Homework

Yu Zhao

February 29, 2016

- First, we prove that there exist $a_0 < a \leq b < b_0$, s.t. $y \in (a_0, b_0)$ regular for f.If not, we assume $\{x_i\}$ are a sequence of critical values of f with $f(x_i) \to a_0$. Notice f is proper then $f^{-1}[a_0 - \epsilon, a_0]$ is bounded for $\epsilon > 0$. Hence x_i are bounded. So we can find a cauchy sequence of $\{x_i\}$, denoted by $\{y_i\}$, which converges to y, the $f(y) = a_0$, which contradicts to the fact that the regular values form an open set.

- Second, we prove that there exists a vector field Y in $\mathbb{R}^n$, s.t $\exists a_1, b_1$, s.t. $a_0 < a_1 < a \leq b < b_1 < b_0$ and $\forall x \in f^{-1}[a_1, b_1]$, we have $Tf|_x(Y) = \frac{\partial}{\partial t}$.

  To prove this, for any $x \in f^{-1}(a_0, b_0)$, we can find a neighboorhood U_x of x with another local coordinate s.t. $f|U_x(x_1, x_2, \ldots, x_n) = x_1$. then let $Y_x = \frac{\partial}{\partial x_i}$.Then let ϕ_x a partition of unity of Y_x on $f^{-1}(a_0, b_o)$. Then we have $Y_1 = \sum_x \phi_x Y_x$ and $Tf|_x(Y_1) = \frac{\partial}{\partial t}$ for $\forall x \in f^{-1}(a_0, b_0)$. Extend $Y_1|f^{-1}[a_1, b_1]$ to a global vector field Y on $\mathbb{R}^n$, then Y satisfy the following condition.

- Let $\epsilon = min\{a - a_0, b_0 - b\}$,then for $\forall 0 < \delta < \epsilon$, $X_\delta f^{-1}(a) = f^{-1}(a + \delta)$ for all $a \in [a, b]$. And notice X_t is an isomorphism for all t. Let $d = (b-a)/n$ for asufficient large integer n. Then $f^{-1}(b) = X_d^n(f^{-1}(a))$ and hence the conclusion is proved.