A criterion for semi-simple Lie algebra

For a semisimple Lie algebra \mathfrak{g} over a field K of chracteristic 0, we know that for all finite \mathfrak{g} module , we have $H^1(\mathfrak{g},A)=0$. And now we are going to prove the converse of this result is also true:

Theorem 0.1. For a Lie algebra \mathfrak{g} , if for all finite \mathfrak{g} -module A, we have that $H^1(\mathfrak{g}, A) = 0$ (*), then \mathfrak{g} is semi-simple.

Instead, we prove that if (*) condition holds for \mathfrak{g} , then for any exact sequence of \mathfrak{g} -module $0 \to A' \to A \to A'' \to 0$ will split. Thus we can decompose \mathfrak{g} itself into simple Lie algebras with the representation ad.

For \mathfrak{g} -module A, B, consider $Hom_K(A, B)$ with the \mathfrak{g} -module structure $(g\phi)(a) = g(\phi(a)) - \phi(ga)$. Then $Hom_{\mathfrak{g}}(A, B) = \{a \in Hom_K(A, B) | ga = 0, \forall g\} = Hom_{\mathfrak{g}}(K, Hom_K(A, B)) = H^0(\mathfrak{g}, Hom_K(A, B))$

Notice that $0 \to Hom_K(A'', A') \to Hom_K(A, A') \to Hom_K(A, A') \to 0$. So we have the following exact sequence:

 $0 \to H^0(\mathfrak{g}, Hom_K(A'', A')) \to H^0(\mathfrak{g}, Hom_K(A, A')) \to H^0(\mathfrak{g}, Hom_K(A', A')) \to 0.$

Since $H^1(\mathfrak{g}, Hom_K(A'', A')) = 0$. Thus $Hom_{\mathfrak{g}}(A, A') \to Hom_{\mathfrak{g}}(A', A')$ is surjective, consdier one element of $Hom_{\mathfrak{g}}(A, A')$ which maps to identity of A', then it forms a split of the exact sequence.

The first extension of this theorem is that we can consider the cohomology of simple \mathfrak{g} -modules instead of all \mathfrak{g} -modules. For a module A which is not simple, we can consider a submodule A' of A and th exact sequence $0 \to A' \to A \to A/A'$. Then we have $H^1(\mathfrak{g}, A') \to H^1(\mathfrak{g}, A) \to H^1(\mathfrak{g}, A/A')$. Thus $H^1(\mathfrak{g}, A) = 0$ if $H^1(\mathfrak{g}, A') = 0$ and $H^1(\mathfrak{g}, A/A') = 0$. With the induction on the dimension, we have the following result:

Lemma 0.2. If for any maximal ideal \mathfrak{h} of \mathfrak{g} , we have $H^1(\mathfrak{g}, \mathfrak{g}/\mathfrak{h}) = 0$, then for all finite modules A, we have $H^1(\mathfrak{g}, A) = 0$

Corollary. For a Lie algebra \mathfrak{g} , if for all maximal ideals \mathfrak{h} , we have that $H^1(\mathfrak{g},\mathfrak{g}/\mathfrak{h})=0$, then \mathfrak{g} is semi-simple.