NA 565 - Fall 2023

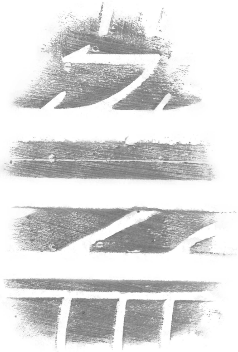
Vehicle Dynamics & Control

September 27, 2023

UNIVERSITY OF MICHIGAN

Tire-Contact Patch

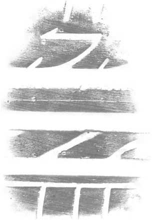
Tire-contact patch of the Volkswagen Golf GTI.



Tire-Contact Patch

- All that is available to control a passenger vehicle through its environment safely are the forces transmitted through the contact patches between the four tires and the road.

- Each of these is no larger than the size of a hand.



https://www.youtube.com/watch?v=3x3SqeSdrAE

Trajectory planning for automated vehicles is still challenging, as it is computationally complex, while highly dynamic environments require fast optimization for real-time applications.

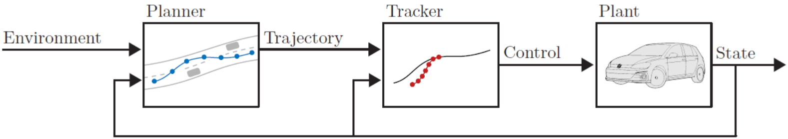
Vehicle Motion Planning and Control

1. Route planner, including graph search (e.g., A^*), Sampling-based (e.g., RRT), Optimization (vehicle routing via linear programming).
2. Trajectory optimization (often nonlinear programming).
3. Tracking controller (MPC).

1. Route planner, including graph search (e.g., A*), Sampling-based (e.g., RRT), Optimization (vehicle routing via linear programming).

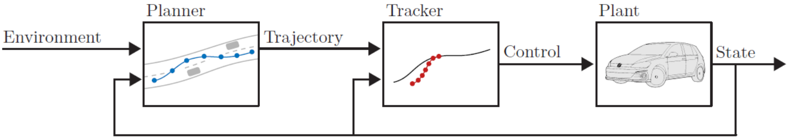
At this level, the planner can be agnostic of the vehicle dynamics. The algorithms can work on a variety of vehicles with no to minimal changes, e.g., Google Maps Platform.

2. Trajectory optimization (often nonlinear programming).
3. Tracking controller (MPC).



2 Trajectory optimization (often nonlinear programming).

Offline or Online?



2 Trajectory optimization.

- Given start and end points, this level is responsible for generating reference trajectories that are kinematically and dynamically (kinodynamic) feasible for the vehicle to track.
- High-fidelity models are often complicated and lead to nonlinear nonconvex problems.
- Solving nonlinear nonconvex problems is difficult both in terms of runtime and finding a satisfactory solution (can get stuck in local minima).

Let the state be $s := [x, y, \psi, v]^\mathsf{T}$ where (x, y) is the position, ψ is the yaw angle, and v is the velocity. The vehicle is controlled by the steering angle of the front wheel δ and the acceleration a.

The equations of motion of the vehicle are given by

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \psi \\ v \end{bmatrix} = \begin{bmatrix} v\cos(\psi + \beta) \\ v\sin(\psi + \beta) \\ \frac{v}{L_r}\sin\beta \\ a \end{bmatrix}, \text{with } \beta := \arctan(\frac{L_r}{L_r + L_f}\arctan\delta),$$

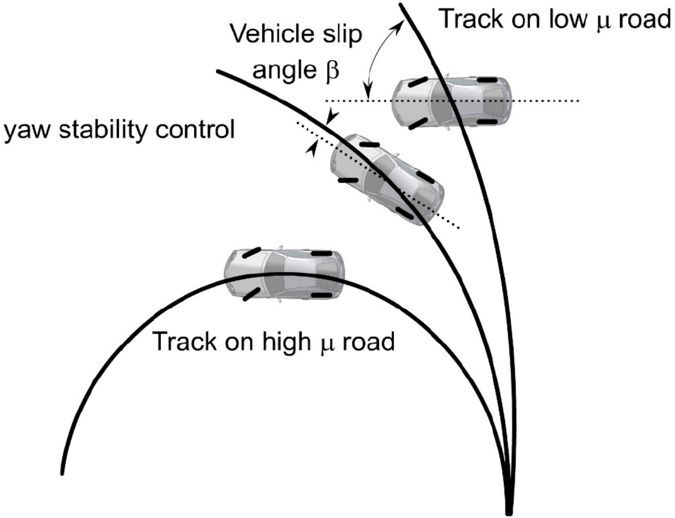
where L_r and L_f are the distance from the rear or front axes to the center of the vehicle (given data).

What's missing!? The real world has dynamics.

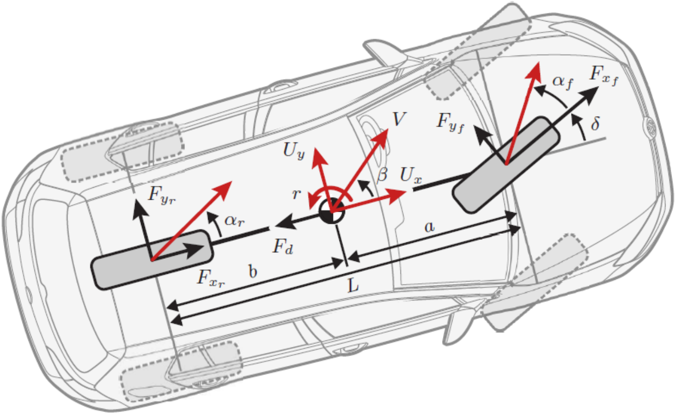
$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \psi \\ v \end{bmatrix} = \begin{bmatrix} v\cos(\psi+\beta) \\ v\sin(\psi+\beta) \\ \frac{v}{L_r}\sin\beta \\ a \end{bmatrix}, \text{with } \beta := \arctan(\frac{L_r}{L_r+L_f}\arctan\delta),$$

- https://www.youtube.com/watch?v=RqajKat0v-4
- https://www.youtube.com/watch?v=Aup4W1s1otk
- https://www.youtube.com/watch?v=tsnYqCRWTbE

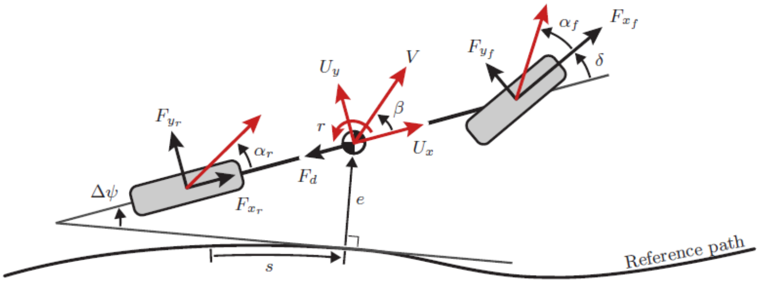
Vehicle Slip Angle



Vehicle Modeling: Dynamic Single-Track Vehicle Model

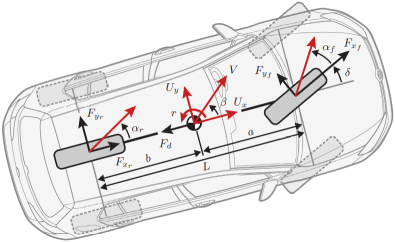


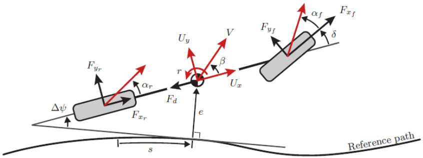
Vehicle Modeling: Dynamic Single-Track Vehicle Model



Vehicle Modeling: Dynamic Single-Track Vehicle Model

- Longitudinal speed U_x
- Lateral speed U_y
- Yaw rate r
- Total velocity V
- Sideslip angle β
- Longitudinal force on the front and rear tire F_{x_f}, F_{x_r}
- Lateral axle forces F_{y_f}, F_{y_r}
- Steering angle δ
- Disturbance force F_d
- Mass m, inertial I_{zz}
- distance along a reference path s
- Lateral distance to the path e
- Heading error $\Delta\psi$



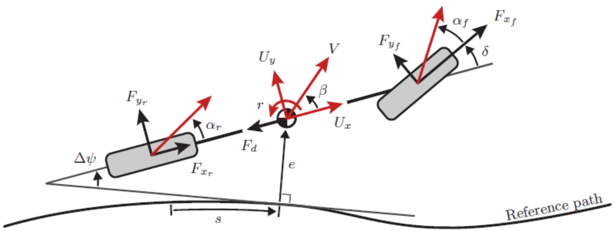


Vehicle Modeling: Dynamic Single-Track Vehicle Model

Equation of motion (recall "$F = ma$ and $\tau = I\alpha$"):

$$
\begin{aligned}
\dot{U}_x &= \frac{-F_{y_f}\sin\delta + F_{x_f}\cos\delta + F_{x_r} - F_d}{m} + rU_y \\
\dot{U}_y &= \frac{F_{y_f}\cos\delta + F_{x_f}\sin\delta + F_{y_r}}{m} - rU_x \\
\dot{r} &= \frac{a(F_{y_f}\cos\delta + F_{x_f}\sin\delta) - bF_{y_r}}{I_{zz}} \\
\dot{s} &= \frac{U_x\cos\Delta\psi - U_y\sin\Delta\psi}{1 - \kappa e} \\
\dot{e} &= U_x\sin\Delta\psi + U_y\cos\Delta\psi \\
\Delta\dot{\psi} &= r - \kappa\dot{s}
\end{aligned}
$$

Vehicle Modeling: Dynamic Single-Track Vehicle Model



$$\dot{U}_x = \frac{-F_{y_f}\sin\delta + F_{x_f}\cos\delta + F_{x_r} - F_d}{m} + rU_y$$

$$\dot{U}_y = \frac{F_{y_f}\cos\delta + F_{x_f}\sin\delta + F_{y_r}}{m} - rU_x$$

$$\dot{r} = \frac{a(F_{y_f}\cos\delta + F_{x_f}\sin\delta) - bF_{y_r}}{I_{zz}}$$

$$\dot{s} = \frac{U_x\cos\Delta\psi - U_y\sin\Delta\psi}{1-\kappa e}$$

$$\dot{e} = U_x\sin\Delta\psi + U_y\cos\Delta\psi, \quad \Delta\dot{\psi} = r - \kappa\dot{s}$$

Remark

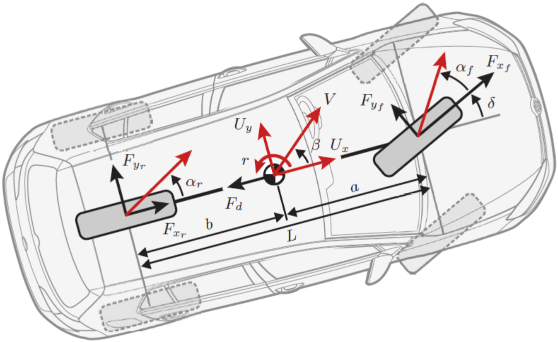
The controlled inputs are steering angle δ and longitudinal (tractive) force F_x. The distribution of F_x on the front and rear wheels, i.e., F_{x_f} and F_{x_r}, are vehicle-dependent, e.g., FWD, RWD, AWD, or 4WD.

- Tire forces and moments are highly nonlinear and difficult to model.
- Common empirical models are the Brush or Pacejka "Magic Formula" models.

Brush-Tire Model for Pure Lateral Forces

This model derives the lateral tire force from the lateral deflection and sliding of the rubber elements in a tire.
Slip angle is the angle between the tire's heading and its direction of travel.

$$\alpha_f = \arctan\left(\frac{U_y + ar}{U_x}\right) - \delta, \quad \alpha_r = \arctan\left(\frac{U_y - br}{U_x}\right)$$



Brush-Tire Model for Pure Lateral Forces

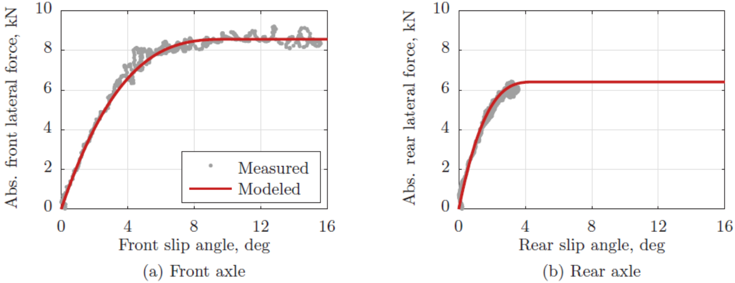
Pure lateral tire force as a function of slip angle α, cornering stiffness C_α, normal load F_z, and tire-road friction coefficient μ:

$$F_y = \begin{cases} -C_\alpha \tan\alpha + \frac{C_\alpha^2}{3\mu F_z}|\tan\alpha|\tan\alpha - \frac{C_\alpha^3}{27\mu^2 F_z^2}\tan^3\alpha & \text{if } |\alpha| < \alpha_{sl} \\ -\mu F_z \operatorname{sgn}\alpha & \text{otherwise} \end{cases}$$

where α_{sl} is the peak slip angle at which total sliding of the tire-contact patch occurs:

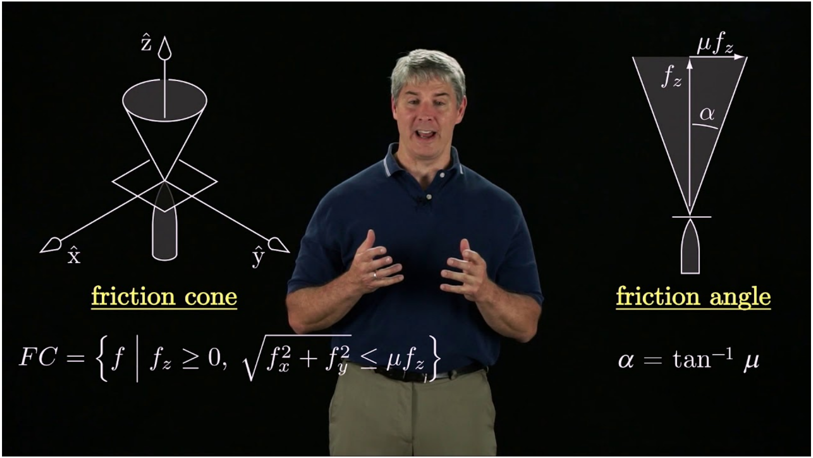
$$\alpha_{sl} = \arctan\left(\frac{3\mu F_z}{C_\alpha}\right)$$

Brush-Tire Model for Pure Lateral Forces

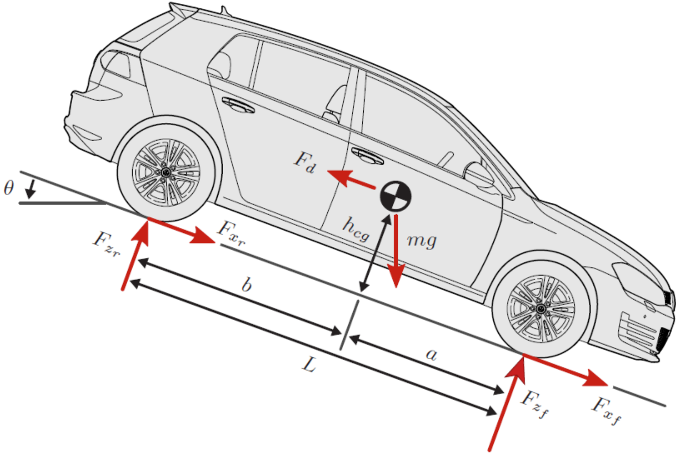


Measured and modeled lateral forces at (a) the front axle and (b) the rear axle (b) in a ramp-steer maneuver with the Audi TTS research vehicle

Friction


$\hat{z}$
$\hat{x}$
$\hat{y}$
friction cone
$FC = \left\{ f \;\middle|\; f_z \geq 0,\ \sqrt{f_x^2 + f_y^2} \leq \mu f_z \right\}$
μf_z
f_z
α
friction angle
$\alpha = \tan^{-1} \mu$

Drag Forces and the Effect of Road Topography

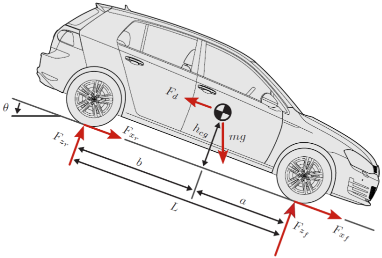


Drag Forces and the Effect of Road Topography

- Normal loads at the axles F_{z_f}, F_{z_r}
- Total commanded longitudinal force F_x
- Distance of the center of gravity h_{cg}
- Road grade (vehicle pitch) θ
- Road bank (vehicle roll) ϕ
- A_{V^2} captures the speed effect on the vehicle's total normal load

$$A_{V^2} = -\frac{d\theta}{ds}\cos\varphi - \kappa\sin\varphi\cos\theta$$

- Rolling resistance F_{rr}
- Aerodynamic drag $F_{aero} = C_D U_x^2$
- Aerodynamic drag coefficient C_D



Drag Forces and the Effect of Road Topography

$$F_{z_f} = \frac{b}{L} m \left(g \cos\theta \cos\varphi + A_{V^2} U_x^2 \right) - \frac{h_{cg}}{L} F_x$$

$$F_{z_r} = \frac{a}{L} m \left(g \cos\theta \cos\varphi + A_{V^2} U_x^2 \right) + \frac{h_{cg}}{L} F_x$$

$$\dot{U}_y = \frac{F_{y_f} \cos\delta + F_{x_f} \sin\delta + F_{y_r} + F_l}{m} - rU_x$$

$$\begin{bmatrix} F_{g_x} \\ F_{g_y} \\ F_{g_z} \end{bmatrix} = {}^{P}R^{N} \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix}$$

$$F_l = F_{g_y} = -mg \cos\theta \sin\varphi$$

$$F_d = F_{rr} + F_{aero} + F_{g_x} = F_{rr} + C_D U_x^2 - mg \sin\theta$$

Nonlinear MPC

minimize	Objective
subject to	State limits
	Input limits
	Tire model
	Initial state
	Dynamics
	Friction limits

Next Time

- CasADi; supports automatic differentiation.
- Nonlinear MPC Examples in Python with CasADi

References and Further Reading

- V. A. Laurense, Integrated motion planning and control for automated vehicles up to the limits of handling. Stanford University, 2019. (and references therein).
- Rajamani, R., 2011. Vehicle dynamics and control. Springer Science & Business Media.