Asymptotic Behaviour of Solitons with a Double Spectral Parameter for the Bogomolny Equation in (2+1)-Dimensional Anti de Sitter Space *

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(Received 22 April 2005)

The asymptotic behaviour of the solitons with a double spectral parameter for the Bogomolny equation in (2+1)-dimensional anti de Sitter space is obtained. The asymptotic solution has two ridges close to each other which locates beside the geodesic of the Poincaré half-plane.

PACS: 02.30.Jr, 05.45.Yv

The Bogomolny equation is a system of partial differential equations describing a typical Yang–Mills–Higgs field. It is integrable in both (2+1)-dimensional Minkowski and anti de Sitter spaces. Many works have been carried out on this equation.[1−8] In the Minkowski space, the solitons behave as peaks asymptotically. The velocities of solitons may change the angle after collision.[1,5] This is a very interesting phenomenon in soliton interaction. In the anti de Sitter space (AdS), the asymptotic behaviour of solitons is much more complicated. For example, when a polynomial in constructing a single soliton solution has only real roots, the solution looks like a ridge locating beside a geodesic.[7]

In Ref. [6], the asymptotic behaviour of solitons with a double spectral parameter was discussed when t is not very large. In that case, the solution looks like a peak with a hole inside. In this Letter, we show the asymptotic behaviour of the solution when $t \to \infty$. The solution behaves like two ridges close to each other which locates beside a geodesic (half circle on the Poincáre half-plane). These two ridges get merged on both the sides of the half circle. Moreover, the directions where two ridges get merged and the phase shift are computed explicitly.

The (2+1)-dimensional AdS is the hyperboloid in $\boldsymbol{R}^{2,2}$ with constant curvature -1, whose metric can be written locally as the Poincaré metric

$$ds^2 = r^{-2}(-dt^2 + dr^2 + dx^2) = r^{-2}(dr^2 + du\,dv), \quad (1)$$

where $u = x + t$ and $v = x - t$.

A Yang–Mills–Higgs field in the (2+1)-dimensional AdS is given by the Yang–Mills potentials A_u, A_v, A_t and the Higgs field Φ, which are valued in the Lie algebra of a Lie group G. Here we only consider $G = SU(2)$. The intensity of the Yang–Mills field is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and the covariant derivative on Φ is defined by $D_i\Phi = \partial_i\Phi + [A_i, \Phi]$ $(i = u, v, t)$. After the gauge transformation given by a G-valued function g, $A_i \to gA_ig^{-1} - (\partial_i g)g^{-1}$, $F_{ij} \to gF_{ij}g^{-1}$, $\Phi \to g\Phi g^{-1}$.

The Bogomolny equation[2]

$$D_u\Phi = rF_{ur}, \quad D_v\Phi = -rF_{vr}, \quad D_r\Phi = -2rF_{uv} \quad (2)$$

describes a typical Yang–Mills–Higgs field, which is invariant under any gauge transformation.

It was proposed in Ref. [2] that Eq. (2) has a Lax pair

$$\begin{aligned}&(rD_r + \Phi - 2(\lambda - u)D_u)\psi = 0,\\ &\Big(2D_v + \frac{\lambda - u}{r}D_r - \frac{\lambda - u}{r^2}\Phi\Big)\psi = 0,\end{aligned} \quad (3)$$

where $D_\mu\psi = \partial_\mu\psi + A_\mu\psi$ and λ is a complex spectral parameter. That is, Eq. (2) is the integrability condition of the over-determined system (3).

We can always choose a gauge so that the Yang–Mills potentials and the Higgs field satisfy[6]

$$A_u = 0, \quad A_r = \frac{1}{r}\Phi. \quad (4)$$

Now we construct the Darboux transformation with a double spectral parameter. A Darboux transformation of degree two with two different spectral parameters is obtained by the composition of two Darboux transformations with single spectral parameter. It is constructed explicitly as follows. Choose two non-real complex constants λ_1 and λ_2 such that $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq \bar\lambda_2$. Take $h_i = (a_i, b_i)^T$ to be the corresponding column solutions of Eq. (3) with $\lambda = \bar\lambda_i$ $(i = 1, 2)$, and

$$\Lambda_i = \begin{pmatrix}\bar\lambda_i & 0\\ 0 & \lambda_i\end{pmatrix}, \quad H_i = \begin{pmatrix}a_i & -\bar b_i\\ b_i & \bar a_i\end{pmatrix}, \quad (5)$$

$S_i = H_i\Lambda_iH_i^{-1}$. Then the Darboux transformation is[9] $\psi \to \widetilde\psi = G\psi$, where

$$G = \frac{(\lambda I - (S_2 - S_1)S_2(S_2 - S_1)^{-1})(\lambda I - S_1)}{(\lambda - \lambda_2)(\lambda - \lambda_1)}. \quad (6)$$

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For the seed solution $\Phi = 0$, $A_u = A_v = A_r = 0$, the components of any solution $(a,b)^T$ of Eq. (3) are meromorphic functions of

$$\omega(\lambda) = v - r^2(\lambda - u)^{-1}. \tag{7}$$

Hence $F = b/a$ is also a meromorphic function of $\omega(\lambda)$.

When the spectral parameter and the solution of the Lax pair depend on a parameter ϵ, we can write $F = F(\omega(\lambda(\epsilon)), \epsilon)$. Let

$$f(\omega(\lambda)) = F(\omega(\lambda), 0), \quad h(\omega(\lambda)) = \partial F(\omega(\lambda), \epsilon)/\partial\epsilon|_{\epsilon=0}.$$

Now take $\mu = \alpha + i\beta$, where α and β are real constants with $\beta \neq 0$. Let $\lambda_1 = \mu$, $\lambda_2 = \mu + \epsilon$, where ϵ is real. The solution of Eq. (3) corresponding to λ_2 depends on ϵ. From Ref. [6], the limit Darboux transformation is

$$\widetilde{\psi} = \Big(I + \frac{2i\beta}{\lambda - \mu}\frac{q^* q}{|q|^2}\Big)\Big(I + \frac{2i\beta}{\lambda - \mu}\frac{p^* p}{|p|^2}\Big)\psi, \tag{8}$$

where $p = (1, f)$,

$$q = (1 + |f|^2)(1, f) + (\bar{\mu} - \mu)\Big(\frac{r^2 f'}{(\mu - u)^2} + h\Big)(\bar{f}, -1). \tag{9}$$

Take $\lambda = 0$ and write $J = \psi|_{\lambda=0}$. By considering the gauge (4), the Lax pair (3) leads to

$$\widetilde{\Phi} = -\big(rJ_r + 2uJ_u\big)J^{-1}. \tag{10}$$

The energy density is defined by $E = -\dfrac{1}{4}\mathrm{tr}^2\widetilde{\Phi}$.

Now we consider the symptotic behaviour of solutions in the case $f(\lambda) = \omega(\lambda)$, $h(\lambda) = 0$. When t is not large enough, the soliton interaction was discussed in Refs. [5,6]. The shape of the solution looks like a peak with a hole inside. However, when t is very large, a single soliton may not look like a peak. Instead, it looks like a ridge along the geodesic (half circle) on the Poincaré half plane[7] if the polynomial in constructing single soliton solution has only real roots.

For the soliton solution with a double spectral parameter, the situation is more complicated. We consider its asymptotic behaviour as $t \to \infty$ here. From Eq. (7), the energy density E is a rational function of t.

First, consider the case $t \to +\infty$. Let

$$r = (t + R)\cos\theta, \quad x = (t + R)\sin\theta \tag{11}$$

and $s = \sin\theta$. Let

$$R = -\alpha(1 - \sin\theta)/2 + z/2. \tag{12}$$

With the help of computer, we obtain

$$E = \frac{32\beta^2(1 - s^2)^2 A t^8 + \cdots}{(Bt^4 + \cdots)^2}, \tag{13}$$

where

$$\begin{aligned} A = {}& z^4 + 2(\beta^2(1 - s)^2 + (1 + s)^2)z^2 \\ & + \beta^4(1 - s)^4 - 2\beta^2(1 - s^2)^2 + (1 + s)^4 \\ B = {}& z^4 + 2\Big(\beta^2(1 - s)^2 + (1 + s)^2\Big)z^2 \\ & + \beta^4(1 - s)^4 + 6\beta^2(1 - s^2)^2 + (1 + s)^4, \end{aligned} \tag{14}$$

and $\cdots$ refers to the lower order terms of t. Then

$$\lim_{t \to +\infty} E = E_0 \equiv 32\beta^2(1 - s^2)^2 AB^{-2}. \tag{15}$$

By computing $\dfrac{\partial E_0}{\partial R}$, we obtain the extremum of E along a direction specified by the angle θ. In this case, z satisfies

$$\begin{aligned} & z(z^6 + 3(\beta^2(1 - s)^2 + (1 + s)^2)z^4 \\ & \quad + 3((\beta^2(1 - s)^2 + (1 + s))^2 z^2 \\ & \quad + (\beta^6(1 - s)^6 - 9\beta^4(1 - s)^4(1 - s)^2 \\ & \quad - 9\beta^2(1 - s)^2(1 + s)^4 + (1 + s)^6)) = 0. \end{aligned} \tag{16}$$

The only possible real roots are $z = 0$ and $z = \pm z_0(s)$ with

$$z_0(s) = \sqrt{-\beta^2(1 - s)^2 - (1 + s)^2 + 2\sqrt{3}|\beta|(1 - s^2)}. \tag{17}$$

Here $z_0(s)$ is real if and only if s satisfies

$$\frac{|\beta| - \sqrt{3} - \sqrt{2}}{|\beta| + \sqrt{3} + \sqrt{2}} \leq s \leq \frac{|\beta| - \sqrt{3} + \sqrt{2}}{|\beta| + \sqrt{3} - \sqrt{2}}. \tag{18}$$

When Eq. (18) holds, two maxima appear at $z = \pm z_0(s)$ and one minimum appears at $z = 0$. By Eqs. (12) and (17), when $t \to +\infty$, the distance between the two maxima tends to a constant $z_0(\sin\theta)$ for fixed θ. Substituting the expression of $z_0(s)$ into Eq. (15), we know that $E_0 = 1$ holds at both the maxima.

When Eq. (18) fails, there is only one maximum at $z = 0$. At this maximum,

$$E_0 = 32\beta^2(1 - s^2)^2 \cdot \frac{\beta^4(1 - s)^4 - 2\beta^2(1 - s^2)^2 + (1 + s)^4}{(\beta^4(1 - s)^4 + 6\beta^2(1 - s^2)^2 + (1 + s)^4)^2}. \tag{19}$$

It tends to zero as $s \to \pm 1$ ($\theta \to \pm\pi/2$) and equals to 1 at

$$s = \frac{|\beta| - \sqrt{3} \mp \sqrt{2}}{|\beta| + \sqrt{3} \pm \sqrt{2}}, \tag{20}$$

which is the boundary of the range (18).

According to Eqs. (12) and (17), for fixed s (or θ), the distance between the two maxima is $z_0(s)$.

Take $\mu = -10 + i$, $t = 40$. The figure of the energy density is shown in Fig. 1 and its vertical projection is shown in Fig. 2.

When θ is fixed, the relation between R and E is shown in Fig. 3, where $\mu = -10 + i$, $t = 2000$, $\theta = 15°$. Two peaks correspond to two ridges close to each other in Figs. 1 and 2. The distance between the two ridges is $z_0(\sin 15°) \approx 1.05$.

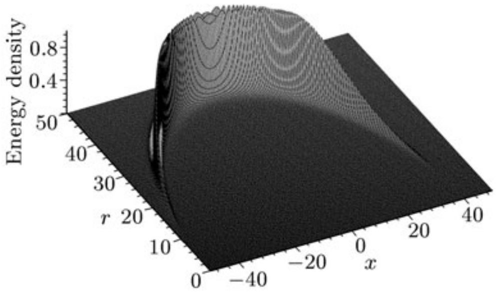

Fig. 1. The energy density for $\mu = -10 + i$ and $t = 40$.

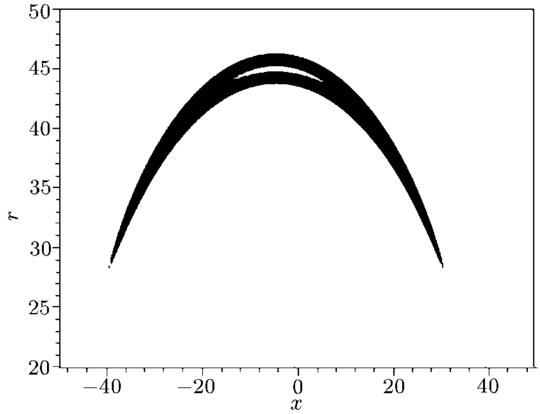

Fig. 2. The same as Fig. 1 but the vertical projection.

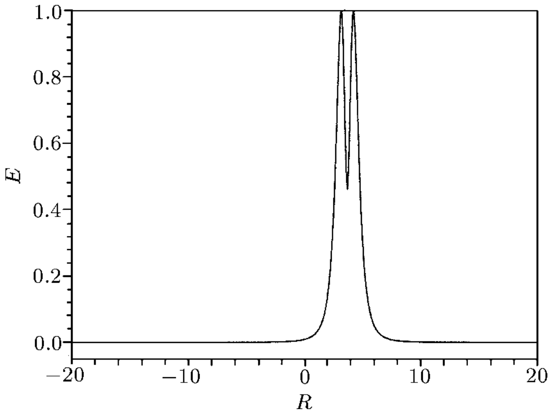

Fig. 3. The relation between R and E for $\mu = -10 + i$ and $t = 2000$ when θ is fixed.

Figure 4 shows the relation between θ and the maximal value of E in the direction specified by θ. The parameters are still $\mu = -10 + i$ and $t = 2000$. When $|\theta| < 33^\circ$, Eq. (18) holds. In this case $E_0 = 1$ at the maximum in any direction. When $|\theta| > 33^\circ$, Eq. (18) fails. The maximum (at $z = 0$) changes when $|\theta|$ changes and the expression is still given by Eq. (19). The maximal value tends to zero when $\theta \to \pm 90^\circ$.

Similar to the case $t \to +\infty$, we can obtain the asymptotic behaviour of the solution as $t \to -\infty$. In this case, let

$$R = \alpha(1 + \sin\theta)/2 + z/2 \tag{21}$$

and $s = -\sin\theta$, then Eqs. (14)–(18) still hold. Comparing Eq. (12) with Eq. (21), we know that there is a phase shift $\Delta R = \alpha$ between $t \to -\infty$ and $t \to +\infty$.

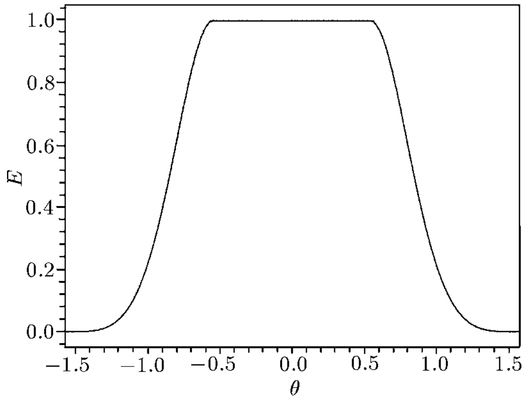

Fig. 4. The relation between θ and the maximal value of E in the direction specified by θ. The parameters are still $\mu = -10 + i$ and $t = 2000$.

In summary, the asymptotic behaviour of the solution as $t \to \pm\infty$ is as follows: When $t \to \pm\infty$, the solution is asymptotic to two ridges close to each other and locating beside a geodesic, which is a half-circle with radius $|t|$. In each direction, there are two peaks of height $E_0 = 1$ with distance $z_0(s)$ if $s = \pm \sin\theta$ satisfies Eq. (18), or there is one peak with lower height E_0 if s does not satisfy Eq. (18). What is more, there is a phase shift $\Delta R = \alpha$ between $t \to -\infty$ and $t \to +\infty$.

Here we only consider the case $f = \omega$. The situation when f is a polynomial with only real roots is similar, but only numerical result can be obtained in this complicated case. On the other hand, if f has a complex root, the situation is quite different. Even for a soliton solution with a single spectral parameter, the asymptotic behaviour is not simply a ridge. Hence the above results are invalid to the case that f has complex roots.

References

[1] Ward R S 1995 *Phys. Lett.* A **208** 203
[2] Ward R S 1999 *Asian J. Math.* **3** 325
[3] Hitchin N J, Segal G B and Ward R S 1999 *Integrable Systems, Twistors, Loop Groups and Riemann Surfaces* (Oxford: Clarendon)
[4] Gu C H 2002 *Sci. China* A **45** 706
[5] Ioannidou T 2002 *Nonlinearity* **15** 1489
[6] Ioannidou T 2003 *Int. J. Mod. Phys.* **18** 4975
[7] Zhou Z X 2001 *J. Math. Phys.* **42** 1085
[8] Zhou Z X 2001 *J. Math. Phys.* **42** 4938
[9] Gu C H, Hu H S and Zhou Z X 2005 *Darboux Transformations in Integrable Systems* (Dordrecht: Springer)