Fast Searching The Densest Subgraph And Decomposition With Local Optimality

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I. Appendix

Proof of Corollary 1. For DkS, we set $|\mathcal{S}| = k$, it has an upper bound as follows:

$$\begin{aligned}
\rho(\mathcal{S}) &= \frac{\sum_{e\in\mathcal{E}(\mathcal{S})} w_e}{|\mathcal{S}|} = \frac{\sum_{e=(u,v),\, u,v\in\mathcal{S}} f_e(u) + f_e(v)}{|\mathcal{S}|} \\
&\le \frac{\sum_{e=(u,v),\, u,v\in\mathcal{S}} (f_e(u) + f_e(v)) + \sum_{e=(u,v),\, u\in\mathcal{S}, v\notin\mathcal{S}} f_e(u)}{|\mathcal{S}|} \\
&= \frac{\sum_{u\in\mathcal{S}} \sum_{e\ni u} f_e(u)}{|\mathcal{S}|} = \frac{\sum_{u\in\mathcal{S}} l_u}{|\mathcal{S}|} \\
&= \frac{\sum_{i=0}^{j-1} \lambda_i * |B_i| + (k - |B_{j-1}|) * \lambda_j}{k}
\end{aligned}$$

Therefore, result 2 holds up. As for result 1, if $k = |B_j|$, according to property 3 the equivalency condition in line 2 holds up because $f_e(u) = 0$ if $u \in \mathcal{S}$ and $v \notin \mathcal{S}$, then result 1 holds up. These results also hold up for DalkS because if $|S| > k$, it will add more nodes into $\mathcal{S}$ with lower upper bounds on their loads. □

Before proofs of Lemma 2, we introduce the definition of k-core from [?]: k-core is the maximal subgraph G_k in graph G, the degree of where any vertex v in G_k is satisfied with $\mathrm{d}_{G_k}(v) \ge k$.

proof of Lemma 2. The Greedy algorithm just moves any node whose degree is the lowest in the remaining graph $\mathcal{H}$. Let us remark $A(k)$ as the nodeset of k-core for specific k. We claim that when deleting a node $u \in A(k)$, there mustn't be any node $v \notin A(k)$ in the remaining graph $\mathcal{H}$. We prove it by way of contradiction, w.l.o.g, we set u as the first node to be deleted in $A(k)$ in Greedy, therefore u has the lowest degree in $\mathcal{H}$ and $\mathrm{d}_{\mathcal{H}}(v) \ge \mathrm{d}_{\mathcal{H}}(u) \ge \mathrm{d}_{A(k)}(u) \ge k$ for any node v in $\mathcal{H}$, which produces a k-core subgraph with a larger size, and it leads to a contradiction. □

Proof of Theorem 5. We set $\mathcal{H}_k (k > 0)$ is the remaining graph after k iterations in Algorithm 2 and $\mathcal{H}_0$ is the initial whole graph, $\mathcal{H}'_{k+1}$ is the nodeset to be deleted in k+1 iteration. Therefore, $\mathcal{H}_{k+1} = \mathcal{H}_k \setminus \mathcal{H}'_{k+1}$ Then:

$$\begin{aligned}
\rho(\mathcal{H}_{k+1}) &= \rho(\mathcal{H}_k \setminus \mathcal{H}'_{k+1}) \\
&= \frac{\mathcal{W}(\mathcal{E}(\mathcal{H}_k)) - \mathcal{W}(\mathcal{E}(\mathcal{H}'_{k+1}))}{|\mathcal{H}_k| - |\mathcal{H}'_{k+1}|} \\
&\ge \frac{\rho(\mathcal{H}_k)\cdot|\mathcal{H}_k| - \sum_{v\in\mathcal{H}'_{k+1}} \mathrm{d}_{\mathcal{H}_k}(v)}{|\mathcal{H}_k| - |\mathcal{H}'_{k+1}|} \\
&> \frac{\rho(\mathcal{H}_k)\cdot|\mathcal{H}_k| - \sum_{v\in\mathcal{H}'_{k+1}} \rho(\mathcal{H}_k)}{|\mathcal{H}_k| - |\mathcal{H}'_{k+1}|} \\
&= \frac{\rho(\mathcal{H}_k)\cdot|\mathcal{H}_k| - \rho(\mathcal{H}_k)\cdot|\mathcal{H}'_{k+1}|}{|\mathcal{H}_k| - |\mathcal{H}'_{k+1}|} \\
&= \rho(\mathcal{H}_k)
\end{aligned}$$

That means the density of graph $\mathcal{H}$ monotonically increases in iterations, then any deleted node has a lower degree (when it is being deleted) than the final density, i.e., δ. Therefore, the remaining graph $\mathcal{H}$ is a δ-core and it is the graph of some time of the greedy search according to Lemma 2.

We claim that the process before getting the δ-core is a monotonic increasing phase of density in Greedy. We can confirm two facts:

1. During Greedy, if there is a node $u \in \mathcal{H}'_1$ existing in the remaining graph, the deletion in Greedy will increase the density of the remaining graph. That's because if Greedy deletes a node $v \notin \mathcal{H}'_1$, then $\mathrm{d}_{\mathcal{H}}(v) \le \mathrm{d}_{\mathcal{H}}(u) < \rho(\mathcal{G}) \le \rho(\mathcal{H})$. $\rho(\mathcal{H})$ will increase and $\rho(\mathcal{G}) \le \rho(\mathcal{H})$ still holds up. If Greedy deletes the node u, now that $\mathrm{d}_{\mathcal{H}}(u) \le \rho(\mathcal{H})$, then $\rho(\mathcal{H})$ will also increase and $\rho(\mathcal{G}) \le \rho(\mathcal{H})$ holds up.
2. During Greedy, if there is a node $u \in \mathcal{H}'_{k+1}$ existing in the remaining graph $\mathcal{H} > \mathcal{H}_k$, and there isn't any node belonging to $\mathcal{H}'_k$. Then: $\rho(\mathcal{H}) \ge \rho(\mathcal{H}_k)$ because we delete more nodes with lower degrees. Therefore, when we deletes a node $v \notin \mathcal{H}'_{k+1}$, then $\mathrm{d}_{\mathcal{H}}(v) \le \mathrm{d}_{\mathcal{H}}(u) < \rho(\mathcal{G}) \le \rho(\mathcal{H}_k) \le \rho(\mathcal{H})$, then $\rho(\mathcal{H})$ will increase and $\rho(\mathcal{G}) \le \rho(\mathcal{H})$ holds up. If Greedy deletes the node u, now that $\mathrm{d}_{\mathcal{H}}(v) \le \rho(\mathcal{H})$, $\rho(\mathcal{H})$ will also increase and $\rho(\mathcal{G}) \le \rho(\mathcal{H})$ holds up.

Therefore, the density monotonically increases in Greedy before getting δ-core. □

TABLE I
DATASET SOURCE AND DENSITY OF ALGORITHMS

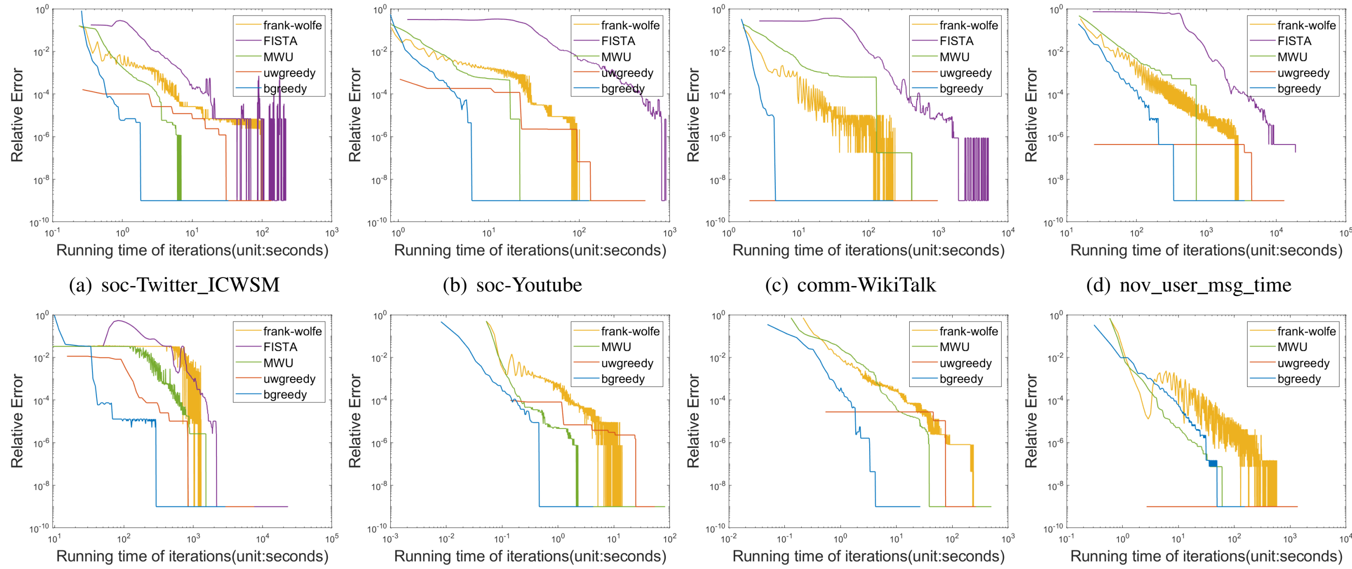
Dataset	Source	Type	Pruning	w_app	exact	DLL	uw_Pruning+DLL
ca-HepPh	Stanford's SNAP database	scholar collaboration network	119	119	119	119	119
comm-EmailEnron	Stanford's SNAP database	communication	37.316	37.344	37.344	37.337	37.337
ca-AstroPh	Stanford's SNAP database	scholar collaboration network	28.481	29.616	32.11	29.552	29.552
PP-Pathways	Stanford's SNAP database	protein interaction network	74.159	77.995	77.995	77.995	77.995
soc-Twitter_ICWSM	konect	social network	25.678	25.683	25.69	25.686	25.685
soc-sign_slashdot	Stanford's SNAP database	social network	39.376	42.132	42.132	42.132	42.132
rating-StackOverflow	konect	social network	20.209	20.209	20.21	20.209	20.209
soc-sign_epinion	Stanford's SNAP database	social network	80.168	85.599	85.637	85.589	85.589
ego-twitter	Stanford's SNAP database	social network	59.281	68.414	69.622	68.414	68.414
soc-Youtube	Stanford's SNAP database	social network	45.545	45.58	45.599	45.576	45.577
comm-WikiTalk	Stanford's SNAP database	communication	114.139	114.139	114.139	114.139	114.139
nov_user_msg_time	We own it privately.	social network	278.815	278.815	278.815	278.815	278.815
cit-Patents	AMiner scholar datasets	scholar collaboration network	132.776	135.706	137.261	135.706	135.706
soc-Twitter_ASU	ASU	social network	593.847	593.847	593.847	593.847	593.847
soc-Livejournal	Livejournal	social network	104.596	104.601	104.609	104.603	104.603
soc-Orkut	Stanford's SNAP database	social network	227.861	227.872	227.874	227.872	227.872
soc-SinaWeibo	Network Repository	social network	164.967	165.193	165.415	165.196	165.191
wang-tripadvisor	konect	rating network	13.442	13.873	14.082	–	–
rec-YelpUserBusiness	Network Repository	rating network	87.825	87.912	87.921	–	–
bookcrossing	konect	rating network	92.148	92.322	92.374	–	–
librec-ciaodvd-review	konect	rating network	233.553	233.59	233.597	–	–
movielens-10m	konect	rating network	1351.35	1351.35	1351.35	–	–
epinions	konect	rating network	595.302	595.314	595.316	–	–
libimseti	konect	social network	1645.71	1645.73	1645.73	–	–
rec-movielens	Network Repository	rating network	1801.16	1801.16	1801.16	–	–
yahoo-song	konect	rating network	46725.2	46725.2	46725.2	–	–

note: : **Pruning** (w_Pruning,uw_Pruning). **w_app**: approximation algorihtms on weighted graph(Priority Tree,Pruning+Priority Tree,BBST). **exact**: exact algorithms(maxflow,w_Pruning+maxflow). **DLL**:Doubly-linked list.

TABLE II
COMPARISON BETWEEN LOWD AND BASELINES WITHOUT THE PRUNING.

	Running time					The number of iteration rounds				
Dataset	**LOWD**	**Greedy++**	**FW**	**FISTA**	**MWU**	**LOWD**	**Greedy++**	**FW**	**FISTA**	**MWU**
ca-HepPh	0.0168	**0.0159**	0.0208	0.0411	0.0161	1	1	2	2	1
comm-EmailEnron	0.1516	**0.0714**	0.5551	4.9384	0.3012	19	2	88	255	46
ca-AstroPh	**0.3158**	0.8396	2.312	4.4843	0.6593	36	33	343	211	92
PP-Pathways	0.1174	**0.0333**	0.3214	2.8722	1.7086	9	1	28	79	164
soc-Twitter_ICWSM	**1.8343**	30.6654	96.3405	43.7426	6.1467	49	114	2816	401	175
soc-sign_slashdot	0.2275	**0.0704**	0.7881	9.033	1.1051	11	1	44	162	64
rating-StackOverflow	2.1887	**1.0537**	10.0728	227.1931	18.7819	32	2	178	1266	341
soc-sign_epinion	**0.3213**	1.4706	1.1853	8.0435	0.9581	9	12	40	90	33
ego-twitter	**1.6001**	2.9768	22.0971	51.1176	1.7266	34	21	550	384	41
soc-Youtube	**6.5253**	133.3086	82.3855	812.4268	22.0866	46	129	612	1792	161
comm-WikiTalk	4.6758	**1.984**	118.307	1942.7338	414.9261	15	1	407	2022	1386
nov_user_msg_time	**341.1144**	4464.4998	2581.9086	>16117.208	718.8805	97	174	624	>1200	177
cit-Patents_AMINER	**291.4452**	846.3628	837.8446	2172.7049	1523.2698	104	56	259	196	471
soc-Twitter_ASU	120.8906	**19.2912**	511.3170	15208.9590	2047.9598	40	1	156	1244	603
soc-Livejournal	–	–	–	–	–	–	–	–	–	–
soc-Orkut	–	–	–	–	–	–	–	–	–	–
soc-SinaWeibo_NETREP	–	–	–	–	–	–	–	–	–	–
wang-tripadvisor	**0.6686**	23.6761	30.209	–	15.6705	93	187	3894	–	2007
rec-YelpUserBusiness	**0.4621**	24.5879	5.1698	–	2.1192	54	228	626	–	239
bookcrossing	**1.5177**	92.9406	27.4916	–	8.877	78	259	1398	–	448
librec-ciaodvd-review	**4.2248**	75.7599	231.5822	–	38.6405	79	143	4635	–	773
movielens-10m	49.2341	**2.7029**	129.6344	–	60.719	159	1	450	–	211
epinions	**30.4071**	3957.2229	4302.242	–	743.6429	64	688	8893	–	1582
libimseti	**25.7101**	5276.5399	751.3750	–	146.2627	39	710	1196	–	234
rec-movielens	**50.4427**	6928.9143	306.0513	–	148.2539	54	733	355	–	172
yahoo-song	–	–	–	–	–	–	–	–	–	–

note: We ignore some datasets which are very large.">1200" and ">16117.208" means we run 1200 iterations(running time: 16117.208s) and can't still detect the densest subgraph.



(a) soc-Twitter_ICWSM (b) soc-Youtube (c) comm-WikiTalk (d) nov_user_msg_time

(e) cit-Patents_AMINER (f) rec-YelpUserBusiness (g) librec-ciaodvd-review (h) movielens-10m

Fig. 1. densest

Algorithm 1: GREEDY DSPSOLVER

Input: Undirected graph $\mathcal{G}$; density metric $\rho(\cdot)$

Output: $\mathcal{S}^*$: the nodeset of the densest subgraph of $\mathcal{G}$.

$\mathcal{S}$, $\mathcal{S}^* \leftarrow \mathcal{V}$

while $\mathcal{S} \neq \emptyset$ **do**

  ▷ *find the vertex u^* with the lowest degree in $\mathcal{S}$*

  $u^* \leftarrow \arg\min_{u \in \mathcal{S}} \mathrm{d}_{\mathcal{S}}(u))$

  Remove u^* and all its adjacent edges from $\mathcal{G}$.

  ▷ *$\mathcal{S} \backslash \{u\}$: the remaining nodeset without u*

  $\mathcal{S} \leftarrow \mathcal{S} \backslash \{u\}$

  if $\rho(\mathcal{S}) > \rho(\mathcal{S}^*)$ **then**

    $\mathcal{S}^* \leftarrow \mathcal{S}$

return $\mathcal{S}^*$.

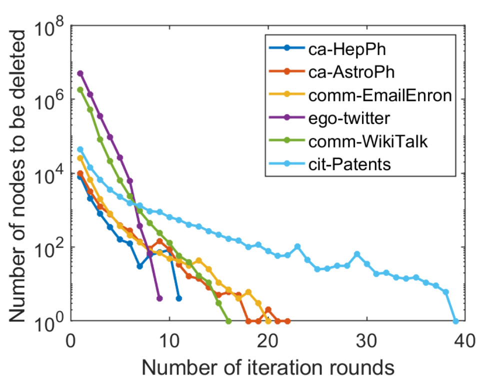


Fig. 2. Exponential decrease in the number of deleted nodes.