Learning Objectives

By the end of this lecture, you should be able to:

- Understand the meaning of the moment.
- See the summarize a distribution.
- Understand the calculation of moment generating function.

Text book sections Ross' book: Section 7.7

Moment of a Random Variable

The k-th moment of an random variable X is defined as $E(X^k)$. The k-th central moment of a random variable X is defined as $E[(X-\mu)^k]$, where $\mu = E(X)$. For example,

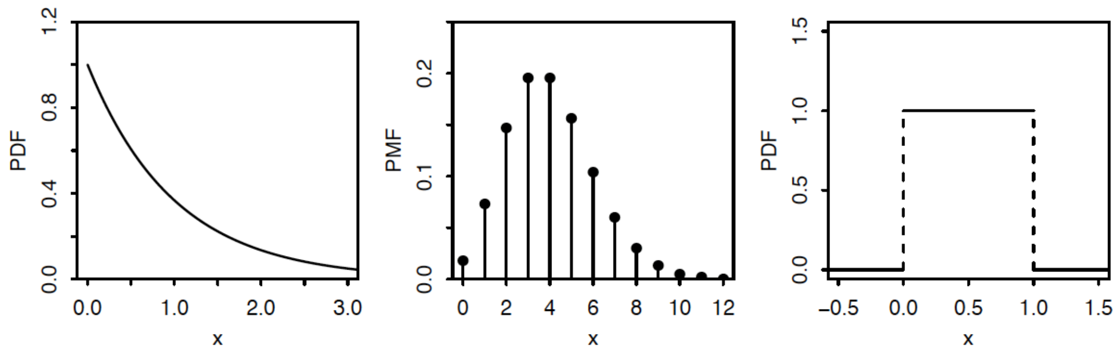
Example 1.1. Suppose that $X \sim \text{Expo}(\lambda)$ and $k \in \mathbb{N}$. Find $E(X^k)$.

Previously we introduced that mean and variance (or standard deviation) are very useful summarize of a distribution. Here, we introduce two more measures. The skewness of an random variable X with mean μ and variance σ^2 is the third standardized moment of X

$$\text{Skew}(X) = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$$

The kurtosis of an random variable X with mean μ and variance σ^2 is a shifted version of the fourth standardized moment of X

$$\text{Kurt}(X) = E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] - 3$$



A moment generating function, as its name suggests, is a generating function that encodes the moments of a distribution. The moment generating function (mgf) of an random variable X is defined as

$$M(t) = E(e^{tX}).$$

As a function of t, it must be finite on some open interval containing 0 such as $(-a, b)$. For example, $M_X(t)$ exists if $M(t) < \infty$ on a range of $t \in (-0.01, 0.0001)$. Otherwise, we say the mgf of X does not exist. Note that $M(0) = 1$ for any valid mfg. Hence, whenever you compute an mgf, plug 0 and see if you get 1, as a quick check.

Example 1.2. Find the mgf of $X \sim \text{Bern}(p)$.

Example 1.3. Find the mgf of $X \sim \text{Geom}(p)$.

Example 1.4. Find the mgf of $X \sim \text{Unif}(a, b)$.

Example 1.5. Find the mgf of $X \sim \text{Pois}(\lambda)$.

Example 1.6. Let $Z \sim N(0, 1)$. Find the mgf of Z.

Learning Objectives

By the end of this lecture, you should be able to:

- Understand the meaning of the each moment.
- Use mgf to find the mean and variance.

Text book sections Ross' book: Section 7.7

Moment Generating Function

Recall the Taylor expansion of e^x.

Then the mgf of random variable X, $M_X(t)$, can be written as

We take the derivative of the mgf with respects to t,

Finally, we evaluate the derivatives of the mgf when $t = 0$

To find the k-th moment of X using its mgf, we take the k-th derivative of the mgf with respects to t and set $t = 0$.

Remark:

- Not all random variables have an mgf. Some random variables X don't even have $E(X)$ exist, or don't have $E(X^k)$ exist for some $k > 1$. The mgf may not exist if the moments grow too quickly.
- Inserting an imaginary number is a way to fix that. The function $\psi(t) = E(e^{itX})$ with $i = \sqrt{-1}$ is called characteristic function.

Example 2.1. Find the first two moments of $X \sim \text{Bern}(p)$.

Example 2.2. Find the first two moments of $X \sim N(0, 1)$.

If X has mgf $M(t)$, then the mgf of $a + bX$ is

$$E\left[e^{t(a+bX)}\right] = e^{at}E(e^{btX}) = e^{at}M_X(bt)$$

Example 2.3. Find the mgf of $X \sim N(\mu, \sigma^2)$.

Example 2.4. Find the mgf of $X \sim \text{Expo}(1)$ and $X \sim \text{Expo}(\lambda)$.

Example 2.5. Use the mgf to find the $E(X)$, $\text{Var}(X)$ of $X \sim \text{Gamma}(\alpha, \lambda)$.

Theorem 2.1. Let $M_X(t)$ and $M_Y(t)$ denote the mgf of random variables X and Y, respectively. If both mgf exist and $M_X(t) = M_Y(t)$ for all values of t, then X and Y have the same distribution.

This theorem is a difficult result in analysis, and the proof is beyond the scope of this course. We will skip the proof here.

Example 2.6. If the mgf of X is given by $M_X(t) = e^{-5t+6t^2}$, what is the distribution of X?

Learning Objectives

By the end of this lecture, you should be able to:

- Understand the sums of random variables.

Text book sections Ross' book: Section 7.7

Sum of independent random variables

Let $X_1, X_2, \ldots, X_n$ be independent random variables with mgf $M_{X_1}(t), M_{X_2}(t), \ldots, M_{X_n}(t)$, respectively. If $U = X_1 + X_2 + \cdots + X_n$, then

$$M_U(t) = M_{X_1}(t) \times M_{X_2}(t) \times \cdots \times M_{X_n}(t)$$

Example 3.1. Suppose we have $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$. Assume that X_1 and X_2 are independent, find the mgf of $X_1 + X_2$.

Example 3.2. Using the mgf to show that the sum of independent Poisson is Poisson.

Example 3.3. Find the mgf of $X \sim \chi^2_\nu$

Example 3.4. Let $Z \sim N(0, 1)$. Use the mgf to find the distribution of Z^2.

Example 3.5. Let $X_1, X_2, \ldots, X_n$ be independent normally distributed random variables with $E(X_i) = \mu_i$ and $\mathrm{Var}(X_i) = \sigma_i^2$. Define Z_i as

$$Z_i = \frac{X_i - \mu_i}{\sigma_i}, \quad i = 1, 2, \ldots, n$$

Define $V = \sum_{i=1}^n Z_i^2$, find the mgf of V.

Example 3.6. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables such that $X_i \sim \text{Bern}(p)$. Define $Y = \sum_{i=1}^n X_i$, find the mgf of Y.

Instructions

This Homework contains *5 questions* and *20 points* in total. Show all work for full credit. Partial credit will be given if your process is clearly shown and mathematically reasonable. If the question is theoretical, please provide a written explanation.
Due date: Friday, November 7th by 11:59pm.

1. (3 points) Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables such that Y_i has a gamma distribution with parameters α_i and λ. That is, the distribution of Y's might have different α's, but all have the same value for λ. That is, for each Y_i, $Y_i \sim \text{Gamma}(\alpha_i, \lambda)$. Prove that $U = Y_1 + Y_2 + \cdots + Y_n$ has a gamma distribution with parameters $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ and λ.

2. (4 points) Suppose $X \sim \chi^2_\nu$, and the moment generating function (mgf) of X is $M_X(t) = (1-2t)^{-\frac{\nu}{2}}$. Use the mgf to find $E(X)$ and $\text{Var}(X)$. *Hint: Use the chain rule to find the derivative of $M_X(t)$.*

3. (3 points) Suppose $X \sim \text{Geom}(p)$ and the moment generating function (mgf) of X is $M_X(t) = \frac{pe^t}{1-(1-p)e^t}$. Use the mgf to find the $E(X)$. *Hint: Use quotient rule to find the derivative of $M_X(t)$.*

4. Let us consider a Log-Normal distribution. We say that X is a Log-Normal random variable with parameters μ and σ^2, denoted as $X \sim \text{LogN}(\mu, \sigma^2)$, if X can be represented by $X = e^Y$, where $Y \sim N(\mu, \sigma^2)$ and its pdf is

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{(\log(x)-\mu)^2}{2\sigma^2}\right], \quad x > 0$$

Note that $\exp(a) = e^a$ is another expression.

   (a) (2 points) Find the $k-$th moment of Log-Normal random variable X. *Hint: Use the mgf of normal distribution to derive the $k-$th moment of Log-Normal distribution under $X = e^Y$.*

   (b) (2 points) Find the mean of X, $E(X)$.

   (c) (2 points) Find the variance of X, $\text{Var}(X)$.

5. (4 points) Let $Y_1, Y_2, \ldots, Y_n$ be independent normally distributed random variables with $E(Y_i) = \mu_i$ and $\text{Var}(Y_i) = \sigma_i^2$ for $i = 1, 2, \ldots, n$. That means, they are not identically distributed. Let $a_1, a_2, \ldots, a_n$ be constant. Define

$$U = \sum_{i=1}^{n} a_i Y_i = a_1 Y_1 + a_2 Y_2 + \cdots + a_n Y_n$$

Find the mgf of U and named the distribution with its parameter values.