Weekly Homework 1

Yu Zhao
Algebraic geometry

February 16, 2016

- 1.
  - (Define the presheaf $\mathcal{F}$). We define the presheaf

    $$\mathcal{F}(U) = \varinjlim_{\substack{V \in \mathcal{B} \\ V \subseteq U}} F(U)$$

    .

    And the functor is defined naturally by the property of limit. Notice for U in the base $\mathcal{F}(U) = F(U)$.
  - (Axiom1, Locality) U_i is an open covering of an open set U, s.t $\mathcal{F}(U)$ are such that $s|U_i = t|U_i$ for each set U_i of the covering, then we prove $s = t$;
    1. (reduce to $U \in \mathcal{B}$). $s = t \Leftrightarrow s|V = t|V$ for all $V \in \mathcal{B}$. And the locality condition also holds foe $s|V, t|V$.Hence we can reduce U to a set in the base.
    2. (reduce U_i to base) let W_{ij} be a base cover of U_i, then W_{ij} also forms a cover of U, and the locality condition holds for this cover.Hence we can reduce U_{ij} to a set in the base.
    3. (proof) Obvious, by finding a base cover of U_{ij} and using the exact sequence.
  - (Axiom 2, gluing lemma) If U_i is an open covering of an open set U, and if for each i a section $s_i \in \mathcal{F}(U_i)$ s.t $s_i|U_i \cap U_j = s_j|U_i \cap U_j$, then we prove there is a section $s \in \mathcal{F}(U)$ such that $s|U_i = s_i$ for each i.
    1. (reduce U to base). The existence of s is unique by axiom 1. Hence we prove that for every V in base, there is a unique section $s_V \in \mathcal{F}(V)$ such that $s_V|V \cap U_i = s_i|V \cap U_i$ for each i. Then for $W \subset V$, $s_V|W = s_W$. Hence there is a section S of U s.t. $S_V = s_v$ by the property of limit. And it's easy to verify that S satisfy the condition. Hence we reduce U in the base.
    2. (reduce U_i to base) let W_{ij} be a base cover of U_i and $s_{ij} = s_i|W_{ij}$. Then s_{ij} also satisfy gluing condition. And for s glue them together, s also glue s_i by axiom 1. Hence we reduce U_i to the base.
    3. (Proof) Trivial, after finding a base cover W_{ijk} of $U_i \cap U_j$ and using the exact sequence.

- (the morphisms of sheaves are induced by the compatible morphisms on the base). This part is obvious, just using the property of limit.
- (Remark). Another proof is based on the stalks. But it's not necessary since this lemma can be generalized with just the notion of open cover to more objects such as etale topology.

- 2.Let A be a local domain and m its maximal ideal. Let $R = A^*$ and $S = A$ and then consider the map $f : X \to Y$ which maps $\{0\}$ to m and notice $m \notin U$ for any proper openset $U \subseteq X$. So for defining the sheaf morphism, we only need to define the mapping $\mathcal{O}_Y(y) \to f_*\mathcal{O}_x(X)$ by the natural embedding of R in S which is obviously not a local ring homomorphism.

- 2.2.4 If X is affine, this position holds. Otherwise,let U_i be an open affine cover of X, let V_{ijk} be open affine covers of $U_i \cap U_j$. Then the following diagram commutes:

$$\begin{tikzcd} Hom(X, SpecA) \arrow[r] \arrow[d, "\alpha_x"] & \prod Hom(U_i, SpecA) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\alpha_u"] & \prod Hom(V_{ijk}, SpecA) \arrow[d, "\alpha_v"] \\ Hom(A, \Gamma(X, \mathcal{O}_X)) \arrow[r] & \prod Hom(A, \Gamma(U_i, \mathcal{O}_{U_i})) \arrow[r, shift left] \arrow[r, shift right] & \prod Hom(A, \Gamma(V_{ijk}, \mathcal{O}_{V_{ij}})) \end{tikzcd}$$

  And the first row is exact because of the glueing lemma, the second arrow is exact because of the property of sheaves.

  Since α_u and α_v are isomorphisms, α_x is also an isomorphism.

- 2.2.8 Since $k[\epsilon]/(\epsilon^2)$ has only one point, we have $Mor(k[\epsilon]/(\epsilon^2), X) \simeq \{(x, f)|x \in X,\ f \in Hom(\mathcal{O}_x, k[\epsilon]/(\epsilon^2))\}$. And Notice for every local homomorphism $f \in Hom(\mathcal{O}_x, k[\epsilon]/(\epsilon^2))\}$, it will induce an residue homomorphism $k(x) \to k$, and notice $k(x)$ is a k-algebra, so $k(x) \simeq k$. And because $f(m_x^2) = 0$, we have $Mor(k[\epsilon]/(\epsilon^2), X) \simeq \{(x, f)|x\ rational\ over\ X,\ f \in Hom(\mathcal{O}_x/m_x^2, k[\epsilon]/(\epsilon^2))\}$. For $f \in Hom(\mathcal{O}_x/m_x^2, k[\epsilon]/(\epsilon^2))\}$, $f(m_x) \in (\epsilon)$, which defines an vector space homomophism over k. On the otherhand if we have a vector space homomophism f from m_x/m_x^2 to k, add the definition that $f(1) = 1$, we define a vector space morphism from $\mathcal{O}_x/m_x^2$,to $k[\epsilon]/(\epsilon^2)$, which is also a ring homomorphism.Hence we prove the conclusion.

- 2.2.12.

  1. (Topology of X). We define $X = \coprod X_i/\sim$. With $x\ y$ if and only if $\exists i, j$, s.t. $x \in U_{ij}$, $y \in U_{ji}$ and $y = \phi_{ij}(x)$.
     - $\sim$ is a equivalence relation by cocycle conditions.
     - The natural map $\Phi_i : X_i \to X$ is injective.
     - We define U open in X if and only if $U \cap X_i$ is open in X_i.
     - Φ_i is an open embedding, since for every W open in X_i, $\Phi_j^{-1}(W) = \phi_{ij}(W \cap U_{ij})$.

2. (Sheaves on X) Here we can assume $U_{ij} = U_{ji} = X_i \cap X_j$. We define presheaf F, s.t

$$F(V) = \{(s_i \in \Gamma(V \cap X_i, \mathcal{O}_{V \cap X_i}))_{i \in I} | \ \phi_{ij}(s_i|_{V \cap X_i \cap X_j\}}) = s_j|_{V \cap X_i \cap X_j}\}$$

and the restriction map defined on every term i.

Thus for $V \subset X_i$ and $s \in \mathcal{O}_{x_i}(V)$, $(\phi_{ij}(s|_{V \cap X_I}))_{i \in I} \in F(V)$ by the cocycle condition. Hence the map r from s to $(\phi_{ij}(s|_{V \cap X_I}))_{i \in I}$ define a presheaf morphism of O_{X_i} and $F|_{X_i}$. And it's really easy to verify that r is in fact an isomorphism!

Notice $s = t$ if and only if $s_i = t_i$ for all $i \in I$ and $F|_{X_i}$ are sheaves, the locality can be directedly checked by terms. For gluing axiom, let $\{V_\alpha\}$ form a cover of V , and s_α is a bunch of sections matching on $V_{\alpha\beta}$, then s_{α_i} also match with each other, hence there exists s_i, s.t $s_i|_{V_\alpha \cap X_i} = s_{\alpha_i}$. And it's easy to check $(s_i)_{i \in I}$ is compatible with ϕ_{ij} and hence in $F(V)$. So the gluing property is proved. And the ring structure of F can be naturally defined by terms.