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Asymptotic Behavior of Soliton Solutions with a Double Spectral Parameter for Principal Chiral Field*

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Abstract *The soliton solutions with a double spectral parameter for the principal chiral field are derived by Darboux transformation. The asymptotic behavior of the solutions as time tends to infinity is obtained and the speeds of the peaks in the asymptotic solutions are not constants.*

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1 Introduction

The principal chiral field in $\mathbf{R}^{1,1}$ is a well-known integrable system in physics. It is also known as a harmonic map from $\mathbf{R}^{1,1}$ to a Lie group G in mathematics. Since late 1970's, a lot of work has been done for this problem.[1−4] Especially, a series of results have been obtained by Darboux transformations.[4−6] Usually the spectral parameters in constructing Darboux transformation are chosen as distinct numbers. However, when the spectral parameters can be multiple, then the behavior of the solutions may be different. For the harmonic maps, this was introduced in Ref. [1] in considering singular Bäcklund transformation. Its counterpart in Darboux transformation is the singular Darboux transformation.[4,7]

From a trivial solution of the equation of principal chiral field, a usual Darboux transformation of degree two with distinct spectral parameters leads to a double soliton solution, which behaves as two single solitons asymptotically when time tends to infinity. However, when there is a double spectral parameter, we shall show that the solution still has two peaks asymptotically, but these two peaks do not move in constant velocities.

2 Lax Pair and Darboux Transformation

The Lax pair of the principal chiral field in $\mathbf{R}^{1,1}$ is

$$\Phi_x = \frac{U}{\lambda+1}\Phi, \quad \Phi_t = \frac{V}{\lambda-1}\Phi\,, \tag{1}$$

where $U(x,t)$ and $V(x,t)$ are two $n\times n$ matrices valued in the Lie algebra of a Lie group G. The integrability condition of Eq. (1) is just the equation of the principal chiral field

$$U_t = V_x, \quad U_t + V_x - [U,V] = 0\,. \tag{2}$$

It is well known that the soliton solutions of the principal chiral field can be obtained by Darboux transformation and various other methods. For $G=\mathrm{SU}(2)$, the Darboux transformation of degree one with a single spectral parameter is constructed as follows.[4]

Let λ_1 be a non-real complex number, and

$$h_1 = (\phi_1(x,t), \psi_1(x,t))^{\mathrm{T}}$$

be a solution of Eq. (2) with $\lambda=\lambda_1$. Denote

$$\Lambda_1 = \mathrm{diag}(\lambda_1, \bar\lambda_1), \quad H_1 = \begin{pmatrix} \phi_1 & -\bar\psi_1 \\ \psi_1 & \bar\phi_1 \end{pmatrix}.$$

Then, $D=\lambda I - S$ with $S = H_1\Lambda_1 H_1^{-1}$ is a Darboux matrix, in the sense that for any solution Φ of Eq. (1), $\widetilde\Phi = D\Phi$ satisfies

$$\tilde\Phi_x = \frac{\tilde U}{\lambda+1}\tilde\Phi, \quad \tilde\Phi_t = \frac{\tilde V}{\lambda-1}\tilde\Phi\,, \tag{3}$$

where $\tilde U$ and $\tilde V$ can be written explicitly as

$$\tilde U = U - S_x, \quad \tilde V = V - S_t \tag{4}$$

and S satisfies

$$S_x(S+I) = [U,S], \quad S_t(S-I) = [V,S]\,. \tag{5}$$

Likewise, the Darboux transformation of degree two is constructed as follows.

Take λ_1 and λ_2 be two non-real complex numbers such that $\lambda_1\neq\lambda_2$, $\lambda_1\neq\bar\lambda_2$. Let

$$h_j = (\phi_j(x,t), \psi_j(x,t))^{\mathrm{T}}$$

be the solution of Eq. (2) with $\lambda=\lambda_j$. Let

$$\Lambda_j = \mathrm{diag}(\lambda_j, \bar\lambda_j), \quad H_j = \begin{pmatrix} \phi_j & -\bar\psi_j \\ \psi_j & \bar\phi_j \end{pmatrix},$$

$$S_j = H_j\Lambda_j H_j^{-1}\,.$$

Then the Darboux matrix is

$$D = \lambda^2 I - \lambda D_1 + D_2\,, \tag{6}$$

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where

$$D_1 = (S_2^2 - S_1^2)(S_2 - S_1)^{-1}\,,$$
$$D_2 = (S_2 - S_1)S_2(S_2 - S_1)^{-1}S_1\,, \tag{7}$$

and

$$\tilde{U} = U - D_{1,x}\,, \quad \tilde{V} = V - D_{1,t}\,. \tag{8}$$

The Darboux transformation with a double spectral parameter can be obtained by taking a suitable limit $\lambda_2 \to \lambda_1$. Let $\lambda_1 = \mu$, $\lambda_2 = \mu + \epsilon$. Let $h_2 = (\phi_2(x,t,\epsilon), \psi_2(x,t,\epsilon))^{\mathrm{T}}$ depend on ϵ smoothly in such a way that $\phi_2(x,t,0) = \phi_1(x,t)$, $\psi_2(x,t,0) = \psi_1(x,t)$. Then we can construct the Darboux matrix $D(x,t,\epsilon)$ as above. The limit of $D(x,t,\epsilon)$ as ϵ tends to zero will give a Darboux transformation with a double spectral parameter μ.

3 Asymptotic Behavior of Solution with a Double Spectral Parameter

To get the soliton solutions, we start with a simple seed solution of Eq. (1). The seed solution of Eq. (2) can be chosen as

$$U = U_0 = \begin{pmatrix} \mathrm{i}\,p & 0 \\ 0 & -\mathrm{i}\,p \end{pmatrix}, \quad V = V_0 = \begin{pmatrix} \mathrm{i}\,q & 0 \\ 0 & -\mathrm{i}\,q \end{pmatrix}, \tag{9}$$

where p and q are real constants. Moreover, by a dilation of x and t, we can always suppose $p = q = 1$, that is, $U_0 = V_0 = \mathrm{i}J$ with $J = \mathrm{diag}(1, -1)$.

Take $\mu = \alpha + \beta\mathrm{i}$, where α and β are real constants and $\beta \neq 0$. The solution of Eq. (1) is

$$\Phi = \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} C_1 \exp\Bigl(\dfrac{\mathrm{i}}{\mu+1}x + \dfrac{\mathrm{i}}{\mu-1}t\Bigr) \\ C_2 \exp\Bigl(-\dfrac{\mathrm{i}}{\mu+1}x - \dfrac{\mathrm{i}}{\mu-1}t\Bigr) \end{pmatrix}, \tag{10}$$

where C_1 and C_2 are complex constants.

Let

$$\sigma = \frac{\psi}{\phi} = \frac{C_2}{C_1} \exp\Bigl(-\frac{2\mathrm{i}}{\mu+1}x - \frac{2\mathrm{i}}{\mu-1}t\Bigr)\,. \tag{11}$$

Since $\beta \neq 0$, we can always choose the complex number $C_2/C_1 = 1$ by applying a translation of x and t.

Hence

$$\sigma = \exp\Bigl(-\frac{2(\beta + (\alpha+1)\mathrm{i})}{(\alpha+1)^2 + \beta^2}x - \frac{2(\beta + (\alpha-1)\mathrm{i})}{(\alpha-1)^2 + \beta^2}t\Bigr)\,. \tag{12}$$

The velocity v of a single soliton solution is determined by the condition that σ is bounded when $x - vt$ is bounded and $t \to \infty$. Hence the velocity of a single soliton is

$$v = -\frac{(\alpha+1)^2 + \beta^2}{(\alpha-1)^2 + \beta^2}\,. \tag{13}$$

To get the soliton solution with a double spectral parameter, take $\lambda_1 = \mu$, $\lambda_2 = \mu + \epsilon$ with $\mu = \alpha + \beta\mathrm{i}$, then

$$\phi_1 = \exp\Bigl(\frac{\mathrm{i}}{\mu+1}x + \frac{\mathrm{i}}{\mu-1}t\Bigr)\,,$$
$$\psi_1 = \exp\Bigl(-\frac{\mathrm{i}}{\mu+1}x - \frac{\mathrm{i}}{\mu-1}t\Bigr)\,,$$
$$\phi_2 = \exp\Bigl(\frac{\mathrm{i}}{\mu+\epsilon+1}x + \frac{\mathrm{i}}{\mu+\epsilon-1}t\Bigr)\,,$$
$$\psi_2 = \exp\Bigl(-\frac{\mathrm{i}}{\mu+\epsilon+1}x - \frac{\mathrm{i}}{\mu+\epsilon-1}t\Bigr)\,. \tag{14}$$

The Darboux matrix with a double spectral parameter is constructed as in the last section.

Now we consider the asymptotic behavior of D_1. Denote

$$G(x,t) = \lim_{\epsilon \to 0} D(x,t,\epsilon)\,,$$
$$G = \lambda^2 - G_1\lambda + G_2\,. \tag{15}$$

According to Eq. (13), let

$$x = \xi - \frac{(\alpha+1)^2 + \beta^2}{(\alpha-1)^2 + \beta^2}t\,, \tag{16}$$

then after complicated calculation with the help of computer, we get the $(1,1)$ component of G_1 as

$$(G_1)_{11} = 2\alpha + \frac{\mathrm{i}N_1}{D}\,, \tag{17}$$

where

$$D = 256d_+\beta^4 - 256d_-\beta^4\xi t^{-1} + \bigl(d_+^2 d_-^3(P^2 + P^{-2}) + 2d_-^3(d_+^2 + 8\beta^2\xi^2)\bigr)t^{-2}\,,$$
$$N_1 = -128d_+d_-\alpha\beta^4 t^{-1} + \bigl(2\beta d_+^2 d_-^3(P^2 - P^{-2}) + 16d_-^3\beta^2((\alpha+1)^2 - \beta^2)\xi\bigr)t^{-2}\,, \tag{18}$$

and

$$P = \exp(2\beta d_+^{-1}\xi)\,, \tag{19}$$
$$d_\pm = (\alpha \pm 1)^2 + \beta^2\,. \tag{20}$$

From the above expression of $(G_1)_{11}$, we know that $(G_{1,x})_{11}$ always tends to zero when ξ is bounded and $t \to \infty$. However, if ξ changes with t in such a way that $Pt^{\pm 1}$ keeps bounded, then $(G_{1,x})_{11}$ does not tend to zero as $t \to \infty$. Hence we let

$$\xi = \eta + \frac{md_+}{2\beta}\ln|t|\,, \tag{21}$$

where $m=\pm 1$. Then

$$\begin{aligned}(G_1)_{11} &= 2\alpha + \frac{2m\beta\mathrm{i}\exp(4m\beta d_+^{-1}\eta)+\cdots}{256d_+^{-1}d_-^{-3}\beta^4+\exp(4m\beta d_+^{-1}\eta)+\cdots}\\ &= 2(\alpha+m\beta\mathrm{i}) - \frac{512md_+^{-1}d_-^{-3}\beta^5+\cdots}{256d_+^{-1}d_-^{-3}\beta^4+\exp(4m\beta d_+^{-1}\eta)+\cdots}\,.\end{aligned}\tag{22}$$

Here $\cdots$ refers to polynomials of t^{-1} and $t^{-1}\ln t$. Hence, when η is bounded and $t\to\infty$,

$$U_{11}\to \mathrm{i}-(G_{1,x})_{11}\to \mathrm{i}-2\mathrm{i}\beta^2d_+^{-1}\mathrm{sech}^2\theta, \qquad V_{11}\to \mathrm{i}-(G_{1,t})_{11}\to \mathrm{i}-2\mathrm{i}\beta^2d_-^{-1}\mathrm{sech}^2\theta\,,\tag{23}$$

where

$$\theta = 2\beta d_+^{-1}\eta + \frac{m}{2}\ln(d_+d_-^3) - m\ln(16\beta^2)\,.\tag{24}$$

Likewise, we get

$$(G_1)_{12} = \frac{N_2}{D}\exp\bigl(2\mathrm{i}(\alpha+1)d_+^{-1}\xi - 4\mathrm{i}d_-^{-1}t\bigr)\,,\tag{25}$$

where

$$\begin{aligned}N_2 &= 32d_+d_-\beta^3\bigl(((\alpha+\beta\mathrm{i})^2-1)P+((\alpha-\beta\mathrm{i})^2-1)P^{-1}\bigr)t^{-1}\\ &\quad + 4\mathrm{i}d_-^3\beta\bigl((d_+^2-2\beta(\alpha+\beta\mathrm{i}+1)^2\xi)P+(d_+^2+2\beta(\alpha-\beta\mathrm{i}+1)^2\xi)P^{-1}\bigr)t^{-2}\,.\end{aligned}\tag{26}$$

Hence,

$$(G_1)_{12} = 32d_+^{-1}d_-^{-1}\beta^3((\alpha+m\beta\mathrm{i})^2-1)\mathrm{sgn}(t)\frac{\exp(2m\beta d_+^{-1}\eta)+\cdots}{256d_+^{-1}d_-^{-3}\beta^4+\exp(4m\beta d_+^{-1}\eta)+\cdots}\,.\tag{27}$$

When η is bounded and $t\to\infty$,

$$U_{12}\to -(G_{1,x})_{12} = 2\beta^2(d_+^3d_-)^{-1/2}((\alpha+m\beta\mathrm{i})^2-1)\mathrm{sgn}(t)\frac{1}{\cosh\theta}\Bigl(\tanh\theta-\frac{\mathrm{i}(\alpha+1)}{\beta}\Bigr)\exp(\mathrm{i}\omega)\,,\tag{28}$$

where

$$\omega = 2(\alpha+1)d_+^{-1}\xi - 4d_-^{-1}t = 2(\alpha+1)d_+^{-1}\eta - 4d_-^{-1}t + m\frac{\alpha+1}{\beta}\ln|t|\,.\tag{29}$$

Correspondingly,

$$V_{12}\to -(G_{1,t})_{12} = 2\beta^2(d_+d_-^3)^{-1/2}((\alpha+m\beta\mathrm{i})^2-1)\mathrm{sgn}(t)\frac{1}{\cosh\theta}\Bigl(\tanh\theta-\frac{\mathrm{i}(\alpha-1)}{\beta}\Bigr)\exp(\mathrm{i}\omega)\,.\tag{30}$$

Therefore, we get the asymptotic solution which has two peaks centered at $\theta=0$ ($m=\pm1$).

In summary, the asymptotic behavior of the solution with a double spectral parameter is as follows.

When $t\to\pm\infty$, the solution splits up into two peaks, whose centers are at

$$x = -\frac{d_+}{d_-}t + \frac{md_+}{2\beta}\ln|t| - \frac{md_+}{4\beta}\ln(d_+d_-^3) + \frac{md_+}{2\beta}\ln(16\beta^2)\,,\tag{31}$$

($m=\pm1$).

In the coordinate

$$\eta = x + \frac{d_+}{d_-}t - \frac{md_+}{2\beta}\ln|t|\,,\tag{32}$$

the asymptotic solution as η keeps bounded and $t\to\pm\infty$ is

$$U = \begin{pmatrix} \mathrm{i}-\mathrm{i}d_+^{-1}f(\theta) & d_+^{-1}g_+(\theta)\\ -d_+^{-1}\bar g_+(\theta) & -\mathrm{i}+\mathrm{i}d_+^{-1}f(\theta)\end{pmatrix}, \qquad V = \begin{pmatrix} \mathrm{i}-\mathrm{i}d_-^{-1}f(\theta) & d_-^{-1}g_-(\theta)\\ -d_-^{-1}\bar g_-(\theta) & -\mathrm{i}+\mathrm{i}d_-^{-1}f(\theta)\end{pmatrix}\,,\tag{33}$$

where

$$\begin{aligned}&f(\theta) = 2\beta^2\mathrm{sech}^2(\theta)\,,\\ &g_\pm(\theta) = 2\beta^2(d_+d_-)^{-1/2}((\alpha+m\beta\mathrm{i})^2-1)\mathrm{sgn}(t)\frac{1}{\cosh\theta}\Bigl(\tanh(\theta)-\frac{\mathrm{i}(\alpha\pm1)}{\beta}\Bigr)\exp(\mathrm{i}\omega)\,.\end{aligned}\tag{34}$$

As an example, take $\mu=8+\mathrm{i}$, $t=\pm10\,000$, then the centers of the two peaks appear at $\xi\approx\pm160$ according to Eq. (31). The solution is shown in Fig. 1. When $t=100$, which is not large enough, the interaction is shown in Fig. 2. In the figures, the horizontal axes are ξ and the vertical axes are $|U_{11}-\mathrm{i}|$.

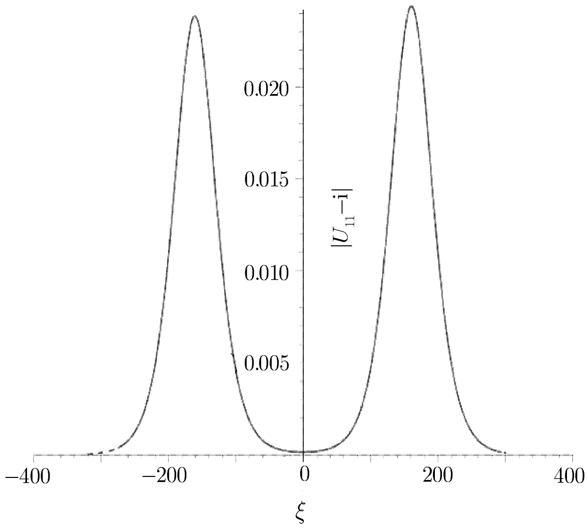

Fig. 1 $|U_{11} - \mathrm{i}|$ at $t = \pm 10\ 000$.

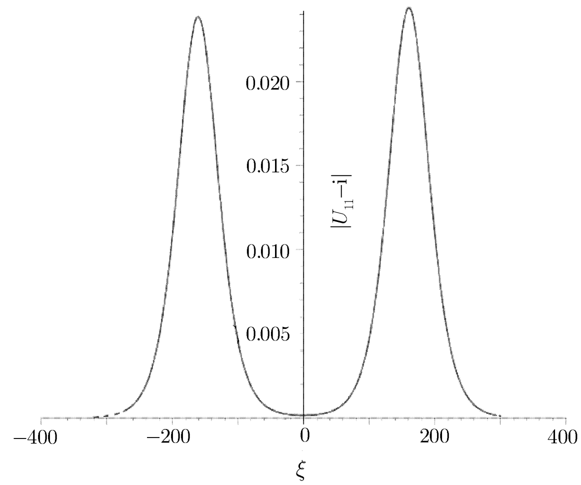

Fig. 2 $|U_{11} - \mathrm{i}|$ at $t = 100$.

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