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PARAMETERS OF DARBOUX TRANSFORMATION FOR REDUCED AKNS, KAUP-NEWELL AND PCF SYSTEMS**

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Abstract

For the integrable system with $u(p,q)$ reduction, there is a well-known sufficient condition to choose the parameters: the spectral parameters only take two mutually conjugate values and the solutions of the Lax pair should satisfy certain orthogonal relations. In this paper, the author proves that, for the AKNS system, the Kaup-Newell system and the principal chiral field (PCF), this condition is also necessary for generic potentials with the $u(p,q)$ reduction. For some other reductions, sufficiency and necessity of more constraints are proved.

Keywords Darboux transformation, Spectral parameters, $u(p,q)$ reduction
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§1. Introduction

Darboux transformation is a powerful method to get explicit solutions of nonlinear PDEs. In 1+1 dimensions, it gives a universal algorithm to get a series of solutions by solving linear ODEs only once. The constructions of Darboux transformations have been widely investigated (see, e.g. [1,3,5,7,8,9,10,11,13]). In 1+1 dimensions, a Darboux transformation is usually given by a Darboux matrix which is a polynomial of the spectral parameter. The most fundamental Darboux matrix is a Darboux matrix of degree one, which is linear in the spectral parameter.

Let $\mathfrak{g}$ be a finite dimensional semi-simple matrix Lie algebra. For the spectral parameter λ, let

$$L(\mathfrak{g}) = \Big\{ \sum_{j=0}^{n} X_j\lambda^{n-j} \,\Big|\, X_j \in \mathfrak{g},\, n \in \mathbf{Z}_+ \cup \{0\} \Big\} \tag{1.1}$$

be a subalgebra of the loop algebra of $\mathfrak{g}$, $L_n(\mathfrak{g}) = \Big\{ \sum\limits_{j=0}^{n} X_j\lambda^{n-j} \,|\, X_j \in \mathfrak{g} \Big\}$. Consider the Lax pair

$$\begin{aligned} \Phi_x &= U(\lambda)\Phi, \\ \Phi_t &= V(\lambda)\Phi, \end{aligned} \tag{1.2}$$

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where

$$
\begin{aligned}
U(\lambda) &\equiv U(x,t,\lambda) = \sum_{i=0}^{m} U_i(x,t)\lambda^{m-i} \in C^\infty(\mathbf{R}^2, L_m(\mathfrak{g})), \\
V(\lambda) &\equiv V(x,t,\lambda) = \sum_{j=0}^{n} V_j(x,t)\lambda^{n-j} \in C^\infty(\mathbf{R}^2, L_n(\mathfrak{g})).
\end{aligned}
\tag{1.3}
$$

The integrability condition of (1.2) gives a system of nonlinear partial differential equations, which comes from the identity

$$
U_t - V_x + UV - VU = 0 \tag{1.4}
$$

for all $\lambda \in \mathbf{C}$. Suppose (1.4) holds for all λ, then (1.2) is completely integrable, and vice versa.

For $\mathfrak{g} = gl(N,\mathbf{C})$ or $sl(N,\mathbf{C})$, diagonalizable Darboux transformation of degree one can be constructed as follows.

Theorem 1.1.[3,13] *Let $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$ where $\lambda_1, \cdots, \lambda_N \in \mathbf{C}$. Let h_i be a column solution of (1.2) with $\lambda = \lambda_i$. $H = (h_1, \cdots, h_N)$ is an $N \times N$ matrix. Take $R(x,t)$ to be an arbitrary invertible matrix function. When $\det H \neq 0$, define $S = RH\Lambda H^{-1}$. Then $T(x,t,\lambda) = \lambda R(x,t) - S(x,t)$ is a Darboux matrix for (1.2). That is, $\widetilde{\Phi} = T\Phi$ satisfies*

$$
\begin{aligned}
\widetilde{\Phi}_x &= \widetilde{U}(\lambda)\widetilde{\Phi}, \\
\widetilde{\Phi}_t &= \widetilde{V}(\lambda)\widetilde{\Phi}
\end{aligned}
\tag{1.5}
$$

for certain $\widetilde{U}(x,t,\lambda)$, $\widetilde{V}(x,t,\lambda) \in L(\mathfrak{g})$.

This is a general scheme to construct diagonalizable Darboux matrices. Any non-diagonalizable Darboux matrix can be obtained by a limit of some diagonalizable Darboux matrices[15].

The matrix H in Theorem 1.1 satisfies

$$
H_x = \sum_{i=0}^{m} U_i H \Lambda^{m-i}, \quad H_t = \sum_{j=0}^{n} V_j H \Lambda^{n-j}. \tag{1.6}
$$

From (1.5), $\widetilde{U}$, $\widetilde{V}$ are given by

$$
\begin{aligned}
\widetilde{U}(\lambda) &= (\lambda R - S)U(\lambda)(\lambda R - S)^{-1} + (\lambda R_x - S_x)(\lambda R - S)^{-1}, \\
\widetilde{V}(\lambda) &= (\lambda R - S)V(\lambda)(\lambda R - S)^{-1} + (\lambda R_t - S_t)(\lambda R - S)^{-1}.
\end{aligned}
\tag{1.7}
$$

Comparing the coefficients, we get

$$
\begin{aligned}
\widetilde{U}_j &= RU_jR^{-1} + \sum_{k=0}^{j-1} R\left[U_k(R^{-1}S)^{j-1-k}, R^{-1}S\right]R^{-1} + R_xR^{-1}\delta_{jm}, \\
\widetilde{V}_j &= RV_jR^{-1} + \sum_{k=0}^{j-1} R\left[V_k(R^{-1}S)^{j-1-k}, R^{-1}S\right]R^{-1} + R_tR^{-1}\delta_{jm},
\end{aligned}
\tag{1.8}
$$

and $R^{-1}S$ satisfies

$$
\begin{aligned}
(R^{-1}S)_x + [R^{-1}S, U(R^{-1}S)] &= 0, \\
(R^{-1}S)_t + [R^{-1}S, V(R^{-1}S)] &= 0,
\end{aligned}
\tag{1.9}
$$

where

$$U(M)=\sum_{j=0}^{m} U_j M^{m-j}$$

for an $N\times N$ matrix M.

When U_i's, V_i's are restricted to smaller Lie subalgebras, special restrictions on $\lambda_1,\cdots,\lambda_N$ and $h_1,\cdots,h_N$ are necessary. For $\mathfrak{g}=u(N)$, a well-known restriction is: $\lambda_i=\mu$ or $\bar\mu$ with $\operatorname{Im}\mu\neq 0$, and $h_i^* h_j=0$ for $\lambda_i\neq\lambda_j$ (see [1,12]). This choice has been applied to various problems[4,6,16].

For some systems like the AKNS system, when the number of spectral parameters is restricted to two, the previous constraint on λ_i and h_i is also necessary, provided that both the seed solution and the derived solution decay at infinity fast enough[12]. A natural question is: if the solutions are not restricted to those which decay at infinity, generally, can the spectral parameters in each Darboux matrix take more than two different values, or can they take two values which are not mutually conjugate? The present paper gives an answer to this question.

Now we consider the Darboux transformation which keeps Lie algebraic reductions. Let $\mathfrak{g}$ be a Lie algebra, $U(\lambda), V(\lambda)\in L(\mathfrak{g})$. Suppose after the Darboux transformation, $\widetilde U(\lambda),\widetilde V(\lambda)\in L(\mathfrak{g})$. In this case, we say that the Darboux transformation keeps the $\mathfrak{g}$-reduction, or the $L(\mathfrak{g})$-reduction. Here we choose $\mathfrak{g}$ as

$$u(p,q)\equiv\{\,X\in gl(p+q,\mathbf{C})\,|\,X^* I_{pq}+I_{pq}X=0\,\},$$
$$su(p,q)\equiv\{\,X\in gl(p+q,\mathbf{C})\,|\,X^* I_{pq}+I_{pq}X=0,\ \operatorname{tr}X=0\}$$

or

$$so(p,q)\equiv\{\,X\in gl(p+q,\mathbf{R})\,|\,X^T I_{pq}+I_{pq}X=0\,\},$$

where $I_{pq}=\operatorname{diag}(\underbrace{1,\cdots,1}_{p},\underbrace{-1,\cdots,-1}_{q})$, and the superscripts "T" and "$*$" refer to the transpose and conjugate transpose of a matrix respectively.

The Darboux transformation which keeps $u(p,q)$ reduction is as follows.

Theorem 1.2.[1,6,12] *Suppose $U(\lambda),V(\lambda)\in L(u(p,q))$. Take $\mu\in\mathbf{C}$ with $\operatorname{Im}\mu\neq 0$. Let $\lambda_i=\mu$ or $\bar\mu$, $h_i^* I_{pq}h_j=0$ for $\lambda_i\neq\lambda_j$ (this always holds identically if it holds at one point (x_0,t_0)). Then after the action of the Darboux matrix $R(\lambda-H\Lambda H^{-1})$, $\widetilde U(\lambda),\widetilde V(\lambda)\in L(u(p,q))$.*

For $su(p,q)$, $so(p,q)$, the situation is similar, which will be discussed in §3.

A twisted reduction with the involution $X\mapsto -I_{pq}^{-1}X^T I_{pq}$ is considered in Theorem 4.1.

For $\mathfrak{g}=u(p,q)$, let $\mathfrak{h}$ be a Cartan subalgebra which contains diagonal matrices in $\mathfrak{g}$, $\mathfrak{h}^\perp$ be the orthogonal of $\mathfrak{h}$ with respect to the Killing form, which contains all the off-diagonal matrices in $u(p,q)$. The regular elements in $\mathfrak{h}$ are the diagonal matrices whose diagonal entries are mutually different.

Due to the integrability condition (1.4), U, V should satisfy a system of PDEs. There are no a priori constraints on U and V which are independent of t.

We call $U(x,t,\lambda)$ generic if for $U\in\mathfrak{g}$ does not satisfy specific constraints which are independent of the derivative with respect to t.

In this paper we show that the conditions in Theorem 1.2 are also necessary for generic $U(\lambda)$ for some systems. To consider this necessity of the restrictions, specific systems should be discussed, because the demand to keep $u(p,q)$ reduction for general system (1.2) is so strong that the problem is almost trivial. If the condition is necessary to the x-part of the system, certainly it is necessary to the whole system. Hence we only consider the x-part here.

We discuss the following three systems:

(A) AKNS system: $m=1$, $U_0=J\in\mathfrak{h}$ is a fixed regular element, $U_1(x,t)\in\mathfrak{h}^\perp$;

(B) Kaup-Newell system: $m=2$, $U_0=J\in\mathfrak{h}$ is a fixed regular element, $U_1(x,t)\in\mathfrak{h}^\perp$, $U_2=0$;

(C) Principal chiral field (PCF) (i.e. harmonic map from $R^{1,1}$ to a Lie group): $m=1$, $U_1=0$.

Without other constraints, the Darboux transformation in Theorem 1.1 should have the following restrictions to guarantee that $\widetilde{U}(\lambda)$ is still in the corresponding system[14].

For (A): R is a constant diagonal matrix;

For (B): R is a diagonal matrix and S is a constant matrix;

For (C): S is a constant diagonal matrix.

Apart from these conditions, to keep $u(p,q)$ reduction, we should have more constraints on Λ and H. Here is our main conclusion, which is the inverse of Theorem 1.2.

Theorem 1.3. *For the systems* (A), (B) *and* (C), *suppose that $U(\lambda)$ is generic, $U(\lambda)$, $\widetilde{U}(\lambda)\in L(u(p,q))$, and $\widetilde{U}(\lambda)\not\equiv U(\lambda)$, then the matrices $\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_{p+q})$ and $H=(h_1,\cdots,h_{p+q})$ in Theorem* 1.1 *should satisfy $\lambda_i=\mu$ or $\bar\mu$ for certain $\mu\in\mathbf{C}$, $\mathrm{Im}\,\mu\neq 0$ and $h_i^*I_{pq}h_j=0$ for $\lambda_i\neq\lambda_j$.*

For $\mathfrak{g}=su(p,q)$, $so(p,q)$, or the twisted case, some more restrictions are needed (see §3, §4).

In §5, we give a simple example to show that for non-generic potential U, spectral parameters can take more than two different values in each Darboux transformation.

§2. General Choice of Parameters for $u(p,q)$ Reduction

In this section, let $\mathfrak{g}=u(p,q)$. We will prove our main theorem—Theorem 1.3. We always suppose that the Darboux matrix of degree one exists and want to determine which kinds of Λ and H are possible to keep the $u(p,q)$ reduction for generic U. Let $\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_{p+q})$, $H=(h_1,\cdots,h_{p+q})$ where h_i is a solution of (1.2) with $\lambda=\lambda_i$. Suppose that after the Darboux transformation, $\widetilde{U}(\lambda),\widetilde{V}(\lambda)\in L(\mathfrak{g})$.

From (1.7),

$$\widetilde{U}(\lambda)=(\lambda R-S)U(\lambda)(\lambda R-S)^{-1}+(\lambda R_x-S_x)(\lambda R-S)^{-1},$$
$$\widetilde{U}^*(\lambda)=-(\lambda R-S)^{*-1}I_{pq}U(\lambda)I_{pq}^{-1}(\lambda R-S)^*+(\lambda R-S)^{*-1}(\lambda R_x^*-S_x^*)$$

for all $\lambda\in\mathbf{R}$. For real λ, $\widetilde{U}^*(\lambda)=-I_{pq}\widetilde{U}(\lambda)I_{pq}^{-1}$ implies

$$\Theta_x(\lambda)=[U(\lambda),\Theta(\lambda)], \tag{2.1}$$

where

$$\begin{aligned}\Theta(\lambda) &= I_{pq}^{-1}(\lambda R - S)^* I_{pq}(\lambda R - S) = \lambda^2 \Gamma + \lambda \Delta + \Omega, \\ \Gamma = I_{pq}^{-1} R^* I_{pq} R, &\quad \Delta = -I_{pq}^{-1} S^* I_{pq} R - I_{pq}^{-1} R^* I_{pq} S, \quad \Omega = I_{pq}^{-1} S^* I_{pq} S.\end{aligned} \tag{2.2}$$

Remark 2.1. In a way similar to (2.1), we have

$$\Theta_t(\lambda) = [V(\lambda), \Theta(\lambda)].$$

Hence, by the uniqueness of solution of this ODE and that of (2.1), if Θ is a scalar for $x = x_0$, $t = t_0$, Θ is that scalar identically.

From now on, we call a matrix to be a scalar if it is a scalar multiple of an identity matrix.

Comparing the coefficients of λ in (2.1), we have

$$[U_{j+2}, \Gamma] + [U_{j+1}, \Delta] + [U_j, \Omega] = \Gamma_x \delta_{j,m-2} + \Delta_x \delta_{j,m-1} + \Omega_x \delta_{j,m} \tag{2.3}$$

with $U_j = 0$ for $j < 0$ or $j \geq m+1$.

Let F_r be the set of all $r \times r$ off-diagonal matrices, D_r be the set of all $r \times r$ diagonal matrices and D_r^0 be the set of all $r \times r$ diagonal matrices whose diagonal entries are mutually different. Suppose $J \in D_r^0$, then $\mathrm{ad}\, J : F_r \to F_r$ is an isomorphism.

Lemma 2.1. *Suppose $P, Q \in F_r$, $J_1, \cdots, J_j \in D_r^0$, $K_1, \cdots, K_k \in D_r$. Let*

$$L = (\mathrm{ad}\, J_1)^{-1} \cdots (\mathrm{ad}\, J_j)^{-1} (\mathrm{ad}\, K_1) \cdots (\mathrm{ad}\, K_k) : F_r \to F_r,$$

then

$$[P, LQ]^{\mathrm{diag}} + (-1)^{p+q} [Q, LP]^{\mathrm{diag}} = 0.$$

In particular,

$$[P, LP]^{\mathrm{diag}} = 0$$

when $j + k$ is even. Moreover, if $j + k$ is odd, J_i, K_i are constant matrices, P is a matrix function of x, then

$$[P, LP_x]^{\mathrm{diag}} = (P \cdot LP)_x^{\mathrm{diag}}.$$

Here the superscripts "diag" and "off" refer to the diagonal and off-diagonal parts of a matrix respectively.

Lemma 2.2. *Suppose $P \in F_r$, $J \in D_r^0$, $K \in D_r$, then*

$$[P, (\mathrm{ad}\, J)^{-1} [(\mathrm{ad}\, J)^{-1}\ \mathrm{ad}\, K(P), P]]^{\mathrm{diag}} = 0.$$

Proof. Both lemmas are derived by direct computation. Note that by Lemma 2.1, $[(\mathrm{ad}\, J)^{-1}\ \mathrm{ad}\, K(P), P]$ is always off-diagonal in Lemma 2.2.

Lemma 2.3. *Suppose A is an $r \times r$ matrix, $[A, X]^{\mathrm{diag}} = 0$ for all $X \in \mathfrak{g}$, then A is a diagonal matrix.*

Lemma 2.4. *For Systems* (A), (B) *and* (C), *Γ, Δ, Ω are all scalars for generic $U(\lambda)$. These scalars are independent of x and t.*

Proof. Denote D to be the set of all constant diagonal matrices.

System (A)

(2.3) gives

$$[J,\Gamma]=0, \tag{2.4}$$

$$[U_1,\Gamma]+[J,\Delta]=\Gamma_x, \tag{2.5}$$

$$[U_1,\Delta]+[J,\Omega]=\Delta_x, \tag{2.6}$$

$$[U_1,\Omega]=\Omega_x. \tag{2.7}$$

$$\begin{aligned}
(2.4),(2.5)^{\text{diag}} &\Rightarrow \quad \Gamma=\gamma \quad (\gamma\in D),\\
(2.5)^{\text{off}} &\Rightarrow \quad \Delta^{\text{off}}=(\operatorname{ad}J)^{-1}\operatorname{ad}\gamma(U_1),\\
(2.6)^{\text{diag}} &\Rightarrow \quad \Delta^{\text{diag}}=\delta \quad (\delta\in D), \quad \text{(by Lemma 2.1)},\\
(2.6)^{\text{off}} &\Rightarrow \quad \Omega^{\text{off}}=(\operatorname{ad}J)^{-2}\operatorname{ad}\gamma(U_{1,x})+(\operatorname{ad}J)^{-1}[(\operatorname{ad}J)^{-1}\operatorname{ad}\gamma(U_1),U_1]\\
&\qquad +(\operatorname{ad}J)^{-1}\operatorname{ad}\delta(U_1),\\
(2.7)^{\text{diag}} &\Rightarrow \quad \Omega^{\text{diag}}=\big(U_1(\operatorname{ad}J)^{-2}\operatorname{ad}\gamma(U_1)\big)^{\text{diag}}+\omega, \quad (\omega\in D),\\
&\qquad \text{(by Lemma 2.1 and Lemma 2.2)}.
\end{aligned}$$

$(2.7)^{\text{off}}$ gives an ODE for U_1 with respect to x:

$$\begin{aligned}
&(\operatorname{ad}J)^{-2}\operatorname{ad}\gamma(U_{1,xx})+(\operatorname{ad}J)^{-1}[(\operatorname{ad}J)^{-1}\operatorname{ad}\gamma(U_1),U_1]_x+(\operatorname{ad}J)^{-1}\operatorname{ad}\delta(U_{1,x}),\\
=&[U_1,(U_1(\operatorname{ad}J)^{-2}\operatorname{ad}\gamma(U_1))^{\text{diag}}]+[U_1,\omega]+[U_1,(\operatorname{ad}J)^{-2}\operatorname{ad}\gamma(U_{1,x})]^{\text{off}}\\
&+[U_1,(\operatorname{ad}J)^{-1}[(\operatorname{ad}J)^{-1}\operatorname{ad}\gamma(U_1),U_1]]^{\text{off}}+[U_1,(\operatorname{ad}J)^{-1}\operatorname{ad}\delta(U_1)]^{\text{off}}.
\end{aligned}$$

The coefficient of $U_{1,xx}$ is zero only when γ is a scalar. Then, the equation becomes

$$(\operatorname{ad}J)^{-1}\operatorname{ad}\delta(U_{1,x})=[U_1,\omega]+[U_1,(\operatorname{ad}J)^{-1}\operatorname{ad}\delta(U_1)]^{\text{off}}.$$

The coefficient of $U_{1,x}$ is zero only when δ is a scalar. If so, $[U_1,\omega]=0$. Since U_1 is generic, ω is also a scalar.

System (B)

(2.3) gives

$$[J,\Gamma]=0,\quad [U_1,\Gamma]+[J,\Delta]=0,\quad [U_1,\Delta]+[J,\Omega]=\Gamma_x,$$
$$[U_1,\Omega]=\Delta_x,\quad \Omega_x=0.$$

In a way similar to the discussion for system (A), we can see that Δ is a scalar.

System (C)

(2.3) gives

$$[U_0,\Gamma]=0, \tag{2.8}$$

$$[U_0,\Delta]=\Gamma_x, \tag{2.9}$$

$$[U_0,\Omega]=\Delta_x, \tag{2.10}$$

$$\Omega_x=0. \tag{2.11}$$

(2.8) implies that U_0 and Γ can be diagonalized simultaneously. Suppose that $U_0(x)=g(x)\widetilde{U}_0(x)g^{-1}(x)$, $\widetilde{U}_0\in\mathfrak{h}$, $g\in U(p,q)$. Since the regular elements are dense in $\mathfrak{h}$ and U_0 is generic, we can suppose, without loss of generality, that $\widetilde{U}_0$ (or U_0) is a regular element. (Otherwise, the conclusion follows by a limit.) Since the eigenvalues of U_0 are purely imaginary, we can want $\operatorname{Im}(\widetilde{U}_0)_{1,1}<\operatorname{Im}(\widetilde{U}_0)_{2,2}<\cdots<\operatorname{Im}(\widetilde{U}_0)_{p+q,p+q}$. Moreover, if $g\widetilde{U}_0g^{-1} = g'\widetilde{U}_0g'^{-1}$, then $g' = g\sigma$ where σ is a diagonal matrix whose diagonal entries are of norm one. Clearly, $(g'^{-1}g'_x)^{\rm diag} = \sigma^{-1}(g^{-1}g_x + \sigma_x\sigma^{-1})^{\rm diag}\sigma = 0$ if and only if $\sigma_x\sigma^{-1} = -(g^{-1}g_x)^{\rm diag} \in \mathfrak{h}$. This is always solvable for σ with $|\sigma| = 1$. Hence we can want $(g^{-1}g_x)^{\rm diag} = 0$. Consequently, for regular $\widetilde{U}_0$,

$$U_0 \to \left(\widetilde{U}_0, g_0 = g(0), X(x) = g^{-1}g_x\right)$$

is a 1-1 correspondence for $\mathrm{Im}(\widetilde{U}_0)_{1,1} < \mathrm{Im}(\widetilde{U}_0)_{2,2} < \cdots < \mathrm{Im}(\widetilde{U}_0)_{p+q,p+q}$, $g_0 \in U(p,q)$, $X(x) \in u(p,q)$, $X^{\rm diag} = 0$. Using this fact, we can consider $(\widetilde{U}_0, g(0), g^{-1}g_x)$ instead of U_0.

Let $\Gamma = g\widetilde{\Gamma}g^{-1}$, $\Delta = g\widetilde{\Delta}g^{-1}$, $\Omega = g\widetilde{\Omega}g^{-1}$, (2.8)–(2.11) become

$$[\widetilde{U}_0, \widetilde{\Gamma}] = 0, \tag{2.12}$$

$$[\widetilde{U}_0, \widetilde{\Delta}] = [g^{-1}g_x, \widetilde{\Gamma}] + \widetilde{\Gamma}_x, \tag{2.13}$$

$$[\widetilde{U}_0, \widetilde{\Omega}] = [g^{-1}g_x, \widetilde{\Delta}] + \widetilde{\Delta}_x, \tag{2.14}$$

$$[g^{-1}g_x, \widetilde{\Omega}] + \widetilde{\Omega}_x = 0. \tag{2.15}$$

$$\begin{aligned}
(2.12) \quad &\Rightarrow \quad \widetilde{\Gamma} \text{ is diagonal},\\
(2.13)^{\rm diag} \quad &\Rightarrow \quad \widetilde{\Gamma} = \gamma, \quad (\gamma \in D),\\
(2.13)^{\rm off} \quad &\Rightarrow \quad \widetilde{\Delta}^{\rm off} = -(\mathrm{ad}\,\widetilde{U}_0)^{-1}\ \mathrm{ad}\,\gamma(g^{-1}g_x),\\
(2.14)^{\rm diag} \quad &\Rightarrow \quad \widetilde{\Delta}^{\rm diag} = \delta \quad (\delta \in D), \quad \text{(by Lemma 2.1)},\\
(2.14)^{\rm off} \quad &\Rightarrow \quad \widetilde{\Omega}^{\rm off} = -(\mathrm{ad}\,\widetilde{U}_0)^{-1}\ \mathrm{ad}\,\delta(g^{-1}g_x)\\
&\qquad - (\mathrm{ad}\,\widetilde{U}_0)^{-1}[g^{-1}g_x, (\mathrm{ad}\,\widetilde{U}_0)^{-1}\ \mathrm{ad}\,\gamma(g^{-1}g_x)]^{\rm off}\\
&\qquad - (\mathrm{ad}\,\widetilde{U}_0)^{-1}((\mathrm{ad}\,\widetilde{U}_0)^{-1}\ \mathrm{ad}\,\gamma(g^{-1}g_x))_x,\\
(2.15)^{\rm diag} \quad &\Rightarrow \quad \Omega^{\rm diag} = -((\mathrm{ad}\,\widetilde{U}_0)^{-1}(g^{-1}g_x)(\mathrm{ad}\,\widetilde{U}_0)^{-1}\ \mathrm{ad}\,\gamma(g^{-1}g_x))^{\rm diag} + \omega,\\
&\qquad (\omega \in D), \quad \text{(by Lemma 2.1 and Lemma 2.2)}.
\end{aligned}$$

$(2.15)^{\rm off}$ gives

$$\widetilde{\Omega}_x^{\rm off} + [g^{-1}g_x, \widetilde{\Omega}^{\rm off}]^{\rm off} + [g^{-1}g_x, \widetilde{\Omega}^{\rm diag}] = 0. \tag{2.16}$$

This is an equation of unknowns $\widetilde{U}_0$ and $g^{-1}g_x$. The only term containing $(g^{-1}g_x)_{xx}$ is

$$-(\mathrm{ad}\,\widetilde{U}_0)^{-2}\ \mathrm{ad}\,\gamma(g^{-1}g_x)_{xx},$$

which is zero only when γ is a scalar. Then (2.16) becomes

$$-((\mathrm{ad}\,\widetilde{U}_0)^{-1}\ \mathrm{ad}\,\delta(g^{-1}g_x))_x - [g^{-1}g_x, (\mathrm{ad}\,\widetilde{U}_0)^{-1}\ \mathrm{ad}\,\delta(g^{-1}g_x)]^{\rm off} + [g^{-1}g_x, \omega] = 0.$$

The term concerning $(g^{-1}g_x)_x$ vanishes only when δ is a scalar. If so, $[g^{-1}g_x, \omega] = 0$. Hence ω is a scalar for generic U_0. Therefore, Γ, Δ, Ω are all scalars.

Till now, we have proved that Γ. Δ, Ω are all scalars for Systems (A), (B) and (C). By (2.1) and (2.2), these scalars are real and independent of x. By Remark 2.1, they are also independent of t. The lemma is proved.

Proof of Theorem 1.3. From Lemma 2.4, Γ, Δ, Ω are real scalars for generic $U(\lambda)$. It is easy to show from (2.2) that $R^{-1}S = H\Lambda H^{-1}$ satisfies

$$\Gamma(R^{-1}S)^2 + \Delta(R^{-1}S) + \Omega = 0,$$

i.e.

$$\Gamma\Lambda^2 + \Delta\Lambda + \Omega = 0.$$

Hence, λ_i can only take two mutually conjugate values, say μ and $\bar{\mu}$. Since $\widetilde{U}(\lambda) \not\equiv U(\lambda)$, $\operatorname{Im}\mu \neq 0$. From (2.2),

$$\Delta/\Gamma \cdot W = -\Lambda^* W - W\Lambda, \qquad \Omega/\Gamma \cdot W = \Lambda^* W \Lambda,$$

where $W = H^* I_{pq} H$. It is easy to show that $\Delta = -\Gamma(\mu + \bar{\mu})$, $\Omega = \Gamma|\mu|^2$ and $W_{ij} = 0$ if $\lambda_i \neq \lambda_j$. The theorem is proved.

§3. General Choice of Parameters for $su(p,q)$ and $so(p,q)$ Reduction

(1) $su(p,q)$ reduction

Suppose $U(\lambda) \in L(su(p,q))$. From (1.7) and (1.9),

$$\begin{aligned}\operatorname{tr}\widetilde{U}(\lambda) &= \operatorname{tr}((\lambda R_x - S_x)(\lambda R - S)^{-1}) = \frac{d}{dx}\ln\det(\lambda R - S)\\ &= \frac{d}{dx}\ln\det R + \frac{d}{dx}\ln\det(\lambda - R^{-1}S)\\ &= \frac{d}{dx}\ln\det R - \operatorname{tr}((R^{-1}S)_x(\lambda - R^{-1}S)) = \frac{d}{dx}\ln\det R.\end{aligned}$$

Hence, to keep $su(p,q)$ reduction, an additional condition that $\det R$ is a constant is necessary and sufficient.

(2) $so(p,q)$ reduction ($p+q$ is even)

An important example using $so(p,q)$ reduction is the $so(p,q)$ principal chiral field.[6] $so(p,q)$ consists of all real matrices in $u(p,q)$. The construction of Darboux transformation is still valid if we can make $\widetilde{U}(\lambda) \in L(so(p,q))$. From (1.9), $R^{-1}S$ is real everywhere if it is real at one point. Since $H\Lambda H^{-1}$ is real and its eigenvalues are non-real, $p+q$ must be even. In this case, we can always take a real initial $R^{-1}S$. The integrability of (1.9) implies that $R^{-1}S$ is always real and $\widetilde{U}(\lambda) \in L(so(p,q))$.

§4. General Choice of Parameters for Twisted $L(su(p,q))$ Reduction with Involution $X \mapsto -I_{pq}^{-1}X^T I_{pq}$

Let σ be an involution of $\mathfrak{g}$, i.e., σ is an isomorphism on $\mathfrak{g}$ with $\sigma^2 = 1$. Let

$$L_\sigma(\mathfrak{g}) = \{\, U(\lambda) \in L(\mathfrak{g}) \,|\, \sigma(U(\lambda)) = U(-\lambda)\,\}$$

be the twisted algebra of $L(\mathfrak{g})$.

Now suppose $U(\lambda) \in L(\mathfrak{g}) = L(su(p,q))$ and $\sigma : \mathfrak{g} \to \mathfrak{g}$, $X \mapsto -I_{pq}^{-1}X^T I_{pq}$. $U \in L_\sigma(u(p,q))$ is equivalent to $\overline{U(\lambda)} = U(-\lambda)$ for real λ. Written in terms of U_j, these conditions are $U_j^T = (-1)^{m-j+1} I_{pq} U_j I_{pq}^{-1}$, $\overline{U_j} = (-1)^{m-j} U_j$.

If we want $\widetilde{U}(\lambda) \in L_\sigma(u(p,q))$, then (1.7) gives

$$\Pi_x(\lambda) = [U(\lambda), \Pi(\lambda)],$$

where

$$\Pi(\lambda) = I_{pq}^{-1}(-\lambda R - S)^T I_{pq}(\lambda R - S) = \lambda^2\Gamma_1 + \lambda\Delta_1 + \Omega_1,$$

$$\Gamma_1 = -I_{pq}^{-1}R^T I_{pq} R, \qquad \Delta_1 = -I_{pq}^{-1}S^T I_{pq} R + I_{pq}^{-1}R^T I_{pq} S, \qquad \Omega_1 = I_{pq}^{-1}S^T I_{pq} S. \quad (4.1)$$

Similar to Lemma 2.4, for Systems (A), (B) and (C), Γ_1, Δ_1 and Ω_1 are scalars for generic $U(\lambda)$. We have the following.

Theorem 4.1. *Suppose $U(\lambda) \in L_\sigma(u(p,q))$. Let $\mu \in \sqrt{-1}\mathbf{R}$, $\lambda_i = \mu$ or $\bar{\mu}$. Let H be given by Theorem 1.1 and satisfy $(H^* I_{pq} H)_{ij} = 0$, $(H^{-1}\bar{H})_{ij} = 0$ for $\lambda_i \neq \lambda_j$. Then after the Darboux transformation $\lambda R - S$, where R is a real scalar multiple of an orthogonal matrix, $\widetilde{U}(\lambda) \in L_\sigma(u(p,q))$. Conversely, for generic $U(\lambda)$, if $\widetilde{U}(\lambda) \not\equiv U(\lambda)$ is given by a Darboux transformation of degree one and $\widetilde{U}(\lambda) \in L_\sigma(u(p,q))$, then that Darboux matrix should be $e^{i\theta}(\lambda R - S)$, where θ is a real constant and R, S satisfy the above conditions.*

Proof. By (4.1), R is a scalar multiple of an orthogonal matrix. Comparing (4.1) with (2.2), we have

$$\bar{R} = e^{i\theta} R, \qquad \bar{S} = -e^{i\theta} S,$$

where $e^{i\theta} = -\bar{\Gamma}_1/\Gamma$ whose norm should be 1. From (1.8), a constant multiple scalar on R does not affect the result of $\widetilde{U}(\lambda)$, $\widetilde{V}(\lambda)$, and we can choose $e^{i\theta} = 1$. This implies

$$\overline{H\Lambda H^{-1}} = -H\Lambda H^{-1},$$

i.e.

$$H^{-1}\bar{H}\bar{\Lambda} + \Lambda H^{-1}\bar{H} = 0.$$

Since $\widetilde{U}(\lambda) \neq U(\lambda)$, $\mu \neq 0$. It is easy to show that $\bar{\mu} = -\mu$, $(H^{-1}\bar{H})_{ij} = 0$ if $\lambda_i \neq \lambda_j$. This proves the necessity of the restrictions.

Conversely, take $\mu \in \sqrt{-1}\mathbf{R}$, $\lambda = \mu$ or $\bar{\mu}$ and solve (1.6), then

$$(H^{-1}\bar{H})_x = \sum_{i=0}^{m} H^{-1} U_i H [H^{-1}\bar{H}, \Lambda^{m-i}],$$

$$(H^{-1}\bar{H})_t = \sum_{j=0}^{n} H^{-1} V_j H [H^{-1}\bar{H}, \Lambda^{n-j}].$$

Hence $[H^{-1}\bar{H}, \Lambda] = 0$ identically if it holds at one point. This means that we can always want $(H^{-1}\bar{H})_{ij} = 0$ if $\lambda_i \neq \lambda_j$. Reversing the discussion on the necessity, we know $\widetilde{U}(\lambda) \in L_\sigma(u(p,q))$. The theorem is proved.

A famous example of this system is the MKdV hierarchy, whose $p = 2$, $q = 0$, $m = 1$, $U(\lambda) = \lambda J + U_1(x,t)$, $J = \begin{pmatrix} i & \\ & -i \end{pmatrix}$, $\bar{U}_1 = U_1$, $U_1^T = -U_1$.

Another example is the $so(n)$ n-wave equation, whose $p = n$, $q = 0$, $m = 1$, $U(\lambda) = \lambda J + U_1(x,t)$, $J = \operatorname{diag}(J_1, \cdots, J_n)$, $\widetilde{U}_1 = U_1$, $U_1^T = -U_1$.

§5. A Remark on the Non-Generic Cases

Let $\mathfrak{g} = u(N)$ $(N \geq 4)$, $2 \leq l \leq N-2$,

$$\mathfrak{g}_1 = \{ X \in U(N) \mid X_{ij} = 0 \text{ for } i \geq l+1 \text{ or } j \geq l+1 \} \cong u(l),$$

$$\mathfrak{g}_2 = \{ X \in U(N) \mid X_{ij} = 0 \text{ for } i \leq l \text{ or } j \leq l \} \cong u(N-l),$$

$\mathfrak{K} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

Suppose $U(\lambda) \in \mathfrak{K}$ for real λ. Let $R = I$,

$$\Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_N) \equiv \operatorname{diag}(\underbrace{\mu, \cdots, \mu}_{p}, \underbrace{\bar{\mu}, \cdots, \bar{\mu}}_{l-p}, \underbrace{\nu, \cdots, \nu}_{q}, \underbrace{\bar{\nu}, \cdots, \bar{\nu}}_{N-l-q})$$

with μ, $\bar{\mu}$, ν, $\bar{\nu}$ mutually different, $1 \le p \le l-1$, $1 \le q \le N-l-1$. Let h_i be a solution of (1.2) such that

(1) the j-th entry of h_i is zero for $j \ge l+1$ if $i \le l$ and for $j \le l$ if $i \ge l+1$;

(2) $h_i^* h_j = 0$ for $\lambda_i \ne \lambda_j$ with $i, j \le l$ or $\lambda_i \ne \lambda_j$ with $i, j \ge l+1$.

Then the Darboux transformation is also divided into two blocks according to the decomposition of $\mathfrak{K} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. It is clear that $\widetilde{U}(\lambda) \in \mathfrak{K}$ for all $\lambda \in \mathbf{R}$.

This example shows that in some reduced cases, the spectral parameters may take more than two values. The corresponding Δ, Ω in (2.1) are diagonal but not scalar.

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