Learning Objectives

By the end of this lecture, you should be able to:

– Apply the basic principle of counting to simple sequential problems.

– Solve problems using the multiplication rule.

Text book sections Ross' book: Section 1.1-1.2

Introduction

Calculating the probability involves counting the number outcomes. We will start with introducing some fundamental methods for counting.

Basic Counting Rule

If an experiment can be broken down into r successive stages, and

Stage i has n_i possible outcomes,

then the total number of different outcomes for the full experiment is:

$$n_1 \cdot n_2 \cdots n_r$$

Example 1.1. Suppose you are buying an ice cream cone. You can choose whether to have a cake cone or a waffle cone, and whether to have chocolate, vanilla, or strawberry as your flavor. How many possible outcomes?

Text

Example 1.2. Suppose that 10 people are running a race. Assume that ties are not possible and that all 10 will complete the race, so there will be well-defined first place, second place, and third place winners. How many possibilities are there for the first, second, and third place winners?

Example 1.3. A student has to select one starter, one main course, and one dessert from a menu containing 4 starters, 5 main courses, and 3 desserts. How many meal combinations are possible?

Example 1.4 A small community contains 10 women, each of whom has 3 children. If one woman and one of her children are chosen. How many different subcommittees are possible?

Summary

The multiplication rule is one of the most fundamental tools in combinatorics. Understanding how to decompose a counting problem into stages allows us to calculate the number of outcomes efficiently. These tools will be built upon when we discuss permutations and combinations in the next lecture.

Learning Objectives

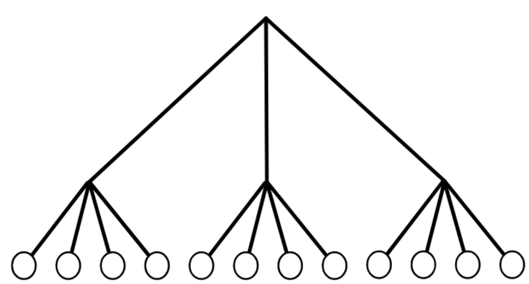
By the end of this lecture, you should be able to:

- Refresh the basic counting rules.
- Distinguish between permutations and combinations.
- Apply formulas to count ordered and unordered selections.
- Solve real-world problems using permutations and combinations.
- Interpret and compute binomial coefficients.

Text book sections Ross' book: Section 1.3-1.5

The basic counting process

Recall the idea about counting rule we discussed in Lecture 1. The following tree diagram illustrating the general idea about the multiplication rule.



Example 2.1. A license plate consists of three uppercase letters followed by three digits. How many such plates can be formed if:

(a) Repetition is allowed?

(b) No repetition is allowed?

Permutations: Order Matters

A permutation is an arrangement of objects in a specific order. The number of permutations of r objects chosen from a set of n distinct objects is:

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Example 2.2. A class in probability theory consists of 6 men and 4 women. An examination is given and the students are ranked according to their performance. Assume that no two students obtain the same score.

(a) How many different ranking s are possible?

(b) If the men are ranked just among themselves and the women just among themselves, how many different rankings are possible?

Example 2.3. Suppose we have 5 applicants and 3 different job positions. How many ways can you assign the 3 jobs?

Example 2.4. How many different 3-digit codes can be created from digits 1 to 5 if no digits repeats?

Combinations: Order Doesn't Matter

A combination is a selection of objects where order does not matter. The number of combinations of r objects chosen from n distinct objects is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

The $\binom{n}{r}$ is called binomial coefficient, read as "*n choose k*", counts the number of ways to choose r elements from a set of n elements. The coefficient is often defined in terms of factorials, but keep in mind that $\binom{n}{k}$ is 0 if $k > n$, even though the factorial of a negative number is undefined.

Example 2.5. How many ways can we choose 3 people out of 10 to form a committee?

Example 2.6. Suppose our class has 70 students enrolled. I need to select three students without replacement for this question. How many possible outcomes of three students could be chosen from our class?

Example 2.7. Out of 8 math problems on an exam, a student must choose 5 to answer. How many ways can this be done?

Example 2.8. Let's select two numbers from set $\{3, 4, 5\}$. How many possible outcomes are there whn the order matters? How many when the order does not matter?

Summary

Permutations count ordered selections, combinations count unordered selections, and binomial coefficients quantify the number of combinations. These tools form the backbone of counting in probability theory.

Learning Objectives

By the end of this lecture, you should be able to:

- Understand and compute the multinomial coefficient.
- Apply the multinomial coefficient to partitioning problems.
- Compare and contrast common counting rules in probability.

Text book sections Ross' book: Section 1.5

Multinomial Coefficient

The number of ways to divide n distinct objects into k non-overlapping groups of sizes $n_1, n_2, \ldots, n_k$, where $n_1 + n_2 + \cdots + n_k = n$, is given by the multinomial coefficient:

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

A multinomial coefficients represents the number of different permutations of n subjects, of which n_1 are alike, n_2 are alike, ..., n_r are alike, where $n_1 + n_2 + \cdots + n_r = n$. Another way to see this is to consider the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where $n_1 + n_2 + \cdots + n_r = n$.

Example 3.1. In how many ways can 12 students be divided into 3 groups of 4 students each?

Example 3.2. A police department consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the 3 groups are possible?

Example 3.3. How many ways to assign 10 balls into 3 boxes such that 4 go into box A,3 balls go into box B, and 3 go into box C?

Example 3.4. How many different ways can the letters of the word `BANANA` be arranged?

Summary: Counting Principles

The following table summarizes the major counting formulas introduced so far.

Concept	Description	Formula
Permutations	Number of ways to choose and arrange r items from n distinct items (order matters)	$P(n, r) = \frac{n!}{(n-r)!}$
Combinations	Number of ways to choose r items from n distinct items (order does not matter)	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$
Multinomial Coefficient	Number of ways to split n items into k labeled groups of sizes $n_1, \ldots, n_k$	$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$

Example 3.5. How many ways can you assign 1st, 2nd, and 3rd place to 10 people?

Example 3.6. How many 3-digit numbers can you form using digits 1-9 if not digit repeats?

Example 3.7. How many ways are there to permute the letters in the word `STATISTICS`?

Learning Objectives

By the end of this lecture, you should be able to:

- Understand the structure of sample spaces and events.
- Understand the operation of events.
- Find the probability for equally likely event.

Text book sections Ross' book: Section 2.1-2.2, 2.5

Sample Space and Events

The experiment is any action or process whose outcome is subject to uncertainty.

The sample space S of an experiment is the set of all possible outcomes. An event is a subset of the sample space S.

Example 4.1. Flipping a coin:

Example 4.2. Rolling a die:

Example 4.3. Taking an exam:

Example 4.4. Drawing one card from a deck:

The union of two events A and B, $A \cup B$ is the event that occurs if and only if at least one of A , B occurs. In another words, A occurs, or B occurs, or both occur.

Example 4.5. Rolling a die, let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$.

Example 4.6. Flipping two coins. Let $A = \{HH\}$ and $B = \{HH, HT\}$.

The intersection of two events A and B, $A \cap B$ is the event that both A and B occur.

Example 4.7. Rolling a die, let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$.

Example 4.8. Flipping two coins. Let $A = \{HH\}$ and $B = \{HH, HT\}$.

The complement of an event A, A^c, is the event that A does not occur.

Example 4.9. Rolling a die, let $A = \{1, 2, 3\}$.

DeMorgan's Law

- $(A \cup B)^c = A^c \cap B^c$.

- $(A \cap B)^c = A^c \cup B^c$.

Example 4.10. Suppose the sample space $S = \{1, 2, 3, 4, 5, 6\}$. Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$.

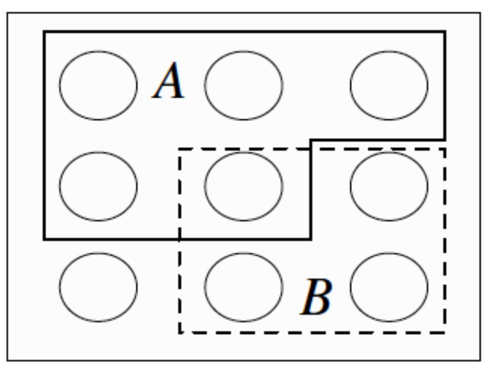
Probability

When every element of your sample space S is equally likely, then the probability of any event $A \subseteq S$ is defined as

$$P(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes in } S}$$

Example 4.11. Suppose we are tossing two fair coins. Denote sample space S contains all possible outcomes. Define an event A be at least one head recorded. Find $P(A)$.

Example 4.12. Randomly select a pebble



Example 4.13. A box contains four \$10 bills, six \$5 bills, and two \$1 bills. Two bills are taken at random from the box without replacement. What is the probability that both bills will be of the same denomination?

Learning Objectives

By the end of this lecture, you should be able to:

- Apply the axioms of probability to compute probabilities.
- Apply useful propositions that follow from the axioms.
- Use these properties to simplify and compute probabilities.

Text book sections Ross' book: Section 2.3-3.4

Axioms of Probability

A probability space contains of a sample space S and a probability function P which takes an event $A \subseteq S$ as input and returns $P(A)$, a real number between 0 and 1, as output. The function P must satisfy the following axioms.

Axiom 1 $0 \leq P(A) \leq 1$.

Axiom 2 $P(S) = 1$

Axiom 3 For any sequence of mutually exclusive events $A_1, A_2, \ldots$,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Example 5.1. Let's consider tolling a based coin. Denote sample space $S = \{H, T\}$. Suppose $P(H) = 0.7$ and $P(T) = 0.3$. Verify those three axioms.

Some Useful Propositions

Proposition 1. The probability of the complement of A is the remaining probability not in A.

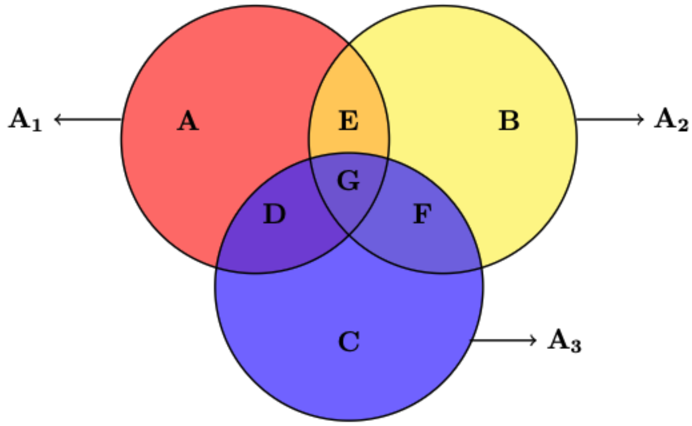
Proposition 2. A smaller subset cannot have a larger probability, that is if $A \subseteq B$, then $P(A) \leq P(B)$.

Proposition 3. For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proposition 4. In general, if we have more than two events $A_1, A_2, \ldots, A_n$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i}^{n} P(A_i) - \sum_{i \leq j} P(A_i \cap A_j) + \sum_{i \leq j \leq k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

Example 5.2. Consider $n = 3$



Example 5.3. Let's flip a coin 10 times, find $P(\text{At least on Head})$.

Example 5.4. Suppose we roll a fair six-sided die once. Let's define $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Find $P(A \cup B)$.

Learning Objectives

- Reinforce your understanding of probability calculations through practice.
- Apply basic counting and probability rules to solve real-world and theoretical problems.
- Build confidence in using axioms and propositions of probability.

Practice Problems

Example 6.1. (Birthday problem) There are k people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year (we exclude February 29), and that people's birthdays are independent (knowing some people's birthdays give us no information about other people's birthday). What is the probability that at least one pair of people in the group have the same birthday?

Example 6.2. (Full house in poker) A 5-card hand is dealt from a standard well-shuffled 52-card deck. The hand is called a *full house* in poker if it consists of three cards of some rank and two cards of another rank, for example, three 7's and two 10's (in any order). What is the probability of a full house?

Example 6.3. (Newton-Pepys problem) Issac Newton was consulted about the following problem by Samuel Pepys, who wanted the information for gambling purposes. Which of the following events has the hightest probability?

- A: At least one 6 appears when 6 fair dice are rolled.
- B: At least two 6's appear when 12 fair dice are rolled.
- C: At least three 6's appear when 18 fair die are rolled.

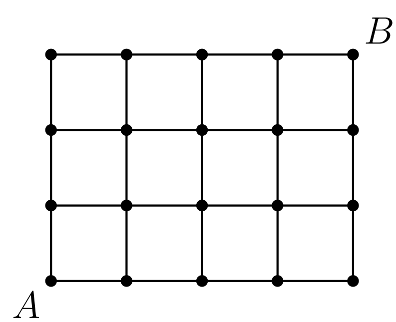
Example 6.4. How many ways are there to select a president, a vice president, a secretary and a treasure from a club of 10 members?

Example 6.5. Suppose a committee of four is needed from the club of ten members. How many committees of four people are possible?

Instructions

This Homework contains *7 questions* and *20 points* in total. Show all work for full credit. Partial credit will be given if your process is clearly shown and mathematically reasonable. If the question is theoretical, please provide a written explanation.
Due date: Friday, September 12th by 11:59pm.

1. (2 points) How many different letter arrangements can be made from the letter `Mississippi`?
2. (2points) If 4 Americans, 3 French people, and 3 British people are to be seated in a row, how many seating arrangements are possible when people of the same nationality must sit next to each other?
3. (2 points) A committee of 6 people is to be chosen from a group consisting of 7 men and 8 women. If the committee must consist of at least 3 women and at least 2 men, how many different committees are possible?
4. (3 points) Consider the grid of points shown here. Suppose that, starting at the point labeled A, you can go one step up or one step to the right at each move. This procedure is continued until the point labeled B is reached. How many different paths from A to B are possible? *Hint: Note that to reach B from A, you must take 4 steps to the right and 3 steps upward.*



5. (3 points) A college has 10 time slots for its courses, and blithely assigns courses to completely random time slots, independently. The college offers exactly 3 statistics courses. What is the probability that 2 or more of the statistics courses are in the same time slot?
6. (3 points) A certain family has 6 children, consisting of 3 boys and 3 girls. Assuming that all birth orders are equally likely, what is the probability that the 3 eldest children are the 3 girls?
7. A club consists of 10 seniors, 12 juniors, and 15 sophomores. An organizing committee of size 5 is chosen randomly (with all subsets of size 5 equally likely).
   (a) (2 points) Find the probability that there are exactly 3 sophomores in the committee?
   (b) (3 points) Find the probability that the committee has at least one representative from each of the senior, junior, and sophomore classes.