

The WKB Method

1 Introduction & Motivation

The WKB method (named after Wentzel, Kramers, and Brillouin) is a powerful technique for finding asymptotic approximations to solutions of linear differential equations, particularly where a small parameter multiplies the highest derivative.

- **Key Idea:** It assumes the solution's rapid variation is **exponential** in character. This is inspired by the exact exponential solutions of constant-coefficient equations.
- **Comparison to Other Methods:**
 - **Boundary Layers:** Solve for fast variation in specific spatial regions.
 - **Multiple Scales:** Introduce multiple temporal scales treated as independent variables.
 - **WKB:** Assume specific exponential dependence from the outset.

Historical Context

The method has multiple independent discoveries:

- **Carlini (1817):** Early use in planetary orbits.
- **Liouville & Green (1837):** Systematic formulation.
- **Quantum Mechanics (1920s):** Rediscovered for Schrödinger's equation.
- **Other Names:** Liouville-Green method, phase integral method, WKBJ method (Jeffreys), geometrical optics approximation.

Fundamental Difference from Multiple Scales

While both methods handle multiple scales of variation, they approach the problem differently:

- **Multiple Scales:** Treats different temporal scales as independent variables, determining the functional form through solvability conditions.
- **WKB:** Assumes an exponential ansatz from the beginning, then determines the phase and amplitude functions.
- **Multiple Scales:** More general but requires solving PDEs.
- **WKB:** More specific but often simpler for suitable problems.

2 Introductory Example

Consider the prototypical WKB problem:

$$\varepsilon^2 y'' - q(x)y = 0 \quad (2.1)$$

where the real-valued function $q(x)$ is smooth and nonzero, and $0 < \varepsilon \ll 1$.

2.1 Step 1: Assume WKB Ansatz

The WKB expansion is:

$$y \sim e^{\theta(x)/\varepsilon^\alpha} (y_0(x) + \varepsilon^\alpha y_1(x) + \varepsilon^{2\alpha} y_2(x) + \dots) \quad (2.2)$$

This captures both rapid exponential variation and slower amplitude modulation.

2.2 Step 2: Compute Derivatives

From (2.2):

$$\begin{aligned} y' &\sim e^{\theta/\varepsilon^\alpha} (\varepsilon^{-\alpha}\theta'y_0 + y'_0 + \theta'y_1 + \varepsilon^\alpha(y'_1 + \theta'y_2) + \dots) \\ y'' &\sim e^{\theta/\varepsilon^\alpha} (\varepsilon^{-2\alpha}(\theta')^2y_0 + \varepsilon^{-\alpha}(\theta''y_0 + 2\theta'y'_0 + (\theta')^2y_1) + (\theta''y_1 + 2\theta'y'_1 + (\theta')^2y_2 + y''_0) + \dots). \end{aligned}$$

2.3 Step 3: Substitute and Balance

Substituting into (2.1) and balancing dominant terms gives $\alpha = 1$. The resulting equations are:

$O(1)$ - Eikonal Equation:

$$(\theta')^2 = q(x) \Rightarrow \theta(x) = \pm \int^x \sqrt{q(s)} ds. \quad (2.3)$$

$O(\varepsilon)$ - Transport Equation:

$$\theta''y_0 + 2\theta'y'_0 = 0 \Rightarrow y_0(x) = \frac{c}{\sqrt{\theta'}} = \frac{c}{q(x)^{1/4}}. \quad (2.4)$$

2.4 Step 4: General Solution

Combining both exponential solutions:

$$y(x) \sim q(x)^{-1/4} \left(a_0 e^{-\frac{1}{\varepsilon} \int^x \sqrt{q(s)} ds} + b_0 e^{\frac{1}{\varepsilon} \int^x \sqrt{q(s)} ds} \right). \quad (2.5)$$

2.5 Verification and Applications From Holmes

- For $q(x) = -e^{2x}$, the WKB approximation matches the exact Bessel function solution remarkably well.
- Can approximate large eigenvalues efficiently.
- Provides physical insight into oscillatory vs. exponential behavior.

3 Second Term and Error Analysis

To improve accuracy and estimate errors, at the $O(\varepsilon^2)$ we collect

$$\theta''y_1 + 2\theta'y'_1 + (\theta')^2y_2 + y''_0 - qy_2 = 0.$$

By eikonal equation $(\theta')^2 = q$, the equation reduces to

$$\theta''y_1 + 2\theta'y'_1 + y''_0 = 0.$$

Setting $y_1 = y_0w$ and using the transport equation, one finds that

$$0 = \theta''y_0w + 2\theta'y'_0w + 2\theta'y_0w' + y''_0 = 2\theta'y_0w' + y''_0$$

Integrating the equation, we find:

$$w(x) = d + \frac{1}{8} \frac{q'}{q^{3/2}} + \frac{1}{32} \int^x \frac{(q')^2}{q^{5/2}} ds. \quad (3.1)$$

The expansion remains well-ordered if $\varepsilon y_1(x) \ll y_0(x)$ or $\varepsilon w(x) \ll 1$ which holds provided that

$$\varepsilon \left[|d| + \frac{1}{32} \left| \frac{q'}{q^{3/2}} \right|_\infty \left(4 + \int_{x_0}^{x_1} \left| \frac{q'}{q} \right| dx \right) \right] \ll 1. \quad (3.2)$$

The expansion fails near **turning points** where $q(x) = 0$.

4 Rigorous Foundation via Fixed Point Theory

Setting $s = \frac{x}{\varepsilon}$ and $u(s) := y(x)$, we find that

$$u'' - q(\varepsilon s)u = 0,$$

where we note the coefficient function $q(\varepsilon s)$ is slowly varying. The second order ODE behaves mostly like a constant-coefficient one. Our objective is to construct a rigorous solution. We first rewrite it as a first order system

$$\begin{pmatrix} u \\ u' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ q(\varepsilon s) & 0 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix},$$

where we note the coefficient matrix can be diagonalized as

$$T \begin{pmatrix} 0 & 1 \\ q(\varepsilon s) & 0 \end{pmatrix} T^{-1} = \Lambda \quad \text{with} \quad T = \begin{pmatrix} -\sqrt{q(\varepsilon s)} & 1 \\ \sqrt{q(\varepsilon s)} & 1 \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} -\sqrt{q(\varepsilon s)} & 0 \\ 0 & \sqrt{q(\varepsilon s)} \end{pmatrix}.$$

Performing a change of unknowns

$$W := T \begin{pmatrix} u \\ u' \end{pmatrix},$$

we find that W solves

$$W' = T \begin{pmatrix} u \\ u' \end{pmatrix}' + T' \begin{pmatrix} u \\ u' \end{pmatrix} = T \begin{pmatrix} 0 & 1 \\ q(\varepsilon s) & 0 \end{pmatrix} T^{-1} W + T' T^{-1} W = \Lambda W + \varepsilon p(\varepsilon s) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} W,$$

where

$$p(x) := \frac{q'(x)}{4q(x)},$$

by the smoothness and non-vanishing assumptions made on $q(x)$, is well-defined and smooth.

Now define $\Phi := w_1/w_2$. We compute

$$\begin{aligned} \Phi' &= \frac{w'_1 w_2 - w_1 w'_2}{w_2^2} \\ &= \frac{\left(-\sqrt{q(\varepsilon s)} w_1 + \varepsilon p(\varepsilon s) w_1 - \varepsilon p(\varepsilon s) w_2 \right) w_2 - w_1 \left(\sqrt{q(\varepsilon s)} w_2 - \varepsilon p(\varepsilon s) w_1 + \varepsilon p(\varepsilon s) w_2 \right)}{w_2^2} \\ &= -2\sqrt{q(\varepsilon s)}\Phi + \varepsilon p(\varepsilon s)\Phi^2 - \varepsilon p(\varepsilon s). \end{aligned}$$

Integrate with initial condition $\Phi(0) = 0$ (corresponding to the solution with initial conditions $w_1(0) = 0, w_2(0) = 1$):

$$\Phi(s) = \varepsilon \int_0^s e^{\int_t^s -2\sqrt{q(\varepsilon \tau)} d\tau} p(\varepsilon t)(\Phi^2(t) - 1) dt.$$

Define the operator on $C^b([0, L/\varepsilon], \mathbb{C})$:

$$(\mathcal{T}\Phi)(s) := \varepsilon \int_0^s K(s, t)(\Phi^2(t) - 1) dt,$$

where the kernel is:

$$K(s, t) = e^{\int_t^s -2\sqrt{q(\varepsilon \tau)} d\tau} p(\varepsilon t).$$

4.1 Contraction Mapping Argument

Step 1: Define the Banach Space

Let $X = C^b([0, S], \mathbb{C})$ with $S = L/\varepsilon$, equipped with the supremum norm:

$$\|\Phi\|_\infty = \sup_{s \in [0, S]} |\Phi(s)|.$$

Step 2: Estimate the Kernel

For real-valued, smooth, non-vanishing $q(\cdot)$, we have:

$$-\operatorname{Re} \sqrt{q(\varepsilon s)} \leq 0 \quad \text{and} \quad |K(s, t)| \leq |p(\varepsilon t)| \leq P(L)$$

where $P(L) = \sup_{x \in [0, L]} |p(x)|$.

Step 3: Invariance of \mathcal{T} on a Ball

Let $B_R = \{\Phi \in X : \|\Phi\|_\infty \leq R\}$. For $\Phi \in B_R$:

$$|(\mathcal{T}\Phi)(s)| \leq \varepsilon \int_0^s P(L)(|\Phi(t)|^2 + 1) dt \leq \varepsilon P(L)(R^2 + 1)S = P(L)L(R^2 + 1).$$

To ensure $\mathcal{T} : B_R \rightarrow B_R$, we require:

$$P(L)L(R^2 + 1) \leq R. \quad (3.1)$$

Step 4: Contraction Property

For $\Phi, \Psi \in B_R$, we estimate that

$$|(\mathcal{T}\Phi)(s) - (\mathcal{T}\Psi)(s)| \leq \varepsilon \int_0^s |K(s, t)| \cdot |\Phi^2(t) - \Psi^2(t)| dt.$$

Since $|\Phi^2 - \Psi^2| = |\Phi - \Psi||\Phi + \Psi| \leq 2R\|\Phi - \Psi\|_\infty$, we get

$$|(\mathcal{T}\Phi)(s) - (\mathcal{T}\Psi)(s)| \leq \varepsilon \int_0^s P(L) \cdot 2R\|\Phi - \Psi\|_\infty dt = 2P(L)LR\|\Phi - \Psi\|_\infty.$$

Thus, \mathcal{T} is a contraction if:

$$2P(L)LR < 1. \quad (3.2)$$

Step 5: Optimal Choices of R

We seek $R > 0$ satisfying both (3.1) and (3.2). The optimal choice is $R = 1$, which set $\frac{R}{R^2+1} = \frac{1}{2R}$. Now we obtain the condition for the choice of L :

$$\alpha(L) := 2LP(L) < 1,$$

which makes \mathcal{T} is a contraction on B_1 .

Theorem 4.1. *If $q(x)$ is real-valued, smooth, and nonzero on $[0, L]$, and if:*

$$L \sup_{x \in [0, L]} \left| \frac{q'(x)}{4q(x)} \right| < \frac{1}{2}$$

then there exists a unique fixed point $\Phi^ \in C^b([0, L/\varepsilon], \mathbb{R})$ with $\|\Phi^*\|_\infty \leq 1$ satisfying the WKB ratio equation. Moreover, the iterative scheme $\Phi_{n+1} = \mathcal{T}\Phi_n$ converges geometrically:*

$$\|\Phi^* - \Phi_n\|_\infty \leq \frac{\alpha(L)^n}{1 - \alpha(L)} \|\Phi_1 - \Phi_0\|_\infty.$$

4.2 Refined Order Analysis of Φ_1

Set $\Phi_0 \equiv 0$. We now demonstrate that the first iterate Φ_1 satisfies $\|\Phi_1\|_\infty = O(\varepsilon)$ in both the exponential ($q(x) > 0$) and oscillatory ($q(x) < 0$) cases.

Case 1: $q(x) > 0$ (Exponential Behavior)

For the case $q(x) > 0$ on $[0, L]$, we have $-2\sqrt{q(\varepsilon\tau)} < -\theta < 0$ for $\tau \in [0, L/\varepsilon]$. Since

$$\Phi_1(s) = -\varepsilon \int_0^s e^{\int_t^s -2\sqrt{q(\varepsilon\tau)} d\tau} p(\varepsilon t) dt,$$

we estimate that for $s \in [0, L/\varepsilon]$

$$|\Phi_1(s)| \leq \varepsilon \int_0^s e^{\int_t^s -2\theta d\tau} P(L) dt = \varepsilon P(L) \frac{1 - e^{-2s\theta}}{2\theta} \leq \varepsilon \frac{P(L)}{2\theta}$$

whence $\|\Phi_1\|_\infty = O(\varepsilon)$.

Case 2: $q(x) < 0$ (Oscillatory Behavior)

For $q(x) < 0$ on $[0, L]$, let $q(x) = -r(x)$ with $r(x) > 0$. Then

$$\Phi_1(s) = -\varepsilon \int_0^s e^{\int_t^s -2i\sqrt{r(\varepsilon\tau)} d\tau} p(\varepsilon t) dt$$

Changing variables $x = \varepsilon s$, $u = \varepsilon t$, $\varepsilon\tau = v$ and denoting $\varphi(x, u) = -2 \int_u^x \sqrt{r(v)} dv$, we obtain that

$$\Phi_1\left(\frac{x}{\varepsilon}\right) = - \int_0^x e^{i\varphi(x, u)/\varepsilon} p(u) du.$$

Now integrate by parts. We compute

$$\frac{d}{du} \left[e^{i\varphi(x, u)/\varepsilon} \right] = e^{i\varphi(x, u)/\varepsilon} \frac{i}{\varepsilon} \frac{\partial \varphi(x, u)}{\partial u} = e^{i\varphi(x, u)/\varepsilon} \frac{i}{\varepsilon} 2\sqrt{r(u)},$$

whence

$$e^{i\varphi(x, u)/\varepsilon} = \frac{\varepsilon}{2i\sqrt{r(u)}} \frac{d}{du} \left[e^{i\varphi(x, u)/\varepsilon} \right]$$

Substitute into the integral:

$$\Phi_1(s) = - \int_0^x p(u) \cdot \frac{\varepsilon}{2i\sqrt{r(u)}} \frac{d}{du} \left[e^{i\varphi(x, u)/\varepsilon} \right] du$$

Now integrate by parts:

$$\Phi_1(s) = - \left[\frac{\varepsilon p(u)}{2i\sqrt{r(u)}} e^{i\varphi(x, u)/\varepsilon} \right]_0^x + \int_0^x \frac{d}{du} \left(\frac{\varepsilon p(u)}{2i\sqrt{r(u)}} \right) e^{i\varphi(x, u)/\varepsilon} du.$$

Then it is easy to see that $\|\Phi_1\|_\infty = O(\varepsilon)$.

In both cases, we have established that:

$$\|\Phi_1\|_\infty = O(\varepsilon).$$

4.3 Recovery of WKB Solutions

Step 1: Reconstruct W from Φ^*

The fixed point Φ^* gives the relationship between components:

$$w_1(s) = \Phi^*(s) w_2(s)$$

The w_2 equation becomes:

$$w'_2(s) = \sqrt{q(\varepsilon s)} w_2(s) - \varepsilon p(\varepsilon s) w_1(s) + \varepsilon p(\varepsilon s) w_2(s) = \sqrt{q(\varepsilon s)} w_2(s) + \varepsilon p(\varepsilon s)(1 - \Phi^*(s)) w_2(s)$$

whose solution reads

$$w_2(s) = ce^{\int_s^x (\sqrt{q(\varepsilon\tau)} + \varepsilon p(\varepsilon\tau)(1 - \Phi^*(\tau))) d\tau}.$$

Changing back to the original coordinate, we find

$$\begin{aligned} w_2(x/\varepsilon) &= ce^{\frac{1}{\varepsilon} \int^x \sqrt{q(x')} dx' + \int^x p(x')(1 - \Phi^*(x'/\varepsilon)) dx'} \\ &= ce^{\frac{1}{\varepsilon} \int^x \sqrt{q(x')} dx'} e^{\int^x p(x') dx'} e^{\int^x p(x') \Phi^*(x'/\varepsilon) dx'} \\ &= ce^{\frac{1}{\varepsilon} \int^x \sqrt{q(x')} dx'} e^{\int^x \frac{q'(x)}{4q(x)} dx'} e^{\int^x p(x') \Phi^*(x'/\varepsilon) dx'} \\ &= ce^{\frac{1}{\varepsilon} \int^x \sqrt{q(x')} dx'} q(x)^{\frac{1}{4}} e^{\int^x p(x') \Phi^*(x'/\varepsilon) dx'} \\ &= ce^{\frac{1}{\varepsilon} \int^x \sqrt{q(x')} dx'} q(x)^{\frac{1}{4}} e^{O(\varepsilon)} \\ &= ce^{\frac{1}{\varepsilon} \int^x \sqrt{q(x')} dx'} q(x)^{\frac{1}{4}} (1 + O(\varepsilon)) \end{aligned}$$

where the $O(\varepsilon)$ term is measured in the Banach space $C^b([0, L], \mathbb{C})$ equipped with L^∞ -norm.

Step 2: Return to Original Variables

Since

$$\begin{pmatrix} u \\ u' \end{pmatrix} = T^{-1}W = \frac{1}{2} \begin{pmatrix} -\frac{1}{\sqrt{q}} & \frac{1}{\sqrt{q}} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\frac{1}{\sqrt{q}} & \frac{1}{\sqrt{q}} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \Phi^* \\ 1 \end{pmatrix} w_2$$

we rigorously justify the validity of the solution of form

$$e^{\frac{1}{\varepsilon} \int^x \sqrt{q(s)} ds} (y_0(x) + O(\varepsilon)).$$

The other part come be obtained by working with $\tilde{\Phi} := w_2/w_1$. We also note that in this case we shall integrate in the backward direction when defining the contraction mapping to gain smallness from the exponential part in the case $q > 0$. Such solution is independent from the foregoing one because at $x = 0$ $|w_2(0)/w_1(0)| < \infty$ while the ratio of the foregoing one is infinite.

Question: How to justify for higher order expansions?

5 Summary

The WKB method is characterized by:

- **Exponential ansatz** for rapid variation
- **Eikonal equation** determines phase (nonlinear)
- **Transport equation** determines amplitude (linear)
- **Turning points** require Airy function matching (Next class)
- **Wide applicability** in quantum mechanics, wave propagation, and eigenvalue problems
- **Historical significance** with multiple independent discoveries across disciplines

The method remains a cornerstone of asymptotic analysis for linear wave-type problems with slowly varying coefficients.