Snowball Option Pricing under Heston Model

CSE 6730 Final Project

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Abstract

With the popularity of financial derivatives, more accurate technical tools are required for the consistent development of financial markets. Traditional models in option pricing fields have caught the behavior of underlying assets with strong assumption that risk-free interest rate and volatility remain constant. In order to release the rigorous assumptions and solve some anomalies, i.e. volatility smile, we adapt the Heston Monte Carlo method to implement option pricing. The system corresponds to the real-world financial market, in which millions of transactions happen every day. In order to make our simulation close to realistic, we abstract the key variables from the financial data from Yahoo. To test the efficiency and practicality of our simulation, we use certain popular exotic option product, which is the Snowball option that enjoys a wide popularity in financial markets nowadays.

The experimental results will be further analyzed and compared to the real-world statistics. We propose to use our enhanced model to replace the old-fashioned Monte Carlo method used in financial companies in order to price the various exotic options more precisely and make it a general way.

Github Repo

1 Project Description

1.1 Research Purpose

We design this project to fulfill two goals:

(1) Compare the results produced by Hetson model to the original model and validate the efficiency of the results.

(2) Considering some of the exotic options are not priced by computation (reasons will be explained in following parts), we want to find a general way to price these options in a reasonable and numerical way, including Snowball options.

1.2 Potential Terms Definition

Here I will try to define some basic terms for who do not have background knowledge about the finance:

1.2.1 Derivatives

The derivatives market refers to the financial market for financial instruments such as futures contracts or options.There are four kinds of participants in a derivatives market: hedgers, speculators, arbitrageurs, and margin traders.There are four major types of derivative contracts: options, futures, forwards, and swaps. We only focus on the option contracts to simplify our life.

Options are financial derivative contracts that give the buyer the right, but not the obligation, to buy or sell an underlying asset at a specific price (referred to as the strike price) during a specific period of time. American options can be exercised at any time before the expiry of its option period. On the other hand, European options can only be exercised on its expiration date. Exotic option is high-leveled and more sophisticated instruments to achieve specific goal as well as expose specific risks.

There are two basic options: call and put. For call options, you will give the buyer to buy the stock at a given price if the stock price rises to reach the strike price, which has the payoff: $P = \max(S - E, 0)$. While for put options, you will give the buyer to sell the stock at a given price if the stock falls to reack the strike price, which is the payoff: $P = \max(E - S, 0)$.

More exotic options refer to complicate the payoff formula by adding different conditions on the movements of the stock. For the Snowball option we are going to focus, more details will be covered in following literature review discussions.

1.2.2 Monte Carlo Method

The basic idea of Monte Carlo method is: in order to solve the problem, first establish a probability model or stochastic process, so that its parameters or numerical characteristics are equal to the solution of the problem; then these parameters or numerical features, finally giving the approximation solved for. The precision of the solution is expressed as the standard error of the estimate. The main theoretical basis of the Monte Carlo method is the theory of probability and statistics, and the main means are random sampling and statistical experiments. The basic steps to solve a practical problem are:

(1) According to the characteristics of the actual problem. A simple and easy-to-implement probability and statistical model is constructed. Make the solution sought exactly the probability distribution or mathematical expectation of the problem;

(2) The sampling methods of various random variables with different distributions in the model are given;

(3) Statistically process the simulation results, and give the statistical estimates and precision estimates of the solution to the problem.

In financial applications, we use Monte Carlo Simulations helping us find the potential stock movements under certain parameters. Then we can add our pricing technique based on this price path. So how to acquire a specific price path is the key to option pricing.

1.3 Realistic Aspects

As we are going to model some real-world phenomenon, there are several conditions we need to consider; and in our project, these conditions also make up some of the parameters we need in our simulation process:

(1) Volatility of the market: not only do we need to care about the volatility of the target stock, but also the market; the correlation between the stock and volatility will also affect the implementation of the model;

(2) Political factors: as we are researching on the Chinese stock market, the politics will play a large role in the impacts of the stock market;

(3) Pandemic influence: as we are experiencing a pandemic period so the stock will possibly move in an unexpected way so when simulating, we need to add some factors representing the effect from Covid-19 to better help us get to the real price of the option in numerical ways.

2 Literature Review

2.1 Background

Exotic options are a category of options contracts that differ from traditional options in their payment structures, expiration dates, and strike prices. A traditional options contract gives a holder a choice or right to buy or sell the underlying assetat an established price before or on the expiration date. However, exotic option can vary in terms of how the payooff is determined and when the option can be exercised. Therefore, these options are generally more complex than plain vanilla call and put options. For plain vanilla options, many pricing theorem are developed. The most comprehensive one is

Black Scholes model. Under some essential assumption, the European option could be formulated as $\frac{df}{dt} + \mu S \frac{df}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2} = rf$. Nevertheless, it relies heavily on a number of assumptions that are, to some extent, unrealistic. Among these are, the ability to trade (hedge) continuously, no transaction costs, constant volatility, continuity of stock price process, independent Gaussian returns, etc.

In order to simulate underlying movement, we need to figure out the volatility. We shall focus first on the issue of "constant volatility". The Black-Scholes model is used to derive implied volatilities from observed option prices. If stock price followed Black-Scholes model, in an arbitrage-free market, these implied volatilities should be independent of exercise prices and time to maturity and also constant over time. However, these volatilities vary systematically, creating phenomena such as "volatility smile (smirk)" and the "term structure of volatility". Black and Scholes themselves found that while their main results seemed to be supported, variance was changing over time. In general, all stochastic models that try to account for a changing variance can be divided into two main categories: deterministic volatility models and stochastic volatility models. A deterministic volatility model is the Black-Scholes-Merton model since it assumes constant volatility of returns over an infinitesimal time period. Stock prices have the Markov property and follow a geometric Brownian motion. The constant elasticity of variance model was developed by Cox and Ross [1] .Volatility is dependent on both the asset price and time and the process followed by stock is,

$$\frac{dS}{S} = \mu dt + \sigma S^{\rho-1} dz$$

In other models, volatility can be just a function of time, . The volatility process can be a simple diffusion, a jump process or a jump-diffusion one. The main difference from the previous deterministic models is that volatility is modeled so that it has a random component of its own.

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t$$

Hull and White proposed a model for stochastic volatility that variance is assumed to follow a log-normal process similar to the one traditionally followed by the asset price.

Another paper proposes the processes followed by the asset price and the volatility are:

$$dS = \mu S dt + \sigma S^{\alpha} dz$$
$$d\sigma = \mu \sigma dt + \xi \sigma^{\beta} dz_S$$

This implementation of the model is based on Monte Carlo. Scott uses an independent diffusion process with mean-reversion to model the random changes in implied volatilities from one day to the other. He also proposes a second Ornstein-Uhlenbeck process for the log volatility.

$$dS = \mu S dt + \sigma S dZ_1$$
$$d\sigma = \mu(\hat{\sigma} - \sigma)dt + \xi S dz_2$$

2.2 Options

We decide to give a brief description about the common exotic options traded in financial markets, which might serve as models for our targeted simulation process.

2.2.1 Chooser options

Chooser options[2] allow an investor to choose whether the option is a put or call during a certain point in the option's life. Both the strike price and the expiration are usually the same, whether it is a put or call. Chooser options are used by investors when there might be an event such as earnings or a product release that could lead to volatility or price fluctuations in the asset price.

2.2.2 Compound options

Compound options[3] are options that give the owner the right, not the obligation, to buy another option at a specific price on or by a specific date. Typically, the underlying asset of a traditional call or put option is an equity security. However, the underlying asset of a compound option is another option.

2.2.3 Barrier options

Barrier options[4] are similar to plain vanilla calls and puts, but only become activated or extinguished when the underlying asset hits a preset price level. In this sense, the value of barrier options jumps up or down in leaps, instead of changing price in small increments.

2.2.4 Binary options

A binary option[5], or digital option, pays a fixed amount only if an event or price movement has occurred. Binary options provide an all-or-nothing payout structure. Unlike traditional call options, in which final payouts increase incrementally with each rise in the underlying asset price above the strike, binaries pay a finite lump sum if the asset is above the strike. Conversely, a buyer of a binary put option is paid the finite lump sum if the asset closes below the stated strike price.

2.2.5 Lock-back options

Look-back options[6] do not have a fixed exercise price at the beginning. Instead, the strike price resets to the best price of the underlying asset as it changes. The holder of a look-back option can choose the most favorable exercise price retrospectively for the period of the option. Look-backs eliminate the risk associated with timing market entry and are typically more expensive than plain vanilla options.

2.2.6 Asian options

Asian options[7] take the average price of the underlying asset to determine if there is a profit as compared to the strike price. For example, an Asian call option might take the average price for 30 days. If the average is less than the strike price at expiration, the option expires worthless.

2.2.7 Range options

Range options[8] have a payoff based on the difference between the maximum and minimum price of the underlying asset during the life of the option. These options eliminate the risks associated with the entry and exit timing, making them more expensive than plain vanilla and look-back options.

3 Conceptual Model

Snowball structures are path-dependent exotic derivatives. It is actually a put option sold with a knock-in structure. As long as the underlying does not fall sharply, the longer you hold the income certificate, the more coupon income you will get. Before pricing this popularized product using our method, we decide to look into details about the payments equations. The payoff will be dependent on these five conditions:

(1) If on a certain monthly observation day, the closing price of the underlying is higher than its knock-out price, the transaction ends early and the client receives an annualized coupon.

(2) If the target does not touch the knock-in and knock-out boundaries on any trading day and observation day during the duration, the contract will automatically end when it expires, and the client will still receive an annualized coupon.

(3) If on any trading day of the duration,the target has fallen below the knock-in limit, and no knock-out event has occurred on all observation days before expiration: in this way, if the price of the target at expiration is higher than the price of the target at the beginning of the period, the customer can get all the principal without other income.

(4) If on any trading day of the duration, the underlying has fallen below the knock-in limit, and no knock-out event has occurred on all observation days before expiration: if the price of the underlying at expiration is lower than the price of the underlying at the beginning of the period, the client will bear the total loss corresponding to the nominal principal caused by the decline of the underlying market value.

(5) If on any trading day of the duration, the underlying has fallen below the knock-in limit, and the knock-out event occurs again on an observation day before the expiration, the transaction will end ahead of schedule and the client will receive an annualized coupon.

In conclusion:

Table 1: Payoffs of the Snowball Option

Payoff Type	Knock-out	Knock-in without knock-out	Nothing happened
Amount	C * P * d / 365	min(S2/S1 - 1, 0) * P	C * P * d / 365

Table 2: Meaning of letters

Detonation	P	C	S1, S2	d
Meaning	Principal	Coupon rate	Start price, End price	Days of trading

Such complicated structures require specific pricing methods. However, in nowadays, when implementing Monte Carlo simulations, analysts usually take a constant volatility without considering the subtle fluctuations that might lead to unpredictable results. In this way, the Heston model which more accurately describes the path of stock prices and volatility will yield a better outcome. What's more, the convenient implementation of simulations will save much time than the traditional PDE method when pricing options. For the next period of this project, we will move on to the details of snowball option pricing with Heston model.

What's more important, nowadays the snowball option makers are willing to sell the option in a nearly 0 price. We are not going into details about how they profit. All we need to know is that this option is make use of volatility to make profits and fortunately, the Heston model we chose just suits our purpose.

4 Simulator

4.1 Theoretical knowledge

As mentioned previously, Heston model was one of the first models that allowed a calibration to real market data using thee semi-closed form solution for European call and put option prices.

In Heston model, one can also consider a correlation between the asset price and the volatility process as for example opposed to Stein and Stein. Furthermore, the model allows for stochastic interest rates and can be used for pricing bond and foreign currency options.

$$dS(t) = \mu S dt + \sqrt{v(t)} S dz_1(t)$$
$$dv(t) = k(\theta - v(t))dt + \gamma\sqrt{v(t)} dz_2(t)$$
$$dz_1(t)dz_2(t) = \rho dt$$

where parameters represent:

$$k - \text{speed of mean reversion}$$

$$\theta - \text{long-term volatility level}$$

$$\gamma - \text{volatility of the volatility}$$

$$\rho - \text{two Brownian motions share a correlation coefficient}$$

One of the process we made is to implement the Heston model as python function. Based on object-oriented programming (OOP) philosophy, we create Monte-Carlo simulator to apply the idea of Heston differential equation, create stock movement path, aggregate the results and finally reach our goal of pricing path-dependent exotic options. The simulator for vanilla (naive) call and put is in 'Simulator.py'. We also make some improvement by implementing the pricing technique on an emerging kind of exotic option, autocallable option (snowball), which is in 'Snowball.py'.

4.2 Simulator construction

So far, we have successfully constructed the Heston model for the prediction of asset prices. Take the European call options for example: we will simulate both the asset prices and the corresponding volatility; then we shall calculate the prices based on the payoff equation. Here are the results:

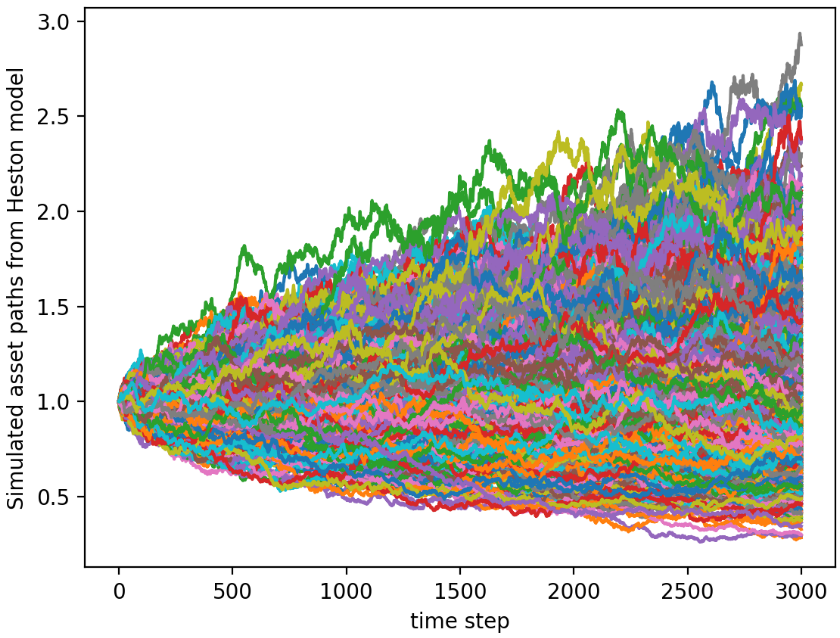

Figure 1: Simulated prices

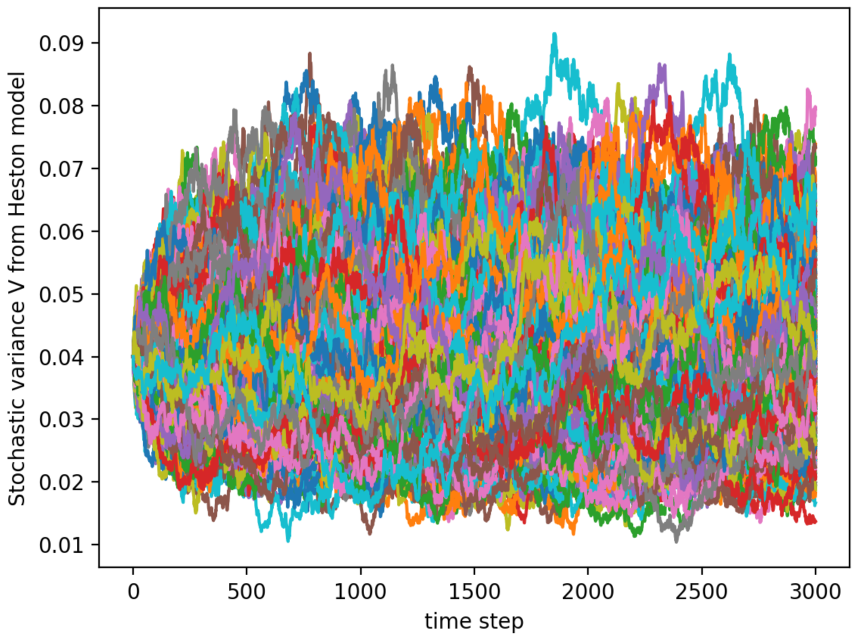

Figure 2: Simulated variances

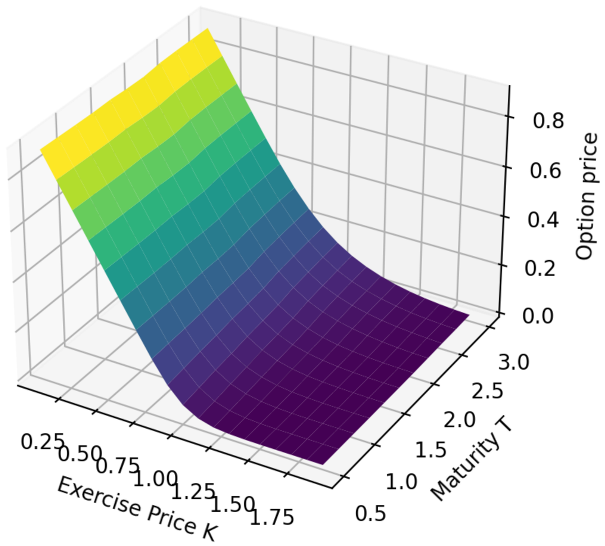

Figure 3: Payoff diagram across exercises and maturities

5 Experimental Results

5.1 Snowball pricing

To conduct our research and use all our previous work, we choose SP500 Index(000905.SH) as our target stock and use last one year's time to help us calculate necessary parameters in the model. Here is a simple glimpse of the stock movements:

Figure 4: SP 500 Index

One important thing worth mentioning: Since there is no price for a snowball option contract, investors focus on the coupon rate to evaluate the premium for it. At the same time, issuers avoid losing money by adjusting the coupon rate. Under the condition that all terms are fixed, after adjusting the discount rate of the underlying, find the coupon rate used to calculate the price of the snowball when it tends to zero, which is the fair price.

So what we are going to achieve is that we want to have a numerical solution of price 0 for the option. Before that, under realistic data, we did a back-test for the snowball option: (numbers represent probabilities)

Table 3: Snowball option back-test results

Index	K-out	K-in	Avg months (K-out)	Avg months (Duration)	Expiration
Number	78.38%	17.12%	4.56	4.76	4.5%

First of all, the buyer of the snowball product is shorting volatility, so choosing a time window during which volatility falls during the duration is more conducive to gaining profits. Secondly, among the five scenarios of the snowball structure listed in the previous chapter, there is only one that will cause a loss of principal, that is, the target falls below the knock-in boundary. Therefore, if investors can predict that the price of the target will not occur during the duration Larger downturns can also be invested to obtain coupon income. In real-world trading, Snowball products with a 12-month maturity will be knocked out in an average of 4.5 months, with only a 4.5% probability of holding to maturity and receiving full coupon income. About 70% of the positive returns of the snowball structure are within 10%, and its distribution is typical with peaks and thick tails, and it is obviously left-biased. Its return characteristics are very similar to that of short-selling volatility-related options strategies, and a large loss occurs with a small probability. , in exchange for stable returns with high probability. This type of product is very suitable for continuous investment under the condition that there is no obvious market trend and long-term fluctuations in a narrow range to obtain coupon income.

So given the condition that our Heston model reveals many details about the volatility, we believe that if we could get a series of accurate number describing the volatility of both the stock and the market, we could yield to a price of 0 for a snowball option.

Table 4: Heston model pricing parameters

Parameter	mean-reversion rate	long-run variance	volatility of volatility	correlation
Number	0.003	0.04	0.3	-0.15

After running 10000 paths and pricing process, we have the following results:

Table 5: Pricing results

Type	Knock out	Knock in	Hold to maturity	Net present value
Number	7089	1765	1146	1.7286

We are very close to the realistic result.

5.2 Model validation

The philosophy of validation for our model is to check whether the price derived by our simulation model close to the real world price as well as analytical price, since the Heston model has exact vanilla option's analytical solution.

Firstly, based on one real world snowball option contract, our goal is to get an 0 net present value (NPV). The reason behind the zero NPV is no-arbitrage principle. This principle asserts that two securities that provide the same future cash flow and have the same level of risk must sell for the same price. Since every contract has two counter-parties, if NPV is not equal to zero, investors will take advantage of this arbitrage opportunity by buying low side and selling high side without taking any risk. In our experiment, we find that the simulated prices converge and close to zero.

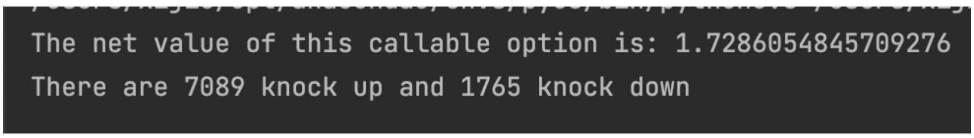

Figure 5: Net present value of snowball contract simulation

Secondly, we use plain vanilla option to check whether the model is suitable to describe the behavior of underlying instruments. The method is called calibration. We implemented the Heston model to fit the implied volatility surface. In vanilla option field, implied volatility represents the characters of the future movement. Results turned out to be in perfect fitting. That is the reason of population of Heston model in modern financial world.

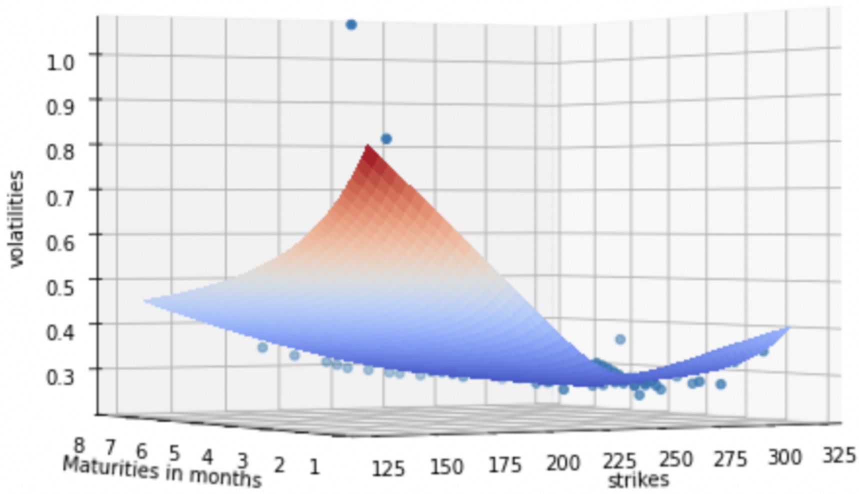

Figure 6: Implied volatility surface fitting

6 Conclusions

The Heston model simulation reaches our goals, i.e. comparing with original model, validating the efficiency and implementing to price exotic options (snowball contract). The experimental results satisfies the real world logic and are more efficient and versatile. We get zero net present value and prove the absence of arbitrage opportunities using one of the popular financial derivatives in current market. We show the advantage over original model, GBM, that Heston model can solve volatility smile and simulate more dynamic paths. We have the confidence to use our enhanced model to replace the old-fashioned Monte Carlo method used in financial companies in order to price the various exotic options more precisely and apply it as a general way.

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