The Method of Multiple Scales

1 Introduction

The **Method of Multiple Scales** (MMS) addresses a fundamental limitation of regular perturbation theory: the appearance of **secular terms** that grow unbounded and ruin long-time approximations.

- **Key Idea:** Introduce multiple time scales treated as independent variables
- **Transformation:** ODE → PDE (paradoxically simpler for long-time behavior)
- **Historical Roots:** Lindstedt (1882), Poincaré (1886) for celestial mechanics; modern form by Kuzmak (1959), Cole & Kevorkian (1963)

2 Introductory Example: Weakly Damped Oscillator

2.1 Regular Expansion Failure

Consider:

$$y'' + \varepsilon y' + y = 0, \quad y(0) = 0,\ y'(0) = 1$$

Regular expansion $y \sim y_0 + \varepsilon y_1 + \cdots$ yields:

$$y \sim \sin t - \tfrac{1}{2}\varepsilon t \sin t + O(\varepsilon^2)$$

The term $-\frac{1}{2}\varepsilon t \sin t$ is **secular**—grows unbounded. Compare with exact solution:

$$y = \frac{1}{\sqrt{1-\varepsilon^2/4}} e^{-\varepsilon t/2} \sin\left(t\sqrt{1-\varepsilon^2/4}\right)$$

which decays exponentially.

2.2 Multiple-Scale Solution

Introduce scales: $t_1 = t$ (fast), $t_2 = \varepsilon t$ (slow). Then:

$$\frac{d}{dt} = \partial_{t_1} + \varepsilon \partial_{t_2}, \quad \frac{d^2}{dt^2} = \partial_{t_1}^2 + 2\varepsilon \partial_{t_1}\partial_{t_2} + \varepsilon^2 \partial_{t_2}^2$$

Assume: $y \sim y_0(t_1, t_2) + \varepsilon y_1(t_1, t_2) + \cdots$

$O(1)$ **problem:**

$$(\partial_{t_1}^2 + 1)y_0 = 0 \Rightarrow y_0 = A_0(t_2)\sin t_1 + B_0(t_2)\cos t_1$$

$O(\varepsilon)$ **problem:**

$$(\partial_{t_1}^2 + 1)y_1 = -2\partial_{t_1}\partial_{t_2} y_0 - \partial_{t_1} y_0$$

Substituting y_0 gives right-hand side:

$$-(2A_0' + A_0)\cos t_1 + (2B_0' + B_0)\sin t_1$$

Remove secular terms by setting coefficients to zero:

$$2A_0' + A_0 = 0, \quad 2B_0' + B_0 = 0$$

With ICs: $A_0(0) = 1$, $B_0(0) = 0 \Rightarrow A_0 = e^{-t_2/2}$, $B_0 = 0$

Uniform approximation:

$$y \sim e^{-t_2/2} \sin t_1 = e^{-\varepsilon t/2} \sin t$$

3 Example 2: Duffing Oscillator (Exercise 3.17)

3.1 Problem Setup

The Duffing equation with weak nonlinearity:

$$y'' + y + \varepsilon y^3 = 0, \quad y(0) = a,\ y'(0) = b$$

Regular expansion yields secular terms. Use MMS with $t_1 = t$, $t_2 = \varepsilon t$.

3.2 Multiple-Scale Analysis

Assume: $y \sim y_0(t_1, t_2) + \varepsilon y_1(t_1, t_2) + \cdots$

$O(1)$ **problem:**

$$(\partial_{t_1}^2 + 1)y_0 = 0 \Rightarrow y_0 = A(t_2)\cos(t_1 + \phi(t_2))$$

Better to use complex form: $y_0 = R(t_2)e^{it_1} + \bar{R}(t_2)e^{-it_1}$

$O(\varepsilon)$ **problem:**

$$(\partial_{t_1}^2 + 1)y_1 = -2\partial_{t_1}\partial_{t_2}y_0 - y_0^3$$

Right-hand side contains $e^{\pm it_1}$ and $e^{\pm 3it_1}$ terms. Secular terms come from $e^{\pm it_1}$.
Remove secularity condition:

$$2iR' + 3|R|^2 R = 0$$

Write $R = \frac{1}{2}Ae^{i\theta}$ with A, θ real:

$$A' = 0, \quad \theta' = -\frac{3}{8}A^2$$

Thus: A constant, $\theta = -\frac{3}{8}A^2 t_2 + \theta_0$

Final approximation:

$$y \sim A\cos\left[\left(1 - \frac{3}{8}\varepsilon A^2\right)t + \theta_0\right]$$

where A and θ_0 determined from initial conditions.

4 Example 3: Forced Oscillator Near Resonance

4.1 Problem Setup (Section 3.4)

Damped, forced Duffing oscillator near resonance:

$$y'' + \varepsilon\lambda y' + y + \varepsilon\kappa y^3 = \varepsilon\cos(1 + \varepsilon\omega)t$$

with zero initial conditions. The forcing frequency $1 + \varepsilon\omega$ is close to natural frequency 1.

4.2 Multiple-Scale Analysis

Use scales: $t_1 = t$, $t_2 = \varepsilon t$

$O(1)$ **problem:**

$$(\partial_{t_1}^2 + 1)y_0 = 0 \Rightarrow y_0 = A(t_2)\cos(t_1 + \phi(t_2))$$

$O(\varepsilon)$ **problem:**

$$(\partial_{t_1}^2 + 1)y_1 = -2\partial_{t_1}\partial_{t_2}y_0 - \lambda\partial_{t_1}y_0 - \kappa y_0^3 + \cos(t_1 + \omega t_2)$$

After lengthy algebra, secular removal gives amplitude-phase equations:

$$A' = -\frac{\lambda}{2}A + \frac{1}{2}\sin\psi$$
$$A\psi' = A\omega - \frac{3\kappa}{8}A^3 + \frac{1}{2}\cos\psi$$

where $\psi = \omega t_2 - \phi$.

Steady-state response: Set $A' = \psi' = 0$ to find constant amplitude solutions. These exhibit the **jump phenomenon** and hysteresis typical of nonlinear resonance.

5 Example 4: Van der Pol Oscillator

5.1 Problem Setup

The classic Van der Pol equation:

$$y'' + y = \varepsilon(1 - y^2)y'$$

describes self-sustained oscillations.

5.2 Multiple-Scale Analysis

Use scales: $t_1 = t$, $t_2 = \varepsilon t$, expansion: $y \sim y_0 + \varepsilon y_1 + \cdots$

$O(1)$ **problem:**

$$(\partial_{t_1}^2 + 1)y_0 = 0 \Rightarrow y_0 = A(t_2)\cos(t_1 + \phi(t_2))$$

$O(\varepsilon)$ **problem:**

$$(\partial_{t_1}^2 + 1)y_1 = -2\partial_{t_1}\partial_{t_2}y_0 + (1 - y_0^2)\partial_{t_1}y_0$$

Right-hand side calculation gives secular terms proportional to $e^{\pm it_1}$. Removal yields:

$$A' = \frac{A}{8}(4 - A^2)$$
$$\phi' = 0$$

Amplitude evolution: Solve $A' = \frac{A}{8}(4 - A^2)$:

- Fixed points: $A = 0$ (unstable), $A = 2$ (stable)
- General solution: $A(t_2) = 2[1 + Ce^{-t_2}]^{-1/2}$

Final approximation:

$$y \sim 2\cos(t + \phi_0) + O(\varepsilon)$$

The system approaches a **limit cycle** with amplitude 2, regardless of initial conditions.

6 Example 5: WKB-Type Problem with Slow Frequency

6.1 Problem Setup

Consider oscillator with slowly varying frequency:

$$y'' + \omega^2(\varepsilon t)y = 0, \quad y(0) = a,\ y'(0) = b$$

where $\omega(\tau) > 0$ is a smooth function.

6.2 Multiple-Scale Analysis

Use scales: $t_1 = \frac{1}{\varepsilon}\int_0^t \omega(\varepsilon s)ds$ (fast phase), $t_2 = \varepsilon t$ (slow time)

Derivative transformation:

$$\frac{d}{dt} = \omega(t_2)\partial_{t_1} + \varepsilon\partial_{t_2}, \quad \frac{d^2}{dt^2} = \omega^2\partial_{t_1}^2 + \varepsilon(2\omega\partial_{t_1}\partial_{t_2} + \omega'\partial_{t_1}) + \varepsilon^2\partial_{t_2}^2$$

Assume: $y \sim y_0(t_1, t_2) + \varepsilon y_1(t_1, t_2) + \cdots$

$O(1)$ **problem:**

$$\omega^2(\partial_{t_1}^2 + 1)y_0 = 0 \Rightarrow y_0 = A(t_2)\cos t_1 + B(t_2)\sin t_1$$

$O(\varepsilon)$ **problem:**

$$\omega^2(\partial_{t_1}^2 + 1)y_1 = -2\omega\partial_{t_1}\partial_{t_2}y_0 - \omega'\partial_{t_1}y_0$$

Secular removal gives:

$$(\omega A)' = 0, \quad (\omega B)' = 0 \Rightarrow A(t_2) = \frac{C}{\omega(t_2)}, \; B(t_2) = \frac{D}{\omega(t_2)}$$

Final approximation (leading order):

$$y \sim \frac{1}{\omega(\varepsilon t)}\left[C\cos\theta(t) + D\sin\theta(t)\right], \quad \theta(t) = \frac{1}{\varepsilon}\int_0^t \omega(\varepsilon s)ds$$

This is the **WKB approximation** derived via multiple scales.

7 Labor-Saving Techniques and General Methodology

7.1 Efficient Solution Forms

For $y'' + \omega^2 y = 0$, use:

- $y = A\cos(\omega t + \phi)$ (amplitude-phase form)
- $y = Re^{i\omega t} + \bar{R}e^{-i\omega t}$ (complex form) - **recommended**

These make secular term identification easier.

7.2 Direct Secularity Condition

For $(\partial_{t_1}^2 + \omega^2)y_1 = f(t_1, t_2)$, secular terms arise if f contains $e^{\pm i\omega t_1}$.
Shortcut: Project f onto $e^{\pm i\omega t_1}$ and set coefficients to zero.

7.3 General Procedure

1. Identify relevant time scales from physics or dominant balance
2. Introduce scales as independent variables
3. Transform derivatives using chain rule
4. Assume perturbation expansion
5. Solve order-by-order
6. Remove secular terms at each order
7. Apply initial/boundary conditions
8. Reconstruct composite approximation

8 Extensions and Refinements

8.1 Three Time Scales

For longer validity ($t = O(1/\varepsilon^2)$), use:

$$t_1 = t, \quad t_2 = \varepsilon t, \quad t_3 = \varepsilon^2 t$$

This captures frequency shifts: $\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots$

8.2 Minimum Error Principle

Multiple-scale expansions are not unique. The preferred expansion minimizes:

$$E = \max_{0 \le t \le T/\varepsilon} |y_{\text{exact}}(t) - y_{\text{approx}}(t)|$$

This principle determines otherwise arbitrary functions in higher-order analyses.

9 Summary

The Method of Multiple Scales is essential for:

- Systems with multiple time scales (fast oscillation/slow modulation)
- Removing secular terms in perturbation expansions
- Studying long-time behavior of nonlinear oscillators
- Analyzing resonance phenomena in forced systems
- Deriving WKB-type approximations for slowly varying systems

The method's power lies in treating different time scales as independent variables and using the resulting freedom to eliminate non-uniformities.