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Homework Solutions

- We define the product structure of group-like elements by the action ∇, i.e.

$$\begin{aligned} \cdot : G \times G &\to G \\ a \cdot b &= \nabla(a \otimes b) \end{aligned}$$

  We need to prove that this action is well defined, i.e. if a, b are two group like elements, then $a \cdot b$ is also group like. In fact $\Delta\nabla(a \otimes b) = (\nabla \otimes \nabla)(id \otimes \tau \otimes id)(\Delta \otimes \Delta)(a \otimes b) = (\nabla \otimes \nabla)(a \otimes b) \otimes (a \otimes b)$. And $\epsilon(1)(a \cdot b) = \epsilon(a) \cdot \epsilon(b) = 1$. Hence $a \cdot b$ is also group like.

  Then $\eta(1)$ is a unit in this action and this action is associative since ∇ is a unit associative algebra. Besides, for $a \in G$, $\nabla(Sa \otimes a) = \eta\epsilon(a) = \eta(1)$. Hence S form the inverse operation. Hence all group like elements form a group.

- 3.3 $[a, [b, c]] = [[a, b], c] + [c, [a, b]] \Leftrightarrow [a, [b, c]] - [[a, b], c] - [b, [a, c]] = 0 \Leftrightarrow [a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$ And through the jacobi identity we know that $[c, [a, b]] = [[[c, a], b] + [b, [c, a]]$ and hence $[[a, b], c] = -[b, [a, c]] + [b, [a, c]]$ which is equivalent to the fact that $ad[ab] = ad(a)ad(b) - ad(b)ad(a)$

- 4.11 Notice that $dimV[k]$ is an additive fuction for the representations of $\mathfrak{sl}(2, \mathbb{C})$. Hence $dimV[k] = \sum n_j dimV_j[k] = \sum_{j=0} n_{k+2j}$. Hence $dimV[k+2] - dimV[k] = n_k(*U)$. And by sum formula (*) over k, we got the other two formulas.