Homework Solutions

• 1. Here we can assume X = S by base change. And because Y is separated over X we can assume that Y is a closed subscheme of $Y \times_X Y$. Thus we have the following diagram:

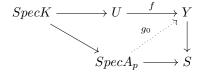
$$U \hookrightarrow X \xrightarrow{f} Y \times_X Y$$

with the image of U is in Y hence the image of X is also in Y. Notice that X is a reduced scheme hence the map factors through Y.

For non-separated scheme. We can assume X is the affine line \mathbb{A}^1 then let Y be the affine line with two points at original. Then let U be the affine line delete the original points. It forms a conterexample.

For non-reduced scheme X, we consider the $U = Speck[x,y,z,y^{-1}]/(xy,x^2)$] and $X = Y = Speck[x,y,z]/(xy,x^2)$. Then U is an open set of X but that the morphism from X to it self by mapping x to -x but map other elements invariant will agree on U.

• 2. Notice this question is a local question, we can assume that X = SpecA is affine and integral and the morphism is defined on a dense open dense subset U of X. Then for any prime ideal $p \subset A$ we have the following diagram:



Here K is the functional field of X and g_0 exists because Y is proper. Since X and Y are varities, we can extend g_0 to be a neigh borhood V of X, with the following diagram commute:

And the map f and g are equal on their on their intersections because they agree on the generic point of X and hence on a dense open subset of X. So the maximal open set must contain all the codim 1 subvarieties.

The maximal domain of definition from $[x:y:z] \rightarrow [1/x:1/y:1/z]$ is $P^2/(\{[x:y:z]|xy=0,yz=0,zx=0\})$

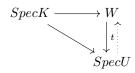
• 4.1 Notice that finte morphism is a property preserved by base change, by using the local criterion. We only need to prove that the following diagram for every valuation ring U:

$$SpecK \longrightarrow Spec \ U \times_Y X \longrightarrow Y$$

$$\downarrow t \qquad \qquad \downarrow \downarrow$$

$$SpecU \longrightarrow X$$

Notice that $Spec\ U \times_Y X$ is also a finite morphism over SpecU, hence we only need to prove the existence of the diagram



for any finite morphism of W = SpecR over SpecU.

Consider the ring homomorphism $U \to R \to K$, then notice that every element of R is integral over U, so its image is also integral over U, but U is integral closed over K, hence the image of R is contained in U, thus this map factors through U, hence t exists form the above diagram.

• 4.7

1. Finding finite affine covers $U_i = SpecA_i$ of X such U_i are invariant under the action σ . Then we define V_i is $SpecB_i$ with $B_i = \{x \in A_i | \sigma(x) = x\}$. Then B_i is a R algebra. And by Noether's Theorem B_i is a finite \mathbb{R} algebra. Another proof is follow: We canchoose two parts of generators $T = \{1, x_j + \sigma(x_j)\}$ and $S = \{\sqrt{-1}, \alpha(x_j) - x_j\}$ with x_j are generators of B_i . Then let $B_i = R[t_i, s_i s_j]$ with $t_i \in T$ and $s_j \in S$. Then B_i is a finite generated subring of a_i and let C_i be the B_i module generated by elements in S. Then $C_i + B_i = A_i$ and $\sigma | C_i = -id$ and $\sigma | B_i = id$. So B_i is the ring of invariant elements.

Consider the ring homomorphism $B_i \otimes C \xrightarrow{p} A_i$. This map is surjective since $2f = (\sigma(f) + f) - i(i(\sigma(f) - f))$. And is injective since $\mathbb{C} = \mathbb{R} + i\mathbb{R}$, and $p(R \otimes C)$ has eigenvalue 1 with σ , and $p(R \otimes iR)$ has eigenvalue -1(*).

For $U_i \cap U_j$ which is also an affine scheme. We can similarly define V_{ij} to be the spectrum of invariant functions over σ . And we can prove that V_{ij} is an open set of V_i . (The keypoint is that $D(f) \cup D(\sigma(f)) = D(f + \sigma(f)) \cup D(f\sigma(f))$, so we can cover U_{ij} by the form od D(f) with $f \in A_i$ invariant under σ , then V_{ijf} form an open cover of V_{ij} and the mapping form V_{ijf} to V_i is also an open immersion. So V_{ij} is also an open imersion of v_i). And then we can glue V_{ij} up to the scheme X_o . And by (*) we have $X = X_0 \times_{\mathbb{R}} \mathbb{C}$.

 X_0 is separated, we only need to point out that if U is a valuation ring over R, then $U \otimes_R C$ is a valuation ring over C.

The uniqueness of X_0 can also be checked locally, and we can check it directly by the ring embedding of B_i in A_i .

- 2. "if" is trivial. And the only if part is natural induced from the construction of X_0 in (*) part.
- 3. $f = f_0 \times id$. Here id is the identity map from C to C. On the other hand if we know f, let Y_i be an affine cover of Y invariant under σ . And let X_{ij} be an affine cover of $f^{-1}(Y_i)$. Then we can naturally induce the mapping from X_{0ij} to Y_{0i} by the mapping of invariant functions which are compatible on each open set. Hence we can induce a map from X_0 to Y_0 with those properties.
- 4. The involution σ from C[t] to C[t] who mapping i to -i can map t to t or t to -t. The ring of invariant function are R[t] in the first situation and R[it] in the second situation which are both isomorphic to A_R^1 . For CP_1 , we consider the morphism of involution on the functional field C(t). σ mapping i to -i, which will map t to t, -t, t^{-1} and $-t^{-1}$. If σ map t to t or -t, then the quotient scheme is RP^1 , if it map t to $-t^{-1}$, then we consider the map from CP_1 to $CP_2/(x^2+y^2+z^2)$, by mapping t to [1/2(1/t-t);1;i/2(1/t+t)], which is an isomorphism, and the responding involution on $CP_2/(x^2+y^2+z^2)$ is induced just by map t to -t. Hence X_0 is $RP_2/(x^2+y^2+z^2)$. It's similar if the map of functional field map t to t^{-1} .

• 5.8

- 1. let $\phi(x) \leq n-1$, then let t_j ($1 \leq j \leq n-1$) form a basis of $\mathcal{F}_x/m\mathcal{F}_x$, then choosing v_j to be representatives of t_j . Then v_j generates \mathcal{F}_x ny nakayama's lemma. Hence v_j generates a neighborhood of \mathcal{F} , because \mathcal{F} is coherent, hence $\phi(x) \leq n-1$ locally holds.
- 2. if \mathcal{F} is locally free, then ϕ is locally constant and hence continuous, hence it is constant.
- 3. Assume X = SpecA with A a reduced notherian local ring. Let p_i be minimal prime ideals of A. Ny nakayama Lemma, we can find a exact sequence:

$$0 \to R \to A^n \to M \to 0$$

With $n = \phi(x)$.

Then localize it at p_i , we get $R_{p_i} = 0$ for all i.

Thus for every q with height 2, we have

$$0 \to R_q \to A_q^n \to M_q \to 0$$

If $R_q \neq 0$, then $Supp(R_q) = Ass(R_q) = q = Ann(R_q)$. Hence $qR_q = 0$. So $R_q = 0$. Contradiction!

Hence $R_q = 0$ for all height 2 ideal. And with induction, we can prove it holds for all finite height prime ideal. And m has finite height, so R = 0. So M is free.

• 3.19 First we reduce X and Y to affine scheme. In fact, let Y_i be an affine cover of Y, and X_{ij} be an affine cover of X, then the image of constructable set D is the union of image $D \cap X_{ij}$, hence we reduce X and Y to be affine. And more we can assume X and Y to be reduced since it will not change the topology.

Assume X = SpecB, Y = SpecH. Next we reduce to prove that the image of X is constructable. In fact, we only need to prove the image of open set and closed set is constructable. For a closed set, we consider it as a closed immersion, then image of a scheme is got via the map of closed immersion and f. And for open set, we can cover it by finite D(f), and only prove its image to be constructable.

Now we prove this result by noethrian induction, i.e. if for every closed subscheme T of closed subset $Y_0 \subset Y$, chevalley holds for Y_0 . Let $Y_0 = SpecS$. First, we can assume the map is dominant, i.e. the ring morphism is injective. In fact the ring morphism $S \to S/ker(f^*) \to B$ factors through $S/ker(f^*)$, so we only need to prove the image of f in $Spec(S/ker(f^*))$ is constructable.

Using the algebra result for b = 1, then for any prime idea $a \notin p$, consider the map from B to the algebraic closure of k(p), which extends to a homomorphism of B which not vanish at 1. Hence it's kernel form a prime ideal, which restrict to p on S.

Thus by removing D(a), and the inverse image of D(a), we can use the induction for $Y_0 - D(a)$, and got the result.

And the map from $Spec \ k[x,y,z]/(z(xy-1))$ to $Spec \ k[x,z]$ is neither open nor closed.

• 3.11(b)

Let α be the sheaf of ideals I of kernel $\mathcal{O}_X \to i^*\mathcal{O}_y$. Then by 5.8 we have $i^*\mathcal{O}_y$ is quasi-coherent and I is also quasi-coherent. Thus we can consider the sheaf \mathcal{O}_X/I on $Y_0 = Supp(I)$. Then it's easy to verify that Y_0 forms a closed subscheme and the map from Y to Y_0 is acturally an isomorphism.

- (a) It is a local question for X and X'. Hence we assume X = Spec(A, Y) = Spec(A/I), X' = Spec(B). Then the result is induced by the fact that torsion functor is right exact.
- (c) We define the sheaf of ideals I_U , to be the set of fuctions that never vanish on any residue field of points of U. Then U is a radical ideal, and thus \mathcal{O}/I has a reduced scheme structure on Y. And for any other closed subschemes Y', the ideal sheaf $I_{Y'}$ is contained in I, So we have the ideal morphism $\mathcal{O}/I_{Y'} \to \mathcal{O}/I$ which induce a map from Y to Y'
- (d)Let $I = ker(\mathcal{O}_x \to f^*\mathcal{O}_y)$, then the quotient ring of I will induce a closed subscheme, which is the threoetic image of f.

• Chow's Lemma

- (a) By using the valuation criterion, we now that the irreducible component are proper over the original scheme and hence proper over S. And thus we can reduce to the situation that X is irreducible.

- (b) Notice that any finite type A_i algebra can be regarded as a closed variety of A_i^n and hence a quasi-projective variety P^n , hence we can deal with the conclusion locally and generates those quasi-projective variety by finite-type A_i algebra while $SpecA_i$ are open affine covers of S.
- (c). Consider $W_i = g^{-1}(U_i)$, here we denote g the projection from X' to P_i . Then we prove it's an open cover of X'. To do this, we only need to show that $f^{-1}(U_i) \subset W_i$. Here f is the map from X' to X. This follows from the fact that $P = \prod P_i$ is seperated over S, and hence the diagram $U_i \to U_i \times_S P_i$ is a closed immersion. In fact, consider the projection z_i from $U_i \times P \to U_i \times P_i$, Then the image of W_i in $U_i \times P_i$ is contained in the closure of the image of $z_i \cdot f$, hence also contained in the graph of U_i in $U_i \times P_i$, which is closed immersion since P_i is separated over S. Notice that the image of X' in g is closed by the propereness of X, then we only need to prove that for $V_i = p_i^{-1}(U_i)$, the map $g: W_i \to V_i$ is closed immersion.

Here we consider the projection map form V_i to X through the projection of U_i . Then V_i is a closed subscheme of $V_i \times X$, hence we only need to prove the map from W_i to $V \times X_i$ is a closed. But in fact, it is just the closed inclusion from W_i to $V \times X_i$. S we prove this lemma.

Now we prove that $g^{-1}(U) = U$, but here we can assume X = U and just check the map form U to $U \times P$ is a closed immersion, which is guranteed by the separated property of X.