Expected Value and Variance (Con't)

Selected Topic from STAT 220/230

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Recap

Recap: Expected Value

Let X be a random variable with $Range(X) = \mathcal{A}$ and probability mass/density function $f(x)$. Let $g(X)$ be some function of X.

The **expected value** of $g(X)$ is given by

$$\mathbb{E}(g(X)) = \begin{cases} \sum_{x \in \mathcal{A}} g(x)\, f(x) & , \quad \text{if } X \text{ is discrete} \\ \\ \int_{x \in \mathcal{A}} g(x)\, f(x) dx & , \quad \text{if } X \text{ is continuous} \end{cases} .$$

(**Law of the Unconscious Statistician**)

We also call it the **mean** or the **expectation** of $g(X)$.

In the special case when $g(x) = x$, the above formula simply gives the expected value of X, denoted by $\mathbb{E}(X)$.

Recap: Variance

The **variance** of X is given by

$$\sigma^2 = \mathrm{Var}(X) = \mathbb{E}\left[\Big(X - \mathbb{E}(X)\Big)^2\right] = \mathbb{E}\left[(X-\mu)^2\right] \, ,$$

where $\mu = \mathbb{E}(X)$.

An often more useful (easier-to-use) formula is

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \mathbb{E}(X^2) - \mu^2 \, ,$$

proven using the linearity of the expected value:

$$\mathbb{E}(aX + Y) = a\mathbb{E}(X) + \mathbb{E}(Y) \, , \text{ with } a \in \mathbb{R} \text{ being not random} \, .$$

Recap: Means and Variances of Common Distributions

$$X \sim \textit{Binomial}(n, p) : \ \mathbb{E}(X) = np \ , \ \mathrm{Var}(X) = np(1-p)$$

$$X \sim \ \textit{Poisson}(\lambda) : \ \mathbb{E}(X) = \mathrm{Var}(X) = \lambda$$

$$X \sim \ \textit{Geometric}(p) : \ \mathbb{E}(X) = \frac{1-p}{p} \ , \ \mathrm{Var}(X) = \frac{1-p}{p^2}$$

$$X \sim \textit{Unif}[a, b] : \ \mathbb{E}(X) = \frac{a+b}{2} \ , \ \mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

$$X \sim \textit{Normal}(\mu, \sigma^2) : \ \mathbb{E}(X) = \mu \ , \ \mathrm{Var}(X) = \sigma^2$$

Application of Expected Value: Winnings in a Lottery Example

Winnings in a Lottery

Example:

A small lottery sells 1000 tickets numbered 000, 001, 002, ..., 997, 998, 999. The tickets cost $10 each. When all the tickets have been sold, a single ticket from 000 to 999 is chosen at random and is used as the winning ticket.

For ticket holders, the prize structure is as follows:

- Your ticket is drawn (same number) - win $5000
- Your ticket has the same first two number as the winning ticket, but the third number is different - win $100
- Your ticket has the same first number as the winning ticket, but the second number is different - win $ 10
- All other cases - win nothing

We are interested in the **expected winnings** per ticket.

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Solution:

Let X represents the winnings from a given ticket.

We are interesting in $\mathbb{E}(X)$.

The possible values for X are 0, 10, 100, 5000 (in dollars).

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$$f(0) = 0.9\ ,\ f(10) = 0.09\ ,\ f(100) = 0.009\ ,\ f(5000) = 0.001\ .$$

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The expected winnings (per ticket) are thus:

$$\mathbb{E}(X) = (0 \cdot 0.9) + (10 \cdot 0.09) + (100 \cdot 0.009) + (5000 \cdot 0.001) = \mathbf{6.80}\ .$$

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Now,...what are the net winnings?

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The expected winnings are thus the expected value of X:

$$\mathbb{E}(X) = (0 \cdot 0.9) + (10 \cdot 0.09) + (100 \cdot 0.009) + (5000 \cdot 0.001) = 6.80\ .$$

The tickets cost \$10 each $\rightarrow$ Your **expected net winnings** are $6.80 - 10 = \mathbf{-3.20}$ (per ticket).

Winnings in a Lottery

Remark:

On average, you lose \$3.20 for each ticket you purchased in this lottery.

For any game of this kind, the expected net winnings per play are crucial. If it is a fair game, the expected net winnings should be 0.

As you can anticipate, casino games and lotteries are almost never fair games.

Applications of Expected Value

Other Examples of Application of Expected Value:

- The average lifetime of a light bulb (Exponential)
- The average waiting time for each customer at a restaurant until they are served (Exponential or Geometric)
- The average number of cracked eggs in a shelf of eggs (Binomial)
- The average magnitude of measurement errors in a scientific equipment (Normal)

Visualizing Mean and Variance using R

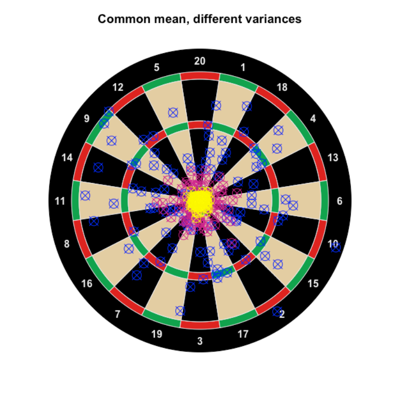
Visualizing Mean and Variance

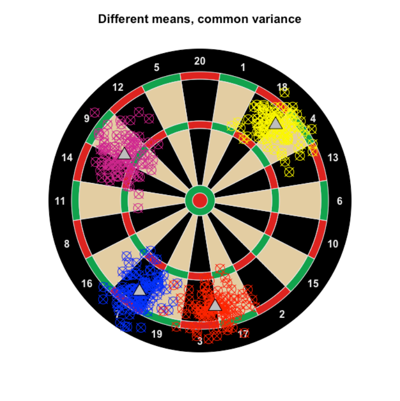
Expected Value: the ”centre” of the data

Variance: the average of squared distances from the ”centre” of the data

Let’s visualize these in R...

Visualizing Mean and Variance





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