A criterion for semi-simple Lie algebra

For a semisimple Lie algebra $\mathfrak{g}$ over a field K of chracteristic 0, we know that for all finite $\mathfrak{g}$ module , we have $H^1(\mathfrak{g}, A) = 0$. And now we are going to prove the converse of this result is also true:

Theorem 0.1. *For a Lie algebra $\mathfrak{g}$, if for all finite $\mathfrak{g}$-module A, we have that $H^1(\mathfrak{g}, A) = 0$(*),then $\mathfrak{g}$ is semi-simple.*

Instead, we prove that if (*) condition holds for $\mathfrak{g}$, then for any exact sequence of $\mathfrak{g}$-module $0 \to A' \to A \to A'' \to 0$ will split. Thus we can decompose $\mathfrak{g}$ itself into simple Lie algebras with the representation ad.

For $\mathfrak{g}-$module A, B, consider $Hom_K(A, B)$ with the $\mathfrak{g}$-module structure $(g\phi)(a) = g(\phi(a)) - \phi(ga)$. Then $Hom_{\mathfrak{g}}(A, B) = \{a \in Hom_K(A, B) | ga = 0 \ , \forall g\} = Hom_{\mathfrak{g}}(K, Hom_K(A, B)) = H^0(\mathfrak{g}, Hom_K(A, B))$

Notice that $0 \to Hom_K(A'', A') \to Hom_K(A, A') \to Hom_K(A, A') \to 0$. So we have the following exact sequence:

$0 \to H^0(\mathfrak{g}, Hom_K(A'', A')) \to H^0(\mathfrak{g}, Hom_K(A, A')) \to H^0(\mathfrak{g}, Hom_K(A', A')) \to 0.$

Since $H^1(\mathfrak{g}, Hom_K(A'', A')) = 0$. Thus $Hom_{\mathfrak{g}}(A, A') \to Hom_{\mathfrak{g}}(A', A')$ is surjective, consdier one element of $Hom_{\mathfrak{g}}(A, A')$ which maps to identity of A', then it forms a split of the exact sequence.

The first extension of this theorem is that we can consider the cohomology of simple $\mathfrak{g}$-modules instead of all $\mathfrak{g}$-modules. For a module A which is not simple, we can consider a submodule A' of A and th exact sequence $0 \to A' \to A \to A/A'$. Then we have $H^1(\mathfrak{g}, A') \to H^1(\mathfrak{g}, A) \to H^1(\mathfrak{g}, A/A')$. Thus $H^1(\mathfrak{g}, A) = 0$ if $H^1(\mathfrak{g}, A') = 0$ and $H^1(\mathfrak{g}, A/A') = 0$. With the induction on the dimension, we have the following result:

Lemma 0.2. *If for any maximal ideal $\mathfrak{h}$ of $\mathfrak{g}$, we have $H^1(\mathfrak{g}, \mathfrak{g}/\mathfrak{h}) = 0$, then for all finite modules A, we have $H^1(\mathfrak{g}, A) = 0$*

Corollary. *For a Lie algebra $\mathfrak{g}$, if for all maximal ideals $\mathfrak{h}$, we have that $H^1(\mathfrak{g}, \mathfrak{g}/\mathfrak{h}) = 0$, then $\mathfrak{g}$ is semi-simple.*