)$$

From (5.126),

$$\pi_\epsilon^\perp = \begin{pmatrix} \pi_{11} + \epsilon(1-\pi_{11}) \\ \pi_{21} - \epsilon\pi_{21} \end{pmatrix} C_1(\epsilon)(\pi_{11} + \bar{\epsilon}(1-\pi_{11}), \pi_{12} - \bar{\epsilon}\pi_{12}). \quad (5.147)$$

From (5.145),

$$\pi_{11}\beta\ \mathrm{diag}(I_r, 0)\beta^* = 0. \qquad (5.148)$$

On the other hand, suppose a k dimensional vector l satisfies $\pi_{11}l = 0$, then by (5.144), $\pi_{12}\pi_{21}l = 0$. Hence

$$(\pi_{21}l)^*(\pi_{21}l) = l^*\pi_{12}\pi_{21}l = l^*(\pi_{11} - \pi_{11}^2)l = 0,$$

which implies $\pi_{21}l = 0$. Therefore,

$$\pi_{21}\beta\ \mathrm{diag}(I_r, 0, \cdots, 0) = 0. \qquad (5.149)$$

Substituting these relations into (5.147) and letting $\epsilon \to 0$, we get

$$\begin{aligned} \lim_{\epsilon\to 0} \pi_\epsilon^\perp &= \pi^\perp \\ &= \begin{pmatrix} \pi_{11} \\ \pi_{21} \end{pmatrix} \beta\ \mathrm{diag}\Big(0, \cdots, 0, \frac{1}{\lambda_{r+1}}, \cdots, \frac{1}{\lambda_k}\Big)\beta^* \begin{pmatrix} \pi_{11} & \pi_{12} \end{pmatrix} \\ &\quad + \begin{pmatrix} \beta\ \mathrm{diag}(I_r, 0)\beta^* & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \qquad (5.150)$$

and

$$\begin{aligned}\lim_{\epsilon\to 0}\pi_\epsilon = \pi = &\begin{pmatrix} -\pi_{12} \\ 1-\pi_{22} \end{pmatrix}\alpha \\ &\cdot \operatorname{diag}\Big(\frac{1}{\lambda_{k+1}},\cdots,\frac{1}{\lambda_{k+s}},0,\cdots,0\Big)\alpha^*\begin{pmatrix} -\pi_{21} & 1-\pi_{22}\end{pmatrix} \\ &+\begin{pmatrix} 0 & 0 \\ 0 & \alpha\operatorname{diag}(0,I_{N-k-s})\alpha^* \end{pmatrix}.\end{aligned} \tag{5.151}$$

Here α is a suitable $(N-k)\times(N-k)$ unitary matrix which can make π_{22} to be diagonal.

Finally, we prove the following theorem.

THEOREM 5.17 *(Theorem of factorization) Any extended solution of uniton $\Psi(\mu)$ of degree n can be factorized as*

$$\Psi(\mu)=(\pi_n+\mu\pi_n^\perp)(\pi_{n-1}+\mu\pi_{n-1}^\perp)\cdots(\pi_1+\mu\pi_1^\perp)C_0, \tag{5.152}$$

where π_i and $\pi_i^\perp = I-\pi_i$ $(i=1,2,\cdots,n)$ are Hermitian projective operators, being analytic functions of (x,y), C_0 is a constant matrix in $U(N)$ which may be chosen as I.

Before proving this theorem, we first give the following two lemmas.

LEMMA 5.18 *Suppose $\Psi(\mu)$ is an extended solution of uniton of degree n, then there exists an Hermite projective operator π such that*

$$\Psi_1(\mu)=(\pi+\mu\pi^\perp)\Psi(\mu)$$

is an extended solution of uniton defined on a dense open subset of Ω and the degree of $\Psi_1(\mu)$ is at most n.

Proof. Suppose

$$\Psi(\mu)=T_0+T_1\mu+\cdots+T_n\mu^n.$$

Since the equations for harmonic map is elliptic and $U(N)$ is a real analytic manifold, A and B are both analytic functions of the real variables x and y, so is T_0. Suppose the maximum rank of T_0 is k, then the points at which the rank of T_0 is k form a dense open subset Ω_1 of Ω. Moreover, suppose L_1 is a constant $N\times(k+a)$ matrix $(a\geq 0)$ such that the maximum rank of T_0L_1 is also k, then the points at with the rank of T_0L_1 is k also form a dense open subset Ω_2 of Ω. We use $L=(L_1,L_2)$ to construct the Darboux transformation. From (5.125) and (5.126),

$$\begin{aligned}&\pi_\epsilon^\perp=\Psi(\epsilon)L_1C_1(\epsilon)L_1^*\Psi(\epsilon)^*,\\ &C_1(\epsilon)L_1^*\Psi_1(\epsilon)^*\Psi_1(\epsilon)L_1=I_k.\end{aligned}$$

When $\epsilon \to 0$,

$$C_1(0) = (L_1^* T_0^* T_0 L_1)^{-1},$$
$$\pi^\perp = T_0 L_1 C(0) L_1^{-1} T_0^*$$

holds in Ω_2. By $\Psi(\mu)^* \Psi(\bar{\mu}^{-1}) = I$, we have

$$T_0^* T_n = 0.$$

Hence $\pi^\perp T_n = 0$, and the degree of $\Psi_1(\mu)$ on Ω_2 does not exceed n. The lemma is proved.

LEMMA 5.19 *Suppose $\Psi(\mu)$ is an extended solution of uniton of degree n, then there exists an Hermitian projective operator τ defined on a dense open subset of Ω such that*

$$\Psi(\mu) = (\tau + \mu \tau^\perp) \Psi_{-1}(\mu),$$

where $\Psi_{-1}(\mu)$ is an extended solution of uniton whose degree does not exceed $n-1$.

Proof. Take τ as $\pi^\perp$ in Lemma 5.18, then $\tau^\perp = \pi$. Let

$$\Psi_{-1}(\mu) = \left(\tau^\perp + \frac{1}{\mu}\tau\right)\Psi(\mu),$$

then

$$\Psi_1(\mu) = (\pi + \mu\pi^\perp)\Psi(\mu) = (\pi + \mu\pi^\perp)(\pi^\perp + \mu\pi)\Psi_{-1}(\mu) = \mu\Psi_{-1}(\mu).$$

Hence the degree of $\Psi_{-1}(\mu)$ does not exceed $n-1$.

On the other hand, it can be verified directly that $\Psi_{-1}(\mu)$ is an extended solution of uniton if and only if

$$\tau^\perp A \tau - 2\tau^\perp \tau_{\bar{\zeta}} = 0,$$
$$\tau A \tau^\perp = 0.$$

Since $\tau = \pi^\perp$, $\tau^\perp = \pi$, these are just the equations (5.129) and (5.130) for π and $\pi^\perp$. The lemma is proved.

Proof of Theorem 5.17. Use mathematical induction. Suppose the theorem is true for all extended solutions of uniton of degree $n-1$. According to Lemma 5.19,

$$\Psi(\mu) = (\tau_n + \mu\tau_n^\perp)\Psi_{-1}(\mu)$$

holds in a dense open subset. By the assumption of induction,

$$\Psi(\mu) = (\pi_n + \mu\pi_n^\perp)(\pi_{n-1} + \mu\pi_{n-1}^\perp)\cdots(\pi_1 + \mu\pi_1^\perp)$$

holds in a dense open subset where τ_n and $\tau_n^\perp$ are denoted by π_n and $\pi_n^\perp$. The left hand side is an analytic function of x and y on Ω, and the right hand side is an analytic function of x and y in a dense open subset of Ω. By analytic continuation, we know that the above equality holds in whole Ω. The theorem is proved.

Remark 45 This theorem was obtained in [102]. Here the construction of factorization is more general. Especially, the rank of each π_i is not fixed. The only restriction is $1 \le \operatorname{rank}\pi_i \le N - k_i$ where k_i is given by Lemma 5.18. Since singular Darboux transformation can be constructed algebraically, the factorization here can also be realized algebraically.

Take $\mu = -1$, then $\Psi(-1)$ can be factorized as

$$\Psi(-1) = (\pi_n - \pi_n^\perp)(\pi_{n-1} - \pi_{n-1}^\perp)\cdots(\pi_1 - \pi_1^\perp).$$

Moreover, the unitons to the Grassmannian have the similar factorization (see [44]).

Remark 46 In the proof of Lemma 5.19, π and $\pi^\perp$ (or $\tau^\perp$ and τ) are constructed from $\Psi(\mu)$. But we have not constructed them from $\Psi_{-1}(\mu)$ algebraically. There was some negligence in [49]. It is still not clear whether one can obtain all unitons by purely algebraic algorithm from one unitons. Partial solution has been obtained in [55].

Chapter 6

GENERALIZED SELF-DUAL YANG-MILLS EQUATIONS AND YANG-MILLS-HIGGS EQUATIONS

6.1 Generalized self-dual Yang-Mills flow

Yang-Mills equations are one of the most important equations in theoretical physics to describe the fundamental interactions in nature. Self-dual Yang-Mills equations are special case of the Yang-Mills equations. They have also great significance in differential topology. Moreover, a lot of soliton equations can be reduced from self-dual Yang-Mills equations. Instead of the general theory of Yang-Mills fields, here we only consider a kind of their generalization in the point view of soliton theory. The Darboux transformation can be applied to this generalized self-dual Yang-Mills flow [40, 41]. Furthermore, a reduction of the generalized self-dual Yang-Mills flow leads to the AKNS system.

6.1.1 Generalized self-dual Yang-Mills flow

First we introduce briefly the self-dual Yang-Mills fields.

Let G be a matrix Lie group, g be its Lie algebra. The Euclidean space $\mathbf{R}^4$ has metric

$$ds^2 = \eta^{ij} dx_i\, dx_j \quad \text{(with summation convention, } \eta^{ij} = \delta^{ij}\text{)}. \tag{6.1}$$

Gauge potential is a set of functions A_i's valued in the Lie algebra g. The strength of the gauge field is the anti-symmetric tensor

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]. \tag{6.2}$$

A Yang-Mills field is defined by the gauge potential satisfying the Yang-Mills equations

$$\eta^{jk}\Big(\frac{\partial F_{ij}}{\partial x_k} + [A_k, F_{ij}]\Big) = 0. \tag{6.3}$$

For any G-valued function $G(x)$, the transformation

$$A'_i = G(x)A_iG(x)^{-1} + (\partial_i G(x))G(x)^{-1}$$

is called a gauge transformation. A gauge transformation does not change the physical essence of the gauge field. Especially, it does not change the Yang-Mills equations.

The dual of the gauge intensity F is a tensor $*F$ defined by

$$(*F)_{ij} = \frac{1}{2}\epsilon_{ijkl}\eta^{ka}\eta^{lb}F_{ab}, \tag{6.4}$$

where

$$\epsilon_{ijkl} = \begin{cases} 1, & (i,j,k,l) \text{ is an even permutation of } (1,2,3,4) \\ -1, & (i,j,k,l) \text{ is an odd permutation of } (1,2,3,4) \\ 0, & \text{otherwise,} \end{cases} \tag{6.5}$$

that is,

$$\begin{aligned} &(*F)_{12} = F_{34}, \quad (*F)_{23} = F_{14}, \quad (*F)_{31} = F_{24}, \\ &(*F)_{34} = F_{12}, \quad (*F)_{14} = F_{23}, \quad (*F)_{24} = F_{31}. \end{aligned} \tag{6.6}$$

If $*F = F$, i.e.,

$$F_{12} = F_{34}, \quad F_{23} = F_{14}, \quad F_{31} = F_{24}, \tag{6.7}$$

this gauge field is called a self-dual Yang-Mills field. An self-dual Yang-Mills field always satisfies the Yang-Mills equation. Similarly, the gauge field satisfying $*F = -F$ is called anti-self-dual. The self-duality and anti-self-duality can be interchanged by changing the orientation of the space-time. Therefore, it is enough for us to discuss only the self-dual Yang-Mills field.

The self-dual Yang-Mills field on $\mathbf{R}^4$ can be complexified to the gauge field on $\mathbf{C}^4$. Besides, $\mathbf{R}^4$ can be replaced by $\mathbf{R}^{2,2}$ with the metric

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \mathrm{d}x_3^2 - \mathrm{d}x_4^2. \tag{6.8}$$

In [98], the self-duality was extended to $\mathbf{R}^{4n}$. Here we will extend it to $\mathbf{R}^{2n}$.

Suppose $(x,p) = (x_1,\cdots,x_n;p_1,\cdots,p_n)$ are the real coordinates of $\mathbf{R}^{2n}$ $(n \geq 2)$, $A_i(x_1,\cdots,x_n,p_1,\cdots,p_n)$'s are a set of $N \times N$ (real or complex) matrix functions of (x,p). Consider the Lax set

$$L_i\Psi \equiv \left(\frac{\partial}{\partial p_i} - \lambda\frac{\partial}{\partial x_i}\right)\Psi = -A_i\Psi, \tag{6.9}$$

where Ψ is an $N \times N$ matrix function, λ is the spectral parameter which appears as the coefficient of $\dfrac{\partial}{\partial x_i}$.

The integrability conditions of (6.9) are

$$\frac{\partial A_i}{\partial p_j} - \frac{\partial A_j}{\partial p_i} - [A_i, A_j] = 0 \tag{6.10}$$

and

$$\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} = 0. \tag{6.11}$$

When $n = 2$, this is just the self-dual Yang-Mills equation [116]. In [98], self-dual Yang-Mills field was generalized to the current form with "space" dimension $4n$ $(n \geq 1)$. We call $\{A_i\}$ satisfying (6.10) and (6.11) a generalized Yang-Mills potential.

Remark 47 We have mentioned that the self-duality depends on the metric of the space. For four dimensional space, here we only consider $\mathbf{R}^{2,2}$, *with the metric*

$$\mathrm{d}s^2 = \mathrm{d}p_1 \, \mathrm{d}x_1 + \mathrm{d}p_2 \, \mathrm{d}x_2.$$

From (6.10), there exists an $N \times N$ matrix J such that

$$A_i = -\frac{\partial J}{\partial p_i} J^{-1}. \tag{6.12}$$

Hence, (6.11) becomes

$$\frac{\partial}{\partial x_i}\left(\frac{\partial J}{\partial p_j} J^{-1}\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial J}{\partial p_i} J^{-1}\right) = 0. \tag{6.13}$$

Clearly, (6.13) and (6.12) are equivalent to (6.10) and (6.11).

We shall consider a flow of the generalized self-dual Yang-Mills field by introducing a "time" variable so that A_i's, J and Ψ depend on t as well as the "space" variables (x, p). Moreover, suppose that Ψ satisfies the evolution equation

$$\frac{\partial \Psi}{\partial t} = V\Psi = \sum_{a=0}^{m+q} V_a \lambda^{m-a} \Psi \quad (q \geq 0), \tag{6.14}$$

where V_a's are $N \times N$-matrix valued functions which are independent of the spectral parameter λ.

(6.14) and (6.9) form a Lax set in $\mathbf{R}^{2n+1}$. Its integrability conditions consist of the recursive relations among V_a's

$$\begin{aligned}
&\frac{\partial V_0}{\partial x_i} = 0, \\
&\frac{\partial V_a}{\partial x_i} = \frac{\partial V_{a-1}}{\partial p_i} + [A_i, V_{a-1}], \quad (a = 1, 2, \cdots, m),
\end{aligned} \tag{6.15}$$

and

$$\begin{aligned}
&\frac{\partial V_{m+q}}{\partial p_i} = [V_{m+q}, A_i], \\
&\frac{\partial V_{a-1}}{\partial p_i} = \frac{\partial V_a}{\partial x_i} + [V_{a-1}, A_i], \quad (a = m+q, \cdots, m+2)
\end{aligned} \tag{6.16}$$

together with the evolution equations

$$\frac{\partial A_i}{\partial t} + \frac{\partial V_m}{\partial p_i} - \frac{\partial V_{m+1}}{\partial x_i} + [A_i, V_m] = 0. \tag{6.17}$$

A solution of (6.10), (6.11) and (6.17) is called a generalized self-dual Yang-Mills flow. The gauge potentials A_i's satisfy not only the self-dual Yang-Mills equation, but also an additional evolution equation (6.17).

Remark 48 There is no direct physical meaning of the generalized self-dual Yang-Mills flow. However, it contains a one parametric family of self-dual Yang-Mills solutions and it will be seen later that almost all known soliton equations are the reductions of the generalized self-dual Yang-Mills equation.

THEOREM 6.1 *If A_i's satisfy the generalized self-dual Yang-Mills equation (6.10) and (6.11), then the equations (6.16) and (6.17) for V_i's are completely integrable and all V_i's are expressed by differentiations and integrations of A_i's.*

Proof. From the first equation of (6.15),

$$V_0 = V_0(p, t). \tag{6.18}$$

The other equations in (6.15) lead to

$$\begin{aligned}
\frac{\partial^2 V_a}{\partial x_i \partial x_j} = & \frac{\partial^2 V_{a-1}}{\partial p_i \partial x_j} + \left[\frac{\partial A_i}{\partial x_j}, V_{a-1}\right] + \left[A_i, \frac{\partial V_{a-1}}{\partial x_j}\right] \\
= & \frac{\partial^2 V_{a-2}}{\partial p_i \partial p_j} + \left[\frac{\partial A_j}{\partial p_i}, V_{a-2}\right] + \left[A_j, \frac{\partial V_{a-2}}{\partial p_i}\right] \\
& + \left[\frac{\partial A_i}{\partial x_j}, V_{a-1}\right] + \left[A_i, \frac{\partial V_{a-2}}{\partial p_j}\right] + [A_i, [A_j, V_{a-2}]].
\end{aligned} \tag{6.19}$$

Using (6.10), (6.11) and the Jacobi identity, we know that the right hand side of (6.19) is symmetric for i and j. Hence (6.15) is completely integrable. Take p_i's as parameters, V_n's can be determined recursively as

$$\begin{aligned} V_0 &= V_0^0(p,t), \\ V_a &= \sum_i \int_{(0,p,t)}^{(x,p,t)} \left(\frac{\partial V_{a-1}}{\partial p_i} + [A_i, V_{a-1}] \right) \mathrm{d}x_i + V_a^0(p,t), \end{aligned} \tag{6.20}$$

where $V_a^0(p,t)$'s are integral "constants" and the integrals are taken along the path in the n-dimensional subspace $p_i = \text{const}$. In fact, these integrals are independent of the path because of the integrability conditions. Moreover, we can choose V_a^0's as arbitrary functions of p and t. If they are independent of t, then the evolution equations (6.17) are a system of integro-differential equations which do not depend on t explicitly.

Let

$$W_a = J^{-1} V_a J \quad (a = m+q, \cdots, m+1), \tag{6.21}$$

then (6.16) implies

$$\frac{\partial W_{a-1}}{\partial p_i} = \frac{\partial W_a}{\partial x_i} + \left[J^{-1} \frac{\partial J}{\partial x_i}, W_a \right] \quad (a = m+q, \cdots, m+2), \tag{6.22}$$

and

$$\frac{\partial W_{m+q}}{\partial p_i} = 0.$$

Hence

$$W_{m+q} = W_{m+q}^0(x,t).$$

Moreover, according to (6.22),

$$\begin{aligned} W_{a-1} = \sum_i \int_{(x,0,t)}^{(x,p,t)} \left(\frac{\partial W_a}{\partial x_i} + \left[J^{-1} \frac{\partial J}{\partial x_i}, W_a \right] \right) \mathrm{d}p_i + W_{a-1}^0(x,t) \\ (a = m+q, \cdots, m+2). \end{aligned} \tag{6.23}$$

The integrals are now taken along a path in the n-dimensional subspace $x_i = \text{constant}$, and independent of the path. Here x_i's are regarded as parameters.

Therefore, when A_i's satisfy (6.10) and (6.11), we can write down V_a $(a = 0, \cdots, m; a = m+q, m+q-1, \cdots, m+1)$ recursively so that (6.15) and (6.16) are satisfied. The theorem is proved.

(6.10) and (6.11) can be regarded as "spatial constraints" among A_i's, and (6.17) are their evolution equations. The meaning of "spatial constraints" is that at any moment A_i's satisfy the generalized self-dual Yang-Mills equation.

6.1.2 Darboux transformation

Let D be an $N \times N$ matrix

$$D = \lambda I - S, \tag{6.24}$$

where S is independent of λ. Let

$$\Psi' = D\Psi \tag{6.25}$$

and we want

$$\begin{aligned} &L_i\Psi' = \left(\frac{\partial}{\partial p_i} - \lambda\frac{\partial}{\partial x_i}\right)\Psi' = -A_i'\Psi', \\ &\frac{\partial \Psi'}{\partial t} = \sum_{a=0}^{m+q} V_a'\lambda^{m-a}\Psi'. \end{aligned} \tag{6.26}$$

As in Section 3.2, direct calculation implies

$$A_i' = A_i - \frac{\partial S}{\partial x_i}, \tag{6.27}$$

$$\frac{\partial S}{\partial p_i} = SA_i - A_i'S = [S, A_i] + \frac{\partial S}{\partial x_i}S, \tag{6.28}$$

$$\begin{aligned} &V_0' = V_0, \\ &V_a' = V_a + V_{a-1}'S - SV_{a-1} \quad (a = 1, 2, \cdots, m), \end{aligned} \tag{6.29}$$

$$\begin{aligned} &V_{m+q}' = SV_{m+q}S^{-1}, \\ &V_{m+k}' = SV_{m+k}S^{-1} - (V_{m+k+1} - V_{m+k+1}')S^{-1}, \\ &(k = q-1, \cdots, 1), \end{aligned} \tag{6.30}$$

and

$$\frac{\partial S}{\partial t} + SV_m - V_m'S - V_{m+1} + V_{m+1}' = 0. \tag{6.31}$$

Here we assume that S is non-degenerate. It can be verified that (6.28) and (6.31) are completely integrable, that is, for any given initial data $S(t_0, p_0, x)$ at $t = t_0$, $p = p_0$, there exists $S(t, p, x)$ satisfying (6.28), (6.31) and the initial condition.

THEOREM 6.2 *Suppose the matrix S is non-degenerate and satisfies the equations (6.28) and (6.31), then $D = \lambda I - S$ is a Darboux matrix for (6.9) and (6.14).*

Now we turn to the explicit construction of the matrix S, which is essentially the same as that for the AKNS system. Take λ_1, $\cdots$, λ_N to be N complex numbers such that at least two of them are different. Let h_α ($\alpha = 1, 2, \cdots, N$) be a column solution of (6.9) and (6.14) for $\lambda = \lambda_\alpha$, i.e., h_α satisfies

$$\frac{\partial h_\alpha}{\partial p_i} = \lambda_\alpha \frac{\partial h_\alpha}{\partial x_i} - A_i h_\alpha, \quad \frac{\partial h_\alpha}{\partial t} = \sum_{a=0}^{m+q} V_a \lambda_\alpha^{m-a} h_\alpha. \tag{6.32}$$

Let

$$H = (h_1, h_2, \cdots, h_N), \quad \Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_N). \tag{6.33}$$

Suppose H is non-degenerate. Let

$$S = H\Lambda H^{-1}. \tag{6.34}$$

THEOREM 6.3 *Suppose S is defined by (6.34), then $D = \lambda I - S$ is a Darboux matrix.*

Proof. From the first equation of (6.32), we have

$$\frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial x_i}\Lambda - A_i H. \tag{6.35}$$

Hence

$$\begin{aligned} \frac{\partial S}{\partial p_i} &= \frac{\partial H}{\partial x_i}\Lambda^2 H^{-1} - H\Lambda H^{-1}\frac{\partial H}{\partial x_i}\Lambda H^{-1} - A_i S + S A_i, \\ \frac{\partial S}{\partial x_i} &= \frac{\partial H}{\partial x_i}\Lambda H^{-1} - H\Lambda H^{-1}\frac{\partial H}{\partial x_i} H^{-1}. \end{aligned} \tag{6.36}$$

Therefore,

$$\frac{\partial S}{\partial p_i} = [S, A_i] + \frac{\partial S}{\partial x_i} S. \tag{6.37}$$

This is just (6.28). On the other hand, it is easy to see that

$$\frac{\partial H}{\partial t} = \sum_{a=0}^{m+q} V_a H \Lambda^{m-a}, \tag{6.38}$$

which implies

$$\begin{aligned} \frac{\partial S}{\partial t} &= \sum_{a=0}^{m+q} V_a H \Lambda^{m-a+1} H^{-1} - \sum_{a=0}^{m+q} H\Lambda H^{-1} V_a H \Lambda^{m-a} H^{-1} \\ &= \sum_{a=0}^{m+q} V_a S^{m-a} S - S \sum_{a=0}^{m+q} V_a S^{m-a}. \end{aligned} \tag{6.39}$$

From (6.29) and (6.30), we know that the right hand side of (6.39) is $-SV_m+V'_mS+V_{m+1}-V'_{m+1}$. Hence (6.31) holds. The theorem is proved.

If $\Psi(\lambda)$ is a fundamental solution of the Lax set, then a column solution of the Lax set can be written as $\Psi(\lambda)l$ where $l=l(\lambda x+p)$ is a column and can be an arbitrary function of $\lambda x_1+p_1,\cdots,\lambda x_n+p_n$. It is noted that in the case of AKNS system, l should be a constant column. Hence there is more freedom for constructing the Darboux matrix in the present case.

Thus we have constructed explicitly the Darboux transformation

$$(\Psi,A_i)\to(\Psi',A'_i) \tag{6.40}$$

provided that the gauge group G is $GL(N,\mathbf{C})$. In the construction, there are only algebraic and differential operations. Moreover, the Darboux transformations can be continued successively with the same algorithm.

In many cases, the gauge potentials A_i's should belong to certain subalgebra of $gl(N)$, say $A_i\in u(N)$, i.e., $A_i^*+A_i=0$. We also want $V_a\in u(N)$.

In the construction of Darboux transformation, the equalities $A_i'^*+A'_i=0$ and $V_a'^*+V'_a=0$ should also be satisfied. This can be guaranteed by the following two constraints:

(i) Take

$$\lambda_\alpha=\mu \text{ or } \bar{\mu}\quad(\mu \text{ is not real}), \tag{6.41}$$

(ii) Take h_α's such that

$$h_\alpha^*h_\beta=0 \tag{6.42}$$

whenever $\lambda_\alpha\neq\lambda_\beta$.

In fact, when $\lambda_\alpha\neq\lambda_\beta$ (i.e. $\lambda_\alpha=\bar{\lambda}_\beta$), the Lax set leads to

$$\begin{aligned}&\left(\frac{\partial}{\partial p_i}-\mu\frac{\partial}{\partial x_i}\right)(h_\alpha^*h_\beta)=0,\\&\frac{\partial}{\partial t}(h_\alpha^*h_\beta)=0.\end{aligned} \tag{6.43}$$

Hence, $h_\alpha^*h_\beta$ is a holomorphic function of $x_i+\mu p_i$ and independent of t. If we take the initial values of h_α, h_β such that $h_\alpha^*h_\beta=0$ at $t=0$ and $x_i+\mu p_i=0$, then $h_\alpha^*h_\beta=0$ holds everywhere.

Similar to the discussion in Section 1.4, we have

$$S^*+S=(\mu+\bar{\mu})I,\quad S^*S=|\mu|^2.$$

(6.27) implies that $A_i'^*+A'_i=0$ holds. Moreover, we have $V_a'^*+V'_a=0$ inductively by (6.29) and (6.30). Therefore, the above construction of the Darboux matrix keeps the $u(N)$ reduction.

Remark 49 If A_i's are independent of t, then the above system is reduced to the generalized self-dual Yang-Mills system, and the Darboux transformation is still valid.

6.1.3 Example

Starting from the trivial solution $A_i = 0$, we want to construct solutions by Darboux transformation. We have to determine V_0, V_1, $\cdots$, V_m, V_{m+1}, $\cdots$, V_{m+q} first. When $A_i = 0$, (6.15) and (6.16) become

$$\begin{aligned}&V_0 = V_0(p),\\ &\frac{\partial V_a}{\partial x_i} = \frac{\partial V_{a-1}}{\partial p_i} \qquad (a = 1, 2, \cdots, m; m+q, \cdots, m+2),\\ &V_{m+q} = V^0_{m+q}(x).\end{aligned} \tag{6.44}$$

Hence, the entries of V_m are polynomials of x of degree $\leq m$, the entries of V_{m+1} are polynomials of p of degree $\leq q-1$. Moreover, (6.17) becomes

$$\frac{\partial V_m}{\partial p_i} = \frac{\partial V_{m+1}}{\partial x_i}. \tag{6.45}$$

Thus the entries of V_m are polynomials of p of degree $\leq q$, the entries of V_{m+1} are polynomials of x of degree $\leq m+1$. Therefore, the entries of V_0 are polynomials of p of degree $\leq m+q$ and the entries of V_{m+q} are polynomials of x of degree $\leq m+q$. From the expression of V and (6.44), we have

$$\lambda\frac{\partial V}{\partial x_i} - \frac{\partial V}{\partial p_i} = 0. \tag{6.46}$$

Hence the entries of V are functions of $\lambda p_i + x_i$ $(i = 1, \cdots, n)$. V can be expressed as

$$V = \frac{1}{\lambda^q}P(\lambda p + x). \tag{6.47}$$

Here the entries of $P(\lambda p + x)$ are polynomials of $\lambda p_i + x_i$ $(i = 1, \cdots, n)$ of degree $\leq m+q$. Therefore, the fundamental solution of (6.9) and (6.14) is

$$\Psi = \exp\left(\frac{1}{\lambda^q}P(\lambda p + x)t\right). \tag{6.48}$$

The Darboux transformation

$$(0, \Psi) \to (A', \Psi') \to (A'', \Psi'') \to \cdots, \tag{6.49}$$

can be constructed successively so that a series of explicit solutions are obtained. We still call them soliton solutions.

To write down the solutions more explicitly, take $N=2$,

$$V=\begin{pmatrix} \mathrm{i}v(\lambda) & 0 \\ 0 & -\mathrm{i}v(\lambda) \end{pmatrix}, \tag{6.50}$$

where

$$v(\lambda)=\frac{1}{\lambda^q}u(\lambda p+x), \tag{6.51}$$

u is a real polynomial of degree $m+q$. Then

$$\Psi=\begin{pmatrix} \mathrm{e}^{\mathrm{i}v(\lambda)t} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}v(\lambda)t} \end{pmatrix}. \tag{6.52}$$

Let $\mu=\sigma+\mathrm{i}\tau$ be a complex number where σ, τ are real and $\tau\neq 0$. Take

$$H=\begin{pmatrix} \mathrm{e}^{\mathrm{i}v(\mu)t} & -g^*(\bar{\mu})\mathrm{e}^{\mathrm{i}v(\bar{\mu})t} \\ g(\mu)\mathrm{e}^{-\mathrm{i}v(\mu)t} & \mathrm{e}^{-\mathrm{i}v(\bar{\mu})t} \end{pmatrix}, \tag{6.53}$$

where $g(\mu)=g(\mu p+x)$, $g^*(\bar{\mu})=\overline{g(\mu)}$. For any holomorphic function g, the two columns of H are solutions of the Lax set for $\lambda=\mu$ and $\bar{\mu}$ respectively. The condition (6.42) is also satisfied. Moreover,

$$\Delta=\det H=\mathrm{e}^{wt}+|g(\mu)|^2\mathrm{e}^{-wt}\neq 0, \tag{6.54}$$

where $w=\mathrm{i}(v(\mu)-v(\bar{\mu}))$ is a real-valued function.

Let

$$\begin{aligned} g(\mu)&=\mathrm{e}^{\rho(\mu)+\mathrm{i}\theta(\mu)}, \\ v(\mu)+v(\bar{\mu})&=\kappa(\mu), \end{aligned} \tag{6.55}$$

where $\rho(\mu)$, $\theta(\mu)$ are real-valued functions. Then

$$\begin{aligned} S&=H\Lambda H^{-1} \\ &=\begin{pmatrix} \sigma-\mathrm{i}\tau\tanh(\rho-wt) & \mathrm{i}\tau\ \mathrm{sech}(\rho-wt)\mathrm{e}^{\mathrm{i}(-\theta+\kappa t)} \\ \mathrm{i}\tau\ \mathrm{sech}(\rho-wt)\mathrm{e}^{\mathrm{i}(\theta-\kappa t)} & \sigma+\mathrm{i}\tau\tanh(\rho-wt) \end{pmatrix} \end{aligned} \tag{6.56}$$

and we get the single soliton solution

$$A_i=-\frac{\partial S}{\partial x_i}. \tag{6.57}$$

This kind of soliton solutions may have very complicated behavior because the choice of v and g has large freedom. The appearance of the functions *tanh* and *sech* in S implies that if v and g are chosen suitably, the entries of S tend to constants and the entries of A_i's tend to 0 when $\rho - wt \to \infty$. However, these solutions are not travelling waves. Multi-soliton solutions can be obtained by successive Darboux transformations with long expressions.

6.1.4 Relation with AKNS system

Let J_i's be n $N \times N$ constant matrices and

$$\begin{aligned}\Phi(p,t) &= \exp(-\textstyle\sum x_j J_j)\Psi,\\ P_i(p,t) &= -\exp(-\textstyle\sum x_j J_j) A_i \exp(\textstyle\sum x_j J_j),\\ U_a(p,t) &= \exp(-\textstyle\sum x_j J_j) V_a \exp(\textstyle\sum x_j J_j).\end{aligned} \tag{6.58}$$

Here Ψ, A_i's and V_a's are the matrix valued functions in the Lax set (6.9) and (6.14). Suppose that Φ, P_i's and U_a's are all independent of x_i's. Thus, (6.58) gives constraints for the Lax set (6.9) and (6.14).

Under these constraints, (6.9) and (6.14) become

$$\begin{aligned}\frac{\partial \Phi}{\partial p_i} &= (\lambda J_i + P_i)\Phi,\\ \frac{\partial \Phi}{\partial t} &= \sum_{a=0}^{m+q} U_a \lambda^{m-a}\Phi.\end{aligned} \tag{6.59}$$

This is the AKNS system on $\mathbf{R}^{n+1}$ where the space-time variables are $(p_1, \cdots, p_n, t)$. Therefore we have

THEOREM 6.4 *AKNS system is a kind of constraints of the self-dual Yang-Mills flow.*

R. S. Ward points out that all known soliton equations in lower dimensions can be the constraints of the self-dual Yang-Mills equation on $\mathbf{R}^4$. This means that the self-dual Yang-Mills equation has profound content. Although the high dimensional AKNS system in Chapter 3 cannot become a constraint of the self-dual Yang-Mills equation on $\mathbf{R}^4$, it is a constraint of a high dimensional generalized self-dual Yang-Mills flow.

6.2 Yang-Mills-Higgs field in 2+1 dimensional Minkowski space-time

6.2.1 Yang-Mills-Higgs field

The Yang-Mills-Higgs equations in $R^{2,1}$ are important partial differential equations in mathematical physics. The system of equations is a reduction of the self-dual Yang-Mills equation on $R^{2,2}$ and the Higgs field appears as one of the reduced gauge potential. This means that the Higgs field occurs inside the framework of Yang-Mills theory. This equation is called Bogomolny equation or monopole equation. There were a series of work on the Bogomolny equations [76]. The case of $R^{2,1}$ has the advantage that there is a real time variable and the metric is of Minkowski. The system is also integrable. A systematical sketch with beautiful results on these equations can be found in [108].

Let $R^{2,1}$ be the Minkowski space-time $\{(x,y,t)\}$ with the metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}t^2$, G be a matrix Lie group, g be its Lie algebra. Let $A = (A_t, A_x, A_y)$ be a gauge potential valued in g and Φ a $g-$valued function called Higgs field.

We consider the Yang-Mills-Higgs field satisfying the following Bogomolny equations

$$(*F)_i = D_i\Phi \quad (i = 1,2,3, x_1 = x, x_2 = y, x_3 = t). \tag{6.60}$$

Here F is the field strength, $D_i\Phi$ is the gauge-derivative of Φ, i.e.,

$$F_{ij} = \partial_j A_i - \partial_i A_j + [A_i, A_j], \tag{6.61}$$

$$D_i\Phi = \partial_i\Phi + [A_i, \Phi], \tag{6.62}$$

and $*$ is the Hodge operator.

For the Minkowski space-time $\mathbf{R}^{2,1}$ with oriented coordinate (t,y,x) and metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}t^2$, the Bogomolny equation becomes

$$F_{xy} = D_t\Phi, \quad F_{ty} = D_x\Phi, \quad F_{xt} = D_y\Phi. \tag{6.63}$$

It admits a Lax pair. In fact, let

$$L_1 = -\partial_t - \partial_y + \lambda\partial_x, \quad L_2 = \lambda\partial_t - \lambda\partial_y - \partial_x, \tag{6.64}$$

where λ is a parameter. The Lax pair is

$$\begin{aligned} L_1\Psi &= (A_t + A_y - \lambda A_x - \lambda\Phi)\Psi, \\ L_2\Psi &= (-\lambda A_t + \lambda A_y + A_x - \Phi)\Psi, \end{aligned} \tag{6.65}$$

where Ψ is an $N\times N$ matrix-valued function. The integrability condition of the Lax pair (6.65) is just the Bogomolny equation (6.63) [76]. The Lax pair (6.65) can also be written down in the covariant form

$$\begin{aligned}&(D_t+D_y-\lambda D_x)\Psi=\lambda\Phi\Psi,\\&(\lambda D_t-\lambda D_y-D_x)\Phi=-\Phi\Psi\end{aligned}\tag{6.66}$$

where $D_i\Psi=\partial_i\Psi+A_i\Psi$ $(i=t,x,y)$.

If $\Psi(t,x,y,\lambda)$ is an $N\times N$ matrix solution of Lax pair (6.65) with $\det\Psi\neq 0$, and $(L_1\Psi)\Psi^{-1}$, $(L_2\Psi)\Psi^{-1}$ are of degree one with respect to λ, then Ψ is called a Lax representation of the Yang-Mills-Higgs field, since A_t, A_x, A_y and Φ can be determined by Ψ.

6.2.2 Darboux Transformations

At first we consider the case $G=GL(N,\mathbf{C})$. Let (Ψ,A,Φ) satisfy the Lax pair. It is required to find a matrix $S(t,x,y)$ such that

$$\Psi'=(\lambda I-S)\Psi,\tag{6.67}$$

together with some A', Φ', satisfy the Lax pair (6.65) too. Hence, $A'=(A'_t,A'_x,A'_y)$ and Φ' give a solution of the Bogomolny equation [42].

Substituting Ψ' in (6.67) into the Lax pair (6.65) with (A_i,Φ) replaced by (A'_i,Φ') $(i=t,x,y)$, we obtain

$$\begin{aligned}&A'_t=A_t-(\Phi'S-S\Phi),\quad A'_y=A_y-(\Phi'S-S\Phi),\\&A'_x=A_x-(\Phi'-\Phi)\end{aligned}\tag{6.68}$$

and

$$\begin{aligned}&-S_t-S_y=A_tS-SA_t+A_yS-SA_y-2(\Phi'S-S\Phi)S,\\&S_t-S_y=SA_t-A_tS+A_yS-SA_y+2(\Phi'-\Phi),\\&S_x=SA_x-A_xS+(\Phi'-\Phi)S+(\Phi'S-S\Phi).\end{aligned}\tag{6.69}$$

From the last equation of (6.69), we get

$$\Phi'=\frac{1}{2}(S_xS^{-1}-SA_xS^{-1}+A_x+\Phi+S\Phi S^{-1}).\tag{6.70}$$

Substituting (6.70) into the first two equations of (6.69), we have

$$\begin{aligned}(\partial_t+\partial_y)S=\;&S_xS-(A_tS-SA_t)-(A_yS-SA_y)\\&+A_xS^2-SA_xS+\Phi S^2-S\Phi S,\end{aligned}\tag{6.71}$$

$$\begin{aligned}(\partial_t - \partial_y)S = S_x S^{-1} - (A_t S - S A_t) + (A_y S - S A_y) \\ + A_x - S A_x S^{-1} - \Phi + S\Phi S^{-1}.\end{aligned} \tag{6.72}$$

The problem is reduced to find S such that (6.71) and (6.72) are satisfied. This is a complicated system of nonlinear partial differential equations. Fortunately, as before, the solution S can be constructed explicitly by a "universal" formula.

Let $\lambda_1, \cdots, \lambda_N$ be N complex numbers so that they are not all equal, h_a be a column solution of the Lax pair for $\lambda = \lambda_a$ ($a = 1, 2, \cdots, N$). Construct

$$H = (h_1, h_2, \cdots, h_N) \quad (\det H \neq 0), \tag{6.73}$$

and

$$S = H\Lambda H^{-1}, \quad \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N). \tag{6.74}$$

It is easily seen that

$$(\partial_t + \partial_y)H = H_x\Lambda - A_t H - A_y H + A_x H\Lambda + \Phi H\Lambda \tag{6.75}$$

and

$$(\partial_t - \partial_y)H = H_x\Lambda^{-1} - A_t H + A_y H + A_x H\Lambda^{-1} - \Phi H\Lambda^{-1}. \tag{6.76}$$

Consequently, direct calculation shows that (6.71) and (6.72) are satisfied. Thus, we have

Theorem 6.5 *Let S be defined by (6.74). Then $\lambda I - S$ is a Darboux matrix for the Lax pair (6.65), and $(A_t', A_x', A_y', \Phi')$ given by (6.68) and (6.70) satisfies the Bogomolny equation (6.63).*

The Darboux transformation

$$(\Psi, A, \Phi) \to (\Psi', A', \Phi')$$

can be applied successively to obtain a series of solutions by using the same purely algebraic algorithm.

Remark 50 In general, if $G = GL(N, \mathbf{R})$, the solution (Ψ', A', Φ') is only a local solution, since we cannot make sure that the condition $\det H \neq 0$ holds globally.

Remark 51 h_a ($a = 1, \cdots, N$) may be expressed as

$$h_a = \Psi(\lambda_a) l_a, \tag{6.77}$$

where l_a's are columns satisfying

$$L_1 l_a = 0, \quad L_2 l_a = 0 \tag{6.78}$$

for $\lambda = \lambda_a$. Moreover, if λ_a is not real, then each entry of l_a is a holomorphic function of

$$\omega(\lambda_a) = \lambda_a^2(t+y) + 2\lambda_a x + (t-y). \tag{6.79}$$

When G is not $GL(N, \mathbf{C})$, but its subgroup, some reductions should be added in the construction of Darboux transformation. An interesting case is $G = U(N)$ (or $SU(N)$). In this case, (A, Φ) should be valued in the Lie algebra $u(N)$, i.e.,

$$A_i + A_i^* = 0, \quad \Phi + \Phi^* = 0. \tag{6.80}$$

After Darboux transformation we should have

$$A_i' + A_i'^* = 0, \quad \Phi' + \Phi'^* = 0. \tag{6.81}$$

This can be realized by choosing λ_a's and h_a's such that

$$\lambda_a = \mu \text{ or } \bar{\mu} \ \ (\mu \text{ is not real}), \tag{6.82}$$

$$h_a^* h_b = 0 \text{ if } \lambda_a \neq \lambda_b. \tag{6.83}$$

THEOREM 6.6 *Suppose A_i's and Φ are valued in $u(N)$. If we choose λ_a's and h_a's such that (6.82) and (6.83) hold, then after the Darboux transformation, A_i''s and Φ' are still valued in $u(N)$.*

Proof. As in Section 3.2, with the conditions (6.82) and (6.83), we have

$$S + S^* = (\mu + \bar{\mu})I, \quad S^* S = \mu\mu^* I. \tag{6.84}$$

Thus $\frac{1}{\mu}S$ is valued in $U(N)$, S_x and $S - \mu I$ are valued in $u(N)$. From (6.70) it is seen that $\Phi' \in u(N)$ and

$$\Phi' S - S\Phi = \frac{1}{2}(S_x - SA_x + A_x S - \Phi S + S\Phi) \in u(N). \tag{6.85}$$

Hence A_x', A_y' and A_t' are valued in $u(N)$. The proof is completed.

Remark 52 The solution (Ψ', A', Φ') is global if (Ψ, A, Φ) is global.

Remark 53 The theorem also holds for the case of $SU(N)$.

6.2.3 Soliton solutions

Starting with the trivial solution

$$A_i = \Phi = 0,$$

the first Darboux transformation gives single solitons. Now the Lax pair (6.65) becomes

$$\begin{aligned}&(-\partial_t - \partial_t + \lambda\partial_x)\psi = 0,\\&(\lambda\partial_t - \lambda\partial_t - \partial_x)\psi = 0.\end{aligned}\tag{6.86}$$

The general solution of (6.86) is

$$\psi = f(\omega(\lambda))\tag{6.87}$$

where each entry of f is an arbitrary holomorphic function, ω is defined by (6.79). The Darboux transformation is constructed in the last subsection and the expressions for Φ', A'_i are

$$\begin{aligned}&\Phi' = \tfrac{1}{2}S_xS^{-1}, \quad A'_x = -\tfrac{1}{2}S_xS^{-1},\\&A'_t = -\tfrac{1}{2}S_x, \quad A'_y = -\tfrac{1}{2}S_x.\end{aligned}\tag{6.88}$$

It is obvious that all solutions are $SU(N)$ solutions.

If $\mu = \mathrm{i}$, the solution is static. If $\mu \neq \mathrm{i}$, the line $\omega(\mu) = c$ can be expressed in the form

$$x = \beta t + \beta_0, \quad y = \gamma t + \gamma_0,\tag{6.89}$$

and the solution (Ψ', A', Φ') is a travelling wave. That is, as functions of (x, y, t), (Ψ', A', Φ') depends on $x - \beta t$ and $y - \gamma t$ only.

In the case $N = 2$, we write down the single solitons more explicitly. Let f_1, f_2 be two holomorphic functions without common zero. We have

$$H = \begin{pmatrix} f_1(\omega(\mu)) & -\overline{f_2(\omega(\mu))} \\ f_2(\omega(\mu)) & \overline{f_1(\omega(\mu))} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{pmatrix}.\tag{6.90}$$

Then

$$S = H\Lambda H^{-1} = \frac{1}{|F_1|^2 + |F_2|^2}\begin{pmatrix} \mu|F_1|^2 + \bar{\mu}|F_2|^2 & (\mu - \bar{\mu})F_1\bar{F}_2 \\ (\mu - \bar{\mu})\bar{F}_1F_2 & \mu|F_2|^2 + \bar{\mu}|F_1|^2 \end{pmatrix}.\tag{6.91}$$

Here F_1 and F_2 denote $f_1(\omega(\mu))$ and $f_2(\omega(\mu))$ respectively. Hence

$$||\Phi'||^2 = -\frac{1}{2}\,\mathrm{tr}\,\Phi'^2 = -\frac{1}{8}\,\mathrm{tr}(S_xS^{-1}S_xS^{-1}) = \frac{4(\operatorname{Im}\mu)^2|F_1F'_2 - F_2F'_1|^2}{(|F_1|^2 + |F_2|^2)^2}\tag{6.92}$$

where F'_j denotes $f'_j(\omega(\mu))$. It is noted that if $\mu \neq i$, along any straight line on the plane $t =$ constant, $\omega(\mu) \to \infty$ when $x^2 + y^2 \to \infty$. It

Figure 6.1. Example (1), $||\Phi||^2$ at $t = 10$

Figure 6.2. Example (2), $||\Phi||^2$ at $t = 10$

Figure 6.3. Example (3), $||\Phi||^2$ at $t = 10$

was shown in [108] that if $F_1 = P_1(\omega(\mu))$, $F_2 = P_2(\omega(\mu))$ where P_1 and P_2 are certain polynomials of ω, then the solution is localized. More generally, if $F_1 = P_1(\omega(\mu))$, $F_2 = P_2(\omega(\mu))\exp(Q(w(\mu)))$ where Q is another polynomial of ω with $\deg Q \le \max\{\deg P_1, \deg P_2\}$, then along any straight line on the plane $t = t_0$, we still have $S_x \to 0$ as $x^2+y^2 \to \infty$. However, the solutions may not be completely localized (see example (3)). This class of single solitons is slightly wider than those constructed in [108], where $Q(\omega) = 1$.

We draw the pictures of $||\Phi||^2$ at $t = 10$ for the following examples
(1) $\mu = i/2, \quad f_1 = 1, \quad f_2 = \omega.$
(2) $\mu = i/2, \quad f_1 = 1, \quad f_2 = (\omega - 1)(\omega + 6)(\omega - 4i).$
(3) $\mu = i/2, \quad f_1 = 1, \quad f_2 = \omega e^{\omega/10}$. The solution Φ does not approach to zero if $t = t_0$ and $(x, y) \to \infty$ along the curves $|\omega|^2 e^{\mathrm{Re}\,\omega/10} = \text{constant}$.

By applying Darboux transformations p times to the trivial solution, we obtain p-soliton solutions. Let $\mu_1, \mu_2, \cdots, \mu_p$ be p complex numbers ($\mu_j \neq \text{real}, \mu_j \neq \mathrm{i}, \mu_j \neq \mu_k$ for $j, k = 1, 2, \cdots, p$). The Lax representation of a p-soliton solution is

$$\Psi_p = (\lambda I - S_p)(\lambda I - S_{p-1}) \cdots (\lambda I - S_1) \tag{6.93}$$

At first we consider the asymptotic behavior of the double soliton

$$\Psi_2 = (\lambda I - S_2)(\lambda I - S_1) \tag{6.94}$$

as $t \to \pm\infty$. Let

$$x = a_1 t + b_1, \quad y = a_2 t + b_2 \tag{6.95}$$

be a straight line which is different from $\omega(\mu_1) = c_1$. Then along (6.95), $\omega(\mu_1) \to \infty$ as $t \to \pm\infty$. Hence

$$\lambda I - S_1 \sim \lambda I - S_0 \tag{6.96}$$

where S_0 is the limit of S as $t \to \pm\infty$. Hence the asymptotic behavior of Ψ_2 is the Darboux transformation of the constant matrix $\lambda I - S_0$ by using $\mu = \mu_2$. Therefore, Ψ_2 behaves as the Lax representation of

Figure 6.4. Splitting of $||\Phi||^2$ for a 2-soliton ($t=0$)

Figure 6.5. Splitting of $||\Phi||^2$ for a 2-soliton ($t=10$)

a single-soliton along the straight line $\omega(\mu_2)=$ constant. Besides, the potential A_i's and Higgs field Φ approach to zero along each straight line if it is neither $\omega(\mu_1)=$ constant nor $\omega(\mu_2)=$ constant..

It is remaining to consider the asymptotic behavior of Ψ_2 along the straight line $\omega(\mu_1)=$ constant.. From the theorem of permutability for Darboux transformation,

$$\Psi_2=(\lambda I-S_2)(\lambda I-S_1)=(\lambda I-S_1')(\lambda I-S_2'), \tag{6.97}$$

where S_1', S_2' are defined in the same way as S_1, S_2 but the order of (μ_1,l_1) and (μ_2,l_2) is changed. Thus the above argument implies that Ψ_2 behaves as the Lax representation of the single-soliton defined by (μ_1,l_1) along the straight line $\omega(\mu_1)=$ constant. as $t\to\pm\infty$. The double soliton is splitting up into two single solitons as $t\to\pm\infty$. For general p we have following splitting theorem as well.

THEOREM 6.7 *A p-soliton splits up into p single solitons asymptotically as $t\to\pm\infty$.*

We draw a picture for $||\Phi||^2$ showing the splitting of a 2-soliton with $\mu_1=i/2$, $\mu_2=2-i/2$ and for both μ_1 and μ_2, $f_1=1$, $f_2=\omega$.

Remark 54 The expression (6.93) can be written in the form [108]

$$\Psi_p=I+\sum_{k=1}^{p}\frac{M_k}{\lambda-\mu_k}. \tag{6.98}$$

Remark 55 Consider the relationship between gauge transformation and Darboux transformation. Let g be a function of (x,y,t) valued in the group G and $\Psi(x,y,t,\lambda)$ be a Lax representation of a solution (A,Φ) of the Bogomolny equation (6.63). Then

$$\Psi\to\widetilde{\Psi}=g\Psi \tag{6.99}$$

is a gauge transformation. In fact, from the Lax pair it is easily seen that

$$\widetilde{A}_i=gA_ig^{-1}-g_xg^{-1},\quad \widetilde{\Phi}=g\Phi g^{-1}. \tag{6.100}$$

Moreover,

$$g\Psi'=g(\lambda I-S)\Psi=(\lambda I-\widetilde{S})g\Psi=(\lambda I-\widetilde{S})\widetilde{\Psi} \tag{6.101}$$

with

$$\widetilde{S} = gSg^{-1}. \tag{6.102}$$

(6.101) means that the gauge transformations commute with the Darboux transformation.

6.3 Yang-Mills-Higgs field in 2+1 dimensional anti-de Sitter space-time

6.3.1 Equations and their Lax pair

The Bogomolny equation in 2+1 dimensional anti-de Sitter space-time is also known to be integrable in the sense that it has a Lax pair [108, 107, 109]. Here we use the Darboux transformation method to derive its explicit solutions and discuss the asymptotic behavior of the solutions as time $t \to \infty$. Moreover, the solutions are compared with those in the Minkowski space-time when the curvature of the anti-de Sitter space-time tends to zero [129, 130].

The 2+1 dimensional anti-de Sitter space-time is the hyperboloid

$$U^2 + V^2 - X^2 - Y^2 = \rho^2 \tag{6.103}$$

($\rho > 0$) in $\mathbf{R}^{2,2}$ with the metric

$$\mathrm{d}s^2 = -\mathrm{d}U^2 - \mathrm{d}V^2 + \mathrm{d}X^2 + \mathrm{d}Y^2. \tag{6.104}$$

It has constant Gauss curvature $-1/\rho^2$. Define the local coordinate

$$r = \frac{\rho}{U+X} - \rho + 1, \quad x = \frac{Y}{U+X}, \quad t = -\frac{V}{U+X}, \tag{6.105}$$

then a part $U + X > 0$ on the 2+1 dimensional anti-de Sitter space-time is represented by the Poincaré coordinates (r, x, t) with $r > -\rho + 1$ and the metric is

$$\mathrm{d}s^2 = \frac{\rho^2}{(r+\rho-1)^2}(-\mathrm{d}t^2 + \mathrm{d}r^2 + \mathrm{d}x^2) = \frac{\rho^2}{(r+\rho-1)^2}(\mathrm{d}r^2 + \mathrm{d}u\,\mathrm{d}v) \tag{6.106}$$

where $u = x + t$, $v = x - t$. Clearly, when $\rho \to +\infty$, the metric on the region $r > -\rho + 1$ tends to the flat metric of the Minkowski space-time.

Remark 56 Here we take the half space as $r > -\rho + 1$ rather than $r > 0$ so that the solitons can keep in a bounded region as $\rho \to +\infty$. This will be shown in Subsection 6.3.4.

As in the Minkowski space-time, the Bogomolny equation is also

$$(*F)_i = D_i\Phi \quad (i = u, v, r). \tag{6.107}$$

With the metric (6.106) and the orientation (v, u, r), (6.107) becomes

$$
\begin{aligned}
D_u\Phi &= \frac{r+\rho-1}{\rho}F_{ur},\\
D_v\Phi &= -\frac{r+\rho-1}{\rho}F_{vr},\\
D_r\Phi &= -\frac{2(r+\rho-1)}{\rho}F_{uv}.
\end{aligned}
\tag{6.108}
$$

It has a Lax pair

$$
\begin{aligned}
&((r+\rho-1)D_r+\rho\Phi-2(\rho\lambda-u)D_u)\psi=0,\\
&\left(2D_v+\frac{\rho\lambda-u}{r+\rho-1}D_r-\frac{\rho(\rho\lambda-u)}{(r+\rho-1)^2}\Phi\right)\psi=0
\end{aligned}
\tag{6.109}
$$

where the action of D_μ on ψ is $D_\mu\psi = \partial_\mu\psi + A_\mu\psi$. This Lax pair was first proposed by [107] to $\rho = 1$, which can be easily generalized to arbitrary ρ. With the help of this Lax pair, we can construct Darboux transformation in next subsection.

6.3.2 Darboux transformations

First we consider the case $G = GL(N, \mathbf{C})$. This is a case free of reduction. Let

$$
(\lambda - u/\rho)I - S(u, v, r) \tag{6.110}
$$

be a Darboux matrix, then there exists $(A'_u, \widetilde{A}_v, A'_r, \Phi')$ such that for any solution ψ of (6.109), $\psi' = ((\lambda - u/\rho)I - S)\psi$ satisfies

$$
\begin{aligned}
&((r+\rho-1)D'_r+\rho\Phi'-2(\rho\lambda-u)D'_u)\psi'=0,\\
&\left(2D'_v+\frac{\rho\lambda-u}{r+\rho-1}D'_r-\frac{\rho(\rho\lambda-u)}{(r+\rho-1)^2}\Phi'\right)\psi'=0
\end{aligned}
\tag{6.111}
$$

with $D'_\mu = \partial_\mu + A'_\mu$. Here the term u/ρ in (6.110) is used only to simplify the calculation.

To determine the Darboux transformation, it is sufficient to find the matrix function S.

For given (A, Φ), (A', Φ') and an arbitrary matrix function Q, we define

$$
\begin{aligned}
&\Delta_i Q = \partial_i Q + A'_i Q - Q A_i \quad (i = u, v, r),\\
&\delta Q = \Phi' Q - Q\Phi.
\end{aligned}
\tag{6.112}
$$

Substituting $\widetilde{\psi} = ((\lambda - u/\rho)I - S)\psi$ into (6.111) and considering (6.109), we get

$$\begin{aligned}
&\Delta_u I = 0, \quad (r+\rho-1)\Delta_r I + 2\rho\Delta_u S + \rho\delta I + 2I = 0,\\
&(r+\rho-1)\Delta_r S + \rho\delta S = 0, \quad \Delta_r I - \frac{\rho}{r+\rho-1}\delta I = 0,\\
&2\Delta_v I - \frac{\rho}{r+\rho-1}\Delta_r S + \frac{\rho^2}{(r+\rho-1)^2}\delta S = 0, \quad \Delta_v S = 0.
\end{aligned} \tag{6.113}$$

This system is equivalent to

$$\Delta_u I = 0, \tag{6.114}$$

$$\Delta_v S = 0, \tag{6.115}$$

$$\Delta_v I = \frac{\rho}{r+\rho-1}\Delta_r S, \tag{6.116}$$

$$\Delta_r I + \frac{\rho}{r+\rho-1}\Delta_u S + \frac{1}{r+\rho-1} I = 0, \tag{6.117}$$

$$\Delta_r I - \frac{\rho}{r+\rho-1}\delta I = 0, \tag{6.118}$$

$$\Delta_r S + \frac{\rho}{r+\rho-1}\delta S = 0. \tag{6.119}$$

The new solution $(A'_u, A'_v, \widetilde{A}_r, \Phi')$ is solved from (6.114), (6.115), (6.118) and (6.119) as

$$A'_u = A_u, \tag{6.120}$$

$$A'_v = SA_vS^{-1} - (\partial_v S)S^{-1}, \tag{6.121}$$

$$A'_r = \frac{1}{2}(SA_r - \partial_r S)S^{-1} + \frac{1}{2}A_r + \frac{\rho}{r+\rho-1}(S\Phi S^{-1} - \Phi), \tag{6.122}$$

$$\begin{aligned}
\Phi' = {}& \frac{r+\rho-1}{2\rho}(SA_r - \partial_r S)S^{-1} - \frac{r+\rho-1}{2\rho}A_r\\
&+\frac{1}{2}(S\Phi S^{-1} + \Phi),
\end{aligned} \tag{6.123}$$

while (6.116) and (6.117) give the equations that S should satisfy.

Remark 57 The Bogomolny equation is gauge invariant. We choose the form of the Darboux matrix (6.110) so that A_u keeps unchanged in Darboux transformation.

Using the standard procedure of constructing Darboux transformation, we have

THEOREM 6.8 *Let $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$ where $\lambda_1, \cdots, \lambda_N$ are complex numbers. Let h_j be a column solution of the Lax pair (6.109) with $\lambda = \lambda_j$, $H = (h_1, \cdots, h_N)$. If $\det H \neq 0$, then $S = H\Lambda H^{-1} - (u/\rho)I$ satisfies (6.116) and (6.117). Hence $(\lambda - u/\rho)I - S$ is a Darboux matrix for (6.109).*

This theorem can also be generalized as follows.

THEOREM 6.9 *Let $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$ where $\lambda_j(u, v, r)$ is a solution of*

$$\partial_r\lambda - \frac{2(\rho\lambda - u)}{r+\rho-1}\partial_u\lambda = 0, \qquad \partial_v\lambda + \frac{\rho\lambda - u}{2(r+\rho-1)}\partial_r\lambda = 0. \tag{6.124}$$

Let h_j be a column solution of the Lax pair (6.109) with $\lambda = \lambda_j(u, v, r)$, $H = (h_1, \cdots, h_N)$. If $\det H \neq 0$ and define $S = H\Lambda H^{-1} - (u/\rho)I$, then $(\lambda - u/\rho)I - S$ is a Darboux matrix for (6.109).

The general non-constant solution of (6.124) is given by the implicit function

$$v - \frac{(r+\rho-1)^2}{\rho\lambda - u} + \frac{\rho - 1}{\lambda} = \text{constant}. \tag{6.125}$$

Remark 58 Clearly Theorem 6.9 is a generalization of Theorem 6.8 because constant λ is a solution of (6.124). We shall show later that constant Λ always leads to global solutions on the 2+1 dimensional anti-de Sitter space-time. However, usually a non-constant Λ only leads to local solutions.

According to Theorem 6.8 and Theorem 6.9, we can construct explicit solutions of (6.108) from a known solution of (6.108) and the corresponding solution of the Lax pair (6.109).

When the group $G = U(N)$, the Lie algebra consists of all anti-Hermitian matrices. Hence $A_\mu^* = -A_\mu$, $\Phi^* = -\Phi$. To keep this reduction, there should be more constraints on λ_j's and h_j's in the construction of the Darboux matrix. They are

$$\begin{aligned} &\lambda_j = \lambda_0 \text{ or } \bar\lambda_0 \text{ for certain non-real } \lambda_0, \\ &h_j^* h_k = 0 \text{ if } \lambda_j \neq \lambda_k. \end{aligned} \tag{6.126}$$

The second condition holds identically if it holds on a line $u = u_0 =$ constant, $v = v_0 =$ constant. In fact, $h_j^* h_k$ satisfies

$$\begin{aligned} &(r+\rho-1)\partial_r(h_j^* h_k) - 2(\rho\lambda_k - u)\partial_u(h_j^* h_k) = 0, \\ &2\partial_v(h_j^* h_k) + \frac{\rho\lambda_k - u}{r+\rho-1}\partial_r(h_j^* h_k) = 0. \end{aligned} \tag{6.127}$$

The general solution of this system is $h_j^* h_k = f_{jk}(\omega(\lambda_k))$ where

$$\omega(\lambda) = v - \frac{(r+\rho-1)^2}{\rho\lambda - u} + \frac{\rho - 1}{\lambda} \tag{6.128}$$

and $f_{jk}(z)$ is a holomorphic function of z. Hence $f_{jk}(\omega(\lambda_k)) \equiv 0$ if $f_{jk}(\omega(\lambda_k)) = 0$ on the line $u = u_0$, $v = v_0$. Therefore, $h_j^* h_k = 0$ identically.

With the condition (6.126), the Darboux transformation gives $\Phi' \in u(N)$, $A'_\mu \in u(N)$, provided that $\Phi \in u(N)$, $A_\mu \in u(N)$.

6.3.3 Soliton solutions in $SU(2)$ case

(1) Expressions of the solutions

Single soliton solutions are given by Darboux transformations from the trivial seed solution $A_i = 0$ $(i = u, v, r)$, $\Phi = 0$. Here we only consider the case where all λ_j's are constants. In this case

$$\partial_r h_j - \frac{2(\rho\zeta_j - u)}{r+\rho-1}\partial_u h_j = 0, \qquad \partial_v h_j + \frac{\rho\zeta_j - u}{2(r+\rho-1)}\partial_r h_j = 0. \tag{6.129}$$

The general solution is

$$h_j = f_j(\omega(\zeta_j)) \tag{6.130}$$

where f_j is a column matrix whose entries are all holomorphic functions and ω is defined by (6.128). Then the Darboux matrix is $\lambda I - H\Lambda H^{-1}$ with $H = (h_1, \cdots, h_N)$.

Now suppose $G = SU(2)$. Considering the conditions (6.126), we take $\Lambda = \mathrm{diag}(\lambda_0, \bar\lambda_0)$ where $\lambda_0 \in \mathbf{C}$ is not real, $\tau = \omega(\lambda_0)$,

$$H = \begin{pmatrix} \alpha(\tau) & -\overline{\beta(\tau)} \\ \beta(\tau) & \overline{\alpha(\tau)} \end{pmatrix}$$

where α, β are holomorphic functions of τ. Let $\sigma(\tau) = \beta(\tau)/\alpha(\tau)$. Then

$$S = \frac{\lambda_0 - \bar\lambda_0}{1+|\sigma|^2}\begin{pmatrix} 1 & \bar\sigma \\ \sigma & |\sigma|^2 \end{pmatrix} + \bar\lambda_0 - \frac{u}{\rho}. \tag{6.131}$$

$$\Phi' = \frac{\lambda_0 - \bar\lambda_0}{(1+|\sigma|^2)^2}\begin{pmatrix} (|\sigma|^2)_u & \bar\sigma^2\sigma_u - \bar\sigma_u \\ \sigma^2\bar\sigma_u - \sigma_u & -(|\sigma|^2)_u \end{pmatrix}, \tag{6.132}$$

$$||\Phi'||^2 = -\frac{1}{2}\,\mathrm{tr}\,\Phi'^2 = \frac{4(\mathrm{Im}\,\lambda_0)^2}{(1+|\sigma|^2)^2}|\partial_u\sigma|^2. \tag{6.133}$$

Therefore, a single soliton solution depends on an arbitrary meromorphic function $\sigma(\tau)$ and $||\Phi'||$ is given by (6.133).

Multi-soliton solutions can be constructed by successive actions of Darboux transformations on trivial solution.

(2) Globalness of the solutions

Till now, the solutions are only defined by the local coordinate (u, v, r), or on the part with $U + X > 0$ in the 2+1 dimensional anti-de Sitter space-time (6.103). However, these solutions can be extended to the whole 2+1 dimensional anti-de Sitter space-time in a natural way.

According to (6.105) and (6.128),

$$\begin{aligned}
\tau &= \frac{\rho\lambda_0(Y+V)(U+X) - 1 - Y^2 + V^2}{(\rho\lambda_0(U+X) - Y + V)(U+X)} + \frac{\rho - 1}{\lambda_0} \\
&= \frac{\rho\lambda_0(Y+V) + X - U}{\rho\lambda_0(U+X) - Y + V} + \frac{\rho - 1}{\lambda_0}.
\end{aligned} \tag{6.134}$$

Denote

$$\xi = \rho\lambda_0(Y+V) + X - U, \quad \eta = \rho\lambda_0(X+U) - Y + V, \tag{6.135}$$

then both ξ and η can never be zero on (6.103) when λ_0 is not real. Hence τ is a smooth function of U, V, X, Y on (6.103), so are

$$\begin{aligned}
\partial_u\tau &= -\frac{(r+\rho-1)^2}{(\rho\lambda_0 - u)^2} = -\frac{\rho^2}{\eta^2}, \quad \partial_v\tau = 1, \\
\partial_r\tau &= -\frac{2(r+\rho-1)^2}{\rho\lambda_0 - u} = -\frac{2\rho}{\eta}.
\end{aligned} \tag{6.136}$$

Therefore, all the solutions which are obtained by Darboux transformation from trivial solution can be extended naturally to whole 2+1 dimensional anti-de Sitter space-time, provided that λ_0 is chosen to be a non-real constant.

(3) Localization of the solutions

The infinity of the 2+1 dimensional anti-de Sitter space-time is at $r \to -\rho + 1$. We shall verify that $||\Phi'|| \to 0$ as $r \to -\rho + 1$. In fact,

$$\begin{aligned}
||\Phi'||^2 &= \frac{4(\operatorname{Im}\lambda_0)^2}{(1+|\sigma(\tau)|^2)^2}|\sigma'(\tau)|^2|\partial_u\tau|^2 \\
&= \frac{4(\operatorname{Im}\lambda_0)^2}{(1+|\sigma(\tau)|^2)^2}|\sigma'(\tau)|^2\frac{(r+\rho-1)^4}{|\rho\lambda_0 - u|^4}.
\end{aligned} \tag{6.137}$$

When $r \to -\rho + 1$, $\tau \to v + (\rho - 1)/\lambda_0$. Hence $||\Phi'|| \to 0$. This means that the solutions are all localized on the 2+1 dimensional anti-de Sitter space-time when λ_0 is a constant.

Figure 6.6.

Figure 6.7.

(4) Asymptotic behavior of the solutions when $t \to \infty$

Without loss of generality, suppose $\lambda_0 = \mathrm{i}$. For general λ_0, a transformation of the coordinates will lead to this case.

EXAMPLE 6.10 *$\sigma(\tau) = \tau$, this is just the localized solution (25) of [107], and*

$$||\Phi'||^2 = \frac{4r^4}{((r^2 + x^2 - t^2)^2 + 2x^2 + 2t^2 + 1)^2}. \tag{6.138}$$

Let

$$x = tR\cos\theta, \quad r = tR\sin\theta. \tag{6.139}$$

When t and θ are fixed, $||\Phi'||^2$ is a function of R only. Its maximum appears at $R = \pm\sqrt{t^2+1}/t$. Hence as $t \to \infty$, the solution has a ridge which is located on the circle $x^2 + r^2 = t^2 + 1$. This is quite different from the usual solitons which have only some peaks.

Figure 6.6 shows this solution at $t = 10$. The vertical axis is $||\Phi'||^{1/2}$.

EXAMPLE 6.11 *$\sigma(\tau)$ is a polynomial of τ of degree k ($k \geq 1$) and all the roots $\tau_1, \cdots, \tau_k$ of $\sigma(\tau)$ are simple.*

The asymptotic behavior of the solution is roughly as follows [129].

(1) If $|\operatorname{Im}\tau_j| << 1$, then $||\Phi_j'||$ is not small only near a half circle centered at $r = 0$. (This half circle is a geodesic of the Poincaré plane for $t = \text{constant}$.) We call such a shape as a ridge.

(2) If $\operatorname{Im}\tau_j >> 1$, then $||\Phi_j'||$ is not small only near one point. The shape is a peak.

(3) If $\operatorname{Im}\tau_j << -1$, then $||\Phi_j'||$ is small everywhere. Nothing can be shown in the figure.

For example, let

$$\sigma(\tau) = (\tau - 2)(\tau - 6)(\tau + 6)(\tau - 2\mathrm{i})(\tau - 6\mathrm{i})(\tau + 6\mathrm{i}), \tag{6.140}$$

the shape of $||\Phi||^{1/4}$is shown in Figure 6.7 ($t = 10$). There are three ridges (corresponding to roots $2, 6, 6$ of $\sigma(\tau)$) and two peaks (corresponding to roots $2\mathrm{i}$, $6\mathrm{i}$ of $\sigma(\tau)$).

EXAMPLE 6.12 *$\sigma(\tau) = \sin(\tau/20)$. $||\Phi||^{1/4}$ is shown in Figure 6.8 ($t = 10$). Although it looks complicated, it is still localized on the 2+1 dimensional anti-de Sitter space-time because it decays as $r \to 0$.*

Figure 6.8.

Figure 6.9.

Figure 6.10.

6.3.4 Comparison with the solutions in Minkowski space-time

When $\rho \to +\infty$, the space-time $r > -\rho + 1$ with metric (6.106) tends to whole $\mathbf{R}^{2,1}$ with the Minkowski metric $\mathrm{d}s^2 = \mathrm{d}r^2 + \mathrm{d}u\,\mathrm{d}v$. By taking the limit $\rho \to +\infty$ in (6.108) and (6.109), the Bogomolny equation and its Lax pair tend to

$$D_u\Phi = F_{ur}, \quad D_v\Phi = -F_{vr}, \quad D_r\Phi = -2F_{uv}, \tag{6.141}$$

and

$$\begin{aligned} &(D_r + \Phi - 2\lambda D_u)\psi = 0, \\ &(2D_v + \lambda D_r - \lambda\Phi)\psi = 0, \end{aligned} \tag{6.142}$$

which are the Bogomolny equation (6.63) and the Lax pair (6.66) in the Minkowski space-time $\mathbf{R}^{2,1}$, with the correspondence $(\lambda, r, u, v) \to (\frac{1}{\lambda}, x, y + t, y - t)$. Then the Darboux transformation $(\lambda - u/\rho)I - S = \lambda I - H\Lambda H^{-1}$ also tends to the Darboux transformation in $\mathbf{R}^{2,1}$, so are the solutions $(A'_u, A'_v, A'_r, \Phi')$. The choice of the half space $r > -\rho + 1$ keeps the soliton solutions in a bounded region as $\rho \to +\infty$.

However, the asymptotic behavior of the solutions are quite different for finite ρ and $\rho \to +\infty$. In the Minkowski space-time, the asymptotic behavior of the solutions is quite simple when $\sigma(\tau)$ is a polynomial with only simple roots. In this case, each root of $\sigma(\tau)$ corresponds to a peak in its graph. However, for finite ρ, we have shown that different kind of the root of $\sigma(\tau)$ creates different shape in the graph. For $\sigma(\tau) = (\tau - 2)(\tau - 6)(\tau + 6)(\tau - 2\mathrm{i})(\tau - 6\mathrm{i})(\tau + 6\mathrm{i})$, Figure 6.7 has shown the solution for $\rho = 1$. Figure 6.9 and Figure 6.10 show the solution for $\rho = 2$ and $\rho \to +\infty$ respectively. In Figure 6.9, the ridges are shorter than those in Figure 6.7, and in Figure 6.10, each root of $\sigma(\tau)$ corresponds to a peak.

For $\sigma(\tau) = \sin(\tau/20)$, Figure 6.8 has shown the solution for $\rho = 1$. Here Figure 6.11 and Figure 6.12 show the solution for $\rho = 30$ and $\rho \to +\infty$ respectively.

Figure 6.11.

Figure 6.12.

Chapter 7

TWO DIMENSIONAL TODA EQUATIONS AND LAPLACE SEQUENCES OF SURFACES IN PROJECTIVE SPACE

7.1 Signed Toda equations

Toda equations are a class of important integrable systems [100]. Besides the original one-dimensional Toda equations, the two dimensional Toda equations also attract many authors in recent years [80, 31, 54, 83, 10]. In particular, the elliptic version of two dimensional Toda equations has many applications in differential geometry, such as minimal surfaces, surfaces of constant mean curvature, harmonic maps, etc.

The hyperbolic two dimensional Toda equations can be traced back to G. Darboux [19]. Consider the hyperbolic partial differential equation

$$z_{xt} + az_x + bz_t + cz = 0. \tag{7.1}$$

Define

$$z_1 = z_t + bz, \tag{7.2}$$

then z_1 satisfies a hyperbolic equation

$$z_{1,xt} + a_1 z_{1,x} + b_1 z_{1,t} + c_1 z_1 = 0, \tag{7.3}$$

which is called a Laplace transformation of (7.1) in t-direction. Similarly, the function

$$z_{-1} = z_x + az, \tag{7.4}$$

satisfies

$$z_{-1,xt} + a_{-1} z_{-1,x} + b_{-1} z_{-1,t} + c_{-1} z_{-1} = 0, \tag{7.5}$$

and is called a Laplace transformation of (7.1) in x-direction.

Applying these Laplace transformations successively, we get a series of hyperbolic systems

$$z_{i,xt} + a_i z_{i,x} + b_i z_{i,t} + c_i z_i = 0 \quad (i = 0, \pm 1, \pm 2, \cdots), \tag{7.6}$$

which is called a Laplace sequence of equations starting from (7.1)

G. Darboux derived the equations

$$(\ln h_i)_{xt} = -h_{i-1} + 2h_i - h_{i+1} \quad (h_i = \frac{\partial a_i}{\partial t} + a_i b_i - c_i). \tag{7.7}$$

He set $h_i = e^{\omega_i}$ and obtained the equations

$$\omega_{i,xt} = -e^{\omega_{i-1}} + 2e^{\omega_i} - e^{\omega_{i+1}} \quad (i = 1, 2, \cdots, n) \tag{7.8}$$

which are called hyperbolic Toda equations now.

The original Toda equations are a system of ordinary differential equations in the form

$$\frac{\mathrm{d}^2\omega_i}{\mathrm{d}t^2} = e^{\omega_{i-1}} - 2e^{\omega_i} + e^{\omega_{i+1}} \quad (i = 1, 2, \cdots, n) \tag{7.9}$$

(in the periodic case, $\omega_0 = \omega_n$, $\omega_{n+1} = \omega_1$). It was established by M. Toda [100] in the study of the motion of a chain of particles in which two neighboring particles attract each other in a nonlinear way.

Consider a system of n particles of the same kind with equal mass arranged along a line and jointed by identical springs. Let the position of the n particles be q_1, q_2, $\cdots$, q_n and $q_1 < q_2 < \cdots < q_n$.

Suppose the ith particle is acted by the $i-1$th particle and the $i+1$th particle only. If the force is dependent of the distance exponentially, then we have a system of Toda equations (7.9).

In [64], we noted that h_i may be negative. We should set

$$h_i = \alpha_i e^{\omega_i}. \tag{7.10}$$

Here $\alpha_i = \pm 1$ as $h_i > 0$ or $h_i < 0$ respectively. Thus (7.8) is changed to

$$\omega_{i,xt} = -\alpha_{i-1}e^{\omega_{i-1}} + 2\alpha_i e^{\omega_i} - \alpha_{i+1}e^{\omega_{i+1}}. \tag{7.11}$$

We call (7.11) two dimensional signed Toda equations.

Accordingly, the one dimensional signed Toda equations should be

$$\frac{\mathrm{d}^2\omega_i}{\mathrm{d}t^2} = \alpha_{i-1}e^{\omega_{i-1}} - 2\alpha_i e^{\omega_i} + \alpha_{i+1}e^{\omega_{i+1}} \quad (i = 1, 2, \cdots, n). \tag{7.12}$$

The physical interpretation is that the system describes the motion of a chain of two kinds of particles and the interacting force between two neighboring particles is attractive or repulsive according to that they belong to the different kinds or the same kind respectively [65].

This is a system consisting of two different kinds of particles, positive and negative, with equal mass and the interaction between two neighboring particles of the same (resp. different) kind is repulsive (resp. attractive). The magnitudes of the forces are the same exponential function of the relative displacement $q_{i+1} - q_i$. Consequently, the acting force from the $i+1$th particle to the ith particle is

$$m\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} = \pm \mathrm{e}^{q_{i+1}-q_i}. \tag{7.13}$$

Here we have + sign (resp. − sign) if two particles are of different (resp. same) kinds.

Define $\alpha_i = 1$ if the ith particle and the $i+1$th particle are of different kinds, $\alpha_i = -1$ if the ith particle and the $i+1$th particle are of the same kinds, then the force acting on the ith particle should be

$$\begin{aligned} &m\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} = \alpha_i \mathrm{e}^{q_{i+1}-q_i} - \alpha_{i-1}\mathrm{e}^{q_i - q_{i-1}} \quad (i = 2, \cdots, n-1), \\ &m\frac{\mathrm{d}^2 q_1}{\mathrm{d}t^2} = \alpha_1 \mathrm{e}^{q_2 - q_1}, \quad m\frac{\mathrm{d}^2 q_n}{\mathrm{d}t^2} = -\alpha_{n-1}\mathrm{e}^{q_n - q_{n-1}}. \end{aligned} \tag{7.14}$$

If the particles lie on a closed chain and let $q_1 = q_{n+1}$, $q_0 = q_n$, we have

$$\begin{aligned} &m\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} = \alpha_i \mathrm{e}^{q_{i+1}-q_i} - \alpha_{i-1}\mathrm{e}^{q_i - q_{i-1}} \quad (i = 1, \cdots, n), \\ &m\frac{\mathrm{d}^2 q_{i+1}}{\mathrm{d}t^2} = \alpha_{i+1} \mathrm{e}^{q_{i+2}-q_{i+1}} - \alpha_i \mathrm{e}^{q_{i+1} - q_i} \quad (i = 1, \cdots, n). \end{aligned} \tag{7.15}$$

Then

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(q_{i+1} - q_i) = \alpha_{i+1}\mathrm{e}^{q_{i+2}-q_{i+1}} - 2\alpha_i \mathrm{e}^{q_{i+1}-q_i} + \alpha_{i-1}\mathrm{e}^{q_i - q_{i-1}}. \tag{7.16}$$

Let

$$q_{i+1} - q_i = \omega_i, \tag{7.17}$$

then

$$m\frac{\mathrm{d}^2 \omega_i}{\mathrm{d}t^2} = \alpha_{i+1}\mathrm{e}^{w_{i+1}} - 2\alpha_i \mathrm{e}^{w_i} + \alpha_{i-1}\mathrm{e}^{w_{i-1}}. \tag{7.18}$$

Thus we get the signed Toda equations.

Consequently, we see that

(i) The arrangement of the particles of the two kinds on the line determine the values of α_i $(i = 1, 2, \cdots, n)$.

(ii) The values α_i $(= \pm 1)$ determine the arrangement of the two kinds of particles on the line.

(iii) If the chain is a closed one (i.e. $q_1 = q_{n+1}$), then the number of α_i which equals 1 should be even.

We return to the two dimensional case and treat the signed Toda equations from the point of view of integrable system. It is known that the two dimensional Toda equations can be deduced from their Lax pair

$$\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}_t = \lambda \begin{pmatrix} 0 & 0 & \cdots & 0 & p_1 \\ p_2 & 0 & \cdots & 0 & 0 \\ 0 & p_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p_n & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}, \tag{7.19}$$

$$\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}_x = \begin{pmatrix} \sigma_1 & \frac{1}{\lambda} & 0 & \cdots & 0 \\ 0 & \sigma_2 & \frac{1}{\lambda} & \cdots & 0 \\ 0 & 0 & \sigma_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \frac{1}{\lambda} \\ \frac{1}{\lambda} & 0 & 0 & \cdots & \sigma_n \end{pmatrix} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}, \tag{7.20}$$

i.e., the integrability conditions of (7.19) and (7.20) are

$$\begin{aligned} p_{i,x} &= p_i(\sigma_i - \sigma_{i-1}), \\ \sigma_{i,t} &= p_i - p_{i+1}. \end{aligned} \tag{7.21}$$

In the case of $p_i > 0$, we can write $p_i = e^{\omega_i}$ and (7.21) is equivalent to the two dimensional Toda equations (7.8).

However, in the general cases, we should put $p_i = \alpha_i e^{\omega_i}$ in (7.21) ($\alpha_i = \pm 1$), and the signed Toda equations (7.12) are derived as well. In particular, if

$$\begin{aligned} &p_1 = p_3 = \cdots = p_{2k-1} = e^{\omega}, \\ &p_2 = p_4 = \cdots = p_{2k} = e^{-\omega}, \quad n = 2k, \end{aligned} \tag{7.22}$$

the equations (7.21) are reduced to sinh-Gordon equation

$$\omega_{xt} = 4 \sinh \omega. \tag{7.23}$$

When

$$\begin{aligned} &p_1 = p_3 = \cdots = p_{2k-1} = e^{-\omega}, \\ &p_2 = p_4 = \cdots = p_{2k} = -e^{\omega}, \end{aligned} \tag{7.24}$$

we have the cosh-Gordon equation

$$\omega_{xt} = -4 \cosh \omega. \tag{7.25}$$

If x, t are replaced by ζ, $\bar{\zeta}$, the complex coordinates of the Euclidean plane, then (7.11) becomes the elliptic version of the signed Toda equations

$$\omega_{i,\zeta\bar{\zeta}} = -\alpha_{i-1}e^{\omega_{i-1}} + 2\alpha_i e^{\omega_i} - \alpha_{i+1}e^{\omega_{i+1}}. \tag{7.26}$$

Here the sign on the right hand side of the Toda equations (7.8) and the signed Toda equations (7.11) is different from that in the ordinary differential equations (7.9) and (7.12). This is due to our calculation is according to Darboux's original equation (7.8). If we change t to $-t$, then the sign of (7.8) and (7.11) will be the same as that in (7.9) and (7.12).

7.2 Laplace sequences of surfaces in projective space $\mathbf{P}_{n-1}$

The Laplace sequences of surfaces as an important subject in classical projective differential geometry, have been studied extensively [28, 95, 63, 62]. In this section, we elucidate the relationship between Laplace sequences of surfaces of period n in $\mathbf{P}_{n-1}$ and the Toda equations of period n. Different from Darboux, we consider this problem from the point of view of integrable system.

We start from the fundamental equations of the periodic Laplace sequences of surfaces which are written in the form of first-order partial differential equations. Multiplying suitable factors to the homogeneous coordinates of the points of the surfaces and changing an independent variable of the fundamental equations, we can simplify the fundamental equations of the periodic Laplace sequences of surfaces quite significantly. We find that there are two types of n periodic Laplace sequences of surfaces in $\mathbf{P}_{n-1}$. It is noted that type Ⅱ occurs only for even n and was not mentioned by Darboux and the researchers in this field. Moreover, both types have the integrability conditions of the same form

$$\frac{\partial^2\omega_i}{\partial x\partial t} = -\alpha_{i-1}e^{\omega_{i-1}} + 2\alpha_i e^{\omega_i} - \alpha_{i+1}e^{\omega_{i+1}}, \quad (i = 1, 2, \cdots, n;\ \omega_{n+1} = \omega_1) \tag{7.27}$$

where $\alpha_i = \pm 1$.

Let $\mathbf{P}_{n-1}$ be $n-1$ dimensional projective space, $(x_1, x_2, \cdots, x_n)$ be the homogeneous coordinates of a point $N \in \mathbf{P}_{n-1}$ and

$$N = N(t, x) \quad (\text{i.e. } x_a = x_a(t, x),\ a = 1, 2, \cdots, n) \tag{7.28}$$

be the equations of a surface of $\mathbf{P}_{n-1}$ in homogeneous coordinates. The straight line determined by N and N_t is the tangent line of the t curve (i.e. x = constant) and the straight line passing through N and N_x is the tangent lines of the x-curve. Suppose that N, N_t, N_x are linearly independent, then two tangent lines at N do not coincide and N, N_t, N_x determine the tangent plane of the surface N. This also means that the surface N is regular.

Suppose that there exist parameters (t, x) of the surface $N(t, x)$ such that along each x-curves the tangent lines $T(t)$ of the t curves form a developable surface, i.e., there exists a point $\lambda N + N_t$ on each $T(t)$ generating a curve $C(t)$ parametrized by x such that the tangent line of $C(t)$ is just $T(t)$. Analytically,

$$(\lambda N + N_t)_x = N_{tx} + \lambda N_x + \lambda_x N \tag{7.29}$$

is a linear combination of N and N_t. Hence $N(t, x)$ should satisfy a hyperbolic equation

$$N_{tx} + aN_t + bN_x + cN = 0. \tag{7.30}$$

Conversely, suppose that a surface $N(t, x)$ satisfies hyperbolic equation (7.30). Let

$$N' = N_t + bN \tag{7.31}$$

which is a point on the tangent line $T(t)$. From

$$N'_x = (N_t + bN)_x = N_{tx} + bN_x + b_x N = -aN_t + (b_v - c)N, \tag{7.32}$$

it is seen that when x varies N' generates a curve whose tangent line is $T(t)$, i.e., along each x-curve the tangent lines of the t-curves form a developable surface. If a surface $N(t, x)$ has the above property, we say that the surface $N(t, x)$ admits a conjugate net and (t, x) are called conjugate parameters.

We have

THEOREM 7.1 *A surface $N(t, x)$ in $\mathbf{P}_{n-1}$ admits a conjugate net with conjugate parameters (t, x) if and only if it satisfies a hyperbolic equation (7.30).*

Suppose that $N(t, x)$ satisfies (7.30). The surface

$$N' = N_t + bN \tag{7.33}$$

is called the Laplace transformation of the surface N in the t-direction. It can be verified by calculation that N' satisfies a hyperbolic equation in the form

$$N'_{tx} + a'N'_t + b'N'_x + c'N' = 0. \tag*{(7.30)$'$}$$

Hence N' admits a conjugate net with conjugate parameters (t, x) too. The tangent lines $T(t)$ to the surface N are tangent lines of surface N'

too, i.e., the straight lines $\overline{NN'}$ constitute a line congruence with focal surfaces N and N'.

Similarly, we can define the Laplace transformation of N in the x-direction

$$N'' = N_x + aN \tag{7.34}$$

and N'' satisfies a hyperbolic equation in the form

$$N''_{tx} + a''N''_t + b''N''_x + c''N'' = 0.$$

Any regular surface in $\mathbf{P}_3$ always admits a conjugate net as seen in the elementary theory of surfaces in Euclidean space. In $\mathbf{P}_n$, a surface $N(t, x)$ satisfying (7.30) or equivalently admitting a conjugate net is called a Laplace surface.

Starting from a given Laplace surface S_0, the Laplace transformations in t and x directions give a sequence of surfaces

$$\cdots, N_{-l}, N_{-l+1}, \cdots, N_0, N_1, \cdots, N_m, \cdots. \tag{7.35}$$

It can be easily shown that the Laplace transformation (7.33), (7.34) and the Laplace sequence (7.35) are independent of the change of homogeneous coordinates $N(t, x) \to \mu N(t, x)$ $(\mu \neq 0)$ geometrically.

The Laplace sequences of period n in the projective space $\mathbf{P}_{n-1}$ is of special interest. Let a system of surfaces

$$N_i = N_i(t, x) \quad (i = 1, 2, \cdots, n) \tag{7.36}$$

satisfy

$$\begin{aligned} N_{i,t} &= \mu_i N_i + p_i N_{i-1} \quad (p_i \neq 0), \\ N_{i,x} &= \sigma_i N_i + q_i N_{i+1} \quad (q_i \neq 0), \end{aligned} \tag{7.37}$$

$(N_{n+1} = N_1, N_0 = N_n)$, then the system of surfaces constitute a Laplace sequence of period n, and vice versa. In fact, the line $\overline{N_i N_{i+1}}$ is the common tangent line of surfaces N_i and N_{i+1}. In other words, (7.37) means that the surfaces N_i and N_{i+1} are the two focal surfaces of the line congruence $\{\overline{N_i N_{i+1}}\}$, and hence N_{i+1} (resp. N_i) is the Laplace transformation of N_i (resp. N_{i+1}).

We shall first simplify the fundamental equation (7.37) of Laplace sequences by multiplying a suitable factor on the homogeneous coordinates of each surface $N_i = N_i(t, x)$.

Let

$$\widetilde{N}_i = k_i(x, t) N_i, \qquad (k_i(x, t) \neq 0) \tag{7.38}$$

we have

$$\begin{aligned} \widetilde{N}_{i,t} &= \widetilde{\mu}_i \widetilde{N}_i + \widetilde{p}_i \widetilde{N}_{i-1}, \\ \widetilde{N}_{i,x} &= \widetilde{\sigma}_i \widetilde{N}_i + \widetilde{q}_i \widetilde{N}_{i+1}, \end{aligned} \tag*{(7.37)$'$}$$

where

$$\begin{aligned}\widetilde{\mu}_i &= \frac{k_{i,t}}{k_i} + \mu_i, \quad \widetilde{p}_i = \frac{k_i}{k_{i-1}} p_i,\\ \widetilde{\sigma}_i &= \frac{k_{i,x}}{k_i} + \sigma_i, \quad \widetilde{q}_i = \frac{k_i}{k_{i+1}} q_i.\end{aligned} \tag{7.39}$$

From the expression $\widetilde{\mu}_i$, we choose

$$k_i = k_i^0(x) e^{-\int \mu_i \, dt}, \tag{7.40}$$

then $\widetilde{\mu}_i = 0$ holds true. Here $k_i^0(x)$ is an arbitrary function of x.

From the integrability condition of (7.37)′, we get $\widetilde{q}_{i,t} = \widetilde{q}_i(\widetilde{\mu}_i - \widetilde{\mu}_{i+1}) = 0$. Hence $\widetilde{q}_i$'s depend on x only.

Furthermore, let $N'_i = m_i(x)\widetilde{N}_i$ $(m_i(x) \neq 0)$, we have

$$\begin{aligned}N'_{i,t} &= p'_i N'_{i-1},\\ N'_{i,x} &= \sigma'_i N'_i + q'_i N'_{i+1}\end{aligned} \tag{7.37$''$}$$

where

$$\begin{aligned}&p'_i = \frac{m_i}{m_{i-1}} \widetilde{p}_i, \quad \sigma'_i = \frac{m_{i,x}}{m_i} \widetilde{\sigma}_i, \quad q'_i = \frac{m_i}{m_{i+1}} \widetilde{q}_i,\\ &(m_{n+1} = m_1, m_0 = m_n).\end{aligned} \tag{7.41}$$

From the last equation, we see that

$$q'_1 q'_2 \cdots q'_n = \widetilde{q}_1 \widetilde{q}_2 \cdots \widetilde{q}_n \equiv Q(x) \tag{7.42}$$

is independent of the choice of m_i. When $Q > 0$ or $Q < 0$ and n is odd, let

$$q = Q^{1/n} \tag{7.43}$$

and take arbitrary $m_1 \neq 0$,

$$m_{i+1} = m_i \widetilde{q}_i q^{-1} \quad (i = 1, 2, \cdots, n), \tag{7.44}$$

then from (7.41) we have

$$q'_i = q \quad (i = 1, 2, \cdots, n). \tag{7.45}$$

We should note that the choice of m_{n+1} is consistent with $m_{n+1} = m_1$, since

$$m_{n+1} = m_n \widetilde{q}_n q^{-1} = m_{n-1} \widetilde{q}_{n-1} \widetilde{q}_n q^{-2} = \cdots = m_1 \widetilde{q}_n \widetilde{q}_{n-1} \cdots \widetilde{q}_1 q^{-n} = m_1. \tag{7.46}$$

When $Q < 0$ and n is even, let

$$q = (-Q)^{1/n} \tag{7.47}$$

and take arbitrary $m_1 \neq 0$,

$$m_{i+1} = m_i \widetilde{q}_i q^{-1} \quad (i = 1, 2, \cdots, n-1), \quad m_{n+1} = -m_n \widetilde{q}_n q^{-1}. \quad (7.48)$$

Then we have

$$q_i'(x) = q(x) \quad (i = 1, 2, \cdots, n-1), \quad q_n' = -q(x). \quad (7.49)$$

The choice of m_{n+1} is still consistent with $m_{n+1} = m_1$.

Thus we have the following theorem.

THEOREM 7.2 *There are two types of Laplace sequences of surfaces with period n in projective space $\mathbf{P}_{n-1}$. Their fundamental equations can be written as*

$$\text{Type I:} \quad \begin{cases} N_{i,t} = p_i N_{i-1} \quad (p_i \neq 0), \\ N_{i,x} = \sigma_i N_i + q N_{i+1} \quad (q \neq 0) \end{cases} \quad (7.50)$$

and

$$\text{Type II:} \quad \begin{cases} N_{i,t} = p_i N_{i-1} \quad (p_i \neq 0), \\ N_{i,x} = \sigma_i N_i + c_i q N_{i+1} \quad (q \neq 0) \end{cases} \quad (7.51)$$

respectively. Here $c_i = 1$ for $i \neq n$ and $c_n = -1$. However, the Laplace sequence of Type II can occur only for even n.

By a transformation of the variable x, (7.50) and (7.51) can be reduced to

$$\text{Type I:} \quad \begin{cases} N_{i,t} = p_i N_{i-1}, \\ N_{i,x} = \sigma_i N_i + N_{i+1} \end{cases} \quad (7.52)$$

and

$$\text{Type II:} \quad \begin{cases} N_{i,t} = p_i N_{i-1}, \\ N_{i,x} = \sigma_i N_i + c_i N_{i+1}. \end{cases} \quad (7.53)$$

Remark 59 In the case n being even, sgn(Q) *is a projective invariant of Laplace sequences of surfaces. This is the reason why there are two types of Laplace sequences of surfaces of period n for even n.*

We write

$$\Psi = \begin{pmatrix} N_1 \\ N_2 \\ \vdots \\ N_n \end{pmatrix}. \quad (7.54)$$

By the rescaling $(t, x, \sigma_i, p_i) \to \left(\lambda t, \dfrac{x}{\lambda}, \lambda\sigma_i, p_i\right)$, we can introduce the spectral parameter in (7.50) and (7.51). Thus we obtain the Lax pair.

THEOREM 7.3 *The fundamental equations (7.52) of the Laplace sequence of type I with period n in $\mathbf{P}_{n-1}$ are the Lax pair of two dimensional signed Toda equations.*

Note that the matrix form of (7.52) is (7.19) and (7.20).

The Laplace sequence of surface of type Ⅱ should correspond to the Lax pair

$$\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}_t = \lambda \begin{pmatrix} 0 & 0 & \cdots & 0 & p_1 \\ p_2 & 0 & \cdots & 0 & 0 \\ 0 & p_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p_n & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}, \tag{7.55}$$

$$\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}_x = \begin{pmatrix} \sigma_1 & \frac{1}{\lambda} & 0 & \cdots & 0 \\ 0 & \sigma_2 & \frac{1}{\lambda} & \cdots & 0 \\ 0 & 0 & \sigma_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \frac{1}{\lambda} \\ -\frac{1}{\lambda} & 0 & 0 & \cdots & \sigma_n \end{pmatrix} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}, \tag{7.56}$$

(7.56) differs from (7.20) slightly. However, (7.55) and (7.56) can serve as the Lax pair of the signed Toda equations too.

The integrability conditions of (7.55) and (7.56) are slightly different from (7.21) too. In fact, they are

$$\begin{aligned} &p_{a,x} = p_a(\sigma_a - \sigma_{a-1}), \quad \sigma_{a,t} = p_a - p_{a+1} \quad (a = 2, \cdots, n-1), \\ &p_{1,x} = p_1(\sigma_1 - \sigma_n), \quad \sigma_{1,t} = -p_1 - p_2, \\ &p_{n,x} = p_n(\sigma_n - \sigma_{n-1}), \quad \sigma_{n,t} = p_1 + p_n. \end{aligned} \tag{7.57}$$

Comparing with (7.21), it is seen that the only change is the expressions for $\sigma_{1,t}$ and $\sigma_{n,t}$.

If we set

$$p_a = \alpha_a e^{\omega_a} \quad (a = 2, \cdots, n), \quad -p_1 = \alpha_1 e^{\omega_1} \tag{7.58}$$

where

$$\alpha_a = \mathrm{sgn}(p_a) \quad (a = 2, \cdots, n), \qquad \alpha_1 = \mathrm{sgn}(-p_1), \tag{7.59}$$

then (7.57) is reduced to the signed Toda equations (7.11).

THEOREM 7.2$'$ *The fundamental equations of the Laplace sequences of type II with period n are the Lax pair of the two dimensional signed Toda equations too.*

Remark 60 If the Laplace sequences of surfaces is non-periodic, there are infinite number of surfaces corresponding to the infinite number of Toda equations and the Lax pair contains matrices of infinite order.

7.3 Darboux transformation

In the previous chapters we have applied Darboux transformation to construct new solutions of many integrable systems from some seed solutions. More precisely, the Darboux transformation is an algorithmic method to accomplish the transformation

$$(u, \Phi) \to (u', \Phi').$$

Here u is a known solution and Φ is a fundamental solution of the Lax pair corresponding to u, u' is the new solution and Φ' is the fundamental solution of the Lax pair corresponding to u'. In some geometrical problems, the fundamental solutions are the geometrical objects to be identified. In this chapter, (u, Φ) are solutions of the two dimensional signed Toda equations and the Laplace sequences of surfaces of period n in $\mathbf{P}_{n-1}$. It is very interesting to see how to apply the Darboux transformation to these objects.

Matveev has provided the Darboux transformation successively already [79]. We can use his method to construct periodic Laplace sequences of type I. Besides, Hu has modified Matveev's method to fit the case of type II. The Darboux matrix of the two cases can be derived by using the general formula $D = I - \lambda S$ in a unified way too.

At first, let (Ψ, p_i, σ_i) be the functions appeared in the Lax pair (7.19) and (7.20) in which $\Psi = (N_1, \cdots, N_n)^T$ is the Laplace sequence of type I in $\mathbf{P}_{n-1}$ and $p_i = \alpha_i e^{\omega_i}$.

The Darboux transformation for Laplace sequence of surfaces of type I is

$$N'_a = N_a - \frac{\Psi_a^0}{\Psi_{a-1}^0} N_{a-1} \tag{7.60}$$

where

$$\Psi_a^0 = N_{ai} l^i, \tag{7.61}$$

N_{ai} is the ith homogeneous coordinate of N_a and l^i is a set of constants. The corresponding

$$p'_a = p_a - \frac{\Psi^0_a}{\Psi^0_{a-1}}, \tag{7.62}$$

$$\sigma'_a = \sigma_a + \frac{\Psi^0_{a-1}}{\Psi^0_a} - \frac{\Psi^0_a}{\Psi^0_{a-1}} \quad (a = 1, \cdots, n). \tag{7.63}$$

To construct the Darboux transformation for the Laplace sequence of type Ⅱ we can still use (7.62) and (7.63) for $a = 2, \cdots, n-2$. However, σ'_1, σ'_{n-1} and σ'_n should be changed slightly as

$$\begin{aligned} \sigma'_1 &= \sigma_1 + \frac{\Psi^0_2}{\Psi^0_1} + \frac{\Psi^0_1}{\Psi^0_n}, \\ \sigma'_{n-1} &= \sigma_{n-1} + \frac{\Psi^0_n}{\Psi^0_{n-1}} - \frac{\Psi^0_{n-1}}{\Psi^0_{n-2}}, \\ \sigma'_n &= \sigma_n - \frac{\Psi^0_1}{\Psi^0_n} - \frac{\Psi^0_n}{\Psi^0_{n-1}}. \end{aligned} \tag{7.64}$$

The validity of all these formulae can be verified by direct calculations.

On the other hand, let λ_0 be a special value of the spectral parameter, Ψ_0 be a solution of the Lax pair (7.19) and (7.20) (resp. (7.55) and (7.56)) with $\lambda = \lambda_0$. Let $\omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$ and $\Omega = \mathrm{diag}(1, \omega, \omega^2, \cdots, \omega^{n-1})$. The Darboux matrix is $I - \lambda S$ with

$$S = H\Lambda^{-1}H^{-1}. \tag{7.65}$$

Here

$$\Lambda = \lambda_0 \Omega, \quad H = (\Psi^0, \Omega\Psi^0, \cdots, \Omega^{n-1}\Psi^0). \tag{7.66}$$

The Darboux transformation is

$$\Psi' = (\lambda - \lambda S)\Psi. \tag{7.67}$$

Direct calculations show that the formulae (7.60) and (7.62)–(7.64) hold.

Now we turn to some examples, i.e., use the above Darboux transformation to construct Laplace sequences.

(1) Laplace sequence of type I

Take the trivial solution of two dimensional Toda equation

$$\sigma_a = 0, \quad p_a = 1. \tag{7.68}$$

The fundamental equations (7.52) are reduced to

$$\begin{aligned} &N_{1,t} = N_4, \quad N_{2,t} = N_1, \quad N_{3,t} = N_2, \quad N_{4,t} = N_3, \\ &N_{1,x} = N_2, \quad N_{2,x} = N_3, \quad N_{3,x} = N_4, \quad N_{4,x} = N_1. \end{aligned} \tag{7.69}$$

Figure 7.1. Example of Laplace sequences of surfaces of period 4 (Type I): N_1 and N_3 are on the surface $x^2+y^2+z^2=1$, N_2 and N_4 are on the surface $y^2-x^2-z^2=1$

Solving the equations, we obtain the Laplace sequence of period 4:

$$\begin{array}{ccccc} N_1 & \cosh u & \sinh u & \cos v & \sin v \\ N_2 & \sinh u & \cosh u & \sin v & -\cos v \\ N_3 & \cosh u & \sinh u & -\cos v & -\sin v \\ N_4 & \sinh u & \cosh u & -\sin v & \cos v \end{array} \tag{7.70}$$

Here $u=t+x$, $v=t-x$.

In (7.70), the rows are homogeneous coordinates of the point N_a, and each column is a solution of the Lax pair

$$\frac{\partial \Psi_a}{\partial t} = \Psi_{a-1}, \quad \frac{\partial \Psi_a}{\partial x} = \Psi_{a+1} \quad (a=1,2,3,4; \Psi_0=\Psi_4, \Psi_5=\Psi_1). \tag{7.71}$$

It is seen that N_1 and N_3 generate the surface

$$S_1: \; x_1^2 - x_2^2 = x_3^2 + x_4^2 \quad \text{or} \quad x^2+y^2+z^2=1 \tag{7.72}$$

if we use the inhomogeneous coordinates $x=x_1/x_4$, $y=x_2/x_4$, $z=x_3/x_4$. Similarly, N_2 and N_4 generate the surface

$$S_2: \; x_2^2 - x_1^2 = x_3^2 + x_4^2 \quad \text{or} \quad y^2-x^2-z^2=1. \tag{7.73}$$

A typical quadrilateral together with S_1 and S_2 is shown in Figure 7.1.

By using Darboux transformation, we obtain a new Laplace sequence of period 4

$$\begin{aligned} N_1' &= N_1 - \frac{\Psi_1^0}{\Psi_4^0} N_4, \quad & N_2' &= N_2 - \frac{\Psi_2^0}{\Psi_1^0} N_1, \\ N_3' &= N_3 - \frac{\Psi_3^0}{\Psi_2^0} N_2, \quad & N_4' &= N_4 - \frac{\Psi_4^0}{\Psi_3^0} N_3. \end{aligned} \tag{7.74}$$

Here Ψ_a^0 is a linear combination of the 4 columns in (7.70), i.e.,

$$\begin{aligned} \Psi_1^0 &= a\cosh u + b\sinh u + c\cos v + d\sin v, \\ \Psi_2^0 &= a\sinh u + b\cosh u + c\sin v - d\cos v, \\ \Psi_3^0 &= a\cosh u + b\sinh u - c\cos v - d\sin v, \\ \Psi_4^0 &= a\sinh u + b\cosh u - c\sin v + d\cos v. \end{aligned} \tag{7.75}$$

By a long calculation, we obtain

$$\begin{array}{ccccc} N_1' & b - cz_3 + dz_2 & -a - cz_1 - dz_4 & az_3 + bz_1 + d & -az_2 + bz_4 - c \\ N_2' & -b - cz_4 + dz_1 & a + cz_2 + dz_3 & az_4 - bz_2 + d & -az_1 - bz_3 - c \\ N_3' & b + cz_3 - dz_2 & -a + cz_1 + dz_4 & -az_3 - bz_1 + d & az_2 - bz_4 - c \\ N_4' & -b + cz_4 - dz_1 & a - cz_2 - dz_3 & -az_4 + bz_2 + d & az_1 + bz_3 - c. \end{array} \tag{7.76}$$

Here

$$\begin{aligned} z_1 &= \cosh u \cos v + \sinh u \sin v, \\ z_2 &= \cosh u \cos v - \sinh u \sin v, \\ z_3 &= \cosh u \sin v + \sinh u \cos v, \\ z_4 &= \cosh u \sin v - \sinh u \cos v. \end{aligned} \tag{7.77}$$

It is not difficult to prove that N_1', N_2', N_3', N_4' generate four algebraic varieties in $\mathbf{P}_3$. In fact, among the parameters z_1, z_2, z_3, z_4, there are algebraic relations

$$z_1 z_2 + z_3 z_4 = 0, \quad z_1^2 - z_2^2 - z_3^2 + z_4^2 = 0. \tag{7.78}$$

From these relations and the parametric representation of N_i', we can eliminate these parameters and find that $x = x_1/x_4$, $y = x_2/x_4$, $z = x_3/x_4$ satisfy an algebraic equation $F_i(x, y, z) = 0$. Hence each N_i' lies on an algebraic variety.

On can apply Darboux transformations successively to get an infinite sequence of Laplace sequences of surfaces of period 4 in $\mathbf{P}_3$.

(2) Laplace sequences of Type II

Now we construct the Laplace sequences of surfaces of type II of period 4 in $\mathbf{P}_3$ explicitly. Here we also take $\lambda_0 = 1$.

For the trivial solution of (7.57), we can take

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0, \quad p_1 = -1, \quad p_2 = p_3 = p_4 = 1. \tag{7.79}$$

So the fundamental equations for the Laplace sequences are

$$\begin{aligned} &N_{1,t} = -N_4, \quad N_{2,t} = N_1, \quad N_{3,t} = N_2, \quad N_{4,t} = N_3, \\ &N_{1,x} = N_2, \quad N_{2,x} = N_3, \quad N_{3,x} = N_4, \quad N_{4,x} = -N_1. \end{aligned} \tag{7.80}$$

Solving these equations we obtain

$$\begin{array}{ccccc} N_1 & e^v \cos u & e^v \sin u & e^{-v} \cos v & e^{-v} \sin v \\ N_2 & \dfrac{e^v(\cos u - \sin u)}{\sqrt{2}} & \dfrac{e^v(\cos u + \sin u)}{\sqrt{2}} & -\dfrac{e^{-v}(\cos u + \sin u)}{\sqrt{2}} & \dfrac{e^{-v}(\cos u - \sin u)}{\sqrt{2}} \\ N_3 & -e^v \sin u & e^v \cos u & e^{-v} \sin u & -e^{-v} \cos u \\ N_4 & -\dfrac{e^v(\sin u + \cos u)}{\sqrt{2}} & \dfrac{e^v(\cos u - \sin u)}{\sqrt{2}} & \dfrac{e^{-v}(\cos u - \sin u)}{\sqrt{2}} & \dfrac{e^{-v}(\cos u + \sin u)}{\sqrt{2}} \end{array} \tag{7.81}$$

Figure 7.2. Example of Laplace sequences of surfaces of period 4 (Type II): N_1 and N_3 are on the surface $z = xy$, N_2 and N_4 are on the surface $y = -xz$

Here $u = (x-t)/\sqrt{2}$, $v = (x+t)/\sqrt{2}$, N_1 and N_3 generate the surface $S_1 : x_1x_4 = x_2x_3$ or $x = yz$, N_2 and N_4 generate the surface $S_2 : x_1x_3 = -x_2x_4$ or $y = -xz$.

A typical quadrilateral N_1, N_2, N_3, N_4 together with the surfaces S_1 and S_2 is shown in Figure 7.2.

By using the formula (7.60), we can get

$$N'_a = N_a - \frac{\Psi_a^0}{\Psi_{a-1}^0} N_{a-1} \tag{7.82}$$

as in the case of Type I. Hence the Darboux transformation gives a new Laplace sequence of period 4. These four surfaces are algebraic surfaces too.

Remark 61 The surfaces (7.72), (7.73) do not contain straight lines, while the surfaces S_1 and S_2 appeared in Laplace sequences of surfaces of type II contain two systems of straight lines. This fact reflects that the Laplace sequences of surfaces of type I and those of type II are not equivalent in the real projective geometry. The projective invariant sgn(Q) *of type I and type II are $+1$ and -1 respectively.*

7.4 Su chain (Finikoff configuration)

There are a class of special Laplace sequences of period 4 in the projective space $\mathbf{P}_3$. They were introduced by P. S. Finikoff and B. Q. Su around 1930 with a plenty of geometrical properties but without explicit examples [28, 95]. Su showed that the integrability condition of the fundamental equations is the sinh-Gordon equation. In this section, we simplify the fundamental equations of the Su chain and introduce the spectral parameter. Then the Darboux transformations are constructed and a series of explicit examples are obtained [62].

Following Su's notations, the Laplace sequences of the surfaces is $N_1N_3N_2N_4$ and the fundamental equations are reduced to

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix}_u = \frac{1}{2} \begin{pmatrix} 0 & 0 & -\mathrm{e}^{-\phi} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-\phi} \\ 0 & \mathrm{e}^{\phi} & 0 & 0 \\ \mathrm{e}^{\phi} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix}, \tag{7.83}$$

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix}_v = \frac{1}{2} \begin{pmatrix} -\phi_v & 0 & 0 & 1 \\ 0 & -\phi_v & 1 & 0 \\ -1 & 0 & \phi_v & 0 \\ 0 & 1 & 0 & \phi_v \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix}. \tag{7.84}$$

To introduce the spectral parameter, we use the transformation $u \to \dfrac{u}{\lambda}$, $v \to \lambda v$ and the equations for Su chain lead to the Lax pair

$$\Phi_u = \lambda U \Phi = \frac{\lambda}{2} \begin{pmatrix} 0 & 0 & -\mathrm{e}^{-\phi} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-\phi} \\ 0 & \mathrm{e}^{\phi} & 0 & 0 \\ \mathrm{e}^{\phi} & 0 & 0 & 0 \end{pmatrix} \Phi, \tag{7.85}$$

$$\Phi_v = V \Phi = \frac{1}{2} \begin{pmatrix} -\phi_v & 0 & 0 & 1/\lambda \\ 0 & -\phi_v & 1/\lambda & 0 \\ -1/\lambda & 0 & \phi_v & 0 \\ 0 & 1/\lambda & 0 & \phi_v \end{pmatrix} \Phi. \tag{7.86}$$

The integrability condition of the Lax pair is

$$\phi_{uv} = \sinh \phi. \tag{7.87}$$

We have already known the way of constructing the solutions of (7.87) together with the solution of 2×2 matrix Lax pair. Now we should construct the solution of (7.87) together with 4×4 matrix Lax pair (7.85) and (7.86).

Introduce 2×2 matrices

$$\begin{aligned} a &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{7.88}$$

Write U and V to be block matrices

$$U = \frac{1}{2} \begin{pmatrix} 0 & \mathrm{e}^{-\phi} a \\ \mathrm{e}^{\phi} b & 0 \end{pmatrix}, \quad V = C + \frac{1}{\lambda} D, \tag{7.89}$$

where

$$C=\frac{1}{2}\begin{pmatrix}-\phi_v I & 0\\ 0 & \phi_v I\end{pmatrix},\quad D=\frac{1}{2}\begin{pmatrix}0 & b\\ a & 0\end{pmatrix},\tag{7.90}$$

then the Lax pair (7.85) and (7.86) take the form

$$\Phi_u=\lambda U\Phi,\quad \Phi_v=\Big(C+\frac{1}{\lambda}D\Big)\Phi.\tag{7.91}$$

Consider the 2×2 Lax pair

$$\Psi_u=\lambda\widetilde{U}\Psi,\quad \Psi_v=\widetilde{V}\Psi,\tag{7.92}$$

where

$$\widetilde{U}=\frac{1}{2}\begin{pmatrix}0 & \mathrm{e}^{-\phi}\\ \mathrm{e}^{\phi} & 0\end{pmatrix},\quad \widetilde{V}=\frac{1}{2}\begin{pmatrix}-\phi_v & 1/\lambda\\ 1/\lambda & \phi_v\end{pmatrix}.\tag{7.93}$$

This Lax pair has appeared in Section 4.3. Let $(h_1,h_2)^T$ be a column solution of (7.92) for $\lambda=\lambda_1$, then the Darboux matrix is

$$\widetilde{D}(\lambda)=I-\frac{\lambda}{\lambda_1}\begin{pmatrix}0 & \dfrac{h_1}{h_2}\\ \dfrac{h_2}{h_1} & 0\end{pmatrix}=\begin{pmatrix}1 & -\dfrac{\lambda}{\lambda_1}\dfrac{h_1}{h_2}\\ -\dfrac{\lambda}{\lambda_1}\dfrac{h_2}{h_1} & 1\end{pmatrix},\tag{7.94}$$

and the new solution ϕ_1 of the sinh-Gordon equation is defined by

$$\mathrm{e}^{\phi_1}=\mathrm{e}^{-\phi}\Big(\frac{h_2}{h_1}\Big)^2.\tag{7.95}$$

Now introduce the map E from 2×2 matrices to 4×4 matrices:

$$E:\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\longrightarrow\begin{pmatrix}\alpha I & \beta a\\ \gamma b & \delta I\end{pmatrix}.\tag{7.96}$$

The Darboux matrix $\widetilde{D}(\lambda)$ is mapped to

$$D(\lambda)=\begin{pmatrix}I & -\dfrac{\lambda_1}{\lambda}\dfrac{h_1}{h_2}a\\ -\dfrac{\lambda_1}{\lambda}\dfrac{h_2}{h_1}b & I\end{pmatrix}.\tag{7.97}$$

By using the differential equations for h_1 and h_2, we can see that $\Phi_1(\lambda)=D(\lambda)\Phi(\lambda)$ satisfies

$$\Phi_{1u}(\lambda)=\lambda U_1\Phi_1(\lambda),\quad \Phi_{1v}(\lambda)=V_1\Phi_1(\lambda),\tag{7.98}$$

where U_1 and V_1 are derived from replacing ϕ in U and V by ϕ_1 in (4.175). Φ_1 is a Su chain obtained from the known Su chain by Darboux transformation. In fact,

$$\Phi_1(\lambda) = \begin{pmatrix} I & -\dfrac{\lambda}{\lambda_1}\dfrac{h_1}{h_2}a \\ -\dfrac{\lambda}{\lambda_1}\dfrac{h_2}{h_1}b & I \end{pmatrix} \Phi(\lambda). \tag{7.99}$$

Differentiating it with respect to u, the first equation of (7.98) is

$$\begin{aligned} &\begin{pmatrix} 0 & -\dfrac{\lambda}{\lambda_1}\Big(\dfrac{h_1}{h_2}\Big)_u a \\ -\dfrac{\lambda}{\lambda_1}\Big(\dfrac{h_2}{h_1}\Big)_u b & 0 \end{pmatrix} \\ &\quad + \frac{\lambda}{2}\begin{pmatrix} I & -\dfrac{\lambda}{\lambda_1}\dfrac{h_1}{h_2}a \\ -\dfrac{\lambda}{\lambda_1}\dfrac{h_2}{h_1}b & I \end{pmatrix}\begin{pmatrix} 0 & \mathrm{e}^{-\phi}a \\ \mathrm{e}^{\phi}b & 0 \end{pmatrix} \\ &= \frac{\lambda}{2}\begin{pmatrix} 0 & \mathrm{e}^{-\phi_1}a \\ \mathrm{e}^{\phi_1}b & 0 \end{pmatrix}\begin{pmatrix} I & -\dfrac{\lambda}{\lambda_1}\dfrac{h_1}{h_2}a \\ -\dfrac{\lambda}{\lambda_1}\dfrac{h_2}{h_1}b & I \end{pmatrix}. \end{aligned} \tag{7.100}$$

Equating the terms with λ^2, we get the first equation of (4.172)

$$\mathrm{e}^{\phi_1} = \Big(\frac{h_2}{h_1}\Big)^2 \mathrm{e}^{-\phi}.$$

The terms with λ give

$$\begin{aligned} -\frac{1}{\lambda_1}\Big(\frac{h_1}{h_2}\Big)_u + \frac{\mathrm{e}^{-\phi}}{2} - \frac{\mathrm{e}^{-\phi_1}}{2} = 0, \\ -\frac{1}{\lambda_1}\Big(\frac{h_2}{h_1}\Big)_u + \frac{\mathrm{e}^{-\phi}}{2} - \frac{\mathrm{e}^{-\phi_1}}{2} = 0, \end{aligned} \tag{7.101}$$

which are the equations that $\dfrac{h_1}{h_2}$ and $\dfrac{h_2}{h_1}$ in 2×2 Darboux matrix should satisfy. Hence the first equation of (7.98) holds.

The second equation of (7.98) can be verified in a similar way. Hence the matrix $D(\lambda)$ derived from the Darboux matrix $\widetilde{D}(\lambda)$ for the 2×2 Lax pair is exactly a Darboux matrix for the Su chain. Thus the following theorem holds.

THEOREM 7.4 *Suppose a solution of the sinh-Gordon equation and a fundamental solution of its Lax pair are known. Then a series of Su* *chains can be constructed by the map E and the Darboux transformation successively. The algorithm is purely algebraic.*

Now we give some geometric properties. Let $\mu = \dfrac{h_1}{h_2}$, then

$$D(\lambda) = \begin{pmatrix} 1 & 0 & \dfrac{\lambda}{\lambda_1}\mu & 0 \\ 0 & 1 & 0 & -\dfrac{\lambda}{\lambda_1}\mu \\ -\dfrac{\lambda}{\lambda_1}\dfrac{1}{\mu} & 0 & 1 & 0 \\ 0 & -\dfrac{\lambda}{\lambda_1}\dfrac{1}{\mu} & 0 & 1 \end{pmatrix}. \tag{7.102}$$

Su chain obtained from a known one via Darboux transformation is

$$\begin{pmatrix} N_1' \\ N_2' \\ N_3' \\ N_4' \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dfrac{1}{\lambda_1}\mu & 0 \\ 0 & 1 & 0 & -\dfrac{1}{\lambda_1}\mu \\ -\dfrac{1}{\lambda_1}\dfrac{1}{\mu} & 0 & 1 & 0 \\ 0 & -\dfrac{1}{\lambda_1}\dfrac{1}{\mu} & 0 & 1 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix} = \begin{pmatrix} N_1 + \dfrac{\mu}{\lambda_1}N_3 \\ N_2 - \dfrac{\mu}{\lambda_1}N_4 \\ N_3 - \dfrac{1}{\lambda_1\mu}N_1 \\ N_4 - \dfrac{1}{\lambda_1\mu}N_2 \end{pmatrix}. \tag{7.103}$$

When u and v are fixed, $N_1N_3N_2N_4$ forms a spatial quadrilateral. After the Darboux transformation, N_1', N_3', N_2' and N_4' locate on four sides of the quadrilateral $N_1N_3N_2N_4$. We say that the quadrilateral $N_1'N_3'N_2'N_4'$' is incident to the quadrilateral $N_1N_3N_2N_4$. The following theorem holds.

Theorem 7.5 *Under the Darboux transformation, the spatial quadrilateral $N_1'N_3'N_2'N_4'$ is incident to $N_1N_3N_2N_4$.*

In the construction of Darboux transformation of the sinh-Gordon equation, we want two parameters. One is assigned value of the spectral parameter $\lambda = \mu$, another is a constant column vector l such that $h = \Phi(\mu)l$. Now let (μ_A, l_A) be n sets of parameters, SC_0 be a given Su chain. By applying the Darboux transformation with parameters (μ_A, l_A) $(A = 1, 2, \cdots, n)$ successively, we get a series of Su chains

$$SC_0 \to SC_1 \to \cdots \to SC_{n-1} \to SC_n.$$

If the order of the parameters (μ_A, l_A) is changed, we have another series of Su chains

$$SC_0 \to SC_1' \to \cdots \to SC_{n-1}' \to SC_n'.$$

From the theorem of permutability, $SC_n = SC'_n$. Since the inverse of a Darboux transformation is a Darboux transformation too, we have a periodic chain of Su chains of period $2n$

$$\begin{array}{ccccccccccc} & \nearrow & SC_1 & \rightarrow & SC_2 & \rightarrow & \cdots & \rightarrow & SC_{n-1} & \searrow & \\ SC_0 & & & & & & & & & & SC_n. \\ & \nwarrow & SC'_1 & \leftarrow & SC'_2 & \leftarrow & \cdots & \leftarrow & SC'_{n-1} & \swarrow & \end{array}$$

Therefore, we get a series of solutions with period $2n$ ($n = 2, 3, 4, \cdots$), i.e., with period $4, 6, 8, \cdots$.

THEOREM 7.6 *There exists a periodic chain of Su chains of period $2n$ connected by Darboux transformations and each quadrilateral of these Su chains is incident to the quadrilateral of the neighboring Su chain.*

Especially, the series of Su chains of period 4 has four Su chains. The quadrilaterals of their congruences are incident one another.

Example:

The simplest Su chain corresponds to the solution $\phi = 0$. In this case,

$$\begin{aligned} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix}_u &= \frac{\lambda}{2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix}, \\ \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix}_v &= \frac{1}{2\lambda} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix}. \end{aligned} \tag{7.104}$$

The first equation can be written as

$$N_{1u} = -\frac{\lambda}{2} N_3, \quad N_{3u} = \frac{\lambda}{2} N_2, \quad N_{2u} = \frac{\lambda}{2} N_4, \quad N_{4u} = \frac{\lambda}{2} N_1,$$

hence

$$N_{1uuuu} = -\Big(\frac{\lambda}{2}\Big)^4 N_1. \tag{7.105}$$

Solving this ordinary differential equation,

$$\begin{aligned} N_1 = \ & A_1 \cosh\beta\cos\beta + A_2 \cosh\beta\sin\beta \\ & + A_3 \sinh\beta\cos\beta + A_4 \sinh\beta\sin\beta, \end{aligned} \tag{7.106}$$

where $\beta = \dfrac{\sqrt{2}\lambda}{4}u$, and A_1, A_2, A_3, A_4 are four dimensional row vectors which are independent of u. Then,

$$\begin{aligned}
N_3 = &\; -\frac{2}{\lambda}N_{1u} = \frac{\sqrt{2}}{2}\Big(-(A_2+A_3)\cosh\beta\cos\beta \\
&+(A_1-A_4)\cosh\beta\sin\beta - (A_1+A_4)\sinh\beta\cos\beta \\
&+(A_3-A_2)\sinh\beta\sin\beta\Big), \\
N_2 = &\; \frac{2}{\lambda}N_{3u} = -A_4\cosh\beta\cos\beta \\
&+A_3\cosh\beta\sin\beta - A_2\sinh\beta\cos\beta \\
&+A_1\sinh\beta\sin\beta, \\
N_4 = &\; \frac{2}{\lambda}N_{2u} = \frac{\sqrt{2}}{2}\Big((A_3-A_2)\cosh\beta\cos\beta \\
&+(A_1+A_4)\cosh\beta\sin\beta + (A_1-A_4)\sinh\beta\cos\beta \\
&+(A_2+A_3)\sinh\beta\sin\beta\Big).
\end{aligned} \tag{7.107}$$

From the second equation of (7.104),

$$\begin{aligned}
N_{1v} &= \frac{1}{2\lambda}N_4, \quad N_{4v} = \frac{1}{2\lambda}N_2, \\
N_{2v} &= \frac{1}{2\lambda}N_3, \quad N_{3v} = -\frac{1}{2\lambda}N_1,
\end{aligned}$$

and hence

$$\begin{aligned}
A_{1v} &= \frac{1}{2\lambda}\frac{\sqrt{2}}{2}(A_3-A_2), \quad A_{2v} = \frac{1}{2\lambda}\frac{\sqrt{2}}{2}(A_4+A_1), \\
A_{3v} &= \frac{1}{2\lambda}\frac{\sqrt{2}}{2}(A_1-A_4), \quad A_{4v} = \frac{1}{2\lambda}\frac{\sqrt{2}}{2}(A_2+A_3).
\end{aligned} \tag{7.108}$$

By differentiation, we have $A_{i,vvvv} = -\left(\dfrac{\lambda}{2}\right)^4 A_i$ $(i = 1,2,3,4)$. Hence each component of

$$A_i = (a_i, b_i, c_i, d_i) \quad (i = 1,2,3,4)$$

are linear combinations of $\cosh\alpha\cos\alpha$, $\cosh\alpha\sin\alpha$, $\sinh\alpha\cos\alpha$ and $\sinh\alpha\sin\alpha$ with constant coefficients. Here $\alpha = \dfrac{\sqrt{2}}{4}\lambda v$.

When the initial condition is taken as

$$\begin{aligned}
N_1 &= (1,0,0,0), \quad N_2 = (0,1,0,0), \\
N_3 &= (0,0,1,0), \quad N_4 = (0,0,0,1)
\end{aligned} \tag{7.109}$$

at $u = v = 0$, then

$$A_1 = \begin{pmatrix} \cosh\alpha\cos\alpha \\ \sinh\alpha\sin\alpha \\ \frac{\sqrt{2}}{2}(-\sinh\alpha\cos\alpha + \cosh\alpha\sin\alpha) \\ \frac{\sqrt{2}}{2}(\cosh\alpha\sin\alpha + \sinh\alpha\cos\alpha) \end{pmatrix}^T,$$

$$A_2 = \begin{pmatrix} \cosh\alpha\sin\alpha \\ -\sinh\alpha\cos\alpha \\ \frac{\sqrt{2}}{2}(-\cosh\alpha\cos\alpha - \sinh\alpha\sin\alpha) \\ \frac{\sqrt{2}}{2}(-\cosh\alpha\cos\alpha + \sinh\alpha\sin\alpha) \end{pmatrix}^T,$$

$$A_3 = \begin{pmatrix} \sinh\alpha\cos\alpha \\ \cosh\alpha\sin\alpha \\ \frac{\sqrt{2}}{2}(-\cosh\alpha\cos\alpha + \sinh\alpha\sin\alpha) \\ \frac{\sqrt{2}}{2}(\cosh\alpha\cos\alpha + \sinh\alpha\sin\alpha) \end{pmatrix}^T,$$

$$A_4 = \begin{pmatrix} \sinh\alpha\sin\alpha \\ -\cosh\alpha\cos\alpha \\ -\frac{\sqrt{2}}{2}(\cosh\alpha\sin\alpha + \sinh\alpha\cos\alpha) \\ \frac{\sqrt{2}}{2}(\cosh\alpha\sin\alpha - \sinh\alpha\cos\alpha) \end{pmatrix}^T.$$

Substituting into (7.107) and letting $\lambda = 1$, we can get the first Su chain. Let $\alpha + \beta = \sigma$, $\alpha - \beta = \tau$. When $\lambda = 1$, $\sigma = \dfrac{\sqrt{2}}{4}(u+v)$, $\tau = \dfrac{\sqrt{2}}{4}(u-v)$.

$$N_1 = \begin{pmatrix} \cosh\sigma\cos\tau \\ \sinh\sigma\sin\tau \\ \frac{\sqrt{2}}{2}(\cosh\sigma\sin\tau - \sinh\sigma\cos\tau) \\ \frac{\sqrt{2}}{2}(\cosh\sigma\sin\tau + \sinh\sigma\cos\tau) \end{pmatrix}^T,$$

$$N_2 = \begin{pmatrix} -\sinh\sigma\sin\tau \\ \cosh\sigma\cos\tau \\ \frac{\sqrt{2}}{2}(\cosh\sigma\sin\tau + \sinh\sigma\cos\tau) \\ \frac{\sqrt{2}}{2}(-\cosh\sigma\sin\tau + \sinh\sigma\cos\tau) \end{pmatrix}^T,$$

$$N_3 = \begin{pmatrix} \frac{\sqrt{2}}{2}(-\cosh\sigma\sin\tau - \sinh\sigma\cos\tau) \\ \frac{\sqrt{2}}{2}(-\cosh\sigma\sin\tau + \sinh\sigma\cos\tau) \\ \cosh\sigma\cos\tau \\ -\sinh\sigma\sin\tau \end{pmatrix}^T,$$

$$N_4 = \begin{pmatrix} \frac{\sqrt{2}}{2}(-\cosh\sigma\sin\tau + \sinh\sigma\cos\tau) \\ \frac{\sqrt{2}}{2}(\cosh\sigma\sin\tau + \sinh\sigma\cos\tau) \\ \sinh\sigma\sin\tau \\ \cosh\sigma\cos\tau \end{pmatrix}^T.$$

It is easy to see that N_1 and N_2 are the same quadratic surface

$$x_3^2 - x_4^2 = -2x_1x_2, \tag{7.110}$$

N_3 and N_4 are the same quadratic surface

$$x_1^2 - x_2^2 = -2x_3x_4. \tag{7.111}$$

The second Su chain can be constructed as follows. From (7.94), the 2×2 Darboux matrix is

$$I - \frac{\lambda}{\lambda_1}\begin{pmatrix} 0 & \tanh\gamma_1 \\ \coth\gamma_1 & 0 \end{pmatrix},$$

where $\gamma_1 = \dfrac{\lambda_1}{2}u + \dfrac{1}{2\lambda_1}v$. Hence the Darboux matrix for the Su chain is

$$I - \frac{\lambda}{\lambda_1}\begin{pmatrix} 0 & a\tanh\gamma_1 \\ b\coth\gamma_1 & 0 \end{pmatrix}.$$

Let $\lambda = 1$, then

$$\begin{pmatrix} N_1' \\ N_2' \\ N_3' \\ N_4' \end{pmatrix} = \begin{pmatrix} N_1 + \dfrac{\tanh\gamma_1}{\lambda_1} N_3 \\ N_2 - \dfrac{\tanh\gamma_1}{\lambda_1} N_4 \\ N_3 - \dfrac{\coth\gamma_1}{\lambda_1} N_1 \\ N_4 - \dfrac{\coth\gamma_1}{\lambda_1} N_2 \end{pmatrix}. \tag{7.112}$$

A series of solutions can be obtained successively in this way.

7.5 Elliptic version of Laplace sequence of surfaces in $\mathbf{CP}^n$

In this section, we define the elliptic Laplace sequences and obtain their relations with harmonic sequences. We also get explicit examples of harmonic sequences [21]. Using Darboux transformation, a series of explicit harmonic sequences can be constructed.

7.5.1 Laplace sequence in $\mathbf{CP}^n$

$\mathbf{CP}^n$ is the n dimensional complex projective space. Let $Z = (Z^1, \cdots, Z^{n+1})$ be homogeneous coordinates of a point in $\mathbf{CP}^n$ where Z^i ($i = 1, 2, \cdots, n+1$) are complex numbers which are not all zero. $Z = f(\zeta, \bar\zeta)$ is a map from $\mathbf{R}^2$ (or a region of $\mathbf{R}^2$) to $\mathbf{CP}^n$. Here $\zeta = z + \mathrm{i}y$ is the complex coordinate of $\mathbf{R}^2$. If f satisfies

$$f_{\zeta\bar\zeta} + af_\zeta + bf_{\bar\zeta} + cf = 0 \tag{7.113}$$

and f_ζ, $f_{\bar\zeta}$, f are linearly independent, then f is called an elliptic Laplace surface in $\mathbf{CP}^n$ parametrized by $(\zeta, \bar\zeta)$. We use $[f]$ to denote the surface or the class of functions $\{\lambda f \mid \lambda \neq 0\}$.

As for the case of real projective space, we can define two kinds of Laplace transformations $L_I([f])$ and $L_{II}([f])$ of an elliptic Laplace surface $[f]$ by

$$f_1 = f_\zeta + bf, \quad f_{-1} = f_{\bar\zeta} + af \tag{7.114}$$

respectively. Geometrically, the transformations L_I and L_{II} are independent of the choice of f in $[f]$, i.e., if we replace f by λf ($\lambda \neq 0$), then the surface $L_I([f])$ and $L_{II}([f])$ are unchanged. A sequence of elliptic Laplace surfaces

$$\cdots, [f_{-2}], [f_{-1}], [f], [f_1], [f_2], \cdots \tag{7.115}$$

can be obtained such that

$$[f_{i+1}] = L_I([f_i]), \quad [f_{i-1}] = L_{II}([f_i]). \tag{7.116}$$

The sequence (7.115) is called an elliptic Laplace sequence, which is determined by $[f]$ completely, provided that $f_{i,\zeta}$, $f_{i,\bar{\zeta}}$ and f_i are linearly independent.

An elliptic Laplace sequence of surfaces can be represented in the canonical form

$$\begin{aligned} &f_{i,\zeta\bar{\zeta}} = \sigma_i f_{i,\bar{\zeta}} + p_i f_i, \\ &f_{i,\zeta} = f_{i+1} + \sigma_i f_i, \\ &f_{i+1,\bar{\zeta}} = p_{i+1} f_i \end{aligned} \tag{7.117}$$

and

$$p_{i+1} = p_i - \sigma_{i,\bar{\zeta}}, \quad \sigma_{i+1} = \frac{p_{i+1,\zeta}}{p_{i+1}} + \sigma_i. \tag{7.118}$$

In fact, by changing f to $f_0 = \lambda f$, (7.113) can be written as

$$f_{0,\zeta\bar{\zeta}} = \sigma_0 f_{0,\bar{\zeta}} + p_0 f_0 \quad (p_0 \neq 0) \tag{7.119}$$

and (7.114) takes the form

$$f_{0,\zeta} = f_1 + \sigma_0 f_0, \quad f_{0,\bar{\zeta}} = p_0 f_{-1}. \tag{7.120}$$

All other equations in (7.117) can be obtained by acting L_I and L_{II} successively.

Remark 62 If $p_i = \alpha_i e^{\omega_i}$, (7.118) is equivalent to the signed Toda equations of elliptic version

$$\omega_{i,\zeta\bar{\zeta}} = -\alpha_{i-1} e^{\omega_{i-1}} + 2\alpha_i e^{\omega_i} - \alpha_{i+1} e^{\omega_{i+1}}. \tag{7.121}$$

7.5.2 Equations of harmonic maps from $\mathbf{R}^2$ to $\mathbf{CP}^n$ in homogeneous coordinates

In Chapter 5 we have seen that the energy of the map from a Riemannian manifold (M, g) to another Riemannian manifold (N, a) is defined by

$$\mathcal{E}(\phi) = \int E(\phi)\, \mathrm{d}V_M. \tag{7.122}$$

Here

$$E(\phi) = g^{ij} a_{\alpha\beta} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j}, \quad \mathrm{d}V_M = \sqrt{g}\, \mathrm{d}x^1 \cdots \mathrm{d}x^n \tag{7.123}$$

in local coordinates. The map ϕ is called a harmonic map if it is a critical point of the energy integral and the Euler-Lagrange equations

$$\frac{\partial (E(\phi)\sqrt{g})}{\partial y^\alpha} - \frac{\partial}{\partial x^i}\left(\frac{\partial (E(\phi)\sqrt{g})}{\partial y_i^\alpha}\right) = 0 \tag{7.124}$$

are the equations of harmonic maps.

By introducing the Hermitian metric

$$(Z,W)=\sum_{\alpha=1}^{n+1} Z^\alpha \overline{W}^\alpha,\quad |Z|^2=\sum_{\alpha=1}^{n+1} Z^\alpha \overline{Z}^\alpha \tag{7.125}$$

in $\mathbf{C}^{n+1}$, $\mathbf{CP}^n$ becomes a Riemannian manifold with the Fubini-Study metric [69]

$$\mathrm{d}s^2=\frac{|Z|^2(\mathrm{d}Z,\mathrm{d}Z)-(Z,\mathrm{d}Z)(\mathrm{d}Z,Z)}{|Z|^4}. \tag{7.126}$$

LEMMA 7.7 *The map* $[f]:\ \mathbf{R}^2\to\mathbf{CP}^n$ *is harmonic if and only if*

$$f_{\zeta\bar\zeta}=\sigma f_{\bar\zeta}+pf \tag{7.127}$$

and

$$(f_{\bar\zeta},f)=0, \tag{7.128}$$

$$\sigma=\frac{(|f|^2)_\zeta}{|f|^2} \tag{7.129}$$

hold true for some $f\in[f]$. *Here*

$$p=-\frac{(f_{\bar\zeta},f_{\bar\zeta})}{|f|^2}. \tag{7.130}$$

The proof can be done by writing out and simplifying the Euler-Lagrange equations based on the Fubini-Study metric.

By using this lemma and the canonical form of elliptic Laplace sequences, we obtain the following result.

THEOREM 7.8 *Let* $\{[f_i]\}$ *be an elliptic Laplace sequence. If one of* $[f_i]$ *(say* $[f_0]$*) is harmonic, then all* $[f_i]$ *are harmonic maps.*

Proof. It is sufficient to prove that f_1 and f_{-1} are harmonic if $f_0=f$ is harmonic. At first, it is seen that the condition (7.128) means that $f\perp f_{-1}$ and condition (7.129) means that $f\perp f_1$. From $f_1=f_\zeta-\sigma f$, it follows that

$$f_{1,\bar\zeta}=f_{\zeta\bar\zeta}-\sigma_{\bar\zeta}f-\sigma f_{\bar\zeta}=(p-\sigma_{\bar\zeta})f. \tag{7.131}$$

Hence $(f_{1,\bar\zeta},f_1)=(p-\sigma_{\bar\zeta})(f,f_1)=0$. Besides,

$$p_1=\frac{(f_{1,\zeta\bar\zeta},f_1)}{|f_1|^2}=-\frac{(f_{1,\bar\zeta},f_{1,\bar\zeta})}{|f_1|^2}=-\frac{|p_1|^2|f|^2}{|f_1|^2}. \tag{7.132}$$

Hence

$$p_1 = -\frac{|f_1|^2}{|f|^2}. \tag{7.133}$$

From

$$\sigma_1 = \frac{p_{1,\zeta}}{p_1} + \sigma = \frac{(|f_1|^2)_\zeta}{|f_1|^2} - \frac{(|f|^2)_\zeta}{|f|^2}, \tag{7.134}$$

it follows that condition (7.129) is satisfied. Hence $[f_1]$ is harmonic.

Now turn to f_{-1}. From $f_{\bar\zeta} = pf_{-1}$, it follows $(f_{-1}, f) = 0$. Hence

$$\sigma_{-1} = \frac{(f_{-1,\zeta}, f_{-1})}{(f_{-1}, f_{-1})}. \tag{7.135}$$

Since f is harmonic, we have

$$p|f|^2 = -\frac{(f_{\bar\zeta}, f_{\bar\zeta})}{|f|^2} = -|p|^2|f_{-1}|^2, \tag{7.136}$$

and hence

$$p = -\frac{|f|^2}{|f_{-1}|^2}. \tag{7.137}$$

From

$$\sigma = \frac{p_\zeta}{p} + \sigma_{-1}, \tag{7.138}$$

we have

$$\sigma_{-1} = \frac{(f_{-1}, f_{-1})_\zeta}{(f_{-1}, f_{-1})}. \tag{7.139}$$

Hence

$$(f_{-1,\zeta}, f_{-1}) = (f_{-1}, f_{-1})_\zeta \tag{7.140}$$

and it follows

$$(f_{-1}, f_{-1,\bar\zeta}) = 0. \tag{7.141}$$

Hence f_{-1} is harmonic. The theorem is proved.

Thus, the Laplace sequences of surfaces becomes the harmonic sequences in [16].

THEOREM 7.9 *Let $\{[f_i]\}$ be an elliptic Laplace sequence. If $f_{-1} \perp f_0$ and $f_0 \perp f_1$, the $\{[f_i]\}$ is a harmonic sequence.*

From the properties of harmonic sequences or by direct calculation, there are following facts.

(1) Let

$$e^{\omega_i} = \frac{|f_{i+1}|^2}{|f_i|^2}. \tag{7.142}$$

The elliptic Toda equations

$$\omega_{i,\zeta\bar{\zeta}} = e^{\omega_{i-1}} - 2e^{\omega_i} + e^{\omega_{i+1}} \tag{7.143}$$

are satisfied. Hence from a harmonic sequence, a solution of two-dimensional elliptic Toda equations can be constructed.

(2) If $f_{-1} \perp f_0$, $f_0 \perp f_1$ and $f_{-1} \perp f_1$, then $f_{i-1} \perp f_i$, $f_i \perp f_{i+1}$ and $f_{i-1} \perp f_{i+1}$ holds true for all i, and these f_i's are minimal surfaces in $\mathbf{CP}^n$.

For the proof of $f_{i-1} \perp f_{i+1}$, it is sufficient to verify $f \perp f_2$ and $f \perp f_{-2}$. From

$$0 = (f, f_{-1})_\zeta = (f_\zeta, f_{-1}) + (f, f_{-1,\bar{\zeta}}) = (f_1, f_{-1}) - (\sigma f, f_{-1}) + \bar{p}_{-1}(f, f_{-2}), \tag{7.144}$$

it is seen that $(f, f_{-2}) = 0$. From

$$0 = (f_1, f)_\zeta = (f_{1,\zeta}, f) + (f_1, f_{\bar{\zeta}}) = (f_2, f) + (\sigma_1 f_1, f_{-1}) + \bar{p}(f_1, f_{-1}), \tag{7.145}$$

we can see $(f_2, f) = 0$. Moreover, the metric on the surface $[f_i]$ induced from the Fubini-Study metric is proportional to the metric $\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta}$, i.e., the map $[f_i]$ is conformal. It is well-known that a harmonic map is minimal if it is conformal [114].

7.5.3 Cases of indefinite metric

If we use the indefinite metric

$$\begin{aligned} (Z, W)_J &= \sum_{a=1}^{J} Z^a \overline{W}^a - \sum_{b=J+1}^{n+1} Z^b \overline{W}^b, \\ |Z|_J^2 &= \sum_{a=1}^{J} Z^a \overline{Z}^a - \sum_{b=J+1}^{n+1} Z^b \overline{Z}^b \end{aligned} \tag{7.146}$$

in $\mathbf{C}^{n+1}$ to replace the positive definite metric (Z, W), we obtain the submanifolds

$$\begin{aligned} \mathbf{CP}^n_{J_+} &= \{[Z] \,|\, |Z|_J^2 > 0\}, \\ \mathbf{CP}^n_{J_-} &= \{[Z] \,|\, |Z|_J^2 < 0\}, \\ \mathbf{CP}^n_{J_0} &= \{[Z] \,|\, |Z|_J^2 = 0\}, \end{aligned} \tag{7.147}$$

and the Fubini-Study metric can be extended to $\mathbf{CP}^n_{J_+}$ and $\mathbf{CP}^n_{J_-}$.

The above results can be extended to $\mathbf{CP}^n_{J_+}$ and $\mathbf{CP}^n_{J_-}$ with some modifications, provided that $|f_i|_J^2 \neq 0$. In these cases the two-dimensional signed Toda equations of elliptic version

$$\omega_{p,\zeta\bar{\zeta}} = \alpha_{p-1} e^{\omega_{p-1}} - 2\alpha_p e^{\omega_p} + \alpha_{p+1} e^{\omega_{p+1}} \tag{7.148}$$

can be obtained.

7.5.4 Harmonic maps from $\mathrm{R}^{1,1}$

Instead of $\mathbf{R}^2$ (or S^2), we take the Minkowski plane $\mathbf{R}^{1,1} = \{(\xi, \eta)\}$ with $\mathrm{d}s^2 = \mathrm{d}\xi\,\mathrm{d}\eta$, then the Laplace sequence can be written in the form

$$\begin{aligned} &f_{i,\xi\eta} = \sigma_i f_{i,\eta} + p_i f_i, \\ &f_{i,\xi} = f_{n+1} + \sigma_i f_i, \\ &f_{i+1,\eta} = p_{i+1} f_i \end{aligned} \tag{7.149}$$

and

$$\begin{aligned} &p_{i+1} = p_i - \sigma_{i,\eta}, \\ &\sigma_{i+1} = \frac{p_{i+1,\eta}}{p_{i+1}} + \sigma_i. \end{aligned} \tag{7.150}$$

Moreover, it is seen that if $f_{i-1} \perp f_i$ and $f_i \perp f_{i+1}$, then f_i is a harmonic map (wave map) from $\mathbf{R}^{1,1}$ to $\mathbf{CP}^n$ (or $\mathbf{CP}^n_{J_+}$, $\mathbf{CP}^n_{J_-}$). However, in general $f_{i-1} \perp f_i$ and $f_i \perp f_{i+1}$ do not imply $f_{i+1} \perp f_{i+2}$, i.e., starting with a harmonic map, the Laplace sequence may not be a harmonic map.

7.5.5 Examples of harmonic sequences from R^2 to CP^n or $\mathrm{R}^{1,1}$ to CP^n

Example 1. Let

$$\begin{aligned} f \ &= (f^1, f^2, \cdots, f^{n+1}) \\ &= \Big(\alpha_1 \exp\Big(\lambda_1 \xi - \frac{\bar{\xi}}{\lambda_1}\Big), \cdots, \alpha_{n+1} \exp\Big(\lambda_{n+1}\xi - \frac{\bar{\xi}}{\lambda_{n+1}}\Big)\Big). \end{aligned} \tag{7.151}$$

We take λ_k $(k = 1, 2, \cdots, n+1)$ such that $|\lambda_k|^2 = 1$ and

$$|\alpha_1|^2 \lambda_1 + \cdots + |\alpha_{n+1}|^2 \lambda_{n+1} = 0. \tag{7.152}$$

Then

$$\cdots, f_{\bar{z}\bar{z}}, f_{\bar{z}}, f, f_z, f_{zz}, \cdots \tag{7.153}$$

is a harmonic sequence in $\mathbf{CP}^n$ since $f_{\zeta\bar{\zeta}} + f = 0$.

In particular, if we take

$$|\alpha_k|^2 = 1, \quad \lambda_k = \mathrm{e}^{\frac{2k\pi}{n+1}\mathrm{i}}, \tag{7.154}$$

then the harmonic sequence is of period n.

Example 2. For a harmonic sequence from $\mathbf{R}^{1,1}$ to $\mathbf{CP}^{n-1}$ $(n = 2k)$

$$
\begin{aligned}
& f = (f^1, f^2, \cdots, f^n), \\
& f^1 = \alpha_1 \cos\Big(\lambda_1 \xi - \frac{\eta}{\lambda_1}\Big), \quad f^2 = \alpha_1 \sin\Big(\lambda_1 \xi - \frac{\eta}{\lambda_1}\Big), \\
& f^3 = \alpha_2 \cos\Big(\lambda_2 \xi - \frac{\eta}{\lambda_2}\Big), \quad f^4 = \alpha_2 \sin\Big(\lambda_2 \xi - \frac{\eta}{\lambda_2}\Big), \\
& \cdots \\
& f^{2k-1} = \alpha_k \cos\Big(\lambda_k \xi - \frac{\eta}{\lambda_k}\Big), \quad f^{2k} = \alpha_k \sin\Big(\lambda_k \xi - \frac{\eta}{\lambda_k}\Big),
\end{aligned}
\tag{7.155}
$$

where λ_1, $\cdots$, λ_k are real numbers, we have

$$
\begin{aligned}
& (f, f) = |\alpha_1|^2 + \cdots + |\alpha_n|^2 = \text{constant}, \\
& (f, f_\xi) = 0, \quad (f, f_\eta) = 0, \\
& f_{\xi\eta} + f = 0.
\end{aligned}
\tag{7.156}
$$

The Laplace sequence is

$$
\cdots, f_{\eta\eta}, f_\eta, f, f_\xi, f_{\xi\xi}, \cdots. \tag{7.157}
$$

It is easily verified that

$$
(f_k, f_{k+1}) = 0, \quad (f_{k-1}, f_k) = 0. \tag{7.158}
$$

Hence f_k is a harmonic map.

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