Learning Objectives

By the end of this lecture, you should be able to:

- Understand the multivariate analogs of the cdf and pmf.
- Find the marginal distribution from a joint distribution for discrete case.
- Find the conditional distribution from a joint distribution for discrete case.

Text book sections Ross' book: Section 6.1, 6.2, 6.4

Joint, Marginal, and Conditional

Previously, we introduced random variables and their distributions and noted that the individual distribution of two random variables do not tell us anything about whether the random variables are independent or dependent.

For example, two Bern($\frac{1}{2}$) random variables X and Y could be independent if they indicate Heads on two different coin flips, or dependent if they indicate Heads and Tails, respectively, on the same coin flip. These individual pmfs are missing important information about how the two random variables are related. In the real life, we usually care about the relationship between multiple random variables in the same experiment.

For example, in Genetics study, to study the relationships between various genetic markers and a particular disease, if we only looked separately at distributions for each genetic marker, we could fail to learn about whether an interaction between markers is related to the disease.

This chapter considers joint distributions, also called multivariate distributions, which capture the previously missing information about how multiple random variables interact.

Joint distribution

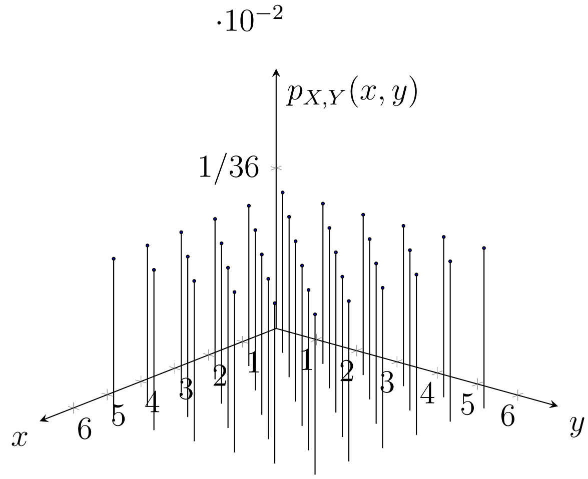
The joint distribution of two random variables X and Y provides complete information about the probability of the vector (X, Y) falling into any subset of the plane.

Consider the experiment of tossing a pair dice. The sample space contains 36 sample points corresponding to 36 ways in which numbers may appear on the faces of dice. We consider two events:

- X : The number of dots appearing on die 1;
- Y : The number of dots appearing on die 2.

Throwing a pair of 1s is the $(1, 1)$, throwing a 2 on die 1 and a 3 on die 2 is $(2, 3)$. Assuming equally chance probability, we have

$$p(x, y) = P(X = x, Y = y) = \frac{1}{36}$$



Let X and Y be discrete random variables. The joint pmf for X and Y is given by

Remarks:

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For discrete random variables X and Y, their joint cdf is defined as

Example 1.1. Let X and Y be predictions about the temperature for yesterday and today respectively.

$$X = \begin{cases} 2 & \text{high} \\ 1 & \text{average} \\ 0 & \text{low} \end{cases} \qquad Y = \begin{cases} 2 & \text{high} \\ 1 & \text{average} \\ 0 & \text{low} \end{cases}$$

Their joint pmf is Their joint pmf is

	$y = 0$	$y = 1$	$y = 2$
$x = 0$	0.20	0.10	0.01
$x = 1$	0.10	0.40	0.04
$x = 2$	0.01	0.04	0.10

What is the probability that the prediction for both days will be high?

What is the probability that the prediction for both days will be the same?

Marginal distribution

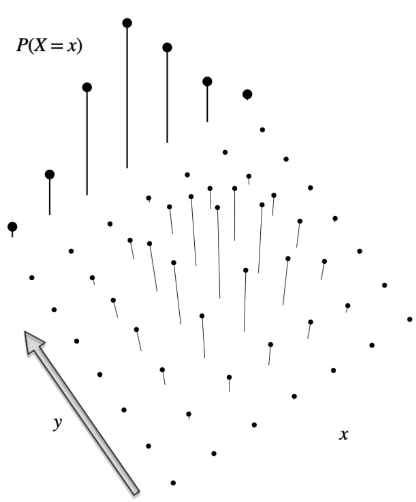
From the joint distribution of X and Y, we can get the distribution of X along by summing over the possible values of Y. This gives us the familiar pmf of X that we have seen in previous module, and we will call it the marginal distribution of X, to make it clear that we are referring to the distribution of X along, without regard for the value of Y.

For discrete random variables X and Y, the marginal pmf of X is given by

$$P(X = x) = \sum_{y} P(X = x, Y = y)$$

and the marginal pmf of Y is given by

$$P(Y = y) = \sum_{x} P(X = x, Y = y)$$



Example 1.2. What is the probability that yesterday's temperature was high?

Conditional distribution

Now suppose that we observe the value of X and want to update our distribution of Y to reflect this information. If we use the marginal pmf $P(Y = y)$, it doesn't take into account any information about X. We should use a pmf that conditions on the event $X = x$, where x is the value we observed for X. This is called conditional pmfs.

Recall the conditional probability in Module 2

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

We use the same logic to find the conditional probabilities for joint random variables. For discrete random variables X and Y, the conditional pmf of Y given $X = x$ is given by

$$P(Y = y|X = x) = \frac{P(X = x, Y = y)}{P(X = x)} = \frac{\text{Joint pmf}}{\text{Marginal pmf of given event}}$$

Remarks:

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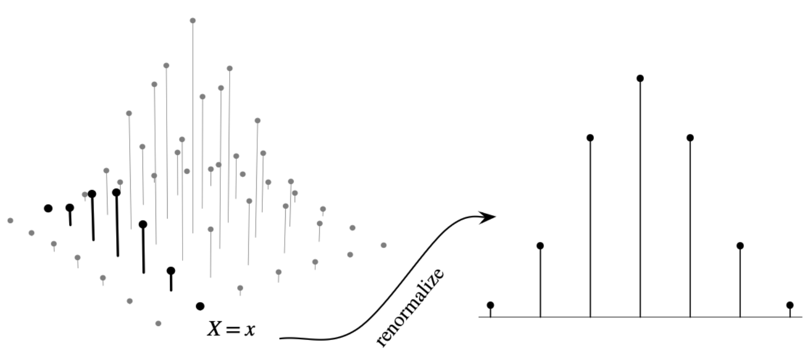


Example 1.3. Let X and Y be as previously defined. What is the probability we would predict a high temperature today given we predicted a low temperature yesterday?

Learning Objectives

By the end of this lecture, you should be able to:

- Understand the use of joint pmf, marginal pmf, and conditional pmf.
- Understand the independent event in multivariate case.
- Find the expected value of discrete multivariate random variables.

Text book sections Ross' book: Section 6.1, 6.2, 6.4

Independence

Previously, we introduced the independence for two events A and B such that

Similarly, two random variables X and Y are independent if and only if

Note that X and Y are dependent (i.e., not independent) if one of the possible pairs (x, y) violates one of the three conditions.

Example 2.1. Let X and Y be predictions about the temperature for yesterday and today respectively.

$$X = \begin{cases} 2 & \text{high} \\ 1 & \text{average} \\ 0 & \text{low} \end{cases} \qquad Y = \begin{cases} 2 & \text{high} \\ 1 & \text{average} \\ 0 & \text{low} \end{cases}$$

Are these random variables independent? Their joint pmf is

	$y = 0$	$y = 1$	$y = 2$
$x = 0$	0.20	0.10	0.01
$x = 1$	0.10	0.40	0.04
$x = 2$	0.01	0.04	0.10

Example 2.2. Let X and Y be random variables with joint pmf such that

$$p(x,y) = P(X = x, Y = y) = \frac{\lambda^{x+y}e^{-2\lambda}}{x!y!}, \quad x, y \in \{0, 1, 2, \dots\}$$

Are X and Y independent?

Expected value

Previously we introduced the expectation of a function of random variables in the univariate situation. Similarly, we can extend this to the multivariate situation.

Let g be a function and if X and Y are discrete, then

$$E[g(X,Y)] = \sum_x \sum_y g(x,y)P(X = x, Y = y)$$

The difference between univariate and multivariate is that we will now need to sum over all possible values of X and Y.

Example 2.3. Let X and Y be discrete random variables with joint pmf $p(x,y)$ and let $g(X,Y) = X$. What is the expected value of $g(X,Y)$?

Covariance

Previously, we introduced mean and variance as the summaries for the univariate distribution. Covariance is a summary of the joint distribution. Roughly speaking, covariance measures a tendency of two random variables to go up or down together, relative to their means.

The covariance between random variables X and Y is defined as

$$\mathrm{Cov}(X,Y) = E[(X - E(X))(Y - E(Y))]$$

Remarks:

(a) Using the formula of variance $\mathrm{Var}(X) = E(X^2) - E^2(X)$, the $\mathrm{Cov}(X,Y)$ can be written as

(b) $\text{Cov}(X, X) =$

Properties of Covariance

- $\text{Cov}(a + bX, c + dY) = bd\text{Cov}(X, Y)$

- For any constant a, b, c, d, and any random variables X, Y, Z, U

  $$\text{Cov}(aX + bY, cZ + dU) = ac\text{Cov}(X, Z) + ad\text{Cov}(X, U) + bc\text{Cov}(Y, Z) + bd\text{Cov}(Y, U)$$

- If X and Y are independent, then $\text{Cov}(X, Y) = 0$.

- Direction of $\text{Cov}(X, Y)$

Learning Objectives

By the end of this lecture, you should be able to:

– Be able to compute the covariance of joint random variables in discrete case.

– Be able to compute the correlation of joint random variables in discrete case.

– Recall the calculation of double integrals.

Text book sections Ross' book: Section 6.1, 6.2, 6.4

Covariance

Example 3.1. Let X and Y be predictions about the temperature for yesterday and today respectively.

$$X = \begin{cases} 2 & \text{high} \\ 1 & \text{average} \\ 0 & \text{low} \end{cases} \qquad Y = \begin{cases} 2 & \text{high} \\ 1 & \text{average} \\ 0 & \text{low} \end{cases}$$

What is the covariance of X and Y? Their joint pmf is

	$y=0$	$y=1$	$y=2$
$x=0$	0.20	0.10	0.01
$x=1$	0.10	0.40	0.04
$x=2$	0.01	0.04	0.10

Correlation

Correlation is a statistical measure that helps determine the strength of the linear dependence between two random variables X and Y. The correlation of X and Y is defined as

$$\rho = \mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} =$$

Remarks:

- The value of $\text{Corr}(X, Y)$ represents the spread around their linear trend, not the slop of the linear trend.

- The range of $\text{Corr}(X, Y)$ is $-1 \leq \text{Corr}(X, Y) \leq 1$.

- If X and Y are independent then $\text{Corr}(X, Y) = 0$. However, $\text{Corr}(X, Y) = 0$ does not imply X and Y are independent.

Example 3.2. Let X and Y be predictions about the temperature for yesterday and today respectively.

$$X = \begin{cases} 2 & \text{high} \\ 1 & \text{average} \\ 0 & \text{low} \end{cases} \qquad Y = \begin{cases} 2 & \text{high} \\ 1 & \text{average} \\ 0 & \text{low} \end{cases}$$

What is the correlation between X and Y?

Example 3.3. Two marbles are selected at random without replacement from a jar containing 3 red, 2 white, and 4 black marbles. Let X be the number of red marbles and Y be the number of white marbles.

(a) Find the pmf of X and Y

(b) Find the conditional distribution of X given $Y = 1$.

(c) Are X and Y independent?

(d) Find the expected number of red among the two selected and that of white marbles.

(e) Find the $\mathrm{Cov}(X, Y)$ and $\mathrm{Corr}(X, Y)$.

Integrals review

Remarks:

- Assume that $f(x,y)$ and $g(x,y)$ are integrable, then
  (a) $\int\int_R (f(x,y)+g(x,y))dxdy = \int\int_R f(x,y)dxdy + \int\int_R g(x,y)dxdy$.
  (b) For any constant c, $\int\int_R cf(x,y)g(x,y)dxdy = c\int\int_R f(x,y)g(x,y)dxdy$
- The double integral of a continuous function $f(x,y)$ over a range $R=[a,b]\times[c,d]$ is equal to the iterated integral (in either order)

$$\int\int_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Example 3.4. Compute $\int_1^\infty \int_2^\infty \frac{1}{3}e^{-(3x+y)}dydx$

Learning Objectives

By the end of this lecture, you should be able to:

- Find the marginal distribution from a joint distribution for continuous case.
- Find the conditional distribution from a joint distribution for continuous case.

Text book sections Ross' book: Section 6.1, 6.2, 6.4

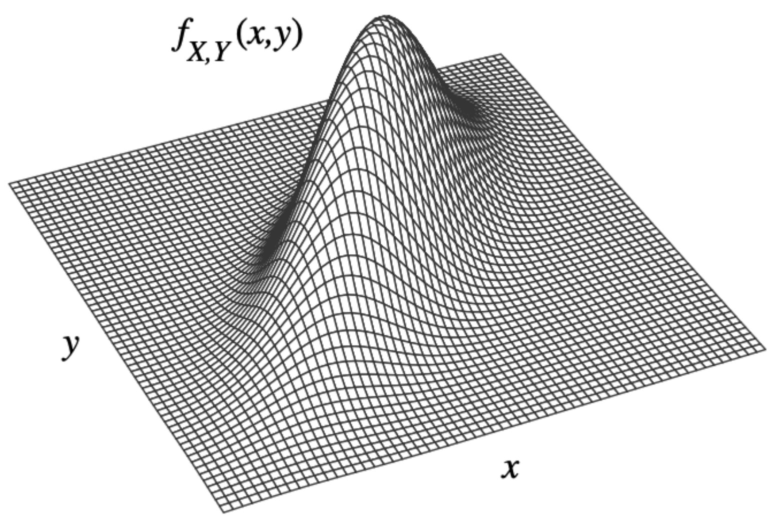
Joint probability density function

Once we have a handle on discrete joint distribution, we simply make the move to the continuous joint distribution by using the integrals.

If X and Y are continuous random variables, their joint cdf $F_{X,Y}(x,y)$ is

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y) = \int_y^\infty \int_x^\infty f_{X,Y}(x,y)dxdy,$$

where $f_{X,Y}(x,y)$ is their joint pdf.



Remarks: We require the joint pdf $f_{X,Y}(x, y)$ to be valid such that

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Example 4.1. Let X and Y have the joint pdf $f(x, y) = kxy$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

(a) Find the value of k that makes this a valid joint pdf.

(b) Find $P(X \leq \frac{1}{2}, Y \leq \frac{3}{4})$.

Marginal, Conditional distribution

In the discrete case, we get the marginal pmf of X by summing over all possible values of Y in the joint pmf. We have similar approach to compute the marginal pdf of X in the continuous case.

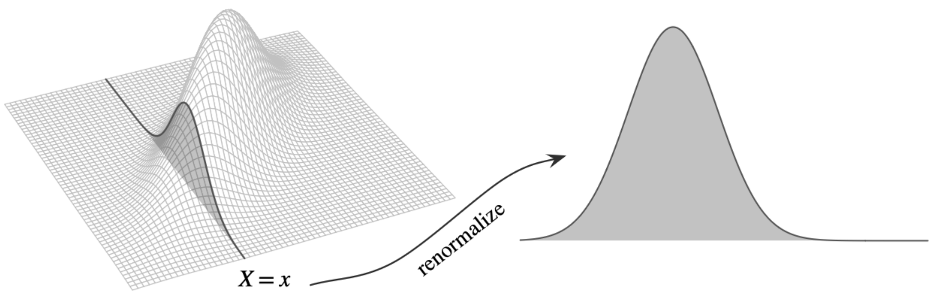
For continuous random variables X and Y with joint pdf $f_{X,Y}(x, y)$, the marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$$

We just integrate over the unwanted variable (Y) to get the joint pdf of the wanted variable (X).

Let X and Y be joint continuous random variables with pdf $f_{X,Y}(x, y)$ and marginal densities $f_X(x)$ and $f_Y(y)$, respectively. The conditional density of Y given $X = x$ is

$$f(y|X = x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = f(y|x)$$



Remarks:

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Independence

Random variables X and Y are independent if for all x and y

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

This is equivalent to the condition

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

for all x and y, and it is also equivalent to the condition

$$f(y|x) = f_Y(y)$$

Example 4.2. Let X and Y have joint pdf as

$$f_{X,Y}(x,y) = \begin{cases} e^{-(x+y)}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the marginal density functions for X and Y.

(b) Find the conditional distribution of X given that $Y = y$.

(c) Find the conditional distribution of Y given that $X = x$.

Learning Objectives

By the end of this lecture, you should be able to:

– Be able to compute the marginal and conditional distribution.

– Be able to compute the expected value in continuous case.

– Understand the covariance in continuous random variables.

Text book sections Ross' book: Section 6.1, 6.2, 6.4

Expected values

Like in the discrete random variables, we introduced the expected value of a function of random variables. We have the similar formula in the continuous random variables.

Let g be a function and if X and Y are continuous, then

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy$$

Covariance and Correlation

The covariance formula is the same as we defined in the discrete case. The covariance between continuous random variables X and Y is defined as

$$\text{Cov}(X,Y) = E[(X - E(X))(Y - E(Y))]$$

Remarks:

(a) Using the formula of variance $\text{Var}(X) = E(X^2) - E^2(X)$, the $\text{Cov}(X,Y)$ can be written as

(b) $\text{Cov}(X,X) =$

Properties of Covariance

- $\text{Cov}(a + bX, c + dY) = bd\text{Cov}(X,Y)$
- For any constant a, b, c, d, and any random variables X, Y, Z, U

$$\text{Cov}(aX + bY, cZ + dU) = ac\text{Cov}(X,Z) + ad\text{Cov}(X,U) + bc\text{Cov}(Y,Z) + bd\text{Cov}(Y,U)$$

- If X and Y are independent, then $\text{Cov}(X,Y) = 0$.

- Direction of $\text{Cov}(X, Y)$

The correlation formula between continuous variables X and Y is defined as

$$\rho = \text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} =$$

Remarks:

- The value of $\text{Corr}(X, Y)$ represents the spread around their linear trend, not the slop of the linear trend.
- The range of $\text{Corr}(X, Y)$ is $-1 \leq \text{Corr}(X, Y) \leq 1$.
- If X and Y are independent then $\text{Corr}(X, Y) = 0$. However, $\text{Corr}(X, Y) = 0$ does not imply X and Y are independent.

Practice Examples

Example 5.2. Let X be the proportion of the capacity of a gas tank that is stocked at the beginning of the week and Y the proportion of the capacity sold during the week. The joint pdf is

$$f_{X,Y}(x, y) = \begin{cases} kx, & 0 \leq y \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of k that makes this a valid joint pdf.

(b) Find $P\left(X \leq \frac{1}{2}, Y \leq \frac{1}{3}\right)$.

(c) Find the marginal density of X and Y.

(d) Find the $\text{Cov}(X, Y)$ and $\text{Corr}(X, Y)$.

Example 5.3. Let $T_1 \sim \text{Expo}(\lambda_1)$ and $T_2 \sim \text{Expo}(\lambda_2)$ be independent. Find $P(T_1 < T_2)$.

Bayes Rule

For continuous random variables X and Y we have the following continuous form of Bayes' rule

$$f(y|x) = \frac{f(x|y)f_Y(y)}{f_X(x)},$$

adn we have the following continuous form from the law of total probability

$$f_X(x) = \int f(x|y)f_Y(y)dy$$

Here are the four versions of Bayes' rule

	Y **discrete**	Y **continuous**
X **discrete**	$P(Y=y\|X=x) = \frac{P(X=x\|Y=y)P(Y=y)}{P(X=x)}$	$f_Y(y\|X=x) = \frac{P(X=x\|Y=y)f_Y(y)}{P(X=x)}$
X **continuous**	$P(Y=y\|X=x) = \frac{f_X(x\|Y=y)P(Y=y)}{f_X(x)}$	$f_{Y\|X}(y\|x) = \frac{f_{X\|Y}(x\|y)f_Y(y)}{f_X(x)}$

Here are the four versions of law of total probability

	Y **discrete**	Y **continuous**
X **discrete**	$\sum_y P(X=x\|Y=y)P(Y=y)$	$\int_{-\infty}^{\infty} P(X=x\|Y=y)f_Y(y)dy$
X **continuous**	$\sum_y f_X(x\|Y=y)P(Y=y)$	$\int_{-\infty}^{\infty} f_{X\|Y}(x\|y)f_Y(y)dy$

Learning Objectives

By the end of this lecture, you should be able to:

- Understand the multivariate transformation using Jacobian.
- Be able to compute the multivariate transformation using Jacobian.

Text book sections Ross' book: Section 6.1, 6.2, 6.4

Bayes Rule

For continuous random variables X and Y we have the following continuous form of Bayes' rule

$$f(y|x) = \frac{f(x|y)f_Y(y)}{f_X(x)},$$

adn we have the following continuous form from the law of total probability

$$f_X(x) = \int f(x|y)f_Y(y)dy$$

Here are the four versions of Bayes' rule

	Y discrete	**Y continuous**
X discrete	$P(Y=y\|X=x) = \frac{P(X=x\|Y=y)P(Y=y)}{P(X=x)}$	$f_Y(y\|X=x) = \frac{P(X=x\|Y=y)f_Y(y)}{P(X=x)}$
X continuous	$P(Y=y\|X=x) = \frac{f_X(x\|Y=y)P(Y=y)}{f_X(x)}$	$f_{Y\|X}(y\|x) = \frac{f_{X\|Y}(x\|y)f_Y(y)}{f_X(x)}$

Here are the four versions of law of total probability

	Y discrete	**Y continuous**
X discrete	$\sum_y P(X=x\|Y=y)P(Y=y)$	$\int_{-\infty}^{\infty} P(X=x\|Y=y)f_Y(y)dy$
X continuous	$\sum_y f_X(x\|Y=y)P(Y=y)$	$\int_{-\infty}^{\infty} f_{X\|Y}(x\|y)f_Y(y)dy$

Functions of joint random variables

In the univariate case, if X is a random variable with pdf $f_X(x)$. In module 4 we introduced how to find the density of functions of random variables such that

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{\partial}{\partial y} g^{-1}(y) \right|$$

Now we can extend this method to the bivariate case. Suppose that X and Y are continuous random variables with joint pdf $f_{X,Y}(x, y)$ and for all (x, y). Let $U = g_1(X, Y)$ and $V = g_2(X, Y)$. The Jacobian can be expressed as

$$J(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

The joint pdf of $f_{U,V}(u, v)$ is given by

$$f_{U,V}(u, v) = f_{X,Y}(\underbrace{h_1(u, v)}_{x}, \underbrace{h_2(u, v)}_{y})|\det(J(x, y))|^{-1}$$

Example 6.2. Suppose that X_1 and X_2 are independent Gamma$(\alpha_i, 1)$ random variables. Define $U = \frac{X_1}{X_1+X_2}$ and $V = X_1 + X_2$.

(a) Find the Jacobian.

(b) Find the joint distribution of X_1 and X_2

(c) Find the joint distribution of U and V.

(d) Find the marginal density of U.

Example 6.3. Let $U \sim \text{Unif}(0, 2\pi)$, and let $T \sim \text{Expo}(1)$ be independent of U. Define

$$X = \sqrt{2T}\cos U$$
$$Y = \sqrt{2T}\sin U$$

Find the joint pdf $f_{X,Y}(x, y)$.

Learning Objectives

By the end of this lecture, you should be able to:

- Be able to compute the joint pdf of bivariate normal distribution

Text book sections Ross' book: Section 6.1, 6.2, 6.4

Bivariate Normal distribution

The multivariate normal distribution is a keystone of modern statistical theory. In general, the multivariate normal density is defined for k continuous random variables $X_1, X_2, \ldots, X_k$. In this course, we will focus on the bivariate case ($k = 2$).

In order to derive the joint pdf of bivariate normal distribution, we will start fromm very beginning. Suppose we the we have to standard normal random variables $Z_1, Z_2 \overset{iid}{\sim} N(0, 1)$

(a) What is the joint pdf of (Z_1, Z_2).

(b) Define $X_1 = \mu_1 + \sigma_1 Z_1$. Show that $E(X_1) = \mu_1$ and $\text{Var}(X_1) = \sigma_1^2$.

(c) Define $X_2 = \mu_2 + \sigma_2(\rho Z_1 + \sqrt{1-\rho^2} Z_2)$. Show that $E(X_2) = \mu_2$ and $\text{Var}(X_2) = \sigma_2^2$.

(d) Compute $\text{Cov}(X_1, X_2)$ and verify that $\text{Corr}(X_1, X_2) = \rho$.

(e) Show that

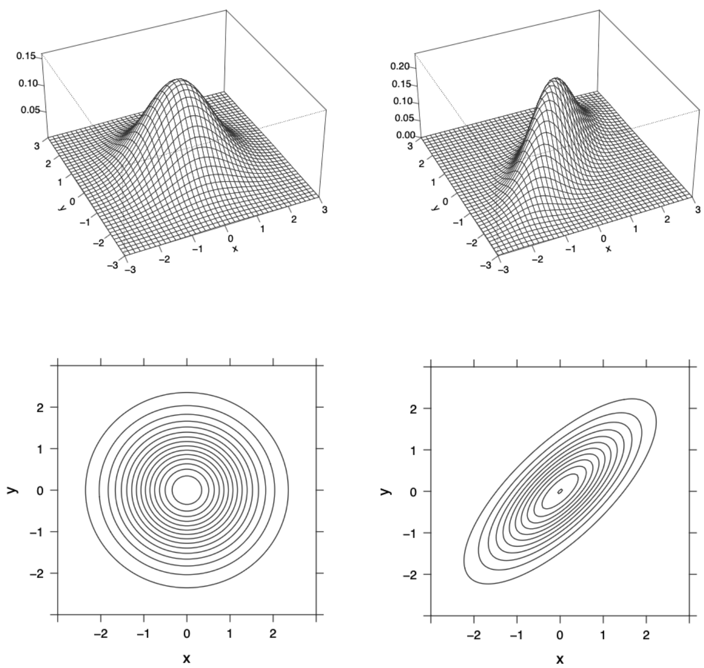
$$z_1 = \frac{x_1 - \mu_1}{\sigma_1},$$

and

$$z_2 = \frac{1}{\sqrt{1-\rho^2}} \left(\frac{x_2 - \mu_2}{\sigma_2} - \rho \frac{x_1 - \mu_1}{\sigma_2} \right)$$

(f) Compute that Jacobian matrix $J(z_1, z_2)$.

(g) Find out the $f_{X_1,X_2}(x_1, x_2)$.



Remarks:

- If (X, Y) follows a bivariate normal distribution, then the marginal distribution of X and Y are also normal. That is, $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$.

- If (X, Y) follows a bivariate normal distribution, and $\rho = 0$, then X and Y are independent.

- If (X, Y) follows a bivariate normal distribution, then the mgf is

$$M_{X,Y}(t_1, t_2) = \exp\left(\mu_x t_1 + \mu_y t_2 + \frac{1}{2}(\sigma_x^2 t_1^2 + 2\rho\sigma_x\sigma_y t_1 t_2 + \sigma_y^2 t^2)\right)$$

- (Optional) The conditional distribution of one variable, given the value of the other, is also a normal distribution.
  - $Y|X = x \sim N(\mu_{y|x}, \sigma_{y|x}^2)$
  - $X|Y = y \sim N(\mu_{x|y}, \sigma_{x|y}^2)$

Practice Examples

Example 8.1. Suppose we have joint pdf for $f_{X,Y}(x,y)$ such that

$$f_{X,Y}(x,y) = \begin{cases} 4xy, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Verify $f_{X,Y}(x,y)$ is a valid joint pdf.

(b) Find the marginal density of x and y.

(c) Find the $P(X \le \frac{1}{2} | Y \ge \frac{3}{4})$.

(d) Find the conditional density function of Y given $X = x$.

Example 8.2. The multivariate generalization of the Beta distribution is called Dirichlet distribution, which as been found to be very important. The bivariate version has the density as

$$f_{X,Y}(x, y) = Cx^{\alpha-1}y^{\beta-1}(1 - x - y)^{\gamma-1}$$

where $x \in (0, 1)$ and $y \in (0, 1 - x)$, where α, β, and γ are constants. Find the value of C in terms of α, β, and γ

Instructions

This Homework contains *4 questions* and *20 points* in total. Show all work for full credit. Partial credit will be given if your process is clearly shown and mathematically reasonable. If the question is theoretical, please provide a written explanation.
Due date: Friday, November 14th by 11:59pm.

1. A fair coin is flipped twice. Let X be the number of Heads in the two tosses, and Y be the indicator random variable for the tosses landing the same way. *Hint: Write out the support of* X*,* $X \in \{0, 1, 2\}$*.* Y *can be defined as*

$$Y = \begin{cases} 1, & \textit{both flips the same} \\ 0, & \textit{otherwise} \end{cases}$$

   (a) (2 points) Find the joint pmf of X and Y.

   (b) (2 points) Find the marginal pmfs of X and Y.

   (c) (1 point) Are X and Y independent?

   (d) (3 points) Find the conditional pmfs of Y given $X = x$ and of X given $Y = y$.

2. Contracts for two construction jobs are randomly assigned to one or more of three firms, A, B, and C. Let X denote the number of contracts assigned to firm A and Y the number of contracts assigned to firm B. Each firm can receive 0,1, or 2 contracts. The joint pmf is

	$y = 0$	$y = 1$	$y = 2$
$x = 0$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$
$x = 1$	$\frac{2}{9}$	$\frac{2}{9}$	0
$x = 2$	$\frac{1}{9}$	0	0

   (a) (2 points) Find the marginal probability of X and Y.

   (b) (2 points) Find $\text{Cov}(X, Y)$.

3. A committee of size k is chosen from a group of n women and m men. All possible committees of size k are equally likely. Let X and Y be the numbers of women and men on the committee, respectively.

   (a) (2 points) Find the joint pmf of X and Y. Be sure to specify the support.

   (b) (2 points) Find the marginal pmf of X by using the pmf.

4. (4 points) Let X and Y be discrete random variables with joint probability function given by the following table

(x, y)	$(0, 0)$	$(0, 1)$	$(1, 0)$	$(1, 1)$	$(2, 0)$	$(2, 1)$
$p_{X,Y}(x, y)$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	0

Find the variance, $\mathrm{Var}(Y - X)$.

Instructions

This Homework contains *4 questions* and *20 points* in total. Show all work for full credit. Partial credit will be given if your process is clearly shown and mathematically reasonable. If the question is theoretical, please provide a written explanation.
Due date: Friday, December 5th by 11:59pm.

1. Let X and Y have joint pdf

$$f_{X,Y}(x,y) = cxy, 0 < x < y < 1$$

   (a) (2 points) Find c to make this a valid joint pdf.

   (b) (2 points) Find the marginal pdfs of X and Y.

   (c) (2 point) Find the conditional pdf of Y given $X = x$.

2. (4 points) Suppose that $X \sim \text{Beta}(\alpha, \beta)$ and $Y \sim \text{Beta}(\alpha+\beta, \gamma)$ are independent random variables. Let $U = XY$ and $V = Y$. Find the joint density of $f_{U,V}(u,v)$. *Hint: Use Jacobian.*

3. Let X and Y have joint pdf as

$$f_{X,Y}(x,y) = \begin{cases} 4e^{-\frac{y}{x^2}-2x}, & x > 0 \quad y > 0 \\ 0, & \text{otherwise} \end{cases}$$

   (a) (3 points) Find the marginal distribution of X and specify the support. *Hint: When you do the integration over y, you may get the following form:*

$$\textit{something of } x \cdot \int_0^\infty e^{-\frac{y}{x^2}} dy$$

   The integration is a known distribution of Y if we make modification as following

$$C \int_0^\infty \frac{1}{C} e^{-\frac{y}{x^2}} dy$$

   That is, you need to identify the distribution of Y and find out the value of C so that $\int_0^\infty \frac{1}{C} e^{-\frac{y}{x^2}} dy = 1$ because the integration over support of a valid pdf is 1.

   (b) (2 points) Find the conditional distribution of Y given $X = x$. That is, $f(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$.

   (c) (1 point) Named the distribution of part (b) and specify the parameter.

4. Let

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi}, & x^2 + y^2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

   (a) (2 points) Find the marginal distribution of X and Y.

   (b) (2 points) Find the conditional pdf $f(x|y)$ given $y \in (-1, 1)$.