Learning Objectives

By the end of this lecture, you should be able to:

- Understand the concept of conditional probability.
- Compute conditional probabilities from data or formulas.
- Apply conditional probability to real-world and theoretical problems.

Text book sections Ross' book: Section 3.1-3.2

Conditional Probability

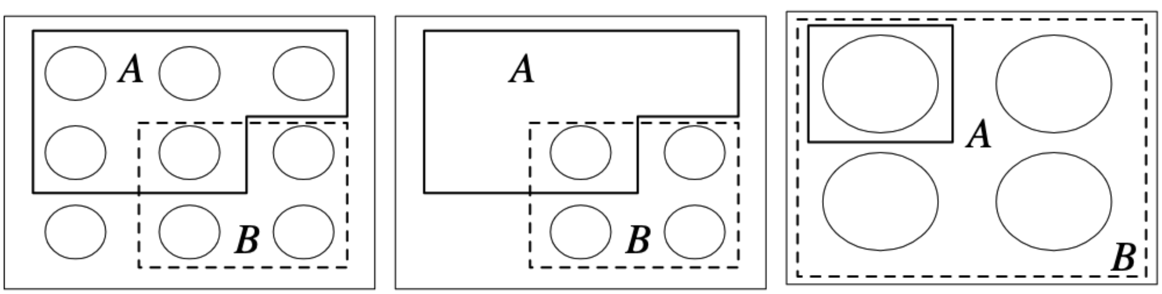
In everyday life, we constantly revise our expectations as new information arrives. Suppose you learn that a friend attended every lecture in my probability course. Naturally, you would expect their chance of passing the course to be higher than that of someone who skipped half the classes. In formal terms, this idea is known as conditional probability.

For two events A and B with $P(B) > 0$, the conditional probability of A given B, denoted as $P(A|B)$, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Here A is the event we want to update, and B is the evidence we observe. We call $P(A)$ as the prior probability and $P(A|B)$ as the posterior probability.

Example 1.1. Pebble world intuition for $P(A|B)$.



Example 1.2. Suppose you roll a fair six-sided die. Let's define an event A be the outcome is even, i.e. $A = \{2, 4, 6\}$; and event B be the outcome is at least 4, i.e. $B = \{4, 5, 6\}$. What is the probability that the roll is even, given that it is at least 4?

Example 1.3. Martin Gardner posed the following puzzle in the 1950s, in his column in Science American.

(a) Mr. Jones has two children, the older child is a girl. What is the probability that both children are girls?

(b) Mr. Smith has two children. At least one of them is a boy. What is the probability that both children are boys?

Example 1.4. Consider a deck of 52 standard playing cards. Let A be the event the card drawn is heart, and B be the event the card is face (Jack, Queen, or King). Compute the probability that the card is heart given that it's a face card.

Example 1.5. Show the conditional probability satisfies the Axioms of Probability.

(i). $0 \leq P(A|B) \leq 1$

(ii). $P(S|B) = 1$.

(iii). For a sequence of disjoint events $A_1, A_2, \ldots, A_n$,

$$P\left(\bigcup_{i=1}^{n} A_i \middle| B\right) = \sum_{i=1}^{n} P(A_i|B)$$

Example 1.6. A standard deck of cards is shuffled well. Two cards are drawn randomly, one at a time without replacement. Let A be the event that the first card is heart, and B be the event that the second card is red. Find $P(A|B)$ and $P(B|A)$

Example 1.7. A family has two children. Find the probability that both children are girls, given that at least one of the two is girl who was born in winter. Assume that the four seasons are equally likely and that gender is independent of season.

Learning Objectives

By the end of this lecture, you should be able to:

- Use the multiplication rule to compute joint probabilities.
- Apply the law of total probability to partitioned sample spaces.

Text book sections Ross' book: Section 3.2-3.3

Multiplication Rule

Probability of the intersections of two events: For any events A and B with positive probabilities,

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

Probability of the intersections of n events: For any events $A_1, A_2, \ldots, A_n$, we have

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1})$$

Example 2.1. Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

Example 2.2. Suppose that when you play a video game for the first time, you have 0.3 probability of winning. Your probability of winning is higher when you play for a second time, but how much higher depends on your first result. If you win on the first game, then your probability of winning on the second game increase to 0.8. But is you lose on the first game, then your probability of winning on the second game increase only to 0.4.

(a). Draw a tree diagram to represent this scenario.

(b). What is the probability that you win on both of your first two plays?

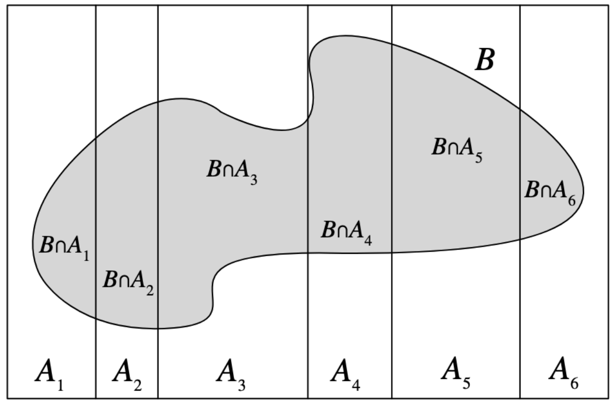
(c). What is the probability that you win on your second play of the game?

The Law of Total Probability

The law of total probability (LOTP) relates conditional probability to unconditional probability. It is essential for fulfilling the promise that conditional probability can be used to decompose complicated probability problems into simpler pieces, and it is often used with Bayes' rule.

Let $A_1, A_2, \ldots, A_n$ be a partition of the sample space S (i.e., the A_i are disjoint events and their union is S), with $P(A_i) > 0$ for all i, then

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$



Example 2.3. Suppose a bag A contains 3 red and 2 blue balls, bag B contains 1 red and 4 blue balls. You randomly pick one bag (with equal probability), then randomly draw one ball from that bag. What is the probability that the ball is red?

Example 2.4. A disease affects 2% of a population. A test detects the disease with 95% accuracy if the person has disease and 10% false positive rate if the person does not have the disease. What is the probability that a randomly selected person tests positive?

Example 2.5. An insurance company believes that people can divided into two classes: those who are accident prone and those who are not. The company's statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability decreases to 0.2 for a person who is not accident prone. If we assume that 30% of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

Example 2.6. Males and females are observed to react differently to a given set of circumstances. It has been observed that 70% of the females react positively to these circumstances, whereas only 40% of males react positively. A group of 20 people, 15 females and 5 males, was subjected to these circumstances, and the subjects were asked to describe their reactions on a written questionnaire. A response picked at random from the 20 was negative. What is the probability that it was that of a male?

Learning Objectives

By the end of this lecture, you should be able to:

- Understand the formulation and interpretation of Bayes' Theorem.
- Apply Bayes' Theorem to compute posterior probabilities.
- Use total probability and tree diagrams to support reasoning.

Text book sections Ross' book: Section 3.3

Bayes' Rule

The Law of total probability allows us to calculate the probability of event by conditioning on other events. These conditional probabilities sometimes are easier to calculate, i.e., it may be easier to find $P(B|A_i)$ for all i than $P(B)$ directly. Similarly, it is much easier to find $P(B|A)$ than $P(A|B)$ (or vice versa). Bayes' Rule (Theorem) can e helpful in this situation.

Let $A_1, \ldots, A_n$ be a partition of the sample space and $P(A_i) > 0$. Then for any other event B, $P(B) > 0$, the probability of A_i given that B has occurred is

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \underline{\hspace{3cm}} = \underline{\hspace{3cm}}$$

Example 3.1. Consider you live in a region where it rains on 30% of all days. A popular weather app claims to predict rain correctly 80% of the time when it does rain, but also "over-forecasts" on dry days 20% of the time. One morning the app issues a rain today, what is the chance you should carry an umbrella?

Example 3.2. Urn A has 5 white and 7 black balls. Urn B has 3 white and 12 black balls. We flip a fair coin. If the outcome is heads, then a ball from urn A is selected whereas if the outcome is tails, then a ball from urn B is selected. Suppose that a white ball is selected, what is the probability that the coin landed tails?

Example 3.3. A patient named Fred is tested for a disease called conditionitis, a medical condition that afflicts 1% of the population. The test result is positive, i.e., the test claims that Fred has the disease. Let D be the event that Fred has the disease and T be the event that he tests positive. Find the conditional probability that Fred has conditionitis, given the evidence provided by the test results.

Example 3.4. In answering a multiple-choice test question, a student either knows the answer or guesses. Let p be the probability that the student knows the answer and $1-p$ be the probability that the student guesses. Assume that a student who guesses at the answer will be correct with probability $\frac{1}{m}$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that he or she answered it correctly?

Example 3.5. A bin contains 3 different types of disposable flashlights. The probability that a type 1 flashlight will give over 100 hours of use is 0.7, with the corresponding probabilities for type 2 and type 3 flashlights being 0.4 and 0.3, respectively. Suppose 20% of the flashlights in the bin are type 1, 30% are type 2, and 50% are type 3.

(a). What is the probability that a randomly chosen flashlight will give more than 100 hours of use?

(b). Given that a flashlight lasted over 100 hours, what is the conditional probability that it was a type j flashlight for $j = 1, 2, 3$?

Example 3.6. (Example from last lecture) Males and females are observed to react differently to a given set of circumstances. It has been observed that 70% of the females react positively to these circumstances, whereas only 40% of males react positively. A group of 20 people, 15 females and 5 males, was subjected to these circumstances, and the subjects were asked to describe their reactions on a written questionnaire. A response picked at random from the 20 was negative. What is the probability that it was that of a male?

Example 3.7. A diagnostic test for a disease is such that it (correctly) detects the disease in 90% of the individuals who actually have the disease. Also, if a person does not have the disease, the test will report that he or she does not have it with probability 0.9. Only 1% of the population has the disease in question. If a person is chosen at random from the population and the diagnostic test indicates she has the disease, what is the conditional probability that she does, in fact, have the disease?

Learning Objectives

By the end of this lecture, you should be able to:

– Understand the definition of independent events.

– Learn how to test for independence.

– Differentiate between disjoint and independent.

Text book sections Ross' book: Section 3.4

Independent events

We have now seen several examples where conditioning on one event changes our beliefs about the probability of another event. The situation where events provide no information about each other is called independence.

Events A and B are independent if

$$P(A \cap B) = P(A)P(B).$$

If $P(A) > 0$ and $P(B) > 0$, then this is equivalent to $P(A|B) = P(A)$ or $P(B|A) = P(B)$.

Remark: Independence is different from disjointness.

Intuitively, it makes sense that if A provides no information about whether or not B occurred, then it also provides nor information about whether or not B^c occurred. Specifically, if A and B are independent, then A and B^c are independent, A^c and B are independent, and A^c and B^c are independent.

Proof:

Example 4.1. Let's draw one card from a standard deck. Consider A be the event that the card is spade and B be the event that the card id an ace.

(Independent of three events). Events A, B, and C are said to be independent if all of the following equations hold:

$$P(A \cap B) = P(A)P(B)$$
$$P(A \cap C) = P(A)P(C)$$
$$P(B \cap C) = P(B)P(C)$$
$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

Pairwise Independent:

Mutually Independent:

Example 4.2. (Pairwise independence doesn't imply mutually independence). Consider two fair, independent coin tosses, and let A be the event that the first is Heads, B the event that the second is Heads, and C the event that both tosses have the same result.

(Conditional independence). Events A and B are said to be conditionally independent given E if $P(A \cap B|E) = P(A|E)P(B|E)$.

Remark: $P(A \cap B) = P(A)P(B)$ __________ $P(A \cap B|E) = P(A|E)P(B|E)$.

Example 4.3. (Conditional independence is not symmetric across conditioning events). Suppose there are two types of classes: good class and bad class. In good class, if you work hard, you are very likely to get an A. In bad classes, the professor randomly assigns grades to students regardless of their effort. Let G be the event that a class is good, W be the event that you work hard, and A be the event that you receive an A. Are W and A conditionally independent given G? What if we condition on G^c?

Example 4.4. (Conditional independence doesn't imply independence) Suppose we have chosen either a fair coin or a biased coin with probability $\frac{3}{4}$ of heads, but we do not know which one we have chosen. We flip the coin a number of times. Let F be the event that we've chosen the fair coin, and let A_1 and A_2 be the events that the first and second coin tosses land Heads. Conditional on F , are A_1 and A_2 independent? Conditional on F^c, are A_1 and A_2 independent? Are A_1 and A_2 are unconditionally independent?

Example 4.5. Considering the following events in the toss of a single die:

$$A = \text{Observe an odd number.}$$
$$B = \text{Observe an even number.}$$
$$C = \text{Observe a 1 or 2.}$$

Are A and B independent events? Are A and C independent events?

Instructions

This Homework contains *6 questions* and *20 points* in total. Show all work for full credit. Partial credit will be given if your process is clearly shown and mathematically reasonable. If the question is theoretical, please provide a written explanation.
Due date: Friday, September 19th by 11:59pm.

1. (3 points) Two fair dice are rolled. What is the conditional probability that at least one hands on 6 given that the dice land on different number?
2. A fair six-sided die is rolled twice. Let A be the first roll is a 4 and B be the sum of the two rolls is at least 8.
   (a) (2 points) Compute $P(A|B)$.
   (b) (2 points) Compute $P(B|A)$.
3. (3 points) If A and B are independent, proof that A^c and B^c are independent. *Hint: Use DeMorgan's Law and Complement Rule.*
4. (4 points) A spam filter is designed by looking at commonly occurring phrases in spam. Suppose that 80% of email is spam. In 10% of the spam emails, the phrase "free money" is used, whereas this phrase is only used in 1% of non-spam emails. A new email has just arrived, which does mention "free money". What is the probability that it is spam?
5. (3 points) Stores A, B, and C have 50, 75, and 100 employees, respectively, and 50, 60, and 70 percent of them respectively are women. Resignations are equally likely among all employees, regardless of sex. One woman employee resigns. What is the probability that she works in store C?
6. (3 points) Consider the Example 4.4 we discussed in Lecture 4. Suppose we have chosen either a fair coin or a biased coin with probability $\frac{3}{4}$ of heads, but we do not know which one we have chosen. We flip the coin a number of times. Let F be the event that we've chosen the fair coin, and let A_1 and A_2 be the events that the first and second coin tosses land Heads. We assume that $P(F) = P(F^c) = \frac{1}{2}$. Are A_1 and A_2 are unconditionally independent? *Hint: Considering Law of Total Probability*