- 4.3: Considering the representation of $\mathfrak{gl}(2,C)$ on the basis $E = \wedge_{i=1}^{n} e_i$, with e_i a basis of C^n. Let $A \in \mathfrak{gl}(2,C)$ and A_{ij} are matrix coefficient of A. Then we have

  $$AE = \sum_{j=1}^{n} e_1 \wedge \ldots \wedge e_{j-1} \wedge \sum A_{jk} e_k \wedge e_{i+1} \wedge \ldots \wedge e_n = \sum_{i=1}^{n} A_{ii} E = Tr(A)E$$

  So the representation of $\mathfrak{gl}(2,C)$ on $\wedge^n C^n$ is the trace map. For $\mathfrak{sl}(2,C)$ since the trace of the matrices are 0, this representation is trivial.

- 4.5(a) Consider the morphism ϕ from $V \otimes W^*$ to $Hom(W,V)$ by:

  $$\phi(v \otimes w^*)(h) = w^*(h)v$$

  This morphism is an isomorphism in the meaning of linear map. We also have $\phi(g(v \otimes w^*))(h) = w^*(g^{-1}h)g(v)$. Hence if we define the action of G on $Hom(W,V)$ by $g(s)(h) = g(s(g^{-1}h))$, then ϕ is a G-isomorphism. Notice

  $$\begin{aligned} Hom(W,V)^G &= \{s \in Hom(W,V) | \forall g \in G, h \in W, g(s(g^{-1}h)) = s(h)\} \\ &= \{s \in Hom(W,V) | \forall g \in G, h \in W, s(gh) = g(s(h))\} \end{aligned}$$

  is just the G-homorphisms of V and W.

  Hence the G-invariant is 0 when W and V are not isomorphic and canonically isomorphic to C when W and V are isomorphic.

- 4.5(b) Let W^* be a proper nonzero sub-representation of V^*, then $ker(W^*)$ is non-zero proper subspace of V and also closed under the action of $\mathfrak{g}$, which is absurd! So V^* is irreducible.

  And we regard the bilnear forms of V as elements of $U = Hom(V, V^*)$, by the mapping ϕ:

  $$(\phi(w)v)(x) = w(v,x)$$

  Then $h \in U$ is $\mathfrak{g}$-bilinear if and only if h is a $\mathfrak{g}$ homomorphism. Hence it has dimension 0 or 1.

- 2.3 $\forall p \in G_1$, let $q = f(p)$. Then let $\bar{p}$ denote the diffemorphism $g \to pg$. Then $f\bar{p} = \bar{q}f$. So $f_*(p)\bar{p}_*(1) = \bar{q}_*(1)f_*(1)$. Notice that $\bar{p}_*(1)$,$\bar{q}_*(1)$ and $f_*(1)$ are isomorphisms. So $f_*(p)$ is also an isomorphism. So f is a local diffemorphism.

- 2.2(a) Let N be a discrete normal subgroup of G. Then $\forall h \in N$, the mapping $\phi_h : g \to ghg^{-1}$ is a continuous map with its image in N. Since its image is connected and contains h, $\phi_h(G) = \{h\}$. Hence h is in the center of G.

- 2.2(b) $\pi_1(G) \simeq ker(\tilde{G} \to G)$ is normal, and is discrete since the covering map is a local diffemorphism.