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Darboux Transformation and Exact Solutions of the Myrzakulov-I Equation *

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The Myrzakulov-I equation is a 2+1-dimensional generalization of the Heisenberg ferromagnetic equation and has a non-isospectral Lax pair. The Darboux transformation with non-constant spectral parameter is constructed and an extra constraint on the spectral parameter for the existence of the Darboux transformation is derived. Explicit expressions of the solutions of the Myrzakulov-I equation are presented.

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There are quite a few 2+1-dimensional generalizations of the Heisenberg ferromagnetic equation such as the Myrzakulov equation, Ishimori equation, etc.,[1,2] Among them, the Myrzakulov-I equation has a non-isospectral Lax pair, and is gauge equivalent to the 2+1-dimensional nonlinear Schrödinger equation. The explicit solutions of the Myrzakulov-I equation has been discussed by many researchers.[3,4]

There have already been quite a lot of works on the non-isospectral problems.[5] As a useful method to obtain explicit solutions of nonlinear partial differential equations, the Darboux transformation of degree one is given by a Darboux matrix $\lambda R - T$, where λ is the spectral parameter, and R and T are matrices such that $R^{-1}T$ can be written explicitly in terms of the solutions of the Lax pair.[6−9] In many isospectral cases, $R = I$ or $T = I$ can be chosen so that the Darboux transformation is constructed directly. However, in non-isospectral cases, in order to keep the reductions of the Lax pair, usually neither R nor T can be chosen as the identity matrix, but should be computed in terms of the concrete system. In this Letter, we construct the Darboux transformation for the (non-isospectral) Lax pair for the Myrzakulov-I equation (1). An extra constraint on the spectral parameter for the existence of the Darboux transformation is derived and all its solutions are obtained. Explicit expressions of the solutions of the Myrzakulov-I equation are presented.

The Myrzakulov-I equation is in the form

$$S_t = \frac{i}{2}[S, S_y]_x + (uS)_x,\ u_x = -\frac{i}{4}\mathrm{tr}(S[S_x, S_y]), \quad (1)$$

where u is an unknown scalar function, $S = \sum_{j=1}^3 S_j(x,y,t)\sigma_j$ is an unknown matrix with $\sum_{j=1}^3 S_j^2 = 1$, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2)$$

are Pauli matrices. Clearly $S^2 = I$ and $\mathrm{tr}S = 0$ hold.

Equation (1) has the Lax pair

$$\Phi_x = M(\lambda)\Phi = \frac{i}{2}\lambda S\Phi,$$

$$\Phi_t = N(\partial_y, \lambda)\Phi = -\lambda\Phi_y + \frac{i}{2}\lambda uS\Phi - \frac{1}{2}\lambda SS_y\Phi, \quad (3)$$

where the spectral parameter λ satisfies

$$\lambda_x = 0, \quad \lambda_t = -\lambda\lambda_y. \quad (4)$$

Here we have used the fact $S_xS = -SS_x$, $S_yS = -SS_y$.

Note that the coefficients of Eq. (3) satisfy the symmetries

$$KSK^{-1} = -\bar{S}, \quad KM(\lambda)K^{-1} = \overline{M(\bar{\lambda})},$$
$$KN(\partial_y, \lambda)K^{-1} = \overline{N(\partial_y, \bar{\lambda})}, \quad (5)$$

with $K = -i\sigma_2 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. Then we know that if $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ is a solution of Eq. (3) with $\lambda = \mu$, then $\begin{pmatrix} -\bar{h}_2 \\ \bar{h}_1 \end{pmatrix}$ is a solution of Eq. (3) with $\lambda = \bar{\mu}$.

The Darboux transformation is constructed as follows. Let $\mu(y,t)$ be a non-zero solution of Eq. (4), $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ be a column solution of Eq. (3) with $\lambda = \mu$,

$$\Lambda = \begin{pmatrix} \mu & \\ & \bar{\mu} \end{pmatrix}, \quad H = \begin{pmatrix} h_1 & -\bar{h}_2 \\ h_2 & \bar{h}_1 \end{pmatrix}. \quad (6)$$

Let

$$N(x,y,t) = H\Lambda^{-1}H^{-1},\ G(x,y,t,\lambda) = A(\lambda N - I), \quad (7)$$

where A is a 2×2 matrix to be determined. We will verify that $G(x,y,t,\lambda)$ is a Darboux matrix for Eq. (3) when A is suitably chosen. That is, for any solution Φ of Eq. (3), $\widetilde{\Phi} = G\Phi$ satisfies

$$\widetilde{\Phi}_x = \frac{i}{2}\lambda\widetilde{S}\widetilde{\Phi},\ \widetilde{\Phi}_t = -\lambda\widetilde{\Phi}_y + \frac{i}{2}\lambda\widetilde{u}\widetilde{S}\widetilde{\Phi} - \frac{1}{2}\lambda\widetilde{S}\widetilde{S}_y\widetilde{\Phi}, \quad (8)$$

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for a certain Hermitian matrix $\widetilde{S}(x,y,t)$ with $\widetilde{S}^2=I$ and $\mathrm{tr}\widetilde{S}=0$, and real function $\widetilde{u}(x,y,t)$.

From Eqs. (4) and (3), Λ and H satisfy

$$\Lambda_x=0,\quad \Lambda_t=-\Lambda\Lambda_y, \tag{9}$$

and

$$H_x=\frac{i}{2}SH\Lambda,\ H_t=-H_y\Lambda+\frac{i}{2}uSH\Lambda-\frac{1}{2}SS_yH\Lambda. \tag{10}$$

Then, $N=H\Lambda^{-1}H^{-1}$ satisfies

$$\begin{aligned}N_x=&\frac{i}{2}(S-NSN^{-1}),\\ N_t=&-N_yN^{-1}+\frac{i}{2}u(S-NSN^{-1})\\ &-\frac{1}{2}(SS_y-NSS_yN^{-1})N.\end{aligned} \tag{11}$$

Substituting $\widetilde{\Phi}=G\Phi$ together with $G=A(\lambda N-I)$ into Eq. (8), we obtain $A_x=0$, $A_t=0$, and

$$\begin{aligned}&\widetilde{S}=ANSN^{-1}A^{-1},\\ &\frac{1}{2}(A^{-1}A_y+NSN^{-1}A^{-1}A_yNSN^{-1})-\frac{i}{2}(\widetilde{u}-u\\ &+iN\{N^{-1}N_y,S\}N^{-1})NSN^{-1}=0,\end{aligned} \tag{12}$$

where $\{M_1,M_2\}=M_1M_2+M_2M_1$.

By Eq. (6),

$$\begin{aligned}&H^*H=(|h_1|^2+|h_2|^2)I,\\ &N^*N=(H\Lambda^{-1}H^{-1})^*(H\Lambda^{-1}H^{-1})=|\mu|^{-2}I,\end{aligned} \tag{13}$$

which leads to

$$N^{-1}N_y+(N^{-1}N_y)^*=-|\mu|^{-2}(|\mu|^2)_yI. \tag{14}$$

Moreover, since $\det N=\det(\Lambda^{-1})=|\mu|^{-2}$,

$$\begin{aligned}\mathrm{tr}(N^{-1}N_y)=&(\det N)^{-1}(\det N)_y\\ =&-|\mu|^{-2}(|\mu|^2)_y.\end{aligned} \tag{15}$$

Hence

$$N^{-1}N_y=i\sum_{j=1}^{3}B_j\sigma_j-\frac{1}{2}|\mu|^{-2}(|\mu|^2)_yI, \tag{16}$$

where B_1,B_2,B_3 are real functions. Then

$$\begin{aligned}\{N^{-1}N_y,S\}=&2i\sum_{j=1}^{3}B_jS_jI-|\mu|^{-2}(|\mu|^2)_yS\\ =&\mathrm{tr}(N^{-1}N_yS)I-|\mu|^{-2}(|\mu|^2)_yS.\end{aligned} \tag{17}$$

Substituting it into Eq. (12) yields

$$\begin{aligned}&\big(A^{-1}A_y+NSN^{-1}A^{-1}A_yNSN^{-1}-|\mu|^{-2}(|\mu|^2)_yI\big)\\ &-i\Big(\widetilde{u}-u-2\sum_{j=1}^{3}B_jS_j\Big)NSN^{-1}=0.\end{aligned} \tag{18}$$

Especially, when $A=aI$ is a scalar function,

$$\begin{aligned}&\big(2a^{-1}a_y-|\mu|^{-2}(|\mu|^2)_y\big)I-i\big(\tilde{u}-u\\ &+i\mathrm{tr}(N^{-1}N_yS)\big)NSN^{-1}=0.\end{aligned} \tag{19}$$

Since NSN^{-1} is traceless, we have

$$\begin{aligned}&\Big(\ln a^2-\ln|\mu|^2\Big)_y=0,\\ &\widetilde{u}=u-i\mathrm{tr}(N^{-1}N_yS).\end{aligned} \tag{20}$$

Note that $\tilde{u}$ given by Eq. (20) is always real since

$$\mathrm{tr}(N^{-1}N_yS)=2i\sum_{j=1}^{3}B_jS_j \tag{21}$$

is purely imaginary.

Since a depends on y only, $\mu(y,t)$ should satisfy an extra equation

$$\big(\ln|\mu|^2\big)_{yt}=0. \tag{22}$$

Considering $\mu_t=-\mu\mu_y$, Eq. (22) is equivalent to $(\mu+\bar{\mu})_{yy}=0$. Thus, μ should be a solution of

$$\mu_t=-\mu\mu_y,\quad (\mu+\bar{\mu})_{yy}=0. \tag{23}$$

Now we solve Eq. (23). From the second equation of Eq. (23), we suppose

$$\mu(y,t)=\alpha(t)y+\beta(t)+iw(y,t), \tag{24}$$

where α,β,w are real-valued functions. Substituting into the first equation of Eq. (23) yields

$$\alpha_ty+\beta_t+\alpha^2y+\alpha\beta=ww_y, \tag{25}$$

$$w_t+(\alpha y+\beta)w_y+\alpha w=0. \tag{26}$$

Here w is solved from Eq. (25) as

$$w=\pm\sqrt{(\alpha_t+\alpha^2)y^2+2(\beta_t+\alpha\beta)y+\gamma(t)}, \tag{27}$$

where $\gamma(t)$ is an arbitrary function. Substituting the expression of w into Eq. (26) and comparing the coefficients of y^2, y^1 and y^0, we obtain

$$\alpha_{tt}+6\alpha\alpha_t+4\alpha^3=0, \tag{28}$$

$$(\beta_t+2\alpha\beta)_t+2\alpha(\beta_t+2\alpha\beta)=0, \tag{29}$$

$$\gamma_t+2\alpha\gamma+2\beta\beta_t+2\alpha\beta^2=0. \tag{30}$$

Note that Eq. (28) can be written as

$$(\alpha^{-1}\alpha_t+2\alpha)_t=-(\alpha^{-1}\alpha_t+2\alpha)^2, \tag{31}$$

when $\alpha\neq0$. With a possible translation of t, the general solution of Eqs. (28)–(30) is

$$\begin{aligned}&\alpha=\frac{t}{t^2+C_1},\quad \beta=\frac{C_2t+C_3}{t^2+C_1},\\ &\gamma=-\frac{2C_2C_3t-C_1C_2^2+C_3^2}{(t^2+C_1)^2}+\frac{C_4}{(t^2+C_1)}.\end{aligned} \tag{32}$$

or

$$\alpha = \frac{1}{2t},\ \beta = \frac{C_2}{t} + C_3,\ \gamma = -\frac{C_2^2}{t^2} - C_3^2 + \frac{C_4}{t}, \quad (33)$$

or

$$\alpha = 0,\ \beta = C_1 t + C_2,\ \gamma = -(C_1 t + C_2)^2 + C_3. \quad (34)$$

The solution (33) is singular at $t = 0$, and the solution (34) makes w in Eq. (27) undefined globally except in the case $C_1 = 0$ where μ is a constant. We will not consider these two cases. For the solution (32),

$$\begin{aligned}\alpha y + \beta &= \frac{(y + C_2)t + C_3}{t^2 + C_1},\\ w^2 &= \frac{1}{(t^2 + C_1)^2}\big(C_1(y + C_2)^2 - 2C_3(y + C_2)t\\ &\quad + C_4 t^2 + C_1 C_4 - C_3^2\big). \end{aligned} \quad (35)$$

Neglecting a translation of y, we can set $C_2 = 0$. Then

$$\begin{aligned}\alpha y + \beta &= \frac{yt + C_3}{t^2 + C_1},\\ w^2 &= \frac{1}{(t^2 + C_1)^2}\big(C_1 y^2 - 2C_3 yt\\ &\quad + C_4 t^2 + C_1 C_4 - C_3^2\big). \end{aligned} \quad (36)$$

Here w^2 is always non-negative ($\omega \not\equiv 0$) if and only if $C_1 > 0$ and $C_1 C_4 \geq C_3^2$. Let $C_1 = \tau^2$, $C_4 = \sigma^2$, $C_3 = \sigma\tau\cos\theta$, where σ, τ, θ are real constants with $\tau \neq 0$, then

$$\begin{aligned}\mu(y,t) &= \frac{yt + \sigma\tau\cos\theta}{t^2 + \tau^2}\\ &\pm i\frac{\sqrt{\tau^2 y^2 - 2\sigma\tau yt\cos\theta + \sigma^2 t^2 + \sigma^2\tau^2\sin^2\theta}}{t^2 + \tau^2}. \end{aligned} \quad (37)$$

Especially, $\mu = \dfrac{y \mp i\sigma}{t \mp i\tau}$ when $\theta = 0$, which is similar to that used in getting explicit solutions in Ref. [3].

From Eq. (37), we have

$$|\mu|^2 = \frac{y^2 + \sigma^2}{t^2 + \tau^2}. \quad (38)$$

The first of Eqs. (20) implies that a can be chosen as $a = \sqrt{y^2 + \sigma^2}$.

Therefore, we have the following result.

Theorem. Suppose that (S, u) is a solution of Eq. (1). Let

$$\begin{aligned}\mu(y,t) &= \frac{yt + \sigma\tau\cos\theta}{t^2 + \tau^2}\\ &\pm i\frac{\sqrt{\tau^2 y^2 - 2\sigma\tau yt\cos\theta + \sigma^2 t^2 + \sigma^2\tau^2\sin^2\theta}}{t^2 + \tau^2}, \end{aligned} \quad (39)$$

where σ, τ, θ are real constants with $\tau \neq 0$. Let $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ be a column solution of Eq. (3) with $\lambda = \mu(y,t)$. Let

$$\Lambda = \begin{pmatrix} \mu & \\ & \bar{\mu} \end{pmatrix}, \quad H = \begin{pmatrix} h_1 & -\bar{h}_2 \\ h_2 & \bar{h}_1 \end{pmatrix}, \quad (40)$$

$$\begin{aligned} N(x,y,t) &= H\Lambda^{-1}H^{-1},\\ G(x,y,t,\lambda) &= \sqrt{y^2 + \sigma^2}(\lambda N - I). \end{aligned} \quad (41)$$

Then $G(x,y,t,\lambda)$ is a Darboux matrix for Eq. (3). The derived solution of Eq. (1) is

$$\widetilde{S} = NSN^{-1}, \quad \widetilde{u} = u - i\,\mathrm{tr}(N^{-1}N_y S), \quad (42)$$

which is globally defined for all x, y, t.

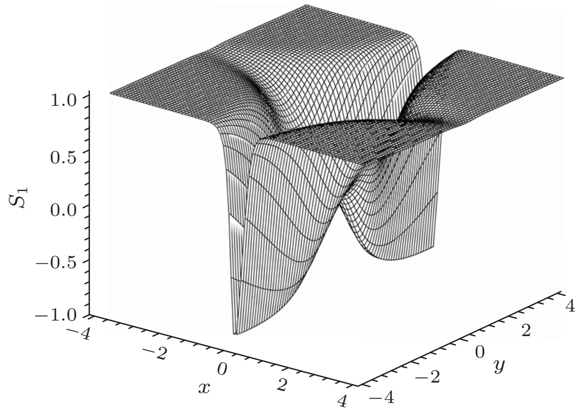

Fig. 1. S_1 of the solution.

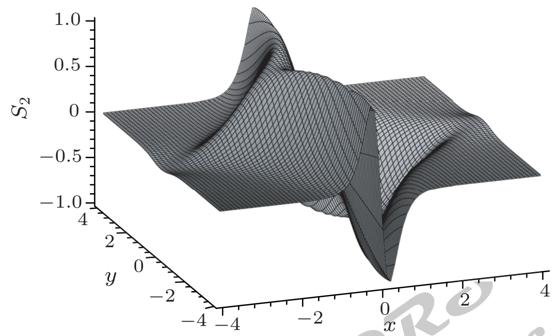

Fig. 2. S_2 of the solution.

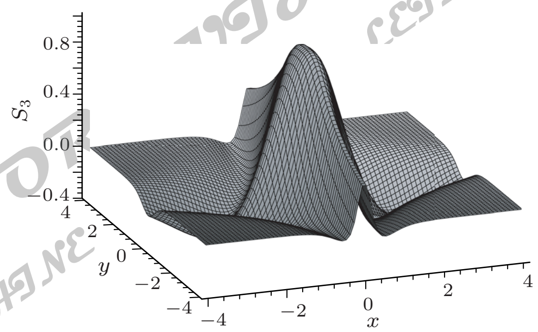

Fig. 3. S_3 of the solution.

Remark. Equation (4) always has a constant solution. Since the explicit solutions with constant spectral parameter have been considered by quite a few authors, we will not discuss this case in the present study.

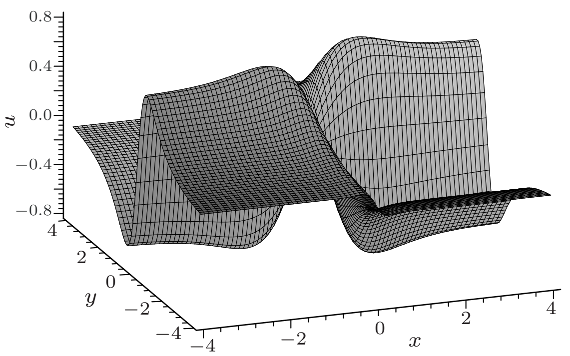

Fig. 4. u of the solution.

Example. To obtain an explicit expression of the solutions, we take $S = \sigma_3$, $u = 0$ as a seed solution of Eq. (1). Let $\mu(y,t)$ be given by (39), then the Lax pair (3) with $\lambda = \mu$ is

$$\Phi_x = \frac{i}{2}\mu\sigma_3\Phi, \quad \Phi_t = -\mu\Phi_y. \tag{43}$$

Its solution is $\Phi = e\frac{i}{2}\mu x\sigma_3\Phi_0(\mu)$, where $\Phi_0(\mu)$ is a 2×2 matrix whose entries are holomorphic functions of μ. Let

$$\Lambda = \begin{pmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{pmatrix}, \ H = \begin{pmatrix} e\frac{i}{2}\mu x & -\overline{b(\mu)}e\frac{i}{2}\bar{\mu}x \\ b(\mu)e-\frac{i}{2}\mu x & e-\frac{i}{2}\bar{\mu}x \end{pmatrix}, \tag{44}$$

where $b(\mu)$ is an analytic function of μ, then

$$N = H\Lambda^{-1}H^{-1} = \frac{1}{1+|b|^2e2\kappa x} \cdot \begin{pmatrix} \mu^{-1}+\bar{\mu}^{-1}|b|^2e2\kappa x & (\mu^{-1}-\bar{\mu}^{-1})\bar{b}ei\bar{\mu}x \\ (\mu^{-1}-\bar{\mu}^{-1})be-i\mu x & \bar{\mu}^{-1}+\mu^{-1}|b|^2e2\kappa x \end{pmatrix}, \tag{45}$$

where $\kappa = \mathrm{Im}\mu$. The new solution of the Myrzakulov-I equation is

$$\widetilde{S} = N\sigma_3N^{-1}, \quad \widetilde{u} = -i\mathrm{tr}(N^{-1}N_y\sigma_3). \tag{46}$$

A solution with parameters $\sigma = 1$, $\tau = 1$, $\theta = \pi/4$ and $b(\mu) = 1$ is plotted in Figs. 1–4, where $t = 0$ is chosen.

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