Learning Objectives

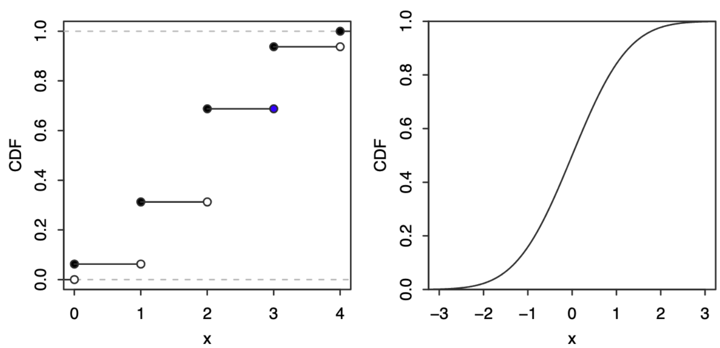
By the end of this lecture, you should be able to:

- Understand the definition of the continuous random variable.
- Learn what a probability density function (pdf) is.
- Explain the concept of a cumulative distribution function (cdf) and its relationship to the pdf.
- Use the PDF to compute probabilities for a continuous random variable

Text book sections Ross' book: Section 5.1

Continuous Random Variable

A random variable that can take on any value in an interval is called continuous. The interval could be $[0, 1]$, or $(-\infty, \infty)$, or a union of disjoint interval $(4, 6) \cup (10, 15)$ on the real line.



Continuous random variables often represent measurements like heights, weights,speeds, etc.

Example 1.1. Example of continuous random variables.

Cumulative distribution function

The cumulative distribution function (cdf) of a continuous random variable X is defined as

$$F_X(c) = P(X \leq c) = \_\_\_\_\_\_\_\_\_\_$$

Properties of cdf (continuous random variable)

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For discrete random variables, the cdf is awkward to work with, its derivative is almost useless since it's undefined at the jumps and 0 everywhere else. For continuous random variables, the cdf is often convenient to work with, and its derivative is a very useful function, called probability density function (pdf).

Probability Density Function

Let $F_X(x)$ be the cdf for a continuous random variable X. Then $f(x)$, given by

$$f(x) = \frac{d}{dx} F_X(x) = F'_X(x)$$

wherever the derivative exists, is called probability density function (pdf) for the random variable X. All probability about X can be answered in terms of f, for example

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

In another words, integrating the pdf over the given the set gives you the probability about the random variable X.

Properties of pdf

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Useful Propositions: For any two constants, a and b, $a \leq b$, if X is a continuous random variable with pdf $f(x)$ and cdf $F_X(x)$, then

1. the probability of constant a, $P(X = a) =$

2.

3. $P(a < X < b) =$

4. the relationship between pdf and cdf

Example 1.2. Suppose that X is a continuous random variable whose pdf is given by

$$f(x) = \begin{cases} C(4x - 2x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the value of C?

(b) Find $P(X > 1)$.

Example 1.3. Let X have the pdf

$$f(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Check $f(x)$ is a valid pdf.

(b) Find the cdf of $f(x)$, $F(x)$.

(c) Compute the probability for $x \in [1, 1.5]$.

Learning Objectives

By the end of this lecture, you should be able to:

- Understand the concept of transforming a continuous random variable using a monotonic increasing function.
- Apply the change of variable technique to find the new PDF.
- Interpret how expectations transform under such functions.
- Recognize the definition of expectation for continuous random variables.

Text book sections Ross' book: Section 5.2, 5.7

Functions of Random Variable

Suppose X is a continuous random variable with pdf $f(x)$, and we are interested in the distribution of a new variable $Y = g(X)$, where g is a differentiable and monotonic function. Our goal is to find the distribution of the transformed random variable Y.

In the discrete case, we get the pmf of $g(X)$ by translating the event $g(X) = y$ into an equivalent event involving X. We look for all values x such that $g(x) = y$; as long as X equals any of these x's, the event $g(X) = y$ will occur and gives the formula

$$P(g(X) = y) = \sum_{x:g(x)=y} P(X = x),$$

or if g is a one-to-one function, we have

$$P(g(X) = y) = P(X = g^{-1}(y))$$

In the continuous case, it is easy to start from the cdf of $g(X)$.

$$F_{g(X)}(y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

We can then differentiate with respect to y to get the pdf of $g(X)$.

Let X be a continuous random variable with pdf f_X, and let $Y = g(X)$, where g is differentiable and strictly increasing (or strictly decreasing). Then the pdf of Y is

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{\partial}{\partial y} g^{-1}(y) \right|$$

Remark: Using the absolute value covers both strictly increasing and decreasing case.

Steps of finding the pdf of Y

Let $Y = g(X)$, where g is differentiable and either increasing or decreasing function, then

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Example 2.1. Suppose X is a continuous random variable with pdf $f_X(x) = 7\exp(-7x)$ for $x \in (0, \infty)$. Find the density of $Y = 4X + 3$.

Example 2.2. Let X be a continuous random variable with pdf $f_X(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2}$. Find the density of $Y = e^X$.

Example 2.3. Suppose that X is a random variable with pdf $f_X(x) = 42x^5(1-x)$ for $x \in (0, 1)$. Using direct calculation, find the distribution function and density of $Y = X^3$.

Expectation of continuous random variable

The definition of expectation for continuous random variables is analogous to the definition of discrete random variables: replace the sum with an integral and the pmf with the pdf.

The expectation of a continuous random variable X with pdf f is

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Note that the integral is taken over the entire real line, but if the support of X is not the entire real line, we can just integrate the support.

Properties of expectation

Let $g(X)$ be a function of a random variable X and c a constant, then we have

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Example 2.4. Daily total solar radiation for a specified location in Florida in October has pdf given by

$$f(y) = \begin{cases} \frac{3}{32}(y-2)(6-y), & 2 \le y \le 6 \\ 0, & \text{otherwise} \end{cases}$$

with measurements in hundreds of calories. Find the expected daily solar radiation for October.

Example 2.5. Find the expected value of random variable X whose pdf is defined as

$$f(x) = \begin{cases} 3x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

and those of $Y = 3X + 2$.

Learning Objectives

By the end of this lecture, you should be able to:

- Review and explore more problems of expectation.
- Define and compute the variance of a continuous random variable.
- Calculate expectation and variance for continuous random variable.
- Review the Calculus.

Text book sections Ross' book: Section 5.2

Variance of continuous random variables

The variance of continuous random variables is defined exactly as it is for a discrete random variable such that

$$\text{Var}(X) =$$

alternative forma

$$\text{Var}(X) =$$

Properties of variance

For any constants a and b and continuous random variables X and Y,

-

- If X and Y are independent,

Example 3.1. Let X be the random variable with pdf

$$f(x) = \begin{cases} \frac{x}{2}, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the mean and variance of X.

Example 3.2. The proportion of time per day that all checkout counters in a supermarket are busy is a random variable Y with density function

$$f(y) = \begin{cases} cy^2(1-y), & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of c that makes $f(y)$ a pdf.

(b) Find the cdf of random variable Y, $F(y)$.

(c) Find $E(Y)$ and $\mathrm{Var}(Y)$.

(d) Find $P(0 \le Y \le \frac{1}{2})$.

(e) Find $P(Y > \frac{3}{4} | Y \ge \frac{1}{2})$

Calculus Review

1. For $k > 0$, $\int_0^\infty x^{k-1}e^{-x}dx =$

2. $\int_0^\infty x\lambda e^{-\lambda x}dx =$

3. $\int_0^\infty x^2\lambda e^{-\lambda x}dx =$

4. $\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-\frac{x^2+y^2}{2}}dxdy =$

4. $\int_{-\infty}^\infty x^2 e^{-\frac{x^2}{2}}dx =$

Learning Objectives

By the end of this lecture, you should be able to:

– Understand the pdf, cdf, mean, and variance of a Uniform distribution.

– Understand the pdf, cdf, mean, and variance of a Exponential distribution.

– Apply these distributions to real world problems.

Text book sections Ross' book: Section 5.3, 5.5

Uniform distribution

A continuous random variable X is said to have the Uniform distribution on the interval (a, b) if its pdf is

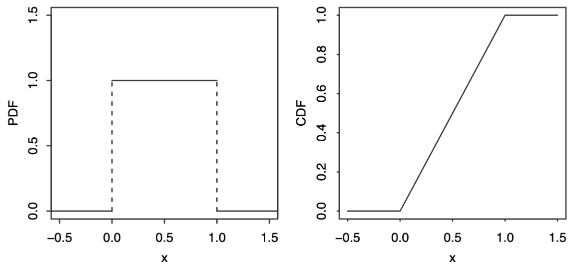
$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

We write it as $X \sim \text{Unif}(a, b)$.

The cdf of $X \sim \text{Unif}(a, b)$, $F(x)$ is

The mean and variance of $X \sim \text{Unif}(a, b)$

Example 4.1. Let $X \sim \text{Unif}(0, 1)$, find the pdf, cdf, mean and variance.



Example 4.2. A circle of radius r has area $A = \pi r^2$. If a random circle has a radius that is uniformly distributed on the interval $(0, 1)$, what are the mean and variance of the area of the circle?

Exponential distribution

A continuous random variable X is said to have the Exponential distribution with rate parameter λ, where $\lambda > 0$, if its pdf is

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

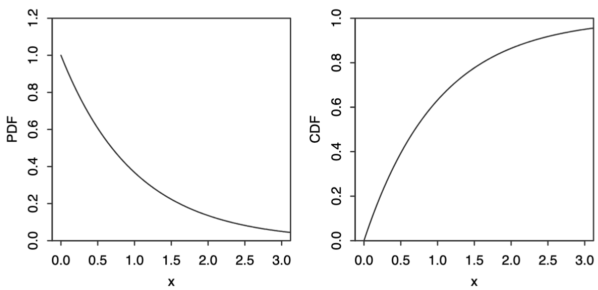
We write $X \sim \text{Expo}(\lambda)$.

The cdf of $X \sim \text{Expo}(\lambda)$, $F(x)$ is

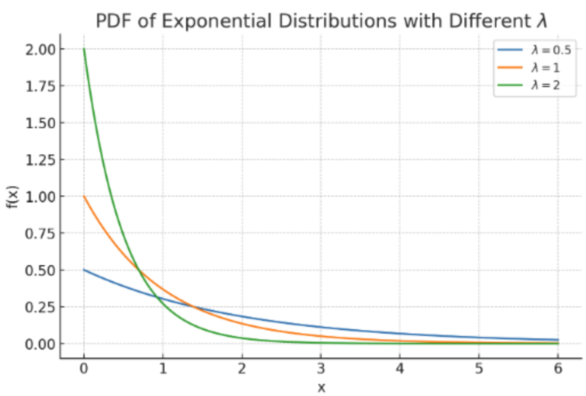
The mean and variance of $X \sim \text{Expo}(\lambda)$

Properties of Exponential distribution

- X can be any non-negative value: $x \geq 0$
- The rate parameter (occurrence) λ can be any positive value: $\lambda > 0$.
- The Exponential distribution is always right-skewed.



Remark: Let $X \sim \text{Expo}(\lambda)$,



There is a very special property of Exponential distribution called memoryless property. It says that even if you've waited for hours or days without success, the success isn't any more likely to arrive soon. Let $X \sim \text{Expo}(\lambda)$, suppose we've already spent s waiting, then

$$P(X \geq s + t | X \geq s) = P(X \geq t)$$

In another words, the probability of waiting an additional t units of time does not depend on how long you have already waited.

Example 4.4. There is a bus stop near Alan's home. When a bus arrives, the time until the next bus arrives is an Exponential random variable with mean 10 minutes. Alan arrives at the bust stop at a random time, not knowing how long ago the previous bus came. What is the distribution of Alan's waiting time for the next bus? What is the average time that Alan has to wait?

Summary: The Exponential distribution is often used to model waiting time until an event of interest occurs. For example, in the medical study, Exponential distribution can be used to model the lifetime (survival time). Since λ is an occurrence rate, higher values of λ indicates shorter survival time.

Learning Objectives

By the end of this lecture, you should be able to:

- Recognize the key features and importance of the normal distribution.
- Understand the probability density function (PDF) of a normal random variable.
- Interpret the parameter μ and σ^2.
- Understand the shape and scale for the normal distribution.

Text book sections Ross' book: Section 5.4

Exponential vs Poisson

The Exponential distribution is closely connected to the Poisson distribution, as suggested by our use of λ for the parameters of both distributions.

Imagine you are monitoring your email inbox. Emails arrive at random time, and you don't know exactly when the next one will come. Suppose on average 5 emails arrive per hour. This means the process has a rate of $\lambda = 5$ emails/hour. The Poisson Process says the number of arrivals in an hour follows a Poisson distribution. The Exponential distribution, instead of asking "how many emails do I get in an hour", asks "How long do I have to wait for the next email?" The waiting time is modeled by the Exponential distribution with parameter λ, let's say the waiting time T,

$$T \sim \text{Expo}(\lambda)$$

If $\lambda = 5$ emails/hour then the average waiting time is $\frac{1}{\lambda} = \frac{1}{5}$ hours $= 12$ minutes. So, the Exponential distribution measures the gaps between Poisson events.

Intuitively, the Poisson answers: "If I check my inbox every hour, how many emails will I hae?", the Exponential answers" "If I sit and wait, how long until the next email arrives". They both describe the same process (email arriving), but from different angles.

Normal distribution

The Normal distribution is the most important and widely used distribution in statistics because of the central limit theorem. The theorem stats that under very weak assumptions, the sum of a large number of i.i.d. random variables has an approximately Normal distribution, regardless of the distribution of the individual random variables.

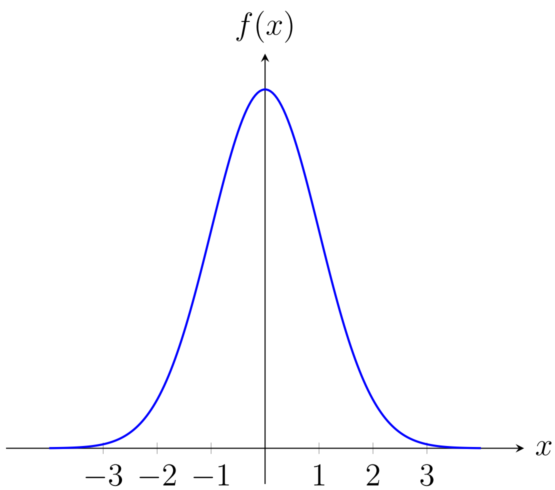
A continuous random variable X is said to have a Normal distribution with two parameters μ and σ^2 if the pdf of X is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

We write as $X \sim N(\mu, \sigma^2)$.

The cdf, $F(x)$, of $X \sim N(\mu, \sigma^2)$ is

The mean and variance of X is



Remark:

- μ is the center (mean) of the distribution. it can take any value, i.e. $\mu \in \mathbb{R}$, and it determines the center (or location) of the distribution.
- σ^2 is the variance of the distribution. It controls the spread around the mean of the distribution.
- Normal distribution is bell-shaped and symmetric around the mean.
- A Normal distribution with $\mu = 0$ and $\sigma^2 = 1$ is called standard Normal distribution, it's typically denoted as Z.

Example 5.1. Let $Z \sim N(0, 1)$ with pdf

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad z \in \mathbb{R}$$

Verify $f(z)$ is a valid pdf.

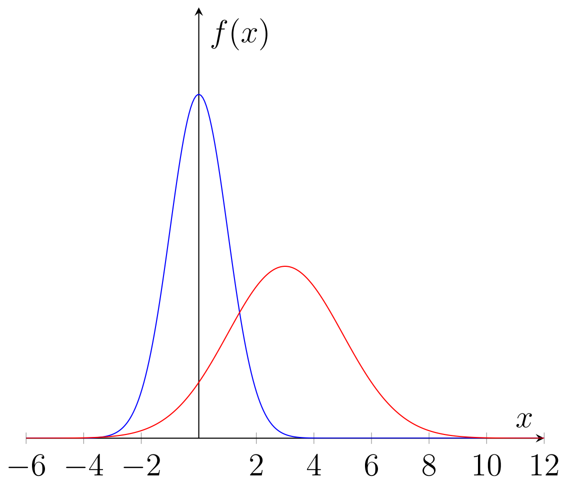
Example 5.2. Let $Z \sim N(0, 1)$ with pdf

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad z \in \mathbb{R}$$

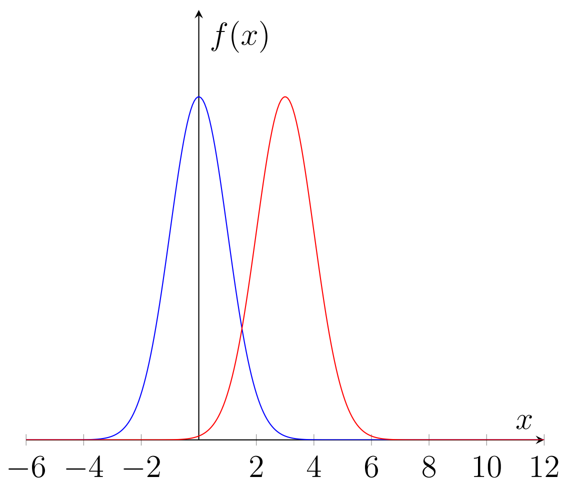
Verify $E(Z) = 0$ and $\text{Var}(Z) = 1$.

Let $X \sim N(\mu, \sigma^2)$ and $Z \sim N(0,1)$. We can use Z to represent X and vice versa. This is called standardization.

Example 5.3. Let $X \sim N(3,4)$ and $Z \sim N(0,1)$



Example 5.4. Let $X \sim N(3,1)$ and $Z \sim N(0,1)$



Remark: The standardization is useful as it allows comparing quantities from different populations.

Learning Objectives

By the end of this lecture, you should be able to:

- Apply the Empirical Rule and standardization for normal probabilities.
- Understand the distribution of sums of squared standard normal variables.
- Recognize the Chi-squared distribution and its properties.

Text book sections Ross' book: Section 5.4

Normal distribution

Previously, we introduced the Normal distribution and the standardization.

Example 6.1. An incoming freshman took her college's placement exams in STAT cna MATH course. In STAT course, she scored 82 and in MATH an 86. The overall results on STAT exam had a mean 72 and a standard deviation of 8, while the mean MATH score was 68 with a standard deviation of 12. Determine which exam she did better.

Now, there are three important benchmarks for the Normal distribution. Let $Z \sim N(0, 1)$, then

- $P(|Z| < 1) \approx 0.683$
- $P(|Z| < 2) \approx 0.954$
- $P(|Z| < 3) \approx 0.997$

This is called empirical rule, which is also called 68-95-99.7 rule. For any Normal random variables, probabilities of falling within one, two, and three standard deviation of the mean are 0.683, 0.954, and 0.997, respectively. The empirical rule applies for any Normal random variables. Often it is easier to apply the rule after standardizing it.

Example 6.2. Let $X \sim N(-1, 4)$. Compute $P(|X| < 3)$.

Chi-squared distribution

A continuous random variable X is said to have a Chi-squared distribution with parameter ν, $\nu > 0$, if the pdf of X is

$$f(x) = \begin{cases} \frac{\left(\frac{1}{2}\right)^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

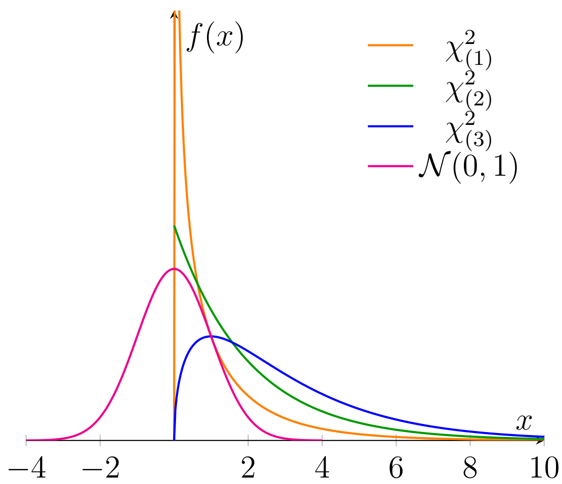
We write as $X \sim \chi^2_\nu$. We will skip the cdf of χ^2_ν as there is no closed form. The mean and variance of X would be

Remarks:

- The parameter ν is called degrees of freedom and $\nu > 0$.
- The χ^2_ν is right skewed.
- The Chi-squared distribution arises from the normal distribution. Let $Z_1, Z_2, \ldots, Z_k$ be independent $N(0, 1)$ random variable. Define

  $$Q = Z_1^2 + Z_2^2 + \cdots + Z_k^2$$

  Then Q follows a Chi-squared distribution with k degrees of freedom $Q \sim \chi^2_k$.



The Chi-squared distribution is fundamental in several statistical test.

- Goodness-of-fit test: Check if observed categorical frequencies match an expected distribution.

- Test of independence: Evaluate whether two categorical variables are independent.

- Sample Variance: Suppose we take a random sample of size n from the population. The sample variance follows Chi-squared distribution.

Relationship to other distribution

- If $Z \sim N(0,1)$, then $Z^2 \sim \chi^2_1$
- If $Z_i \overset{iid}{\sim} N(0,1)$, then $\sum_{i=1}^n Z_i^2 \sim \chi^2_n$

(Optional) Integration by Pattern Recognition

We can always express a pdf $f(x)$ of a named continuous random variable as product of two parts: essential part (depends on x), and the constant part (does not depend on x). For example, if $X \sim N(\mu, \sigma^2)$

$$f(x) = \underbrace{\frac{1}{\sigma\sqrt{2\pi}}}_{\text{constant}} \underbrace{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}_{\text{essential}}$$

We note that the essential part contains all of the information to determine the distribution of X uniquely. For example, in this case, the power of the exponential function is a quadratic function of x with a negative coefficient on the quadratic term, which is a unique property of a Normal distribution.

Learning Objectives

By the end of this lecture, you should be able to:

- Define the Gamma distribution and its parameters.
- Understand the pdf and cdf of Gamma distribution.
- Define the beta distribution and its parameters
- Understand the pdf and cdf of Beta distribution.

Text book sections Ross' book: Section 5.6.1, 5.6.4

Gamma distribution

Before writing down the pdf of Gamma distribution, we first introduce the gamma function, a very famous function in mathematics. The gamma function, Γ, is defined by

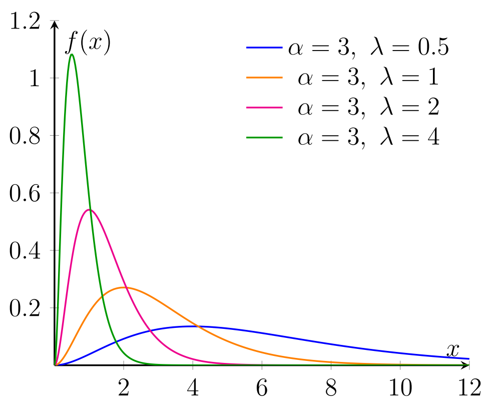
$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx,$$

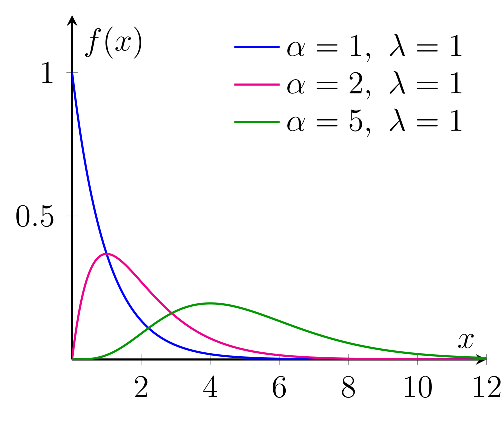
for all real numbers $a > 0$. It should be noted that for a positive integer a, $\Gamma(a) = (\alpha - 1)!$

A continuous random variable X is said to have a Gamma distribution with parameters α and λ if the pdf of X is

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

We write as $X \sim \text{Gamma}(\alpha, \lambda)$.





Remarks:

- α is shape parameter and $\alpha > 0$.
- λ is rate parameter and $\lambda > 0$.
- When $\alpha = 1$, Gamma$(1, \lambda) =$ Expo(λ).
- Let $Y_1, Y_2, \ldots, Y_a \overset{iid}{\sim}$ Expo(λ), then

Properties of Gamma distribution

Let a continuous random variable $X \sim$ Gamma(α, λ),

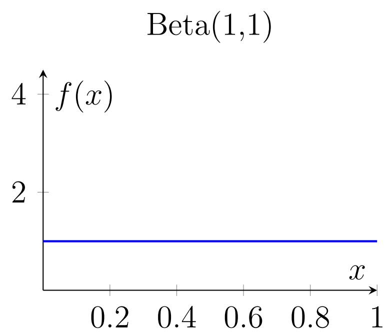
- $E(X) =$
- $\text{Var}(X) =$
- If $X \sim$ Gamma(α, λ) and $Y \sim$ Gamma(β, λ), X and Y are independent, then

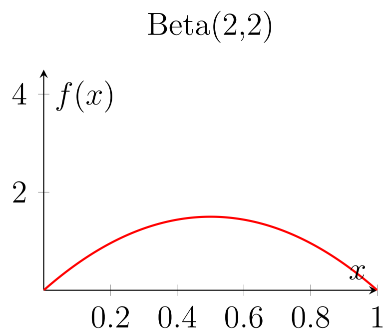
Beta distribution

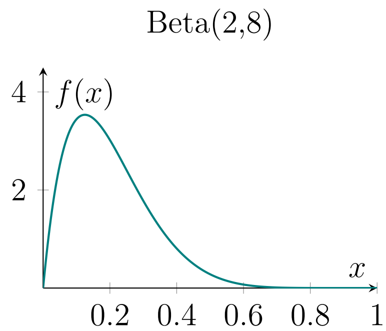
A random variable X is said to have the Beta distribution with parameters α and β if its pdf is

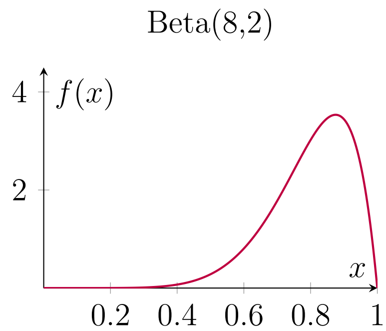
$$f(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

where $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. We write as $X \sim$ Beta(α, β).









Remarks:

- The support of X is

- The distribution is often used to model __________

- α is shape parameter 1 and β is shape parameter 2.
- If $\alpha = \beta = 1$, then

- If $\alpha = \beta$, then
- If $\alpha < \beta$, then
- If $\alpha > \beta$, then

Properties of Beta distribution

- $E(X) =$

- $\text{Var}(X) =$

Relationship between Gamma and Beta distribution

If $X \sim \text{Gamma}(\alpha, \lambda)$, $Y \sim \text{Gamma}(\beta, \lambda)$, and X, Y are independent, then the proportion $\frac{X}{X+Y} \sim \text{Beta}(\alpha, \beta)$ distribution.

Example 7.1. The proportion of gasoline remaining in a suppliers tank at the end of the week follows the Beta distribution with $\alpha = 2$ and $\beta = 1$.

(a) What is the probability that more than 90% of the original amount of gasoline will still be remaining at the end of the week, i.e. $P(X > 0.9)$?

(b) What is the expected proportion remaining?

Summary of Gamma and Beta distribution

- Gamma distribution can be used to model the waiting time (biomedical study), insurance claims, reliability, and service times.
- Beta distribution can be used to model the proportions and probabilities. The most powerful application is the prior information for binomial data $X \sim \text{Bin}(n, p)$.

Learning Objectives

By the end of this lecture, you should be able to:

- Define the Weibull distribution and its parameters.
- Understand the pdf and cdf of Weibull distribution.
- Define the Log-Normal distribution and its parameters
- Understand the pdf and cdf of Log-Normal distribution.

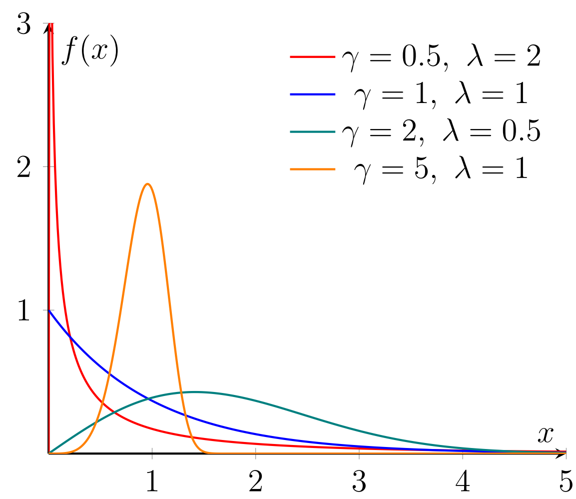
Text book sections Ross' book: Section 5.6.2, 5.6.4

Weibull distribution

The Exponential distribution can be used to model the survival time (lifetime). But since Exponential distribution is memoryless, which makes it unrealistic. Now we introduce another distribution that widely used in modeling the survival time (lifetime). A random variable X with parameters γ and λ is said to have Weibull distribution and its pdf is given by

$$f(x) = \begin{cases} \gamma\lambda e^{-\lambda x^{\gamma}} x^{\gamma-1}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

We write as $X \sim \text{Weib}(\gamma, \lambda)$.



Remarks:

- $E(X) =$

- $\text{Var}(X) =$

- If $k = 1$, then $X \sim$

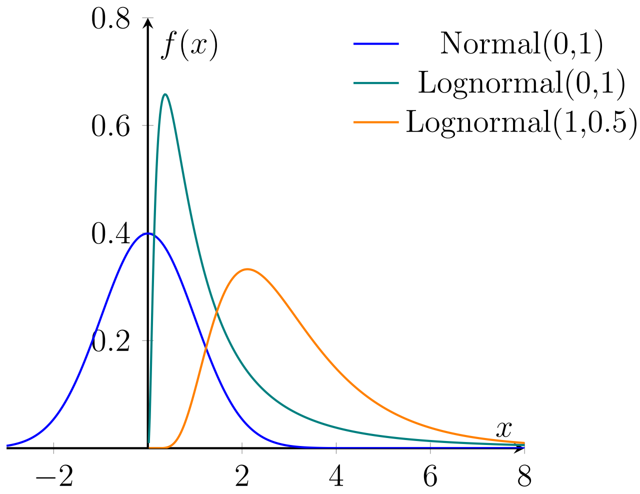
- Weibull distribution is widely used in survival analysis and reliability analysis.

Log-Normal distribution

A random variable X has a Log-Normal distribution and its pdf is given by

$$f(x) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

We write as $X \sim \text{LogN}(\mu, \sigma^2)$. Log-Normal distribution is actually coming from Normal distribution. Let $X \sim N(\mu, \sigma^2)$, let $Y = e^X$, the $Y \sim \text{LogN}(\mu, \sigma^2)$.



Remarks:

- Log-Normal distribution DOES NOT mean "log of a Normal"
- Log-Normal distribution DOES means "log is Normal"
- The shape of Log-Normal distribution is right skewed
- The parameters μ and σ^2 are still the mean and variance of the distribution, but this is underling normal distribution of $\log(X)$, not X itself.
- $E(X) =$
- $\mathrm{Var}(X) =$

Summary of Log-Normal distribution

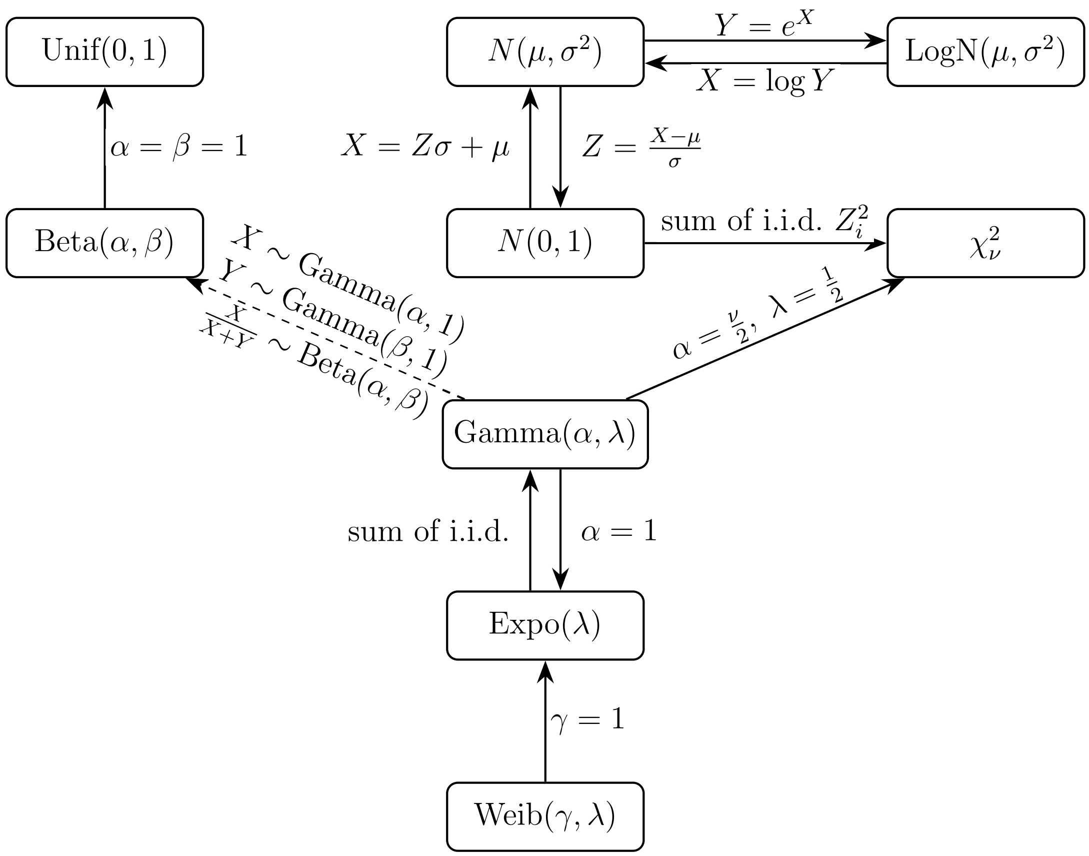
- Log-Normal distribution always used to model the stock prices or financial returns. If stock returns (percent changes) are normally distributed, then the stock price itself follows a Log-Normal distribution. For example, modeling the distribution of Google's stock price after 1 year.
- In general, Log-Normal distribution is useful when a variable is positive and right skewed.

Learning Objectives

By the end of this lecture, you should be able to:

- Review the main continuous distributions introduced so far.
- Understand their key properties.
- Understand the pdf and cdf of Weibull distribution.
- Highlight the relationships between these distributions.
- Provide guidance on when to use each distribution in applications.

Continuous relationship map



Conditions of Use

Distribution	Conditions	Example
Unif(a, b)	All outcomes equally likely on (a, b)	Randomly choosing a real number between 0 and 1
$N(\mu, \sigma^2)$	Continuous data with symmetric, bell-shaped pattern	Heights, measurement errors, IQ scores
χ^2_ν	Sum of squares of ν standard normals	Test statistics in ANOVA or goodness-of-fit tests
Gamma(α, λ)	Waiting time until α events occur (rate λ)	Total rainfall in a season, insurance claim sizes
Expo(λ)	Waiting time between independent events	Time between arrivals at a service counter
Beta(α, β)	Continuous data on $(0, 1)$, often proportions or probabilities	Fraction of defective items in a batch
LogN(μ, σ^2)	Variable whose log is normal; positive and skewed	Income, stock prices, survival times
Weib(γ, λ)	Flexible lifetime model; monotone hazard	Reliability of machines, time-to-failure analysis

Example 9.1. The achievement scores for a college entrance examination are normally distributed with mean 75 and standard deviation 10. What fraction of the scores lies between 80 and 90?

Example 9.2 A circle with a random radius $U \sim \text{Unif}(0,1)$ is generated. Let X be its area.

(a) Find the cdf and pdf of X.

(b) Find the expected value of X.

Example 9.3. The Mental Development Index (MDI) of the Bayley Scales of Infant Development is a standardized measure used in longitudinal follow-up of high-risk infants. The scores on the MDI have an approximately Normal distribution with a mean of 100 and standard deviation of 15.

(a) What proportion of children have MDI of at least 88?

(b) What is the probability a randomly selected infants will get a scores between 100 and 130?

Example 9.4. Let T be the lifetime of a certain person (how long that person lives), and let T have cdf $F(t)$ and pdf $f(t)$. The hazard function of T is defined by

$$h(t) = \frac{f(t)}{1 - F(t)}.$$

Show that an Exponential random variable has constant hazard function and conversely, if the hazard function of T is a constant then T must be $\text{Expo}(\lambda)$.

Example 9.5. The lifetime of a certain brand of light bulb follows the Exponential distribution. It is advertised to have an expected lifetime of 1000 hours. Let X be the lifetime until it fails.

(a) What is the probability that a randomly selected bulb of this brand will last more than 2000 hours?

(b) Given that the bulb has already worked 2000 hours, what is the probability that it survives a further 2000 units of time?

Instructions

This Homework contains *5 questions* and *20 points* in total. Show all work for full credit. Partial credit will be given if your process is clearly shown and mathematically reasonable. If the question is theoretical, please provide a written explanation.
Due date: Friday, October 24th by 11:59pm.

1. Suppose that Y has density function

$$f(y) = \begin{cases} ky(1-y), & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

   (a) (2 points) Find the value of k that makes $f(y)$ a pdf.

   (b) (2 points) Find $P(0.4 \leq Y \leq 1)$.

2. The amount if flour used per day by a bakery is a random variable Y that has an Exponential distribution with mean equal to 4 tons. The cost of the flour is proportional to $U = 3Y + 1$.

   (a) (2 points) Find the pdf of U.

   (b) (2 points) Use the answer in part (a) to find $E(U)$.

3. (3 points) The waiting time Y until delivery of a new component for an industrial operation is uniformly distributed over the interval from 1 to 5 days. The cost of this delay is given by $U = 2Y^2 + 3$. Find the pdf of U.

4. (3 points) A density function sometimes used by engineers to model lengths of life of electronic components is Rayleigh density, given by

$$f(y) = \begin{cases} \left(\frac{2y}{\theta}\right) e^{-\frac{y^2}{\theta}}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

   If Y has the Rayleigh density, find the pdf for $U = Y^2$.

5. The pH of water samples from a specific lake is a random variable Y with pdf given by

$$f(y) = \begin{cases} \frac{3}{8}(7-y)^2, & 5 \leq y \leq 7 \\ 0, & \text{otherwise} \end{cases}$$

   (a) (4 points) Find $E(Y)$ and $\text{Var}(Y)$.

   (b) (2 points) Would you expect to see a pH measurement below 5.5 very often? Why?

Instructions

This Homework contains *5 questions* and *20 points* in total. Show all work for full credit. Partial credit will be given if your process is clearly shown and mathematically reasonable. If the question is theoretical, please provide a written explanation.
Due date: Friday, October 31st by 11:59pm.

1. (3 points) Suppose that with probability p, the random variable X has a normal distribution with mean 0 and variance 1, and with probability $1-p$, X has a normal distribution with mean 1 and variance 1. What is the variance of X? *Hint: Consider two normal distribution such that*

$$\begin{aligned} X_1 &\sim N(0,1) \quad \textit{with probability } p \\ X_2 &\sim N(1,1) \quad \textit{with probability } 1-p. \end{aligned}$$

The random variable $X = pX_1 + (1-p)X_2$. Now use the $Var(X) = E(X^2) - E^2(X)$ to find the variance of X. To find $E(X)$, note that $E(X) = pE(X_1) + (1-p)E(X_2)$ the expectation of X_1 and X_2 is given. Use the similar way to find $E(X^2)$.

2. (4 points) For a certain insurance company, 60% of claims have a normal distribution with mean 5,000 and variance 1,000,000. The remaining 40% have a normal distribution with mean 4,000 and variance 1,000,000. Calculate the probability that a randomly selected claim exceeds 6,000. *Hint: Use the Law of Total Probability. Also, $P(Z > 1.0) = 0.1578$ and $P(Z > 2.0) = 0.0228$.*

3. Let $Y = |Z|$ with $Z \sim N(0,1)$. The distribution of y is called a *Folded Normal* with mean zero and variance 1. This is a well-defined continuous random variable, even thou the absolute value function is not differentiable at 0 (due to the sharp corner).

   (a) (2 points) Find the cdf of Y. *Hint: You can use $\Phi(z)$ to represent the cdf of standard normal distribution. So in your final answer, the cdf of Y, $F_Y(y)$, is representing in terms of $\Phi(y)$. Starting with $F_Y(y) = P(|Z| <\leq y) = P(-y \leq Z \leq y)$.*

   (b) (2 points) Find the pdf of Y. *Hint: In part (a) we used $\Phi(y)$ as the cdf of Y. Taking the derivative of $\Phi(y)$ gives you the pdf of Y. That is, $\frac{d}{dy}\Phi(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$. This is the pdf of standard normal distribution as we mentioned in the lecture.*

   (c) (2 points) Find $E(Y)$.

   (d) (2 points) Find Var(Y). *Hint: Since $Y = |Z|$, then $Y^2 = Z^2$. In another word, $E(Y^2) = E(Z^2)$.*

4. (3 points) A standard Cauchy random variable has density function

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

Show that X is a standard Cauchy random variable, then $\frac{1}{X}$ is also a standard Cauchy random variable. *Hint: Use*

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{\partial}{\partial y} g^{-1}(y) \right|$$

to find the pdf of y. Eventually, you need to verify the form of $f_Y(y)$ with the pdf of Cauchy density $f(x)$.

5. (2 points) Using the Example 9.4 from Module 4 Lecture 9 as a reference. Compute the hazard function of a Weibull random variable, $X \sim \text{Weib}(k, \lambda)$. *Hint:* $h(t) = \frac{f(t)}{1-F(t)}$.