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Darboux Transformation with a Double Spectral Parameter for the Myrzakulov-I Equation *

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The Darboux transformation with a double spectral parameter for the Myrzakulov-I equation is obtained by taking a suitable limit of the parameters. The globalness of the derived solutions is proved.

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The Myrzakulov-I equation is a 2+1-dimensional generalization of the Heisenberg ferromagnetic equation and has a non-isospectral Lax pair.[1,2] The explicit solutions to the Myrzakulov-I equation have been discussed by many researchers.[3,4] Darboux transformation is one of the useful methods to obtain explicit solutions to the nonlinear partial differential equation.[5] The Darboux transformation of degree 1 for this equation has been constructed and exact global 'one-soliton' solutions are derived.[6]

On the other hand, the Darboux transformation of degree 2 usually relates to a double soliton.[5,7,8] Its limit solution is more important in certain cases. Sometimes they can describe a rogue wave.[9,10,11,12,13] However, lots of limit solutions have singularity. Hence searching for a global solution is important in both mathematics and applications.

In this Letter, we construct the Darboux transformation of degree 2 and reduce it to the Darboux transformation with a double spectral parameter by taking suitable limits. Globalness of the limit solutions is proved.

The Myrzakulov-I equation is in the form

$$\begin{aligned} S_t &= \frac{i}{2}[S, S_y]_x + (uS)_x, \\ u_x &= -\frac{i}{4}\mathrm{tr}\,(S[S_x, S_y]), \end{aligned} \tag{1}$$

where u is an unknown scalar function,

$$S = \sum_{j=1}^{3} S_j(x,y,t)\sigma_j \tag{2}$$

is an unknown matrix with $S^2 = \sum_{j=1}^{3} S_j^2 = 1$, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Equation (1) has the Lax pair

$$\begin{aligned} \Phi_x &= M(\lambda)\Phi = \frac{i}{2}\lambda S\Phi, \\ \Phi_t &= N(\partial_y, \lambda)\Phi \\ &= -\lambda\Phi_y + \frac{i}{2}\lambda uS\Phi - \frac{1}{2}\lambda SS_y\Phi, \end{aligned} \tag{3}$$

where the 'spectral parameter' λ satisfies

$$\lambda_x = 0, \ \ \lambda_t = -\lambda\lambda_y.$$

From Ref. [6] we know that the Darboux matrix of degree 1 for Eq. (3) is in the form

$$\begin{aligned} G(x,y,t) &= \sqrt{y^2+\sigma^2}(\lambda N - I), \\ N(x,y,t) &= H\Lambda^{-1}H^{-1}, \end{aligned} \tag{4}$$

where

$$\Lambda = \begin{pmatrix} \mu & 0 \\ 0 & \bar\mu \end{pmatrix}, \ H = \begin{pmatrix} h_1 & -\bar h_2 \\ h_2 & \bar h_1 \end{pmatrix},$$

and $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ is a column solution to Eq. (3) with

$$\begin{aligned} \lambda = \mu(y,t) = &\ \frac{yt + \sigma\tau\cos\theta}{t^2+\tau^2} \\ &\pm i\frac{\sqrt{\tau^2y^2 - 2\sigma\tau yt\cos\theta + \sigma^2t^2 + \sigma^2\tau^2\sin^2\theta}}{t^2+\tau^2}, \end{aligned} \tag{5}$$

where σ, θ, τ are real constants with $\tau \neq 0$. It is easy to prove that $\begin{pmatrix} -\bar h_2 \\ \bar h_1 \end{pmatrix}$ is a solution to Eq. (3) with $\lambda = \bar\mu(y,t)$. The authors of Ref. [7] gave the following result.

Lemma 1: Suppose that (S,u) is a solution to Eq. (1). After the Darboux transformation of $G(x,y,t)$, the derived solution to Eq. (1) is

$$\widetilde S = NSN^{-1}, \ \widetilde u = u - i\mathrm{tr}(N^{-1}N_yS). \tag{6}$$

Now we consider the Darboux matrix of degree 2 for Eq. (3).

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According to Ref. [5], let

$$\Lambda_1 = \begin{pmatrix} \mu_1 & \\ & \overline{\mu}_1 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} \mu_2 & \\ & \overline{\mu}_2 \end{pmatrix},$$

where μ_i is the function in the form of Eq. (5) with $\sigma_i, \tau_i, \theta_i, i = 1, 2$. Let

$$H_1^{(i)} = \begin{pmatrix} h_1^{(i)} \\ h_2^{(i)} \end{pmatrix}, \quad H_2^{(i)} = \begin{pmatrix} -\overline{h}_2^{(i)} \\ \overline{h}_1^{(i)} \end{pmatrix}$$

be solutions to Eq. (3) with $\lambda = \mu_i$ and $\lambda = \overline{\mu}_i$, respectively, and $H_i = (H_1^{(i)}, H_2^{(i)})$.

Let

$$N_1 = H_1 \Lambda_1^{-1} H_1^{-1}, \quad G_1(\lambda) = \sqrt{y^2+\sigma^2}(\lambda N_1 - I), \tag{7}$$

then $G_1(\lambda)$ is a Darboux matrix for Eq. (3).

Next we take

$$\widetilde{H}_1^{(2)} = G_1(\mu_2) H_1^{(2)}, \ \widetilde{H}_2^{(2)} = G_1(\overline{\mu}_2) H_2^{(2)},$$

then $\widetilde{H}_1^{(1)}$ and $\widetilde{H}_2^{(2)}$ are new solutions to the Lax pair of Eq. (3) with $\lambda = \mu_2$ and $\lambda = \bar{\mu}_2$, respectively, where (S, u) is replaced by $(\widetilde{S}, \widetilde{u})$. As a result,

$$\widetilde{H}_2 = (\widetilde{H}_2^{(1)}, \ \widetilde{H}_2^{(2)}) = \sqrt{y^2+\sigma^2}(N_1 - N_2) N_2^{-1} H_2,$$

and

$$\widetilde{N}_2 = \widetilde{H}_2 \Lambda_2^{-1} \widetilde{H}_2^{-1} = (N_2 - N_1) N_2 (N_2 - N_1)^{-1}. \tag{8}$$

This leads to another Darboux matrix

$$\begin{aligned} G_2(\lambda) &= \sqrt{y^2+\sigma^2}(\lambda \widetilde{N}_2 - I) \\ &= \sqrt{y^2+\sigma^2}(\lambda (N_2 - N_1) N_2 (N_2 - N_1)^{-1} - I). \end{aligned}$$

The composition $G(\lambda) = G_2(\lambda) G_1(\lambda)$ is a Darboux matrix G of degree 2. It is

$$\begin{aligned} G &= \sqrt{y^2+\sigma_2^2}(\lambda \widetilde{N}_2 - I)\sqrt{y^2+\sigma_1^2}(\lambda N_1 - I) \\ &= \sqrt{(y^2+\sigma_2^2)(y^2+\sigma_1^2)} \\ &\quad \cdot (\lambda^2 (N_2 - N_1) N_2 (N_2 - N_1)^{-1} N_1 \\ &\quad - \lambda (N_2^2 - N_1^2)(N_2 - N_1)^{-1} + I). \end{aligned} \tag{9}$$

Suppose that (S, u) is a solution to Eq. (1). After the Darboux transformation G of degree 2, the derived solution to Eq. (1) is

$$\begin{aligned} \widetilde{\widetilde{u}} &= u - i\mathrm{tr}\,(N_1^{-1} N_{1y} S) - i\mathrm{tr}\,(\widetilde{N}_2^{-1} \widetilde{N}_{2y} \widetilde{S}), \\ \widetilde{\widetilde{S}} &= \widetilde{N}_2 \widetilde{S} \widetilde{N}_2^{-1} = (N_2 - N_1)(N_1^{-1} - N_2^{-1})^{-1} \\ &\quad \cdot S (N_1^{-1} - N_2^{-1})(N_2 - N_1)^{-1}, \end{aligned} \tag{10}$$

where $\widetilde{S} = N_1 S N_1^{-1}$, and $\widetilde{N}_2 = (N_2 - N_1) N_2 (N_2 - N_1)^{-1}$.

We will discuss the limit of the solution (10) when one parameter tends to another.

For given $\mu(x, y, t)$ and $H(x, y, t)$, let $\mu^{(\varepsilon)}(x, y, t)$ and $H^{(\varepsilon)}(x, y, t)$ be a smooth function and a smooth matrix-valued function such that $\mu^{(0)}(x, y, t) = \mu(x, y, t)$ and $H^{(0)}(x, y, t) = H(x, y, t)$, respectively.

Now $\mu_1 = \mu$, $H_1 = H$, $\mu_2 = \mu^{(\varepsilon)}$, and $H_2 = H^{(\varepsilon)}$ are adopted, and N and $N^{(\varepsilon)}$ are defined as in Eq. (4).

We can construct Darboux matrix $G(x, y, t, \epsilon)$ by Eq. (9). The limit of $G(x, y, t, \epsilon)$ as ϵ tends to zero will give a Darboux transformation with a double spectral parameter μ.

According to Eq. (4),

$$\begin{aligned} N &= \frac{1}{|h_1|^2 + |h_2|^2} \\ &\cdot \begin{pmatrix} \mu^{-1}|h_1|^2 + \bar{\mu}^{-1}|h_2|^2 & (\mu^{-1} - \bar{\mu}^{-1}) h_1 \bar{h}_2 \\ (\mu^{-1} - \bar{\mu}^{-1}) \bar{h}_1 h_2 & \bar{\mu}^{-1}|h_1|^2 + \mu^{-1}|h_2|^2 \end{pmatrix}. \end{aligned} \tag{11}$$

Denote $U = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C} \right\}$, then $AA^* = \det(A) I$ for any $A \in U$. Clearly, $N, N^{(\epsilon)} \in U$, and $N^{(\epsilon)} - N \in U$. Equation (8) becomes

$$\widetilde{N} = \frac{(N^{(\epsilon)} - N) N^{(\epsilon)} (N^{(\epsilon)} - N)^*}{\det(N^{(\epsilon)} - N)}. \tag{12}$$

For any smooth function $f^{(\epsilon)}(x, y, t)$, denote

$$Df(x, y, t) = \frac{\partial}{\partial \epsilon} f^{(\epsilon)}(x, y, t)|_{\epsilon=0}.$$

Suppose $DN \neq 0$, then Eq. (12) becomes

$$\begin{aligned} \widetilde{N} &= \frac{(DN\epsilon + O(\epsilon^2)) N^{(\epsilon)} (DN\epsilon + O(\epsilon^2))^*}{\det(DN\epsilon + O(\epsilon^2))} \\ &= \frac{(DN + O(\epsilon)) N^{(\epsilon)} (DN + O(\epsilon))^*}{\det(DN + O(\epsilon))}. \end{aligned} \tag{13}$$

Thus as $\epsilon \to 0$,

$$\widetilde{N} \to \frac{(DN) N (DN)^*}{\det(DN)} = (DN) N (DN)^{-1}.$$

Lemma 2: If $DN \neq 0$, the Darboux transformation with a double spectral parameter is

$$\begin{aligned} \overline{G} = (y^2 + \sigma^2)\{ &\lambda^2 (DN) N (DN)^{-1} N \\ &- \lambda [N + (DN) N (DN)^{-1}] + I\} \end{aligned} \tag{14}$$

and the solutions (10) become

$$\begin{aligned} \overline{u} &= u - i\mathrm{tr}\,(N^{-1} N_y S) \\ &\quad - i\mathrm{tr}\,\{(DN) N^{-1} (DN)^{-1} \\ &\quad \cdot [(DN) N (DN)^{-1}]_y N S N^{-1}\}, \\ \overline{S} &= (DN) N (DN)^{-1} N S N^{-1} (DN) N^{-1} (DN)^{-1}. \end{aligned} \tag{15}$$

What we care about in the following is whether Eq. (15) is defined globally. The key point is to find out whether $(DN)^{-1}$ exists globally.

Theorem 1: Let (S, u) be a solution to Eq. (1) such that the Lax pair Eq. (3) is uniquely solvable. Let $(h_1, h_2)^T$ be a non-zero global solution to Eq. (3) and N is given by Eq. (11). Then $\bar{G}$ given by Eq. (14) is globally defined if $D\mu \neq 0$.

Proof: By the unique solvability of Eq. (3), $(h_1, h_2)^T$ is non-zero everywhere if it is non-zero at one point. According to Eq. (11),

$$DN = \begin{pmatrix} Da & -D\bar{c} \\ Dc & D\bar{a} \end{pmatrix},$$

where

$$\begin{aligned} Da =& -\frac{\mu^{-2}D\mu|h_1|^2 + \overline{\mu^{-2}D\mu}|h_2|^2}{|h_1|^2 + |h_2|^2} \\ &- \frac{\mu^{-1} - \bar{\mu}^{-1}}{(|h_1|^2 + |h_2|^2)^2}(Y + \bar{Y}), \end{aligned} \tag{16}$$

$$\begin{aligned} Dc =& \frac{(-\mu^{-2}D\mu + \overline{\mu^{-2}D\mu})\bar{h}_1 h_2}{|h_1|^2 + |h_2|^2} \\ &+ \frac{(\mu^{-1} - \bar{\mu}^{-1})}{(|h_1|^2 + |h_2|^2)^2}M, \end{aligned} \tag{17}$$

$$Y = |h_1|^2\bar{h}_2 Dh_2 - |h_2|^2\bar{h}_1 Dh_1, \tag{18}$$

$$\begin{aligned} M =& |h_1|^2\bar{h}_1 Dh_2 - \bar{h}_1^2 Dh_1 \\ &+ |h_1|^2 h_2 D\bar{h}_1 - \bar{h}_1 h_2^2 D\bar{h}_2, \end{aligned} \tag{19}$$

or

$$h_1\bar{h}_2 M = |h_1|^2 Y - |h_2|^2\bar{Y}. \tag{20}$$

Here $\det(DN) = 0$ if and only if $Da = Dc = 0$. Considering the real and imaginary parts of Eqs. (16) and (17), we have

$$\mathrm{Re}\Big(\frac{D\mu}{\mu^2}\Big) = 0, \tag{21}$$

$$\Big(\mathrm{Im}\frac{D\mu}{\mu^2}\Big)(|h_1|^4 - |h_2|^4) + 4\Big(\mathrm{Im}\frac{1}{\mu}\Big)(\mathrm{Re}Y) = 0, \tag{22}$$

$$\mathrm{Im}Y = 0, \tag{23}$$

$$\begin{aligned} &|h_1|^2|h_2|^2\mathrm{Im}\frac{D\mu}{\mu^2}(|h_1|^2 + |h_2|^2) \\ &- \Big(\mathrm{Im}\frac{1}{\mu}\Big)\mathrm{Re}Y(|h_1|^2 - |h_2|^2) = 0 \end{aligned} \tag{24}$$

when $h_1 \neq 0$ and $h_2 \neq 0$. It is easy to verify that these equations also hold when $h_1 = 0$ or $h_2 = 0$. Eliminating $\mathrm{Re}Y$ in Eqs. (22) and (24), we obtain

$$\mathrm{Im}\frac{D\mu}{\mu^2}(|h_1|^2 + |h_2|^2)^2 = 0.$$

Considering Eq. (21), we have $D\mu = 0$. This contradicts the assumption $D\mu \neq 0$. The theorem is proved.

Remark 1: The condition $D\mu \neq 0$ holds in quite general cases, which will be shown in the following examples.

To obtain an explicit expression of the solution, we take $S = \sigma_3$, $u = 0$ as a seed solution to Eq. (1). Let $\mu(y, t)$ be given by Eq. (5), then the solution to the Lax pair in Eq. (3) with $\lambda = \mu$ is

$$\varPhi = e^{\frac{i}{2}\mu x\sigma_3}\varPhi_0(\mu),$$

where $\varPhi_0(\mu)$ is a matrix whose entries are holomorphic functions of μ. Since μ is globally defined, so is $\varPhi$.

Since μ depends on three parameters σ, τ, θ, we can discuss the change of σ, τ, θ, respectively.

First we consider the limit of σ. Let

$$\mu = \mu(y, t, \theta, \tau, \sigma),\ \mu^{(\epsilon)} = \mu(y, t, \theta, \tau, \sigma + \epsilon),$$

then $DN = \partial_\sigma N$. The solution is given by Eq. (15) where DN is replaced by $\partial_\sigma N$.

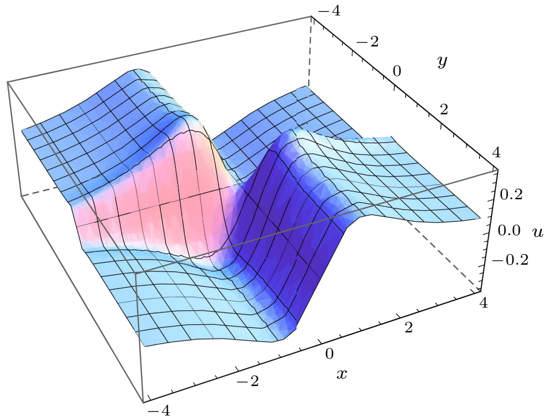

Fig. 1. The value of u of the solution with single spectral parameter.

Similarly, replacing DN by $\partial_\tau N$ or $\partial_\theta N$ respectively in Eq. (15), we obtain the limit solution for τ or θ.

Remark 2: For the limit of θ, since

$$\det N = \det N^{(\epsilon)} = \frac{y^2 + \sigma^2}{t^2 + \tau^2},$$

is independent of θ, the expression (15) for $\bar{S}$ can be simplified. According to Eq. (10),

$$\begin{aligned} \tilde{\bar{S}} =& (N^{(\epsilon)} - N)(N^{-1} - (N^{(\epsilon)})^{-1})^{-1}S \\ &\cdot (N^{-1} - (N^{(\epsilon)})^{-1})(N^{(\epsilon)} - N)^{-1} \\ =& (N^{(\epsilon)} - N)(N^* - (N^{(\epsilon)})^*)^{-1}S \\ &\cdot (N^* - (N^{(\epsilon)})^*)(N^{(\epsilon)} - N)^{-1} \\ =& (N^{(\epsilon)} - N)^2 S(N^{(\epsilon)} - N)^{-2}, \end{aligned} \tag{25}$$

hence the limit solution is

$$\bar{S} = (\partial_\theta N)^2 S(\partial_\theta N)^{-2}. \tag{26}$$

For the limit of θ,

$$\mathrm{Re}(D\mu) = \mathrm{Re}(\partial_\theta\mu) = -\frac{\sigma\tau\sin\theta}{t^2 + \tau^2}.$$

According to theorem 1, Eq. (15) can be defined globally if $\sigma \neq 0$, $\tau \neq 0$ and $\sin\theta \neq 0$.

For the limit of σ,

$$\mathrm{Re}(D\mu) = \mathrm{Re}(\partial_\sigma \mu) = \frac{\tau \cos\theta}{t^2 + \tau^2},$$

Eq. (15) is defined globally if $\tau \neq 0$ and $\cos\theta \neq 0$.

For the limit of τ, it is easier to compute $\frac{D\mu}{\mu^2}$ than $D\mu$. Hence

$$\mathrm{Re}\Big(\frac{D\mu}{\mu^2}\Big) = -\mathrm{Re}(\partial_\tau \mu^{-1}) = -\frac{\sigma \cos\theta}{y^2 + \tau^2}.$$

The solution is defined globally if we choose $\sigma \neq 0$ and $\cos\theta \neq 0$.

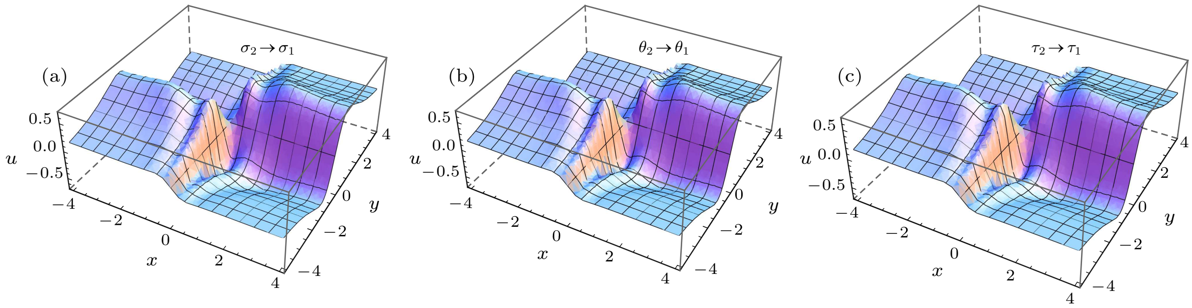

Fig. 2. The value of u of the solution with a double spectral parameter

Taking $\sigma = 1$, $\tau = 1$, $\theta = \pi/4$, $t = 0$ and $\varPhi(\mu) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, here we only plot the u part of the solutions.

The solution with a single spectral parameter is plotted in Fig. 1 and the solutions with double spectral parameters (by the limit of θ, σ, τ respectively) are plotted in Fig. 2. From these figures, we can see that the solutions with a double spectral parameter look like perturbations of the solutions with a single spectral parameter, and the shapes of solutions with a double spectral parameter by different limits of θ, σ or τ are similar.

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