**Algebraic Geometry,**

# Homework Solutions

- 1. Here we can assume $X = S$ by base change. And because $Y$ is seperated over $X$ we can assume that $Y$ is a closed subscheme of $Y \times_X Y$.Thus we have the following diagram:

$$U \hookrightarrow X \xrightarrow{f} Y \times_X Y$$

with the image of $U$ is in $Y$ hence the image of $X$ is also in $Y$ . Notice that $X$ is a reduced scheme hence the map factors through $Y$.

For non-seperated scheme. We can assume $X$ is the affine line $\mathbb{A}^1$ then let $Y$ be the affine line with two points at original. Then let $U$ be the affine line delete the original points. It forms a conterexample.

For non-reduced scheme $X$ , we consider the $U = Speck[x, y, z, y^{-1}]/(xy, x^2)]$ and $X = Y = Speck[x, y, z]/(xy, x^2)$. Then $U$ is an open set of $X$ but that the morphism from $X$ to it self by mapping $x$ to $-x$ but map other elements invariant will agree on $U$.

- 2. Notice this question is a local question, we can aassume that $X = SpecA$ is affine and integral and the morphism is defined on a dense open dense subset $U$ of $X$. Then for any prime ideal $p \subset A$ we have the following diagram:

$$\begin{tikzcd}
SpecK \arrow[r] \arrow[dr] & U \arrow[r, "f"] & Y \arrow[d] \\
& SpecA_p \arrow[r] \arrow[ur, dotted, "g_0"] & S
\end{tikzcd}$$

Here $K$ is the functional field of $X$ and $g_0$ exists because $Y$ is proper. Since $X$ and $Y$ are varities, we can extend $g_0$ to be a neigh borhood $V$ of $X$, with the following diagram commute:

$$\begin{tikzcd}
SpecK \arrow[r] \arrow[d] & U \arrow[r, "f"] & Y \arrow[d] \\
SpecA_p \arrow[r] & V \arrow[r] \arrow[ur, dotted, "g"] & S
\end{tikzcd}$$

And the map $f$ and $g$ are equal on their on their intersections because they agree on the generic point of $X$ and hence on a dense open subset of $X$. So the maximal open set must contains all the codim 1 subvarieties.

The maximal domain of definition from $[x : y : z] \to [1/x : 1/y : 1/z]$ is $P^2/(\{[x : y : z]|xy = 0, yz = 0, zx = 0\})$

- 4.1 Notice that finte morphism is a property preserved by base change, by using the local criterion. We only need to prove that the following diagram for every valuation ring $U$:

$$\begin{tikzcd}
SpecK \arrow[r] \arrow[dr] & Spec\ U \times_Y X \arrow[r] \arrow[d, shift right] & Y \arrow[d] \\
& SpecU \arrow[r] \arrow[u, dotted, shift right, "t"'] & X
\end{tikzcd}$$

Notice that $Spec\ U \times_Y X$ is also a finite morphism over $SpecU$, hence we only need to prove the existence of the diagram

$$\begin{tikzcd}
SpecK \arrow[r] \arrow[dr] & W \arrow[d, shift right] \\
& SpecU \arrow[u, dotted, shift right, "t"']
\end{tikzcd}$$

for any finite morphism of $W = SpecR$ over $SpecU$.

Consider the ring homomorphism $U \to R \to K$, then notice that every element of $R$ is integral over $U$, so its image is also integral over $U$, but $U$ is integral closed over $K$, hence the image of $R$ is contained in $U$, thus this map factors through $U$, hence $t$ exists form the above diagram.

- 4.7

  1. Finding finite affine covers $U_i = SpecA_i$ of $X$ such $U_i$ are invariant under the action $\sigma$. Then we define $V_i$ is $SpecB_i$ with $B_i = \{x \in A_i | \sigma(x) = x\}$. Then $B_i$ is a $R$ algebra. And by Noether's Theorem $B_i$ is a finite $\mathbb{R}$ algebra. Another proof is follow: We canchoose two parts of generators $T = \{1, x_j + \sigma(x_j)\}$ and $S = \{\sqrt{-1}, \alpha(x_j) - x_j\}$ with $x_j$ are generators of $B_i$. Then let $B_i = R[t_i, s_i s_j]$ with $t_i \in T$ and $s_j \in S$. Then $B_i$ is a finite generated subring of $a_i$ and let $C_i$ be the $B_i$ module generated by elements in $S$. Then $C_i + B_i = A_i$ and $\sigma | C_i = -id$ and $\sigma | B_i = id$. So $B_i$ is the ring of invariant elements.

     Consider the ring homomorphism $B_i \otimes C \xrightarrow{p} A_i$. This map is surjective since $2f = (\sigma(f) + f) - i(i(\sigma(f) - f))$. And is injective since $\mathbb{C} = \mathbb{R} + i\mathbb{R}$, and $p(R \otimes C)$ has eigenvalue 1 with $\sigma$, and $p(R \otimes iR)$ has eigenvalue -1(*).

     For $U_i \cap U_j$ which is also an affine scheme. We can similarly define $V_{ij}$ to be the spectrum of invariant functions over $\sigma$. And we can prove that $V_{ij}$ is an open set of $V_i$. (The keypoint is that $D(f) \cup D(\sigma(f)) = D(f + \sigma(f)) \cup D(f\sigma(f))$, so we can cover $U_{ij}$ by the form od $D(f)$ with $f \in A_i$ invariant under $\sigma$, then $V_{ijf}$ form an open cover of $V_{ij}$ and the mapping form $V_{ijf}$ to $V_i$ is also an open immersion. So $V_{ij}$ is also an open imersion of $v_i$). And then we can glue $V_{ij}$ up to the scheme $X_o$. And by (*) we have $X = X_0 \times_{\mathbb{R}} \mathbb{C}$.

     $X_0$ is seperated, we only need to point out that if $U$ is a valuation ring over $R$, then $U \otimes_R C$ is a valuation ring over $C$.

     The uniqueness of $X_0$ can also be checked locally, and we can check it directly by the ring embedding of $B_i$ in $A_i$.

  2. "if" is trivial. And the only if part is natural induced from the construction of $X_0$ in (*) part.

  3. $f = f_0 \times id$. Here $id$ is the identity map from $C$ to $C$. On the other hand if we know $f$, let $Y_i$ be an affine cover of $Y$ invariant under $\sigma$. And let $X_{ij}$ be an affine cover of $f^{-1}(Y_i)$. Then we can naturally induce the mapping from $X_{0_{ij}}$ to $Y_{0_i}$ by the mapping of invariant functions which are compatible on each open set. Hence we can induce a map from $X_0$ to $Y_0$ with those properties.

  4. The involution $\sigma$ from $C[t]$ to $C[t]$ who mapping $i$ to $-i$ can map $t$ to $t$ or $t$ to $-t$. The ring of invariant function are $R[t]$ in the first situation and $R[it]$ in the second situation which are both isomorphic to $A^1_R$. For $CP_1$, we consider the morphism of involution on the functional field $C(t)$. $\sigma$ mapping $i$ to $-i$, which will map $t$ to $t$, $-t$, $t^{-1}$ and $-t^{-1}$. If $\sigma$ map $t$ to $t$ or $-t$, then the quotient scheme is $RP^1$, if it map $t$ to $-t^{-1}$, then we consider the map from $CP_1$ to $CP_2/(x^2 + y^2 + z^2)$, by mapping $t$ to $[1/2(1/t - t); 1; i/2(1/t + t)]$, which is an isomorphism, and the responding involution on $CP_2/(x^2 + y^2 + z^2)$ is induced just by map $i$ to $-i$. Hence $X_0$ is $RP_2/(x^2 + y^2 + z^2)$. It's similar if the map of functional field map $t$ to $t^{-1}$.

- 5.8

  1. let $\phi(x) \leq n - 1$, then let $t_j(\ 1 \leq j \leq n - 1)$ form a basis of $\mathcal{F}_x / m\mathcal{F}_x$, then choosing $v_j$ to be representatives of $t_j$. Then $v_j$ generates $\mathcal{F}_x$ ny nakayama's lemma. Hence $v_j$ generates a neighborhood of $\mathcal{F}$, because $\mathcal{F}$ is coherent, hence $\phi(x) \leq n - 1$ locally holds.

  2. if $\mathcal{F}$ is locally free, then $\phi$ is locally constant and hence continuous, hence it is constant.

  3. Assume $X = SpecA$ with $A$ a reduced notherian local ring. Let $p_i$ be minimal prime ideals of $A$. Ny nakayama Lemma, we can find a exact sequence:

$$0 \to R \to A^n \to M \to 0$$

With $n = \phi(x)$.

Then localize it at $p_i$, we get $R_{p_i} = 0$ for all $i$.

Thus for every $q$ with height 2, we have

$$0 \to R_q \to A_q^n \to M_q \to 0$$

If $R_q \neq 0$, then $Supp(R_q) = Ass(R_q) = q = Ann(R_q)$. Hence $qR_q = 0$. So $R_q = 0$.Contradiction!

Hence $R_q = 0$ for all height 2 ideal. And with induction, we can prove it holds for all finite height prime ideal. And $m$ has finite height, so $R = 0$. So $M$is free.

- 3.19 First we reduce $X$ and $Y$ to affine scheme. In fact, let $Y_i$ be an affine cover of $Y$, and $X_{ij}$ be an affine cover of $X$, then the image of constructable set $D$ is the union of image $D \cap X_{ij}$, hence we reduce $X$ and $Y$ to be affine. And more we can assume $X$ and $Y$ to be reduced since it will not change the topology.

  Assume $X = SpecB$, $Y = SpecH$. Next we reduce to prove that the image of $X$ is constructable. In fact, we only need to prove the image of open set and closed set is constructable. For a closed set, we consider it as a closed immersion, then image of a scheme is got via the map of closed immersion and $f$. And for open set, we can cover it by finite $D(f)$, and only prove its image to be constructable.

  Now we prove this result by noethrian induction, i.e. if for every closed subscheme $T$of closed subeset $Y_0 \subset Y$, chevalley holds for $Y_0$. Let $Y_0 = SpecS$. First, we can assume the map is dominant, i.e. the ring morphism is injective. In fact the ring morphism $S \to S/ker(f^*) \to B$ factors through $S/ker(f^*)$, so we only need to prove the image of $f$ in $Spec(S/ker(f^*))$ is constructable.

  Using the algebra result for $b = 1$, then for any prime idea $a \notin p$, considerthe map from $B$ to the algebraic closure of $k(p)$, which extends to a homomorphism of $B$ which not vanish at 1. Hence it's kernel form a prime ideal, which restrict to $p$ on $S$.

  Thus by removing $D(a)$, and the inverse image of $D(a)$, we can use the induction for $Y_0 - D(a)$, and got the result.

  And the map from $Spec\ k[x, y, z]/(z(xy - 1))$ to $Spec\ k[x, z]$ is neither open nor closed.

- 3.11(b)

  Let $\alpha$ be the sheaf of ideals $I$ of kernel $\mathcal{O}_X \to i^*\mathcal{O}_y$. Then by 5.8 we have $i^*\mathcal{O}_y$ is quasi-coherent and $I$ is also quasi-coherent. Thus we can consider the sheaf $\mathcal{O}_X/I$ on $Y_0 = Supp(I)$. Then it's easy to verify that $Y_0$ forms a closed subscheme and the map from $Y$ to $Y_0$ is acturally an isomorphism.

  (a) It is a local question for $X$ and $X'$. Hence we assume $X = SpecA$, $Y = Spec(A/I)$, $X' = Spec(B)$. Then the result is induced by the fact that torsion functor is right exact.

  (c) We define the sheaf of ideals $I_U$, to be the set of fuctions that never vanish on any residue field of points of $U$. Then $U$ is a radical ideal, and thus $\mathcal{O}/I$ has a reduced scheme structure on $Y$. And for any other closed subschemes $Y'$, the ideal sheaf $I_{Y'}$ is contained in $I$, So we have the ideal morphism $\mathcal{O}/I_{Y'} \to O/I$ which induce a map from $Y$ to $Y'$

  (d)Let $I = ker(\mathcal{O}_x \to f^*\mathcal{O}_y)$, then the quotient ring of $I$ will induce a closed subscheme, which is the threoetic image of $f$.

- Chow's Lemma

  - (a) By using the valuation criterion, we now that the irreducible component are proper over the original scheme and hence proper over $S$. And thus we can reduce to the situation that $X$ is irreducible.

- (b) Notice that any finite type $A_i$ algebra can be regarded as a closed variety of $A_i^n$ and hence a quasi-projective variety $P^n$, hence we can deal with the conclusion locally and generates those quasi-projective variety by finite-type $A_i$ algebra while $SpecA_i$ are open affine covers of $S$.
- (c). Consider $W_i = g^{-1}(U_i)$, here we denote $g$ the projection from $X'$ to $P_i$. Then we prove it's an open cover of $X'$. To do this, we only need to show that $f^{-1}(U_i) \subset W_i$. Here $f$ is the map from $X'$ to $X$ . This follows from the fact that $P = \prod P_i$ is seperated over $S$, and hence the diagram $U_i \to U_i \times_S P_i$ is a closed immersion. In fact, consider the projection $z_i$ from $U_i \times P \to U_i \times P_i$, Then the image of $W_i$ in $U_i \times P_i$ is contained in the closure of the image of $z_i \cdot f$, hence also contained in the graph of $U_i$ in $U_i \times P_i$, which is closed immersion since $P_i$ is seperated over $S$. Notice that the image of $X'$ in $g$ is closed by the propereness of $X$, then we only need to prove that for $V_i = p_i^{-1}(U_i)$, the map $g : W_i \to V_i$ is closed immersion.

  Here we consider the projection map form $V_i$ to $X$ through the projection of $U_i$.Then $V_i$ is a closed subscheme of $V_i \times X$, hence we only need to prove the map from $W_i$ to $V \times X_i$ is a closed. But in fact, it is just the closed inclusion from $W_i$ to $V \times X_i$. S we prove this lemma.

  Now we prove that $g^{-1}(U) = U$, but here we can assume $X = U$ and just check the map form $U$ to $U \times P$ is a closed immersion, which is guranteed by the seperated property of $X$.