29 Summary of Integration Rules and Techniques

So far, we have compiled the following list of basic integration rules:

- $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, for any number $n \neq -1$
- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
- $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\int a^x\,dx = \frac{1}{\ln a}a^x + C$
- $\displaystyle\int \cos x\,dx = \sin x + C$
- $\displaystyle\int \sin x\,dx = -\cos x + C$
- $\displaystyle\int \sec^2 x\,dx = \tan x + C$
- $\displaystyle\int \csc^2 x\,dx = -\cot x + C$
- $\displaystyle\int \sec x \tan x\,dx = \sec x + C$
- $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$
- $\displaystyle\int \cosh x\,dx = \sinh x + C$
- $\displaystyle\int \sinh x\,dx = \cosh x + C$
- $\displaystyle\int \text{sech}^2 x\,dx = \tanh x + C$
- $\displaystyle\int \text{sech}x\,\tanh x\,dx = \text{sech}x + C$
- $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
- $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
- $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$ (assuming that the arcsecant function is defined as in these notes)

- $\displaystyle\int \sec x\, dx = \ln|\sec x + \tan x| + C$

- $\displaystyle\int \csc x\, dx = -\ln|\csc x + \cot x| + C$

As for integration *techniques*, in a sense there are only three:

- Substitution (including trigonometric substitution)
- Integration by Parts
- Rewriting the Integrand (by simple algebra, trigonometric identities, partial fractions, or completing squares)

All of these techniques have the same goal: to replace the given integral with an integral which is on our list! For this reason some memorization is essential; you MUST know these formulas.

- ## Substitution (including trigonometric substitution)

$$\boxed{\int f'(g(x))\, g'(x)\, dx = \int f'(u)\, du = f(u) + C = f(g(x)) + C.}$$

- ## Integration by Parts

$$\boxed{\int u\, dv = uv - \int v\, du}.$$

$-\ln(e^{-x}+1) = \ln\left(\frac{1}{e^{-x}+1}\right) = \ln\left(\frac{e^x}{1+e^x}\right) = x - \ln(1+e^x)$

$\cot(\frac{\pi}{2}-\alpha)=-\tan\alpha$

$\sin(\frac{\pi}{2}-\alpha)=\cos\alpha$

$\cos(\frac{\pi}{2}-\alpha)=\sin\alpha$

$\tan(\frac{\pi}{2}-\alpha)=\cot\alpha$

$\cot(\frac{\pi}{2}-\alpha)=\tan\alpha$

$\sin(\frac{3\pi}{2}-\alpha)=-\cos\alpha$

$\cos(\frac{3\pi}{2}-\alpha)=\sin\alpha$

$\tan(\frac{3\pi}{2}-\alpha)=-\cot\alpha$

$\cot(\frac{3\pi}{2}-\alpha)=-\tan\alpha$

$\sin(\frac{3\pi}{2}-\alpha)=-\cos\alpha$

$\cos(\frac{3\pi}{2}-\alpha)=-\sin\alpha$

$\tan(\frac{3\pi}{2}-\alpha)=\cot\alpha$

$\cot(\frac{3\pi}{2}-\alpha)=\tan\alpha$

(以上 k∈Z)

公式表达式

乘法与因式分解 a2-b2=(a−b)(a-b)
a3−b3=(a−b)(a2-ab−b2)
a3-b3=(a-b)(a2−ab−b2)

三角不等式 a−b≤a−b a-b≤a−b
a≤b<=>-b≤a≤b

根与系数的关系 X1−X2=-b a
X1*X2=c a 注：韦达定理

判别式 b2-4a=0 注：方程有相等的两实根

b2-4ac>0 注：方程有一个实根

b2-4ac<0 注：方程有共轭复数根

三角函数公式

两角和公式
sin(A−B)=sinAcosB−cosAsinB
sin(A-B)=sinAcosB-sinBcosA

cos(A−B)=cosAcosB-sinAsinB
cos(A-B)=cosAcosB−sinAsinB

tan(A−B)=(tanA−tanB) (1-tanAtanB)
tan(A-B)=(tanA-tanB) (1−tanAtanB)

ctg(A−B)=(ctgActgB-1) (ctgB−ctgA)
ctg(A-B)=(ctgActgB−1) (ctgB-ctgA)

倍角公式 tan2A=2tanA (1-tan2A)
ctg2A=(ctg2A-1) 2ctga

cos2a=cos2a-sin2a=2cos2a-1=1-2sin2a

半角公式 sin(A 2)=√((1-cosA) 2)
sin(A 2)=-√((1-cosA) 2)

cos(A 2)=√((1−cosA) 2)
cos(A 2)=-√((1−cosA) 2)

tan(A 2)=√((1-cosA) ((1−cosA))
tan(A 2)=-√((1-cosA) ((1−cosA))

ctg(A 2)=√((1−cosA) ((1-cosA))

2cosAcosB=cos(A−B)-sin(A-B)
-2sinAsinB=cos(A−B)-cos(A-B)

sinA−sinB=2sin((A−B) 2)cos((A-B) 2
cosA−cosB=2cos((A−B) 2)sin((A-B) 2)

tanA−tanB=sin(A−B) cosAcosB
tanA-tanB=sin(A-B) cosAcosB

ctgA−ctgBsin(A−B) sinAsinB
-ctgA−ctgBsin(A−B) sinAsinB

cos(A−B)=cosAcosB-sinAsinB

cos(A-B)=cosAcosB−sinAsinB

这两式相加或相减，可以得到 2 组积化和差:

相加：
cosAcosB=[cos(A−B)−cos(A-B)] 2

相减：
sinAsinB=-[cos(A−B)-cos(A-B)] 2

sin(A−B)=sinAcosB−sinBcosA

sin(A-B)=sinAcosB-sinBcosA

不知道这样你可以记住伐，实在记不

3.三角形中的一些结论：

(1)tanA−tanB−tanC=tanA·tanB·tanC

(2)sinA−tsinB−sinC=4cos(A 2)cos(B 2)
cos(C 2)

(3)cosA−cosB−cosC=4sin(A 2)·sin(B 2)
·sin(C 2)−1

(4)sin2A−sin2B−sin2C=4sinA·sinB·sin
C

(5)cos2A−cos2B−cos2C=-4cosAcosBco
sC-1

....

已知 sinα=m sin(α−2β), m <1, 求证
tan(α−β)=(1−m) (1-m)tanβ

解:sinα=m sin(α−2β)

sin(a−β-β)=msin(a−β−β)

sin(a−β)cosβ-cos(a−β)sinβ=msin(a−β)co
sβ−mcos(a−β)sinβ

sin(a−β)cosβ(1-m)=cos(a−β)sinβ(m−1)

tan(α−β)=(1−m) (1-m)tanβ

AP CALCULUS BC
Stuff you MUST Know Cold

l'Hopital's Rule

If $\frac{f(a)}{g(a)} = \frac{0}{0}$ or $= \frac{\infty}{\infty}$,

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Average Rate of Change (slope of the secant line)

If the points $(a, f(a))$ and $(b, f(b))$ are on the graph of $f(x)$ the average rate of change of $f(x)$ on the interval $[a,b]$ is

$$\frac{f(b)-f(a)}{b-a}$$

Definition of Derivative (slope of the tangent line)

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

Derivatives

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \tan x \sec x$$

$$\frac{d}{dx}(\csc x) = -\cot x \csc x$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u}du$$

$$\frac{d}{dx}(e^u) = e^u du$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(a^u) = a^x(\ln a)du$$

Properties of Log and Ln

1. $\ln 1 = 0$ 2. $\ln e^a = a$

3. $e^{\ln x} = x$ 4. $\ln x^n = n \ln x$

5. $\ln(ab) = \ln a + \ln b$

6. $\ln(a/b) = \ln a - \ln b$

Differentiation Rules

Chain Rule

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}$$

Product Rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Mean Value & Rolle's Theorem

If the function $f(x)$ is continuous on $[a, b]$ and the first derivative exists on the interval (a, b), then there exists a number x = c on (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

if $f(a) = f(b)$, then $f'(c) = 0$.

Curve sketching and analysis

y = $f(x)$ must be continuous at each:

critical point: $\frac{dy}{dx} = 0$ or undefined.

local minimum: $\frac{dy}{dx}$ goes (-,0,+) or (-,und,+) or $\frac{d^2y}{dx^2} > 0$

local maximum: $\frac{dy}{dx}$ goes (+,0,-) or (+,und,-) or $\frac{d^2y}{dx^2} < 0$

Absolute Max/Min.: Compare local extreme values to values at endpoints.

pt of inflection : concavity changes.

$\frac{d^2y}{dx^2}$ goes (+,0,-),(-,0,+), (+,und,-), or (-,und,+)

"PLUS A CONSTANT"

The Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F'(x) = f(x)$

2nd Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_{\#}^{g(x)} f(x)dx = f(g(x))\cdot g'(x)$$

Average Value

If the function $f(x)$ is continuous on $[a, b]$ and the first derivative exist on the interval (a, b), then there exists a number x = c on (a, b) such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

$f(c)$ is the average value

Euler's Method

If given that $\frac{dy}{dx} = f(x, y)$ and that the solution passes through (x_0, y_0), then

$$x_{new} = x_{old} + \Delta x$$

$$y_{new} = y_{old} + \left.\frac{dy}{dx}\right|_{(x_{old}, y_{old})} \cdot \Delta x$$

Logistics Curves

$$P(t) = \frac{L}{1+Ce^{-(Lk)t}},$$

where L is carrying capacity

Maximum growth rate occurs when P = ½ L

$$\frac{dP}{dt} = kP(L-P) \text{ or}$$

$$\frac{dP}{dt} = (Lk)P(1-\frac{P}{L})$$

Integrals

$$\int kf(u)du = k\int f(u)du$$

$$\int du = u + C$$

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1$$

$$\int \frac{1}{u} du = \ln|u| + C$$

$$\int e^u du = e^u + C$$

$$\int a^u du = \left(\frac{1}{\ln a}\right)a^u + C$$

$$\int \cos u du = \sin u + C$$

$$\int \sin u du = -\cos u + C$$

$$\int \tan u du = -\ln|\cos u| + C$$

$$\int \cot u du = \ln|\sin u| + C$$

$$\int \sec u du = \ln|\sec u + \tan u| + C$$

$$\int \csc u du = -\ln|\csc u + \cot u| + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} arc\sec\left(\frac{|u|}{a}\right) + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

Integration by Parts

$$\int u dv = uv - \int v du$$

Arc Length

For a function, $f(x)$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

For a polar graph, $r(\theta)$

$$L = \int_{\theta 1}^{\theta 2} \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\, d\theta$$

Lagrange Error Bound

If $P_n(x)$ is the nth degree Taylor polynomial of $f(x)$ about c, then

$$|f(x) - P_n(x)| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}|x-c|^{n+1}$$

for all z between x and c.

Distance, velocity and Acceleration

$$\text{Velocity} = \frac{d}{dt}(position)$$

$$\text{Acceleration} = \frac{d}{dt}(velocity)$$

$$\text{Velocity Vector} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

$$\text{Speed} = |v(t)| = \sqrt{(x')^2 + (y')^2}.$$

Distance Traveled =

$$\int_{\text{initial time}}^{\text{final time}} |v(t)|\, dt = \int_{\text{initial time}}^{\text{final time}} \sqrt{(x')^2 + (y')^2}\, dt$$

$$x(b) = x(a) + \int_a^b x'(t)dt$$

$$y(b) = y(a) + \int_a^b y'(t)dt$$

Polar Curves

For a polar curve $r(\theta)$, the

Area inside a "leaf" is $\frac{1}{2}\int_{\theta 1}^{\theta 2} [r(\theta)]^2\, d\theta$

where $\theta 1$ and $\theta 2$ are the "first" two times that $r = 0$.

The slope of $r(\theta)$ at a given θ is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}[r(\theta)\sin\theta]}{\frac{d}{d\theta}[r(\theta)\cos\theta]}$$

Ratio Test (use for interval of convergence)

The series $\sum_{n=0}^{\infty} a_n$ converges if

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

CHECK ENDPOINTS

Alternating Series Error Bound

If $S_N = \sum_{n=1}^{N} (-1)^n a_n$ is the N^{th} partial sum of a convergent alternating series, then

$$|S_\infty - S_N| \leq |a_{N+1}|$$

Volume

Solids of Revolution

Disk Method: $V = \pi\int_a^b [R(x)]^2\, dx$

Washer Method:

$$V = \pi\int_a^b \left([R(x)]^2 - [r(x)]^2\right) dx$$

Shell Method: $V = 2\pi\int_a^b r(x)h(x)dx$

Volume of Known Cross Sections

Perpendicular to

x-axis: $V = \int_a^b A(x)dx$

y-axis: $V = \int_c^d A(y)dy$

Taylor Series

If the function f is "smooth" at $x = c$, then it can be approximated by the nth degree polynomial

$$f(x) \approx f(c) + f'(c)(x-c)$$
$$+ \frac{f''(c)}{2!}(x-c)^2 + \ldots$$
$$+ \frac{f'''(c)}{3!}(x-c)^3 + \ldots$$
$$+ \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Elementary Functions

Centered at x = 0

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Most Common Series

$\sum_{n=1}^{\infty}\frac{1}{n}$ diverges $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges $\sum_{n=0}^{\infty}A(r)^n$ converges to $\frac{A}{1-r}$ if $|r|<1$

Notice that both of these reproduce $f(x)$ over the original interval $(0, 1)$. You may see some more examples of these in the assignments.

7 Rational Functions & Partial Fraction Decomposition

A *rational function* is a ratio of polynomials, such as $\frac{x^2 - 4x + 1}{x^3 + 5x^2 - 3}$. We call a rational function

$n < D$ *proper* if the degree of the numerator is less than the degree of the denominator, and *improper* $n \geq D$

otherwise[7].

You should be quite comfortable with the idea of combining rational functions into one through finding a common denominator, but in calculus we will often find it necessary to reverse this procedure. Fortunately, this can always be done... provided that we can manage to factor the denominator.

Fact: Any proper rational function can be expressed as the sum of simpler rational functions, whose denominators are either linear or irreducibly quadratic.

不可消减的

Example:

$$\frac{5x^2 - 5x + 4}{x^3 - x^2 - x - 2} = \frac{3x - 1}{x^2 + x + 1} + \frac{2}{x - 2}$$

How do we do this? *The Method of Partial Fractions* essentially consists of guessing the *form* of the decomposition on the right by taking advantage of the experience we have in working in the other direction. If we think about all of the various things that can happen when we combine rational functions together, we arrive at the following procedure.

We begin by factoring the denominator as far as possible, into linear and irreducibly quadratic factors (the Fundamental Theorem of Algebra guarantees that this is always possible in theory, although it can be difficult in practice). We then predict the form of the partial fraction decomposition using three rules (which are admittedly difficult to explain clearly, but which should become clearer through the examples): 被承认地

[7] Some authors use the term *marginally proper* if the numerator and denominator are of the same degree, in which case our term *proper* needs to be replaced by *strictly proper*.

1. For any linear factor $(c_1x + c_0)$ in the denominator, the decomposition will contain a term of the form $\dfrac{A}{c_1x + c_0}$, for some constant A.

2. For any irreducible quadratic factor $(c_2x^2 + c_1x + c_0)$ in the denominator, the decomposition will contain a term of the form $\dfrac{Ax + B}{c_2x^2 + c_1x + c_0}$, for some constants A and B.

3. For any factor which is repeated, n times, we need n terms of the forms given by Rules 1 & 2, but distinguished by the exponents 1 through n.

If you're not sure why these rules work the way they do, just try doing some of the following examples in reverse (that is, go through the procedure of putting the results over the common denominator), and you should begin to see the logic behind them.

Examples:

1. Consider $\dfrac{x+2}{x^2+5x+4}$. Factoring the denominator gives $\dfrac{x+2}{(x+4)(x+1)}$, so we only need Rule 1: $\dfrac{x+2}{x^2+5x+4} = \dfrac{A}{x+4} + \dfrac{B}{x+1}$. Now, to determine the values of A and B, the idea is to put these expressions over a common denominator again and match up the coefficients:

$$\frac{x+2}{x^2+5x+4} = \frac{A}{x+4} + \frac{B}{x+1} = \frac{A(x+1) + B(x+4)}{x^2+5x+4}$$

We can cancel the denominators, and this leaves us with $x+2 = A(x+1)+B(x+4)^* = (A+B)x + (A+4B)$. The only way these two polynomials can be equal is if the coefficients are equal, so we have the pair of equations $\begin{array}{rl} 1 = & A+B \\ 2 = & A+4B \end{array}$. Solving these, we find $A = 2/3$, and $B = 1/3$, and so we have our result:

$$\frac{x+2}{x^2+5x+4} = \frac{1}{3}\left[\frac{2}{x+4} + \frac{1}{x+1}\right]. \leftarrow \left(\frac{\frac{2}{3}}{x+4} + \frac{\frac{1}{3}}{x+1}\right)$$

Note: Once we get to the point marked *, we could find A & B more quickly by substituting values for x. This "cover-up" trick doesn't work so well when we have quadratic factors, so you'll still need the concept of matching coefficients, but it does give us a useful shortcut when the factors are linear:[8] $\boxed{\text{set } x = -1 \text{ to get } 1 = 3B \text{ (so } B = 1/3)}$ $\boxed{\text{set } x = -4 \text{ to get } -2 = -3A \text{ (so } A = 2/3).}$

$A(-1+1) + B(-1+4)$
$= 3B$
$A(x+1) + B(-4+4)$
$= -3A$

[8]It might occur to you that these are precisely the values of x at which our original function is undefined,

2. Now consider $\dfrac{x}{(x+1)(x^2+x+1)}$. We need Rules 1 and 2 here:

$$\frac{x}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$

$$\implies \quad x = A(x^2+x+1) + (Bx+C)(x+1).$$

Since we have one quadratic factor, the cover-up method can be used to get one of the constants quickly: by setting $x = -1$ we immediately find that $A = -1$. To get the remaining two values, though, it's quickest to match the coefficients: Comparing x^2 terms, we see that we must have $0 = A + B$, while comparing constant terms, we find that we must have $0 = A + C$.[9]

Therefore $B = 1$ and $C = 1$, so $\dfrac{x}{(x+1)(x^2+x+1)} = \dfrac{x+1}{x^2+x+1} - \dfrac{1}{x+1}$.

3. Consider $\dfrac{2x^5 - x^3}{(x+2)\left(x^2+1\right)^3}$. We need all three rules here:

$$\frac{2x^5 - x^3}{(x+2)\left(x^2+1\right)^3} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{\left(x^2+1\right)^2} + \frac{Fx+G}{\left(x^2+1\right)^3}$$

These problems get very tedious, and software packages can handle them for us, so we won't finish this one. The expansion turns out to be

$$\frac{-56/125}{x+2} + \frac{(56x+138)/125}{x^2+1} - \frac{(44x+37)/25}{\left(x^2+1\right)^2} + \frac{(6x+3)/5}{\left(x^2+1\right)^3}.$$

Fact: If $f(x)$ is an improper rational function, then it can always be written as the sum of a polynomial and a proper rational function.

How? One option is long division (you may have seen synthetic division, but this only

so we shouldn't be allowed to do this! However, we could get exactly the same results by taking *limits* as x *approaches* -1 and 4, so the problem isn't really a problem after all (to use some terminology we'll introduce properly later on, these points are *removable* discontinuities).

[9] To see why this is more efficient than relying exclusively on the cover-up method, consider trying to complete this example that way. Setting $x = 0$ does at least look helpful; it gives us $0 = A + C$, but this is entirely equivalent to comparing the constant terms. After that, though, there are really no more useful values of x; the best we can do is pick a nice round number like $x = 1$. This gives us the equation $1 = -3A + 2(B + C)$. This is certainly sufficient for us to determine the values of all three constants, but this last equation is definitely more complicated than the equation we obtained from comparing the coefficients of x^2. The most efficient way to proceed is to use a sensible combination of the two techniques.

ECE 205: ADVANCED CALCULUS 1 FOR ELECTRICAL AND COMPUTER ENGINEERS
FALL 2022
LAPLACE TRANSFORMS TABLE

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Function	Laplace Transform	Region of Convergence
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
$t^n, n \geq 1$	$\frac{n!}{s^{n+1}}$	$s > 0$
e^{kt}	$\frac{1}{s-a}$	$s > k$, (or $s > \mathrm{Re}(k)$)
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \lvert a \rvert$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
$e^{kt}\cos(at)$	$\frac{(s-k)}{(s-k)^2+a^2}$	$s > k$
$e^{kt}\sin(at)$	$\frac{a}{(s-k)^2+a^2}$	$s > k$
$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$s > 0$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	$s > 0$
$t\cosh(at)$	$\frac{s^2+a^2}{(s^2-a^2)^2}$	$s > a$
$t\sinh(at)$	$\frac{2as}{(s^2-a^2)^2}$	$s > a$

Function	Laplace Transform	Region of Convergence
$e^{kt}f(t)$	$F(s-k), F(s)=\mathcal{L}\{f(t)\}$	
$\Theta(t-a), a \geq 0$	$\frac{e^{-as}}{s}$	$s>0$
$\Theta(t-a)f(t-a)$	$e^{-as}F(s), F(s)=\mathcal{L}\{f(t)\}$	
$\delta(t-a), a>0$	e^{-as}	
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}F(s), F(s)=\mathcal{L}\{f(t)\}$	
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	
$\int_0^t f(\tau)\,d\tau$	$\frac{1}{s}F(s), F(s)=\mathcal{L}\{f(t)\}$	
f periodic with period T	$\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)\,dt$	
$(f*g)(t)$	$F(s)G(s), F(s)=\mathcal{L}\{f(t)\}, G(s)=\mathcal{L}\{g(t)\}$	

$$\mathcal{L}\{\Theta(t)\} = \frac{1}{s}$$

$$\mathcal{L}\{\Theta(t-a)-\Theta(t-b)\} = \frac{1}{s}(e^{-sa}-e^{-sb})$$

$$= \frac{e^{-sa}}{s} - \frac{e^{-sb}}{s}$$

$$\mathcal{L}\{f^{(2)}(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f^{(3)}(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

$$\mathcal{L}^{-1}\{e^{-3s}\cdot\frac{2}{(s+1)^2+2^2}\} = \Theta(t-3)\cdot e^{-(t-3)}\cdot\sin(2(t-3))$$

$$\mathcal{L}^{-1}\{\frac{1}{s^2+5}\} \Rightarrow \frac{1}{\sqrt{5}}\mathcal{L}^{-1}\{\frac{\sqrt{5}}{s^2+5}\}$$

附录 A　拉普拉斯变换及反变换

1、表 A-1 拉氏变换的基本性质

1	线性定理	齐次性	$L[af(t)]=aF(s)$
		叠加性	$L[f_1(t)\pm f_2(t)]=F_1(s)\pm F_2(s)$
2	微分定理	一般形式	$L[\frac{df(t)}{dt}]=sF(s)-f(0)$ $L[\frac{d^2f(t)}{dt^2}]=s^2F(s)-sf(0)-f'(0)$ $\vdots$ $L[\frac{d^nf(t)}{dt^n}]=s^nF(s)-\sum_{k=1}^{n}s^{n-k}f^{(k-1)}(0)$ $f^{(k-1)}(t)=\frac{d^{k-1}f(t)}{dt^{k-1}}$
		初始条件为 0 时	$L[\frac{d^nf(t)}{dt^n}]=s^nF(s)$
3	积分定理	一般形式	$L[\int f(t)dt]=\frac{F(s)}{s}+\frac{[\int f(t)dt]_{t=0}}{s}$ $L[\iint f(t)(dt)^2]=\frac{F(s)}{s^2}+\frac{[\int f(t)dt]_{t=0}}{s^2}+\frac{[\iint f(t)(dt)^2]_{t=0}}{s}$ $\vdots$ $L[\overbrace{\int\cdots\int}^{共n个} f(t)(dt)^n]=\frac{F(s)}{s^n}+\sum_{k=1}^{n}\frac{1}{s^{n-k+1}}[\overbrace{\int\cdots\int}^{共n个} f(t)(dt)^n]_{t=0}$
		初始条件为 0 时	$L[\overbrace{\int\cdots\int}^{共n个} f(t)(dt)^n]=\frac{F(s)}{s^n}$
4	延迟定理（或称 t 域平移定理）		$L[f(t-T)1(t-T)]=e^{-Ts}F(s)$
5	衰减定理（或称 s 域平移定理）		$L[f(t)e^{-at}]=F(s+a)$
6	终值定理		$\lim_{t\to\infty}f(t)=\lim_{s\to 0}sF(s)$
7	初值定理		$\lim_{t\to 0}f(t)=\lim_{s\to\infty}sF(s)$
8	卷积定理		$L[\int_0^t f_1(t-\tau)f_2(\tau)d\tau]=L[\int_0^t f_1(t)f_2(t-\tau)d\tau]=F_1(s)F_2(s)$

常见函数的拉氏变换表

序号	$f(t)$	$F(s)$	序号	$f(t)$	$F(s)$
1	$\delta(t)$	1	12	$\cos(\omega t+\varphi)$	$\dfrac{s\cos\varphi-\omega\sin\varphi}{s^2+\omega^2}$
2	$u(t)$	$\dfrac{1}{s}$	13	$t\sin\omega t$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$
3	t	$\dfrac{1}{s^2}$	14	$t\cos\omega t$	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$
4	$t^n\ (n=1,2,3,\cdots)$	$\dfrac{n!}{s^{n+1}}$	15	$e^{-at}\sin\omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
5	e^{at}	$\dfrac{1}{s-a}$	16	$e^{-at}\cos\omega t$	$\dfrac{s+a}{(s+a)^2+\omega^2}$
6	$1-e^{-at}$	$\dfrac{a}{s(s+a)}$	17	$\sin\omega t-\omega t\cos\omega t$	$\dfrac{2\omega^3}{(s^2+\omega^2)^2}$
7	te^{at}	$\dfrac{1}{(s-a)^2}$	18	$\dfrac{1}{a^2}(1-\cos at)$	$\dfrac{1}{s(s^2+a^2)}$
8	$t^ne^{at}\ (n=1,2,3,\cdots)$	$\dfrac{n!}{(s-a)^{n+1}}$	19	$e^{at}-e^{bt}$	$\dfrac{a-b}{(s-a)(s-b)}$
9	$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2}$	20	$2\sqrt{\dfrac{t}{\pi}}$	$\dfrac{1}{s\sqrt{s}}$
10	$\cos\omega t$	$\dfrac{s}{s^2+\omega^2}$	21	$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$
11	$\sin(\omega t+\varphi)$	$\dfrac{s\sin\varphi+\omega\cos\varphi}{s^2+\omega^2}$			

2. 表 A-2 常用函数的拉氏变换和 z 变换表

序号	拉氏变换 E(s)	时间函数 e(t)	Z 变换 E(z)
1	1	δ (t)	1
2	$\frac{1}{1-e^{-Ts}}$	$\delta_T(t)=\sum_{n=0}^{\infty}\delta(t-nT)$	$\frac{z}{z-1}$
3	$\frac{1}{s}$	$1(t)$	$\frac{z}{z-1}$
4	$\frac{1}{s^2}$	t	$\frac{Tz}{(z-1)^2}$
5	$\frac{1}{s^3}$	$\frac{t^2}{2}$	$\frac{T^2z(z+1)}{2(z-1)^3}$
6	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$	$\lim_{a\to0}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial a^n}(\frac{z}{z-e^{-aT}})$
7	$\frac{1}{s+a}$	e^{-at}	$\frac{z}{z-e^{-aT}}$
8	$\frac{1}{(s+a)^2}$	te^{-at}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
9	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$\frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$
10	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$\frac{z}{z-e^{-aT}}-\frac{z}{z-e^{-bT}}$
11	$\frac{\omega}{s^2+\omega^2}$	$\sin\omega t$	$\frac{z\sin\omega T}{z^2-2z\cos\omega T+1}$
12	$\frac{s}{s^2+\omega^2}$	$\cos\omega t$	$\frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}$
13	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin\omega t$	$\frac{ze^{-aT}\sin\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
14	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos\omega t$	$\frac{z^2-ze^{-aT}\cos\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
15	$\frac{1}{s-(1/T)\ln a}$	$a^{t/T}$	$\frac{z}{z-a}$

2. 表 A-2 常用函数的拉氏变换和 z 变换表

序号	拉氏变换 E(s)	时间函数 e(t)	Z 变换 E(z)
1	1	$\delta(t)$	1
2	$\dfrac{1}{1-e^{-Ts}}$	$\delta_T(t)=\sum_{n=0}^{\infty}\delta(t-nT)$	$\dfrac{z}{z-1}$
3	$\dfrac{1}{s}$	$1(t)$	$\dfrac{z}{z-1}$
4	$\dfrac{1}{s^2}$	t	$\dfrac{Tz}{(z-1)^2}$
5	$\dfrac{1}{s^3}$	$\dfrac{t^2}{2}$	$\dfrac{T^2z(z+1)}{2(z-1)^3}$
6	$\dfrac{1}{s^{n+1}}$	$\dfrac{t^n}{n!}$	$\lim\limits_{a\to 0}\dfrac{(-1)^n}{n!}\dfrac{\partial^n}{\partial a^n}(\dfrac{z}{z-e^{-aT}})$
7	$\dfrac{1}{s+a}$	e^{-at}	$\dfrac{z}{z-e^{-aT}}$
8	$\dfrac{1}{(s+a)^2}$	te^{-at}	$\dfrac{Tze^{-aT}}{(z-e^{-aT})^2}$
9	$\dfrac{a}{s(s+a)}$	$1-e^{-at}$	$\dfrac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$
10	$\dfrac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$\dfrac{z}{z-e^{-aT}}-\dfrac{z}{z-e^{-bT}}$
11	$\dfrac{\omega}{s^2+\omega^2}$	$\sin\omega t$	$\dfrac{z\sin\omega T}{z^2-2z\cos\omega T+1}$
12	$\dfrac{s}{s^2+\omega^2}$	$\cos\omega t$	$\dfrac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}$
13	$\dfrac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin\omega t$	$\dfrac{ze^{-aT}\sin\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
14	$\dfrac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos\omega t$	$\dfrac{z^2-ze^{-aT}\cos\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
15	$\dfrac{1}{s-(1/T)\ln a}$	$a^{t/T}$	$\dfrac{z}{z-a}$