Divide-and-Conquer: Quick Sort

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Outline

Quick Sort

- comparison based algorithm
- running time: $O(n \log n)$ (on average)
- efficient in practice

Example: quick sort

6	4	8	2	9	3	9	4	7	6	1

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partition with respect to $x = A[1]$

in particular, x is in its final position

1	4	2	3	4	6	6	9	7	8	9

≤ 6 $\qquad\qquad$ > 6

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in particular, x is in its final position

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sort the two parts recursively

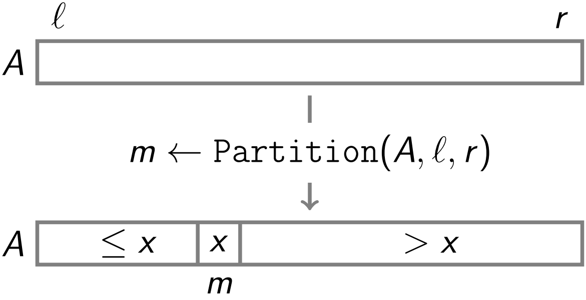
1	2	3	4	4	6	6	7	8	9	9

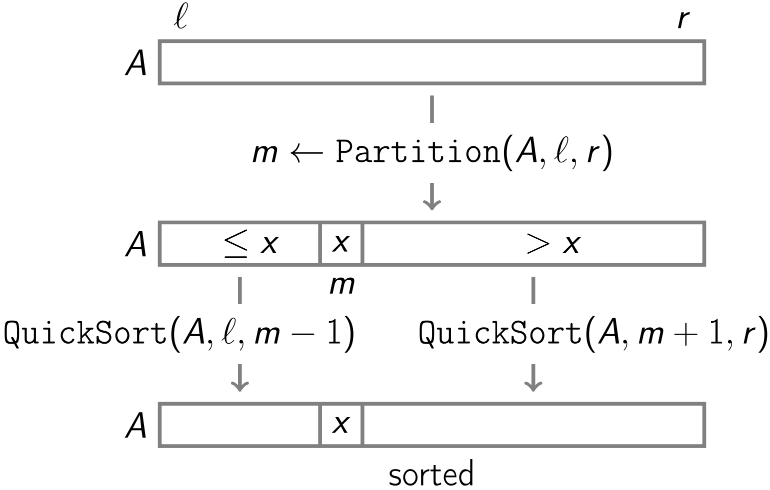
Outline

QuickSort(A, ℓ, r)

```
if ℓ ≥ r:
  return
m ← Partition(A, ℓ, r)
{A[m] is in the final position}
QuickSort(A, ℓ, m − 1)
QuickSort(A, m + 1, r)
```


ℓ
r
A


ℓ
r
A
$m \leftarrow \texttt{Partition}(A, \ell, r)$
A
$\leq x$
x
$> x$
m


ℓ
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$m \leftarrow \texttt{Partition}(A, \ell, r)$
A
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x
$> x$
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$\texttt{QuickSort}(A, \ell, m-1)$
$\texttt{QuickSort}(A, m+1, r)$
A
x
sorted

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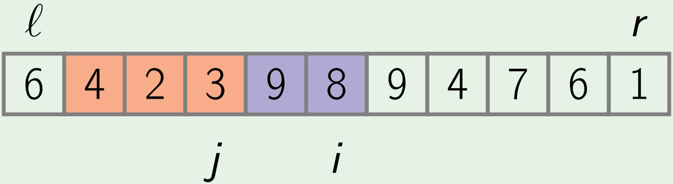

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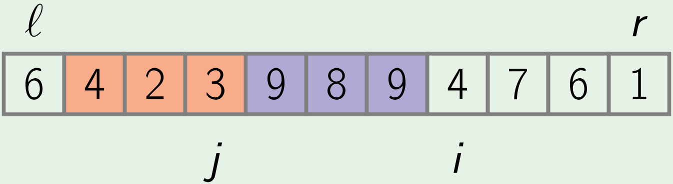

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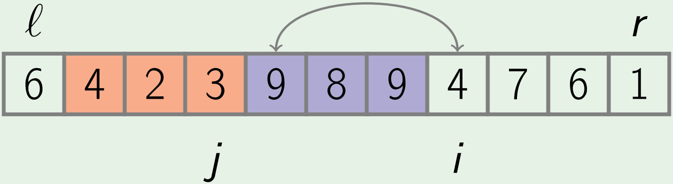

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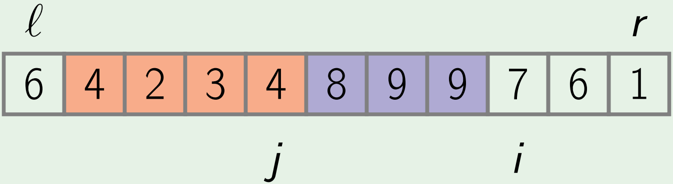

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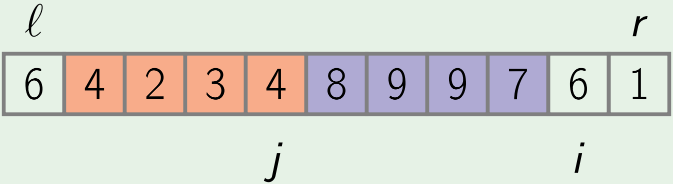

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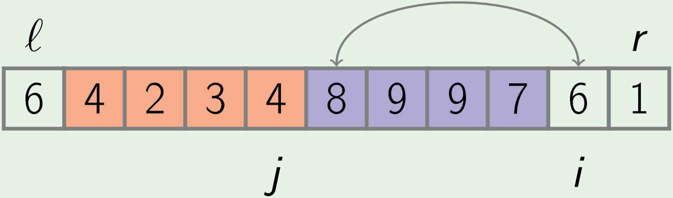

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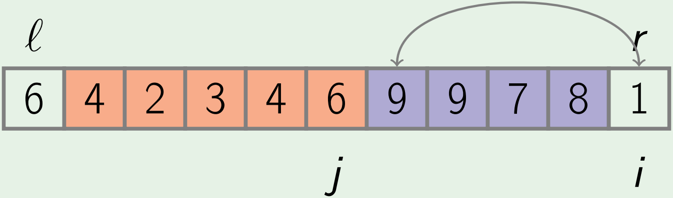

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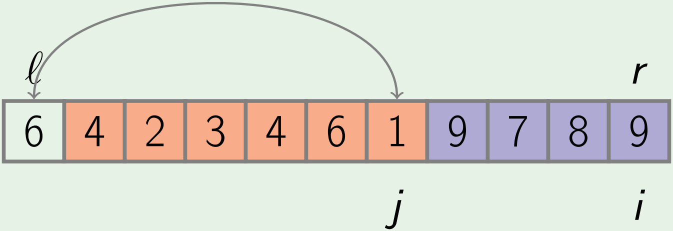

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Partition(A, ℓ, r)

$x \leftarrow A[\ell]$ {pivot}
$j \leftarrow \ell$
for i from $\ell + 1$ to r:
    if $A[i] \leq x$:
        $j \leftarrow j + 1$
        swap $A[j]$ and $A[i]$
    $\{A[\ell + 1 \ldots j] \leq x,\ A[j + 1 \ldots i] > x\}$
swap $A[\ell]$ and $A[j]$
return j

Outline

Unbalanced Partitions

- $T(n) = n + T(n-1)$:

  $T(n) = n+(n-1)+(n-2)+\cdots = \Theta(n^2)$

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- $T(n) = n + T(n-5) + T(4)$:

  $T(n) \geq n + (n-5) + (n-10) + \cdots = \Theta(n^2)$

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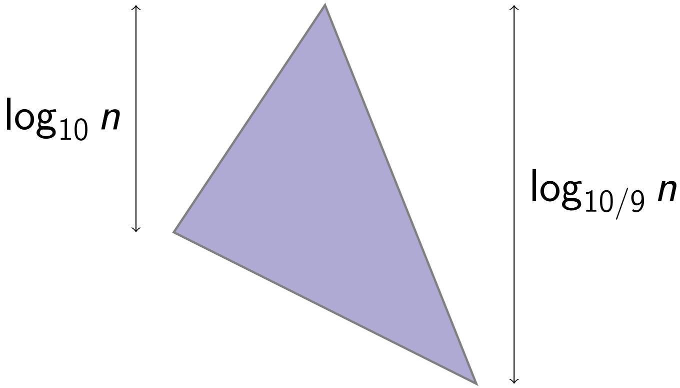
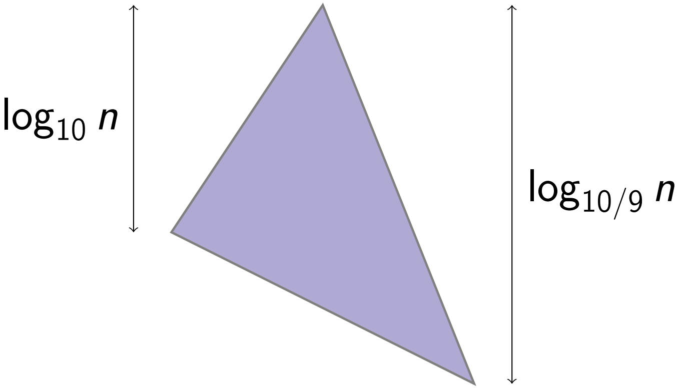


Balanced Partitions

$$T(n) = T(n/10) + T(9n/10) + O(n)$$



$$T(n) = O(n \log n)$$

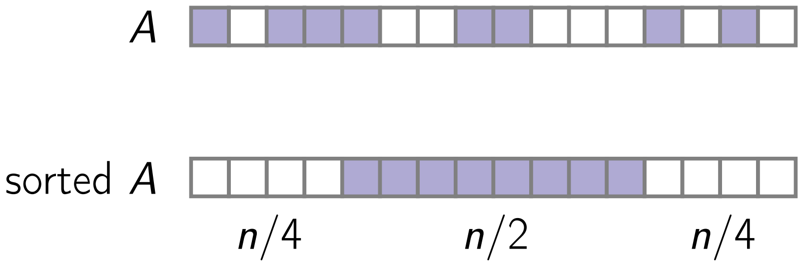
Random Pivot

RandomizedQuickSort(A, ℓ, r)

```
if ℓ ≥ r:
  return
k ← random number between ℓ and r
swap A[ℓ] and A[k]
m ← Partition(A, ℓ, r)
{A[m] is in the final position}
RandomizedQuickSort(A, ℓ, m − 1)
RandomizedQuickSort(A, m + 1, r)
```

Why Random?

half of the elements of A guarantees a balanced partition:



Theorem

Assume that all the elements of $A[1 \dots n]$ are pairwise different. Then the average running time of `RandomizedQuickSort`(A) is $O(n \log n)$ while the worst case running time is $O(n^2)$.

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Remark

Averaging is over random numbers used by the algorithm, but not over the inputs.

Outline

1. Overview
2. Algorithm
3. Random Pivot
4. Running Time Analysis
5. Equal Elements
6. Final Remarks

Proof Ideas: Comparisons

- the running time is proportional to the number of comparisons made

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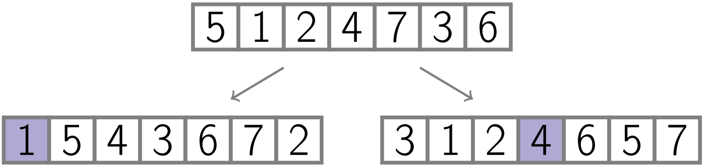
- balanced partition are better since they reduce the number of comparisons needed:



1 is min $\qquad$ $3, 1, 2 < 6, 5, 7$

Proof Ideas: Probability

A	5	1	8	9	2	4	7	3	6
A'	1	2	3	4	5	6	7	8	9

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$$\text{Prob}\,(3 \text{ and } 4 \text{ are compared}) =$$

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A'	1	2	3	4	5	6	7	8	9

$$\text{Prob}\left(1 \text{ and } 9 \text{ are compared}\right) = \frac{2}{9}$$

$$\text{Prob}\left(3 \text{ and } 4 \text{ are compared}\right) = 1$$

Proof

- let, for $i < j$,

$$\chi_{ij} = \begin{cases} 1 & A'[i] \text{ and } A'[j] \text{ are compared} \\ 0 & \text{otherwise} \end{cases}$$

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- for all $i < j$, $A'[i]$ and $A'[j]$ are either compared exactly once or not compared at all (as we compare with a pivot)
- this, in particular, implies that the worst case running time is $O(n^2)$

Proof (continued)

- crucial observation: $\chi_{ij} = 1$ iff the first selected pivot in $A'[i \ldots j]$ is $A'[i]$ or $A'[j]$

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- crucial observation: $\chi_{ij} = 1$ iff the first selected pivot in $A'[i \dots j]$ is $A'[i]$ or $A'[j]$
- then $\mathsf{Prob}(\chi_{ij}) = \frac{2}{j-i+1}$ and $\mathsf{E}(\chi_{ij}) = \frac{2}{j-i+1}$

Proof (continued)

Then (the expected value of) the running time is

$$\begin{aligned}
\mathrm{E}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\chi_{ij} &= \sum_{i=1}^{n}\sum_{j=i+1}^{n}\mathrm{E}(\chi_{ij}) \\
&= \sum_{i<j}\frac{2}{j-i+1} \\
&\le 2n\cdot\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right) \\
&= \Theta(n\log n)
\end{aligned}$$

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- the array is always split into two parts of size 0 and $n-1$
- $T(n) = n + T(n-1) + T(0)$ and hence $T(n) = \Theta(n^2)$!

To handle equal elements, we replace the line

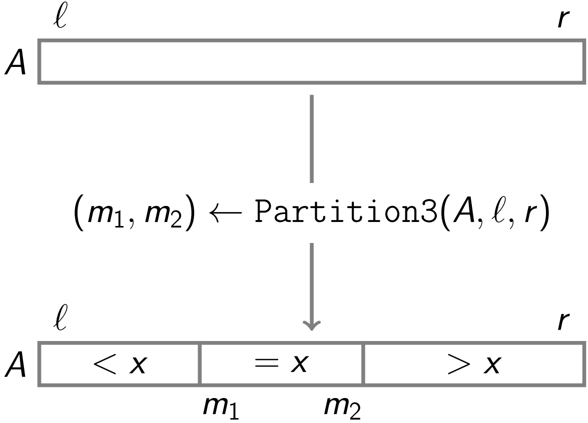
$$m \leftarrow \texttt{Partition}(A, \ell, r)$$

with the line

$$(m_1, m_2) \leftarrow \texttt{Partition3}(A, \ell, r)$$

such that

- for all $\ell \leq k \leq m_1 - 1$, $A[k] < x$
- for all $m_1 \leq k \leq m_2$, $A[k] = x$
- for all $m_2 + 1 \leq k \leq r$, $A[k] > x$


ℓ
r
A
(m1, m2) ← Partition3(A, ℓ, r)
ℓ
r
A
< x
= x
> x
m1
m2

$\texttt{RandomizedQuickSort}(A, \ell, r)$

```
if ℓ ≥ r:
  return
k ← random number between ℓ and r
swap A[ℓ] and A[k]
(m₁, m₂) ← Partition3(A, ℓ, r)
{A[m₁ ... m₂] is in final position}
RandomizedQuickSort(A, ℓ, m₁ − 1)
RandomizedQuickSort(A, m₂ + 1, r)
```

Outline

Tail Recursion Elimination

$\texttt{QuickSort}(A, \ell, r)$

```
while ℓ < r:
    m ← Partition(A, ℓ, r)
    QuickSort(A, ℓ, m − 1)
    ℓ ← m + 1
```

$\texttt{QuickSort}(A, \ell, r)$

```
while ℓ < r:
    m ← Partition(A, ℓ, r)
    if (m − ℓ) < (r − m):
        QuickSort(A, ℓ, m − 1)
        ℓ ← m + 1
    else:
        QuickSort(A, m + 1, r)
        r ← m − 1
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Worst-case space requirement: $O(\log n)$

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- if the recursion depth exceeds a certain threshold $c \log n$ the algorithm switches to heap sort
- the running time is $O(n \log n)$ in the worst case

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