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On Darboux Transformations for Soliton Equations in High-Dimensional Spacetime★

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Abstract. The Darboux transformations for a class of completely integrable systems in the spacetime $\mathbf{R}^{n+1}$, which are much more general than the systems in *Lett. Math. Phys.* **26**, 199–209 (1989), are considered. The structure of the nonlinear evolution equations with space constraints is elucidated. It is pointed out that the inverse scattering method can be used to solve the Cauchy problem with initial data given on a noncharacteristic line.

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1. Introduction

In a previous paper [7], the first author considered a class of completely integrable systems in high-dimensional spacetime $\mathbf{R}^{n+1}$, which extends the AKNS systems in $\mathbf{R}^{1+1}$ to the high-dimensional case. A Darboux matrix method is established for constructing exact solutions with algebraic algorithms. Multi-soliton solutions were obtained as Darboux transformations of trivial solutions. It was proved that as $t \to \pm\infty$, a k multi-soliton splits up into k single solitons and their interaction is elastic. However, in that paper, the order m of the matrices in the Lax set is equal to the space dimensions n. Besides, the function V in the time part of the Lax set is a polynomial of the spectral parameter λ of second degree.

In this Letter we treat the most general case, i.e. we remove the restriction $N = n$ in [7] and consider that V is a polynomial of λ of arbitrary degree. It is proved that all results of [7] are still valid. Of course, the proof is more complicated and technical. Moreover, the structure of the derived overdetermined system is elucidated. It is shown that the initial data given on a noncharacteristic straight line are sufficient to determine the solution. Such a kind of initial value problem can be

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solved by the inverse scattering method. It should be noted that for the generalized wave equation and generalized sine–Gordon equation, which are special cases of our system and free from the time part, the inverse scattering method was considered in [1].

The systems considered in this Letter have quite a lot of degrees of freedom. They contain many special cases. One can use the framework to obtain special exact solutions for some famous equations such as the KP equation, DS equation, etc. In Section 2, we introduce the general AKNS system in $\mathbf{R}^{n+1}$ for arbitrary n. We reduce the integrability condition to a part of them. We point out that the coefficients of the t-part equations are differential polynomials of the unknown P and put the strict proof in the Appendix. Section 3 is devoted to the Darboux transformations. It is shown that most of the results in [7], in particular, the universal formula for the Darboux matrices, are still valid. In Section 4, we analyse the structure of the evolution equations with space constraints, showing that the initial data on a one-dimensional noncharacteristic line are sufficient to determine the solution. It is also pointed out that the Cauchy problem in such a form can be solved by using the inverse scattering method for $\mathbf{R}^{1+1}$ case.

2. General AKNS System in $\mathbf{R}^{n+1}$

We consider the following general AKNS system in the spacetime $\mathbf{R}^{n+1}$

$$\partial_i \Psi = (\lambda J_i + P_i)\Psi \quad (i = 1, \ldots, n)$$

$$\partial_t \Psi = V\Psi = \sum_{\alpha=0}^{m} V_\alpha \lambda^{m-\alpha} \Psi \tag{2.1}$$

where $\partial_i \Psi = \partial\Psi/\partial x_i$, $\partial_t \Psi = \partial\Psi/\partial t$, J_i's are $N \times N$ constant diagonal matrices, P_i's, V_α's are $N \times N$ matrix functions and P_i's are off-diagonal. Moreover, we suppose that J_i's satisfy the conditions

(J): $[J_i, A] = 0$ for all i implies A is diagonal, and all the J_i's are linearly independent with respect to real numbers.

The integrability conditions of (2.1) are

$$[J_i, P_j] = [J_j, P_i], \tag{2.2}$$

$$\partial_j P_i - \partial_i P_j + [P_i, P_j] = 0, \tag{2.3}$$

$$[J_i, V_{\alpha+1}] = \partial_i V_\alpha - [P_i, V_\alpha], \tag{2.4}$$

$$\partial_t P_i - \partial_i V_m + [P_i, V_m] = 0. \tag{2.5}$$

From (2.2) and condition (J), there exists an off-diagonal matrix P such that

$$P_i = [P, J_i] \tag{2.6}$$

for all i.

Equations (2.3) are space-constraints of P, (2.4) together with the diagonal part of (2.5) are equations for V_α's and the off-diagonal part of (2.5) are the evolution equations for P. Thus, we obtain an overdetermined system of evolution equations for P and (2.1) is its Lax set (or Lax pair for the case $n = 1$).

The number of equations (2.3)–(2.5) can be reduced. At first, we may assume that the diagonal entries of J_1 are different from each other. In fact, this can be realized by a suitable linear transformation of x_i.

LEMMA 1. *Equations* (2.2)–(2.5) *are equivalent to*

$$\partial_1 P_i - \partial_i P_1 + [P_i, P_1] = 0, \tag{2.7-i}$$

$$[J_i, V_{\alpha+1}^{\mathrm{off}}] = \partial_i V_\alpha^{\mathrm{off}} - [P_i, V_\alpha]^{\mathrm{off}}, \tag{2.8-i-α}$$

$$\partial_i V_\alpha^{\mathrm{diag}} = [P_i, V_\alpha^{\mathrm{off}}]^{\mathrm{diag}}, \tag{2.9-i-α}$$

$$\partial_t P_1 - \partial_1 V_m^{\mathrm{off}} + [P_1, V_m]^{\mathrm{off}} = 0. \tag{2.10}$$

Proof. Clearly, (2.8) and (2.9) are just (2.4) in another form. We only need to prove that (2.7)–(2.10) imply (2.3) with $j \geqslant 2$ and (2.5) with $i \geqslant 2$.

Let

$$\Delta_{ij} = \partial_j P_i - \partial_i P_j + [P_i, P_j], \tag{2.11}$$

then direct calculation shows

$$\begin{aligned}
&[J_1, \Delta_{ij}] \\
&\quad = \partial_j [J_i, P_1] - \partial_i [J_j, P_1] + [J_1, [P_i, P_j]] \\
&\quad = [J_i, \Delta_{1j}] - [J_j, \Delta_{1i}]
\end{aligned}$$

by Jacobi identity.

Since $P_i = [P, J_i], P_j = [P, J_j]$, it can be directly checked that $[P_i, P_j]$ is off-diagonal. Hence, Δ_{ij} is off-diagonal. $\Delta_{1i} = 0$ $(i \geqslant 1)$ implies $\Delta_{ij} = 0$ for all i, j. This proves (2.3).

Now consider (2.5). For any $i > 1$,

$$\begin{aligned}
&[\partial_t P_i - \partial_i V_m + [P_i, V_m], J_1] \\
&\quad = [\partial_1 V_m - [P_1, V_m], J_i] - [\partial_i V_m - [P_i, V_m], J_1] \\
&\quad = \partial_1 [V_m, J_i] - \partial_i [V_m, J_1] + [[P_i, V_m], J_1] - [[P_1, V_m], J_i].
\end{aligned}$$

By (2.3), (2.4) and Jacobi identity, it equals to

$$\begin{aligned}
&\partial_1 [P_i, V_{m-1}] - \partial_i [P_1, V_{m-1}] - [[V_m, J_1], P_i] + [[V_m, J_i], P_1] \\
&\quad = [\partial_1 P_i - \partial_i P_1, V_{m-1}] + [P_i, [P_1, V_{m-1}]] - [[P_1, [P_i, V_{m-1}]] \\
&\quad = 0.
\end{aligned}$$

Hence (2.5) holds.

Therefore, the space-constraints are reduced to (2.7) and the evolution equations are reduced to (2.10).

For the matrix functions V_α's, we have

THEOREM 1. *V_α's are polynomials of P and its x-derivatives, denoted by $V_\alpha = V_\alpha[P]$, and $V_\alpha[P]$ are determined uniquely if $V_\alpha[0] = V_\alpha^0(t)$ are given, where V_α^0's are diagonal and independent of x_i's.*

Idea of proof. V_0, V_1, V_2 can be calculated directly from (2.8) and (2.9) just as in [7]. For a general V_α, we use mathematical induction. We use the result in [10] to show that the V_α's are polynomials of $P, \partial_1 P, \ldots, \partial_1^\alpha P$, considering $x_2, \ldots, x_n$ as parameters. Then we show that the V_α's should be differential polynomials of P whose coefficients are determined by (2.8), (2.9) and the 'constants of integration V_α^0'. For details of the proof, we need some ideas on differentiable rings and the procedure is very technical. The complete proof appears in the Appendix.

Therefore, all the equations generated from the Lax set (2.1) are differential equations.

3. Darboux Transformations

Similar to [6, 7], we can also use Darboux transformations to construct the solutions of Equations (2.7), (2.10) and (2.1) from a known solution.

THEOREM 2. *Let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ be a constant diagonal matrix. Let (Ψ, P) be a known solution to (2.7), (2.10) and (2.1), h_a be a column solution of (2.1) with $\lambda = \lambda_a$, and $H = (h_1, \ldots, h_N)$. Suppose $\det H \neq 0$. Let $G = -H\Lambda H^{-1}$, then*

$$\tilde{P} = P + G^{\mathrm{off}}, \tag{3.1}$$

$$\tilde{\Psi} = (\lambda + G)\Psi \tag{3.2}$$

satisfy (2.1), (2.7) and (2.10) too.

Proof. It is sufficient to prove

$$\begin{aligned} &\partial_i \tilde{\Psi} = (\lambda J_i + [\tilde{P}, J_i])\tilde{\Psi} \quad (i = 1, \ldots, n), \\ &\partial_t \tilde{\Psi} = \sum_{\alpha=0}^{m} \tilde{V}_\alpha \lambda^{m-\alpha} \tilde{\Psi} \end{aligned} \tag{3.3}$$

with

$$\tilde{P} = P + G^{\mathrm{off}}, \qquad \tilde{V}_\alpha = V_\alpha[\tilde{P}]. \tag{3.4}$$

The first equation of (3.3) can be checked directly as in [7], by using the fact that

$$\partial_i G = [G, J_i G - [P, J_i]]. \tag{3.5-i}$$

On the other hand, direct calculation gives

$$\tilde{V}_0 = V_0, \tag{3.6-0}$$

$$\tilde{V}_{\alpha+1} = V_{\alpha+1} + GV_\alpha - \tilde{V}_\alpha G, \tag{3.6-α}$$

and

$$\partial_t G = -\left[G, \sum_{\alpha=0}^{m} V_\alpha(-G)^{n-\alpha}\right]. \tag{3.7}$$

The second equation of (3.4) can be proved in the following way. First we prove the identity

$$[J_i, \tilde{V}_{\alpha+1}] = \partial_i \tilde{V}_\alpha - [\tilde{P}_i, \tilde{V}_\alpha]. \tag{3.8-α}$$

Here $\tilde{V}_\alpha$ ($\alpha = 0, 1, \ldots, m-1$) are considered as differential polynomials of P and G of degree $\beta \leqslant \alpha$ and 0, respectively, and the coefficients are functions of t, provided that (2.3) and (3.5) hold true. Assume that (3.8-(α-1)) is true. Substituting (3.6-α) and (3.6-(α-1)) in the left and right sides of (3.8-α), respectively, using (2.4) and (3.5-i), we see that (3.8-α) is true after long but routine calculations. Hence, (3.8-α) holds by induction. Similarly, we can prove

$$\partial_i \tilde{V}_m^{\text{diag}} = [\tilde{P}_i, V_m]^{\text{diag}}. \tag{3.9}$$

From (2.4) and the diagonal part of (2.5), it is easily seen that

$$[J_i, \tilde{V}_{\alpha+1}[\tilde{P}]] = \partial_i \tilde{V}_\alpha[\tilde{P}] - [\tilde{P}_i, V_\alpha[\tilde{P}]], \tag{3.10-α}$$

$$\partial_i^{\text{diag}} V_m[\tilde{P}] = [\tilde{P}_i, V_m^{\text{diag}}[\tilde{P}]]. \tag{3.11}$$

Let

$$\Delta_\alpha = V_\alpha[\tilde{P}] - \tilde{V}_\alpha = V_\alpha[P + G^{\text{off}}] - V_\alpha[P] - GV_{\alpha-1}[P] + \tilde{V}_{\alpha-1}G. \tag{3.12}$$

By induction, it is seen that Δ_α are differential polynomials of P and G with orders less than α and 0, respectively. From the constraint (2.3), Δ_α contains only P and their x_1 derivatives. By (3.6), we see that $\Delta_0 = 0$. Suppose that $\Delta_\alpha = 0$. From (3.8-α) and (3.10-α), we have $\Delta_{\alpha+1}^{\text{off}} = 0$. From (3.8-($\alpha + 1$)) and (3.10-($\alpha + 1$)) (or (3.9) and (3.11)), we have $\partial_i \Delta_{\alpha+1}^{\text{diag}} = 0$, in particular $\partial_1 \Delta_{\alpha+1}^{\text{diag}} = 0$. It is easily seen that $\Delta_{\alpha+1}^{\text{diag}}$ cannot contain P, its x_1-derivatives, and G. Hence, $\Delta_{\alpha+1}^{\text{diag}} = f(t)$. Let $P = G = 0$. From (3.6) we have $\Delta_{\alpha+1}^{\text{diag}} = 0$. Hence, $\Delta_\alpha = 0$ ($\alpha = 0, 1, \ldots, m$). (3.4) is proved.

Theorem 2 gives a way for constructing an explicit solution $(\tilde{P}, \tilde{\Psi})$ from a given solution (P, Ψ). Here $\tilde{P}_i = [\tilde{P}, J_i]$ and $P_i = [P, J_i]$.

Remark 1. For $\tilde{P}$, the corresponding solution $\tilde{\Psi}$ of the Lax set is also constructed explicitly. The new Darboux matrix can be constructed as in Theorem 2 without solving any differential equations. This procedure can be done successively by a purely algebraic algorithm and an infinite sequence of solutions can be obtained.

Remark 2. The permutability theorem also holds in the $\mathbf{R}^{n+1}$ case.

Remark 3. For $P_i \in \mathrm{su}(N)$, a suitable choice of λ_a, h_a [7] can give a Darboux transformation which insures $\tilde{P}_i \in \mathrm{su}(N)$.

Remark 4. The behaviour of the interaction of solutions shown in [7] still holds. More precisely, a kth multi-soliton solution, which is defined as the generic kth Darboux transformation of the trivial solution $P = 0$, asymptotically splits to k single solitons as $t \to \pm\infty$. Moreover, their shapes and velocities are not changed asymptotically after the interaction.

Remark 5. The Darboux transformation discussed here can be used to the reduction of some $1 + 2$ dimensional problems, such as N-wave equation and DS equation [4, 11, 12].

4. The Structure of the Evolution Equations and 'Cauchy Problem'

It is easily seen that by using a suitable affine transformation, we can make the diagonal elements of each J_i different. Then the space constraints (2.7) are in the form

$$\partial_i P_{\alpha\beta} - C_{i\alpha\beta}\partial_1 p_{\alpha\beta} + \sum_{\delta\varepsilon\rho\tau} B_{i\alpha\beta\delta\varepsilon\rho\tau} P_{\delta\varepsilon} P_{\rho\tau} = 0. \tag{4.1}$$

Here $C_{i\alpha\beta}$ and $B_{i\alpha\beta\delta\varepsilon\rho\tau}$ are constants and can be expressed explicitly via the diagonal elements of J_i's. These equations can be solved at least locally in the class of analytic functions or in the case that the system is hyperbolic (in fact, when each J_i is real and $P \in \mathrm{su}(N)$ the system has global solution), if the initial data is given at $x_2 = x_3 = \cdots = x_n = 0$. That is to say, if at $t = t_0$, the Cauchy data

$$P(t_0, x_1, 0, \ldots, 0) = P_{t_0}(x_1) \tag{4.2}$$

is given, then the solution to (4.1) exists uniquely, at least locally. We should note that (4.1) is still overdetermined. However, from the complete integrability, the solution can be determined in the following procedure. For $t = t_0$, by solving the Cauchy problems in 2-variables step by step, we will obtain★

$$\begin{aligned} P_{\alpha\beta}(t_0, x_1, x_2, 0, \ldots, 0) &\to P_{\alpha\beta}(t_0, x_1, x_2, x_3, 0, \ldots, 0) \\ &\to \cdots \to P_{\alpha\beta}(t_0, x_1, x_2, \ldots, x_{n-1}, 0) \\ &\to P_{\alpha\beta}(t_0, x_1, \ldots, x_n). \end{aligned} \tag{4.3}$$

If we can solve the Cauchy problem for the evolution equation (2.10) on the $(1 + 1)$-dimensional subspace $x_2 = x_3 = \cdots = x_n = 0$ with the initial data

$$P(0, x_1, 0, \ldots, 0) = P_0(x_1), \tag{4.4}$$

★Note: Owing to the complete integrability, the result is independent of the choice of the order of $x_2, x_3, \ldots, x_n$.

then the solution to the whole system (2.7) and (2.10) are determined uniquely. Hence, according to the structure of the system (2.7) and (2.10), the initial condition should be put on a line (say $x_2 = x_3 = \cdots = x_n = 0$) at $t = 0$ and then the solution (if it exists) is determined uniquely.

This Cauchy problem can be solved by the inverse scattering method. Here we formally state the method. Let Σ be the set

$$\Sigma = \{z \in \mathbf{C} \mid \exists a \neq b, \mathrm{Re}(zJ_a^{(1)}) = \mathrm{Re}(zJ_b^{(1)})\}. \tag{4.5}$$

Here $J_a^{(1)}$'s are diagonal elements of J_1 and $\Sigma_1, \ldots, \Sigma_s$ are the rays in Σ arranged counterclockwisely and Ω_ρ is the region bounded by Σ_ρ and $\Sigma_{\rho+1}$ ($\Sigma_{s+1} = \Sigma_1$). According to the general theory [2], the scattering data for the initial data on $x_2 = \cdots = x_n = 0$, $t = 0$ are

$$\{z_\alpha, \quad v_{0\alpha}(z), \quad v_\rho(z)\}. \tag{4.6}$$

Here the z_α's are complex numbers, the $v_{0\alpha}(z)$'s are polynomials of $(z - z_\alpha)^{-1}$, and the $v_\rho(z)$'s are analytic functions on Σ_ρ. Moreover, the scattering data on the line $x_2 = x_2, \ldots, x_n = x_n, t = t$ should be

$$\begin{aligned} z_\alpha(z, x_2, \ldots, x_n, t) &= z_\alpha, \\ v_\rho(z, x_2, \ldots, x_n, t) &= \mathrm{e}^{\omega} v_\rho \mathrm{e}^{-\omega}, \\ v_{0\alpha}(z, x_2, \ldots, x_n, t) &= \mathscr{P}_0(\mathrm{e}^{\omega} v_{0\alpha} \mathrm{e}^{-\omega}). \end{aligned} \tag{4.7}$$

Here

$$\omega = \sum_{i=2}^{n} zJ_i x_i + \sum_{\alpha=0}^{m} z^{m-\alpha} \int V_\alpha^0(t)\,\mathrm{d}t,$$

$\mathscr{P}_0$ is the notation of principal part [9]. Hence, the inverse scattering transformation gives the solution $P(x_1, \ldots, x_n, t)$.

Remark. After Darboux transformation, the scattering data transforms in the same way as in $\mathbf{R}^{1+1}$ in [9].

Appendix. Proof of Theorem 1

To prove Theorem 1, we shall show the following facts. (i) (2.8) and (2.9) determine each V_α recursively up to m arbitrary functions of t, provided that P satisfies (2.7). (ii) V_α's given by (2.8) and (2.9) are differential polynomials of P, i.e. each entry of V_α is a polynomial of the entries of P and their derivatives with respect to x_i ($i = 1, \ldots, n$).

We first introduce some definitions and two lemmas.

Let $\mathfrak{F}_0$ be the ring of all smooth functions of t, $\mathfrak{F}_0$ be the ring of all smooth functions of $x_1, \ldots, x_n, t$. Let $\{X_{ab}^A\}$ be indeterminates where $a, b = 1, \ldots, N$ and A's are multi-indices, $A = (A_1, \ldots, A_n)$ where $A_1, \ldots, A_n$ are nonnegative integers.

Denote $\mathfrak{F}_s = \mathfrak{F}_0[X^A_{ab}]$ and $\bar{\mathfrak{F}}_s = \bar{\mathfrak{F}}_0[X^A_{ab}]$ be the rings of polynomials of X^A_{ab} over the ring $\mathfrak{F}_0$ and $\bar{\mathfrak{F}}_0$, respectively, with $|A| \leqslant s$. Here $|A| = \Sigma A_i$ for $A = (A_1, \ldots, A_n)$. Moreover, denote $\mathfrak{F} = \bigcup_s \mathfrak{F}_s$, $\bar{\mathfrak{F}} = \bigcup_s \bar{\mathfrak{F}}_s$. It is clear that $\mathfrak{F}_s \subset \bar{\mathfrak{F}}_s$.

Define the derivative operators ∂_i $(i = 1, 2, \ldots, n)$ in $\bar{\mathfrak{F}}$ as follows. (i) For $f \in \bar{\mathfrak{F}}_0$, $\partial_i f$ is as usual. (ii) $\partial_i X^A_{ab} = X^{A+e_i}_{ab}$ $(i = 1, \ldots, n)$, where e_i is the vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ whose ith component is 1 and the others are 0. (iii) The derivative of a general element of $\bar{\mathfrak{F}}$ is defined by Leibnitz rule. Of course, the derivative operators are linear.

We have

LEMMA A.1. *Suppose $F \in \bar{\mathfrak{F}}$, $\partial_i F \in \mathfrak{F}$ for some i, and $F[0] = 0$, then $F \in \mathfrak{F}$. Here $F[0]$ is the value of F when all X^A_{ab} are zero.*

Proof. Let $\delta_i \colon \bar{\mathfrak{F}} \to \bar{\mathfrak{F}}$ satisfy (i) $\delta_i f = \partial_i f$ for any $f \in \bar{\mathfrak{F}}_0$, (ii) $\delta_i X^A_{ab} = 0$, (iii) δ_i satisfies Leibnitz rule, then for $F \in \bar{\mathfrak{F}}_s$,

$$\partial_i F = \delta_i F + \sum_{a,b=1,\ldots,N; |A| \leqslant s} \frac{\partial F}{\partial X^A_{ab}} X^{A+e_i}_{ab} \tag{A.1}$$

where $\partial F / \partial X^A_{ab}$ is defined as a usual derivative of a polynomial.

If $F \in \bar{\mathfrak{F}}_0$, $\partial_i F \in \mathfrak{F}_1$ and $F[0] = 0$, then it is clear that $F \in \mathfrak{F}_0$. Suppose for any $F \in \bar{\mathfrak{F}}_{s-1}$ satisfying $\partial_i F \in \mathfrak{F}_s$ and $F[0] = 0$, $F \in \mathfrak{F}$ holds. Now consider the element $F \in \bar{\mathfrak{F}}_s$ satisfying $\partial_i F \in \mathfrak{F}_{s+1}$ and $F[0] = 0$.

For $|A| = s$, $\partial_i F$ is linear with respect to $X^{A+e_i}_{ab}$. On the other hand, $\partial_i F \in \mathfrak{F}$. Comparing the terms containing $X^{A+e_i}_{ab}$ $(|A| = s)$ in both sides of (A.1), we have $\partial F / \partial X^A_{ab} \in \mathfrak{F}$. Therefore, if G denotes the sum of all the monomials in F containing $X^A_{ab}(|A| = s)$, $G \in \mathfrak{F}$. Let $H = F - G$, then H satisfies $H \in \bar{\mathfrak{F}}_{s-1}$, $\partial_i H \in \mathfrak{F}_s$, $H[0] = 0$. By the assumption of induction, $H \in \mathfrak{F}$. Hence $F \in \mathfrak{F}$. The lemma is proved.

LEMMA A.2. *Let M_N be the linear space of $N \times N$ matrices. Let X^A be the matrix whose entries are (X^A_{ab}), $X = X^0$. $V^0_\alpha(t)$ $(\alpha = 0, \ldots, m)$ are $N \times N$ diagonal matrix functions of t. Let $\bar{\mathfrak{G}}$ be the ideal of $\bar{\mathfrak{F}}$ generated by*

$$\begin{aligned} &\{\partial_1^{A_1} \cdots \partial_n^{A_n}([X^{e_j}, J_i] - [X^{e_i}, J_j] + [[X, J_i], [X, J_j]]) \\ &\quad | 1 \leqslant i, j \leqslant n, A_1, \ldots, A_n \text{ are nonnegative integers}\}, \end{aligned} \tag{A.2}$$

$\mathfrak{G} = \bar{\mathfrak{G}} \cap \mathfrak{F}$, then there exist unique $V_\alpha[X^A_{ab}] \in \mathfrak{F}/\mathfrak{G} \otimes M_N$ satisfying

$$[J_i, V^{\text{off}}_\alpha] = \partial_i V^{\text{off}}_{\alpha-1} - [[X, J_i], V_{\alpha-1}]^{\text{off}}, \tag{A.3-i-α}$$

$$\partial_i V^{\text{diag}}_\alpha = [[X, J_i], V^{\text{off}}_\alpha]^{\text{diag}}, \tag{A.4-i-α}$$

$$V_\alpha[0] = V^0_\alpha(t). \tag{A.5-α}$$

Proof. Uniqueness is trivial.

We use induction to prove the existence. Clearly, $V^0_0(t)$ satisfies (A.3-i-0) and (A.4-i-0). Here we define $V_{-1} = 0$.

Suppose (A.3-i-β), (A.4-i-β) hold for $\beta \leqslant \alpha$. From (A.3-1-($\alpha+1$)), $V_{\alpha+1}^{\mathrm{off}} \in \mathfrak{F}/\mathfrak{G} \otimes M_N$. We need to check that $V_{\alpha+1}^{\mathrm{off}}$ satisfies (A.3-i-($\alpha+1$)) for $i \geqslant 2$. Equivalently, we need to check

$$[J_1, [J_i, V_{\alpha+1}^{\mathrm{off}}] - \partial_i V_\alpha + [[X, J_i], V_\alpha]] = 0. \tag{A.6}$$

The left-hand side is equal to

$$\begin{aligned}
&[J_i, [J_1, V_{\alpha+1}^{\mathrm{off}}]] - [J_1, \partial_i V_\alpha] + [J_1, [[X, J_i], V_\alpha]\\
&\quad = [J_i, \partial_1 V_\alpha] - [J_1, \partial_i V_\alpha] - [J_i, [[X, J_1], V_\alpha]] + [J_1, [[X, J_i], V_\alpha]]\\
&\quad = (\partial_1 \partial_i V_{\alpha-1}^{\mathrm{off}} - [[X^{e_1}, J_i], V_{\alpha-1}]^{\mathrm{off}} - [[X, J_i], \partial_1 V_{\alpha-1}]^{\mathrm{off}}) -\\
&\qquad - (\partial_1 \partial_i V_{\alpha-1}^{\mathrm{off}} - [[X^{e_i}, J_1], V_{\alpha-1}]^{\mathrm{off}} - [[X, J_1], \partial_i V_{\alpha-1}]^{\mathrm{off}}) -\\
&\qquad - [J_i, [[X, J_1], V_\alpha]] + [J_1, [[X, J_i], V_\alpha]]\\
&\quad = [[X, J_i], [X, J_1]], V_{\alpha-1}]^{\mathrm{off}} - [[X, J_i], [[X, J_1], V_{\alpha-1}]]]^{\mathrm{off}} +\\
&\qquad + [[X, J_1], [[X, J_i], V_{\alpha-1}]]]^{\mathrm{off}} - [[X, J_i], [J_1, V_\alpha]]^{\mathrm{off}} +\\
&\qquad + [[X, J_1], [J_i, V_\alpha]]^{\mathrm{off}} - [J_i, [[X, J_1], V_\alpha]] + [J_1, [[X, J_i], V_\alpha]].
\end{aligned}$$

By the Jacobi identity, the sum of first three terms is zero, and the sum of the rest four terms is also zero. Hence, (A.6) holds. $V_{\alpha+1}^{\mathrm{off}}$ satisfies (A.4-i-α) for all i.

Now consider the diagonal part of $V_{\alpha+1}$. From [10], there is a unique $V_{\alpha+1}^{\mathrm{diag}(i)} \in \mathfrak{F}/\mathfrak{G} \otimes M_N$ satisfying (A.4-i-($\alpha+1$)) $(i = 1, \ldots, m)$ and $V_{\alpha+1}^{\mathrm{diag}(i)}[0] = V_{\alpha+1}^{0}$.

In order to prove $V_{\alpha+1}^{\mathrm{diag}(i)} = V_{\alpha+1}^{\mathrm{diag}(j)}$ for all i, j, we need to verify

$$\partial_j [[X, J_i], V_{\alpha+1}^{\mathrm{off}}]^{\mathrm{diag}} - \partial_i [[X, J_j], V_{\alpha+1}^{\mathrm{off}}]^{\mathrm{diag}} = 0. \tag{A.7}$$

The left-hand side is equal to

$$\begin{aligned}
&[[X^{e_j}, J_i], V_{\alpha+1}^{\mathrm{off}}]^{\mathrm{diag}} + [[X, J_i], \partial_j V_{\alpha+1}^{\mathrm{off}}]^{\mathrm{diag}} -\\
&\qquad - [[X^{e_i}, J_j], V_{\alpha+1}^{\mathrm{off}}]^{\mathrm{diag}} - [[X, J_j], \partial_i V_{\alpha+1}^{\mathrm{off}}]^{\mathrm{diag}}\\
&\quad = - [[[X, J_i], [X, J_j]], V_{\alpha+1}^{\mathrm{off}}]^{\mathrm{diag}} + [X, [J_i, \partial_j V_{\alpha+1}^{\mathrm{off}}]]^{\mathrm{diag}} -\\
&\qquad - [X, [J_j, \partial_i V_{\alpha+1}^{\mathrm{off}}]]^{\mathrm{diag}}\\
&\quad = - [[[X, J_i], [X, J_j]], V_{\alpha+1}^{\mathrm{off}}]^{\mathrm{diag}} +\\
&\qquad + [X, \partial_i \partial_j V_\alpha^{\mathrm{off}} - [[X^{e_j}, J_i], V_\alpha] - [[X, J_i], \partial_j V_\alpha]]^{\mathrm{diag}} -\\
&\qquad - [X, \partial_i \partial_j V_\alpha^{\mathrm{off}} - [[X^{e_i}, J_j], V_\alpha] - [[X, J_j], \partial_i V_\alpha]]^{\mathrm{diag}}\\
&\quad = - [[[X, J_i], [X, J_j]], V_{\alpha+1}^{\mathrm{off}}]^{\mathrm{diag}} - [X, [[X, J_i], [J_j, V_{\alpha+1}]]]^{\mathrm{diag}} +\\
&\qquad + [X, [[X, J_j], [J_i, V_{\alpha+1}]]]^{\mathrm{diag}} + [X, [[[X, J_i], [X, J_j]], V_\alpha]]^{\mathrm{diag}} -\\
&\qquad - [X, [[X, J_i], [[X, J_j], V_\alpha]]]^{\mathrm{diag}} + [X, [[X, J_j], [[X, J_i], V_\alpha]]]^{\mathrm{diag}}.
\end{aligned}$$

The Jacobi identity implies that the sum of the last three terms is zero, and direct calculation shows that the sum of the first three terms is zero. This proves (A.7).

Let

$$\pi_{ij} = V_{\alpha+1}^{\mathrm{diag}(i)} - V_{\alpha+1}^{\mathrm{diag}(j)} \in \mathfrak{F}/\mathfrak{G} \otimes M_N.$$

(A.7) implies that $\partial_i \partial_j \pi_{ij} = 0$. Hence $\partial_i \pi = c(x_1, \ldots, \hat{x}_j, \ldots, x_n, t) \in \mathfrak{F}_0$ where $\hat{x}_j$ implies that x_j does not appear. From the condition $\pi[0] = 0, (\partial_i \pi)[0] = \partial_i(\pi[0]) = 0$. Hence $\partial_i \pi \equiv 0$. $\pi[0] = 0$ implies $\pi \equiv 0$. Therefore, $V_{\alpha+1}^{\mathrm{diag}} = V_{\alpha+1}^{\mathrm{diag}(i)}$ satisfies (A.4-i-(α + 1)) for all i. Since

$$\partial_i(V_{\alpha+1}^{\mathrm{diag}} - V_{\alpha+1}^0) \in \mathfrak{F}/\mathfrak{G} \otimes M_N, \qquad (V_{\alpha+1}^{\mathrm{diag}} - V_{\alpha+1}^0)[0] = 0,$$

Lemma A.1 implies $V_{\alpha+1}^{\mathrm{diag}} \in \mathfrak{F}/\mathfrak{G} \otimes M_N$. Lemma A.2 is proved.

From Lemma A.2, we immediately obtain Theorem 1.

References

1. Ablowitz, M. J., Beals, R., and Tenenblat, K., On the solution of the generalized wave and generalized sine-Gordon equation, *Stud. Appl. Math.* **74**, 177 (1986).
2. Beals, R. and Coifman, R. R., Scattering and inverse scattering for first order systems, *Comm. Pure Appl. Math.* **37**, 39 (1984).
3. Beals, R. and Coifman, R. R., Inverse scattering and evolution equations, *Comm. Pure Appl. Math.* **38**, 29 (1985).
4. Cheng, Y., Constraint of integrable systems: From higher to lower dimensions, *Phys. Lett.* **A166**, 217 (1992).
5. Gu, C. H., On the Darboux form of Bäcklund transformations, in *Integrable System, Nankai Lectures on Math. Phys.*, World Scientific, Singapore, 1989, pp. 162–168.
6. Gu, C. H., Darboux transformations for a class of integrable systems in n variables, to appear in *Colloque International, Analyse, Variétés et Physique*, 1992.
7. Gu, C. H., On the interaction of solitons for a class of integrable systems in the spacetime $\mathbf{R}^{n+1}$, *Lett. Math. Phys.* **26**, 199–209 (1992).
8. Konopelchenko, B., Sidorenko, J., and Strampp, W., (1 + 1)-dimensional integrable systems as symmetry constraints of (2 + 1) dimensional systems, *Phys. Lett.* **A157**, 17 (1991).
9. Sattinger, D. H. and Zurkowski, V. D., Gauge theory of Bäcklund transformations II, *Physica* **26D**, 225 (1987).
10. Wilson, G., Commuting flows and conservation laws for Lax equations, *Math. Proc. Camb. Phil. Soc.* **86**, 131 (1979).
11. Zhou, Z. X., Explicit Solutions of N-Wave Equation in 1 + 2 Dimensions, *Phys. Lett.* **A168**, 370 (1992).
12. Zhou, Z. X., A method to obtain explicit solutions of 1 + 2 dimensional AKNS system, *Internat. J. Modern Phys. A* **3**, 565 (1993).