Learning Objectives

By the end of this lecture, you should be able to:

- Understand the definition of the discrete random variable.
- Learn what a probability mass function (pmf) is.
- Work through examples to connect definition with calculations.

Text book sections Ross' book: Section 4.1-4.2

Random Variable

Given an experiment with sample space S, a random variable is a function from the sample space S to the real numbers $\mathbb{R}$. Intuitively, a random variable is a mapping rule to assign a number to each outcome in S.

Suppose we are interested in the HW score of a randomly selected person. Here is a workflow of this Module.

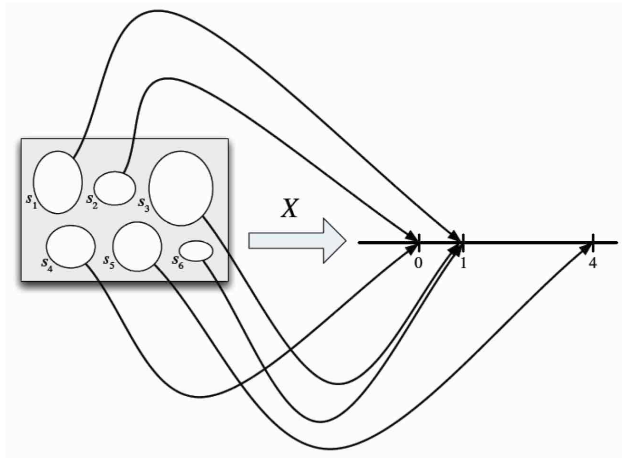
(a) Define X. We define a function X that returns the HW score of selected person.

(b) Express X's distribution. Once X is defined, we will have 70 values of X. The distribution of 70 values is called the probability distribution of X, showing frequencies of possible values of X.

(c) Summarize. We will summarize the distribution numerically.

Remarks:

- A random variable is a deterministic function from sample space S to the real numbers $\mathbb{R}$.
- We often use capital letters X, Y, Z to denote the random variables.
- Random variables take on numerical values. This is convenient simplification compared to having to work with the full complexity of S at all times.



A random variable X depicted here is defined on a sample space with 6 elements, and has possible values 0, 1, and 4. The randomness comes from choosing a random pebble according to the probability function P for the sample space.

Remarks:

- X is an random variable but an equality or inequality of an random variable is an event.
- This is important as we can only calculate the probability of an event.

Example 1.1. Consider an experiment where we toss a fair coin twice. The sample space consists of four possible outcomes: $S = \{HH, HT, TH, TT\}$.

Random variables can be broadly classified as discrete and continuous, depending on their possible values. A random variable X is said to be discrete if the set of values are finite or countably infinite. If X is a discrete random variable, then the finite or countably infinite set of values x such that $P(X = x) > 0$ is called the support of X.

Probability mass function

Given a random variable, we would like to be able to describe its behavior using the probability language. For example, if M is the number of major earthquakes in California in the next five years, what is the probability that $M = 0$? The distribution of a random variable provides the answers to this question. It specifies the probabilities of all events associated with the random variable.

In writing $P(X = x)$, we are using $\underline{X = x}$ to denote an <u>event</u>, consisting all outcomes <u>s</u> to which $\underline{X}$ assigns the number $\underline{x}$. This event is also written as $\underline{X = x}$; formally, $\underline{\{X = x\}}$ is defined as $\underline{\{s \in S : X(s) = x\}}$. In short, we write as $\underline{\{X = x\}}$.

For a discrete random variable X, a probability mass function (pmf) p_X is the function given by $p_X(x) = P(X = x)$. The pmf describe how likely the possible values of X are to occur.

Example 1.2. Consider flipping a coin twice. Let X denote an event observing the number of heads and Y be observing the number of tails. Find the pmfs of X and Y respectively.

Example 1,3, Consider the experiment of tossing 3 fair coins and let X be the number of heads observed.

If X is a discrete random variables with pmf as $p_X(x) = P(X = x)$ for all real number. For the pmf to be valid, it must satisfy two conditions

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Example 1.4. We roll two fair 6-sided dice. Let $T = X + Y$ be the total of two rolls, where X and Y are the individual rolls.

(a) Write down the pmf $p_T(t)$.

(b) Verify this is a valid pmf.

(c) What is the probability that T is greater than 7?

Learning Objectives

– Define the cumulative distribution function (CDF) for discrete random variables.

– Understand the relationship between the pmf and cdf.

– Practice computing CDF values from a given PMF.

– Learn how to find the distribution of a function of a discrete random variable.

Text book sections Ross' book: Section 4.2

Cumulative distribution function

Another function that describes the distribution of an random variable is called cumulative distribution function (cdf). Unlike the pmf, which only works for discrete random variables, the cdf is defined for all random variables.

The cdf of an random variable X is the function F_X given by $F_X(x) = P(X \leq x)$. We sometimes just write F for a cdf.

Properties of cdf

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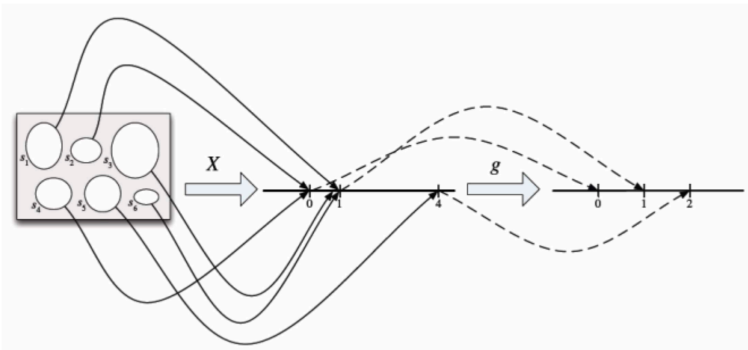
Example 2.1. Suppose you are drawing a ball from a bag. The bag contains 1 red ball, 2 blue balls and 1 green balls. Let random variable X be defined numerically as $X = 0$ if the ball is red; $X = 1$ if the ball is blue; and $X = 2$ if the ball is green. Assume there are 4 balls total, each ball has an equal chance of being selected. Find the pmf and the cdf. Graph the cdf plot.

Function of random variables

For an experiment with sample space S, a random variable X and a function $g : \mathbb{R} \to \mathbb{R}$, $g(X)$ is the random variable that maps s to $g(X(s))$ for all $s \in S$.



For example, let's define a function of a random variable X as $g(X) = \sqrt{X}$. This is the composition of the functions X and g, saying "first apply X, then apply $g(X)$.

Remarks:

- Any function of an random variable is still an random variable.
- The pmf of $Y = g(X)$ is

$$P(Y = y) = P(g(X) = y) = P(X = g^{-1}(y)) = \sum_{x:g(x)=y} P(X = x)$$

- The support of $g(X)$ is the set of all y such that $g(x) = y$ for x in the support of X.

Example 2.3. Let X be the outcome when rolling a fair six-sided die. Define a new random variable $Y = X^2$.

(a) Find the support of Y.

(b) Find the pmf of Y.

(c) Find the cdf of Y.

Example 2.4. Let the discrete random variable X take values in $\{-2, -1, 0, 1, 3\}$ with the following pmf. Define $Y = g(X) = |X|$.

$$p_X(x) = \begin{cases} 0.10, & x = -2 \\ 0.20, & x = -1 \\ 0.30, & x = 0 \\ 0.25, & x = 1 \\ 0.15, & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

(a) Verify $p_X(x)$ is a valid pmf.

(b) Find the pmf of Y, $p_Y(y)$.

(c) Verify $p_Y(y)$ is a valid pmf.

Learning Objectives

- Define the expectation (mean) and variance of a discrete random variable.
- Compute the expectation and variance from the pmf.
- Interpret the expectation and variance
- Practice expectation and variance calculations for various discrete distributions.

Text book sections Ross' book: Section 4.3-4.4

Expectation

One of the most important concepts in probability theory is that of the expectation of a random variable. Suppose we repeat a random experiment many times - such as rolling a die. Each time we record the value of a random variable X. Over time, we may observe a pattern: the values of X vary from trail to trail, but they tend to cluster around a certain point. The expected value formalizes this idea by providing the long-run average outcome of X over many repetitions of the experiment.

If X is a discrete random variable having a pmf $p_X(x)$, the expectation (or expected value, or mean) of X, denoted by $E(X)$, is defined by

$$E(X) = \sum_{\text{all } x} x p_X(x)$$

In words, the expected value of X is a weighted average of the possible values that X can take on, each value being weighted by their probability.

Example 3.1. Recall the previous example from Lecture 2. Suppose you are drawing a ball from a bag. The bag contains 1 red ball, 2 blue balls and 1 green balls. Let random variable X be defined numerically as $X = 0$ if the ball is red; $X = 1$ if the ball is blue; and $X = 2$ if the ball is green. Assume there are 4 balls total, each ball has an equal chance of being selected. Find the expectation of X.

Expectation of a function of a random variable

Suppose that we are given a discrete random variable along with its pmf and we want to compute the expected value of some function of X, say $g(X)$. We need to extend the idea of expected value to functions of random variable.

Let X be a discrete random variable with pmf $p_X(x)$ and let g be any real-valued function defined on the values of X. Then the expected value of $g(X)$ is

$$E[g(X)] = \sum_{\text{all } x} g(x) p_X(x)$$

Example 3.3. In a classroom quiz game, students earn points based on a spinner with 1, 3, 5 point with respective probabilities

$$P(X=1) = 0.5 \quad P(X=3) = 0.3 \quad P(X=5) = 0.2$$

Suppose the actual prize awarded is the square of the points obtained, i.e. the reward is $g(X) = X^2$. Computing the expected prize value $E[g(X)] = E(X)$.

Linearity of expectation

The most important property of expectation is linearity. For any constant a, b, and c, and any random variables X and Y,

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- (Not linearity of expectation, but useful) If X and Y are independent

Example 3.4. In a soccer league, a player's performance in a match is evaluated by a scoring system. The player earns 5 points for every goal, and loses 2 points for every foul committed. Let G be the number of goals scored in a match and F be the number of fouls committed in the match. The player's total score for the match is given by $S = 5G - 2F$. Find the expected score $E(S)$.

Variance

Previously we introduced the expected value of a random variable. The expected value $E(X)$ gives us a measure of the center of the distribution. But the expected value does not tell us the variation, or spread of the random variable.

Let X be a random variable with expected value as $E(X)$. The variance of X, denoted as $\text{Var}(X)$, is defined as

$$\text{Var}(X) = E(X - E(X))^2,$$

the alternative form is

The variance of X measures how far X is from its mean on average, but instead of simply taking the average difference between X and its mean $E(X)$, we take the average squared difference.

The standard deviation is the square root of the variance

$$\text{SD}(X) = \sqrt{\text{Var}(X)}$$

Similar with expectation, variance also has some useful properties.

- For any constant a, b, and a random variable X,

- If X and Y are independent,

Example 3.3. In the Gregorian calendar, each year has either 365 days (a normal year) or a 366 days (a leap year). A year is randomly chosen, with probability $\frac{3}{4}$ of being a normal year and $\frac{1}{4}$ of being a lead year. Find the mean and variance of the number of days in the chosen year.

Example 3.4. Find the mean and variance for the random variable X.

x	1	2	3
$P(X = x)$	0.3	0.4	0.3

Learning Objectives

– Define independence for random variables.

– Recognize the Bernoulli distribution and its properties.

– Understand how the Binomial distribution generalizes the Bernoulli.

– Connect the Bernoulli and Binomial distributions through examples.

Text book sections Ross' book: Section 6.2, 4.6

Independence of random variables

As we had the notion of independence of events, we can define independence of random variable. Intuitively, if two random variables X and Y are independent, then knowing the value of X gives no information about the value of Y, and vice versa.

In discrete case, random variables X and Y are said to be independent if

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

for all x, y with x in the support of X and y in the support of Y.

Example 4.1. A fair die is rolled twice. Let X be the outcome of the first roll and Y the outcome of the second roll.

Other than independence of random variable, we will often work with random variables that are independent and have the same distribution. We call such random variables independent and identically distributed, or i.i.d. for short.

Remark: "Independent" and "identically distributed" are two often confused but completely different concepts.

- Random variables are independent if they provide no information about each other.
- Random variables are identically distributed if they have the same pmf.

For example

- Let X be the result of a die roll and Y the result of a second, independent die roll.
- Let X be the result of a die roll and Y the closing price of the Dow Jones (a stock market index) a month from now.

Bernoulli Distribution

We will now start exploring specific families of discrete distributions that are widely used in modeling real-world random phenomena.

Let X be a random variable that models a single trail with success probability p. Then X is said to follow a Bernoulli distribution with parameter p, $0 < p < 1$ and pmf

$$p_X(x) = \begin{cases} p, & x = 1 \\ 1 - p, & x = 0 \end{cases}$$

Remark:

- Any random variable values are 0 and 1 has Bern(p) distribution with p the probability of the random variable equaling 1 (or success probability).
- This number p in Bern(p) is called the parameter of the distribution.

The indicator random variable of an event A is the random variable which equals 1 if A occurs and 0 otherwise. We denote the indicator random variable of A by I_A or $I(A)$ such that

Properties of Bernoulli random variable

Let $X \sim \text{Bern}(p)$, then

- $E(X) =$
- $\text{Var}(X) =$

Example 4.2. A die is rolled, and $X = 1$ if the outcome is even, otherwise $X = 0$.

Binomial distribution

An experiment that can result in either success or failure is called Bernoulli trial. The parameter p is called success probability of the Bern(p) distribution. It's hard not to starting thinking about what happens when we have more than one trial.

Suppose that n independent Bernoulli trials are performed, each with the same success success probability p. Let X be the number of success. The distribution of X is called Binomial distribution with parameter n and p.

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 0, 1, \dots, n$. We write $X \sim \text{Bin}(n, p)$ to mean that X has the Binomial distribution with $\underline{n}$ trials and success probability $\underline{p}$, where n is a positive integer and $0 < p < 1$.

Example 4.3. Consider tossing a fair coin 3 times. Let $X_i = 1$ if the i th toss is heads and $X_i = 0$ if it is tails. Suppose this is not a fair coin with $P(X_i = 1) = \frac{1}{3}$ and X_i's be independent. Let $Y = X_1 + X_2 + X_3$ be the total number of heads in the 5 tosses.

(a) What is the probability of all 3 times are heads?

(b) What is the probability of exactly 2 heads and 1 tails?

(c) In general, for n independent coin tosses, each with success probability p. What is the probability of getting exactly k heads?

Learning Objectives

- Recognize situations where the Binomial distribution applies.
- Identify the parameters n and p from a problem description.
- Use the Binomial pmf to compute probabilities for specific outcomes.
- Apply complement rules to find probabilities like "at least one" or "at most k".

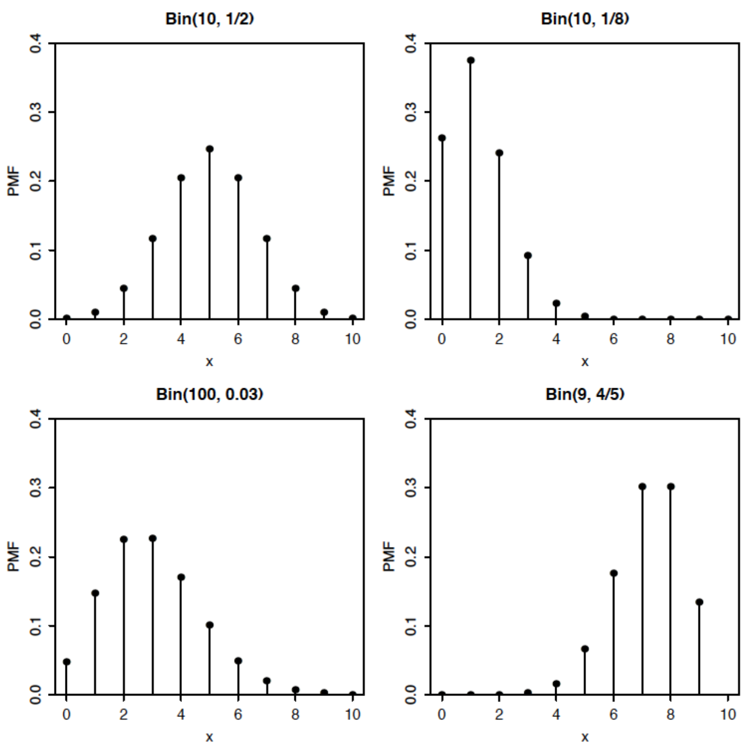
Text book sections Ross' book: Section 4.6

Binomial distribution

A random variable $X \sim \text{Bin}(n, p)$ is a Binomial distribution with pmf

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 0, 1, \ldots, n$.



Remark:

- When $n = 1$, the Binomial distribution reduce to Bernoulli distribution.
- Suppose $X_i \sim \text{Bern}(p)$ for $i = 1, \ldots, n$, then $Y = \sum_{i=1}^{n} X_i \sim \text{Bin}(n, p)$.

Properties of Binomial distribution

Let $X \sim \text{Bin}(n, p)$, then

- $E(X) =$
- $\text{Var}(X) =$
- If $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Bin}(m, p)$

Example 5.1. A factory produces batteries. Historically, 95% of the batteries pass quality inspection. An inspector randomly selects 20 batteries for testing. Let X be the number of batteries that pass inspection, then $X \sim \text{Bin}(20, p = 0.95)$. Compute the probability that at least 18 batteries pass.

Example 5.2. A player makes free throws with probability $p = 0.8$. In a game, the player attempts 5 free throws. Let X be the number of shots made.

(a) What is the probability of making all shots?

(b) What is the probability of making at least 4 shots?

Example 5.3. A complex electronic system is built with a certain number of backup components in its subsystems. One subsystem has four identical components, each with a probability of 0.2 of failing in less than 1000 hours. The subsystem will operate if any two of the four components are operating. Assume that the components operate independently.

(a) Find the probability that exactly two of the four components last longer than 1000 hours.

(b) Find the probability that the subsystem operates longer than 1000 hours.

Summary

An experiment is a Binomial experiment if it meets each of the following conditions:

1. The number of trials, n, is known and fixed.
2. There are two possible outcomes of each trial: 0 or 1
3. Outcomes are independent from one trial to next.
4. The probability of success, p, remains the same from one trial to the next.

The corresponding random variable X is a Binomial random variable.

Learning Objectives

- Recognize when the Poisson distribution is appropriate and identify its parameter.
- Derive and use the Poisson pmf to compute probabilities.
- Recognize when the Negative Binomial distribution applies and identify its parameters.
- Compute probabilities for the Negative Binomial and interpret them in real-world contexts.

Text book sections Ross' book: Section 4.7, 4.8.2

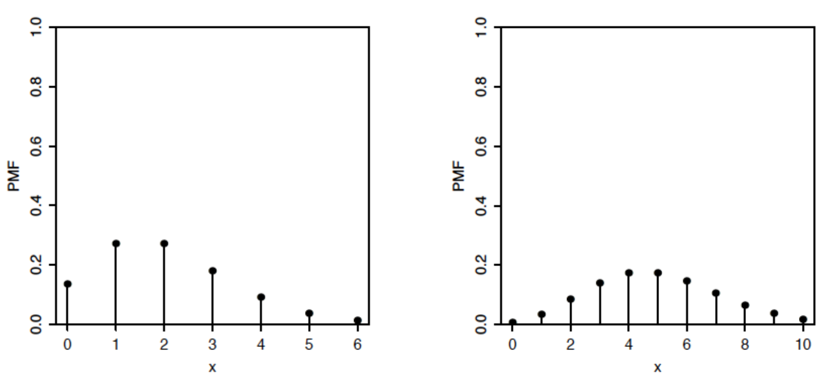
Poisson distribution

In many real-world situations, we are interested in the number of times an event occurs within a fixed interval, whether it be time, space, volume, or distance. Poisson distribution models the number of success in a particular region or interval of time, and is more appropriate for modeling a rare event.

A random variable X has Poisson distribution with parameter λ, where $\lambda > 0$, if the pmf of X is

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

We write this as $X \sim \text{Pois}(\lambda)$.



Example 6.1. Examples of Poisson random variables.

Properties of Poisson distribution

Let $X \sim \text{Pois}(\lambda)$,

- $E(X) =$

- $\text{Var}(X) =$

- If $X \sim \text{Pois}(\lambda_1)$ and $Y \sim \text{Pois}(\lambda_2)$ and X is independent of Y, then $X + Y \sim \text{Pois}(\lambda_1 + \lambda_2)$

- Suppose $Y \sim \text{Bin}(n, p)$, if n is very large and p is small, then Y is approximately follows a $\text{Pois}(\lambda = np)$.

Example 6.2. Suppose that a random system of police patrol is devised so that a patrol officer may visit a given beat location $Y = 0, 1, 2, \ldots$ times per half-hour period, with each location being visited an average of once per time period. Assume Y possesses, approximately, a Poisson distribution. Calculate the probability that the patrol office will miss a given location during a half-hour period. What is the probability that it will be visited once? Twice? At least once?

Negative Binomial distribution

Poisson distribution is wildly used in the counting process when assuming that mean equals to variance. There is another useful distribution of modeling the counting process called Negative Binomial distribution.

Imagining you are flipping a coin over and over. Every flip is independent, and the chance of heads is p (probability of success), and you stop when you've seen r heads (number of success). Let X denote the number of coins flipped until we receive r heads, then X is said to have the Negative Binomial distribution with parameters r and p, the pmf of X is

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^n - r, \quad n = r, r+1, r+2, \ldots$$

We write this as $X \sim \text{NBin}(r, p)$.

Example 6.3. Suppose we are interested in an random variable Y defined as the number of iid Bernoulli trials on which the second success occurs. What is $P(X = 4)$?

In this problem we are interested in the event $\{Y = 4\}$, meaning there were 2 failures and 2 success. Note that the last trial always has to be a success. There are 3 possible sequences that would satisfy this

Remarks:

- In the Negative Binomial distribution, the number of trials is random variable and the number of success if fixed and known.

Properties of Negative Binomial distribution If $X \sim \text{NBin}(r, p)$,

- $E(X) =$
- $\text{Var}(X) =$

Example 6.4. A geological study indicates that an exploratory oil well drilled in a particular region should strike oil with probability 0.2. Find the probability that the third oil strike comes on the fifth well drilled.

Summary:

Conditions for a Negative Binomial random variable:

1. Trials are independent of each other.
2. There are two possible outcomes of each trial: success and failure.
3. The probability of success, p, remains the same from one trial to the next.
4. The trial is repeated until we see the r-th success.

Learning Objectives

– Recognize when the Geometric distribution applies.

– Derive and use the pmf of the Geometric distribution.

– Recognize when the Hypergeometric distribution applies.

– Distinguish between Binomial and Hypergeometric settings.

Text book sections Ross' book: Section 4.8.1, 4.8.3

Geometric distribution

Consider a sequence of independent Bernoulli trials, each with the same success probability p with trials performed until a success occurs. Let X be the number of failures before the first successful trial, then X has the Geometric distribution with parameter p. The pmf of X is

$$P(X = k) = (1 - p)^k p, \quad k = 1, 2, \ldots$$

We write as $X \sim \text{Geom}(p)$.

Example 7.1. Suppose that the probability of engine malfunction during any one-hour period is $p = 0.02$. Find the probability that a given engine survival until the 4th hour and then malfunctions.

Properties of Geometric distribution

If $X \sim \text{Geom}(p)$,

- $E(X) =$

- $\text{Var}(X) =$

Example 7.2. If the probability of engine malfunction during any one-hour period is $p = 0.02$ and Y denotes the number of one-hour interval until the first malfunction, find the mean and variance of Y.

Example 7.3. Suppose that 30% of the applicants for a certain industrial job possess advanced training in computer programming. Applicants are interviewed sequentially and are selected at random from the pool.

(a) Find the probability that the first applicant with advanced training in programming is found on the fifth interview.

(b) What is the expected number of applicants who need to be interviewed in order to find the first one with advanced training?

Summary:

Conditions for a Geometric random variable:

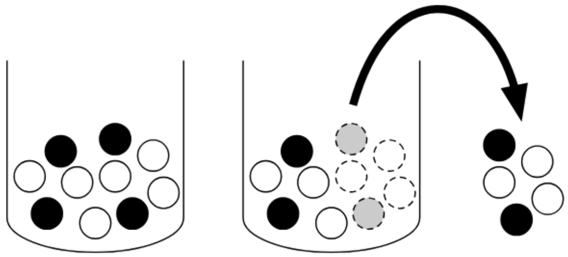
1. Trials of the experiment are independent of each other.
2. There are two possible outcomes of each trials: success and failure.
3. The probability of success, p, remains the same from one to the next.
4. The trial is repeated until we see the first success.

Hypergeometric distribution

Consider an urn with N balls, out of m are white balls. We draw n balls out of the urn at random without replacement, such that all $\binom{N}{n}$ samples are equally likely. Let X be the number of white balls in the sample. Then X is said to have Hypergeometric distribution with parameters n, m, N, then the pmf of X is

$$P(X = x) = \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}},$$

where $x \le m$ and $n - x \le N - m$. We write as $X \sim \text{HGeom}(n, m, N)$.



Summary:

Conditions for Hypergeometric random variable:

1. The population is finite and consists of N objects.
2. There are 2 possible outcomes for each individual, success and failure.
3. There are m individuals with successes in the population.
4. A sample of n individual are randomly sampled without replacement.

Properties of Hypergeometric distribution

- $E(X) =$
- $\mathrm{Var}(X) =$

Example 7.4. An important problem encountered by personnel directors and others faced with the selection of the best finite set of elements is exemplified by the following scenario. From a group of 20 Ph.D. engineers, 10 are randomly selected for employment. What is the probability tha the 10 selected include all the best 5 best engineers in the group of 20?

Example 7.5. An industrial product is shipped in lots of 20. Testing to determine whether an item is defective is costly, and hence the manufacturer samples his production rather than using a 100% inspection plan. A sampling plan, constructed to minimize the number of defectives shipped to customers, call for sampling five items from each lot and rejecting the log if more than one defective is observed. (If the lot is rejected, each item in it is later tested.)

(a) If a lot contains four defectives, what is the probability that it will be rejected?

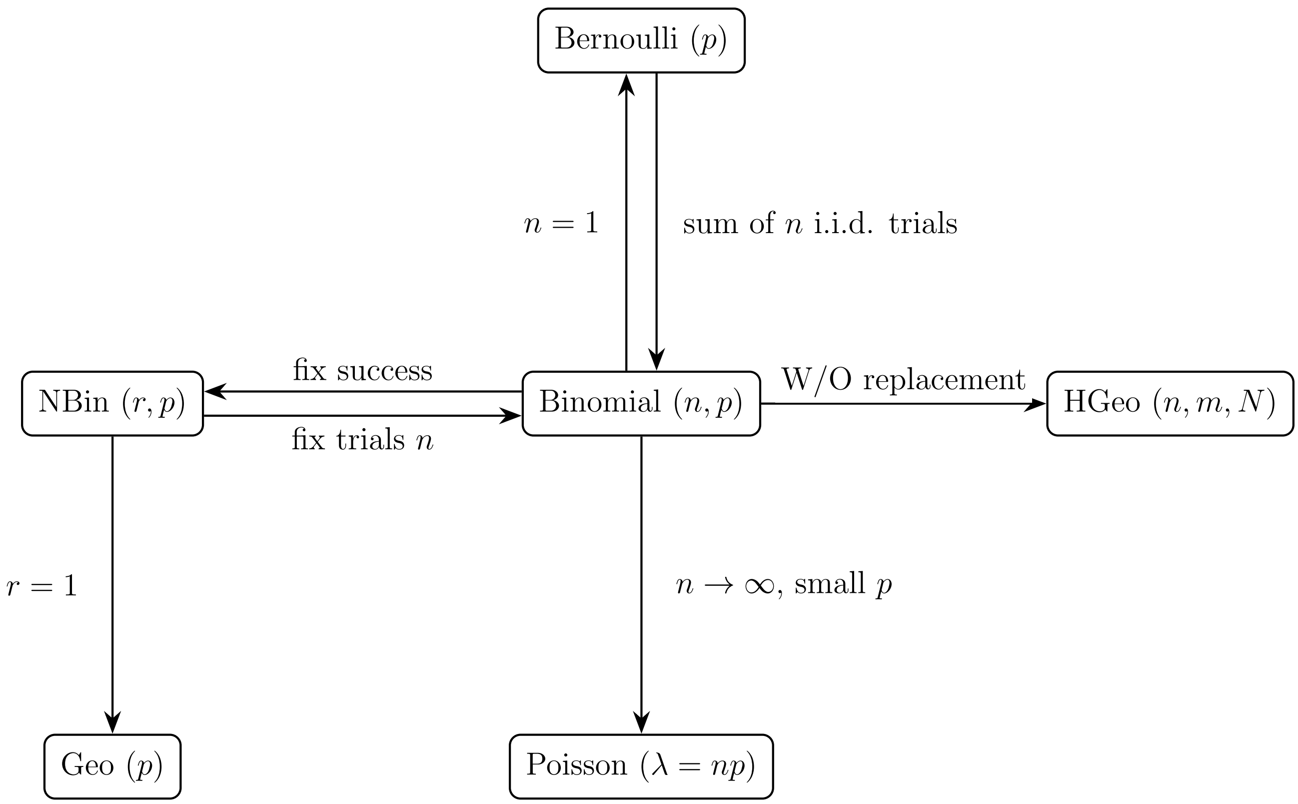
(b) What is the expected number of defectives in the sample of size 5?

(c) What is the variance of the number of defectives in the sample of size 5?

Learning Objectives

- Identify which discrete distribution applies to a given scenario.
- Recognize the parameters and write the pmf for each distribution.
- Understand the relationships between the distributions.

Discrete distribution relationship map



Conditions of Use

Distribution	**Description**	**Conditions**	**Example**
Bernoulli (p)	One trial with two possible outcomes (success/failure)	Single binary trial	Tossing a coin
Binomial (n, p)	Number of successes in n independent Bernoulli trials	Fixed n, independent trials, constant p	Number of heads in 10 coin tosses
Poisson (λ)	Number of events in a fixed interval of time or space	Events are rare, occur independently, at constant average rate	Number of calls per hour at a call center
Negative Binomial (r, p)	Number of trials until r successes	Trials are independent, constant p, stop after r successes	Basketball shots until 3 baskets are made
Geometric (p)	Number of trials until first success	Special case of Negative Binomial with $r = 1$	Rolls of a die until a "6" appears
Hypergeometric (n, m, N)	Number of successes in m draws without replacement from finite population	Population finite, sampling without replacement	Number of aces when drawing 5 cards from a deck

Example 8.1. Experience has shown that 30% of all persons afflicted by a certain illness recover. A drug company has developed a new medication. Ten people with the illness were selected at random and receive the medication; nine recovered shortly thereafter. Suppose that the medication was absolutely worthless.

(a) What distribution we should use and why.

(b) What is the probability that at least nie of then receiving the medication will recover?

Example 8.2. The mean number of automobiles entering a mountain tunnel per two-minute period is one. An excessive number of cars entering the tunnel during a brief period of time produces a hazardous situation.

(a) What distribution should we use and why.

(b) Find the probability that the number of autos entering the tunnel during a two minute period exceeds three.

Example 8.3. In southern California, a growing number of individuals pursuing teaching credentials are choosing paid internships over traditional student teaching programs. A group of eight candidates for three local teaching positions consisted of five who had enrolled in paid internships and three who enrolled in traditional student teaching programs. All eight candidates appear to be equally qualified, so three are randomly selected to fill the open position. Let Y be the number of internship trained candidates who are hired.

(a) Does Y have a Binomial or HyperGeometric distribution? Why? Include the support of Y.

(b) Find the probability that two or more internship trained candidates are hired.

(c) What are the mean and standard deviation of Y?

Example 8.4. Ten percent of the engines manufactured on an assembly line are defective. If engines are randomly selected one at a time and tested.

(a) What is the distribution we need to use if we wish to find the probability tha the third nondefective engine will be found on the fifth trial and why? Include the support of the random variable.

(b) Find the mean and variance of the trial on which the third nondefective engine is found.

Instructions

This Homework contains *6 questions* and *20 points* in total. Show all work for full credit. Partial credit will be given if your process is clearly shown and mathematically reasonable. If the question is theoretical, please provide a written explanation.
Due date: Friday, September 26th by 11:59pm.

1. (3 points) Consider the experiment of tossing 3 fair coins and let X be the number of heads observed. In the lecture we showed the pmf of X is

$$p_X(x) = \begin{cases} \frac{1}{8}, & x = 0, \\ \frac{3}{8}, & x = 1, \\ \frac{3}{8}, & x = 2, \\ \frac{1}{8}, & x = 3, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) (2 points) Find the cdf of X. (Show the calculation process)

   (b) (1 point) Graph the cdf.

2. (3 points) Let Y be a random variable with $p(y)$ given in the accompanying table. Find $E(Y)$, $E(\frac{1}{Y})$, $E(Y^2 - 1)$, and $\text{Var}(Y)$.

y	1	2	3	4
$p(y)$	0.4	0.3	0.2	0.1

3. The maximum patent life for a new drug is 17 years. Subtracting the length of time required by the FDA for testing and approval of the drug provides the actual patent life for the drug, that is, the length of time that the company has to recover research and development costs and to make a profit. The distribution of the lengths of actual patent lives for new drugs is give below:

Years, y	3	4	5	6	7	8	9	10	11	12	13
$p(y)$	0.03	0.05	0.07	0.10	0.14	0.20	0.18	0.12	0.07	0.03	0.01

   (a) (2 points) Find the mean patent life for a new drug.

   (b) (2 points) Find the variance and standard deviation of Y = the length of life of a randomly selected new drug.

4. (3 points) In a gambling game a person draws a single card from an ordinary 52-card playing deck. A person is paid \$15 for drawing a jack or a queen and \$5 for drawing a king or an ace. A person who draws any other card pays \$4. If a person plays this game, what is the expected gain?

5. A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win \$1.10; if they are different colors, then you win -\$1.00. (That is, you lose \$1.00).

   (a) (2 points) Calculate the expected value of the amount you win.

   (b) (2 points) Calculate the variance of the amount you win.

6. (3 points) Suppose we are rolling a fair die. Denote X be the outcomes we observed. Find $E(X)$ and $\mathrm{Var}(X)$.

Instructions

This Homework contains *7 questions* and *20 points* in total. Show all work for full credit. Partial credit will be given if your process is clearly shown and mathematically reasonable. If the question is theoretical, please provide a written explanation.
Due date: Friday, October 10th by 11:59pm.

1. A manufacturer of a low-calorie dairy drink wishes to compare the taste appeal of a new formula (formula B) with that of the standard formula (formula A). Each of four judges is given three glasses in random order, two containing formula A and the other containing formula B. Each judge is asked to state which glass he or she most enjoyed. Suppose that the two formulas are equally attractive. Let Y be the number of judges stating a preference for the new formula.

   (a) (2 points) Find the distribution function for Y.

   (b) (2 points) What is the probability that at least three of the four judges state a preference for the new formula?

   (c) (2 points) Find the expected value and variance for Y.

2. (2 points) The probability of a customer arrival at a grocery service counter in any one second is equal to 0.1. Assume that customers arrive in a random stream and hence that an arrival in any one second is independent of all others. Find the probability that the first arrival will occur during the third one-second interval. What distribution should you use and why.

3. (2 points) A parking lot has two entrances. Cars arrive at entrance I according to Poisson distribution at an average of three per hour and at entrance II according a Poisson distribution at an average of four per hour. What is the probability that a total of three cars will arrive at the parking lot in a given hour? (Assume that the numbers of cars arriving at the two entrances are independent.) *Hint: Consider the sum of two independent Poisson is still Poisson.*

4. (2 points) Suppose that a radio contains six transistors, two of which are defective. Three transistors are selected at random, removed from the radio, and inspected. Let Y equal the number of defectives observed. Find the probability distribution of Y. Explain the reason of the distribution you selected and provide the distribution table.

5. (3 points) A driver is selected at random. If the driver is a good driver, he is from a Poisson population with a mean of 1 claim per year. If the driver is a bad driver, he is from a Poisson population with a mean of 5 claims per year. There is equal probability that the driver is either a good driver or a bad driver. If the driver had three claims last year, calculate the probability that the driver is a good driver. *Hint: Consider Bayes' Rule and Law of Total Probability.*

6. (3 points) A company takes out an insurance policy to cover accidents that occur at its manufacturing plant. The probability that one or more accidents will occur during any

given month is $\frac{3}{5}$. The number of accidents that occur in any given month is independent of the number of accidents that occur in all other months. Calculate the probability that there will be at least four months in which no accidents occur before the fourth month in which at least one accident occurs. Explain the reason of using that distribution.

7. (2 points) A small commuter plane has 30 seats. The probability that any particular passenger will not show up for a flight is 0.10, independent of other passengers. The airline sells 32 tickets for the flight. Calculate the probability that more passengers show up for the flight than there are seats available.