Homework

Yu Zhao

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1. Consider the map f from C to C^2 by $f(t) = (t^3, t^5)$, which is obviously a bijective map. In fact if $t_1 \neq t_2$ but $f(t_1) = f(t_2)$, then we have t_1 and t_2 are all nonzero, and induce that $t_1^2 = t_2^2$ and hence $t_1 = t_2$. Similarly, this map is surjective. And this map is continuous, hence we only need to check it is an open map.

   For $t \neq 0$, f is a local isomorphism, and for $t = 0$, then $f(B_\epsilon) = C \cap U(\epsilon^3, \epsilon^5)$, here C is the algebraic curve $\{(x, y)|x^5 = y^3\}$ and $U(a, b) = \{(x, y)||x| < a,\ |y| < b\}$.

2. Consider the action Z/pZ acts on C^2 by $[1](z_1, z_2) = (e^{2\pi i/p}z_1, e^{2\pi iq/p}z_2)$. with q, p coprime with q. Then the quotient of the action on S^3 is lens space $L(p, q)$. And if this quotient space is a manifold, it must be dimension 4. Consider the neighborhood of 0 into an open subset of R^4, then we can induce a embedding of $L(p, q)$ in to S^4, which will not exist.

   A example is that we can assume $p = 2$ and $q = 1$, then the lens space is $P = RP^3$, if it can be embedded into S^4. We consider the Z_2 cohomology. By Alexander's duailty theorem, we have $\widetilde{H^0}(S^4 - P) = Z_2^2$, thus it has two connected component and we denote them A, B to be there closure. Then we have $A \cap B = P$ and $A \cup B = S^4$, by Lefshetz duality and Mayer-Viectoris sequence we have $H_4(A) \oplus H_4(B) = H^4(S^4)/P = \tilde{H}^0(P) = 0$.

   By the Mayer-Viectoris sequence again, we have $H^1(A) + H^1(B) = Z_2$. We assume $H^1(A) = 0$ and $H^1(B) = Z_2$ generated by b. and hence we use the $Z/2Z$ cohomology, and thus found b^3 generate $H^3(P)$. Hence the map from $H^3(P)$ to $H^4(S^4)$ is a zero map. Thus we have the following exact sequence: $Z_2 = H^4(S^4) \simeq H^4(A) \oplus H^4(B)$ contradiction!