Theory Lecture 5

Sorting Part 2

Learning Objectives

- Study some more sorting algorithms
- Understand the design principles behind them
- Look at the application of sorting to improving other algorithms

Heap Sort

Selection sort

Selection sort

```
SelectionSort(l : list)

for k=0…l.length-1
   i=IndexOfMin(l(k…end))
   swap(l(i),l(j))
end for
```

Recall selection sort from lecture 3.

Our analysis showed that its complexity was $O(n^2)$, poor for a sorting algorithm

Selection sort

Selection sort

```
SelectionSort(l : list)

for k=0…l.length-1
   i=IndexOfMin(l(k…end))
   swap(l(i),l(j))
end for
```

Heap Data Structure

Organization

Binary Tree

Common operations

`Insert(v)`	Insert element v
`ExtractMin()`	Remove and return the minimum element

Recall selection sort from lecture 3.

Our analysis showed that its complexity was $O(n^2)$, poor for a sorting algorithm.

In lecture 4, we looked at the heap data structure.

These two operations were $O(\log n)$

Note we can execute ‘**`IndexOfMin`**’ for a heap in $O(\log n)$

Heap sort

Heap sort

```
HeapSort(l : list)

h = new heap

for k=0…l.length-1
   h.Insert(l(k))
end for
for k=0…l.length-1
   l(k)=ExtractMin()
end for
```

HeapSort is an implementation of SelectionSort, but using a heap to get the smallest element.

`Insert(.)` in the first loop takes $O(\log n)$, hence this loop is $O(n \log n)$

`ExtractMin()` in the second loop takes $O(\log n)$ and so the second loop is $O(n \log n)$

Overall complexity of HeapSort is $O(n \log n)$

Just choosing the right data structure makes SelectionSort competitive

QuickSort

Partitioning

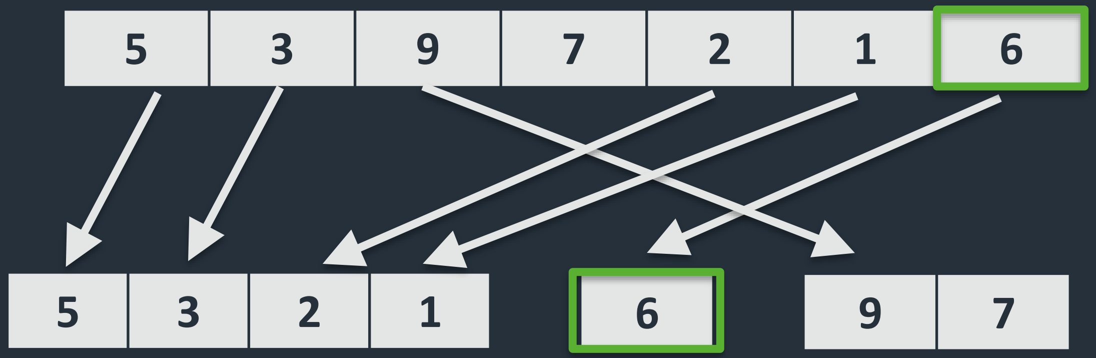
QuickSort is an example of the divide-and-conquer algorithm strategy

Idea:

Choose a pivot

Anything less than or equal to the pivot goes in a lower list

Anything greater than the pivot goes in an upper list

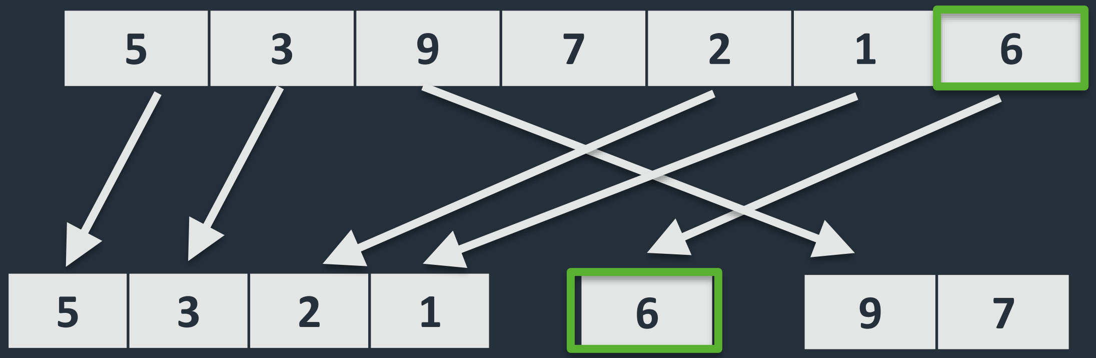


Recursion

Now the pivot (6) is in the correct place.

Sort the lower and upper lists individually.

If we chose the ideal pivot, these would be half the size (c.f. MergeSort)



QuickSort

QuickSort

```
QuickSort(s : list, l : int, u : int)
// l,u indicate which part of the list to
// sort. Initially l=0, u=n-1
if l<u
   p = partition(s,l,u)
   QuickSort(s,l,p-1)
   QuickSort(s,p+1,u)
end if
```

`partition` does the main work. It chooses a pivot and divides the list into the two parts.

We could implement this by creating two new lists at each partition (lower and upper).

But with good choice of pivot and using minimum swaps, we can work in the original list

Partitioning in place

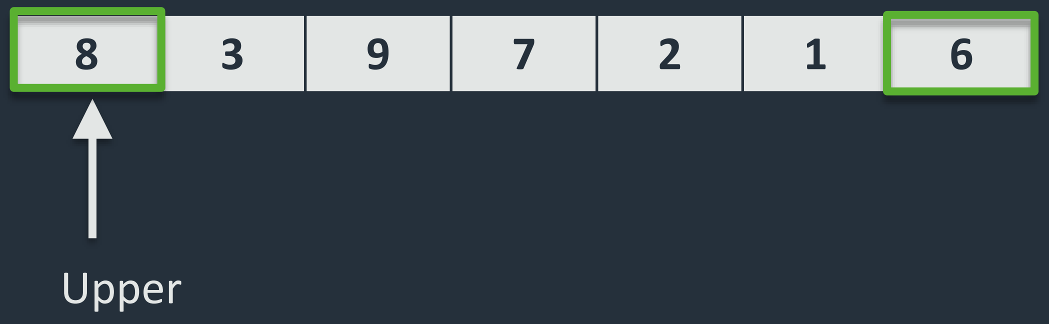
To partition in place, we choose the last item as the pivot.

We put the lower elements at the start.

A index marks the start of the upper list

Compare each element with the pivot

If lower, swap with first upper element, then move up upper index



8≥6, so no swap

Partitioning in place

To partition in place, we choose the last item as the pivot.

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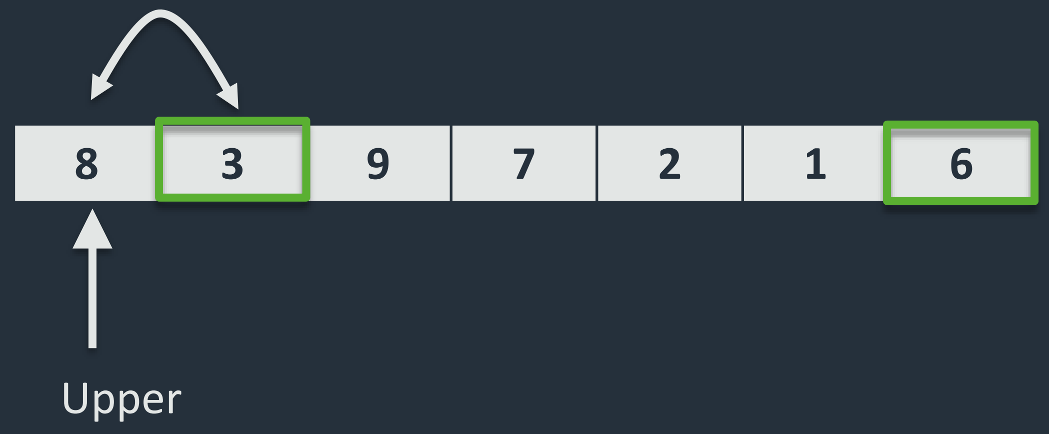


3<6, so swap and move up

Partitioning in place

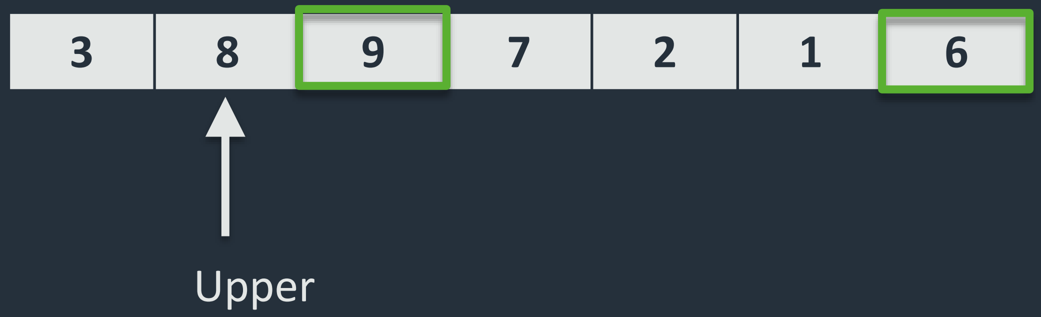
To partition in place, we choose the last item as the pivot.

We put the lower elements at the start.

A index marks the start of the upper list

Compare each element with the pivot

If lower, swap with first upper element, then move up upper index



9≥6, so no swap

Partitioning in place

To partition in place, we choose the last item as the pivot.

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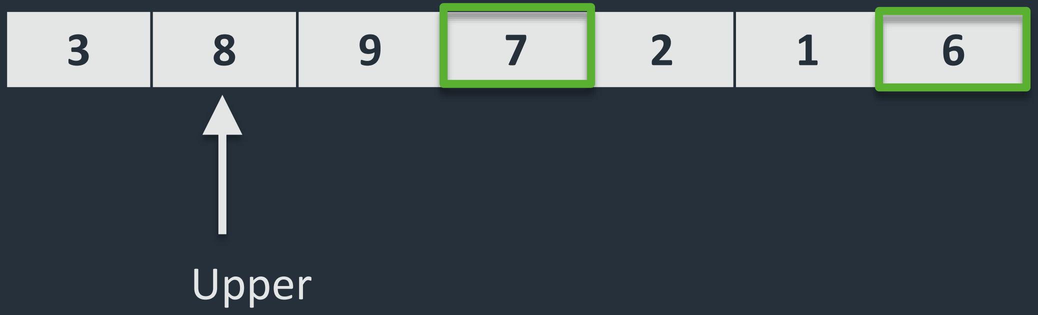
Compare each element with the pivot

If lower, swap with first upper element, then move up upper index



7≥6, so no swap

Partitioning in place

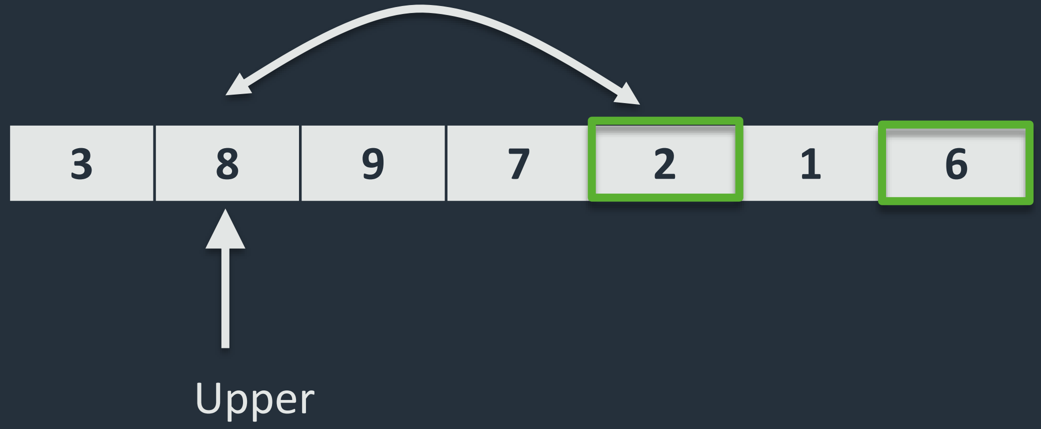
To partition in place, we choose the last item as the pivot.

We put the lower elements at the start.

A index marks the start of the upper list

Compare each element with the pivot

If lower, swap with first upper element, then move up upper index



2≥6, so swap and move up

Partitioning in place

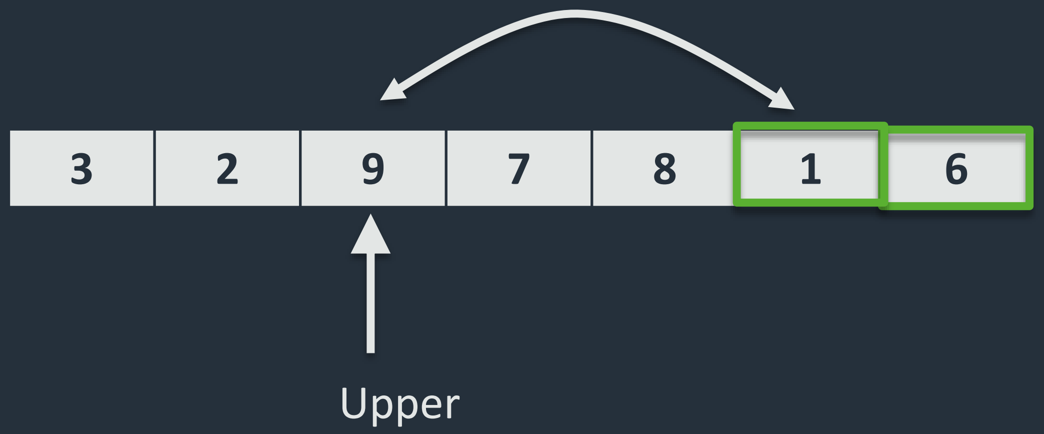
To partition in place, we choose the last item as the pivot.

We put the lower elements at the start.

A index marks the start of the upper list

Compare each element with the pivot

If lower, swap with first upper element, then move up upper index



1≥6, so swap and move up

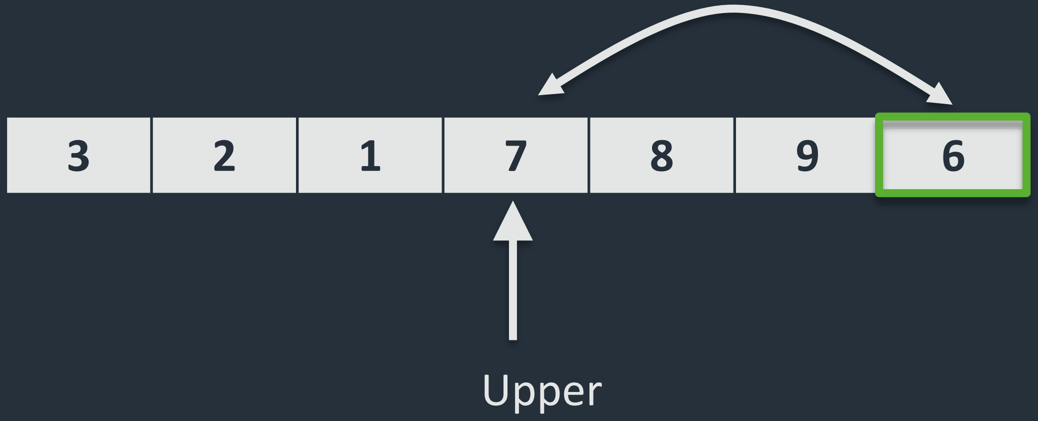
Partitioning in place

To partition in place, we choose the last item as the pivot.

We put the lower elements at the start.

A index marks the start of the upper list

Finally, swap pivot and move up upper index



Swap pivot

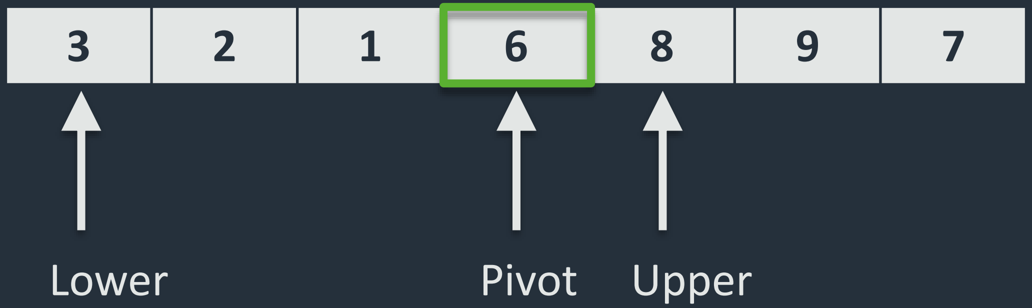
Partitioning in place

To partition in place, we choose the last item as the pivot.

We put the lower elements at the start.

A index marks the start of the upper list

Finally, swap pivot and move up upper index



Partition Algorithm

Partition

```
Partition(s : list, l : int, u : int)
// l,u indicate which part of the list to
// partition.
p = u
upperindex = l
for i=l…u-1
   if s(i)<s(p)
      swap(i,upperindex)
      upperindex = upperindex + 1
   end if
end for
swap(p,upperindex)
return upperindex    // This is the new
                     // pivot position
```

`partition` does one pass through the list, hence it is $O(n)$.

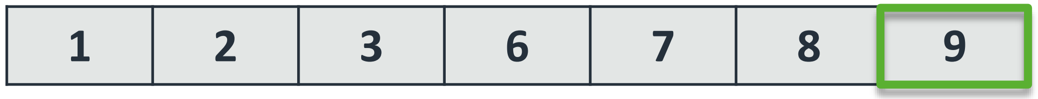
QuickSort Complexity(1)

What is the complexity of QuickSort?

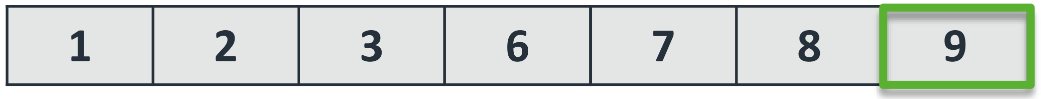
Remember this is the worst-case time taken.

If the pivot splits the list exactly in the middle, each sublist is around half the size (like MergeSort), so we would expect $O(n \log n)$ complexity.

The pivot could be right at the end, (e.g. if the list were sorted)



QuickSort Complexity(2)



The lower list is size $n - 1$, and this must now be sorted.

It takes n recursive calls to QuickSort to finish the sorting, each calling Partition, which itself is $O(n)$

Hence QuickSort is $O(n^2)$, worse than MergeSort and HeapSort!

Why is it called QuickSort, and why is it the standard library sorting routine?

QuickSort Expected Complexity(1)

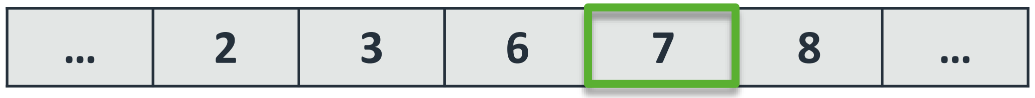
To study expected complexity, we need to make assumptions about the expected input.

Assumption

The input is in a uniformly random order. Any element has equal chance to appear at any position.

Then our choice of the last element as pivot is equivalent to choosing a random element.

QuickSort Expected Complexity(2)



Since the pivot has equal probability to be anywhere, all lengths of lower and upper list are equally likely.

However, the longest of the two dominates our expected time, because it takes longer to sort

The expected length of the longest list can be calculated by

$$E[l] = \sum_{l_l=\frac{n}{2}}^{n} P(l_l)l_l + \sum_{l_u=\frac{n}{2}}^{n} P(l_u)l_u$$

QuickSort Expected Complexity(3)

The length probabilities are all equal, so $P(l) = \frac{1}{n}$

$$E[l] = \frac{1}{n}\sum_{l_l=\frac{n}{2}}^{n} l_l + \frac{1}{n}\sum_{l_u=\frac{n}{2}}^{n} l_u = \frac{2}{n}\sum_{l_l=\frac{n}{2}}^{n} l_l \simeq \frac{3}{4}n$$

The expected size of the longest partition is $\frac{3}{4}n$

After *k* recursions, the longest partition will then be

$$\left(\frac{3}{4}\right)^k n$$

The algorithm terminates when the partition is size 1

$$\left(\frac{3}{4}\right)^k n = 1, \qquad k = \log_{\frac{4}{3}} n \in O(\log n)$$

QuickSort Expected Complexity(4)

In summary, QuickSort recurses to expected depth $O(\log n)$, and executes $O(n)$ operations (in Partition) at each level, so the expected complexity is

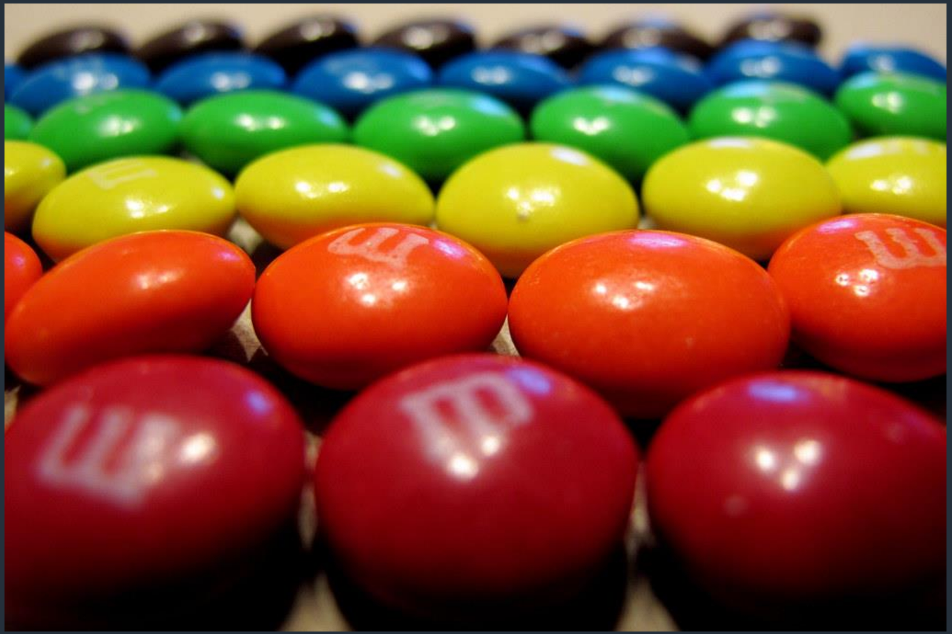
$$O(n \log n)$$

The same as MergeSort, HeapSort, but it has an inferior worst case complexity. So which is best?

Depends on the exact situation, but the partition operations of QuickSort are in-place with minimum swaps, so very efficient. Typically it is several times quicker than the others.

This expected complexity analysis is only an estimation, we should really compute the probability of every configuration, then average the runtime.

Applications of Sorting



Improved Search

Sorted list

Compare the query with the middle item in the list.

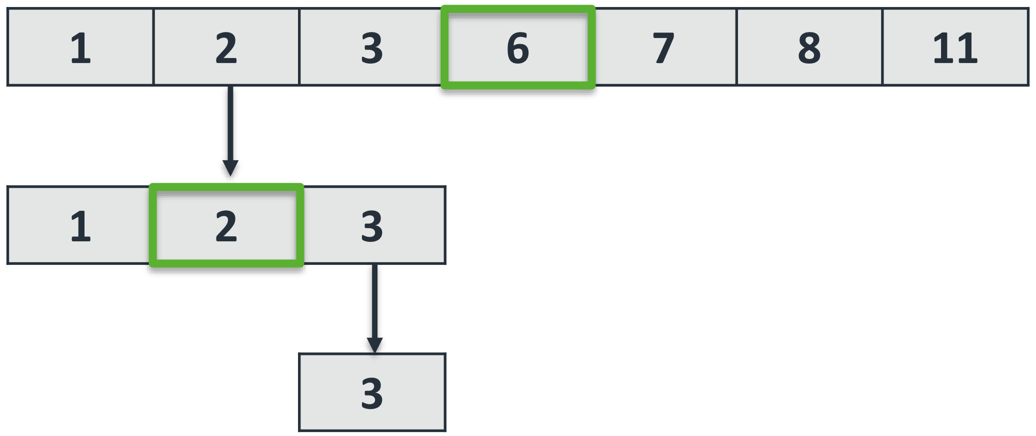
If it is less, search the first half

If it is greater, search the second half

This is Binary Search

$$O(\log n)$$

Query 3:



Unsorted list

Search every item, return if equal to query

$$O(n)$$

Since sorting is $O(n \log n)$, we gain if we intend to make more than $O(\log n)$ queries.

Number of Occurrences

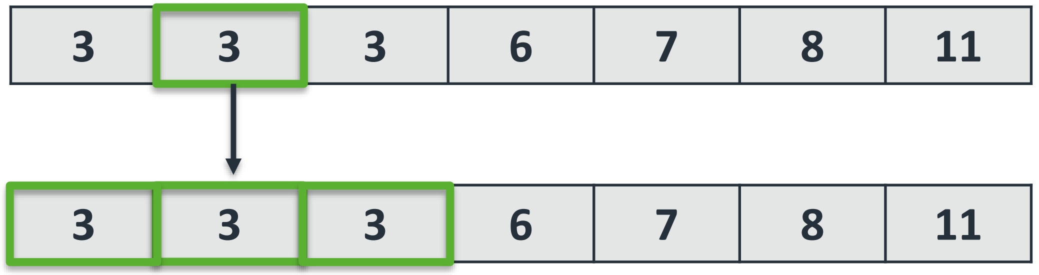
Sorted list

To find the number of occurrences of an item *x*, first search for the item in $O(\log n)$ time.

Then scan the list locally – all occurrences of x will be next to each other.

This takes $O(\log n + c)$ where *c* is the number of occurrences.

Query 3:



Unsorted list

Scan the list, keeping a count of the item

$$O(n)$$

Since sorting is $O(n \log n)$, we gain if we intend to make more than $O(\log n)$ queries.

Selection

Sorted list

Find the k^{th} largest item in the list.

This is simply the k^{th} entry, and takes $O(1)$ to recover.

For example, the median is the $(n/2)^{\text{th}}$ largest item.

Unsorted list

k largest

```
kLargest(l : list, k: int)
Initialise list m with k items all -∞
for j=0…l.length-1
   i=IndexOfMin(m)
   if l(j)>m(i)
      m(i)=l(j)
   end if
end for
return ExtractMin(m)
```

This algorithm is $O(n^2)$

Sorting always quicker

Summary

Understand:

- HeapSort and QuickSort
- The difference between worse-case and expected-time
- Simple expected-time analysis
- How sorting can speed up other algorithms

Read

- Skiena, Sections 3.5, 4.1, 4.2, 4.6
- Attempt exercises 4-2, 4-20, 4-21

Next

- Algorithms for trees