7 APPENDIX

Proof of Corollary 1. For DkS, we set $|\mathcal{S}| = k$, it has an upper bound as follows:

$$\begin{aligned}
\rho(\mathcal{S}) &= \frac{\sum_{e \in \mathcal{E}(\mathcal{S})} w_e}{|\mathcal{S}|} = \frac{\sum_{e=(u,v),\, u,v \in \mathcal{S}} f_e(u) + f_e(v)}{|\mathcal{S}|} \\
&\leq \frac{\sum_{e=(u,v),\, u,v \in \mathcal{S}} (f_e(u) + f_e(v)) + \sum_{e=(u,v),\, u \in \mathcal{S}, v \notin \mathcal{S}} f_e(u)}{|\mathcal{S}|} \\
&= \frac{\sum_{u \in \mathcal{S}} \sum_{e \ni u} f_e(u)}{|\mathcal{S}|} = \frac{\sum_{u \in \mathcal{S}} l_u}{|\mathcal{S}|} \\
&= \frac{\sum_{i=0}^{j-1} \lambda_i * |B_i| + (k - |B_{j-1}|) * \lambda_j}{k}
\end{aligned}$$

Therefore, result 2 holds up. As for result 1, if $k = |B_j|$, according to property 3 the equivalency condition in line 2 holds up because $f_e(u) = 0$ if $u \in \mathcal{S}$ and $v \notin \mathcal{S}$, then result 1 holds up. These results also hold up for DalkS because if $|S| > k$, it will add more nodes into $\mathcal{S}$ with lower upper bounds on their loads. □

Before proofs of Lemma 2, we introduce the definition of k-core from [43]: k-core is the maximal subgraph G_k in graph G, the degree of where any vertex v in G_k is satisfied with $\mathrm{d}_{G_k}(v) \geq k$.

proof of Lemma 2. The Greedy algorithm just moves any node whose degree is the lowest in the remaining graph $\mathcal{H}$. Let us remark $A(k)$ as the nodeset of k-core for specific k. We claim that when deleting a node $u \in A(k)$, there mustn't be any node $v \notin A(k)$ in the remaining graph $\mathcal{H}$. We prove it by way of contradiction, w.l.o.g, we set u as the first node to be deleted in $A(k)$ in Greedy, therefore u has the lowest degree in $\mathcal{H}$ and $\mathrm{d}_{\mathcal{H}}(v) \geq \mathrm{d}_{\mathcal{H}}(u) \geq \mathrm{d}_{A(k)}(u) \geq k$ for any node v in $\mathcal{H}$, which produces a k-core subgraph with a larger size, and it leads to a contradiction. □

Proof of Theorem 5. We set $\mathcal{H}_k (k > 0)$ is the remaining graph after k iterations in Algorithm 2 and $\mathcal{H}_0$ is the initial whole graph, $\mathcal{H}'_{k+1}$ is the nodeset to be deleted in k+1 iteration. Therefore, $\mathcal{H}_{k+1} = \mathcal{H}_k \setminus \mathcal{H}'_{k+1}$,

$$\begin{aligned}
\rho(\mathcal{H}_{k+1}) &= \rho(\mathcal{H}_k \setminus \mathcal{H}'_{k+1}) \\
&= \frac{\mathcal{W}(\mathcal{E}(\mathcal{H}_k)) - \mathcal{W}(\mathcal{E}(\mathcal{H}'_{k+1}))}{|\mathcal{H}_k| - |\mathcal{H}'_{k+1}|} \\
&\geq \frac{\rho(\mathcal{H}_k) \cdot |\mathcal{H}_k| - \sum_{v \in \mathcal{H}'_{k+1}} \mathrm{d}_{\mathcal{H}_k}(v)}{|\mathcal{H}_k| - |\mathcal{H}'_{k+1}|} \\
&> \frac{\rho(\mathcal{H}_k) \cdot |\mathcal{H}_k| - \sum_{v \in \mathcal{H}'_{k+1}} \rho(\mathcal{H}_k)}{|\mathcal{H}_k| - |\mathcal{H}'_{k+1}|} \\
&= \frac{\rho(\mathcal{H}_k) \cdot |\mathcal{H}_k| - \rho(\mathcal{H}_k) \cdot |\mathcal{H}'_{k+1}|}{|\mathcal{H}_k| - |\mathcal{H}'_{k+1}|} \\
&= \rho(\mathcal{H}_k)
\end{aligned}$$

That means the density of graph $\mathcal{H}$ monotonically increases in iterations, then any deleted node has a lower degree (when it is being deleted) than the final density, i.e., δ. Therefore, the remaining graph $\mathcal{H}$ is a δ-core and it is the graph of some time of the greedy search according to Lemma 2.

We claim that the process before getting the δ-core is a monotonic increasing phase of density in Greedy. We can confirm two facts:

1. During Greedy, if there is a node $u \in \mathcal{H}'_1$ existing in the remaining graph, the deletion in Greedy will increase the density of the remaining graph. That's because if Greedy deletes a node $v \notin \mathcal{H}'_1$, then $\mathrm{d}_{\mathcal{H}}(v) \leq \mathrm{d}_{\mathcal{H}}(u) < \rho(\mathcal{G}) \leq \rho(\mathcal{H})$. $\rho(\mathcal{H})$ will increase and $\rho(\mathcal{G}) \leq \rho(\mathcal{H})$ still holds up. If Greedy deletes the node u, now that $\mathrm{d}_{\mathcal{H}}(u) \leq \rho(\mathcal{H})$, then $\rho(\mathcal{H})$ will also increase and $\rho(\mathcal{G}) \leq \rho(\mathcal{H})$ holds up.
2. During Greedy, if there is a node $u \in \mathcal{H}'_{k+1}$ existing in the remaining graph $\mathcal{H} > \mathcal{H}_k$, and there isn't any node belonging to $\mathcal{H}'_k$. Then: $\rho(\mathcal{H}) \geq \rho(\mathcal{H}_k)$ because we delete more nodes with lower degrees. Therefore, when we deletes a node $v \notin \mathcal{H}'_{k+1}$, then $\mathrm{d}_{\mathcal{H}}(v) \leq \mathrm{d}_{\mathcal{H}}(u) < \rho(\mathcal{G}) \leq \rho(\mathcal{H}_k) \leq \rho(\mathcal{H})$, then $\rho(\mathcal{H})$ will increase and $\rho(\mathcal{G}) \leq \rho(\mathcal{H})$ holds up. If Greedy deletes the node u, now that $\mathrm{d}_{\mathcal{H}}(v) \leq \rho(\mathcal{H})$, $\rho(\mathcal{H})$ will also increase and $\rho(\mathcal{G}) \leq \rho(\mathcal{H})$ holds up.

Therefore, the density monotonically increases in Greedy before getting δ-core. □

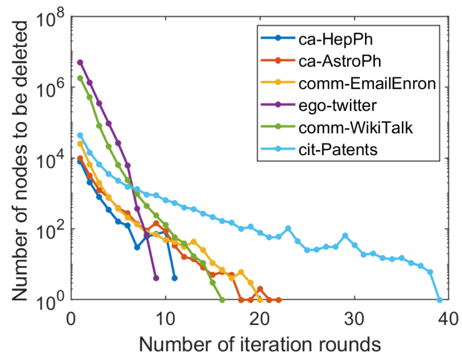


Figure 7: Exponential decrease in the number of deleted nodes.

Algorithm 3: Greedy DSPSolver

Input: Undirected graph $\mathcal{G}$; density metric $\rho(\cdot)$

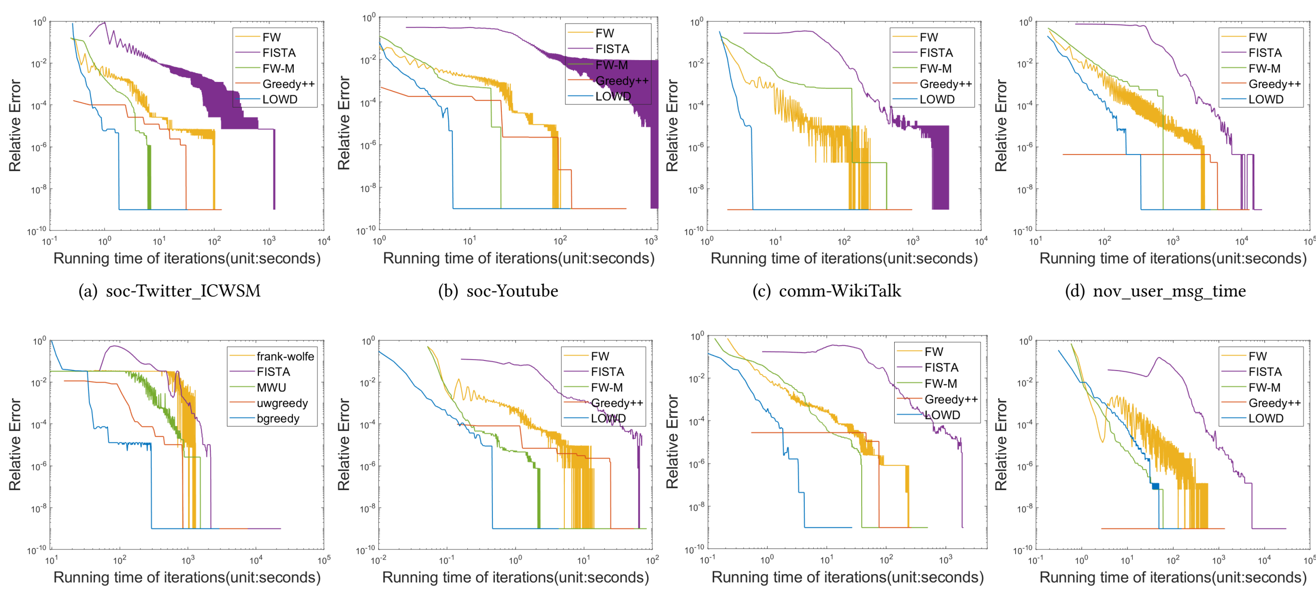
Output: $\mathcal{S}^*$: the nodeset of the densest subgraph of $\mathcal{G}$.

```
S, S* ← V
while S ≠ ∅ do
    ▷ find the vertex u* with the lowest degree in S
    u* ← arg min_{u∈S} d_S(u))
    Remove u* and all its adjacent edges from G.
    ▷ S\{u}: the remaining nodeset without u
    S ← S\{u}
    if ρ(S) > ρ(S*) then
        S* ← S
return S*.
```

Table 5: Dataset source and density of algorithms

Dataset	Source	Type	Pruning	w_app	exact	DLL	uw_Pruning+DLL
ca-HepPh	Stanford's SNAP database	scholar collaboration network	119	119	119	119	119
comm-EmailEnron	Stanford's SNAP database	communication	37.316	37.344	37.344	37.337	37.337
ca-AstroPh	Stanford's SNAP database	scholar collaboration network	28.481	29.616	32.11	29.552	29.552
PP-Pathways	Stanford's SNAP database	protein interaction network	74.159	77.995	77.995	77.995	77.995
soc-Twitter_ICWSM	konect	social network	25.678	25.683	25.69	25.686	25.685
soc-sign_slashdot	Stanford's SNAP database	social network	39.376	42.132	42.132	42.132	42.132
rating-StackOverflow	konect	social network	20.209	20.209	20.21	20.209	20.209
soc-sign_epinion	Stanford's SNAP database	social network	80.168	85.599	85.637	85.589	85.589
ego-twitter	Stanford's SNAP database	social network	59.281	68.414	69.622	68.414	68.414
soc-Youtube	Stanford's SNAP database	social network	45.545	45.58	45.599	45.576	45.577
comm-WikiTalk	Stanford's SNAP database	communication	114.139	114.139	114.139	114.139	114.139
nov_user_msg_time	We own it privately.	social network	278.815	278.815	278.815	278.815	278.815
cit-Patents	AMiner scholar datasets	scholar collaboration network	132.776	135.706	137.261	135.706	135.706
soc-Twitter_ASU	ASU	social network	593.847	593.847	593.847	593.847	593.847
soc-Livejournal	Livejournal	social network	104.596	104.601	104.609	104.603	104.603
soc-Orkut	Stanford's SNAP database	social network	227.861	227.872	227.874	227.872	227.872
soc-SinaWeibo	Network Repository	social network	164.967	165.193	165.415	165.196	165.191
wang-tripadvisor	konect	rating network	13.442	13.873	14.082	–	–
rec-YelpUserBusiness	Network Repository	rating network	87.825	87.912	87.921	–	–
bookcrossing	konect	rating network	92.148	92.322	92.374	–	–
librec-ciaodvd-review	konect	rating network	233.553	233.59	233.597	–	–
movielens-10m	konect	rating network	1351.35	1351.35	1351.35	–	–
epinions	konect	rating network	595.302	595.314	595.316	–	–
libimseti	konect	social network	1645.71	1645.73	1645.73	–	–
rec-movielens	Network Repository	rating network	1801.16	1801.16	1801.16	–	–
yahoo-song	konect	rating network	46725.2	46725.2	46725.2	–	–

note: : **Pruning** (w_Pruning,uw_Pruning). **w_app**: approximation algorihtms on weighted graph(Priority Tree,Pruning+Priority Tree,BBST). **exact**: exact algorithms(maxflow,w_Pruning+maxflow). **DLL**:Doubly-linked list.



(a) soc-Twitter_ICWSM (b) soc-Youtube (c) comm-WikiTalk (d) nov_user_msg_time

(e) cit-Patents_AMINER (f) rec-YelpUserBusiness (g) librec-ciaodvd-review (h) movielens-10m

Figure 6: Comparison on detecting the densest subgraph without pruning.

Table 6: Comparison on detecting the densest subgraph without pruning.

	Running time/s					iteration count T				
Dataset	**LOWD**	**Greedy++**	**FW**	**FISTA**	**FW_M**	**LOWD**	**Greedy++**	**FW**	**FISTA**	**FW_M**
ca-HepPh	0.0168	<u>**0.0159**</u>	0.0208	0.1109	<u>0.0161</u>	1	1	2	3	1
comm-EmailEnron	<u>0.1516</u>	<u>**0.0714**</u>	0.5551	27.5346	0.3012	19	2	88	567	46
ca-AstroPh	<u>**0.3158**</u>	0.8396	2.312	24.2470	<u>0.6593</u>	36	33	343	375	92
PP-Pathways	<u>0.1174</u>	<u>**0.0333**</u>	0.3214	12.6981	1.7086	8	1	18	131	167
soc-sign_slashdot	<u>0.2275</u>	<u>**0.0704**</u>	0.7881	11.6623	1.1051	11	1	44	81	64
soc-sign_epinion	<u>**0.3213**</u>	1.4706	1.1853	11.1319	<u>0.9581</u>	9	12	40	46	33
soc-Twitter_ICWSM	<u>**1.8343**</u>	30.6654	96.3405	1207.2601	<u>6.1467</u>	49	114	2816	4901	175
rating-StackOverflow	<u>2.1887</u>	<u>**1.0537**</u>	10.0728	567.1578	18.7819	32	2	178	1125	341
ego-twitter	<u>**1.6001**</u>	2.9768	22.0971	2567.2520	<u>1.7266</u>	34	21	550	8115	41
soc-Youtube	<u>**6.5253**</u>	133.3086	82.3855	1001.4894	<u>22.0866</u>	46	129	612	929	161
comm-WikiTalk	<u>4.6758</u>	<u>**1.984**</u>	118.307	1933.6041	414.9261	15	1	407	1151	1386
nov_user_msg_time	<u>**341.1144**</u>	4464.4998	2581.9086	9826.2465	<u>718.8805</u>	97	174	624	454	177
cit-Patents_AMINER	<u>**291.4452**</u>	846.3628	<u>837.8446</u>	27554.332	1523.2698	104	56	259	1913	471
soc-Twitter_ASU	<u>120.8906</u>	<u>**19.2912**</u>	511.3170	12174.952	2047.9598	40	1	156	1244	603
soc-Livejournal	–	–	–	–	–	–	–	–	–	–
soc-Orkut	–	–	–	–	–	–	–	–	–	–
soc-SinaWeibo_NETREP	–	–	–	–	–	–	–	–	–	–
wang-tripadvisor	<u>**0.6686**</u>	23.6761	30.209	198.1408	<u>15.6705</u>	93	187	3894	4512	2007
rec-YelpUserBusiness	<u>**0.4621**</u>	24.5879	5.1698	62.8514	<u>2.1192</u>	54	228	626	886	239
bookcrossing	<u>**1.5177**</u>	92.9406	27.4916	910.8345	<u>8.877</u>	78	259	1398	5510	448
librec-ciaodvd-review	<u>**4.2248**</u>	75.7599	231.5822	1882.2756	<u>38.6405</u>	79	143	4635	4339	773
movielens-10m	<u>49.2341</u>	<u>**2.7029**</u>	129.6344	5329.5159	60.719	159	1	450	1611	211
epinions	<u>**30.4071**</u>	3957.2229	4302.242	>37786.731	<u>743.6429</u>	64	688	8893	>10000	1582
libimseti	<u>**25.7101**</u>	5276.5399	751.3750	24832.745	<u>146.2627</u>	39	710	1196	4632	234
rec-movielens	<u>**50.4427**</u>	6928.9143	306.0513	24923.281	<u>148.2539</u>	54	733	355	3195	172
yahoo-song	–	–	–	–	–	–	–	–	–	–

note: We ignore some datasets which are very large.">10000" and ">37786.731" means we run 10000 iterations(running time: 37786.731s) and can't still detect the densest subgraph.