UNIT I: FUNCTIONS AND LIMITS

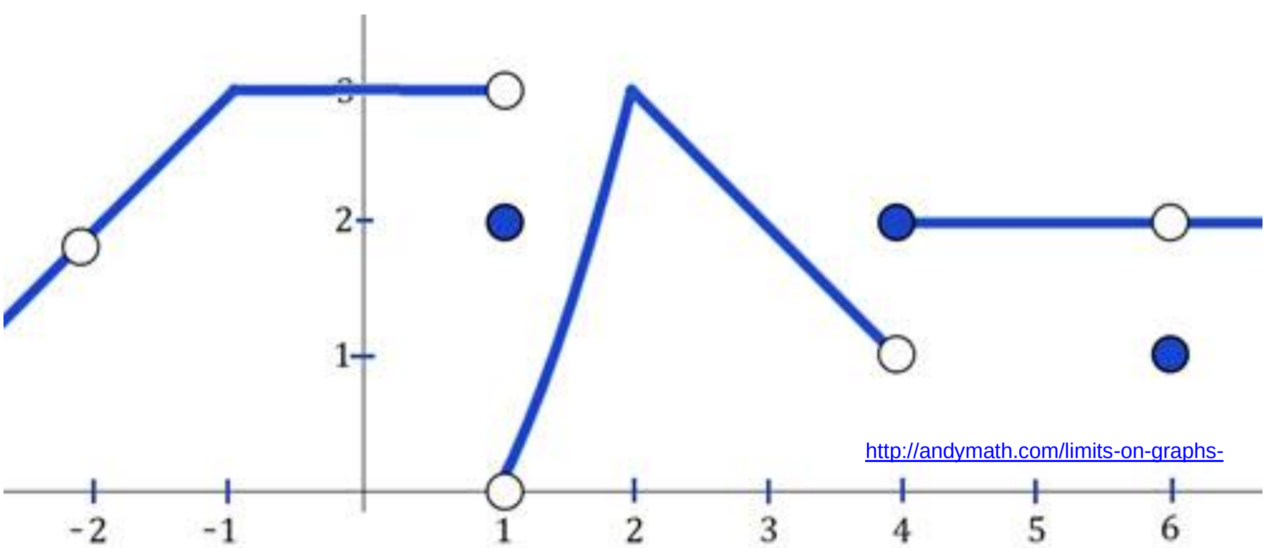
OVERVIEW

This unit will serve as an introduction in the study of Calculus, where the prerequisite knowledge in developing its first concepts will be described here. We start with the treatment of a certain type of a relation which we will call as a function. Using the concept of functions as a basis, we will describe next all of the possible methods to evaluate the limit of any function. The understanding of functions and limits will be very vital in the formulation of the concept of the derivative in the next unit.

Overall, the study of Calculus is attributed to two mathematicians, namely *Isaac Newton* and *Gottfried Wilhelm von Leibniz*, wherein they work independently of each other. Surprisingly enough, they discovered a single concept, but off course with different notations. All those possible notations will be tackled as we go along with our discussion of Calculus.

LEARNING OUTCOMES: After the completion of this unit, the students are expected to:

1. Determine whether a given relation is a function or not.
2. Evaluate the value of a given function.
3. Sketch the graphs of basic functions (linear, quadratic, piecewise function)
4. Define and discuss the concept of the limit of a function.
5. Familiarize and apply correctly each of the limit theorems to evaluate the limit of basic functions.
6. Evaluate the limit of a function based on its graph.
7. Describe the condition for a given limit to exist using the definition of one-sided limits.
8. Evaluate limits of trigonometric functions using certain limit theorems.
9. Differentiate the concept of evaluating infinite limits and limits at infinity.



http://andymath.com/limits-on-graphs-

1.1. FUNCTIONS AND THEIR GRAPHS

Most people think of the possible answers on the question "*Why study calculus, if it is not that useful in our everyday lives?*". While it is true that applications of calculus is not that evident and cannot be seen by our naked eyes, the study of calculus *can help develop our critical and analytical thinking skills*, which is a need for every engineer or science major. Also, the *understanding of concepts of calculus will serve as a very important component in the thorough study of higher mathematics, chemistry, physics and specialized subjects in any engineering fields*. Therefore, the study of calculus is very vital for an engineering student, a math or a science major.

To introduce ourselves with the study of this broad field of mathematics, we start with some introduction to the study of calculus. That is, our first concern must be on the type of expressions where all the definitions and concepts of calculus is based upon. By principle, the concepts of calculus are based on a certain group of expression or relation called a **function**.

For us to explore all the possible properties of a function to formulate its formal definition, let us have first a preliminary description of a function.

Function – can be thought of as a correspondence from a set X or real numbers x to a set Y of real numbers y, where the number y is unique for a specific value of x.

To illustrate and explain this preliminary description, suppose that we are given with a square having a side that measures x units. Recall that its area A can be computed using the formula:

$$A = x^2 . \qquad (1)$$

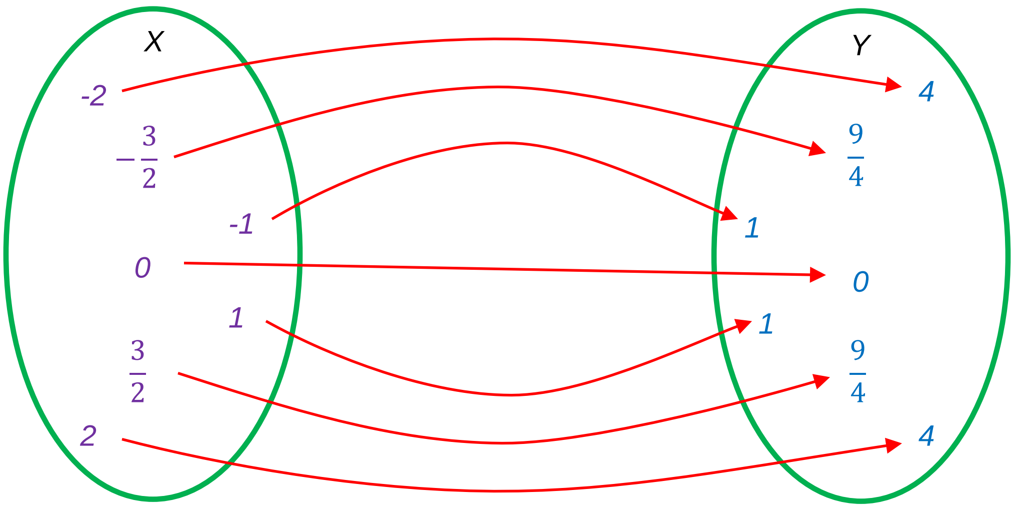
Given this formula, note that it is only possible to compute for the area A of the square if we are given with the measure of its side x. That is, for every value of x, there corresponds a **single value** of A. The description "single value" is synonymous to the word "unique value" meaning, there must be ONLY ONE value of A for a specific value of x. It is not possible given this expression that there will be more than one value of A for a value of x. This part of the concept falls under the 2^{nd} part of our preliminary description of a function, that is, *the number A is unique for a specific value of x*.

Also, note from the expression that the value of the area of the square A *depends* on the measure of its side x. From this reason, we can also call A as our **dependent variable**, and x as our **independent variable**. The terms are defined from the reason that the value of A is "dependent" on the value of x.

To fully describe the properties of functions, let's denote the dependent variable to be equal to y in general and the independent variable as x in (1). Doing this, we have:

$$y = x^2 \qquad (2)$$

Since we already replaced *A* by *y*, (2) is now a different expression, and we can now allow negative values for *x*. To explore some other properties of functions, we let set *X* be the possible values of *x* and set *Y* be the possible values of *y*. For simplicity, we give sample values of *x* and we find corresponding values of *y* using the equation $y = x^2$. Placing those values of *x* and *y* in a diagram and writing the elements of sets *X* and *Y* in roster method (enumerating the elements), we have:



$X = \{\ldots, -2, -3/2, -1, 0, 1, 3/2, 2, \ldots\}$ or $Y = \{0, 1, 9/4, 4, \ldots\}$ or

$X = (-\infty, +\infty)$ $Y = [0, +\infty)$

From the diagram, note that *every value of x gives us a single or unique value of y*. There are no instances that a value of *x* in the equation $y = x^2$ gives us two or more values of *y*. Again, this concept serves as a very important part of the definition of a function based on our previous discussion.

In our diagram, we set here the values of *x* to be equal to -2, -3/2, -1, 0, 1, 3/2 and 2 respectively. But note that these are just representative values of x, that is there are many other possible values of x given the equation $y = x^2$. In general, the possible values of *x* in $y = x^2$ that we can set can be *any positive or negative number*, without any restrictions. Any real number ($\mathbb{R}$) can be substituted to *x*. To summarize those mentioned possible values, we write it in *interval notation* as $(-\infty, +\infty)$ which means that we can substitute in *x* any number ranging from the lowest possible number ($-\infty$) up to the largest possible number ($+\infty$). We call the possible values of *x* of a function as the **domain** of that function.

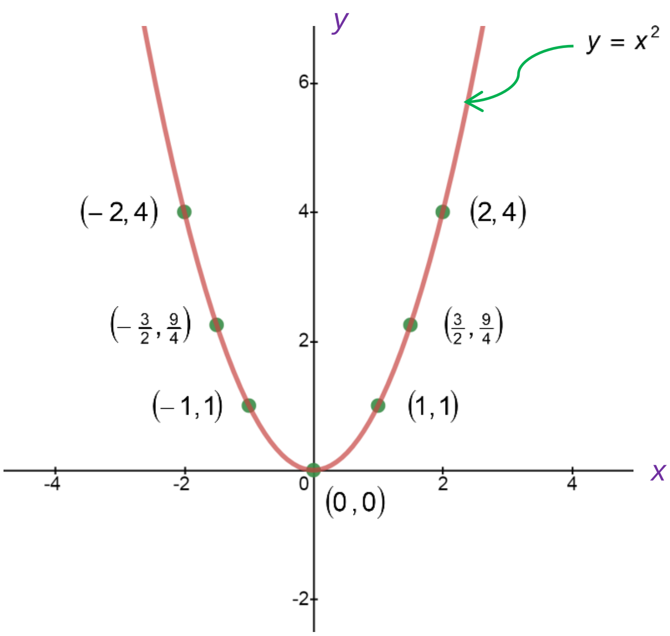
NOTE: By definition, **the symbol ∞ is not a symbol for a real number with a specified exact value**. We just use this symbol to denote or mean a very large positive number ($+\infty$) or a very large negative number ($-\infty$), so large that we cannot define how large that ∞ is. *Therefore in mathematics, when we say that the answer is undefined, it means that the answer is very large, either positive or negative*. Thus, when we divide any number (except 0) by 0, we describe the answer as **undefined** because it will give us a very large positive or negative number, depending on the sign of the numerator. That is, for example:

$$\frac{8}{0} = +\infty \qquad ; \qquad \frac{-8}{0} = -\infty$$

For the values of *x* that we set, we obtained the resulting values in the figure by using the expression $y = x^2$. Note that for every value of *x* that we set, *the resulting values of y is always either a positive number or 0*. This is so because any number (positive, negative or 0) raised to the second power will always yield a positive number or 0. To describe all these possible values of *y* in interval notation we write it as $[0, +\infty)$, which means that the values range from 0 up to the largest possible number ($+\infty$). The open bracket notation before 0 signifies that *0 is included in the possible values of y*. *If ever we have an instance that the lower or upper boundary must not be included, we must place an open or close parenthesis*. In terms of a formal term, we call all resulting values of *y* given the possible values of *x* as the **range** of that function. We will illustrate further on how to find the domain and range of a function in the examples that we will encounter later in this section, either if we are given with just the equation itself or given its graph.

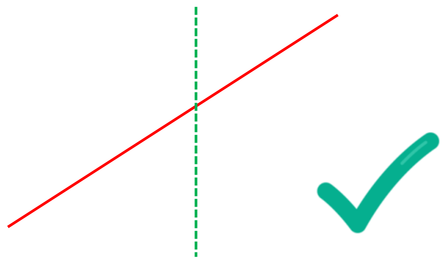
To further discover some other properties of functions; let us summarize the values of *x* and the corresponding values of *y* that we obtain, in tabular form which we call as the *table of values*. We will use the pairs of *x* and *y* values in the table of values to sketch the graph of the function $y = x^2$.

x	-2	$-\frac{3}{2}$	-1	0	1	$\frac{3}{2}$	2
y	4	$\frac{9}{4}$	1	0	1	$\frac{9}{4}$	4

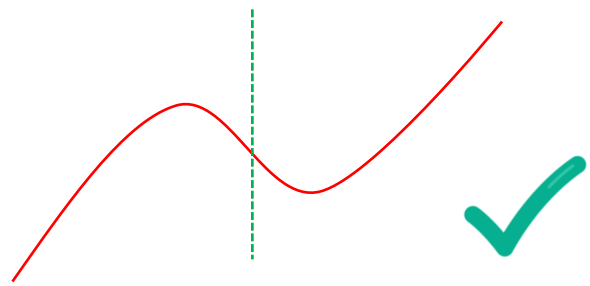


As stated in the concept of a function, *every value of x must give only one value of y*. This is evident in the ordered pairs formed in the table of values. That is, given (-2, 4), $\left(-\frac{3}{2}, \frac{9}{4}\right)$, (-1, 1), (0, 0), (1, 1), $\left(\frac{3}{2}, \frac{9}{4}\right)$, and (2, 4) notice that **no two distinct ordered pairs have the same first number**. The first number described corresponds to the *x*-coordinate of each ordered pair. The moment we are involved with two ordered pairs having the same *x*-coordinate, automatically those ordered pairs will not resemble a function.

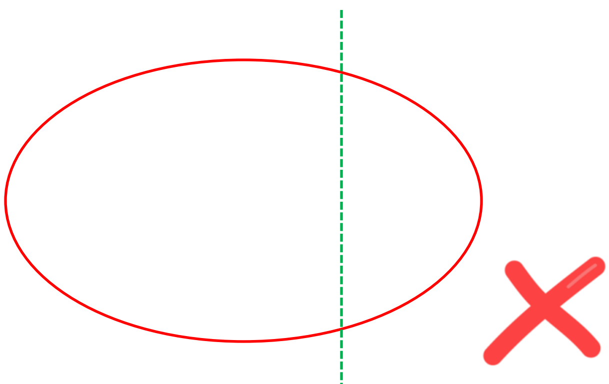
The consequence of this concept can be seen if these ordered pairs are plotted in the rectangular coordinate system to form the graph of the function $y = x^2$. Since no two *x*-coordinates are the same, then notice from the graph that **there are no two points in the graph that are vertically collinear with each other**. That is, if we are going to plot a vertical line anywhere in the graph, there will be no such line that will intersect two or more points in the graph. This concept, commonly known as the **vertical line test** determines if the graph of a certain equation resembles a function. Some illustrations of the vertical line test are shown below.



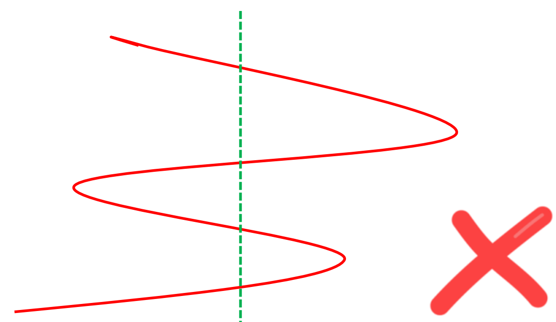
A straight-line graph is a function, since any vertical line plotted will give **only one** point of intersection. Since the equation of this is a *1st degree equation*, **all first degree equations are functions**.



Equations having graphs like this are functions, since again any vertical line plotted anywhere along its graph has only one point of intersection. In general, this graph is an example of the graph of a *polynomial function*, therefore **all polynomial equations are functions**. (Recall: polynomial equations must satisfy the definition of a polynomial from algebra)



A vertical line plotted in a part of the graph has two points of intersection. Therefore this graph does not resemble a function. In general, **all graphs of conic sections (except a parabola opening upward / downward) are not functions**.



The graph illustrated is not a graph of a function since a vertical line intersects the graph at more than one point.

Based on all of the concepts illustrated, we are now ready to state the formal definition of a function. Note that this formal definition is just a summary of all of the concepts that we talked about in this section.

Definition: **Function**

A **function** is a set of ordered pairs (x, y) in which no two distinct ordered pairs have the same first number. The set of all possible values of x is called the *domain* of the function, and the set of all resulting values of y is called the *range* of the function.

In the examples that will follow, we will focus on evaluating function values, graphing functions and determining their domain and range. Again, *for an expression or relation to be considered a function, only one value of y must correspond to each value of x and in terms of its graph, it must pass the vertical line test*. In terms of a formal notation, we can denote the dependent variable *y* in a function as

$$y = f(x)$$

where $f(x)$ is read as "*f* of *x*" or "function of *x*". That is, when we say that we have a function of x, it means that the function expression is in terms of the *x* variable only. The notation $f(x)$ is first used by the Swiss mathematician Leonhard Euler (1707 – 1783). Therefore using the function notation, we can write $y = x^2$ as

$$f(x) = x^2.$$

All throughout our study of calculus, we will acquaint ourselves to the function notation $f(x)$ and as long as the relation / expression can be considered a function, all *y* variables can be replaced by $f(x)$.

Example 1: Given $f(x) = x^2 + 3x - 4$, evaluate: (a) $f(0)$; (b) $f(2)$; (c) $f(2h)$; (d) $f(x+h)$; (e) $f(x) + f(h)$.

This example illustrates on how to find what we call as **function values**. Recalling the function notation $f(x)$, let us say we compare this with $f(0)$. It seems here that the value of *x* is 0. Therefore we can develop this procedure for evaluating function values in general: *we replace or substitute the number / expression inside the pair of parenthesis in the function notation to the variable x in the given expression for* $f(x)$*, then we simplify*.

Illustrating this procedure we have this solution, that is if $f(x) = x^2 + 3x - 4$:

a. $f(0) = 0^2 + 3(0) - 4 \quad \Rightarrow \quad f(0) = 0 + 0 - 4 \quad \Rightarrow \quad \boxed{f(0) = -4}$

b. $f(2) = 2^2 + 3(2) - 4 \quad \Rightarrow \quad f(2) = 4 + 6 - 4 \quad \Rightarrow \quad \boxed{f(2) = 6}$

c. $f(2h) = (2h)^2 + 3(2h) - 4 \quad \Rightarrow \quad \boxed{f(2h) = 4h^2 + 6h - 4}$

d. $f(x+h) = (x+h)^2 + 3(x+h) - 4 \quad \Rightarrow \quad \boxed{f(x+h) = x^2 + 2xh + h^2 + 3x + 3h - 4}$

e. Evaluate $f(x)$ and $f(h)$ first, then combine to obtain $f(x)+f(h)$

$$f(x)=x^2+3x-4 \qquad ; \qquad f(h)=h^2+3h-4$$

$$\therefore f(x)+f(h)=\left(x^2+3x-4\right)+\left(h^2+3h-4\right) \quad \Rightarrow \quad \boxed{f(x)+f(h)=x^2+3x+h^2+3h-8}$$

Note in items (c), (d) and (e) that the final expressions for function values are in terms of h and x, where in (d), we applied the concept of *square of a binomial* from algebra to simplify $(x+h)^2$. As a strong warning, WE MUST NOT DISTRIBUTE THE "2" EXPONENT to x and h, since it will violate concepts of algebra. The expansion of $(x+h)^2$ must have a middle term. Our solution must be in conformity with the existing or known concepts of algebra, trigonometry or geometry, whenever applicable.

Also, note that from our final answers for (d) and (e) that $f(x+h)\neq f(x)+f(h)$. There is no such concept that "*f*" can be distributed to x and h to form $f(x)+f(h)$. These warnings must be kept in mind to avoid possible errors in evaluating function values.

Example 2: Given (a) $f(x)=4x^2-5x+7$; (b) $f(x)=\sqrt{x+1}$, evaluate $\dfrac{f(x+h)-f(x)}{h}$; $h\neq 0$

This example is also an illustration of evaluating function values, where we are required to find an expression for $\dfrac{f(x+h)-f(x)}{h}$, given two different functions in (a) and (b). By concept, the value of h in the expression cannot be equal to 0, because division by zero is not defined. Note that this condition for the value of h is stated in the given. From now on, we must be critical about the given expressions that we will encounter, that is *we must take note if there are restricted values in our involved variables*.

As a strategy to evaluate $\dfrac{f(x+h)-f(x)}{h}$, we apply the technique that we employ in Example 1 (e), that is we find first expressions for $f(x+h)$ and $f(x)$. Then we substitute and simplify. Executing this in the two functions involved:

a. $f(x)=4x^2-5x+7$

$$f(x+h): \quad f(x+h)=4(x+h)^2-5(x+h)+7$$

$$=4\left(x^2+2xh+h^2\right)-5(x+h)+7$$

$$f(x+h)=4x^2+8xh+4h^2-5x-5h+7$$

$f(x)$: $f(x)=4x^2-5x+7$

Since $f(x+h)=4x^2+8xh+4h^2-5x-5h+7$ and $f(x)=4x^2-5x+7$;

$$\Rightarrow \frac{f(x+h)-f(x)}{h}=\frac{(4x^2+8xh+4h^2-5x-5h+7)-(4x^2-5x+7)}{h}$$

$$=\frac{\cancel{4x^2}+8xh+4h^2-\cancel{5x}-5h+\cancel{7}-\cancel{4x^2}+\cancel{5x}-\cancel{7}}{h}$$

$$=\frac{8xh+4h^2-5h}{h}$$; **factor out *h*** in numerator to cancel common factors in the numerator and denominator to simplify further

$$=\frac{\cancel{h}(8x+4h-5)}{\cancel{h}}$$

$$\boxed{\therefore \frac{f(x+h)-f(x)}{h}=8x+4h-5}$$

b. $f(x)=\sqrt{x+1}$

$f(x+h)$: $f(x+h)=\sqrt{(x+h)+1}$ $f(x)$: $f(x)=\sqrt{x+1}$

$f(x+h)=\sqrt{x+h+1}$

$$\Rightarrow \frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}$$

It seems that no further simplification process can be applied to our expression for $\frac{f(x+h)-f(x)}{h}$ because from algebra, radicals in our expression is not in the denominator. But in certain parts of our study of calculus, **it is more advisable to transfer the radicals in the denominator**. For us to do this, *we are to multiply both the numerator and denominator by the conjugate of the numerator*. Recall that the **conjugate** of a binomial of the form $a+b$ is $a-b$, and vice-versa. That is, **we just change the sign of the second term** to form the conjugate of a binomial.

Then in our expression so far, the conjugate of $\sqrt{x+h+1}-\sqrt{x+1}$ must be $\sqrt{x+h+1}+\sqrt{x+1}$, and we multiply this in the numerator and denominator. Executing this and simplifying:

$$\Rightarrow \frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h+1}-\sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1}+\sqrt{x+1}}{\sqrt{x+h+1}+\sqrt{x+1}}$$

$$= \frac{(x+h+1)-(x+1)}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)} = \frac{\cancel{x}+h+\cancel{1}-\cancel{x}-\cancel{1}}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}.$$

$$= \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}$$

$$\boxed{\therefore \frac{f(x+h)-f(x)}{h} = \frac{1}{\sqrt{x+h+1}+\sqrt{x+1}}}$$

The manner on how we evaluate the expression $\frac{f(x+h)-f(x)}{h}$ where $h \neq 0$ must be familiarized because we will commonly encounter this expression, especially in finding what we call as the *derivative* of a function. The topic regarding derivatives will be covered in the next unit. In general, if we encounter radicals in the numerator, **we must ALWAYS apply the procedure of multiplying the numerator and denominator by the conjugate of the numerator** and simplifying the resulting expression.

Test Your Understanding: Evaluate the following function values:

1. Given $f(x) = 2x^2 + 5x - 3$, evaluate: (a) $f(-2)$; (b) $f(-1)$; (c) $f(0)$; (d) $f(3)$; (e) $f(h+1)$; (f) $f(2x^2)$; (g) $f(x^2-3)$; (h) $f(x+h)$; (i) $f(x)+f(h)$; (j) $\frac{f(x+h)-f(x)}{h}$; $h \neq 0$

2. Given $F(x) = \sqrt{x+9}$, evaluate: (a) $F(x+9)$; (b) $F(x^2-9)$; (c) $F(x^4-9)$; (d) $F(x^2+6x)$; (e) $F(x^4-6x^2)$; (f) $\frac{F(x+h)-F(x)}{h}$; $h \neq 0$

The next set of examples will illustrate on how to find the domain and range of some functions, without sketching their graphs.

Example 3: Find the domain and range of the following functions: (a) $f(x) = x+1$; (b) $f(x) = x^2+2$; (c) $f(x) = \sqrt{x-3}$; (d) $f(x) = \sqrt{x^2-9}$

Note that we are to determine the domain and range given with these model functions. Recall that *the domain corresponds to the possible values of x* while *the range corresponds to the corresponding values of y, given each value of x.* For each function, we will think first of the possible values of *x*, that is *if there are any value of x that are not allowed in the function*. Often times, these restrictions for the value of *x* can occur if we are given with a *fraction* (denominator cannot be zero) or a *radical expression* (negative radicands are not allowed for even indices). After determining those possible values of *x*, we substitute some of them in the function to determine all of the possible values of *y* or $f(x)$.

a. Given $f(x) = x + 1$, we first think if there are any restrictions for the possible values of *x*. Since we are just given with a *linear function*, all possible values of *x* are allowed starting from the smallest possible number ($-\infty$) up to the largest possible number ($+\infty$). Therefore, **the domain of $f(x)$ consists of the set of real numbers**.

   After determining the domain, we substitute those possible values of x from $-\infty$ to $+\infty$ to determine the range. Note that the corresponding values of $f(x)$ upon substitution ranges from the smallest possible number ($-\infty$) up to the largest possible number ($+\infty$). Therefore, **the range of $f(x)$ also consists of the set of real numbers**. Writing these results in interval notation and denoting the domain as set D and the range as set R, we have:

   D: $(-\infty, +\infty)$ or $\mathbb{R}$; R: $(-\infty, +\infty)$ or $\mathbb{R}$

NOTE: *Given any polynomial function, all values of x are possible to be substituted with no restrictions; therefore its domain in general is always at the interval* $(-\infty, +\infty)$*.* The previous example of a linear function is an example of a polynomial function. As an illustration, the domain of the functions $f(x) = x^2 - 5x + 8$, $f(x) = 4x^4 - 5x^3 - x^2 + 1$ and $f(x) = -x^6 + 7x^2 - 9$ are all $(-\infty, +\infty)$ since they are all polynomial functions having *non-negative and non-fractional exponents*. The range of these functions is in general not $(-\infty, +\infty)$, as will be illustrated in (b).

b. For the function $f(x) = x^2 + 2$, note that it is a polynomial function because the exponents are not negative or a fraction. Then **its domain is automatically $(-\infty, +\infty)$ or the set of all real numbers**.

   To think of all the corresponding values of $f(x)$, notice that the variable *x* is raised to the second power. Recall that *any number raised to an even exponent will always give us a positive number or zero (if the base is 0)*. Therefore we have an assurance that the term x^2 will always give us a positive (non-negative) number or zero. Zero so far is the lowest possible answer for x^2. Then considering the second term which is 2, the lowest possible answer for $x^2 + 2$ will give us 2 ($0 + 2 = 2$). We conclude therefore that **the range of $f(x)$ starts from 2 (where 2 is included) up to the largest possible number ($+\infty$)**. In interval notation:

D: $(-\infty, +\infty)$ or $\mathbb{R}$	;	R: $[2, +\infty)$

Note that since 2 is a possible value (included in the range), we place an *open bracket* before it. By notation, *an open or close parenthesis must be placed before or after $-\infty$ or $+\infty$ since it is not a symbol for a real number*.

c. For the possible values of *x* of the function $f(x)=\sqrt{x-3}$, *we must see to it that the radicand x – 3 is not negative* (square root of a negative number will yield an imaginary number). From this, note that the lowest possible allowable value of *x* is 3. Any value of *x* less than 3 will make the radical imaginary. Therefore, **the domain of $f(x)$ starts from 3 up to the largest possible number ($+\infty$)**.

Given that 3 is the lowest possible value of *x*, note that substituting 3 to $f(x)=\sqrt{x-3}$ will yield an answer of 0 for $f(x)$. Therefore **we can consider 0 as the lowest possible value of $f(x)$, up to the largest possible number ($+\infty$)**. To write the domain and range in interval notation:

D: $[3, +\infty)$	;	R: $[0, +\infty)$

NOTE: By definition, finding the square root of a number must result to a positive and a negative root. But **considering both square roots will violate the definition of a function**. For example, substituting x = 4 to $f(x)=\sqrt{x-3}$ will give us $f(x)=y=\pm 1$. Forming the corresponding ordered pairs, we have (4, 1) and (4, -1). But recall that for an expression to be a function, *there must be no two ordered pairs that has the same first number* or *x*-coordinates. For this reason, *we just take the answer for the square root of a number to be equal to its positive root or zero* (this is called the **principal square root**). This explains the justification on the range of $f(x)=\sqrt{x-3}$ that starts from 0 up to a very large positive number.

d. Same as (c), we must see to it that given $f(x)=\sqrt{x^2-9}$, *the answer for the radical must not be imaginary*. Setting this condition for the possible value of *x*, note that *the value of x^2 must not be less than 9*. From this, one possible group for the value of *x* is that *x* must be greater than or equal to 3 and in interval form, we have $[3, +\infty)$. But taking note again that x^2 always yields a positive number, negative values of *x* can also be considered as long as the answer for x^2 is not less than 9. Therefore, an another possible group for allowable values of *x* is that *x* can also be less than or equal to -3, that is $(-\infty,-3]$. For us to combine the two groups of possible values of x, we place a union symbol ($\cup$) in between the two interval forms. Therefore**, the domain of the function has the form** $(-\infty,-3]\cup[3,+\infty)$.

For the resulting values of y or $f(x)$, substituting any value of x along $(-\infty,-3]\cup[3,+\infty)$ will result to zero or any positive number (principal square root). Therefore, **the range of the function is** $[0,+\infty)$**.**

$$\boxed{\text{D: } (-\infty,-3]\cup[3,+\infty); \qquad \text{R: } [0,+\infty)}$$

The next set of examples will illustrate the process of graphing some basic functions and using the graph to determine their domain and range.

Example 4: Sketch the graph of the function $f(x)=\dfrac{2x^2+7x+3}{x+3}$ and determine its domain and range.

Before we construct the graph of any function, **we must first determine if there are any restrictions for the value of x in the given expression for f(x)**. Since we have a presence of a denominator in the given function, *we must see to it that it must not be equal to zero*. Note that if x = -3, the expression for the denominator will be zero. Then automatically, *-3 will be our restricted value of x*, we discard it automatically to the domain of $f(x)$, and we write:

$$f(x)=\frac{2x^2+7x+3}{x+3} \;;\; x\neq -3$$

To continue in thinking on how to graph the given function, *notice that the numerator is factorable*. Upon factoring*, a common factor in the numerator and denominator can be cancelled out*. Executing the simplification process:

$$\Rightarrow f(x)=\frac{2x^2+7x+3}{x+3} \;;\; x\neq -3$$

$$=\frac{\cancel{(x+3)}(2x+1)}{\cancel{x+3}}$$

$$\therefore f(x)=2x+1 \quad ; \quad x\neq -3$$

The condition that $x \neq -3$ must STILL be applied to our simplified expression for $f(x)$ which is $f(x) = 2x + 1$ since *this function just came from the original function having* $x \neq -3$ *as a preliminary condition*.

Notice that our simplified expression is a *linear function*, where the technique of graphing is discussed from junior or senior high school algebra. By concept, *the graph of a linear function must be a **straight line***. In particular, we will apply the method of transforming the equation to its **slope-intercept form**. Recalling that $f(x)$ can be replaced by *y* and the slope-intercept form is given by $y = mx + b$ (*m* = slope of the line and *b* = y-intercept):

$$\Rightarrow f(x) = 2x + 1 \qquad ; \qquad x \neq -3$$

$$y = 2x + 1 \qquad ; \qquad m = 2 \text{ and } b = 1$$

Since b = 1, we obtained one point in the graph of $f(x)$, which is (0, 1). Since the graph is just a straight line, *we just need to find an another point in the graph and just connect them and extend to form its graph*.

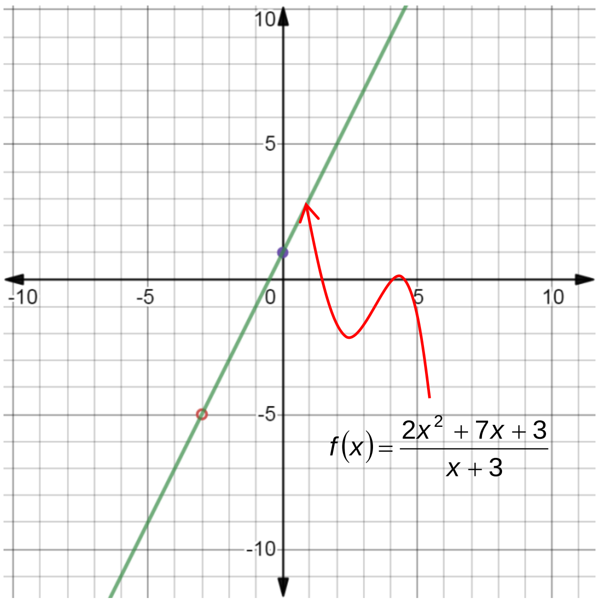
Now, recall that *x* cannot take the value of -3. We must also take this into consideration in the graph of $f(x)$, that is, we must also find the restricted value of y by substitution to $f(x) = 2x + 1$:

*if $x \neq -3$; $y \neq 2(-3) + 1 \quad \Rightarrow \quad y \neq -6 + 1$

$y \neq -5$; *restricted point* is at (-3, -5)

From the word "restricted", *the point (-3, -5) must not be a part of the graph* of the original function $f(x) = \dfrac{2x^2 + 7x + 3}{x + 3}$. To emphasize this, we plot this point as a **hollow point (hole)** in the graph.

Now, we have two points, the point (0, 1) obtained from the *y*-intercept *b* = 1, and the restricted point (-3, -5). Plotting these points on the Cartesian plane noting that (-3, -5) must be plotted as a hole and joining these points to form a line, we obtain the graph in the next page.



For the *domain and range* of the function, recall that *x* cannot take a value of -3 and this is the single value of *x* that serves as its restricted value. Then in this case, **the domain of the function can be written as the combination of the intervals** $(-\infty,-3)$ **and** $(-3,+\infty)$ **or** $(-\infty,-3)\cup(-3,+\infty)$.

Upon substituting -3 to $f(x)$, recall that we obtained -5 as the restricted value of y. Therefore, **the range of the function can be written as the combination of the intervals** $(-\infty,-5)$ **and** $(-5,+\infty)$ **or** $(-\infty,-5)\cup(-5,+\infty)$.

D: $(-\infty,-3)\cup(-3,+\infty)$ or $\mathbb{R}$ except -3 ; R: $(-\infty,-5)\cup(-5,+\infty)$ or $\mathbb{R}$ except -5

Example 5: Sketch the graph of the function $f(x)=\begin{cases} 9-x^2 & if\ x\neq -3 \\ 4 & if\ x=-3 \end{cases}$ and determine its domain and range.

The function that we are going to graph is an example of a **piecewise function**. From the root word "piece", *a piecewise function is a type of a function that is subdivided into two or more parts*. In the example, the function is composed of two parts such that $f(x)$ is $9 - x^2$ for values of $x\neq -3$, and $f(x)$ is equal to 4 if $x=-3$.

To sketch the graph of a piecewise function, we will graph these two parts in a single Cartesian plane, taking note of the values of *x* that each part represents.

Starting with $f(x)=9-x^2$ at values of $x\neq -3$, note that the restricted value for *x* is already given, which is -3. Later, we can find the corresponding restricted value for *y* by substituting -3 to *x* in $f(x)=9-x^2$. Recalling that the graph of $f(x)$ here is a ***parabola*** ($f(x)$ is a second-degree equation), *we transform first the equation to its standard form for us to find the coordinates of its vertex* upon replacing $f(x)$ by *y*. Doing this:

(i) for $f(x)=9-x^2$; $x\neq -3$

$y=9-x^2$; Recall (standard form of a parabola): $(x-h)^2=4a(y-k)$

$x^2=9-y \quad\Rightarrow\quad x^2=-y+9 \quad\Rightarrow\quad x^2=-(y-9)$

Note that in transforming the equation of a parabola to its standard form, *all terms with x variables must be transferred to the left side*, and we *apply the concept of completing the square* (if possible). Then *all terms with y variables and constant terms must be at the right side, and we always factor out the coefficient of the y variable*. The parabola obtained here *must either open upward or downward*, depending on the sign of the coefficient of *y* – *k*. Parabolas opening to the left or right will not be encountered here for it violates the vertical line test for a function.

Therefore, from our obtained standard form, the *parabola must open downward* (coefficient of *y* – 9 is negative). Upon comparison of $x^2 = -(y-9)$ with the standard form $(x-h)^2 = 4a(y-k)$, the value of *h* is 0 and the value of *k* is 9. Then *the vertex of the involved parabola must be located at (0, 9)*.

For the restricted point ($x \neq -3$):

*if $x \neq -3$; $y \neq 9-(-3)^2 \Rightarrow y \neq 9-9$

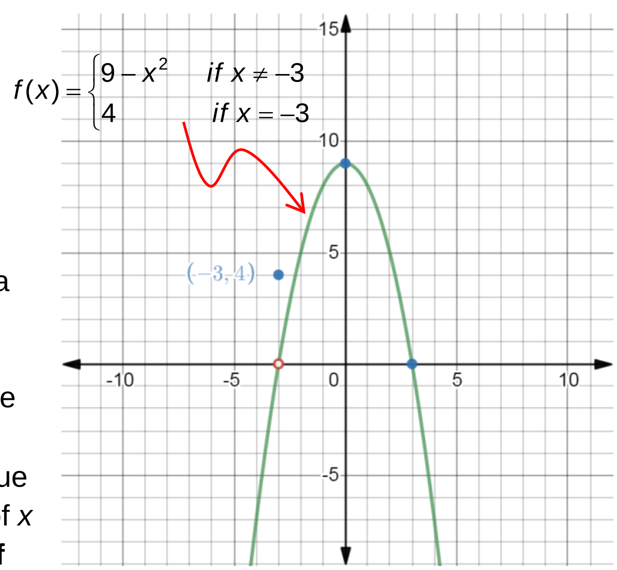
$y \neq 0$; *restricted point* is at (-3, 0)

Since (-3, 0) is a restricted point, *it must be plotted as a hole* since it must not be included in the graph of $f(x)$.We can also use this point as a guide point to sketch the graph of the parabola involved, opening downward having a vertex of (0, 9). To sketch the right branch of the parabola, we plot the "mirror image" of (-3, 0) which is (3, 0) and we connect the dots to draw the parabola as shown below. We are now through with the first part of the graph of the piecewise function.

For the second part, it states that $f(x)=4$ if $x=-3$. Replacing $f(x)$ by *y*, it turns out that if $x=-3$, the value of *y* is 4. The graph therefore of the second part just *corresponds to a **point***, located at (-3, 4). Writing the formal solution and plotting the said point:

(ii) for $f(x)=4$; $x=-3$

$y=4$; point: (-3, 4)



Since we are now through with the graph, let us next note of its domain and range.

The first part of the piecewise function is a 2nd-degree polynomial function, therefore its domain must be the set of all real numbers except -3 because it is applicable only for values where $x \neq -3$. But the second part defines the function if the value of *x* is 3. Combining the possible values of *x* in the two parts, **the domain therefore of**

the piecewise function is composed of the set of all possible numbers or $(-\infty, +\infty)$

For the range of the piecewise function, we just combine the range of its 1st and 2nd parts. Given $f(x) = 9 - x^2$, its range is at $(-\infty, 9]$ since x^2 will always yield a positive number and in most instances, the difference between 9 and x^2 is negative. The highest possible value of $9 - x^2$ is 9 and it occurs if $x = 0$. For the second part, automatically $f(x) = y = 4$. Since this value of 4 already lies at the interval $(-\infty, 9]$, we conclude that **the range of the piecewise function is** $(-\infty, 9]$.

D: $(-\infty, +\infty)$ or $\mathbb{R}$; R: $(-\infty, 9]$

As an alternative, we can also directly determine from a mere look of the graph that we constructed the expressions for the domain and range of the function.

Example 6: Sketch the graph of the function $f(x) = \begin{cases} x - 1 & \text{if } x < 3 \\ 5 & \text{if } x = 3 \\ 2x + 1 & \text{if } x > 3 \end{cases}$ and determine its domain and range.

The piecewise function involved *consists of three parts*. Just like our previous example, we consider and graph each part one by one. For each portion, we must take note and understand very carefully on what values of *x* each portion will be applicable.

The first part reads $f(x) = x - 1$ that is applicable for values of *x* that are less than 3. Understanding very carefully the meaning of $x < 3$, it turns out that the graph is only applicable for values of *x* that is less than 3. *The graph of* $f(x) = x - 1$ *then is only applicable for the* ***left side*** *of x = 3 in the Cartesian plane*, where 3 is NOT included. The value *x* = 3 can be considered here as the restricted value of *x*. *To determine on what point the graph will end or terminate, we must find also the corresponding restricted value of y, and plot the point obtained as a hole*. Noting that the graph of $f(x) = x - 1$ is a ***straight line***, we find first its slope and y-intercept:

(i) for $f(x) = x - 1$; $x < 3$

$y = x - 1$; $m = 1$ and $b = -1$

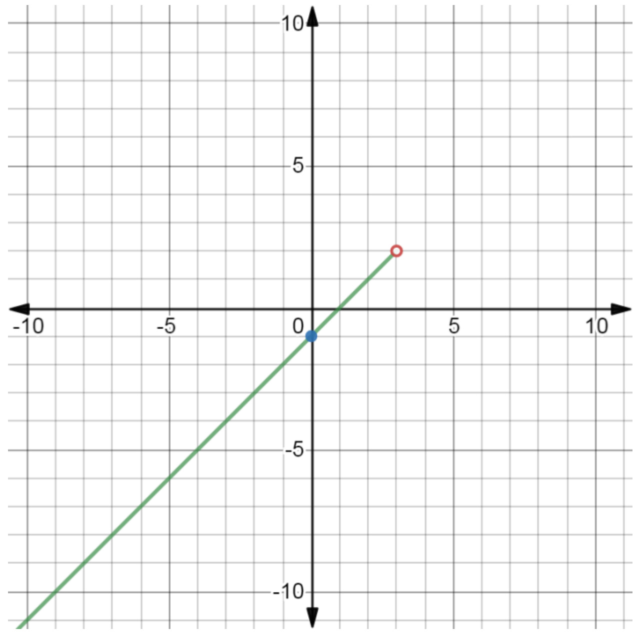
Again, for us to sketch a line, *we need to determine any two points* that passes through the graph of $f(x) = x - 1$. To see if we can use the *y*-intercept, we note that the value of *x* there is 0. Since 0 < 3, then the y-intercept satisfies the condition x < 3 and we can use (0, -1) as our first point.

For the second point, we find the point where the first part of the graph ends, by using the restricted value of *x*. Solving for the restricted value of y:

*if $x \neq 3$; $y \neq 3 - 1$

$y \neq 2$;

restricted point is at (3, 2)



The restricted point must be plotted as a hollow point or a hole. We note that since the part of the graph is only applicable for values of *x* that are less than 3, the graph must be at the right side of *x* = 3 and *the point (3, 2) will serve as the terminating point*. This portion of the graph is shown in the right.

Note that we are not yet through with the graph of the piecewise function itself. We still have two portions of the piecewise function that we will plot on the same coordinates. To proceed with the second portion, we have $f(x) = 5$ applicable if $x = 3$. Replacing $f(x)$ by y, it turns out that $y = 5$ if $x = 3$. Therefore *the second portion corresponds to a **point**, located at (3, 5)*. In terms of a formal solution:

(ii) for $f(x) = 5$; $x = 3$

$y = 5$; point: (3, 5)

For the third part of the function, we are given that $f(x) = 2x + 1$ is applicable for values of *x* that are greater than 3. If x > 3, then it means that *the graph of $f(x)$ here must be located at the **right side** of x = 3*. Unlike the first part where the restricted point is our terminating point, here *the restricted point will serve as on where the graph will start*. The same, the restricted point is found by substituting *x* = 3 to $f(x) = 2x + 1$.

To help us sketch the graph of the third part, we note that the function is ***linear*** and *we need to determine two points in its graph.* Transforming to slope-intercept form:

(iii) for $f(x) = 2x + 1$; $x > 3$

$y = 2x + 1$; *m* = 2 and *b* = 1

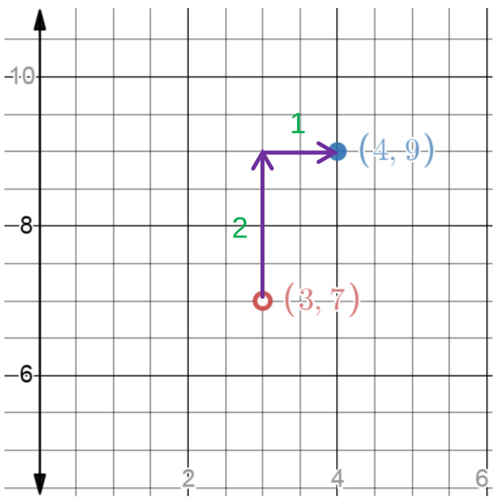
In the first part of the piecewise function, we recall that the *y*-intercept is included in the graph since it satisfies the condition *x* > 3. For this third part, note that it is only applicable for values of *x* that is greater than 3. The *y*-intercept has an *x*-coordinate of 0 but it is not greater than 3. Then in this case, *we cannot use the y-intercept as our first point because it is not a part of the graph*.

Let us determine first the restricted point which will serve as our first point. Substituting $x = 3$ to $f(x) = 2x + 1$:

*if $x \neq 3$; $\quad y \neq 2(3)+1 \quad \Rightarrow \quad y \neq 6+1$

$y \neq 7$; *restricted point* is at (3, 7)

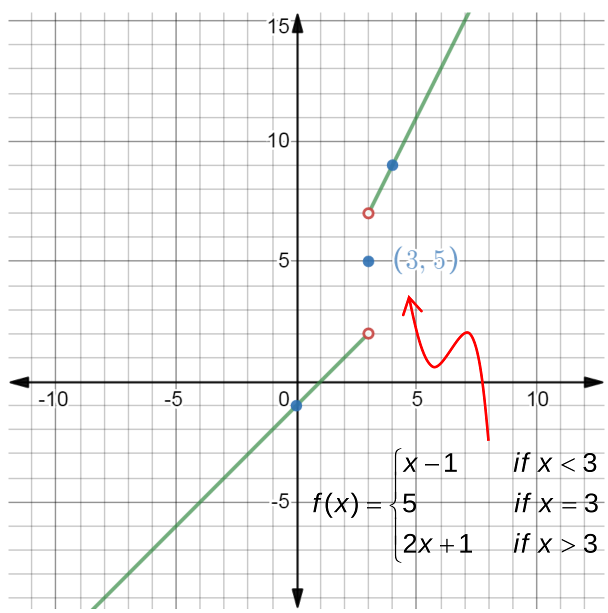
The restricted point must be plotted as a hole. Since we cannot use the *y*-intercept to plot a second point, *we can use the value of slope* (*m*) that we obtained. Recalling that slope is the ratio of rise and run, we apply it using the restricted point as a reference. With our value of *m* = 2, we can write it to be equal to the ratio of 2 and 1, that is:



$$m = 2 = \frac{2}{1} = \frac{rise}{run}$$

From this, the value of rise is 2 and the value of run is 1. To obtain our second point, *we must go 2 steps up (rise) and 1 step to the right (run) from the restricted point (3, 7)* as illustrated. Doing this, we obtained an another point (4, 9) where this point is included in the graph because its *x*-coordinate of 4 is greater than 3. In instances that the point formed is not included in the graph, we can also write 2 as -2 / -1, where we must go 2 steps down and 1 step to the left.

We can now connect the two points obtained noting that (3, 7) is our starting point (but not included in the graph) and extend to form the graph of the third part of the piecewise function. After plotting all the three portions of the given piecewise function, we obtain this graph as shown.



After completing the graph of the graph of the function, we next determine its domain and range. Note that the function is defined for all cases of values of *x*, starting with *x* < 3, *x* = 3 and *x* > 3. Therefore, **the domain of $f(x)$ is composed of the set of real numbers or $(-\infty, +\infty)$.**

For the range, we will just *combine the range of the three parts of the piecewise function*. Recall in the first part that if x = 3, y = 2 for $f(x) = x - 1$. *The value of y = 2 will serve as an upper boundary for the range of the first part*, taking note that 2 is NOT included because it is a restricted value. Then the range for this part is $(-\infty, 2)$. For the second part, the only value of y here is 5. Since it is not included in our previous range of $(-\infty, 2)$, we must consider the value of 5 separately, by writing it as $[5, 5]$. This notation of $[5, 5]$ means that *only the number 5 is considered*. Lastly for the 3rd part, recall that *the graph starts at the point (3, 7), where 7 here is not included and then proceeding up to the largest possible number (*$+\infty$*)*. Therefore, the range for this part is $(7, +\infty)$.
Combining the range from the three portions, we have $(-\infty, 2) \cup [5, 5] \cup (7, +\infty)$**.**

D: $(-\infty, +\infty)$ or $\mathbb{R}$; R: $(-\infty, 2) \cup [5, 5] \cup (7, +\infty)$

Example 7: Sketch the graph of the function $f(x) = \begin{cases} -4 & if\ x < -2 \\ -1 & if\ -2 \le x \le 2 \\ 3 & if\ 2 < x \end{cases}$ and determine its domain and range.

This is an another example of a piecewise function. Given three portions of the graph of $f(x)$, we plot them one by one on a single coordinate system.

Starting with the first part, we have $f(x) = -4$. Since it is applicable for $x < -2$, *the function is applicable to the left side of x = -2*. Replacing $f(x)$ by y, we obtain $y = -4$ where its graph is a **horizontal line** *extending from the smallest possible number (*$-\infty$*) up to -2* (but -2 is NOT included). The point (-2, -4) serves as a restricted point. In terms of a formal solution:

(i) for $f(x) = -4$; $x < -2$

$y = -4$; restricted point: (-2, -4)

The graph of the second part is *still a horizontal line* since the form of the function is the same. The difference is that the line involved is applicable for $-2 \le x \le 2$ which means that the horizontal line defined by $f(x) = -1$ is only applicable between x = -2 and x = 2 where both -2 and 2 are included (no restricted point).

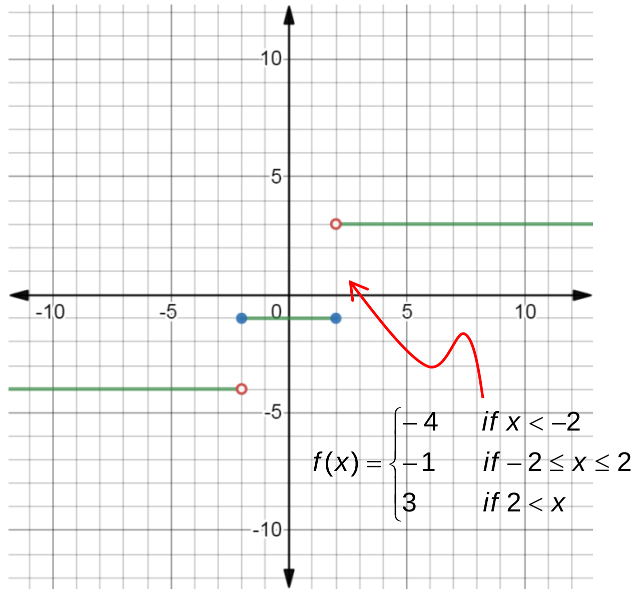
(ii) for $f(x) = -1$; $-2 \le x \le 2$

$y = -1$

For the third portion, still the graph of $f(x)=3$ is a *horizontal line*, where it is applicable *starting with x = 2 (NOT included) up to the largest possible number* $(+\infty)$. The point (2, 3) serves as a restriction point and also, a starting point.

(i) for $f(x)=3$; $2<x$ or $x>2$

$y=3$; restricted point: (2, 3)



The graph of the function is shown on the right. Since the function is defined for all values of *x*, **its domain is the set of real numbers or** $(-\infty,+\infty)$**.**

The range of this function is just the *right sides of* $f(x)$ since these are the only values of *y* where the graph has corresponding points. We can just emphasize and enumerate them by enclosing them in a pair of braces. Therefore **the range of** $f(x)$ **is given by** $\{-4,-1,3\}$**.**

$$\boxed{\text{D: } (-\infty,+\infty) \text{ or } \mathbb{R} \quad ; \quad \text{R: } \{-4,-1,3\} \text{ or } [-4,-4]\cup[-1,-1]\cup[3,3]}$$

Test Your Understanding: Sketch the graphs of the following functions and determine their domain and range.

1. $f(x)=\dfrac{x^2-25}{x+5}$

2. $f(x)=\dfrac{x^2-4x+3}{x-1}$

3. $f(x)=\begin{cases}-2 & \text{if } x\le 3\\ 2 & \text{if } x>3\end{cases}$

4. $f(x)=\begin{cases}2x-1 & \text{if } x\ne 2\\ 0 & \text{if } x=2\end{cases}$

5. $f(x)=\begin{cases}x^2-4 & \text{if } x\ne 3\\ -2 & \text{if } x=3\end{cases}$

6. $f(x)=\begin{cases}1-x^2 & \text{if } x<0\\ 3x+1 & \text{if } x\ge 0\end{cases}$