Algebraic Geometry,

Homework Solutions

- 1. Here we can assume $X = S$ by base change. And because Y is seperated over X we can assume that Y is a closed subscheme of $Y \times_X Y$.Thus we have the following diagram:

  $$U \hookrightarrow X \xrightarrow{f} Y \times_X Y$$

  with the image of U is in Y hence the image of X is also in Y . Notice that X is a reduced scheme hence the map factors through Y.

  For non-seperated scheme. We can assume X is the affine line $\mathbb{A}^1$ then let Y be the affine line with two points at original. Then let U be the affine line delete the original points. It forms a conterexample.

  For non-reduced scheme X , we consider the $U = Speck[x, y, z, y^{-1}]/(xy, x^2)]$ and $X = Y = Speck[x, y, z]/(xy, x^2)$. Then U is an open set of X but that the morphism from X to it self by mapping x to $-x$ but map other elements invariant will agree on U.

- 2. Notice this question is a local question, we can aassume that $X = SpecA$ is affine and integral and the morphism is defined on a dense open dense subset U of X. Then for any prime ideal $p \subset A$ we have the following diagram:

  $$\begin{tikzcd} SpecK \arrow[r] \arrow[dr] & U \arrow[r, "f"] & Y \arrow[d] \\ & SpecA_p \arrow[r] \arrow[ur, dotted, "g_0"] & S \end{tikzcd}$$

  Here K is the functional field of X and g_0 exists because Y is proper. Since X and Y are varities, we can extend g_0 to be a neigh borhood V of X, with the following diagram commute:

  $$\begin{tikzcd} SpecK \arrow[r] \arrow[d] & U \arrow[r, "f"] & Y \arrow[d] \\ SpecA_p \arrow[r] & V \arrow[r] \arrow[ur, dotted, "g"] & S \end{tikzcd}$$

  And the map f and g are equal on their on their intersections because they agree on the generic point of X and hence on a dense open subset of X. So the maximal open set must contains all the codim 1 subvarieties.

  The maximal domain of definition from $[x : y : z] \to [1/x : 1/y : 1/z]$ is $P^2/(\{[x : y : z]|xy = 0, yz = 0, zx = 0\})$

- 4.1 Notice that finte morphism is a property preserved by base change, by using the local criterion. We only need to prove that the following diagram for every valuation ring U:

  $$\begin{tikzcd} SpecK \arrow[r] \arrow[dr] & Spec\ U \times_Y X \arrow[r] \arrow[d, shift right] & Y \arrow[d] \\ & SpecU \arrow[r] \arrow[u, dotted, shift right, "t"'] & X \end{tikzcd}$$

  Notice that $Spec\ U \times_Y X$ is also a finite morphism over $SpecU$, hence we only need to prove the existence of the diagram

  $$\begin{tikzcd} SpecK \arrow[r] \arrow[dr] & W \arrow[d, shift right] \\ & SpecU \arrow[u, dotted, shift right, "t"'] \end{tikzcd}$$

for any finite morphism of $W = SpecR$ over $SpecU$.

Consider the ring homomorphism $U \to R \to K$, then notice that every element of R is integral over U, so its image is also integral over U, but U is integral closed over K, hence the image of R is contained in U, thus this map factors through U, hence t exists form the above diagram.

- 4.7

  1. Finding finite affine covers $U_i = SpecA_i$ of X such U_i are invariant under the action σ. Then we define V_i is $SpecB_i$ with $B_i = \{x \in A_i | \sigma(x) = x\}$. Then B_i is a R algebra. And by Noether's Theorem B_i is a finite $\mathbb{R}$ algebra. Another proof is follow: We canchoose two parts of generators $T = \{1, x_j + \sigma(x_j)\}$ and $S = \{\sqrt{-1}, \alpha(x_j) - x_j\}$ with x_j are generators of B_i. Then let $B_i = R[t_i, s_i s_j]$ with $t_i \in T$ and $s_j \in S$. Then B_i is a finite generated subring of a_i and let C_i be the B_i module generated by elements in S. Then $C_i + B_i = A_i$ and $\sigma | C_i = -id$ and $\sigma | B_i = id$. So B_i is the ring of invariant elements.

     Consider the ring homomorphism $B_i \otimes C \xrightarrow{p} A_i$. This map is surjective since $2f = (\sigma(f) + f) - i(i(\sigma(f) - f))$. And is injective since $\mathbb{C} = \mathbb{R} + i\mathbb{R}$, and $p(R \otimes C)$ has eigenvalue 1 with σ, and $p(R \otimes iR)$ has eigenvalue -1(*).

     For $U_i \cap U_j$ which is also an affine scheme. We can similarly define V_{ij} to be the spectrum of invariant functions over σ. And we can prove that V_{ij} is an open set of V_i. (The keypoint is that $D(f) \cup D(\sigma(f)) = D(f + \sigma(f)) \cup D(f\sigma(f))$, so we can cover U_{ij} by the form od $D(f)$ with $f \in A_i$ invariant under σ, then V_{ijf} form an open cover of V_{ij} and the mapping form V_{ijf} to V_i is also an open immersion. So V_{ij} is also an open imersion of v_i). And then we can glue V_{ij} up to the scheme X_o. And by (*) we have $X = X_0 \times_{\mathbb{R}} \mathbb{C}$.

     X_0 is seperated, we only need to point out that if U is a valuation ring over R, then $U \otimes_R C$ is a valuation ring over C.

     The uniqueness of X_0 can also be checked locally, and we can check it directly by the ring embedding of B_i in A_i.

  2. "if" is trivial. And the only if part is natural induced from the construction of X_0 in (*) part.

  3. $f = f_0 \times id$. Here id is the identity map from C to C. On the other hand if we know f, let Y_i be an affine cover of Y invariant under σ. And let X_{ij} be an affine cover of $f^{-1}(Y_i)$. Then we can naturally induce the mapping from $X_{0_{ij}}$ to Y_{0_i} by the mapping of invariant functions which are compatible on each open set. Hence we can induce a map from X_0 to Y_0 with those properties.

  4. The involution σ from $C[t]$ to $C[t]$ who mapping i to $-i$ can map t to t or t to $-t$. The ring of invariant function are $R[t]$ in the first situation and $R[it]$ in the second situation which are both isomorphic to A^1_R. For CP_1, we consider the morphism of involution on the functional field $C(t)$. σ mapping i to $-i$, which will map t to t, $-t$, t^{-1} and $-t^{-1}$. If σ map t to t or $-t$, then the quotient scheme is RP^1, if it map t to $-t^{-1}$, then we consider the map from CP_1 to $CP_2/(x^2 + y^2 + z^2)$, by mapping t to $[1/2(1/t - t); 1; i/2(1/t + t)]$, which is an isomorphism, and the responding involution on $CP_2/(x^2 + y^2 + z^2)$ is induced just by map i to $-i$. Hence X_0 is $RP_2/(x^2 + y^2 + z^2)$. It's similar if the map of functional field map t to t^{-1}.

- 5.8

  1. let $\phi(x) \leq n - 1$, then let $t_j(1 \leq j \leq n - 1)$ form a basis of $\mathcal{F}_x / m\mathcal{F}_x$, then choosing v_j to be representatives of t_j. Then v_j generates $\mathcal{F}_x$ ny nakayama's lemma. Hence v_j generates a neighborhood of $\mathcal{F}$, because $\mathcal{F}$ is coherent, hence $\phi(x) \leq n - 1$ locally holds.

  2. if $\mathcal{F}$ is locally free, then ϕ is locally constant and hence continuous, hence it is constant.

  3. Assume $X = SpecA$ with A a reduced notherian local ring. Let p_i be minimal prime ideals of A. Ny nakayama Lemma, we can find a exact sequence:

$$0 \to R \to A^n \to M \to 0$$

With $n = \phi(x)$.

Then localize it at p_i, we get $R_{p_i} = 0$ for all i.

Thus for every q with height 2, we have

$$0 \to R_q \to A_q^n \to M_q \to 0$$

If $R_q \neq 0$, then $Supp(R_q) = Ass(R_q) = q = Ann(R_q)$. Hence $qR_q = 0$. So $R_q = 0$.Contradiction!

Hence $R_q = 0$ for all height 2 ideal. And with induction, we can prove it holds for all finite height prime ideal. And m has finite height, so $R = 0$. So $M$is free.

- 3.19 First we reduce X and Y to affine scheme. In fact, let Y_i be an affine cover of Y, and X_{ij} be an affine cover of X, then the image of constructable set D is the union of image $D \cap X_{ij}$, hence we reduce X and Y to be affine. And more we can assume X and Y to be reduced since it will not change the topology.

  Assume $X = SpecB$, $Y = SpecH$. Next we reduce to prove that the image of X is constructable. In fact, we only need to prove the image of open set and closed set is constructable. For a closed set, we consider it as a closed immersion, then image of a scheme is got via the map of closed immersion and f. And for open set, we can cover it by finite $D(f)$, and only prove its image to be constructable.

  Now we prove this result by noethrian induction, i.e. if for every closed subscheme $T$of closed subeset $Y_0 \subset Y$, chevalley holds for Y_0. Let $Y_0 = SpecS$. First, we can assume the map is dominant, i.e. the ring morphism is injective. In fact the ring morphism $S \to S/ker(f^*) \to B$ factors through $S/ker(f^*)$, so we only need to prove the image of f in $Spec(S/ker(f^*))$ is constructable.

  Using the algebra result for $b = 1$, then for any prime idea $a \notin p$, considerthe map from B to the algebraic closure of $k(p)$, which extends to a homomorphism of B which not vanish at 1. Hence it's kernel form a prime ideal, which restrict to p on S.

  Thus by removing $D(a)$, and the inverse image of $D(a)$, we can use the induction for $Y_0 - D(a)$, and got the result.

  And the map from $Spec\ k[x, y, z]/(z(xy - 1))$ to $Spec\ k[x, z]$ is neither open nor closed.

- 3.11(b)

  Let α be the sheaf of ideals I of kernel $\mathcal{O}_X \to i^*\mathcal{O}_y$. Then by 5.8 we have $i^*\mathcal{O}_y$ is quasi-coherent and I is also quasi-coherent. Thus we can consider the sheaf $\mathcal{O}_X/I$ on $Y_0 = Supp(I)$. Then it's easy to verify that Y_0 forms a closed subscheme and the map from Y to Y_0 is acturally an isomorphism.

  (a) It is a local question for X and X'. Hence we assume $X = SpecA$, $Y = Spec(A/I)$, $X' = Spec(B)$. Then the result is induced by the fact that torsion functor is right exact.

  (c) We define the sheaf of ideals I_U, to be the set of fuctions that never vanish on any residue field of points of U. Then U is a radical ideal, and thus $\mathcal{O}/I$ has a reduced scheme structure on Y. And for any other closed subschemes Y', the ideal sheaf $I_{Y'}$ is contained in I, So we have the ideal morphism $\mathcal{O}/I_{Y'} \to O/I$ which induce a map from Y to Y'

  (d)Let $I = ker(\mathcal{O}_x \to f^*\mathcal{O}_y)$, then the quotient ring of I will induce a closed subscheme, which is the threoetic image of f.

- Chow's Lemma

  - (a) By using the valuation criterion, we now that the irreducible component are proper over the original scheme and hence proper over S. And thus we can reduce to the situation that X is irreducible.

- (b) Notice that any finite type A_i algebra can be regarded as a closed variety of A_i^n and hence a quasi-projective variety P^n, hence we can deal with the conclusion locally and generates those quasi-projective variety by finite-type A_i algebra while $SpecA_i$ are open affine covers of S.
- (c). Consider $W_i = g^{-1}(U_i)$, here we denote g the projection from X' to P_i. Then we prove it's an open cover of X'. To do this, we only need to show that $f^{-1}(U_i) \subset W_i$. Here f is the map from X' to X . This follows from the fact that $P = \prod P_i$ is seperated over S, and hence the diagram $U_i \to U_i \times_S P_i$ is a closed immersion. In fact, consider the projection z_i from $U_i \times P \to U_i \times P_i$, Then the image of W_i in $U_i \times P_i$ is contained in the closure of the image of $z_i \cdot f$, hence also contained in the graph of U_i in $U_i \times P_i$, which is closed immersion since P_i is seperated over S. Notice that the image of X' in g is closed by the propereness of X, then we only need to prove that for $V_i = p_i^{-1}(U_i)$, the map $g : W_i \to V_i$ is closed immersion.

  Here we consider the projection map form V_i to X through the projection of U_i.Then V_i is a closed subscheme of $V_i \times X$, hence we only need to prove the map from W_i to $V \times X_i$ is a closed. But in fact, it is just the closed inclusion from W_i to $V \times X_i$. S we prove this lemma.

  Now we prove that $g^{-1}(U) = U$, but here we can assume $X = U$ and just check the map form U to $U \times P$ is a closed immersion, which is guranteed by the seperated property of X.